

2. Synthetic and analytic geometry

Human activity and thought are extremely complex historical processes, in which many conflicting tendencies unfold. This also holds for mathematics, in which the unfolding of these conflicts is an important element pushing development forwards. In the process of mathematical research, as well as in the mathematical method itself, dialectical conflicts are of fundamental significance : analytic-synthetic, axiomatic-constructive, exact-intuitive, abstract-concrete, special-general, simple-complex, finite-infinite, regular-singular, algebraic-geometric, qualitative-quantitative. All these conflicts have had a marked influence on many fields of mathematics. I cannot go further into details here, so I shall refer to my essay on dialectic in mathematics [B5], as well as to the article by Alexandroff in the same book.

In what follows we shall confine ourselves to showing the unfolding of the conflict between analytic and synthetic in the development of the theory of plane curves.

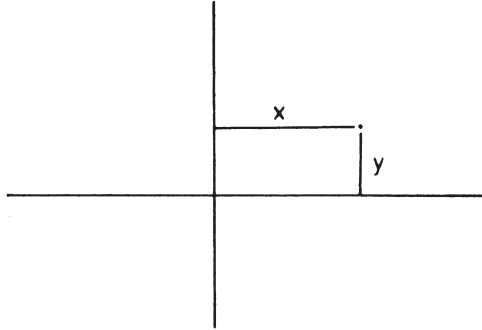
2.1 Coordinates

The introduction of coordinates into geometry and the development of analytic geometry are usually attributed to Descartes and Fermat. This is, in the main, correct. However, it is important to emphasize that this brilliant achievement, which decisively affected the further development of mathematics, was made possible by earlier works which prepared the way for it – as indeed is the case with all brilliant achievements in human history. This is not to deny the novelty, the jump in evolution, but merely to conceive it as part of a historical process.

The idea of using coordinates in land measurement and city planning appears to have been the basis of Egyptian and Roman land surveying (for this and what follows see, e.g., Smith, History of Mathematics [S6], II, p. 316). The Greek geographers and astronomers, e.g. Hipparchus and Ptolemy, used degrees of longitude and latitude to describe points on the surface of the earth and in the sky.

The Greek geometers, in discussing curves, used relations between segments appearing in the construction which amount, from a present-day standpoint, to equations for the curves in cartesian coordinates. However, this analytic element was not fully developed : the basic idea of

each analytic method is the reduction of a system to a few basic elements. The advantage of simplification gained in this way may possibly be opposed by the disadvantage of complexity in the reconstruction of the system from the basic elements. In the case of analytic geometry the reduction consists in choosing two perpendicular lines in the plane and determining the position of points, on a curve for example, by their distances x , y from these lines.



More generally, one can take any two intersecting lines as coordinate axes.

The simplicity of the reduction of the description to a position relative to only two lines is opposed by the complexity of the equation $f(x,y) = 0$ for the coordinates x , y of the points of the curve. The Greeks, whose thinking was perhaps more synthetic than analytic, preferred to use many auxiliary lines, in order to attain simple relationships.

The use of coordinates developed further in the Middle Ages, particularly in the work of Oresme (c. 1360), where there are already attempts to represent functions by graphs. Kepler and Galilei were influenced by this.

The decisive advantage of the analytic method, the reduction of qualitative geometric relations to complex quantitative relations – equations between coordinates – could only come to light after methods for handling such quantitative relations, i.e. algebra, had been developed. This development was due mainly to the eastern mathema-

ticians of the Middle Ages, Arabs, Persians and Indians.

With the decline of the Roman empire came a period of socio-economic and cultural degeneration and poorer communication with the Orient, then trade began to develop between the blossoming Italian merchant cities and the Arab world, European scholars and traders such as Leonardo of Pisa (known as Fibonacci) (c. 1200-1220) studied the eastern culture, among other reasons "to put to use in their mercantile civilisation, which already in the twelfth and thirteenth century had seen the growth of banking and the beginnings of a capitalistic form of industry" (for this and what follows cf. D. Struik [S8], pp. 86-129*). Fibonacci cited the Arab algebraists and contributed to the gradual spread of Arabic numerical calculation in the account books of European merchants and bankers. This development strengthened in the next 300 years in the mercantile towns, growing under the direct influence of trade, navigation, astronomy and surveying. The townspeople were interested in numbers, arithmetic and calculation. "The fall of Constantinople in 1453, which ended the Byzantine empire, led many Greek scholars to the western cities. Interest in the original Greek texts increased and it became easier to satisfy this interest. University professors joined with cultured laymen in studying the texts, ambitious reckon masters listened and tried to understand the new knowledge in their own way" (quotation from Struik). The first objective was to pick up the old knowledge of the Greeks and Arabs. But the Renaissance was also a new age : "Characteristic of the new age was the desire not only to absorb classical information but also to create new things, to penetrate beyond the boundaries set by the classics". In mathematics, algebra was developed far beyond that of the Arabs.

At the beginning of the 16th century, Scipio del Ferro at the University of Bologna solved the cubic equation

$$x^3 + px = q$$

for non-negative p, q by

$$x = \sqrt[3]{\frac{p}{27} + \frac{q^2}{4}} + \frac{q}{2} - \sqrt[3]{\frac{p}{27} + \frac{q^2}{4}} - \frac{q}{2} .$$

* In the English edition, pp. 98-123 (Translator's note).

Ferrari reduced the equation of degree 4 to a cubic, Bombelli introduced a theory of pure imaginary numbers in his "Algebra" of 1572, in connection with the investigation of cubic equations, and Vieta (1540-1603) perfected the theory of equations, and was one of the first to denote the constants and unknowns in equations by letters. This brought algebra to a level of development at which it could be applied to geometry.

2.2 The development of analytic geometry

In Europe, the value of algebra for the solution of geometric problems was already known to Leonardo of Pisa in his "Practica geometriae" (1220), following the Arab mathematicians who had earlier known and used the relation between algebraic and geometric problems. Conversely, geometric methods were used for the solution of equations, e.g. by Cardano (1545). The relation between algebra and geometry later became common knowledge, e.g. with Vieta and Ghetaldi (1630).

But it was Fermat and Descartes who first arrived at a programme, a general method for the treatment of geometric problems by algebraic-analytic methods, i.e. at analytic geometry.

Fermat had the idea of analytic geometry in 1629. The details were first published posthumously. He had the rectangular coordinate axes as well as the equations $y = mx$ for lines and $x^2 + ay^2 = b^2$ for the conic sections. He knew that the extrema of a function $y = f(x)$ lie where the tangents to the corresponding curve in the (x,y) -plane are parallel to the x -axis.

Descartes published his geometry in 1637, but had worked on it earlier, perhaps since 1619. This is not the place to thoroughly assess Descartes' achievement within the general social development of his time. I content myself with a few quotations from the book of Struik already cited : Descartes' "Geometry" "brought the whole field of classical geometry within the scope of the algebraists". The book was originally published as an appendix to the "Discours de la Méthode"*, the "discourse on reason in which the author explained his rationalistic approach to the study of nature". ([S8])

In accordance with many other great thinkers of the seventeenth

* Complete title : Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences.

century, Descartes searched for a general method of thinking in order to be able to facilitate inventions and to find the truth in the sciences. Since the only known natural science with some degree of systematic coherence was mechanics, and the key to understanding of mechanics was mathematics, mathematics became the most important means for the understanding of the universe. Moreover, mathematics with its convincing statements was itself a brilliant example that truth could be found in science "... Cartesians, believing in reason ... found in mathematics the queen of the sciences".

His "Geometry" actually contains little analytic geometry in the modern sense, no "cartesian" axes and no derivations of the equations of conic sections as in Fermat. Nevertheless, the influence of this work on the development of geometry has not been overestimated. Descartes' programme and merit is the "consistent application of the well-developed algebra of the early seventeenth century to the geometrical analysis of the ancients, and by this, an enormous widening of its applicability".

A further merit of Descartes is the following : the Greeks and subsequent mathematicians had indeed considered not only lengths x , y of segments but also their products, such as x^2 , x^3 , xy etc., but the latter were not regarded as numbers of the same type, i.e. segments. Descartes abolished this distinction : "An algebraic equation became a relation between numbers, a new advance in mathematical abstraction necessary for the general treatment of algebraic curves, which one can regard as the final adoption of the algorithmic-algebraic tradition of the east by the west".

Thus Descartes' achievement lay in a unification of the numerous attempts already existing and in the conception of a general, quasi-mechanical method for the solution of geometric problems. Unlike the mathematicians before him he does not want to investigate just individual curves, he wants a general method for the investigation and classification of all curves. This is clearly expressed in the following quotation from him :

"I could give here several other ways of tracing and conceiving curved lines, of ever-increasing complexity ; but in order to comprehend all those which occur in nature and to separate them by order into certain genera, I know of no better way than to say that, for those we may call "geometric", that is those which are determined by some precise and exact measure, their points must bear a certain relation to the points of

a straight line which can be expressed by a single uniform equation."

Thus Descartes sees the equation of a curve as the starting point of this general method. The method itself consists in the application of algebra to this equation.

The execution of this programme by Descartes, Fermat and their contemporaries, by Wallis, Pascal, Newton and Leibniz, was however not carried out by algebraic methods alone, but also by those which led to infinitesimal calculus – infinitesimal calculations in which limit processes complemented the algebraic methods.

The development of infinitesimal calculus was made possible by numerous and very varied earlier developments, which we cannot describe adequately here. These preparatory developments included investigations of curves such as the calculation of arc lengths (rectification), areas and volumes, as well as centres of mass, which go back to ancient traditions and from a modern viewpoint amount to calculation of integrals.

To these were added the investigations of Fermat and others on problems such as the tangent problem, which from our viewpoint amount to calculation of derivatives. Descartes' method consists in transforming these problems, which had previously been handled by geometric methods and more or less strict limit arguments, into problems which could be solved by algebraic calculations and, if necessary, by limit arguments. In the case of limits, each separate case was handled by a new and different argument, until Leibniz and Newton, with the differential and integral calculus, found a uniform method for handling all these problems. Leibniz conceived infinitesimal calculus to be, among other things, a method for unifying the different investigations of curves. Struik : "The search for a universal method by which he could obtain knowledge, make inventions, and understand the essential unity of the universe was the mainspring of his life."

Thus it was that the geometry of plane curves became analytic geometry, the investigation of the equations defining curves by algebraic and analytic methods. This was the starting point for two hundred years of development, after which the tendency to synthesis again came to the fore.

2.3 Equations for curves

We shall present equations for some of the curves considered earlier.

The equation for a line is of course

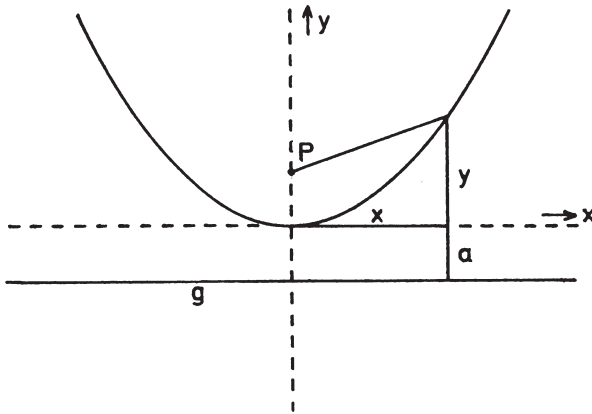
$$ax + by + c = 0.$$

The equation for a circle with centre (x_0, y_0) and radius r is

$$(x-x_0)^2 + (y-y_0)^2 = r^2,$$

as follows immediately from Pythagoras' theorem.

The parabola is the locus of all points equidistant from a line g and a point P . We choose the line through P and perpendicular to g as the y -axis and the perpendicular bisector of the perpendicular from P to g as the x -axis.



When the distance from P to g equals $2a$, the points of the parabola satisfy

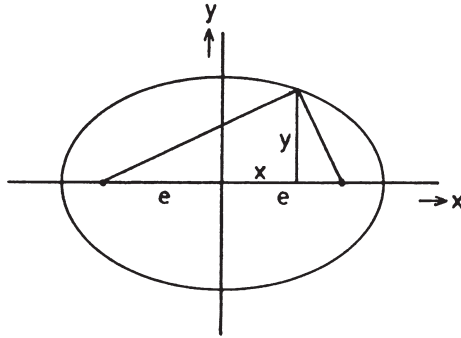
$$(y+a)^2 = (y-a)^2 + x^2$$

by Pythagoras' theorem, and we obtain the equation of the parabola as:

$$x^2 - 4ay = 0.$$

For the ellipse and hyperbola we choose the x -axis to be the line

through the foci, and the y -axis as their perpendicular bisector. Let $2e$ be the distance between the foci, and let c be the sum resp. difference of the distances from the foci.



It follows immediately from Pythagoras' theorem and the curve definition that:

$$\sqrt{(x+e)^2 + y^2} \pm \sqrt{(e-x)^2 + y^2} = c.$$

For the ellipse one has the plus sign and $c > 2e$, for the hyperbola the minus sign and $c < 2e$. Squaring both sides one obtains

$$(x+e)^2 + y^2 + (x-e)^2 + y^2 - c^2 = \mp 2 \sqrt{(x^2 + y^2 + e^2)^2 - 4x^2 e^2},$$

and squaring again :

$$(4c^2 - 16e^2)x^2 + 4c^2 y^2 = c^2(c^2 - 4e^2).$$

If one sets $c = 2a$ and $4c^2 - 16e^2 = \pm 16b^2$, one obtains the equation for the ellipse resp. hyperbola

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

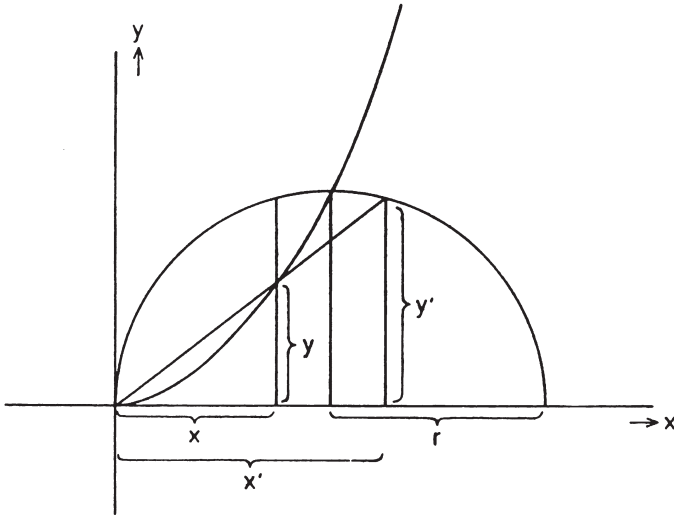
If one does not choose the axes in the special way described above, but arbitrarily, and if one also admits degenerate conic sections, i.e. line pairs or double lines, then one obtains the general equation of a

conic section as

$$ax^2 + by^2 + cxy + dx + ey + f = 0,$$

i.e. an equation $f(x,y) = 0$ where $f(x,y)$ is a polynomial of degree 2 in the variables x and y .

To present the equation of the cissoïd we recall the figure appearing in the definition, to which we have added coordinate axes.



With the notation shown on the figure, the coordinates x, y of a point on the cissoïd satisfy :

$$\frac{y}{x} = \frac{y'}{x'}$$

$$x' = 2r - x$$

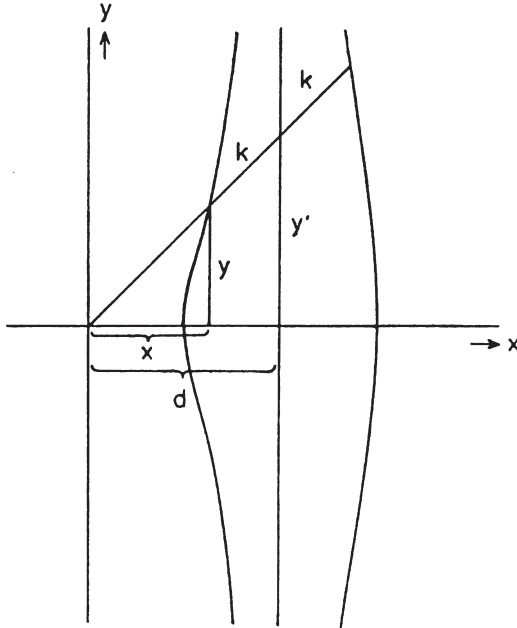
$$y' = \sqrt{r^2 - (r-x)^2}.$$

Squaring both sides of the first equation and substituting the other two yields the equation of the cissoïd :

$$y^2(2r-x) - x^3 = 0,$$

and thus an equation $f(x,y) = 0$, where $f(x,y)$ is a polynomial of degree 3 in x and y .

We establish the equation of the conchoid in quite an analogous way. We choose coordinate axes and notations as in the following figure.



Then :

$$\frac{y}{x} = \frac{y'}{d}$$

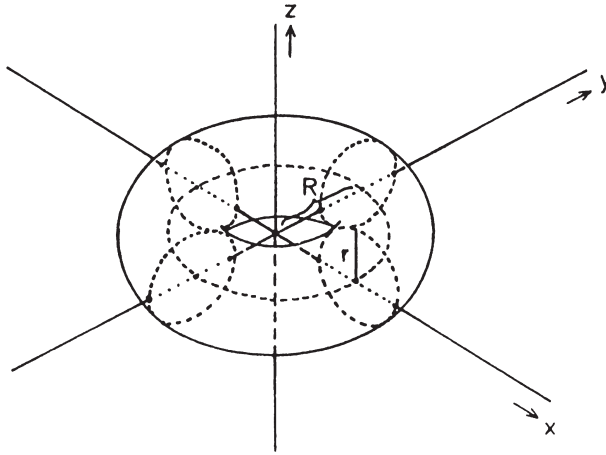
$$(y' - y)^2 + (d - x)^2 = k^2.$$

An easy calculation gives the equation of the conchoid :

$$(y^2 + x^2)(d - x)^2 - k^2 x^2 = 0.$$

Thus it is an equation $f(x,y) = 0$ where f is a polynomial of degree 4 in x and y .

The spiric sections of Perseus result from cutting a torus by a plane. We therefore begin by setting up the equation of a torus relative to cartesian space coordinates x, y, z . The torus results from rotation of a circle about the z -axis. The circle lies in a plane which contains this axis, and has radius r . Its centre lies in the (x,y) -plane and has distance R from the z -axis.



Then we obviously have

$$(R - \sqrt{x^2 + y^2})^2 + z^2 = r^2,$$

hence

$$R^2 + x^2 + y^2 + z^2 - r^2 = 2R\sqrt{x^2 + y^2}.$$

Thus one obtains the equation

$$(R^2 - r^2 + x^2 + y^2 + z^2)^2 - 4R^2(x^2 + y^2) = 0$$

for the torus. If one now sets y equal to a constant c , then one obtains the equation of the spiric sections :

$$(R^2 - r^2 + c^2 + x^2 + z^2)^2 - 4R^2(x^2 + c^2) = 0.$$

This is an equation $f(x,y) = 0$ where $f(x,y)$ is a polynomial of degree 4.

It is likewise easy to set up the equation of the Cassini curves on the basis of the definition. When one does this one sees that the equation coincides with the one above for $c = r$. In this way one easily sees that the Cassini curves are special cases of the spiric sections and with this we have the first example of the effectiveness of the analytic method.

The epicyclic curves are most simply described by a parametric representation

$$\begin{aligned} x &= R \cos m\phi + r \cos n\phi \\ y &= R \sin m\phi + r \sin n\phi. \end{aligned}$$

When $\frac{m}{n}$ is rational one can easily derive an equation for the curve from this. We can assume without loss of generality that m, n are non-negative integers, because if not there are integers m', n', q with $m = \frac{m'}{q}$, $n = \frac{n'}{q}$, and when we use $\phi' = \phi/q$ as parameter we obtain a parametric representation with integral m', n' .

It now follows easily by iteration of the addition theorems for trigonometric functions that $\cos m\phi$ and $\sin m\phi$ are polynomial functions of $\cos \phi, \sin \phi$. For example

$$\begin{aligned} \cos 2\phi &= \cos^2 \phi - \sin^2 \phi \\ \sin 2\phi &= 2 \sin \phi \cos \phi. \end{aligned}$$

If one then sets $t = \tan \frac{\phi}{2}$,

$$\sin \phi = \frac{2t}{1+t^2}, \quad \cos \phi = \frac{1-t^2}{1+t^2}.$$

Using the parameter t in place of ϕ , one now obtains a parametrisation for the epicyclic curve :

$$\begin{aligned} x &= R_1(t) \\ y &= R_2(t) \end{aligned}$$

where the $R_i(t)$ are rational functions of t , i.e. quotients $P_i(t)/Q_i(t)$ of polynomials. The points of our curve are therefore the points (x,y) for which there is a t which is a common zero of the polynomial equations

$$\begin{aligned} P_1(t) - xQ_1(t) &= 0 \\ P_2(t) - yQ_2(t) &= 0. \end{aligned}$$

But it is well known that this holds precisely for the (x,y) for which the resultant of these two polynomials, which is of course a polynomial $f(x,y)$ in x and y , vanishes. Thus $f(x,y) = 0$ is the desired equation for our epicyclic curve for rational m/n .

For irrational m/n and arbitrary coordinates ξ, η in the plane there can be no continuous equation $f(\xi, \eta) = 0$ whose zero set is the epicyclic curve, because the points of this curve lie densely in an annulus. But when one works with polar coordinates, which are essentially coordinates in the universal covering of the original punctured plane, then the curve has an equation of the form

$$\phi^2 - a + b \cos \phi = 0.$$

Thus one sees that these curves, despite having the same type of generation and trigonometric parametrisation, are very different from the point of view of equations. One satisfies a polynomial equation and has a very nice, namely rational, parametrisation, the other has no reasonable equation at all in the plane, and only a transcendental equation in the covering.

Similar remarks apply to the Lissajou curves. For a rational frequency ratio the two vibrations satisfy an algebraic equation, which one can derive by the method above, for an irrational ratio they do not.

Finally we mention the equation of the cycloid in cartesian coordinates:

$$x = r \arccos \frac{r-y}{r} - \sqrt{2ry-y^2}$$

for a circle of radius r which rolls on the x -axis.

Thus we see that many interesting curves can be defined as the zero sets of equations $f(x,y) = 0$, where $f(x,y)$ is a polynomial in

x and y . Such curves were called algebraic curves by Leibniz.

Some curves have an analytic equation $f(x,y) = 0$ in cartesian coordinates, where $f(x,y)$ is not a polynomial, and hence a transcendental, function. Leibniz called such curves transcendental curves. (Example : the cycloid.)

Other curves satisfy, e.g., a transcendental equation relative to polar coordinates. Often these are also called transcendental curves.

According to Loria [L4], Descartes considered a somewhat different class of curves, which he called geometric curves. In our language they are defined as those which satisfy a differential equation $f(x,y,y') = 0$, where $f(x,y,y')$ is a polynomial in all three variables. This class includes the algebraic curves and many special transcendental ones. I do not know whether Loria's assertion is historically correct.

It is clear that algebraic methods will apply most successfully when $f(x,y)$ is a polynomial. Only for this class, the algebraic curves, is there a general and self-contained algebraic theory. In what follows we will therefore confine ourselves to the investigation of algebraic curves.

2.4 Examples of the application of analytic methods

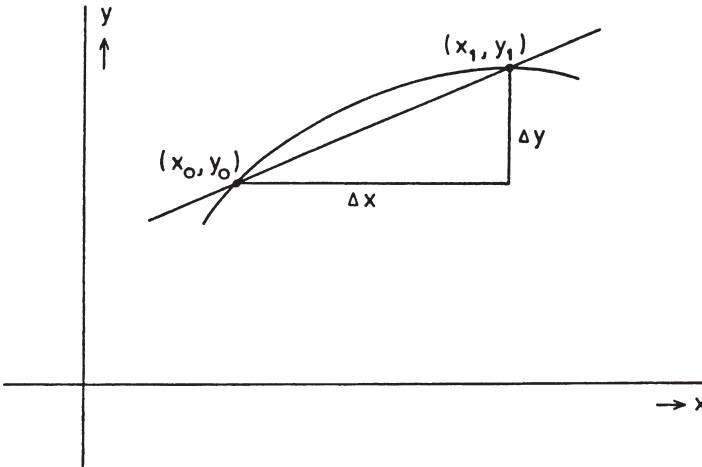
The tangent problem consists in constructing the tangent to a given plane curve at a given point.

The meaning of "constructing" remains open at first. The mathematicians of the 17th century gave solutions to this problem in many individual cases. The unification of these results was a very important element in the development of differential calculus. After the completion of this development the solution of the tangent problem could be presented as follows.

The curve in the plane is given by an equation $f(x,y) = 0$ in cartesian coordinates x, y . If (x_0, y_0) is a given point on the curve, then $f(x_0, y_0) = 0$. It may be that the curve does not have a uniquely defined tangent at this point, e.g. in the case of an ordinary double point, as we have seen in many examples. One hypothesis which guarantees the existence and uniqueness of the tangent at a point, is that the curve be smooth, or regular, at this point. Analytically, this is equivalent to the hypothesis of the implicit function theorem, i.e. that the partial derivatives of f do not both vanish at (x_0, y_0) .

Thus we can assume without loss of generality that $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Then, by the implicit function theorem, there is a unique solution $y = y(x)$ with $y(x_0) = y_0$ and $f(x, y(x)) \equiv 0$ in an interval containing x_0 , and $\frac{dy}{dx} = -\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y}$.

This function $y(x)$ therefore gives a local parametrisation $x \mapsto (x, y(x))$ for our curve in the neighbourhood of the given point.



If one considers a secant of the curve through a neighbouring point (x_1, y_1) on the curve, then this has the equation

$$(y - y_0) = \frac{y_1 - y_0}{x_1 - x_0} (x - x_0) = \frac{\Delta y}{\Delta x} (x - x_0).$$

Here one sees the advantage of the application of algebra to geometry made possible by Descartes : the description of the secant is reduced to the simplest algebraic operations on the differences between the coordinates of the points. Now comes the infinitesimal part of the analysis : if one lets (x_1, y_1) tend toward (x_0, y_0) , then the secant becomes the desired tangent, and correspondingly the difference quotient $\frac{\Delta y}{\Delta x}$ becomes the differential quotient $\frac{dy}{dx}$. Thus the computation above and this differential quotient finally give us the equation of the tangent at the point (x_0, y_0) :

$$\frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0) = 0.$$

This determines the tangent analytically. Now when our curve is algebraic, $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ may be computed from x_0, y_0 and the coefficients of the polynomial $f(x, y)$ by rational operations, i.e. the corresponding segments may be constructed by ruler and compass and one has thereby - in principle - a construction of the tangent in an entirely classical sense.

We have just defined regular points. A non-regular point is a singular point. Thus it is a point (x_0, y_0) with $f(x_0, y_0) = 0$, $\frac{\partial f}{\partial x}(x_0, y_0) = 0$, $\frac{\partial f}{\partial y}(x_0, y_0) = 0$. Examples of such points that we have seen previously (in non-analytic description) are the cusps and ordinary double points.

We give an example of the way in which the analytic description of singular points enables us to determine them by algebraic operations : we determine the singular points of the astroid.

With respect to suitable cartesian coordinates the astroid has the equation

$$(x^2+y^2-1)^3 + 27x^2y^2 = 0,$$

as we shall see later. Thus the partial derivatives, which we shall denote by f_x and f_y instead of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, satisfy

$$3(x^2+y^2-1)^2 2x + 54xy^2 = 0$$

$$3(x^2+y^2-1)^2 2y + 54x^2y = 0$$

at the singular points. The three equations imply

either (a) $x = 0$ and $y = \pm 1$

or (b) $y = 0$ and $x = \pm 1$

or (c) $(x^2+y^2-1)^2 + 9y^2 = 0$ and $(x^2+y^2-1)^2 + 9x^2 = 0$.

From (c) it follows that $x^2 = y^2$ and hence

$$(2x^2-1)^2 + 9x^2 = 0.$$

If one sets $\xi = x^2$, this gives the equation

$$4\xi^2 + 5\xi + 1 = 0.$$

The solutions are $\xi = -1$ and $\xi = -\frac{1}{4}$.

Thus one obtains $(x, y) = (\pm i, \pm i)$ resp. $(x, y) = (\pm \frac{i}{2}, \pm \frac{i}{2})$ as solutions of (c). The first of these solutions are also solutions

of the astroid equation, the latter are not.

Thus we have obtained exactly 8 points as solutions $f = 0$,
 $f_x = 0, f_y = 0$:

$$\begin{aligned} &(0, \pm 1) \\ &(\pm 1, 0) \\ &(\pm i, \pm i). \end{aligned}$$

The 4 points $(0, \pm 1)$ and $(\pm 1, 0)$ are the 4 cusps of the astroid. These are precisely the singular points of these real curves. What is the meaning of the other four points? This cannot be understood until one replaces the real curve $f(x,y) = 0$ by a complex curve, in which points (x,y) with complex coordinates are also admitted as solutions of $f(x,y) = 0$, and therefore we shall do this later.

The singular points of the astroid are thus determined. The real ones are simple cusps. But curves can have much more complicated "higher" singularities, and since Cramer, around 1750, the investigation of such singularities has been a permanently important theme of this theory.

In the pair of opposites "regular-singular" we have before us another dialectical pair of opposites which plays a role in all fields of mathematics and relates closely to other pairs of opposites such as "special-general" and "finite-infinite". A singularity within a totality is a place of uniqueness, of speciality, of degeneration, of indeterminacy or infinity. All these basic meanings are closely connected. I cannot go into more detail here and refer to my lecture on singularities [B7].

In various examples we have seen that the construction of the envelope, the curve enveloping a given family of curves, can lead to interesting and important new curves. We shall now show in an example how one can conceive this process, or construction, analytically.

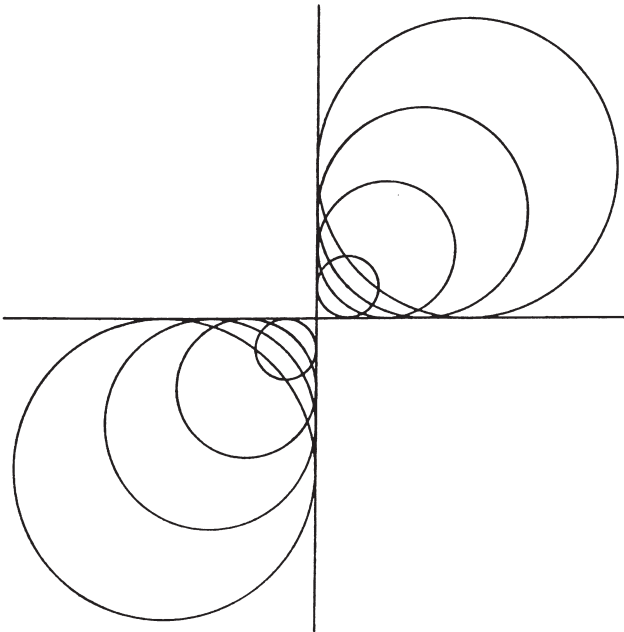
A one-parameter family of curves is given by a one-parameter family of equations $f_a(x,y) = 0$, where a is the parameter and x,y are cartesian coordinates in the plane. One can best express the fact that this family depends on the parameter a in a reasonable way by saying $f_a(x,y) = f(x,y,a)$, where the function $f(x,y,z)$ depends on x,y,z in the desired way, e.g. differentiably or analytically. Now how does one describe resp. define the envelope? Intuitively speaking,

it should be a curve which touches all curves of the family and which is touched at each of its points by some curve of the family. One possible way to make this precise is the following:

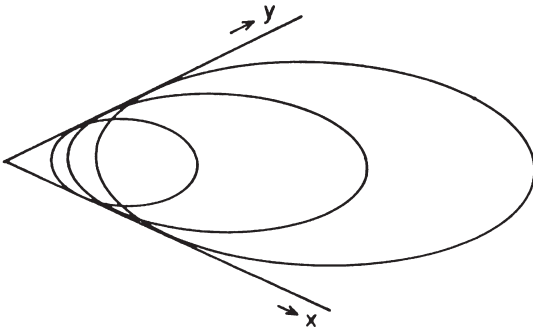
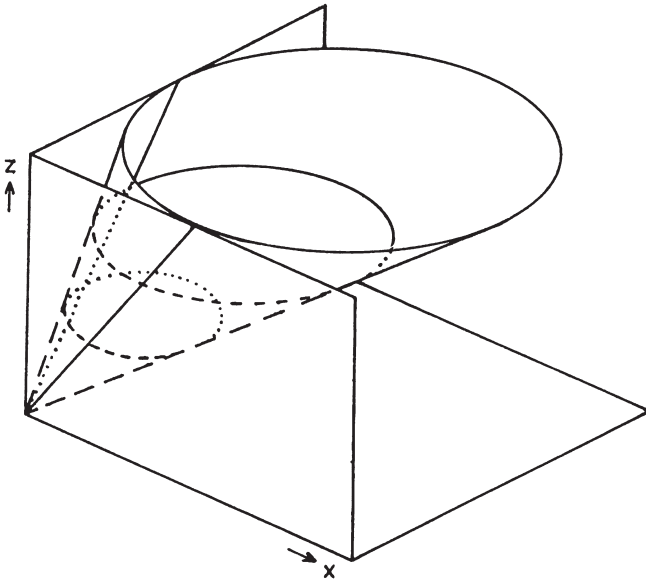
We consider the surface F in 3-dimensional space, with coordinates x, y, z given by $f(x, y, z) = 0$. When we cut F with the plane $z = a$ and project the curve of intersection onto the (x, y) -plane by $(x, y, z) \mapsto (x, y)$ we obtain precisely the curve $f_a(x, y) = 0$. This leads us to consider the mapping $F \rightarrow \mathbb{R}^2$ defined by projection.

We look at an example. Let the family of curves be the family of circles

$$(x-a)^2 + (y-a)^2 = a^2$$



The enveloping curve is obviously just the pair of axes, the surface F is an oblique cone, and the axis pair is obviously its outline under projection onto the (x, y) -plane.



This brings us to the following definition :

The envelope of the family of curves given by $f_a(x,y) = 0$ is the outline of the surface with equation $f(x,y,z) = 0$ under projection onto the (x,y) -plane (where $f_a(x,y) = f(x,y,a)$.)

This definition does not correspond entirely to the intuitive definition given above, but it has the advantage that it immediately gives us an analytic method for the calculation of the envelope. For the outline is precisely the image in the (x,y) -plane of all points of F at which there is a vertical tangent. These are precisely the points at which $\frac{\partial f}{\partial z} = 0$. Thus we obtain the equation of the envelope by elimination between the equations

$$f(x,y,z) = 0$$

$$\frac{\partial f}{\partial z}(x,y,z) = 0.$$

We remark that not every family of curves has a curve as envelope. For example, one sees immediately that for the family of all lines through the origin the envelope in the above sense consists only of the origin itself.

The families of curves which appear most frequently in our context are not parametrised by a real parameter a , but by the points (a,b) of a curve which itself is given by an equation $g(a,b) = 0$. Thus one has a family of equations $f(x,y,a,b) = 0$ for the family of curves. By carrying over the considerations above one now sees easily that :

One obtains the equations of the envelope by elimination of a,b from the equations

$$f(x,y,a,b) = 0 \quad (1)$$

$$g(a,b) = 0 \quad (2)$$

$$f_a g_b - f_b g_a = 0 \quad (3)$$

As an example, we shall compute the equation of the astroid. The astroid was the envelope of the family of lines resulting from the motion of a line with two points constrained to slide on the axes. We choose the distance between the points equal to one. When a,b are the intercepts of the line on the axes, the equations then read:

$$(1) \quad \frac{x}{a} + \frac{y}{b} - 1 = 0$$

$$(2) \quad a^2 + b^2 - 1 = 0$$

$$(3) \quad \frac{b}{a^2} x - \frac{a}{b^2} y = 0.$$

Multiplication of (1) by b^3 yields

$$\frac{b^3}{a} x + b^2 y - b^3 = 0.$$

Substitution of (3) in this yields

$$(a^2 + b^2)y - b^3 = 0.$$

Hence, by (2)

$$y = b^3$$

and similarly $x = a^3$.

Substitution in (1) gives the natural equation of the astroid

$$x^{2/3} + y^{2/3} = 1.$$

When one raises both sides to the power 3, carries roots to the right-hand side and again raises to the power 3, one finally obtains the equation of the astroid

$$(x^2 + y^2 - 1)^3 + 27x^2 y^2 = 0.$$

We remark that any curve can be regarded as the envelope of the family of its tangents. This fact plays an important role in Plücker's idea of regarding the lines as elements of a new manifold, and taking the coefficients of their equations as coordinates in this manifold. The totality of tangents to a curve then constitutes a new curve in this manifold of lines, the dual curve of the original.

After these examples of the application of differential calculus to the analytic geometry of curves, we shall now discuss in detail one example among the generally known applications of integral calculus (such as computation of area), namely the problem of rectifying a curve, i.e. calculating arc length.

Suppose a curve in cartesian coordinates is given by a parametrisation

$$x = x(t)$$

$$y = y(t).$$

We know that such a parametrisation always exists locally, in the neighbourhood of a regular point. Then the arc length of the curve from $(x(t_0), y(t_0))$ to $(x(t_1), y(t_1))$ is

$$\int_{t_0}^{t_1} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt.$$

Even for simple curves, e.g. the ellipse, such an integral is in general not elementarily computable, i.e. expressible in terms of elementary functions, and this was a factor leading to the theory of elliptic integrals and the development of function theory.

We shall look at quite a simple example. We consider a generalised parabola

$$x^p - y^q = 0.$$

Incidentally, one sees here again how the analytic base allows interesting curves to be defined through mere generalisation of the form of equations. A global parametrisation of this curve (for p, q relatively prime) is

$$\begin{aligned} x &= t^q \\ y &= t^p. \end{aligned}$$

However, we prefer to choose the parametrisation by x

$$y = x^{p/q}.$$

We then find the arc length to be

$$\int_0^x \sqrt{1 + \frac{p^2}{q^2} x^{2(p/q) - 1}} dx.$$

This cannot be elementarily evaluated for general p, q . However, for the special case $p = 3, q = 2$, i.e. for Neil's parabola (or semicubical parabola)

$$x^3 - y^2 = 0$$

we obtain

$$\int_0^x \sqrt{1 + \frac{9}{4} x} dx = \frac{8}{27} \left(1 + \frac{9}{4} x\right)^{3/2} \Big|_0^x.$$

And hence in this case we can construct a segment of the same length as a given arc of the curve by ruler and compasses. Neil's parabola was one of the first curves to be rectified, by Neil himself in 1657.

2.5 Newton's investigation of cubic curves

The analytic method of Descartes is in a certain sense the most general method for the generation of curves, because it allows the concept of plane algebraic curve to be defined in full generality, as the zero set of an equation $f(x,y) = 0$, where $f(x,y)$ is an arbitrary polynomial in x and y . Each particular choice of equation defines

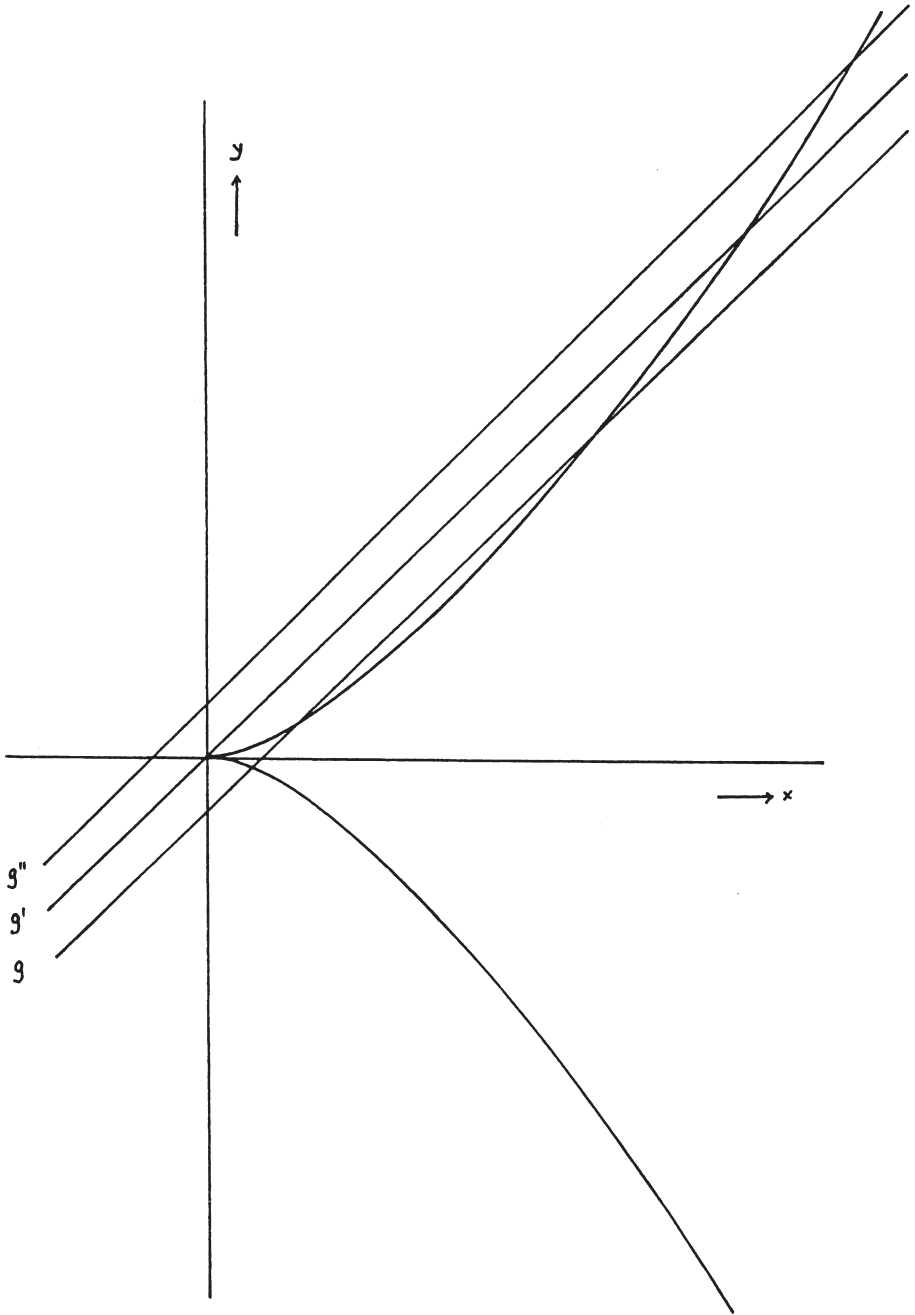
a curve, whose particular properties can be investigated, e.g. the generalised parabola $x^p - y^q = 0$ or the folium of Descartes
 $x^3 + y^3 + axy = 0$.

The generality underlying the definition of algebraic curves makes for a corresponding generality in the resulting problem of bringing order into the totality of all these curves, of classifying them in a suitable sense.

A first approach to a classification was already found by Newton. It comes directly from the form of the equation : obviously the equation $f(x,y) = 0$, and hence the curve itself, can be more complex as the degree of the polynomial increases. This number, the order of the curve, is therefore a certain quantitative measure for the geometric, qualitative complexity of curves. It is easy to see that, under a change of cartesian coordinates, the equation is transformed into another of the same order, and thus in the order we have found a measure invariant under such transformations, an invariant for the qualitative geometric properties of the curve.

It is easy to capture the geometric meaning of this invariant more precisely. Let $f(x,y) = 0$ be the equation of a given curve of order d , and let $ax + by + c = 0$ be the equation of an arbitrary line. We can eliminate one of the variables from the line equation, say $y = \alpha x + \beta$ for $b \neq 0$. If we substitute this in the polynomial $f(x,y)$, then in $g(x) = f(x, \alpha x + \beta)$ we obtain a polynomial in x of degree $\leq d$, and for all but at most finitely many lines this polynomial is exactly of degree d .

Now the zeroes of $g(x)$ are obviously just the abscissas of the intersections of the line with the curve, and hence we obtain : a curve of order d is cut by an arbitrary line in at most d points. Is it perhaps exactly d ? We consider an example, say the semicubical parabola $y^2 - x^3 = 0$.



The figure above shows the semicubical parabola and some of the lines intersecting it. The line g cuts the curve at three points, as we would expect on the basis of the preceding considerations, and each line which results from g by a sufficiently small displacement has the same property. But when we displace g so far that the line goes through the origin, say at position g' , two of the three points of intersection coincide. If one assigns this intersection a multiplicity 2, then g' still cuts in three points. Algebraically, this means that in the polynomial equation for the intersections of the curve with the line $y = ax$, namely $a^2x^2 - x^3 = 0$, the zero $x = 0$ has multiplicity 2. If we displace our line still further, say to the position g'' , then the two intersections at the origin vanish! Algebraically, this corresponds to the fact that for $y = ax + b$ with $a \cdot b > 0$ the intersection equation $x^3 - (ax+b)^2 = 0$ has two non-real roots. By the fundamental theorem of algebra, each polynomial of degree d has d zeros - with multiplicities counted - but of course these need not all be real. We see here that it will be useful to also admit complex solutions for the equation $f(x,y) = 0$ of our curve.

Let us consider again the intersection of the semicubical parabola with all lines through the origin. All of these lines, apart from the two coordinate axes, cut the curve in three points: the origin with multiplicity two, and a further point P with multiplicity one. When the line moves to the position of the x -axis, P also moves to the origin, the single intersection of the curve with the x -axis. For this reason, one should now count this intersection with multiplicity 3! We see from this that it will become necessary to develop a theory of the multiplicities with which curves are cut at their singular points.

If our line now moves to the other distinguished position, that of the y -axis, then P moves to infinity, so that the only intersection with the curve which remains is the origin, counted with multiplicity two.

We see from this that it will become necessary to develop a systematic theory of infinitely distant points, in order to capture the infinitely distant intersections as well. This is a factor which led to the development of projective geometry.

We shall in fact develop a satisfactory theory of multiplicity and intersections in complex projective geometry.

We have seen that with the order d of curves we have at our disposal the first invariant which can be applied to the classification of curves. It is now quite natural to proceed with the classification problem by obtaining an overview of all curves of lower order, say by considering the cases $d = 1, 2, 3, 4$.

The case of curves of order 1 is trivial : these are just all lines $ax + by + c = 0$.

The curves of order 2 are, when one admits degenerate cases such as line pairs and double lines, just the conic sections

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

One also calls these curves of order 2 quadrics.

The first interesting case is that of the cubics, i.e. the curves of order 3.* This case was first investigated systematically by Newton in his article on curves, [N2]. We shall give a brief survey of the results which Newton obtained in it.

In his investigation of cubic curves, Newton systematically applies the analytic method : he investigates the equation $f(x,y) = 0$, where $f(x,y)$ is a polynomial of degree 3. He starts with the asymptotic behaviour of curves. For "large" x,y this behaviour is of course determined by the terms of highest, i.e. third, degree in f , and hence by a homogeneous polynomial of third degree, $\tilde{f}(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$. Now there are obviously two possible cases. Case A : \tilde{f} is the third power of a linear form. By a change of cartesian coordinates (not necessarily rectangular) one then has $\tilde{f}(x,y) = ax^3$ with $a \neq 0$. Case B : \tilde{f} is not a third power. Then one sees that with a suitable change of coordinates \tilde{f} can be brought into the form $\tilde{f}(x,y) = xy^2 - ax^3$. In the remaining terms of f , i.e. in $f - \tilde{f}$, the monomials y^2 , xy and y can appear, in addition to powers of x . In case B one can make the y^2 and xy terms vanish by displacement of the origin, and in case A one can make xy and y vanish, by a new change of coordinates, when y^2 appears. If y^2 does not appear, but xy does, one can make y vanish. In summary, one has : a cubic curve which contains no lines has an equation

* Particular examples of cubics had already long been known, e.g. the cissoid of Diocles, the folium of Descartes and Neil's parabola.

of one of the 4 normal forms which follow, relative to suitable cartesian coordinates :

- I. $xy^2 + ey = ax^3 + bx^2 + cx + d.$
- II. $xy = ax^3 + bx^2 + cx + d.$
- III. $y^2 = ax^3 + bx^2 + cx + d.$
- IV. $y = ax^3 + bx^2 + cx + d.$

Newton now discusses these equations, having to make case distinctions according to the roots of $ax^3 + bx^2 + cx + d = 0$. In this way he obtains 72 cases, and there are six cases that he overlooked. The discussion of cases II and IV is of course trivial, case I is the most complicated. We shall not go into this discussion in detail here, but in order to give an impression of it we reproduce the first pages of Newton's work and the section in which he discusses case III. In this case he obtains 5 types of "diverging parabolas" : there are two which are non-singular, when all three roots of $ax^3 + bx^2 + cx + d = 0$ are different, with two curved paths in the case of three real zeros, and one in the case of one. If two of the three roots coincide one obtains a curve with an isolated singular point or an ordinary double point. If all three roots coincide one of course obtains the semicubical parabola. Newton's manuscript also contains pictures of all these curves, so that we can simply refer to it.

CURVES. *The incomparable Sir Isaac Newton gives this following Ennumeration of Geometrical Lines of the Third or Cubick Order; in which you have an admirable account of many Species of Curves which exceed the Conick-Sections, for they go no higher than the Quadratick or Second Order.*

The Orders of Geometrick Lines.

1: GEOMETRICK-LINES, are best distinguish'd into *Classes, Genders, or Orders*, according to the Number of the Dimensions of an Equation, expressing the relation between the *Ordinates* and the *Abscisse*; or which is much at one, according to the Number of Points in which they may be cut by a Right Line. Wherefore, a Line of the *First Order* will be only a *Right Line*: These of the *Second* or *Quadratick Order*, will be the *Circle* and the *Conick-Sections*; and these of the *Third* or *Cubick Order*, will be the *Cubical* and *Nelian Parabola's*, the *Cissoid* of the Antients, and the rest as below enumerated. But a *Curve* of the *First Gender* (because a *Right Line* can't be reckoned among the *Curves*) is the same with a Line of the *Second Order*, and a *Curve* of the *Second Gender*; the same with a Line of the *Third Order*, and a Line of an *Infiniteſimal Order*, is that which a *Right Line* may cut in infinite Points, as the *Spiral*, *Cycloid*, the *Quadratrix*, and every Line generated by the Infinite Revolutions of a *Radius* or *Rota*.

2. The chief Properties of the *Conick-Sections* are every where treated of by Geometers; and of the same Nature are the Properties of the *Curves* of the *Second Gender*, and of the rest, as from the following Ennumeration of their Principle Properties will appear.

3. For if any right and parallel Lines be drawn and terminated on both Sides by one and the same *Conick-Section*; and a *Right Line* bisecting any

two of them, shall bisect all the rest; and therefore such a Line is called the *Diameter* of the Figure; and all the *Right Lines* so bisected, are called *Ordinate Applicates* to that *Diameter*, and the Point of Concourse to all the *Diameters* is called the *Center* of the Figure; as the Intersection of the *Curve* and of the *Diameter*, is called the *Vertex*, and that *Diameter* the *Axis* to which the *Ordinates* are *Normally* applied. And so in *Curves* of the *Second Gender*, if any two right and parallel Lines are drawn occurring to the *Curve* in Three Points; a right Line which shall so cut those Parallels, that the Sum of Two Parts terminated at the *Curve* on one Side of the Intersecting Line shall be equal to the third Part terminated at the *Curve* on the other side, this Line shall cut, after the same manner, all others parallel to these, and occurring to the *Curve* in Three-Points; that is, shall so cut them that the Sum of the Two Parts on one Side of it, shall be equal to the Third Part on the other.

And therefore these Three Parts one of which is thus every where equal to the Sum of the other two, may be called *Ordinate Applicates* also: And the Intersecting Line to which the *Ordinates* are applied, may be called the *Diameter*; the Intersection of the *Diameter* and the *Curve*, may be called the *Vertex*, and the Point of Concourse of any two *Diameters*, the *Center*.

And if the *Diameter* be *Normal* to the *Ordinates*, it may be called the *Axis*; and that Point where all the *Diameters* terminate, the *General Centre*.

Affymptotes and their Properties.

4. The *Hyperbola* of the *First Gender* has Two *Affymptotes*, that of the *Second*, Three; that of the *Third*, Four, and it can have no more, and so of the rest. And as the Parts of any *Right Line* lying between the *Conical Hyperbola* and its Two *Affymptotes* are every where equal; so in the *Hyperbola's* of the *Second Gender*, if any *Right Line* be drawn, cutting both the *Curve* and its Three *Affymptotes*, in Three Points; the Sum of the Two Parts of that *Right Line*, being drawn the same way from any Two *Affymptotes* to Two Points of the *Curve*, will be equal to the Third Part drawn a contrary way from the Third *Affymptote* to a Third Point of the *Curve*.

Latera Transversa and Recta.

C U R

C U R

5. And as in *Non Parabolick Conick Sections*, the Square of the Ordinate Applicatè, that is, the Rectangle under the Ordinates, drawn at contrary Sides of the Diameter, is to the Rectangle of the Parts of the Diameter, which are terminated at the Vertex's of the *Ellipsis* or *Hyperbola*; as a certain Given Line which is called the *Latus Rectum*, is to that Part of the Diameter which lies between the Vertex's, and is called the *Latus Transversum*: so in *Non-Parabolick Curves* of the Second Gender, a Paralleloiped under the Three Ordinate Applicatès, is to a Paralleloiped, under the Parts of the Diameter terminated at the Ordinates, and the Three Vertex's of the Figure in a certain Given Ratio; in which Ratio, if you take Three Right Lines to the Three Parts of a Diameter situated between the Vertex's of the Figure, one answering to another, then these Three Right Lines may be called the *Latera Recta* of the Figure, and the Parts of the Diameter between the *Vertices*, the *Latera Transversa*. And as in the Conick Parabola having to one and the same Diameter but one only Vertex, the Rectangle under the Ordinates is equal to that under the Part of the Diameter cut off between the Ordinates and the Vertex, and a certain Line called the *Latus Rectum*: So in the Curves of the Second Gender, which have but two Vertex's to the same Diameter; the Paralleloiped under Three Ordinates, is equal to the Paralleloiped under the Two Parts of the Diameter cut off between the Ordinates and those Two Vertexes, and a given Right Line, which therefore may be called the *Latus Rectum*.

The Ratio of the Rectangles under the Segments of Parallels.

Lastly, As in the Conick-Sections when two parallels terminated on each side at the Curve, are cut by two other Parallels terminated on each side by the Curve, the First being cut by the Third, and the Second by the Fourth; as here the Rectangle under the Parts of the First, is to the Rectangle under the Parts of the Third; as the Rectangle under the Parts of the Second is to that under the Parts of the Fourth: So when Four such Right Lines occur to a Curve of the Second Gender, each one in Three Points, then shall the Paralleloiped under the Parts of the First right Line be to that under the Parts of the Third, and as the

Paralleloiped under the Parts of the Second Line into that under the Parts of the Fourth.

Hyperbolick and Parabolick Legs.

All the Legs of Curves of the second and higher Genders, as well as of the first, infinitely drawn out, will be of the *Hyperbolick* or *Parabolick* Gender; and I call that an *Hyperbolick Leg*, which infinitely approaches to some *Asymptote*; and that a *Parabolick* one, which hath no *Asymptote*. And these Legs are best known from the Tangents: For if the Point of Contact be at an infinite Distance, the Tangent of an *Hyperbolick Leg* will coincide with the *Asymptote*, and the Tangent of a *Parabolick Leg* will recede *in infinitum*, will vanish and no where be found. Wherefore the *Asymptote* of any Leg is found, by seeking the Tangent to that Leg at a Point infinitely distant: And the *Course*, *Place* or *Way* of an infinite Leg, is found by seeking the Position of any Right Line, which is parallel to the Tangent where the Point of Contact goes off *in infinitum*: For this Right Line is directed towards the same way with the infinite Leg.

The Reduction of all Curves of the Second Gender, to Four Cases of Equations.

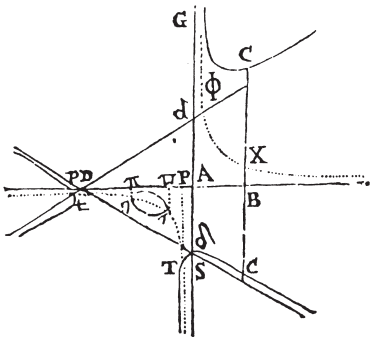
C A S E I

All Lines of the First, Third, Fifth and Seventh Order, and so of any one, proceeding in the Order of the odd Numbers, have at least two Legs or Sides proceeding on *ad infinitum*, and towards contrary ways. And all Lines of the Third Order have two such Legs or Sides running out contrary ways, and towards which no other of their infinite Legs (except in the *Cartesian Parabola*) do tend. If the Legs are of the *Hyperbolick* Gender, let *CAS* be their *Asymptote*; and to it let the Parallel *Cbc* be drawn, terminated (if possible) at both Ends at the Curve. Let this Parallel be bisected in *X*; and then will the Place of that Point *X*

C U R

C U R

Fig. 10



be the *Conical Hyperbola* $X\phi$, one of whose *Asymptotes* is AS : Let its other *Asymptote* be AB ; then the *Equation* by which the Relation between the *Ordinate* BC and the *Abscissa* AB is determined, if AB be put $= x$ and $BC = y$, will always be in this Form, $xy + ey = ax^3 + bxx + cx + d$, where the Terms e, a, b, c and d denote given Quantities, affected with their Signs $+$ and $-$; of which any one may be wanting, so the Figure, through their Defect, don't turn into a *Conick-Section*. And this *Conical Hyperbola* may coincide with its *Asymptotes*, that is, the Point X may come to be in the Line AB , and then the Term $+ey$ will be wanting.

C A S E II.

9. But if the *Right Line* CBc cannot be terminated both ways at the *Curve*, but will occur to the *Curve* only in one Point; then draw any *Line* in a given Position, which shall cut the *Asymptote* AS in A ; as also any other *Right Line*, as BC , parallel to the *Asymptote*, and meeting the *Curve* in the Point C : And then the *Equation* by which the Relation between the *Ordinate* BC and the *Abscissa* AB is determined, will always put on this Form, $xy = ax^3 + bxx + cx + d$.

C A S E III.

10. But if the opposite *Legs* are of the *Parabolic Gender*, draw the *Right Line* CBc , terminated at both Ends, if it's possible, at the *Curve*;

and running according to the *Courfe* of the *Legs*; which bisect in B : Then shall the *Place* of B be a *Right Line*. Let that *Right Line* be AB , terminated at any given Point, as A ; and then the *Equation* by which the Relation between the *Ordinate* BC and the *Abscissa* AB is determined, will always be in this Form, $yy = ax^3 + bxx + cx + d$.

C A S E IV.

11. But if the *Right Line* CBc meet the *Curve* but in one Point, and therefore can't be terminated at the *Curve* at both Ends; let the Point where it occurs to the *Curve* be C ; and let that *Right Line* at the Point B , fall on any other *Right Line* given in Position, as AB , and terminated at any given Point, as A : Then will the *Equation*, by which the Relation between the *Ordinate* BC and the *Abscissa* AB is determined, always be in this Form, $yy = ax^3 + bxx + cx + d$.

The Names of the Forms.

12. In the Enumeration of *Curves* of these Cases, we call that an *Inscribed Hyperbola*, which lies entirely within the Angle of the *Asymptotes*, like the *Conical Hyperbola*; and that a *Circumscribed* one, which cuts the *Asymptotes*, and contains the Parts cut off within its own proper Space; and that an *Ambigenal* one, which hath one of its infinite *Legs* inscribing it, and the other circumscribing it. I call that a *Converging* one, whose *Concave* *Legs* bend inwards towards one another, and run both the same way; but that I call a *Diverging* one, whose *Legs* turn their *Convexities* towards each other, and tend towards quite contrary ways. I call that *Hyperbola contrary leg'd*, whose *Legs* are *Convex* towards contrary Parts, and run infinitely on towards contrary ways; and that a *Conchoidal* one, which is applied to its *Asymptote* with its *Concave* *Vertex* and *diverging* *Legs*; and that an *Anguineal* or *Eel-like* one, which cuts its *Asymptote* with contrary *Flexions*, and is produced both ways into contrary *Legs*. I call that a *Cruciform* or *Cross-like* one, which cuts its *Conjugate* cross wise; and that *Nodate*, which, by returning round into, decussates it self. I call that a *Cuspidate*, whose two Parts meet and terminate in the Angle of Contact; and that *Punctate*, whose *Oval Conjugate* is infinitely small, or a Point: And that *Hyperbola I*

C U R

C U R

gles of the Asymptotes, but in the contiguous or adjoining ones, and that on each Side the Abcissa

Fig. 68.

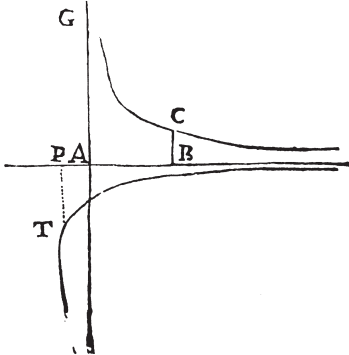
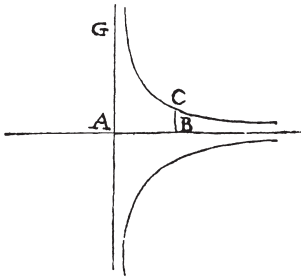


Fig. 69.

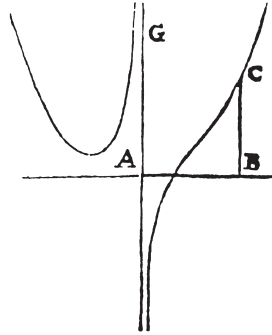


AB; and even without any Diameter, if the Term cy be there, but with one if that be wanting. Which two Species are the Sixty fourth and Sixty fifth.

A Trident.

26. In the second Case of the Equations there is $xy = ax^3 + bxx + cx + d$: And the Figure in this Case will have four infinite Legs, of which

Fig. 76.



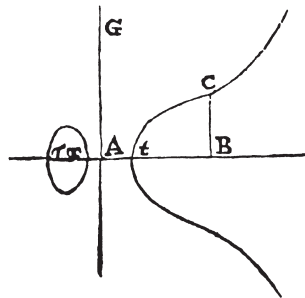
two are Hyperbola's about the Asymptote AG tending towards contrary Parts; and two converging Parabola's, and, with the Former, making as it were the Figure of a Trident. And this Figure is that Parabola by which *D. Cartes* constructed Equations of six Dimensions. This therefore is the Sixty sixth Species.

Five Diverging Parabola's.

27. In the third Case the Equation was $yy = ax^3 + bxx + cx + d$; and designs a Parabola, whose Legs diverge from one another, and run out infinitely contrary ways. The Abcissa AB is its Diameter, and its five Species are these:

If, of the Equation $ax^3 + bxx + cx + d = 0$, all the Roots At, AT, At , are real and unequal; then the Figure is a diverging Parabola

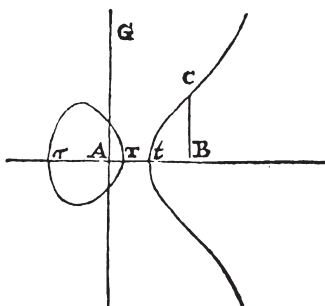
Fig. 70.



C U R

C U R

Fig. 71.



of the Form of a Bell, with an Oval at its Vertex. And this makes a *Sixty seventh Species*.

If two of the Roots are equal, a Parabola will be formed, either *Nodated* by touching an Oval,

Fig. 72.

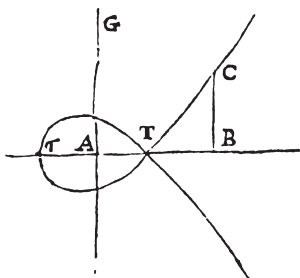
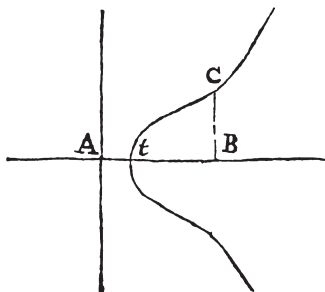


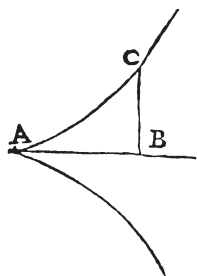
Fig. 73.



or *Punctate*, by having the Oval infinitely small. Which two *Species* are the *Sixty eighth* and *Sixty ninth*.

If three of the Roots are equal, the Parabola will be *Cuspidate* at the Vertex. And this is the

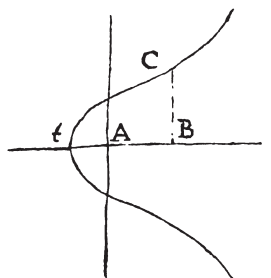
Fig. 75.



Neilian Parabola, commonly called *Semi-cubical*. Which makes the *Seventieth Species*.

If two of the Roots are impossible, there will (See Fig. 73.)

Fig. 73.



be a *Pure* Parabola of a Bell-like Form. And this makes the *Seventy first Species*.

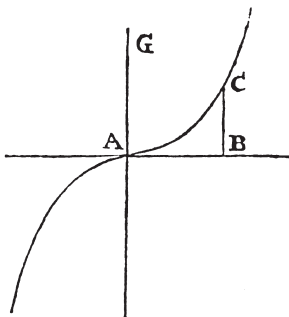
C U R

C U R

The Cubical Parabola.

28. in the Fourth Case, let the Equation be $y = ax + bxx + cx + d$; then will it denote

Fig. 77.



the Cubical Parabola with contrary turn'd Legs. And this makes up, or compleats, the Number of these Curves to be in all Seventy two.

Of the Genesifs of Curves by Shadows.

29. If the Shadows of Figures are projected on an infinite Plane illuminated from a lucid Point, the Shadows of the Conick-Sections will always be Conick-Sections; those of the Curves of the Second Gender, will always be Curves of the Second Gender; and the Shadows of Curves of the Third Gender, will themselves be of the same Gender, and soon in infinitum. And as a Circle, by the Projection of its Shade, generates all the Conick-Sections; so will the five diverging Parabola's spoken of in ch. 28. by their Shadows generate and exhibit all Curves of the Second Gender; and so some more simple Curves of other Genders may be found, which, by the Projection of their Shadows from a lucid Point upon a Plane, shall from all other Curves of the same kinds.

Of the double Points of Curves.

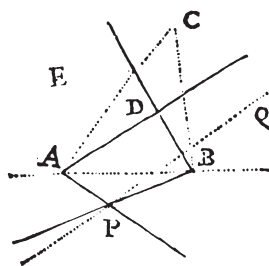
30. I said above, that Curves of the Second Gender might be cut by a Right Line in three Points; but two of those Points are sometimes co-

incident. As when the Right Line passes by an Oval infinitely small, or by the Concourse of two Parts of a Curve mutually intersecting each other, or running together into a Cuspis. And if at any time all the Right Lines tending the same way with the infinite Leg of any Curve, do cut it in one only Point, (as happens in the Ordinates of the Cartesian, and in the Cubical Parabola, and in the Right Lines which are parallel to the Abscissa of the Hyperbolisms of Hyperbola's and Parabola's;) then you are to conceive that those Right Lines pass through two other Points of the Curve (as I may say) placed at an infinite Distance; and these two co-incident Intersections, whether they be at a finite or an infinite Distance, I call the Double Point. And such Curves as have this Double Point, may be described by the following Theorems.

Theorems for the Organical Description of Curves.

31. Theor. I. If two Angles, as PAD and PBD, whose Magnitude is given, be turned round the Poles A and B, given also in Position; and their Legs AP, BP, by their Point of Concourse

Fig. 78.



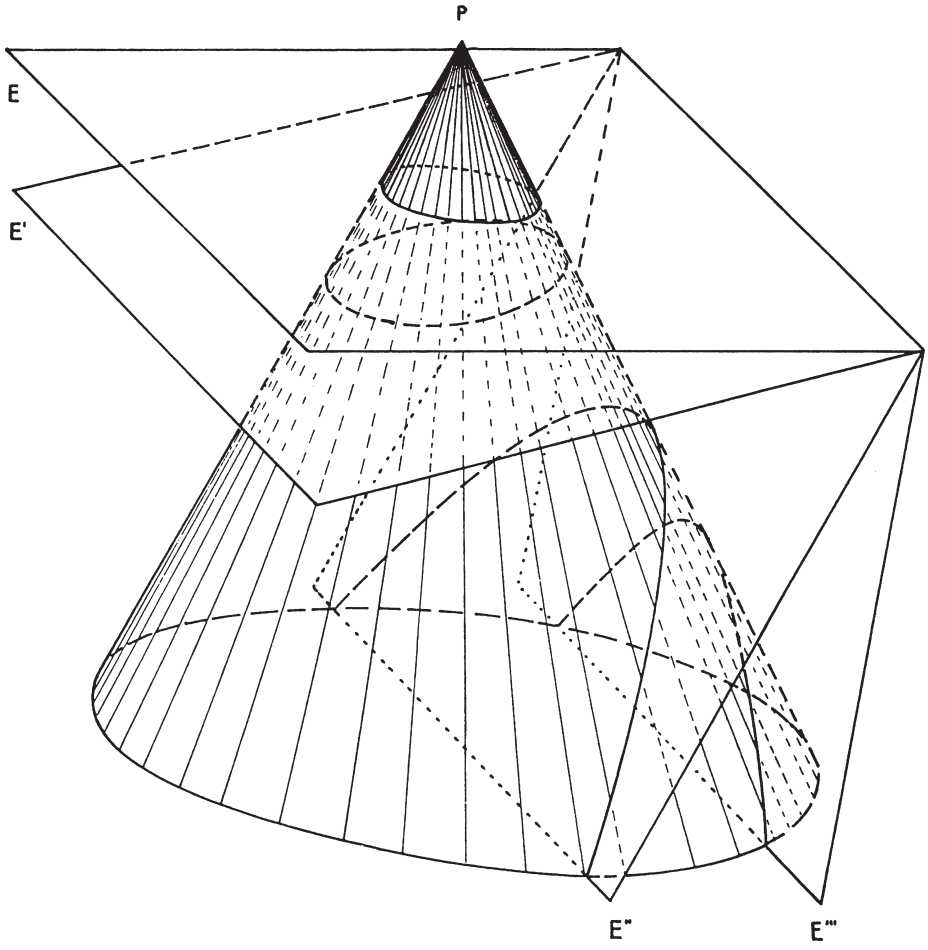
P, describe a Conick-Section passing thro' their Poles A and B; except when that Right Line happens to pass through either of the Poles A or B; or when the Angles BAD and ABD vanish together into nothing; for in such Cases the Point will describe a Right Line.

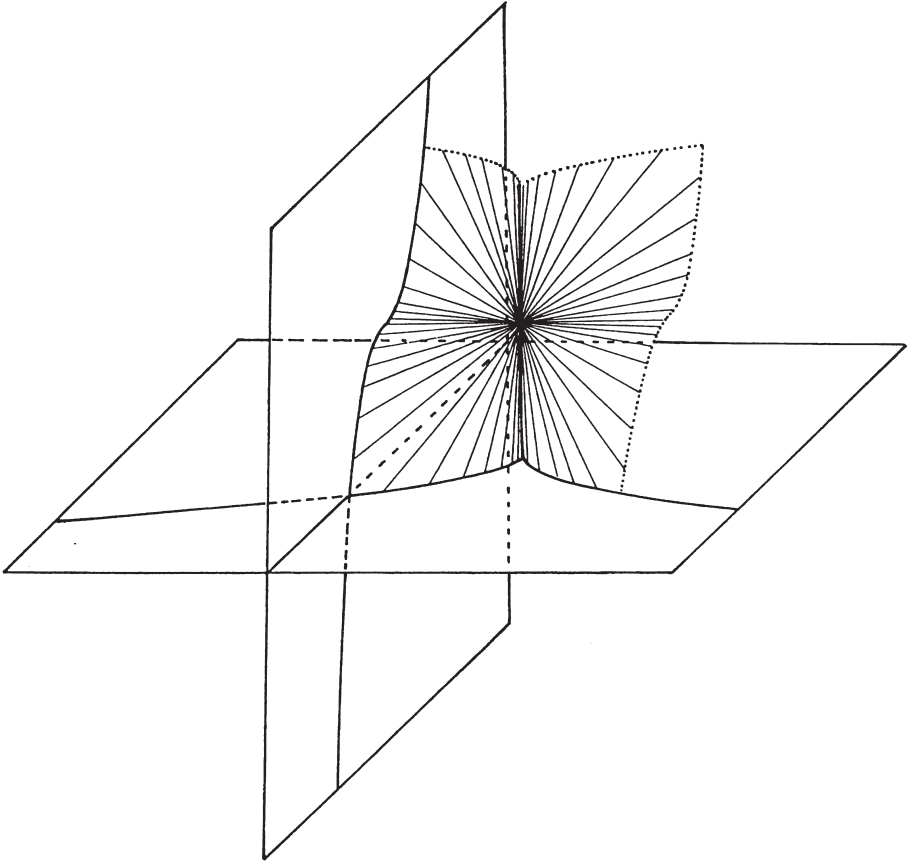
II. If the first Legs AP, BP, by their Point of Concourse P, do describe a Conick-Section pas-

Newton's description of all the cubics was later criticised by some mathematicians, e.g. Euler, because this classification lacks a general principle. Plücker later gave a more refined classification, in which he arrives at 219 types.

In the face of this multiplicity of types, the question of a unifying principle naturally arises. And such a principle already has its basis in the work of Newton, namely the short section (29) : Of the Genesis of Curves by Shadows, reproduced on the previous page.

The following two pictures illustrate these ideas of Newton. The first shows how projection of a circle in a plane E from a point P onto the planes E' , E'' , E''' yields ellipse, parabola and hyperbola. The second picture shows the projection of a cubical parabola onto a semicubical parabola, and conversely.







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