1
The Standard Model

1.1
Introduction

The electroweak theory (also known as the Glashow–Weinberg–Salam theory) is a unified theory describing the weak and electromagnetic interaction of elementary particles. Its prototype was Glashow's [34] model to combine the weak and electromagnetic interaction in the framework of $SU(2) \times U(1)$ symmetry. Weinberg and Salam [35, 36] supplemented the Higgs mechanism [37] to generate masses of gauge particles and fermions, and succeeded in placing the model in the mathematical framework of gauge theories. Its renormalizability was proved by 't Hooft [38, 39], completing it as a self-consistent mathematical theory.

The quantum chromodynamics (QCD) is a gauge theory based on color $SU(3)$ symmetry. Its prototype was an idea that the color charge which comes in three kinds is the source of the strong interaction acting among the quarks [40, 41]. It has been elevated to the theory of strong interaction when the asymptotic freedom was discovered by Gross, Politzer, and Wilczek [42–44].

The essence of the Standard Model, a name to denote GWS theory and/or QCD, can be summarized in the following phrases.

1. Building blocks of matter are quarks and leptons.
2. Their interactions are described in the mathematical framework of the gauge field theory.
3. The vacuum is in a sort of superconducting phase.

This chapter gives a simple introduction to axioms of the electroweak theory and prepares tools for calculating cross sections at least at the tree level. For understanding the physical (or rather geometrical) picture of the gauge theory, we refer the reader to Vol. 1, Chapt. 18.

Starting from the fundamental Lagrangian based on a gauge symmetry and applying spontaneous symmetry breaking, one obtains a Lagrangian which is more relevant for describing observed phenomena and which can provide associated Feynman rules. A simple description on the role of gauge symmetry as well as the Higgs

Elementary Particle Physics, Volume 2: Foundations of the Standard Model, First Edition. Yorikiyo Nagashima.

field to maintain unitarity in the framework of the spontaneously broken gauge theory is also given.

1.2
Weak Charge and $SU(2) \times U(1)$ Symmetry

Based on the accumulated evidence described in Vol. 1 of this book, we choose to start from axioms of the electroweak theory and derive equations of motion. We will use the word "charge" for an object that is capable of producing a force field. This is in analogy to the electric charge which is the source of the electromagnetic force. Then, the source of the weak force may be called the weak charge. It has remarkable characteristics similar to the electric charge in the sense that the static weak charge produces Coulomb-like weak force potential and the moving charge, that is, the weak current, like the electric current produces "weak" magnetic fields whose dynamic characteristic is very similar to that in QED. However, there are two conspicuous differences between the two forces.

The weak charge appears in two kinds and the symmetry they obey is $SU(2)$, in contrast to $U(1)$ of the electric charge. Its value could be, and generally are, different for the left- and right-handed particles in contrast to the electric charge which does not differentiate between the two. This is referred to as the chiral symmetry. Understanding its concept is fundamental for clarification of the weak phenomena.

The electromagnetic force is the essential force at molecular and atomic levels. The electromagnetic phenomena at their fundamental level do not differentiate left from right unless we elaborate in order to closely examine it. We have been accustomed to this concept for over a hundred years. Besides, a massive particle can be in either state depending on an observer's position. Accordingly, we take it for granted that the left- and right-handed particles are the same particle, though just in different states. The discovery of the different weak charge carried by the left- and right-handed particles made us recognize that they could be different particles. Since the fundamental forces which we know respect the local gauge symmetry, we need to apply gauge transformations differently for the left- and right-handed particles, which is referred to as the chiral gauge transformation. The name is a misnomer because it connotes a different operation from the ordinary gauge transformation. Indeed, for the fermion, one can define the chiral gauge transformation by

$$\psi \to \psi' = e^{-i\alpha\gamma^5}\psi \tag{1.1}$$

Since $\gamma^5\psi_{\substack{L\\R}} = \gamma^5[(1 \mp \gamma^5)/2]\psi = \mp\psi_{\substack{L\\R}}$, it certainly produces different transformations for left- and right-handed fields. However, the weak charge carriers are not restricted to fermions. We had better avoid the use of γ^5 in the gauge operator. Instead, we shall consider that the difference lies in the operand field. We must consider that the operation is just the same old gauge transformation, but the operand acts differently depending on the weak charge it carries. All we need to recognize is that fermions of different chirality carry different charges.

As an example, let us consider the chiral gauge transformation which follows the $U(1)$ symmetry and denote the charge operator as "Y" (call it hypercharge). Then, the gauge transformation for the fermion field $f = \psi_f$ or scalar ϕ is[1)]

$$
\begin{aligned}
f_L &\to f'_L = e^{-i\alpha Y} f_L = e^{-i\alpha Y(f_L)} f_L \\
f_R &\to f'_R = e^{-i\alpha Y} f_R = e^{-i\alpha Y(f_R)} f_R \\
\phi &\to \phi' = e^{-i\alpha Y} \phi = e^{-\alpha Y(\phi)} \phi
\end{aligned}
\tag{1.3}
$$

where $Y(f_L)$, and so on, are the hypercharges that each field carries.

We said that the weak charge comes in two varieties and respects the $SU(2)$ symmetry. Actually, the true weak and the electromagnetic force are based on a mixture of $SU(2)$ and $U(1)$, and we need to be careful about the terminology. The weak force in its original form, that is, before mixing and spontaneous symmetry breakdown, has chiral $SU(2)$ symmetry. In $SU(2)$ terminology, all the weak force carriers constitute isospin multiplets. All the left-handed fermions constitute doublets. For instance, the electric-type neutrino ν_e and the electron e^- are members of a doublet. $\Psi_L^T \equiv (\nu_{eL}, e_L^-)$. All the right-handed particles belong to $SU(2)$ singlets ($I = I_3 = 0$), namely, they do not carry weak charges. In the Standard Model, all the leptons can be classified by their isospin component as

$$
\text{Leptons} \quad
\begin{cases} I_3 = +1/2 \\ I_3 = -1/2 \end{cases}
\quad \Psi_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L
\tag{1.4}
$$
$$
I = I_3 = 0 \qquad e_R^-, \mu_R^-, \tau_R^-
$$

The leptons which have $I_3 = 1/2$, that is, the neutrinos are electrically neutral and those which have $I_3 = -1/2$ have electric charge $Q = -1$ in units of the positron charge. In the Standard Model, the right-handed neutrinos do not exist.[2)] For the quarks

$$
\text{Quarks} \quad
\begin{cases} I_3 = +1/2 \\ I_3 = -1/2 \end{cases}
\quad \begin{pmatrix} u \\ d' \end{pmatrix}_L , \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L , \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L
\tag{1.5}
$$
$$
I = I_3 = 0 \qquad u_R, d_R, c_R, s_R, t_R, b_R
$$

where $D'^T \equiv (d', s', b')$ are Cabibbo–Kobayashi–Maskawa rotated fields:

$$
D' = V_{CKM} D , \quad \to \quad \begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}
\tag{1.6}
$$

1) The equation is an abbreviated version. In the quantum field theoretical treatment, the transformation equation should read

$$
e^{i\alpha Y} \psi e^{-i\alpha Y} = e^{-i\alpha Y(\psi)} \psi \tag{1.2}
$$

where Y is the generator of the gauge transformation (see Vol. 1, Eq. (9.159)) and $Y(\psi)$ is the eigenvalue of Y that the field carries.

2) In reality, they do exist as demonstrated by the discovery of the neutrino oscillation. In the context of this textbook, it is more simple and no inconvenience is encountered by assuming the massless neutrino. The neutrino oscillation phenomena will be treated separately in Vol. 3.

The quarks with $I_3 = 1/2$ have $Q = 2/3$ and those with $I_3 = -1/2$ have $Q = -1/3$. Each quark carries another degree of freedom, that is, three colors which are the source of the strong interaction. Its dynamics will be treated in detail later, as now we must put aside the strong interaction in the discussion of the electroweak force and simply consider that they have an extra three degrees of freedom.

The original Lagrangian for the weak force is invariant under $SU(2)$ gauge transformations which are generically denoted as U. Denoting the isospin operator as $\boldsymbol{t}$, we have

$$\begin{aligned}\Psi_{\rm L}(x) &\to \Psi'_{\rm L}(x) = U\Psi_{\rm L}(x) = \exp(-ig_W \boldsymbol{\alpha}(x)\cdot\boldsymbol{t})\Psi_{\rm L}(x) = \exp[-ig_W \boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2]\Psi_{\rm L}(x)\\ \Psi_{\rm R}(x) &\to \Psi'_{\rm R}(x) = U\Psi_{\rm R}(x) = \exp(-ig_W \boldsymbol{\alpha}(x)\cdot\boldsymbol{t})\Psi_{\rm R}(x) = \Psi_{\rm R}(x)\end{aligned} \tag{1.7}$$

where $\Psi_{\rm L}$ is any fermion doublet in Eqs. (1.4) and (1.5), $\boldsymbol{\alpha} = (\alpha_1(x), \alpha_2(x), \alpha_3(x))$ is a set of three independent continuous variables and x is a simplified notation for the Lorentz coordinate variables $x \equiv (x^0, \boldsymbol{x})$. $\boldsymbol{\tau}$ is the Pauli 2×2 matrix that operates on the isospin components of doublets.

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.8}$$

The field $\Psi_{\rm R}$ does not receive any change by $SU(2)$ transformation simply because $I(\Psi_{\rm R}) = 0$. If we denote the gauge bosons of $SU(2)$ as

$$\boldsymbol{W}(x) = (W_1(x), W_2(x), W_3(x)) \tag{1.9}$$

which constitute an isospin triplet ($I = 1$), the gauge transformation changes them to

$$\boldsymbol{W}_\mu \cdot \boldsymbol{t} \to \boldsymbol{W}'_\mu \cdot \boldsymbol{t} = U\boldsymbol{W}_\mu \cdot \boldsymbol{t}U^{-1} + \frac{i}{g_W} U\partial_\mu U^{-1} \tag{1.10}$$

but keeps the covariant derivative

$$D_\mu = \mathbf{1}\partial_\mu + ig_W \boldsymbol{W}_\mu \cdot \boldsymbol{t} \tag{1.11}$$

invariant, that is, $D' = \mathbf{1}\partial_\mu + ig_W \boldsymbol{W}'_\mu \cdot \boldsymbol{t}$.

Now, we must treat the $U(1)$ part of $SU(2) \times U(1)$ in the GWS theory. It acts on all the leptons and quarks. We tentatively call it the B-force. It respects chiral $U(1)$ symmetry whose gauge transformation was given by Eq. (1.3). Each left- or right-handed fermion has its own hypercharge in addition to isospin component I_3 due to $SU(2)$. The weak isospin and the hypercharge satisfy Nishijima–Gell–Mann's law:

$$Q = I_3 + Y/2 \tag{1.12}$$

From this relation, one deduces that

$$\begin{aligned} &Y(\nu_{e\rm L}) = Y(e^-_{\rm L}) = -1\,, \quad Y(\nu_{e\rm R}) = 0\,, \quad Y(e^-_{\rm R}) = -2\\ &Y(u_{\rm L}) = Y(d_{\rm L}) = 1/3\,, \quad Y(u_{\rm R}) = 4/3\,, \quad Y(d_{\rm R}) = -2/3\end{aligned} \tag{1.13}$$

Assignment of Y to other fermions is obtained similarly.

We denote the gauge bosons of $U(1)$ as B_μ and the covariant derivative as $D_\mu = \partial_\mu + i(g_B/2)B_\mu$, where the factor 1/2 is introduced for later convenience. Then, the covariant derivative including both W and B is given by

$$\begin{aligned} D_\mu &= \partial_\mu + i g_W \boldsymbol{W}_\mu \cdot \boldsymbol{t} + \frac{i g_B}{2} B_\mu \\ &= \partial_\mu + \frac{i g_W}{2}(W_\mu^1 \tau_1 + W_\mu^2 \tau_2 + W_\mu^3 \tau_3) + \frac{i g_B}{2} B_\mu \end{aligned} \tag{1.14}$$

There is another important player of the electroweak force, namely, the Higgs field,

$$\Phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \tag{1.15}$$

in the electroweak interaction. ϕ^+, ϕ^0 are two complex scalar fields. Together, they constitute an isospin doublet ($I = 1/2$, $I_3 = \pm 1/2$) and carry hypercharge $Y(\phi^+) = Y(\phi^0) = 1$. The self-interaction of the Higgs field is the cause of the spontaneously symmetry breaking of the $SU(2)_L \times U(1)$, giving mass to the gauge bosons as well as to the fermions. The original Lagrangian before mixing and spontaneous symmetry breaking is given by

$$\begin{aligned} \mathcal{L}_{EW} &= \overline{\Psi} i \gamma^\mu D_\mu \Psi - \frac{1}{4} \boldsymbol{F}_{\mu\nu} \cdot \boldsymbol{F}^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ &\quad + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) - G_f [\overline{e}_R (\Phi^\dagger \Psi_L) + (\overline{\Psi}_L \Phi) e_R] \end{aligned} \tag{1.16}$$

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu - g_W \boldsymbol{W}_\mu \times \boldsymbol{W}_\nu \tag{1.17a}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.17b}$$

$$D_\mu = \partial_\mu + i g_W \boldsymbol{W}_\mu \cdot \boldsymbol{t} + i \left(\frac{g_B}{2}\right) Y B_\mu \tag{1.17c}$$

$$V(\Phi) = \lambda \left(|\Phi|^2 + \frac{\mu^2}{2\lambda} \right)^2 \qquad \lambda > 0 \tag{1.17d}$$

This is the master equation to calculate reaction rates of any electroweak processes. We shall use, for example, ν_e, e^- to denote quantized fields, that is, $\nu_e(x) = \psi_{\nu_e}(x)$, $e^-(x) = \psi_e(x)$, where there is no confusion. Here, we have only written down the Lagrangian of the $\Psi = (\nu_e, e^-)$ which will be needed in the following discussions. The Lagrangian for other fermions can be written down similarly. The first line of Eq. (1.16) is referred to as the gauge sector and the second line as the Higgs sector. $V(\Phi)$ is the self-interacting potential of the Higgs field. The whole expression satisfies the $SU(2) \times U(1)$ gauge symmetry manifestly. It is important to remember that both the gauge and the Higgs sectors are constructed to respect the gauge symmetry separately. The last term of Eq. (1.16) referred to as the Yukawa interaction was added to generate fermion masses. It can be written down as

$$\overline{e}_R(\Phi^\dagger \Psi_L) + (\overline{\Psi}_L \Phi) e_R = \overline{e}_R \nu_{eL} \phi^- + \overline{\nu}_{eL} e_R \phi^+ + \overline{e}_R e_L \phi^{0\dagger} + \overline{e}_L e_R \phi^0 \tag{1.18}$$

The case that this term is also $SU(2) \times U(1)$ invariant can be illustrated as follows. Because we constructed the Higgs field as belonging to the $I = 1/2$ doublet, and the product with another isospin doublet Ψ_L, then $(\Phi^\dagger \Psi_L)$ is therefore isospin rotation invariant. By multiplying e_R which is a scalar in $SU(2)$ transformation, the expression becomes Lorentz invariant because each term is of the form $\overline{\psi}_R \psi_R \phi$ or $\overline{\psi}_L \psi_R \phi$. One can show that each term is also chiral $U(1)$ invariant by referring to the hypercharges in Eq. (1.13). For instance, take a look at the first term on the right hand side of Eq. (1.18). By the chiral $U(1)$ gauge transformation

$$\begin{aligned} &e_R \to e_R' = e^{-iY(e_R)\beta} e_R\,, \quad \nu_{eL} \to \nu_{eL}' = e^{-iY(\nu_{eL})\beta} \nu_{eL}\,, \\ &\phi^- \to \phi^{-\prime} = e^{-iY(\phi^-)\beta} \phi^0 \end{aligned} \tag{1.19}$$

Considering $Y(e_R) = -2$, $Y(\nu_{eL}) = -1$, $Y(\phi^-) = -1$, one sees that the product $(\overline{e}_R \nu_{eL})\phi^-$ is $U(1)$ gauge invariant. Other terms can be proven similarly.

Mixing of *SU*(2) and *U*(1) The interaction of the gauge boson with the fermion is contained in the covariant derivative of Eq. (1.16). It is given by

$$-\mathcal{L}_{Wff} = g_W \overline{\Psi} \gamma^\mu \boldsymbol{W}_\mu \cdot \boldsymbol{t} \Psi + \frac{g_B}{2} \overline{\Psi} \gamma^\mu \Psi B_\mu \tag{1.20}$$

Since $\boldsymbol{t} = \boldsymbol{\tau}/2$ acts only on the left-handed fields and B_μ couples to the hypercharge Y which acts on the left- and right-handed fields differently, we have

$$\begin{aligned} -\mathcal{L}_{Wff} = {} & \frac{g_W}{2} \overline{\Psi}_L \gamma^\mu \left(W_\mu^1 \tau_1 + W_\mu^2 \tau_2 + W_\mu^3 \tau_3 \right) \Psi_L \\ & + \frac{g_B}{2} B_\mu \left(\overline{\Psi}_L \gamma^\mu Y \Psi_L + \overline{\Psi}_R \gamma^\mu Y \Psi_R \right) \end{aligned} \tag{1.21}$$

We wrote "Y" explicitly to remind the reader that B_μ is acting on the hypercharge.

Since both W_3 and B can couple to the same fermion, they mix and constitute the electromagnetic field (the photon) and the neutral gauge boson Z. To determine how they mix, we consider their coupling to the electron doublet.

$$\Psi_L(x) = \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L \tag{1.22}$$

Then, the coupling to the neutral boson part (terms containing W_μ^3 and B_μ in Eq. (1.21)) is expressed as

$$\frac{g_W}{2} (\overline{\nu}_{eL} \gamma^\mu \nu_{eL} - \overline{e}_L \gamma^\mu e_L) W_\mu^3 + \frac{g_B}{2} (-\overline{\nu}_{eL} \gamma^\mu \nu_{eL} - \overline{e}_L \gamma^\mu e_L - 2\overline{e}_R \gamma^\mu e_R) B_\mu \tag{1.23a}$$

Rearranging the equation, we obtain

$$\frac{1}{2} \left[\overline{\nu}_{eL} \gamma^\mu \nu_{eL} \left(g_W W_\mu^3 - g_B B_\mu \right) - \overline{e}_L \gamma^\mu e_L \left(g_W W_\mu^3 + g_B B_\mu \right) - 2 g_B (\overline{e}_R \gamma^\mu e_R) B_\mu \right] \tag{1.23b}$$

Since the electromagnetic field should not couple to the neutrino, we define the weak neutral boson Z[3] by $Z \sim g_W W_\mu^3 - g_B B_\mu$ and its orthogonal component as the electromagnetic field. With suitable normalization, they can be expressed as

$$A^\mu = \cos\theta_W B^\mu + \sin\theta_W W_3^\mu = \frac{1}{\sqrt{g_W^2 + g_B^2}} \left[g_W B^\mu + g_B W_3^\mu \right] \quad (1.24a)$$

$$Z^\mu = -\sin\theta_W B^\mu + \cos\theta_W W_3^\mu = \frac{1}{\sqrt{g_W^2 + g_B^2}} \left[-g_B B^\mu + g_W W_3^\mu \right] \quad (1.24b)$$

which also defines the Weinberg angle θ_W[4]. The Weinberg angle is related to the gauge coupling strength of the $SU(2)$ (g_W) and $U(1)$ (g_B) by

$$\tan\theta_W = \frac{g_B}{g_W} \quad (1.25)$$

As a result, the Z boson couples to the right-handed component of the fermion (via B^μ) as well as to their left-handed component (via W_3^μ). This is the reason that we have to be careful about saying that the weak force works only on the left-handed particles. The statement is true only for charged current reactions which couple to $W^\pm$. The neutral current which couples to Z contains the right-handed components. The photon couples to I_3 component and hypercharge of the particle which is the origin of the Nishijima–Gell–Mann's law.[5]

The kinetic energy part (derivatives of the fields) of the gauge fields can be rewritten as

$$\begin{aligned} \mathcal{L}_{KE} &= -\frac{1}{4} \sum_{a=1}^{3} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ &= -\frac{1}{4} \left(2F_{\mu\nu}^- F^{+\mu\nu} + F_{\mu\nu}^Z F^{Z\mu\nu} + F_{\mu\nu}^A F^{A\mu\nu} \right) \\ F_{\mu\nu}^\mp &= \partial_\mu W_\nu^\mp - \partial_\nu W_\mu^\mp\,, \quad F_{\mu\nu}^Z = \partial_\mu Z_\nu - \partial_\nu Z^\mu\,, \quad F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A^\mu \end{aligned} \quad (1.26)$$

where $W_\mu^\mp = (W_\mu^1 \pm i W_\mu^2)/\sqrt{2}$ are charged boson field operators.

3) The nomenclature "neutral boson" is used in general to mean any charge neutral member including the photon, but hereafter we use the word "neutral gauge boson" to specifically mean the Z unless otherwise noted.

4) In the following, $\sin\theta_W$ appears more often than θ_W itself. It will also be referred to as the Weinberg angle.

5) This is a circular logic. Historically, the hypercharge is assigned to satisfy the Nishijima–Gell–Mann law. Here, however, we are starting from an axiom that the hypercharge is the fundamental constants that all the particles possess transcendentally.

1.3 Spontaneous Symmetry Breaking

When the equation of motion satisfies a certain symmetry, its solution generally possesses the same symmetry. However, the solution is not necessarily stable. In such a case, the chosen state could break the symmetry. When the ground state (or vacuum in the field theory) does not respect the symmetry which the equation of motion has, we have the spontaneous symmetry breaking. Let us see how it happens. The Higgs potential Eq. (1.17d) contains a quartic term as well as the quadratic term.

$$V(\Phi) = V(0) + \mu^2|\Phi|^2 + \lambda|\Phi|^4 , \quad |\Phi| = \sqrt{|\phi^+|^2 + |\phi^0|^2} \tag{1.27}$$

$V(0)$ is arbitrary and usually assumes a value to make the vacuum energy vanish. Only the energy difference from the vacuum matters unless we deal with the gravity.

1.3.1 An Intuitive Picture of Spontaneous Symmetry Breaking

Let us consider what happens to the field after the symmetry breakdown. For simplicity, let us consider a case $V(0) = 0$ and Φ is an isospin singlet complex field $\Phi = \phi = (\phi_1 + i\phi_2)/\sqrt{2}$. The potential is expressed as

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \tag{1.28}$$

The symmetry which the field satisfies is the Abelian $U(1)$ gauge symmetry, namely, the Lagrangian is invariant under the transformation $\phi \to \phi' = e^{-i\alpha}\phi$. If $\lambda = 0$ and $\mu^2 > 0$, the potential represents a harmonic oscillator. Energy excited states appear as particles in the quantized field theory and μ represents the mass of the quanta. If $\lambda \neq 0$[6], the potential still has a minimum at $|\phi| = 0$ as described by the curve denoted $T > T_c$ in Figure 1.1a. In this case, the particle picture still holds, the quartic term represents self-interactions of the particles.

Let us consider μ^2 not as an a priori given mass squared, but some dynamical object that changes with temperature, that is, $\mu^2 = C(T - T_c)$, where T_c is a critical temperature whose meaning will soon be clarified. By doing so, we regard the vacuum not as an empty space, but some dynamical object which changes its characteristics with temperature just like any medium in condensed matter physics. As the temperature goes down, the μ^2 changes sign at $T = T_c$, and below T_c, the potential develops minima at $\phi \neq 0$. The μ^2 being negative cannot be interpreted as the mass squared, but should be regarded as a part of the potential. If the field ϕ is complex as we assumed, the potential shape is like a Mexican hat (Figure 1.1b). The old vacuum $|\phi| = 0$ is no longer a stable point and the vacuum moves to a stable point, the minimum at $|\phi| = \sqrt{-\mu^2/2\lambda} \equiv v/\sqrt{2}$. The field ϕ is not zero at the new vacuum, or the vacuum expectation value of the field is finite.

6) We do not consider the case $\lambda < 0$ because the vacuum becomes unstable.

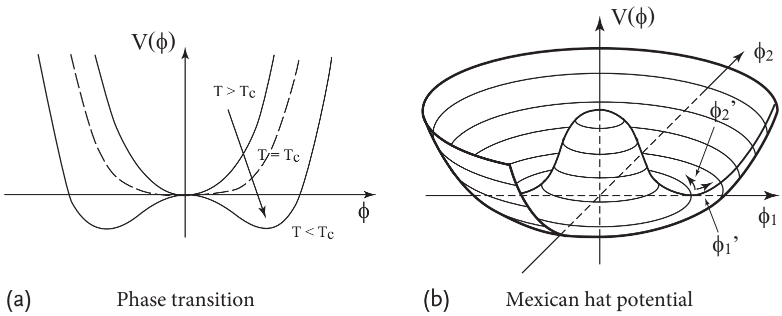


Figure 1.1 (a) Depending on the temperature, the potential changes its shape. (b) Illustration of a Mexican hat potential as required for the Higgs field. The ground state is degenerate and the choice of one specific value spontaneously breaks the symmetry.

For the complex ϕ, the new vacuum is infinitely degenerate because any point on the circle that satisfies the above condition is eligible as a vacuum. The vacuum the nature picks up can be anywhere on the circle. We conveniently set a new coordinate system such that the new vacuum is at $\phi_1 = \nu/\sqrt{2}, \phi_2 = 0$. As the experimentally observed energy is the excitation from the new vacuum, we are now dealing with new fields

$$\phi_1' = \phi_1 - \nu\,, \quad \phi_2' = \phi_2 \tag{1.29}$$

and the Higgs Lagrangian is rewritten as

$$\begin{aligned}\mathcal{L} &= \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi) \Rightarrow \partial_\mu \phi'^\dagger \partial^\mu \phi' - V(\nu + \phi') \\ &= \frac{1}{2}\left(\partial_\mu \phi_1' \partial^\mu \phi_1' - 2\lambda \nu^2 \phi_1' \phi_1'\right) + \frac{1}{2}\left(\partial_\mu \phi_2' \partial^\mu \phi_2'\right) \\ &\quad - \left[\lambda \phi_1' \left\{(\phi_1')^2 + (\phi_2')^2\right\} + \frac{\lambda}{4}\left\{(\phi_1')^2 + (\phi_2')^2\right\}^2\right]\end{aligned} \tag{1.30}$$

As one can see, the terms in the penultimate line show that the new field ϕ_1' has mass $m_H^2 = 2\lambda \nu^2$ and the ϕ_2' has vanishing mass. The last line describes the interaction between the two fields and among themselves. The different mass of the two fields can be pictorially understood from Figure 1.1b. ϕ_1' is in the bottom of the potential valley and it takes energy to excite the field, that is, to climb up the potential. However, for ϕ_2', there is no potential to hinder the motion, that is, it takes no inertia to move on the circle. The massless field ϕ_2' is referred as the Goldstone boson. They no longer satisfy the gauge symmetry. Remnants of the original symmetry can be seen in the fact that any point on the circle is equivalent and could have been chosen as the new vacuum.

The important point is that the new vacuum, once chosen, is fixed there. This is because of the infinite degrees of freedom that the field possesses. It may not be apparent intuitively, so perhaps an example in condensed matter physics that obeys the same mathematics may help to clarify it. ϕ is referred to as the order parameter there. Think of a ferromagnet which we discussed in Vol. 1, Chapt. 18. The order parameter in this case represents the magnetization. Above the critical temperature, the spin is not aligned because of thermal motion and there is no magnetization, namely, the medium is a paramagnet. Naturally, the order parameter vanishes in the ground state of the paramagnet. The medium respects the rotational symmetry because the spin has no preferred orientation. Below the Curie temperature, the spins begin to be aligned due to spin-spin force, and at absolute zero temperature, the whole medium is aligned. The magnetization in the ground state is finite and the medium is a ferromagnet. The spontaneous symmetry breaking is nothing but the phase transition in condensed matter physics and it is a metamorphose of the entire medium.

In a medium where all the spins are aligned in the same direction, it is possible to excite a spin or two by giving a small energy ΔE to each spin to change its direction. The turbulence propagates as a wave which, if quantized, is the Goldstone boson represented by ϕ_2'. However, changing the whole ground state, namely, changing the spin orientation of the entire medium, takes energy $\Delta E \times N_A$ where N_A is of the order of the Avogadro number which is large but finite. Macroscopically, it is possible to heat the medium above the critical temperature to transfer the ferromagnet back to paramagnet. However, the vacuum in the field theory is the vacuum of the universe. It has infinite degrees of freedom. The vacuum, once chosen, is impossible to change. We cannot heat up the whole universe!

1.3.2 Higgs Mechanism

Now, we go back to the original $SU(2)$ non-Abelian Higgs field Eq. (1.15) and we include the gauge field. The reason to consider a doublet Higgs field is that we want to use three Goldstone bosons later and an isospin doublet of the Higgs fields is the minimum requirement within the $SU(2)$ symmetry. The physics aspect of the theory changes from that of the ferromagnet, but the role of the Higgs field in inducing the spontaneous symmetry breaking is the same. Below the critical temperature, the Higgs field acquires the vacuum expectation value (vev)

$$|\Phi^2| = |\phi^+|^2 + |\phi^0|^2 = v^2/2 \tag{1.31}$$

Notice that there are three fields that can vanish. The vacuum is infinitely degenerate and we choose the new vacuum at $\mathrm{Re}[\phi^0] = v/\sqrt{2}$, $\mathrm{Im}[\phi^0] = \phi^+ = 0$, namely,

$$\Phi' = \begin{bmatrix} 0 \\ (v + H)/\sqrt{2} \end{bmatrix} \tag{1.32}$$

where we use nomenclature $H = \mathrm{Re}[\phi^0] - v\sqrt{2}$ to denote it as a real observable Higgs field. This is the spontaneous symmetry breakdown of the $SU(2)$ symmetry (or $SU(2) \times U(1)$ if we include Z instead of W_3). The Higgs field before the symmetry breaking can be parametrized as

$$\Phi(x) = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (\omega_2 + i\omega_1)/2 \\ v + H - i\omega_3/2 \end{bmatrix} \tag{1.33}$$

Without loss of generality, it can be rewritten as

$$\Phi = \exp\left(i\frac{\boldsymbol{\omega}}{v}\cdot\frac{\boldsymbol{\tau}}{2}\right)\begin{bmatrix} 0 \\ \frac{v+H}{\sqrt{2}} \end{bmatrix} \tag{1.34}$$

In fact, in the vicinity of the chosen vacuum where $v \gg |H|, |\boldsymbol{\omega}|$, Eq. (1.34) reduces to Eq. (1.33). The reason to place three of the fields in an exponent is because we can realize the new Higgs field Eq. (1.32) by a gauge transformation

$$\Phi \to \Phi' = U\Phi = \exp[-i\boldsymbol{\omega}\cdot\boldsymbol{\tau}/(2v)]\Phi \tag{1.35}$$

The $\boldsymbol{\omega}$ components disappear from the Higgs field. As the gauge transformation also changes the gauge field $\boldsymbol{W}_\mu$, they are transformed to $\boldsymbol{W}'_\mu$.

$$\begin{aligned}\boldsymbol{W}_\mu\cdot\boldsymbol{t} \to \boldsymbol{W}'_\mu\cdot\boldsymbol{t} &= U\boldsymbol{W}_\mu\cdot\boldsymbol{t}U^{-1} + \frac{i}{g_W}U\partial_\mu U^{-1} \\ &= U\boldsymbol{W}_\mu\cdot\boldsymbol{t}U^{-1} - \frac{1}{g_W v}\partial_\mu\boldsymbol{\omega}\cdot\boldsymbol{t}\end{aligned} \tag{1.36}$$

with $\boldsymbol{t} = \boldsymbol{\tau}/2$. The field $\boldsymbol{\omega}$ has reappeared as the third (longitudinal) component of the gauge bosons. Namely, three of the Higgs fields are absorbed by the gauge field and have become their third component.[7] We rename the new fields Φ' and $\boldsymbol{W}'_\mu$ again as Φ, $\boldsymbol{W}_\mu$. They are the physical observables because we live in a vacuum where the gauge symmetry has been spontaneously broken.

Now, we realize that when the broken symmetry is a local gauge symmetry, the Goldstone bosons do not appear. Three Goldstone bosons are produced by a doublet Higgs, but they are absorbed by the three gauge bosons ($W^\pm$, Z). The new gauge bosons acquire the third component which means they have become massive and the electroweak force has become short ranged. The appearance of the mass term will be explicitly shown later in Eq. (1.37). Analogous phenomenon actually happens in the superconductor below the critical temperature where the gauge symmetry is spontaneously broken and the electromagnetic force is converted to a short ranged force. The phenomenon that the magnetic force cannot penetrate into the superconducting medium and is wholly repelled is known as the Meissner effect. This is the reason that we said the physics outcome is different if the broken symmetry is the local gauge symmetry. The whole mechanism, the spontaneous breakdown of the gauge symmetry and associated mass generation of the gauge bosons, is referred to as the Higgs mechanism. The mathematics we developed is more appropriate to the superconductivity rather than the ferromagnetism. Indeed, the nonrelativistic Hamiltonian derived from the Abelian Lagrangian after spontaneous symmetry breakdown has identical form to the Ginzburg–Landau free energy of the superconductivity (see the boxed paragraph in Vol. 1, Sect. 18.5.2). In this case, the order parameter is the wave function of the Cooper pair whose condensation induces the superconductivity. This is the reason we said that the vacuum where we live is in a sort of superconducting phase.

7) They are sometimes referred as the would-be-Goldstone bosons because if the symmetry is not the gauge symmetry, they would have manifested themselves as physically observable Goldstone bosons.

1.3.3 Unitary Gauge

We have seen that the Higgs field after the spontaneous symmetry breaking ($= \Phi'$) is equivalent to that obtained as the result of the gauge transformation, that is, $\Phi' = \exp[-i\boldsymbol{\omega} \cdot \boldsymbol{t}/v]\Phi$. Since we chose the vacuum to fix the Higgs field in this form, this simply means a special gauge is chosen and fixed. This gauge is referred to as the unitary gauge or *U*-gauge. The unitary gauge is one where the physical meaning of the various fields is clearest.

Notice that the new fields do not respect the gauge symmetry, not because the symmetry was broken, but because we chose the new field (old Φ') as the excitation from a specific vacuum point to describe phenomena in our world. In doing so, we also fixed the gauge. Mathematically, the whole procedure is just choosing an appropriate gauge and fixing it. We can equally describe nature using old variables in the original gauge. From a theoretical point of view, it is much more convenient because it respects the gauge symmetry manifestly. However, then it is hard to make connections with observables and the physical interpretation of mathematics gets complicated. The symmetry was not really broken, rather it was hidden as a result of choosing new variables.

What is the physical meaning of the phase field?

In the *U*-gauge, the Higgs field takes the form given in Eq. (1.34). Mathematically, it removes extra Higgs components beautifully which are absorbed by the gauge bosons. It can also be cast in a form given by Eq. (1.33) which looks more familiar. What is the difference between the two choices? Let us remember that a particle picture of the quantized field is obtained by quantizing a harmonic potential which is quadratic in the field variables. The potential of the field in general, however, is not necessarily quadratic. It may have quite a complicated structure depending on the property of the field. In the framework of the quantum field theory, we refer to particles as excited states, or small perturbations around a stable environment which we call the vacuum. Mathematically, it may be a local minimum and we always expand the potential in a Taylor power series. If we use only up to quadratic terms, we get the particle picture of the quantized field.

Higher order terms are treated as the interaction among particles. The interaction is different depending on where and how we expand the potential. The expanded power series contains hints for the global structure of the whole potential. Inclusion of the higher order terms is to consider more global characteristics of the field which may not behave like particles. Consider, for instance, a superconducting object which we modeled in developing the Higgs formalism. It behaves as a macroscopic quantum fluid rather than as particles. The vacuum we developed as the result of the spontaneously broken symmetry is in a state of a superconducting phase. In the local vicinity of which we call the vacuum, the potential may well be approximated by the quadratic potential and the familiar particle picture is valid.

However, globally, it may be a part of circulating field or may be spatially expanding just as our universe. If it is happening in the cosmic scale, we will not notice it because we are only dealing with particles which are simply small excitations of a local minimum. The $\boldsymbol{\omega}$ field in Eq. (1.34) is referred to as the phase filed. Locally, however, it is equivalent to linear fields in Eq. (1.33), that is, from a phenomenological point of view, there is no difference. The potential form given in Eq. (1.30) could be considered as representing the global circle structure of the vacuum, but for a description of phenomena we observe locally, it only appears as higher order corrections. However, a glimpse of the global structure may be obtained by investigating the particle interactions.

Our next task is to find what the Higgs sector of the Lagrangian have produced after the spontaneous symmetry breakdown.

1.3.4 Mass Generation

Mass of the Gauge Boson First, we take a look at the kinetic energy part of the Higgs sector. Substituting Eq. (1.32) in the first term of the second line of Eq. (1.16), we obtain

$$\begin{aligned}(D_\mu \Phi)^\dagger (D^\mu \Phi) &= (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + \Phi^\dagger \left(g_W \boldsymbol{W}_\mu \cdot \boldsymbol{t} + (g_B/2) B_\mu \cdot Y\right)^2 \Phi \\ &= (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + \frac{(v+H)^2}{8}\left[2 g_W^2 W_\mu^- W^{+\mu} + (-g_W W_\mu^0 + g_B B_\mu)^2\right] \\ &= (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) + \left(\frac{g_W(v+H)}{2}\right)^2 W_\mu^- W^{+\mu} + \frac{1}{2}\left(\frac{g_Z(v+H)}{2}\right)^2 Z_\mu Z^\mu\end{aligned} \tag{1.37}$$

Linear terms in the derivative vanish and we used

$$g_W W_\mu^0 - g_B B_\mu = g_Z Z_\mu\,, \quad g_Z = \sqrt{g_W^2 + g_B^2} \tag{1.38}$$

In the absence of the interaction with the Higgs field (i.e., when $H = 0$), the second and third term in the last line of Eq. (1.37) gives the mass terms for W and Z. The H term gives the interaction of the Higgs field with W and Z. Considering that there is a factor 1/2 difference between the charged and neutral vector bosons, we have

$$m_W = \frac{g_W v}{2}\,, \quad m_Z = \frac{g_Z v}{2} \tag{1.39}$$

We have just explicitly shown that the gauge bosons have acquired mass after symmetry breaking. Notice that Eq. (1.37) does not contain the electromagnetic field. Therefore, the photon does not acquire mass. This is not accidental. Remember that the symmetry was broken by the vacuum. This means that the vacuum expectation value of both the isospin ($SU(2)$) operator $\boldsymbol{t}$ and the hypercharge operators Y

do not vanish.

$$t\langle\Phi\rangle = \frac{\boldsymbol{\tau}}{2}\begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} \neq 0\,, \quad Y\langle\Phi\rangle = \langle\Phi\rangle \neq 0 \tag{1.40}$$

which is another statement of the symmetry breaking in the new vacuum. On the other hand, as $Q = I_3 + Y/2$ and $Y(\Phi) = 1$,

$$Q\langle\Phi\rangle = \frac{1}{2}(1+\tau_3)\begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} = 0 \tag{1.41}$$

Therefore, the vacuum is not invariant under $SU(2)_{\mathrm{L}}$ or $U(1)_{Y}$ gauge transformation separately, but is invariant under the combined gauge transformation $U = \exp[-Qe\alpha(x)]$. The photon field which is generated by the charge operator Q has kept its freedom of the gauge transformation which means the gauge field has vanishing mass.

Careful readers might have noticed that a massive vector has three degrees of freedom (two transverse polarizations and one longitudinal polarization), whereas the massless vector boson has only two degrees of freedom. Is it not a contradiction? As a matter of fact, the extra degree of freedom was provided by the Higgs components $(\omega_1, \omega_2, \omega_3)$ that had disappeared. This can be seen in the expression of Eq. (1.36) in which the second term provides the longitudinal component. Originally, there were four components in the Higgs field. Three of them are absorbed by the three gauge bosons, and the degree of freedom that the gauge boson had, increased from 2×3 to 3×3. The total number of the degrees of freedom is invariant.

The Vacuum Expectation Value We can determine the value of the vacuum expectation value by comparing the muon decay amplitude with the conventional four-Fermi interaction Lagrangian.

The fermion interaction with charged gauge bosons can be extracted by inspecting the covariant derivative of the fermion (the first term in the first line of Eq. (1.16)). By including the second fermion doublet with the muon-flavor, the interaction Lagrangian takes the form

$$\mathcal{L}_{Wff} = -\frac{g_W}{\sqrt{2}}(\overline{\nu}_{e\mathrm{L}}\gamma^{\mu}e_{\mathrm{L}} + \overline{\nu}_{\mu\mathrm{L}}\gamma^{\mu}\mu_{\mathrm{L}})\,W_{\mu}^{-} + h.c. \tag{1.42}$$

Here *h.c.* means hermitian conjugate. From Eq. (1.42), one can extract the tree decay amplitude for the $\mu^-(p_1) \to e^-(p_4) + \nu_{\mu}(p_3) + \nu_e(p_2)$ (see Feynman rules in Section 1.6 or Vol. 1, Chapt. 6 for more details)

$$\begin{aligned} S_{fi} - \delta_{fi} = &-(2\pi)^4\delta^4(p_1 - p_2 - p_3 - p_4)\overline{u}(p_3)\left(\frac{ig_W}{\sqrt{2}}\gamma^{\mu}\right)\frac{(1-\gamma^5)}{2}u(p_1) \\ &\times \frac{-i\left(g_{\mu\nu} - \frac{q^{\mu}q^{\nu}}{m_W^2}\right)}{q^2 - m_W^2 + i\epsilon}\overline{u}(p_4)\left(\frac{ig_W}{\sqrt{2}}\gamma^{\nu}\right)\frac{(1-\gamma^5)}{2}v(p_2) \\ &\xrightarrow{m_W^2 \gg q^2} \sim i\frac{g_W^2}{8m_W^2}\overline{u}(p_3)\gamma^{\mu}(1-\gamma^5)u(p_1)\overline{u}(p_4)\gamma_{\mu}(1-\gamma^5)v(p_2) \end{aligned} \tag{1.43}$$

The expression agrees with the transition amplitude obtained from the four-Fermi theory provided (compare this with Vol. 1, Eq. (15.75))

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2} = \frac{1}{2v^2} \tag{1.44}$$

Inserting the value of $G_F = 1.16637 \times 10^{-5}\ \text{GeV}^{-2}$, we obtain $v = 246\,\text{GeV}$.

Mass of the Fermion Let us take a look at the Higgs-fermion interaction (the third term in the second line of Eq. (1.16)).

$$-\mathcal{L}_{Hee} \equiv G_e[\overline{e}_R(\Phi^\dagger \Psi_L) + (\overline{\Psi}_L \Phi)e_R] \tag{1.45}$$

After the symmetry breakdown, the Higgs-fermion interaction becomes

$$-\mathcal{L}_{Hee} \to \frac{G_e}{\sqrt{2}}(v + H)(\overline{e}_R e_L + \overline{e}_L e_R) = m_e \overline{e} e + \frac{m_e}{v}(\overline{e} e)H\,, \quad m_e = \frac{G_e v}{\sqrt{2}} \tag{1.46}$$

The first term gives the mass and the second, the Higgs interaction with the fermion. As a by-product of obtaining the fermion mass term, a new interaction of the Higgs with the fermion has appeared. Its interaction can be obtained by replacing the mass term v by $v + H$, just as the interaction with the gauge bosons in Eq. (1.37). Notice that the coupling strength is proportional to the mass of the fermion. Feynman diagrams for the Higgs-fermion interaction are given in Section 1.6.

One important side effect of the fermion mass generation is that the axial current no longer conserves.

$$\partial_\mu A^\mu(x) = \partial_\mu \overline{\psi}(x)\gamma^5\gamma^\mu\psi(x) = 2m\overline{\psi}(x)\psi(x) \neq 0 \tag{1.47}$$

This applies to the charged current as well as to the neutral current. Therefore, the chiral gauge symmetry is broken by the mass term and only the vector current conserves.[8] One refers to the phenomenon as the symmetry breakdown of $SU(2)_L \times SU(2)_R \to SU(2)_V$.

Mass of the Higgs The Lagrangian of the Higgs field can be extracted from the first and second term on the second line of Eq. (1.16). If we use the transformed Φ, we note that $\Phi^\dagger\Phi = (v + H)^2/2$ and $(\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) = \partial_\mu H \partial^\mu H/2$. Putting aside

8) This is not to be confused with the chiral symmetry breaking in QCD. In QCD, the chiral symmetry breaking also occurs due to spontaneous breakdown of the QCD vacuum. More details will be given in Section 7.1.5. It is a global chiral symmetry breaking caused by the strong interaction among the quarks, and $q\overline{q}'$ plays the role of Higgs. Incidentally, this was the Nambu's original proposal of the spontaneous symmetry breaking [45, 46]. In QCD, additional explicit chiral symmetry breaking is induced by the fermion mass term. It is treated as an external perturbation because it is a product of the electroweak symmetry breaking.

the interactions with the gauge field and fermions, the Higgs part of the Lagrangian is expressed as

$$\begin{aligned}\mathcal{L}_H &= \frac{1}{2}\partial_\mu H \partial^\mu H - \lambda v^2 H^2 - \left(\lambda v H^3 + \frac{\lambda}{4} H^4\right) \\ &= \frac{1}{2}\left(\partial_\mu H \partial^\mu H - m_H^2 H^2\right) - \frac{g_W}{4 m_W} m_H^2 H^3 - \frac{1}{32}\frac{g_W^2}{m_W^2} m_H^2 H^4 \end{aligned} \tag{1.48a}$$

$$m_H^2 = 2\lambda v^2 \tag{1.48b}$$

where Eq. (1.39) was used to express the coupling constants in terms of masses and gauge couplings. The coupling strength, again, is proportional to the masses of the interacting particles. This is a conspicuous feature of the Standard Model. From this Lagrangian, one can construct the Higgs propagator and Feynman rules for the interactions of the Higgs field. They are listed in Section 1.6.

1.4 Gauge Interactions

Coupling with Fermions Interactions of the gauge boson with fermions are contained in the covariant derivative and are given by Eq. (1.21). As it is the starting point to derive the Feynman rules for the gauge interactions, we reproduce it here.

$$\begin{aligned}-\mathcal{L}_{Wff} &= g_W \overline{\Psi}\gamma^\mu \boldsymbol{W}_\mu \cdot \boldsymbol{t}\Psi + \frac{g_B}{2}\overline{\Psi}\gamma^\mu \Psi B_\mu \\ &= \frac{g_W}{2}\overline{\Psi}_\mathrm{L}\gamma^\mu \left(W_\mu^1 \tau_1 + W_\mu^2 \tau_2 + W_\mu^3 \tau_3\right)\Psi_\mathrm{L} \\ &\quad + \frac{g_B}{2} B_\mu \left(\overline{\Psi}_\mathrm{L}\gamma^\mu Y_\mathrm{L}\Psi_\mathrm{L} + \overline{\Psi}_\mathrm{R}\gamma^\mu Y_\mathrm{R}\Psi_\mathrm{R}\right)\end{aligned} \tag{1.49a}$$

where $\boldsymbol{t} = \boldsymbol{\tau}/2$ was used. We removed the right-handed fields from the W interaction and added the hypercharge operator "Y" explicitly to remind the reader that both $\boldsymbol{W}$ and B_μ act differently on the left- and right-handed fields. Using Eq. (1.24) to rewrite W_μ^3 and B_μ in terms of A_μ and Z_μ, and denoting the charged W bosons as $W^\mp = (W_1 \pm W_2)/\sqrt{2}$, we have

$$\begin{aligned}-\mathcal{L}_{Wff} &= \frac{g_W}{\sqrt{2}}\overline{\Psi}_\mathrm{L}\gamma^\mu \left(W_\mu^+ \tau_+ + W_\mu^- \tau_-\right)\Psi_\mathrm{L} \\ &\quad + \overline{\Psi}_\mathrm{L}\gamma^\mu \left[g_W I_3 (c_W Z_\mu + s_W A_\mu) + \frac{g_B}{2} Y_\mathrm{L}(-s_W Z_\mu + c_W A_\mu)\right]\Psi_\mathrm{L} \\ &\quad + \overline{\Psi}_\mathrm{R}\gamma^\mu \left[\frac{g_B}{2} Y_\mathrm{R}(-s_W Z_\mu + c_W A_\mu)\right]\Psi_\mathrm{R}\end{aligned} \tag{1.49b}$$

where we abbreviated $s_W = \sin\theta_W$, $c_W = \cos\theta_W$. Using $Q = I_3 + Y/2$, $I_3\Psi_R = 0$, we finally obtain

$$
\begin{aligned}
-\mathcal{L}_{Wff} &= \frac{g_W}{\sqrt{2}}\overline{\Psi}_L\gamma^\mu\left(W_\mu^+\tau_+ + W_\mu^-\tau_-\right)\Psi_L + g_Z\overline{\Psi}\gamma^\mu(I_{3L} - Qs_W^2)\Psi Z_\mu \\
&\quad + e\overline{\Psi}\gamma^\mu Q\Psi A_\mu \\
e &= g_W\sin\theta_W\,, \quad g_Z = g_W/\cos\theta_W = e/\sin\theta_W\cos\theta_W
\end{aligned}
\tag{1.50}
$$

which defines the electromagnetic coupling constant (i. e. the electric charge) in terms of g_W and the Weinberg angle. Here, L in I_{3L} is there to remind the reader that it only acts on the left-handed components.

For actual calculations, it is more convenient to separate couplings to the left and right or alternatively, to vector and axial vector parts. Coupling types which appear in the Feynman amplitude rule are given by the matrix element $i\mathcal{L}_{Wff}$. Therefore, omitting the field operators and attaching a suffix to differentiate the fermion flavor, they are expressed as

$$\gamma - ff: \quad -iQ_f e\gamma^\mu \tag{1.51a}$$

$$W^\pm - ff: \quad -i\frac{g_W}{2\sqrt{2}}\gamma^\mu(1-\gamma^5) \tag{1.51b}$$

$$
\begin{aligned}
Z - ff: \quad & -i\frac{g_Z}{2}\gamma^\mu\left[\epsilon_L(f)(1-\gamma^5) + \epsilon_R(f)(1+\gamma^5)\right] \\
&= -i\frac{g_Z}{2}\gamma^\mu\left(v_f - a_f\gamma^5\right)
\end{aligned}
\tag{1.51c}
$$

$$\epsilon_L(f) = I_3 - Q_f s_W^2\,, \quad \epsilon_R(f) = -Q_f s_W^2 \tag{1.51d}$$

$$v_f = I_3 - 2Q_f s_W^2\,, \quad a_f = I_3 \tag{1.51e}$$

The Feynman rules are given in Section 1.6.

Self-interactions of the Gauge Boson The Lagrangian of the non-Abelian gauge field contains, in addition to the quadratic kinetic term, higher powers of the field's operator which represent self-interactions.

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} = -\frac{1}{4}\sum_A F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} \tag{1.52a}$$

$$
\begin{aligned}
\boldsymbol{F}_{A\mu\nu} &= \partial_\mu\boldsymbol{W}_{A\nu} - \partial_\nu\boldsymbol{W}_{A\mu} - g_W(\boldsymbol{W}_\mu\times\boldsymbol{W}_\nu)_A \\
(\boldsymbol{W}_\mu\times\boldsymbol{W}_\nu)_A &= \sum_{B,C}\epsilon_{ABC}W_{B\mu}W_{C\nu}
\end{aligned}
\tag{1.52b}
$$

$$F_{B\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.52c}$$

The self energy of the gauge boson is given by its triple and quartic terms in Eq. (1.52a). It can be unpacked into

$$\begin{aligned}\mathcal{L}_{\rm GWS} &= \mathcal{L}_{\rm KE} + \mathcal{L}_{\rm SE} \\ &= -\frac{1}{2}(\partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu)\cdot(\partial^\mu \boldsymbol{W}^\nu) \\ &\quad + g_W(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu \cdot \partial^\mu \boldsymbol{W}^\nu) - \frac{g_W^2}{4}\left[(\boldsymbol{W}_\mu \cdot \boldsymbol{W}^\mu)^2 - (\boldsymbol{W}_\mu \cdot \boldsymbol{W}_\nu)(\boldsymbol{W}^\mu \cdot \boldsymbol{W}^\nu)\right]\end{aligned} \tag{1.52d}$$

From this Lagrangian, one can derive Feynman amplitudes corresponding to 3-gauge and 4-gauge boson vertices.

3-W vertex: Let us first treat the 3-W vertex matrix elements. Rewriting

$$\begin{aligned}\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu \cdot \partial^\mu \boldsymbol{W}^\nu &= (W_{1\mu} W_{2\nu} - W_{2\mu} W_{1\nu})\partial^\mu W_3^\nu + (\text{cyclic}) \\ &= i(W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^-)\partial^\mu (c_W Z + s_W A)^\nu \\ &\quad + (\text{cyclic})\end{aligned} \tag{1.53}$$

where we used $W^\pm = (W_1 \mp i W_2)/\sqrt{2}$ and Eq. (1.24). Feynman diagrams of the interaction can be obtained by calculating matrix elements of Eq. (1.53).

As an example, let us consider the $\gamma \to W^+ W^-$ vertex. For symmetry reasons, we treat all the momenta as incoming (see Figure 1.5). Denoting the polarization and momentum of the incoming photon as ϵ_3, k_3 and those of outgoing W's as $\epsilon_n, -k_n, n = 1, 2$, the scattering matrix element of the 3-W vertex is given by

$$\begin{aligned}S_{fi} - \delta_{fi} &= -(2\pi)^4 \delta^4(k_1 + k_2 + k_3) i \mathcal{M}_{fi} \\ &= i \int d^4x \langle \epsilon_1(W^-, -k_1); \epsilon_2(W^+, -k_2)|[\cdots]|\epsilon_3(\gamma, k_3)\rangle \\ &= (2\pi)^4 \delta^4(k_1 + k_2 + k_3)(i)(i g_W s_W) \\ &\quad \times \left[\{(\epsilon_{1\mu}\epsilon_{2\nu}) - (\epsilon_{2\mu}\epsilon_{1\nu})\}(-i k_3^\mu)\epsilon_3^\nu + (\text{cyclic})\right]\end{aligned} \tag{1.54a}$$

$$\begin{aligned}[\cdots] = \Big[(i g_W)\left\{W_\mu^-(x) W_\nu^+(x) - W_\mu^+(x) W_\nu^-(x)\right\} \\ \times \partial^\mu \{c_W Z(x) + s_W A(x)\}^\nu\Big]\end{aligned} \tag{1.54b}$$

Omitting the ever present δ function, the matrix element becomes

$$\begin{aligned}-i\mathcal{M}_{fi} &= (-i g_W s_W)[(k_3 \cdot \epsilon_1)(\epsilon_2 \cdot \epsilon_3) - (k_3 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_1) + (\text{cyclic})] \\ &= i e[\{(k_1 - k_2)\cdot \epsilon_3\}(\epsilon_1 \cdot \epsilon_2) + \{(k_2 - k_3)\cdot \epsilon_1\}(\epsilon_2 \cdot \epsilon_3) \\ &\quad + \{(k_3 - k_1)\cdot \epsilon_2\}(\epsilon_3 \cdot \epsilon_1)] \\ &= \varepsilon^\mu(W^-)\varepsilon^\nu(W^+) \\ &\quad \times [(i e)[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu]]\varepsilon^\lambda(\gamma)\end{aligned} \tag{1.55}$$

The Feynman amplitude for the vertex is given by the content in the bracket.

$$ie[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu] \tag{1.56}$$

4-W vertex: The Lagrangian of quartic interaction is given by

$$\begin{aligned}
\mathcal{L}_{4W} &= -\frac{g_W^2}{4}(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu)\cdot(\boldsymbol{W}^\mu \times \boldsymbol{W}^\nu)\\
&= -\frac{g_W^2}{4}\left[(\boldsymbol{W}_\mu \cdot \boldsymbol{W}^\mu)(\boldsymbol{W}_\nu \cdot \boldsymbol{W}^\nu) - (\boldsymbol{W}_\mu \cdot \boldsymbol{W}_\nu)(\boldsymbol{W}^\mu \cdot \boldsymbol{W}^\nu)\right] \qquad (1.57a)\\
&= -\frac{g_W^2}{4}\Big[(W_1^2 + W_2^2 + W_3^2)^2\\
&\quad - (W_{1\mu}W_{1\nu} + W_{2\mu}W_{2\nu} + W_{3\mu}W_{3\nu})\left(W_1^\mu W_1^\nu + W_2^\mu W_2^\nu + W_3^\mu W_3^\nu\right)\Big]\\
&= -\frac{g_W^2}{2}\Big[\{W_1^2W_2^2 - (W_1\cdot W_2)^2\}\\
&\quad + \{(W_1^2 + W_2^2)\,W_3^2 - (W_1\cdot W_3)^2 - (W_2\cdot W_3)^2\}\Big]\\
&= g_W^2\left[\frac{1}{2}\left\{(W^+)^2(W^-)^2 - (W^+\cdot W^-)^2\right\}\right.\\
&\quad \left. - \left\{(W^+\cdot W^-)X^2 - (W^+\cdot X)(W^-\cdot X)\right\}\right] \qquad (1.57b)
\end{aligned}$$

$$X = c_W Z + s_W A \tag{1.57c}$$

where $A^2, (A\cdot B)$ stand for the Lorentz scalar product $A_\mu A^\mu, (A_\mu B^\mu)$. The first bracket describes the $W^+W^- \leftrightarrow W^+W^-$ process and the second $A(Z)A(Z) \leftrightarrow W^+W^-$.

Next, we consider $A_\alpha Z_\beta \to W_\mu^+ W_\nu^-$. By taking the matrix element and removing the δ function, we can get the scattering matrix amplitude

$$\begin{aligned}
-i\mathcal{M} &\sim i\int d^4x\,\mathcal{L}_{4W}(x)\\
&\sim -ig_W^2 s_W c_W\langle\varepsilon(W^-)\varepsilon(W^+)|[(W^+\cdot W^-)X^2\\
&\quad - (W^+\cdot X)(W^-\cdot X)]|\varepsilon(\gamma)\varepsilon(Z)\rangle\\
&= -ie^2\cot\theta_W[2\{\varepsilon(W^-)\cdot\varepsilon(W^+)\}\{\varepsilon(\gamma)\cdot\varepsilon(Z)\}\\
&\quad - \{\varepsilon(W^-)\cdot\varepsilon(Z)\}\{\varepsilon(W^+)\cdot\varepsilon(\gamma)\} - \{\varepsilon(W^-)\cdot\varepsilon(\gamma)\}\{\varepsilon(W^+)\cdot\varepsilon(Z)\}]\\
&= \varepsilon^\mu(W^+)\varepsilon^\nu(W^-)\left[-ie^2\cot\theta_W(2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})\right]\\
&\quad \times\varepsilon^\alpha(\gamma)\varepsilon^\beta(Z) \qquad (1.58)
\end{aligned}$$

Other processes can be calculated similarly. The Feynman diagrams for the triple and quartic coupling of the gauge bosons are summarized in Section 1.6.

1.5 Higgs Interactions

Coupling with Gauge Bosons We have seen that the mass is generated in the Higgs sector and that its value is proportional to the vacuum expectation value "v." As the physical Higgs always appears in combination with the vacuum expectation value, its coupling is closely related with the mass of particles which it couples. It is obtained by replacing v by $(v + H)$ as is seen in Eq. (1.37) and Eq. (1.46). The interaction Lagrangian for the Higgs-gauge coupling is obtained by expanding Eq. (1.37) and picking terms that contain the Higgs field H, that is,

$$\begin{aligned}\mathcal{L}_{W-H} &= \left(\frac{g_W(v+H)}{2}\right)^2 W_\mu^- W^{+\mu} + \frac{1}{2}\left(\frac{g_Z(v+H)}{2}\right)^2 Z_\mu Z^\mu - \mathcal{L}_{\text{mass}} \\ &= \frac{g_W^2}{2} v H W_\mu^- W^{+\mu} + \frac{g_Z^2}{4} v H Z_\mu Z^\mu + \frac{g_W^2}{4} H^2 W_\mu^- W^{+\mu} \\ &\quad + \frac{g_Z^2}{8} H^2 Z_\mu Z^\mu \\ &= \left[g_W m_W W_\mu^- W^{+\mu} + \frac{1}{2} g_Z m_Z Z_\mu Z^\mu\right] H \\ &\quad + \left[\frac{g_W^2}{4} W_\mu^- W^{+\mu} + \frac{1}{2}\frac{g_Z^2}{4} Z_\mu Z^\mu\right] H^2 \end{aligned} \tag{1.59}$$

It contains interactions of the type HWW, HZZ and H^2W^2, H^2Z^2. Because of Eq. (1.39), the coupling strength can be expressed in terms of the mass of the particle.

$$\begin{aligned} HWW &: \tfrac{g_W^2 v}{2} = g_W m_W\,, & HZZ &: \tfrac{g_Z^2 v}{2} = g_Z m_Z \\ HHWW &: \tfrac{g_W^2}{4} = \tfrac{m_W^2}{v^2}\,, & HHZZ &: \tfrac{g_Z^2}{4} = \tfrac{m_Z^2}{v^2} \end{aligned} \tag{1.60}$$

Feynman diagrams for these interactions are given in Section 1.6.

Coupling with Fermions The coupling is again obtained from the fermion mass term by replacing $v \to v + H$ which is given by the second term of the last equality of Eq. (1.46).

$$\mathcal{L}_{Hff} = -\frac{m_f}{v}\overline{\psi}_f \psi H = -g_W \frac{m_f}{2m_W}\overline{\psi}_f \psi_f H \tag{1.61}$$

Self Coupling The self-interaction of the Higgs is already given in Eq. (1.48).

$$\mathcal{L}_{\text{H-self}} = -\frac{g_W}{4m_W} m_H^2 H^3 - \frac{1}{32}\frac{g_W^2}{m_W^2} m_H^2 H^4 \tag{1.62}$$

The expression is simple, but in extracting matrix elements, one needs to count the symmetry factor carefully, for instance, $\langle 0|H^3|h_1h_2h_3\rangle \to 3!$ and $\langle 0|H^4|h_1h_2h_3h_4\rangle \to 4!$ where h_i's are the ith Higgs particles. Then, the Feynman rules are given by

$$-i\frac{3}{2}\frac{g_W}{m_W}m_H^2: \quad \text{triple-Higgs interaction} \tag{1.63a}$$

$$-i\frac{3}{4}\frac{g_W^2}{m_W^2}m_H^2: \quad \text{quartic-Higgs interaction} \tag{1.63b}$$

1.6
Feynman Rules of Electroweak Theory

Now that we have given all the rules to calculate transition matrix elements for the electroweak interaction, we summarize the Feynman rules in the unitary gauge below.

Feynman Rule 1: External Lines: We attach wave functions to fermions or polarizations to bosons for each incoming or outgoing particle (Fig. 1.2). Spinor indices for fermions are sometimes omitted.

Feynman Rule 2: Internal Lines To each internal line, we attach one of the propagators depicted in Figure 1.3, depending on the particle species. For fermions, the sign of momentum follows that of an arrow.

Feynman Rule 3: Fermion-Gauge Boson Vertices For vertices of fermions and gauge bosons, we attach coupling constants and appropriate γ factors (Fig. 1.4). The photon couples to the electromagnetic current with charge $Q_f e$ and is of the vector

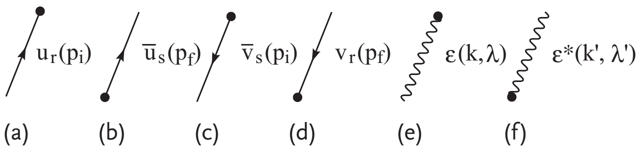


Figure 1.2 Wave functions to fermions and polarization vectors to bosons are to be attached to each external line.

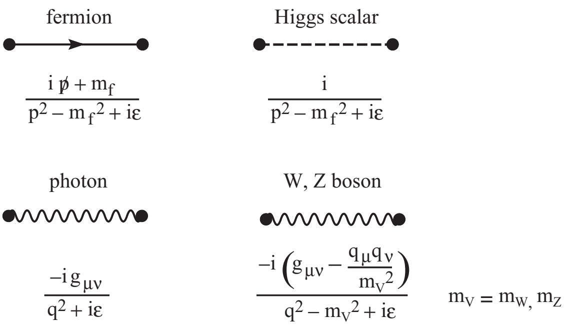


Figure 1.3 Propagators are to be attached to each internal line.

type. Q_fs are given by

$$\text{Leptons}: \begin{cases} Q_{\nu_e} = Q_{\nu_\mu} = Q_{\nu_\tau} = 0 \\ Q_e = Q_\mu = Q_\tau = -1 \end{cases}$$

$$\text{Quarks}: \begin{cases} Q_u = Q_c = Q_t = +2/3, \\ Q_d = Q_s = Q_b = -1/3 \end{cases} \tag{1.64}$$

with opposite charge assigned to the antifermions. The neutral Z boson couples to the neutral current which is a mixture of the left- and right-handed fermions. Its coupling constant is a product of a common constant

$$g_Z = \frac{e}{\sin\theta_W \cos\theta_W} \tag{1.65}$$

and flavor dependent constants

$$\epsilon_L(f) = I_{3f} - Q_f \sin^2\theta_W\,, \quad \epsilon_R(f) = -Q_f \sin^2\theta_W \tag{1.66}$$

where $f = e, \mu, \tau$ or u, d, s, c, b, t. An alternative expression in terms of the vector and axial vector couplings is also used. Using

$$g_Z \gamma^\mu \left(\varepsilon_L(f) \frac{1-\gamma^5}{2} + \varepsilon_R(f) \frac{1+\gamma^5}{2} \right) = \frac{g_Z}{2} \gamma^\mu (v_f - a_f \gamma^5) \tag{1.67}$$

the vector and the axial vector couplings are expressed as

$$v_f = I_3 - 2Q_f \sin^2\theta_W\,, \quad a_f = I_3 \tag{1.68}$$

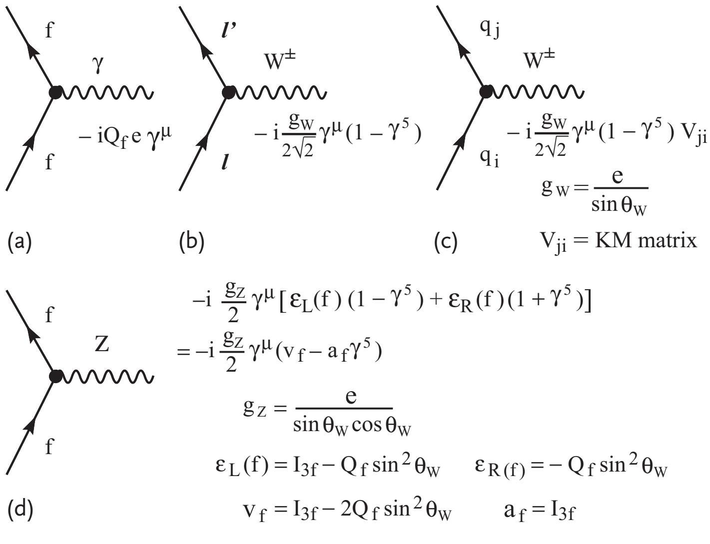


Figure 1.4 Vertices of fermions with gauge bosons.

They are mutually related by

$$v_f = \epsilon_{\mathrm{L}}(f) + \epsilon_{\mathrm{R}}(f)\,, \quad a_f = \epsilon_{\mathrm{L}}(f) - \epsilon_{\mathrm{R}}(f) \tag{1.69}$$

The charged W boson couples to left-handed fermions and its strength is given by

$$g_W = \frac{e}{\sin\theta_{\mathrm{W}}} \tag{1.70}$$

Notice that the fields that appear in the original Lagrangian are so-called weak eigenstates. However, when we calculate cross sections, we use mass eigenstates. For the electromagnetic and neutral current interactions, we do not need to differentiate the mass eigenstates from the weak eigenstates, but for the charged current interactions they are different. They are related by the Cabibbo–Kobayashi–Maskawa (CKM) matrix V_{ji}[9]. Therefore, for the $W^\pm$-fermion interaction with up-quark j ($= u, c, t$) and down-quark i ($= d, s, b$), the CKM elements V_{ji} have to be attached.

Feynman Rule 4: Nonlinear Couplings of the Gauge Bosons: Because of the non-Abelian nature of the electroweak theory, the gauge bosons have self couplings which were given in Eqs. (1.56) and (1.58). Their Feynman graphs are shown in Figure 1.5. Note that there are no γZZ or ZZZ couplings. In the figure, all the momenta are taken to be inward going.

Feynman Rule 5: Higgs Couplings: In Figure 1.6 we list vertices which include at least one Higgs particle. Notice, the coupling strength is proportional to the mass of the connecting particles.

Feynman Rule 6: Momentum Assignment and Loops The momenta of external lines are fixed by experimental conditions. Then, at each vertex, the energy-momenta have to conserve. The energy-momentum conservation constrains that sum of all energy-momenta of external lines have to vanish assuming all the external momenta are inward going. It also fixes all the momenta for tree diagrams which do not contain loops. Each loop leaves one momentum unconstrained and has to be integrated, leading to divergent integrals. The integration includes a sum over spinor indices and polarizations, depending on the particle species that form the loop. For each closed fermion loop, an extra sign (–) has to be attached. It is a result of the anticommutativity of the fermion fields.

Amplitude for $e^-e^+ \to f\overline{f}$: Once all the Feynman diagrams are given, calculation of scattering amplitudes can be carried out in a straightforward way. As an example, we construct an amplitude for the reaction $e^-e^+ \to f\overline{f}$ in the $O(\alpha^2)$ process where f is any of the leptons or quarks.

9) Details of the CKM matrix elements are discussed in Chapter 6.

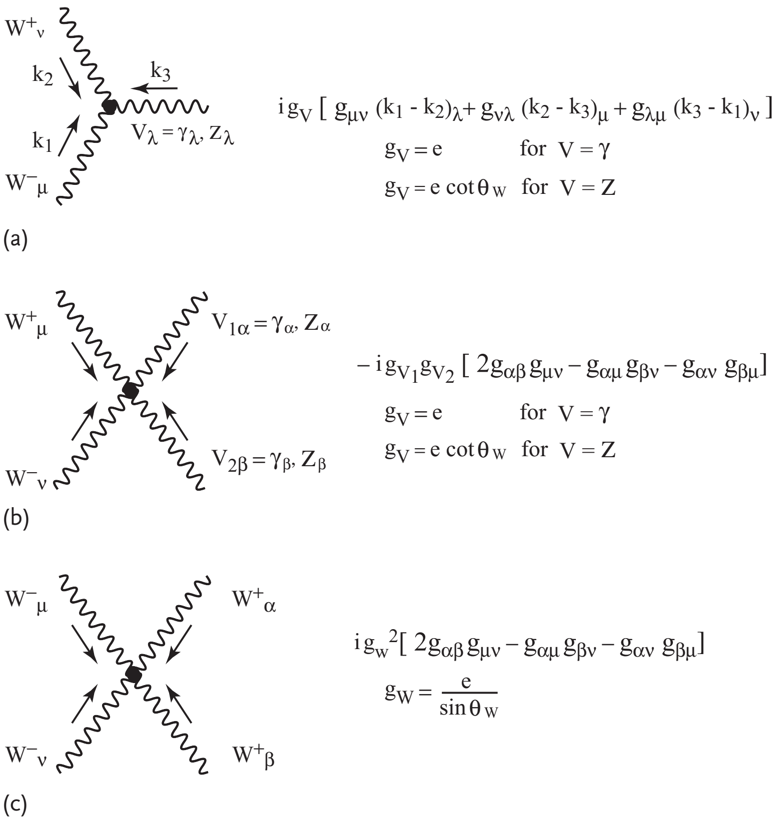


Figure 1.5 Nonlinear gauge boson couplings.

According to the Feynman rules we just described, we attach appropriate functions to every part of the Feynman diagram as shown in Figure 1.7. The S-matrix and the cross section is written as

$$S_{fi} = \delta_{fi} - (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) i\mathcal{M} \tag{1.71a}$$

$$d\sigma = \frac{1}{F} \overline{\sum}_{\substack{\text{spin}\\ \text{pol}}} |\mathcal{M}|^2 dLIPS \tag{1.71b}$$

$$dLIPS = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \tag{1.71c}$$

$$F = 4\left[(p_1 \cdot p_2)^2 - (m_1 m_2)^2\right] \simeq 2s \quad \text{for} \quad s = (p_1 + p_2)^2 \gg m_1^2, m_2^2 \tag{1.71d}$$

where F is the initial flux and $dLIPS$ is the Lorentz invariant phase space of the final state. $\overline{\sum}$ denotes the average of the initial state and sum over final state degrees of freedom which is valid when polarizations are not observed. Referring to the

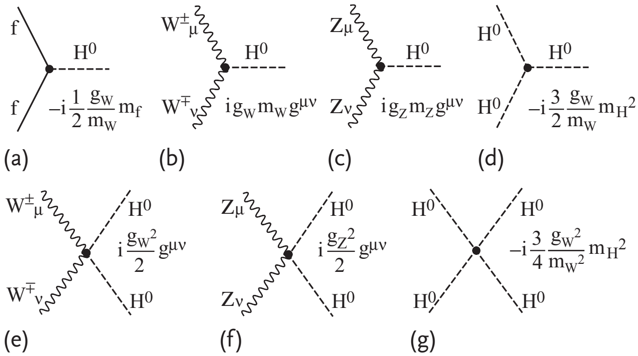


Figure 1.6 Vertices that include the Higgs boson.

Feynman diagram in Figure 1.7, the transition amplitude $\mathcal{M}$ can be written as

$$-i\mathcal{M} = \left[\overline{u}(p_3)\left\{-i\frac{g_Z}{2}\gamma^\mu(v_f - a_f\gamma^5)\right\} v(p_4)\right] \frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_Z^2}\right)}{q^2 - m_Z^2 + i\epsilon}$$

$$\times \left(-i\Sigma^{\nu\lambda}_{\gamma Z}(q^2)\right)\frac{-ig_{\lambda\rho}}{q^2+i\epsilon}\left[\overline{v}(p_2)\,(-iQ_i e\gamma^\rho)\,u(p_1)\right] \tag{1.72a}$$

$$-i\Sigma^{\nu\lambda}_{\gamma Z}(q^2) = -\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i(\not q - \not p)+m_f}{(q-p)^2 - m_f^2 + i\epsilon}\left\{-i\frac{g_Z}{2}\gamma^\nu(v_f - a_f\gamma^5)\right\}\right.$$

$$\left.\times \frac{i\not p + m_f}{p^2 - m_f^2+i\epsilon}(-iQ_f e\gamma^\lambda)\right] \tag{1.72b}$$

where $\overline{u}(p_3)$, $u(p_1)$, ... are plane wave solutions of the Dirac equation [see Appendix A]. We have separated the fermion loop part of the Feynman diagram because it has to be integrated over the internal momentum and an extra (–) sign has been attached according to the rule (6). $\Sigma^{\nu\lambda}_{\gamma Z}(q^2)$ is a diverging integral and has to be treated with a renormalization prescription which will be discussed in detail in Chapter 5 and in Appendix C[10].

A Note on the Ghosts The loop Feynman diagram in Figure 1.7 was given just to illustrate how the Feynman rules work in the unitary gauge. In general, once we go to higher order diagrams which contain loops, things are more complicated and we need to consider ghost's contributions. We did not include the ghosts in our Lagrangian and their associated Feynman rules were not given either because technical details of higher order calculations are beyond the scope of this book. We only mention their role in the non-Abelian gauge theories.[11]

10) Generally, in this book, we derive cross sections only at the tree level and describe higher order corrections qualitatively. The only exception is the description of precision Z resonance data in Chapter 5 which are compared with theoretical calculations including radiative corrections.

11) Some notes are given in the Appendix D for settings of the R-gauge and Feynman rules for the ghosts.

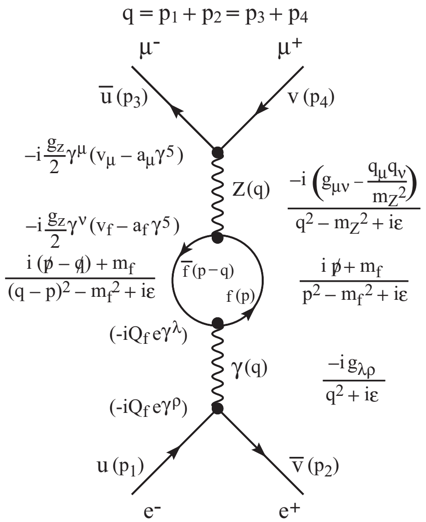


Figure 1.7 An example of the Feynman diagram in order $O(\alpha^2)$ for the process $e^- e^+ \rightarrow \mu^- \mu^+$. To every element of the Feynman diagram (wave functions, vertices and propagators), corresponding functions are attached.

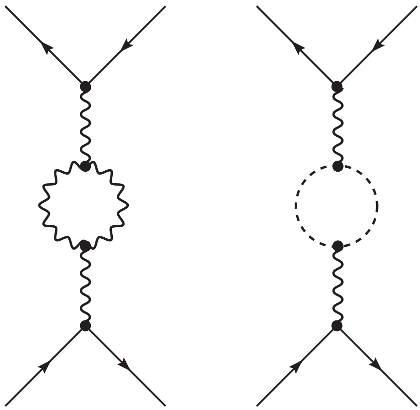
Ghosts are fictitious scalar fields having the same isospin degrees of freedom as the gauge particles, but obeying the Fermi-Dirac statistics. They are mathematical artifacts that appear in the covariant gauge and only appear in the internal lines of the Feynman diagrams. They do not appear in the physical gauge[12]. They only couple to the gauge fields. Their sole role is to compensate the unphysical degrees of contributions in the loop generated by self-interactions of the non-Abelian gauge fields in the internal lines. Unphysical contributions are generated by unphysical components, that is, scalar components of the massive gauge particle. Therefore, whenever loop diagrams of the non-Abelian gauge fields like that in Figure 1.8a appear, the ghosts (Figure 1.8b) have to be included to compensate unphysical contributions.

1.7
Roles of the Higgs in Gauge Theory

Unitary Gauge and R-Gauge So far, we emphasized the role of the Higgs field in attaching mass to the gauge as well as matter particles. Here, we describe another role in maintaining renormalizability of spontaneously broken gauge theories.

A major difficulty in the theory of weak interaction is the existence of massive gauge bosons because they violate the gauge symmetry. In the GWS theory, it has been shown that the symmetry is not broken, but hidden. In the unitary gauge, the dynamical variables are chosen to match observed phenomena. However, in this

12) Coulomb gauge in QED and axial gauge in non-Abelian gauge. See Section 7.1.1.



(a) (b)

Figure 1.8 The non-Abelian ghost field which only appears as internal lines. The propagation of the ghosts are denoted by dashed lines. Feynman diagrams containing a ghost loop (b) cancels unphysical contributions created by a self-interacting non-Abelian gauge loop (a).

gauge, the massive vector bosons have three degrees of freedom corresponding to three polarization states. The longitudinal polarization has components

$$\epsilon^\mu(3) = (0, 0, 0, 1) \tag{1.73}$$

in the particle's rest frame. When it is in motion, it is Lorentz boosted and has components

$$\epsilon_\mathrm{L} = \left(\frac{|\boldsymbol{k}|}{m}, 0, 0, \frac{\omega}{m}\right) = \frac{k^\mu}{m} + \frac{m}{\omega + |\boldsymbol{k}|}(-1, \hat{k}), \quad \hat{k} = \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \tag{1.74}$$

in a coordinate system where the particle momentum is expressed as $k^\mu = (\omega, 0, 0, k)$, $\omega = \sqrt{|\boldsymbol{k}|^2 + m^2}$. Accordingly, the propagator of a massive vector meson takes a form

$$i\Delta_F(k) = \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_V^2}\right)}{k^2 - m_V^2} \tag{1.75}$$

At high energy as $k \to \infty$, the value of the propagator approaches a constant and is the cause of bad divergences in the loop integral (see Appendix C or Vol. 1, Sect. 15.8). In QED, we saw that the gauge invariance controlled the divergence in order to not grow faster than the logarithm of the momenta. There, the gauge propagator behaved like $\sim 1/k^2$ and the divergences were removed by introducing a few number of counter terms, in other words, the theory was renormalizable (see Vol. 1, Chapt. 8). However, presence of the longitudinal polarization adds an extra diverging contribution. This is why the unitarity of the process involving the massive gauge boson is broken. However, if the gauge symmetry is not really broken, but merely hidden, there must be a mechanism in the framework of spontaneous symmetry breaking to guarantee that the divergence created by the longitudinal polarization of the gauge boson is somehow canceled.

'’t Hooft conceived of a clever gauge containing ξ as a parameter. Its formal setting is given in Appendix D. Here, we only mention its usefulness for higher order calculations and for the role of the longitudinal polarization. In this gauge (referred to as the R-gauge), the vector boson propagator is expressed as

$$i\Delta_F(k) = -\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2 - \xi m^2}\right]\frac{i}{k^2 - m^2} \tag{1.76}$$

For $m \to 0$, it reproduces the photon propagator and by setting $\xi = 1$, it becomes the Feynman gauge in QED. The ordinary massive vector propagator can be reproduced by setting $\xi = \infty$. However, for such a setting, which is the case in the U-gauge, many divergent terms appear. If the theory is convergent as claimed, one has to carry out the algebra very carefully, otherwise one easily gets lost.

As long as ξ is kept finite, the propagator has a built-in cut-off and the longitudinal part can be calculated without difficulty. Setting $\xi = 1$, which is referred to as the ’t Hooft Feynman gauge, makes the calculation especially simple. Therefore, it is the preferred setting for most theoretical calculations. Only logarithmic divergences appear in the R-gauge and the calculation can be carried out in a straightforward way.

From a renormalization point of view, one sees that $\Delta_F \to 1/k^2$ as $k^2 \to \infty$ and is assured of the healthy theory applying the same logic to prove the renormalizability of the massless gauge boson, that is, the QED. Since it includes the U-gauge as a special choice of $\xi \to \infty$, the gauge invariance assures the renormalizability of the spontaneously broken gauge symmetry.

The price to pay is the appearance of the ξ dependent poles which has to be removed because it is not physical. Also, would-be-Goldstone bosons reappear. They vanish in the U-gauge because they are absorbed by the gauge bosons to become their longitudinal component. In the R-gauge, they appear as redundant degrees of freedom. However, it has been shown that the redundant would-be-Goldstone bosons exactly cancels the unwanted fictitious pole of the gauge bosons. The second price is that dynamical properties of each chosen variable in this gauge are not directly connected to observable quantities. Obtained mathematical results are hard to interpret in physical terms. Thus, the usual convention is to use the unitary gauge for physical interpretation, but rely on the R-gauge for actual calculation in order to address various theoretical technicalities.

Calculations are not difficult as long as one stays in the tree approximation. As we do not want to get involved in the higher order loop calculations too much, we will work in the unitary gauge in the following and restrict ourselves to qualitative discussions using a simple example. We calculate certain tree processes faithfully a’la Feynman rules and show how the renormalizability is restored in the spontaneously broken symmetry frame. Hopefully, we can obtain intuitive and clear insight by staying in the unitary gauge.

How is the Unitarity Maintained? Postulating the $SU(2)$ gauge symmetry for the weak interaction, the existence of the neutral vector boson W^0 was required. In the spontaneously broken symmetry, the existence of the Higgs field is also required. We shall see that they are necessary ingredients to keep the unitarity. Specifically, we show in qualitative arguments that their role is to eliminate badly diverging integrals induced by the massive gauge bosons. We take a simple example and see how the unitarity of the theory can be maintained.

$\boldsymbol{\nu\overline{\nu} \rightarrow W^+W^-}$: Let us consider the process of $\nu\overline{\nu} \rightarrow W^+W^-$ (Figure 1.9a). This is not a practically doable process, but serves as an illustration where the theoretical problem lies.

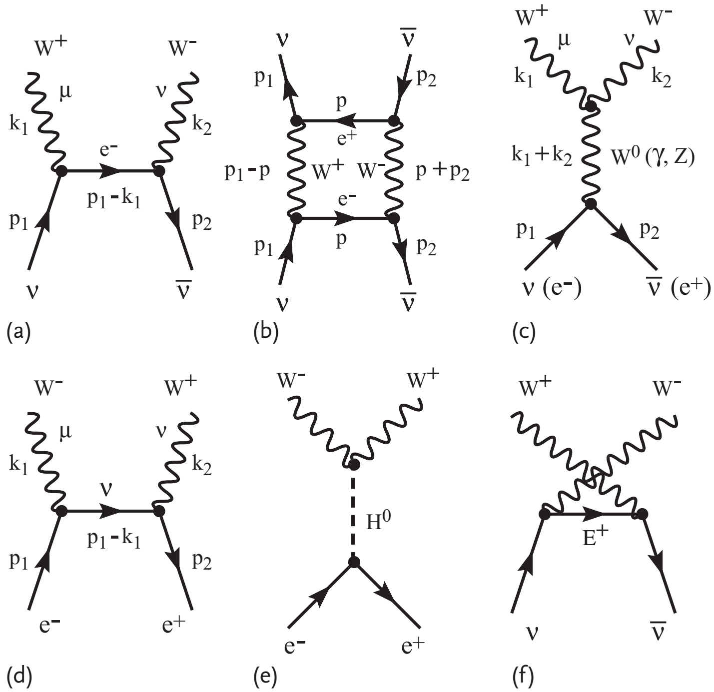


Figure 1.9 The role of various processes in the non-Abelian gauge theory. (a) Cross section for $\nu\overline{\nu} \rightarrow W^+W^-$ increases in proportion to the energy squared ($\sim s$). (b) Because of (a), integration over the W^+W^- intermediate state diverges badly. The remedy is two-fold: by introducing an additional contribution due to either a neutral W^0 in the s-channel (c) or a new charged E^+ in the u-channel (f), one possibly compensates the leading divergent term. (d) A massive fermion introduces another divergence. The wrong helicity component associated with massive fermions diverges. As it contributes to the $J = 0$ partial wave, it can be compensated by introducing a scalar particle (the Higgs) intermediate state (e).

Using the Feynman rules listed in Section 1.6, the scattering amplitude for the process is given by

$$\begin{aligned}
-i\mathcal{M}_a &= \epsilon^*_{2\mu}(\lambda_2)\overline{v}(p_2)\gamma^\mu\left(-i\frac{g_W}{2\sqrt{2}}(1-\gamma^5)\right)\\
&\quad\times\frac{i(\not{p}_1-\not{k}_1)+m_e}{(p_1-k_1)^2-m_e^2}\gamma^\nu\left(-i\frac{g_W}{2\sqrt{2}}(1-\gamma^5)\right)u(p_1)\epsilon_{1\nu}(\lambda_1)\\
&= -i\frac{g_W^2}{4}\overline{v}(p_2)\frac{\not{\epsilon}_2(\not{p}_1-\not{k}_1)\not{\epsilon}_1(1-\gamma^5)}{m_W^2-2(p_1\cdot k_1)}u(p_1)
\end{aligned} \tag{1.77}$$

The polarization vectors $\epsilon_\mu(\lambda), \lambda = 1-3$ satisfy $\epsilon(\lambda)\cdot\epsilon(\lambda') = -\delta_{\lambda\lambda'},\quad k\cdot\epsilon(\lambda) = 0$. An expression for the longitudinal polarization was given in Eq. (1.74). It shows that at high energy ($|\boldsymbol{k}| \gg m_W$), where we are interested in, the longitudinal polarization can be approximated as $\epsilon^\mu \simeq k^\mu/m_W$. Replacing ϵ^μ with k^μ/m_W, we have

$$\begin{aligned}
&\not{\epsilon}_2(\not{p}_1-\not{k}_1)\not{\epsilon}_1(1-\gamma^5) \sim \not{k}_2(\not{p}_1-\not{k}_1)\not{k}_1(1-\gamma^5)\\
&= \not{k}_2\left\{2(p_1\cdot k_1)-\not{k}_1\not{p}_1-m_W^2\right\}(1-\gamma^5)\\
&= -D\not{k}_2(1-\gamma^5)-m\not{k}_2\not{k}_1(1+\gamma^5)
\end{aligned} \tag{1.78a}$$

where $D = m_W^2 - 2(p_1\cdot k_1)$ and we used $\not{p}_1 u(p_1) = mu(p_1), \overline{v}(p_2)\not{p}_2 = -\overline{v}(p_2)m$. For $\nu\overline{\nu}$ reactions, $m = 0$, but we retain it for later discussions when the neutrino is replaced with the electron. Using $p_1 - k_1 = k_2 - p_2$, it can be rewritten as

$$\begin{aligned}
&= \not{k}_2(\not{k}_2-\not{p}_2)\not{k}_1(1-\gamma^5) = \left\{m_W^2-2(p_2\cdot k_2)+\not{p}_2\not{k}_2\right\}\not{k}_1(1-\gamma^5)\\
&= D\not{k}_1(1-\gamma^5)-m\not{k}_2\not{k}_1(1-\gamma^5)
\end{aligned} \tag{1.78b}$$

$$= \frac{1}{2}D(\not{k}_1-\not{k}_2)(1-\gamma^5)-m\not{k}_2\not{k}_1 \tag{1.78c}$$

Equation (1.78c) is obtained by taking average of Eq. (1.78a) and Eq. (1.78b). The second term in Eq. (1.78c) can further be rewritten as

$$-m\not{k}_2\not{k}_1 = mD + m^2(\not{k}_1-\not{k}_2) \tag{1.78d}$$

The second term is $O(m^2/s)$ compared to the first and can be neglected. Substituting Eqs. (1.78c) and (1.78d) in Eq. (1.77), the amplitude for $\nu\overline{\nu} \to W^+W^-$ at high energy is given by

$$-i\mathcal{M}_a = -i\frac{g_W^2}{8m_W^2}\overline{v}(p_2)[(\not{k}_1-\not{k}_2)(1-\gamma^5)+2m]u(p_1) \tag{1.79}$$

It rises linearly with $|\boldsymbol{k}|$. As $\overline{v}(p)u(p) \sim E$ in the relativistic normalization, the cross section grows like $d\sigma \sim |\mathcal{M}|^2/s \sim s = (k_1+k_2)^2$.

Let us first consider the massless neutrino case. Then, the second term in the bracket is absent. We remark that the first term is a pure $J = 1$ amplitude. To prove it, we insert an explicit representation for the γ matrices and the plane wave solution for the Dirac particle (see Appendix A).

$$u(p) = \begin{bmatrix} \sqrt{E - \boldsymbol{p}\cdot\boldsymbol{\sigma}}\,\xi_r \\ \sqrt{E + \boldsymbol{p}\cdot\boldsymbol{\sigma}}\,\xi_r \end{bmatrix}, \quad v(p) = \begin{bmatrix} \sqrt{E - \boldsymbol{p}\cdot\boldsymbol{\sigma}}\,\eta_r \\ -\sqrt{E + \boldsymbol{p}\cdot\boldsymbol{\sigma}}\,\eta_r \end{bmatrix}, \quad \bar{v}(p) = v^\dagger(p)\gamma^0 \tag{1.80a}$$

$$\not{k} = \begin{bmatrix} 0 & \omega - \boldsymbol{k}\cdot\boldsymbol{\sigma} \\ \omega + \boldsymbol{k}\cdot\boldsymbol{\sigma} & 0 \end{bmatrix}, \quad \gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.80b}$$

$$\xi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \eta = i\sigma_2\xi^* \tag{1.80c}$$

Choosing helicity eigenstates for ξ_r, η_r, and evaluating in the center of mass frame ($\boldsymbol{k}_1 = -\boldsymbol{k}_2 = \boldsymbol{k}, \boldsymbol{p}_1 = -\boldsymbol{p}_2 = \boldsymbol{p}$), we have for $\nu(h = -1)\overline{\nu}(h = +1) \rightarrow W^+W^-$,

$$\begin{aligned} &\overline{v}(p_2)(\not{k}_1 - \not{k}_2)(1-\gamma^5)u(p_1) = (\sqrt{E+p}\,\xi_+, -\sqrt{E-p}\,\xi_+) \\ &\quad \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2\boldsymbol{k}\cdot\boldsymbol{\sigma} \\ 2\boldsymbol{k}\cdot\boldsymbol{\sigma} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{E+p}\,\xi_- \\ \sqrt{E-p}\,\xi_- \end{bmatrix} \\ &= 4(E+p)\xi_+(\boldsymbol{k}\cdot\boldsymbol{\sigma})\xi_- = 4(E+p)|\boldsymbol{k}|\sin\theta\, e^{-i\phi} \end{aligned} \tag{1.81}$$

If the scattering amplitude is expressed in terms of the Jacob–Wick's partial wave expansion (see Vol. 1, Eq. (9.47)),

$$\begin{aligned} \mathcal{M}_a &= 8\pi\sqrt{s}\, f_{\lambda_3\lambda_4,\lambda_1\lambda_1} \\ f_{\lambda_3\lambda_4,\lambda_1\lambda_1} &= \frac{1}{2i|\boldsymbol{p}|}\sum_J (2J+1)\langle\mu|(S_J - 1)|\lambda\rangle d^J_{\mu,\lambda}(\theta) \\ &\sim \frac{1}{|\boldsymbol{p}|}\sum_J (2J+1)e^{i\delta_J}\sin\delta_J\, d^J_{\mu,\lambda}(\theta) \end{aligned} \tag{1.82}$$

where $d^J_{\mu,\lambda}$ is the rotation matrix elements with angular momentum J. Setting the initial helicity $\lambda = 1/2 - (-1/2) = 1$ and the final helicity $\mu = 0 - 0 = 0$, $d^1_{0,1}(\theta) = \sin\theta$, it proves that the first term of Eq. (1.81) is a pure $J = 1$ contribution. Since the unitarity ($S_J = e^{2i\delta_J}$) constrains $|e^{i\delta_J}\sin\delta_J| \leq 1$ which, in turn, means $|\mathcal{M}|$ should not grow more than a constant. Therefore, Eq. (1.81) violates the unitarity badly at high energy.

It is also the cause of the diverging loop integral. Consider Figure 1.9b. The diagram, if cut in half, is the scattering amplitude $\nu\overline{\nu} \rightarrow W^+W^-$ squared. Indeed, the unitarity of the scattering matrix dictates that the imaginary part of the forward scattering amplitude in Figure 1.9b is proportional to the total cross section of $\nu\overline{\nu} \rightarrow W^+W^-$ (see Eq. (I.36)). Since the intermediate state can have any momentum p as can be seen from Figure 1.9b, it has to be integrated over p which results in a bad divergence, the integrand growing $\sim s$.

To ameliorate the situation, we consider adding other diagrams just to cancel the bad divergence. Cancellations either in the u-channel or in the s-channel are possible. One in the t-channel does not help because it gives a similar amplitude with the same sign. Since Eq. (1.77) is in the pure $J = 1$ state, we consider adding another neutral vector boson V^0_μ in the s-channel.

The corresponding amplitude is depicted in Figure 1.9c. The amplitude can be calculated to give

$$
\begin{aligned}
-i\mathcal{M}_c &= \overline{v}(p_2)\left(-i\gamma^{\delta}\frac{g_{V1}}{2}(1-\gamma^5)\right)u(p_1)\frac{-i\left(g_{\delta\lambda}-\frac{q_\delta q_\lambda}{m_V^2}\right)}{q^2-m_V^2+i\epsilon}\\
&\quad\times(ig_{V2})\epsilon_{1\mu}\epsilon_{2\nu}V^{\nu\mu\lambda}(-k_2,-k_1)\\
V^{\mu\nu\lambda}(k_1,k_2) &= g^{\mu\nu}(k_1-k_2)_\lambda+g^{\nu\lambda}(k_2-k_3)^\mu+g^{\lambda\mu}(k_3-k_1)^\nu\\
q &= k_1+k_2\,,\quad k_3=-(k_1+k_2)
\end{aligned}
\tag{1.83}
$$

The trilinear coupling part can be simplified by using $(\epsilon_1\cdot k_1)=(\epsilon_2\cdot k_2)=0$ and $k_3=(k_1+k_2)$.

$$
\epsilon_{1\mu}\epsilon_{2\nu}V^{\nu\mu\lambda}(-k_2,-k_1)=\left[(\epsilon_1\cdot\epsilon_2)(k_1-k_2)^\lambda-2(\epsilon_2\cdot k_1)\epsilon_1^\lambda+2(\epsilon_1\cdot k_2)\epsilon_2^\lambda\right]
\tag{1.84a}
$$

Inserting $\epsilon_i\sim k_i/m_W$, we have

$$
\epsilon_{1\nu}\epsilon_{2\mu}V^{\mu\nu\lambda}(-k_2,-k_1)\simeq-\frac{1}{m_W^2}(k_1\cdot k_2)(k_1-k_2)^\lambda
\tag{1.84b}
$$

Substituting Eq. (1.84b) in Eq. (1.83), the matrix element becomes

$$
-i\mathcal{M}_c=+i\frac{g_{V1}g_{V2}}{4m_W^2}\overline{v}(p_2)(\not{k}_1-\not{k}_2)(1-\gamma^5)u(p_1)\frac{(k_1\cdot k_2)}{(k_1\cdot k_2)+m_V^2/2}
\tag{1.85}
$$

If $g_{V1}g_{V2}=g_W^2/2$, Eq. (1.85) cancels the first term of Eq. (1.79) for $|\boldsymbol{k}|\gg m_V^2$. This is exactly the case for the V to be I_3 member of the gauge boson in $SU(2)$. This is also true for the case of $SU(2)\times U(1)$ where W^0 is replaced by γ and Z. For the γ, Eq. (1.51) and Figure 1.5 gives

$$
\frac{g_{V1}}{2}(1-\gamma^5)\to Q_f e=Q_f g_W\sin\theta_W\,,\quad g_{V2}=e=g_W\sin\theta_W
\tag{1.86a}
$$

Note, we deliberately retained the charge Q_f to do similar arguments later for the electron. For the neutrino, $Q_\nu=0$ and the photon does not contribute. For the Z, Eq. (1.51e) and Figure 1.5 gives

$$
\begin{aligned}
&\frac{g_{V1}}{2}(1-\gamma^5)\to\frac{g_Z}{2}(v_f-a_f\gamma^5)\,,\quad g_{V2}=e\cot\theta_W=g_W\cos\theta_W\\
&g_Z=\frac{g_W}{\cos\theta_W}\,,\quad v_f=I_{3f}-2Q_f\sin^2\theta_W\,,\quad a_f=I_{3f}
\end{aligned}
\tag{1.86b}
$$

Then,

$$
\begin{aligned}
\sum_{\gamma,Z}g_{V1}g_{V2} &= 2Q_f g_W^2\sin^2\theta_W+g_W^2\left(I_{3f}-2Q_f\sin^2\theta_W^2-I_{3f}\gamma^5\right)\\
&= g_W^2\,I_{3f}(1-\gamma^5)\\
&= \frac{g_W^2}{2}(1-\gamma^5)\quad\text{for}\quad\nu
\end{aligned}
\tag{1.87}
$$

Namely, the gauge invariance provides exactly the proper counter terms in order to not give bad high energy behavior.

In passing, we mention that the divergence can also be compensated by introducing a “heavy electron E^+” in the u-channel, provided it also has the coupling required by the gauge symmetry.[13] The model eliminates the neutral current. It was proposed as an alternative to the GWS theory before the discovery of the neutral current, and hence was ruled out by its discovery. However, the example illustrates the power of the gauge symmetry in controlling the divergence.

Problem 1.1

Prove that the diagram Figure 1.9f cancels the dominant contribution of Figure 1.9a provided the same coupling constant is used.

Higgs in the renormalization When the initial fermion pair is massive as is the case for the electron, the diverging term reappears. This is because the massive fermion can have opposite helicity for the same chirality. The left-handed neutrino is in a pure helicity ($h = -1$) state, but the electron can have a positive helicity component with amplitude proportional to its mass $\sim m_e/p$[14]. This induces an extra component, namely, the second term in Eq. (1.79). Note, the first term also changes sign because $W^\pm$ is interchanged in the transition $e^-e^+ \to W^-W^+$ as shown in Figure 1.9d. Then, the contribution of Figure 1.9c also changes sign because $I_3(e^-) = -1/2$, and thus the compensation mechanism is again valid here.

As is clear from the expression, this term has $J = 0$ and is proportional to the fermion mass. This is not the dominant divergence as that of the $J = 1$ component. Nevertheless, it is a divergence that grows faster than the logarithm. It gives the $O(s)$ term to the loop integral. A scalar meson is necessary to compensate it and this is the place where the Higgs comes in. The coupling of the Higgs field after spontaneous symmetry breakdown is proportional to the fermion's mass (Eq. (1.46)) just as required. It does not couple to the neutrino because it is not needed. The scattering amplitude corresponding to $e^-e^+ \to H^0 \to W^+W^-$ (Figure 1.9e) is given by

$$-i\mathcal{M}_e = \overline{v}(p_2)\left(-\frac{i}{2}\frac{g_W}{m_W}m_e\right)u(p_1)\frac{i}{(k_1+k_2)^2 - m_H^2}(ig_W m_W g^{\mu\nu})\epsilon_{1\mu}\epsilon_{2\nu} \tag{1.88a}$$

13) This is equivalent to asking that E^+ be a member of the multiplet in which ν and e^- belong. In other words, the fermions constitute a triplet as opposed to a doublet in the Standard Model. The model is based on $SU(2)$ gauge theory and identifies I_3 as the electric charge operator [47].

14) See arguments in Vol. 1, Sect. 4.3.5. It is also apparent in Eq. (1.80a) because for $E > |\boldsymbol{p}|$, the opposite helicity component does not vanish.

Inserting $\epsilon_{i\mu} \to k_{i\mu}/m_W$ again and neglecting m_H^2 relative to $(k_1 \cdot k_2)$, we obtain

$$-i\mathcal{M}_e \simeq i\frac{g_W^2}{4m_W^2} m_e \overline{v}(p_2)u(p_1) \tag{1.88b}$$

which cancels the second term of Eq. (1.79).

***W W* → *W W* scattering** The cancellation mechanism we just mentioned is at work for the vector boson scattering as well. The two diagrams with two triple boson vertices in Figure 1.10a,b produces a term $\sim (k_3 \cdot k_4)\{(k_3 - k_4)\cdot(k_1 - k_2)\}(k_1 \cdot k_2) \sim s^3$ which is reduced to $\sim s^2$ by the propagator. Therefore, the amplitude includes terms $O(s^2) + O(s) + O(\ln s)$. The term of $O(s^2)$ is compensated by the quartic coupling diagram of Figure 1.10c if the latter has the right coupling as is required by the gauge theory. The $O(s)$ term can be compensated by a term which includes a scalar field in the intermediate state if it has the right coupling as is given by the second and last term of Eq. (1.37). Thus, the remaining divergence is at most logarithmic and can be handled with the renormalization prescription.

In summary, the gauge theory which has broken spontaneously has a built-in mechanism to compensate all the annoying divergences and make the theory renormalizable [38, 39]. Conversely, if one tries to compensate diverging integrals by introducing additional particles and determines their particle species, coupling

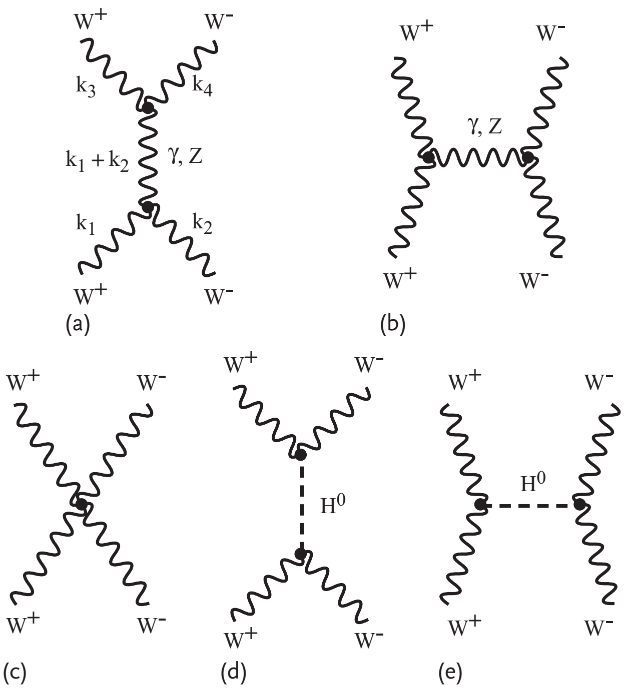


Figure 1.10 *W-W* scattering goes through triple vector coupling (a,b), quartic coupling (c) and Higgs coupling (d,e).

constants and masses, they always end up with the spontaneously broken gauge theory [48–51]. We conclude, therefore, that the spontaneously broken gauge theory is the only renormalizable theory that can handle massive vector bosons.

In retrospect, the role of the Higgs to rescue the divergence problems whenever they occur is clear if one accepts the fundamental role of the gauge symmetry to overcome the difficulties and looks at the original Lagrangian. The gauge sector (the first line of Eq. (1.16)) and the Higgs sector (the second line of Eq. (1.16)) both independently satisfy the $SU(2) \times U(1)$ gauge symmetry. The masses are generated in the Higgs sector. If the symmetry is broken spontaneously, which is equivalent to rewriting fields in a certain gauge, the gauge invariance as a whole is still maintained mathematically. The gauge invariance is broken when one separates the mass terms, add them alone to the gauge sector and discard the rest of the Higgs contributions. Since the whole Lagrangian which includes the Higgs part is gauge invariant, it is no wonder that the difficulty is solved by including the Higgs contributions.