Chapter 2
The different languages of q-calculus

The study of q is often considered to be one of the most difficult subjects to engage in mathematics. This is partly because of the many formulas involved, partly due to the wide range of different notations used. Below is an outline of how and why this wide and confusing variety of notations came into being, a situation which is closely connected to the 300-year old history of q. The history of the study of q may be illustrated by a tree.

This q-tree has 13 distinct roots:

1. the study of elliptic functions from the nineteenth century
2. the development of theta functions
3. additive analytic numbers theory, or theory of partitions
4. the field of hypergeometric functions
5. gamma function theory
6. Bernoulli and Euler polynomials
7. umbral calculus
8. theory of finite differences
9. combinatorics + q-binomial coefficient identities
10. theory of finite fields + primitive roots
11. Mock theta functions
12. multiple hypergeometric functions
13. elliptic integrals and Dedekind eta function

The main trunk of the tree is the subject q. The trunk is divided into two main branches, the two principal Schools in q-analysis: Watson and Austrian. Each of the two main branches bears several smaller twigs, smaller Schools which have sprouted—and still do—somewhat later from one of the principal branches.

Now, the following pages may remind the reader more of a study in languages and their interconnections than an exposition in mathematics. So be it for a while. To fully understand the development and current state of q-analysis, the pitfalls and the problems to be dealt with, it is imperative to establish a 'frame' within which to get a proper general view of the vast area and its many different and connected

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branches. We may indeed use the language metaphor to grasp q at its present, somewhat confusing state:

The various Schools of q to be outlined below have developed over a period of roughly 300 years since the Bernoullis and Euler. These Schools today use and communicate in such different languages, that they have problems understanding each other. The big School of Watson speaks—both literally and metaphorically—English. It has little exact consciousness of, indeed may not find it necessary, to study and know the roots of the vernacular. Still metaphorically speaking, the Watson School does not understand the development of old Icelandic—which was the beginnings of q—into a modern Scandinavian language. Thus, it is difficult for the Watson School practitioners to understand and indeed to communicate with Schools where completely different dialects have evolved. And we have in the q-area, in relation to other areas, a tendency to develop specific, new languages (e.g. notations), which surpasses this tendency in other fields of mathematics. This is either because practitioners find these more easy to speak, because they find them more beautiful or simply because they do not know and cannot pronounce the older forms.

2.1 Schools—traditions

It is today possible, indeed clarifying, to divide q-analysis/calculus into several Schools or traditions. What distinguishes these Schools is first and foremost

1. Their history (their roots).
2. Which specific modern language they write in.
3. Their different notations.

These Schools or traditions should not be seen as iron-fence surrounded exclusive units, but more as blocks with rather fluid and sometimes overlapping boundaries. It is thus sometimes convenient to place a q-scholar in two traditions or Schools. If we take Harold Exton as an example, it is possible to say that he was firmly rooted in the historic tradition of the Austrian School, but he nevertheless used (partly) the notation from the Watson School. He—to use a metaphor again—spoke Watsonian but with a strong Austrian accent. The result was, as is too often the case, a mishmash.

The two main Schools are, as already stated, the Austrian School and the Watson School. In this context, it might be of interest to tell something about the early development of the combinatorics in the German territory. The predecessor of the Austrian School was the combinatorial School of Karl-Friedrich Hindenburg (1741–1808) and Christoph Gudermann. The goal of the combinatorial School was to develop functions in power series by Taylor's formula.

The *Austrian School* is named in honour of one of its main figures, the Berliner Wolfgang Hahn (1911–98), who held a professorship in Graz, Austria, from 1964. Hahn was strongly influenced by Heine. The Austrian School is a continuation of the Heine q-umbral calculus from the mid nineteenth century, which at the time however met with little attention except for Rogers, who introduced the first q-Hermite polynomials and first proved the Rogers-Ramanujan identities.

The *Watson School* takes its name from the English mathematician George Neville Watson (1886–1965), who wrote the famous essay *Treatise on the Theory of Bessel Functions*, and furnished a rigorous proof for the Rogers-Ramanujan identities.

Both Schools or traditions recognize the early legacies of Gauß and Euler. Only the Austrian School, however, represents and incorporates the entire historical background which includes the pre-q mathematics, namely the Bernoulli and Euler numbers, the theta functions and the elliptic functions. The vast area of theta functions and elliptic functions, which will be dealt with in Sections 3.8 and 6.13, is in fact q-analysis before q was really introduced. Pre-q or q-analysis in disguise could perhaps be an appropriate term for this period and its practitioners. The Austrian School takes the development of q all the way from Jacob Bernoulli, Gauß and Euler in the 17th and 18th century, through the central European mathematicians of the nineteenth century: Heine, Thomae, Jacobi, and from the twentieth century: Pringsheim, Lindemann, Hahn, Lesky and Cigler, the Englishman Jackson, the Austrians Peter Paule, Hofbauer, Axel Riese and the Frenchman Appell. The present (Swedish) author of this book confesses to be firmly rooted in the Austrian School.

The Austrian School is little known in the English speaking world, e.g. the USA and the Commonwealth, for two main reasons: Immediately after the first world war, ca. 1920 until 1925, the German and Austrian mathematicians were barred from participation in the big mathematical conferences; they were simply deemed political pariahs by the French, unwanted, and the communications with the English speaking q-analysts were for a period limited, as was the exchange of ideas. The second and more important reason has to do with languages. Most of the mathematicians of the Austrian School wrote in either German or French and, as regards the oldest, namely Euler, Gauß and Jacobi, in Latin. Only few English-speaking mathematicians today master these languages (it is simply not part of the common American curriculum), and few are therefore familiar with the works of these early European scientists.

The Watson School is today the most widespread and influential of the two principal Schools/traditions. But again this is mainly due the language. If English—as is the case today for many mathematicians—is your primary and perhaps only approach to the study of mathematics, then adherence to the Watson School is almost automatic. The Watson School is nowadays the main highway to the study of q. Not necessarily, though, the best, the most correct or even the smoothest or most beautiful route to choose.

The Watson School, in this author's opinion—I hope and I will convince the reader as the book progresses—does not take the full and necessary steps to understand and incorporate the work early q and pre-q analysts into modern q-calculus.

Before Ramanujan, Cambridge had enjoyed quite a strong Austrian School representation with names such as James Glaisher (1848–1928) and Arthur Cayley (1821–95), from the nineteenth century.

F. H. Jackson (1870–1960), James Rogers (1862–1933) and Andrew Russell Forsyth (1858–1942) from the twentieth century represent a kind of transition to the Watson School.

The Watson School may be said to start in earnest with the Indian Cambridge mathematician Srinivasa Ramanujan's discoveries, namely the mock theta functions and the Rogers-Ramanujan identities. The Watson School goes, in a manner of speaking, straight from Gauß to the twentieth century Srinivasa Ramanujan (1887–1920) and thus skips and misses out a 150-years period of fruitful European studies. The early Watson School adherents and practitioners were the mathematicians at Cambridge who Ramanujan's short stay inspired, e.g. Eric Harold Neville (1889–1961) and Wilfrid Norman Bailey (1893–1961). Bailey devoted a lot of time to administration and sports during his academic career. He was an excellent teacher and among his students were L. J. Slater, Jackson, F. Dyson and Ernest Barnes (1874–1953). Interestingly enough, all of these turned into special functions. Barnes became a bishop and wrote a long series of papers on special functions and difference calculus. In a way, Barnes was a predecessor of Nørlund, but they studied different problems. Although Barnes was an excellent mathematician, far better than Jackson, he never returned to the academia after a few years as a teacher. In his youth, Bailey met Jackson in the Navy and they certainly discussed q-calculus already then. In a couple of papers around 1947 Bailey intended to simplify some of Rogers's proofs of generalizations of the Rogers-Ramanujan identities. He then invented a new notation and gave Dyson credit to some of the formulas. The famous Bailey's lemma comes from this time; there are several variations of this, and Wengchang Chu claims that Bailey's lemma is a special case of a generalization of the Carlitz q-Gould-Hsu inversion formula. In the author's opinion, the wisest way is to go back to Rogers's original proofs. After his retirement, in 1958, Bailey intended to write a major work on q-series. For some reasons this failed, although he moved to Eastbourne, where Jackson spent his last years.

2.2 Ramifications and minor Schools

The explanations in Sections 2.2, 12.7–12.9 are written for convenience and give a good account of the current state of affairs. These explanations are not in standard terms and cannot be cited. The study of q can be further divided into sub-groups, traditions or Schools. Some of these, e.g. the Chinese or Japanese ones, can hardly be said to form Schools as such; what they have in common however, is a cultural and linguistic background, which in a certain sense shapes their mathematical work.

The *Carlitz, Gould and Vandiver tradition* is perhaps better termed the American-Austrian School as its practitioners all tend to work in the European Watson School tradition. This School can be traced back to the 1930s.

Isaac Joachim Schwatt (1867–1934), Leonard Carlitz (1907–1999) and Harry Schultz Vandiver (1882–1973) were descendants of European immigrants to the USA and therefore read and worked in the pre-q, or q-in-disguise tradition. Henry

W. Gould (1928–) on the other hand simply enjoyed reading European mathematical literature. This group had a strong interest in Bernoulli numbers, finite differences and combinatorial identities. However, not all practitioners of the American-Austrian School wrote books on their subject—the largest part of their contribution is contained in their lectures and articles. Vandiver and Gould collected card-files on articles about Bernoulli numbers; this work was continued by Karl Dilcher and made available on the Web.

It is also possible that Carlitz was a student of Schwatt at University of Pennsylvania, Philadelphia. Carlitz was born in Philadelphia, and the two have worked on similar mathematical topics. Schwatt, who became PhD in 1893 in Philadelphia, remained in this city during his whole career (1897–1928). We will come back to the q-analogues of the Schwatt formulas in Chapter 5, where q-Stirling numbers are discussed. Carlitz spent a post-doc year 1930–31 with E. T. Bell in Pasadena, and we have come to the next School.

The so-called *E. T. Bell, Riordan, Rota-School* started in 1906 in San Francisco (the year of the big earthquake), when Eric Temple Bell (1883–1960) read some books on number theory [430, p. 109]. The first one was by Paul Bachmann (1837–1920) who enriched the whole theory with detailed proofs. Bachmann had a doctorate from Berlin 1862; his instructors were Ernst Kummer (1810–1893) and Martin Ohm (1792–1872) (Bachmann himself had no doctoral students). The second book Bell read was *Théorie des Nombres* [363] by E. Lucas (1842–1891) [363]; here Bell was initiated into the so-called umbral calculus. Morgan Ward (1901–1963) also belonged to this School; his supervisor was Bell. John Daum recognized the connection between q-series and hypergeometric functions; Daum belonged to the second mathematical generation after Bell.

John Riordan (1903–1988) and Gian-Carlo Rota (1932–1999) were also members of this group. In 1963, Riordan and Rota met in Boston and went to a restaurant where they discussed Riordan's new book *An Introduction to Combinatorial Analysis*; Riordan's book is actually dedicated to E. T. Bell. After 31 years, in the footsteps of Bell, Rota and his student Brian Taylor attempted a rigorous foundation for the umbral calculus in the excellent treatise *The Classical Umbral Calculus* [439]. Unfortunately, Rota learned late of Bell's combinatorial work [430, p. 227], so he could not find the q-analogue of [439], which is presented in this book.

Special functions have always been a major research topic in India, an inheritance from the old nineteenth century Austrian Cambridge School under Glaisher and Forsythe. The *Indian School* is made up of many different branches and can for convenience be divided into at least three different areas of interest: Srivastava, Hahn and Ramanujan. The Srivastavan branch was originally founded in the 1960s by Hari M. Srivastava in the footsteps of Shanti Saran. The city of Lucknow is a centre for experts, and will be of interest in the following. Saran got his PhD in 1955 in Lucknow on a treatise on hypergeometric functions of three variables; his supervisor was R. P. Agarwal (1925–2008). Srivastava received his PhD in 1965 in Jai Narain Vyas on a similar topic. The mathematical journal *Ganita* is printed in Lucknow since 1950; since the sixties, almost every issue contains an article about special functions. In the 1952 issue, there is a report on hypergeometric functions by H. M. Srivastava and A. M. Chak. In the 1956 issue there is a report on the hypergeometric functions of three variables by Krishna Ji Srivastava, which refers to Saran. These two mathematicians were probably unaware of the articles by Horn.

H. M. Srivastava was born in India, settled in Canada in the early seventies, but travelled frequently to India until 1985. He was a good friend of and therefore also inspired by Gould of the American–Austrian School. Srivastava left India permanently in the 1980s and without its founder the School quickly dwindled. The members of the Srivastava School were especially interested and produced works on hypergeometric functions of many variables, on generating functions and on different polynomials, e.g. Laguerre and/or differential operators. Recently (2009), Srivastava again visited Vijay Gupta in India and perhaps something interesting will arise from this cooperation.

H. L. Manocha (Polytechnic Institute of New York University) has written a book about generating functions [484]; one of his graduate students was Vivek Sahai.

Srinivasa Rao, who was earlier in Chennai, is since 2004 at the Ramanujan Center of Sastra University in Kumbakonam, one of his students was V. Rajeswari.

Subuhi Khan, Aligarh University, has done basic studies on connections between Lie algebras and special functions, which she has presented at conferences in Hong Kong and Decin.

Wolfgang Hahn of the Austrian School spent a year in India before taking up as professor at Graz and his Indian pupils, following in his footsteps, are especially interested in q-Laplace transformations and Hahn q-additions.

The Ramanujan branch was and is still a very active and productive group. They are inspired by the great Indian master Ramanujan and his mock theta functions. Brilliant works appear from time to time from this otherwise uneven group of Indian mathematicians.

The *Hungarian School* is yet another European branch with a strong affinity and connection to the Austrian School. This is also a q-School without the q, ... i.e. the work is done primarily on q-related topics such as the theory of orthogonal polynomials and recurrence relations. The Hungarian School emerged in the post WW1 years, the prominent figure being the Hungarian-born Gabor Szegő (1895–1985). Szegő is the father of the so-called Rogers-Szegő polynomials—q-polynomials with many similarities to Hermite polynomials. He later took up work at Stanford and wrote thence in the Watson tradition. One of his successors is Richard Askey; the Askey tableau of orthogonal polynomials stems from him. There is also a q-version of this. Another major contribution is the book *Calculus of Finite Differences* by Charles Jordan [320]. Other strong representatives of this School are Eugene Wigner (1902–1995) and John von Neumann (1903–1957), who also worked without q, but strongly influenced the development of quantum groups. Their works are of high quality, but rather difficult. Wigner introduced the 3-j coefficients, but his formulas could have been greatly improved by using hypergeometric functions in his formulas. This came only in our time, when Joris Van der Jeugt used multiple hypergeometric functions for 9-j coefficients.

The *Danish School*: This School is closely linked with the Austrian School by the close linguistic relationship with German and French. There are two branches: one deals with Stirling numbers and finite difference calculus, the other with multiple hypergeometric functions and gamma functions. In 1909 Thorvald N. Thiele (1838–1910) wrote a book about interpolation theory, containing a table of Stirling numbers. Johan Ludwig Jensen (1859–1925) wrote about gamma functions and thereby influenced Niels Nielsen (1865–1931) in the same area. Nielsen had a strong interest in special functions, in his own way, and introduced Stirling numbers in a paper in French. He has also written biographies on ancient French mathematicians. Nielsen has developed Fakultätenreihen by Bessel functions in his 1904 book on cylinder functions [396]. Here one can also find an excellent bibliography. In the footsteps of Nielsen, Niels Erik Nørlund (1885–1981) in the remarkable work [403] gave the first rigorous treatment of finite differences from the perspective of the mathematician. Nørlund gave lectures on hypergeometric series in Copenhagen until 1955. Nørlund also knew q-calculus; F. Ryde published a thesis on this subject under his supervision in Lund. The next link in the chain is perhaps Per Karlsson (1936–), the expert on multiple hypergeometric functions and friend, among other things, collaborator and model of the author. One can mention also Christian Berg, who works on moment problems and real analysis.

Russian School: Russian mathematicians have greatly influenced the above mentioned Hungarian School through studies of polynomials.

Russia has, in general, a strong tradition in mathematics, which dates back to Euler and the 18th century mathematicians who took up this heritage. Euler has very much contributed to the *Proceedings of the St. Petersburg Academy*. After his death, his successors could not keep the high level from before.

Euler himself died in St. Petersburg, and it is a well-established fact that both his direct influence and also his unpublished papers and work remained in Russia, which explains in part the high level of mathematics in Russia. Many Russian mathematicians from the nineteenth century did excellent work in the area of Bernoulli numbers and umbral calculus, among them Grigoriew, Chistiakov and Imchenetsky.

According to Grigoriew [247, p. 147], the generalized Bernoulli numbers, which Nørlund used in [403], were also used by Blissard (1803–1875) [75] and Imchenetsky [289]. L. Geronimus (1898–1984) wrote about certain Appell polynomials. Leading figures in the tradition of orthogonal polynomials and Bessel functions were P. L. Chebyschew (1821–1894) and Nikolay Yakovlevich Sonine (1849–1915). Their articles are, unfortunately, less accessible, being written in Russian, though some are translated into English. Sonine was one of the last representatives of this School, who could read Euler in the original language.

There is a connection to theoretical physics:

Valentine Bargmann (1908–1989) was born in Berlin to Russian parents. After studying in Berlin and Zurich, he went to Princeton, where he joined Einstein and Wigner. He is famous for the unitary irreducible representations of $SL_2(\mathbb{R})$ and the Lorentz group (1947) and for the Bargmann-Fock space. Russian mathematicians strive to develop, among other things, the theory of Heisenberg ferromagnetic equations and are also actively studying the connection between 3-j coefficients and hypergeometric functions.

The *Italian School* is quite strong. It started when Giuseppe Lauricella (1867–1913) studied hypergeometric functions of many variables in the nineteenth century.

Like Paul Appell (1855–1930), Lauricella focused on symmetric functions, because these provide the most beautiful formulas.

Pia Nalli (1886–1964), who made a highly interesting study of the so-called q-addition, was influenced by the Italian literature about elliptic functions. Nalli was the first one who used the NWA q-addition in her only article on q-calculus.

Letterio Toscano (Messina) (1905–1977), who at the beginning wrote only in Italian, published many interesting articles about Bernoulli, Euler and Stirling numbers in connection with the operator $x\mathrm{D}$.

Due to his friendship with Francesco Tricomi (1897–1978) (Torino), Toscano could write in publications of the Italian academy; the two belonged to the same generation. Tricomi published books on confluent hypergeometric functions and elliptic functions before joining the Bateman project.

In 2005, Donato Trigiante, a student of Tricomi, published elegant matrix representation for the Bernoulli and Euler polynomials.

In 2007, the author presented q-analogues of Trigiante's matrix formulas at the OPSFA Conference in Marseille.

Another Italian mathematician, who, among other things, published on Laguerre polynomials, was Giuseppe Palama (1888–1959) from Milano. In the last decades, Rota and Brini have published works on umbral calculus. We find a q-analogue of Rota's infinite alphabet in this book, that the author introduced (2005).

The *Scottish School* consisted of Thomas Murray MacRobert (1884–1962), who introduced the MacRobert E-function. In this context, MacRobert together with Meijer introduced the $\triangle(l;\lambda)$-operator of Srivastava [475]. Under his leadership began the *Proceedings of the Glasgow Mathematical Association* in 1951, in which many publications on special functions appeared. MacRobert had no doctoral students, but he helped many authors to write on special functions in British journals. After his death, many mathematicians, e.g. from Egypt or India, have followed in his footsteps.

2.2.1 Different notations

This large number of different Schools and their branches or off-springs has also resulted in a profusion of different notations. The scheme below is an attempt to bring some sort of order into the profusion and confusion of notations in the different areas.

The different notations:

1. q-hypergeometric functions

Watson: The q-shifted factorial is denoted by $(a;q)_n$, (1.31).

Austrian:

1. The q-shifted factorial is denoted by $\langle a;q\rangle_n$, (1.30).
2. Cigler only uses the Gaussian q-binomial coefficients, which are nearly equivalent to (1.30) above.

3. Yves André [22, p. 685] uses a notation that is equivalent to $\langle a; q\rangle_n$. André [22, p. 692] also denotes D_q by δ_σ.

Indian:

V. Rajeswari and K. Srinivasa Rao in 1991 [426] and in 1993 [470, p. 72] use my umbral notation in connection with the q-analogues of the 3-j and 6-j coefficients.

Russian:

1. Also Gelfand has used a similar notation in one of his few papers [228, p. 38] on q-calculus. His comment is the following: Let us assume that at first we use Watson's notation (1.31) for the q-hypergeometric series.
   If all α_i and β_i are non-zero, it is convenient to pass to the new parameters a_i, b_i, where $\alpha_i = q^{a_i}$, $\beta_i = q^{b_i}$.
2. Igor Frenkel (MIT) uses none of the above notations. He simply writes the definitions. In the book [186] the symbol $\{a\}$ is used instead of our $\{a\}_q$. In [186] there is a special notation for $(z; q)_a$, but no notation for D_q. Instead of q-factorials, Theta functions are used [186, p. 172].
3. Naum Vilenkin (1920–1991), and Anatoly Klimyk (1939–2008) [523] in their three volumes on representation theory for Lie groups use their own Watson-like notation, as does Boris Kupershmidt.

2.3 Finite differences and Bernoulli numbers

Finite differences and Bernoulli numbers are closely related to q-Analysis. The Bernoulli numbers were first used by Jacob Bernoulli (1654–1705) [70], who calculated the sum:

$$\mathrm{s}_m(n) \equiv \sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} n^{m+1-k} B_k, \tag{2.1}$$

where the *Bernoulli numbers* B_n are defined by:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{\mathrm{B}_{2n} z^{2n}}{(2n)!}. \tag{2.2}$$

The *Bernoulli polynomials* are defined by:

$$\mathrm{B}_n(x) \equiv (\mathrm{B} + x)^n = \sum_{k=0}^{n} \binom{n}{k} \mathrm{B}_k x^{n-k}, \tag{2.3}$$

where B^n must be replaced by B_n on expansion.

In 1738 Euler used the generating functions to study the Bernoulli polynomials. The Bernoulli polynomials were also studied by J.-L. Raabe (1801–1859) [422] and Oskar Xaver Schlömilch (1823–1901) [453, p. 211].

We now write down the basic equations for finite differences where, E is the shift operator and $\triangle \equiv \mathrm{E} - \mathrm{I}$.

Theorem 2.3.1 [525, p. 200], [138, p. 26], [458, p. 9]. *Faulhaber-Newton-Gregory-Taylor series*

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} (\triangle^k f)(0). \tag{2.4}$$

Theorem 2.3.2

$$\triangle^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathrm{E}^{n-k} f(x). \tag{2.5}$$

This formula can be inverted.

Theorem 2.3.3 [460, p. 15, 3.1]

$$\mathrm{E}^n f(x) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f(x). \tag{2.6}$$

The Leibniz rule goes as follows:

Theorem 2.3.4 [320, p. 97, 10], [138, p. 27, 2.13], [385, p. 35, 2], [411, p. 19].

$$\triangle^n (fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f \; (\triangle^{n-i} \mathrm{E}^i) g. \tag{2.7}$$

In Chapter 4, we will retain the binomial coefficients in the corresponding q-formulas, whereas in Chapter 5, q-binomial coefficients for the corresponding formulas will be used.

Karl Weierstraß said that the calculus of finite differences will once play a leading role in mathematics. Two important elements of the calculus of finite differences are the Bernoulli numbers and the Γ function. Nørlund says in a letter to Mittag-Leffler 1919 [183] “Someone who is not an expert in these fields cannot be an expert on calculus of finite differences.” We will show that the Γ function can always be transferred to the Pochhammer-Symbol. The q-factorial and the Γ_q function are the corresponding q-terms. We infer that the (q-)hypergeometric function is also part of the calculus of finite differences. This is no accident, as Douglas Barker Sears and Hjalmar Mellin (1854–1933) have shown.

2.4 Umbral calculus, interpolation theory

The interpolation theory, which was often used by astronomers of the nineteenth century, like Gauß, Bessel, W. Herschel (1738–1822), J. Herschel (1792–1871), is essentially equivalent to the theory of finite differences.

Herschel [277] wrote: "*The want of a regular treatise, on the calculus of finite differences in English, has long been a serious obstacle to the progress of the enquiring student. The Appendix annexed to the translation of Lacroix's Differential and integral calculus, although from the necessity of studying compression it is not so complete as its author could have wished...*"

Computations on elliptic functions with finite differences were made by Jacobi, Weierstraß and Louis Melville Milne-Thomson (1891–1974).

In 1706, John Bernoulli (1667–1748) invented the difference symbol $\triangle$. Fifty years later, in 1755, Leonhard Euler used its inverse, the $\sum$ operator [189, Chapter 1]. Euler was John Bernoulli's student together with Bernoulli's two sons, Nicolaus II and Daniel. Even though John Bernoulli used the symbol $\triangle$ already in 1706, he had in mind not finite differences thereby, but differential quotients. Hence, Euler stands out as the one who devised the designation that has remained in use. Euler's proofs were however not entirely satisfactory from a modern point of view [277, p. 87].

The two symbols, sometimes also called the difference and sum calculus, correspond respectively to differentiation and integration in the continuous calculus. We will find 2 different q-analogues of the inverse operators $\triangle$ and $\sum$ in Sections 4.3 and 5.3.

Euler [189] and Joseph-Louis Lagrange (1736–1813) have introduced the umbral calculus, where operators like (4.126) were used. In 1812, in the footsteps of L. Arbogast (1759–1803), another Frenchman, Jacques-Frédéric Français, wrote E for the forward shift operator [345, p. 163] and reproved the Lagrange formula from 1772,

$$\mathrm{E} = e^{\mathrm{D}}. \tag{2.8}$$

A decade later, Augustin Cauchy (1789–1857) in his *Exercices de mathématiques* [345, p. 163] for the first time found operational formulas like

$$\mathrm{D}\big(e^{rx} f(x)\big) = e^{rx}(r+x) f(x). \tag{2.9}$$

Arbogast and Fourier regarded this umbral method as an elegant way of discovering, expressing, or verifying theorems, rather than as a valid method of proof [345, p. 172]. Cauchy had similar opinions. We will see that this sometimes also obtains for the method of the author.

Textbooks on the subject were written by Andreas von Ettingshausen (1796–1878), J. Herschel, J. Pearson and Augustus De Morgan (1806–1871).

Robert Murphy (1806–1843) was a forerunner of George Boole (1815–1864) and Heaviside, who among other things found nice formulas for derivatives in the spirit of Carlitz.

In 1854 Arthur Cayley introduced the concept of Cayley table in his article *On the theory of groups*. Cayley defined a group as a set of objects with a multiplication and a symbolic equation $\theta^n = 1$. Blissard was obviously influenced by Cayley's work, at least in what concerns his notation.

Another French entry came from E. Lucas, who invented a modern notation for umbral calculus. The Lucas umbral calculus was widespread in Russia, for example one finds the defining formula for the Bernoulli numbers in Chistiakov 1895 [127, p. 105]. The Blissard umbral calculus has attracted attention in Russia; in [127, p. 113] one finds the Blissard Bernoulli number formulas with sine and cosine.

By 1860 two textbooks on finite differences were published in England, one of them by Boole, which covered almost all the theorems that we know now. Heaviside was able to greatly simplify Maxwell's 20 equations in 20 variables to four equations in two variables. This and other articles about electricity problems, which appeared in 1892–98, were severely criticized for their lack of rigour by contemporary mathematicians.

It seems that Heaviside's contribution to mathematics was underestimated by his contemporaries, since in fact he both discussed formal power series and the rudiments of umbral calculus, which we present in this book.

Angelo Genocchi (1817–1889) and Salvatore Pincherle (1853–1936) contributed to the early Italian development of the subject. Alfred Clebsch (1833–1872) and Paul Gordan (1837–1912) continued the theory of invariants that had started with Sylvester and Cayley.

The Heine q-umbral calculus reached its peak in the thesis by Edwin Smith (1879–) [467] 1911, which was supervised by Pringsheim.

F. H. Jackson followed this path in the early twentieth century, and fully understood the symbolic nature of the subject in his first investigations of q-functions. Like Blissard, Jackson worked as a priest his whole life; both of them had studied in Cambridge. To honour Jackson, we will use his notation for $\mathrm{E}_q(x)$.

Steffensen [488], Jordan [320] and Milne-Thomson [385] wrote books about finite differences intended both for mathematicians and statisticians. Johann Cigler wrote an excellent book [138] on finite differences with a view to umbral calculus.

2.5 Elliptic and Theta Schools and notations, the oldest roots–the q-forerunners

Just as the nicest equations in mechanics are connected with the torque (and the angular momentum), the theta functions and the elliptic functions give the most beautiful equations in calculus.

moment	theta functions, elliptic functions

As was the case with q: Schools and traditions abound, so it is the case with the forerunners of q-analysis: different Schools, different notations. The early q-analysis may be defined as the mathematics done in this area even before the q was introduced and properly defined, namely the elliptic functions and theta functions from ca. 1750 and onwards. We may call these the q-forerunners or speak of the pre-q

mathematics or q-analysis in disguise. Euler had in fact already introduced q-series and Jacobi continued to use the letter q, which has survived until today.

Eight Schools stand out:

1. Jacobi theta functions and elliptic functions of one variable.
2. Riemann Theta functions of one or more variables or Abelian functions.
3. Weierstraß elliptic functions and sigma functions.
4. The Glaisher-Neville-School for Jacobi elliptic functions.
5. The Γ_q-School.
6. The Heine Ω function School.
7. The Weierstraß-Mellin School of Gamma functions and hypergeometric functions.
8. The Italian elliptic function School.

School 1 is about the Jacobi theta functions, so called after the German mathematician Carl Gustav Jacob Jacobi. Jacobi's development of the theta functions, of which we have four, was made in parallel to (but before) that of Weierstraß of School 3. This means that if we know all 4 Jacobi Theta functions, we can compute all Weierstraß σ functions by means of 4 linear equations and vice versa.

There were in fact originally three different notations for Jacobi elliptic functions:

Jacobi notation, Abel notation and finally the modern Gudermann notation $\mathrm{sn}u, \mathrm{cn}u, \mathrm{dn}u$ from 1844.

School 2 covers the Riemann Theta functions after G.F.B. Riemann (1826–66). These are functions of one or more variables. Prominent persons working in this School have been Krazer, Rauch and Lebowitz, Thomae, Göpel, Rosenheim and Forsyth.

School 3 deals with the Weierstraß elliptic functions, after Weierstraß. These are more general, defined on a general lattice in the complex plane.

School 4 is just a modern English version of School 1. By 1875 the theory of elliptic functions was very popular; the *Messenger of Mathematics* had since no. 4 of 1875 a separate Section on elliptic functions, where among others Glaisher and Cayley have contributed.

School 5 covers the q-Gamma function. There are, for instance, many modern papers with inequalities for quotients of these functions. Atakishiyev [50, p. 1326] has rediscovered the q-analogue of the Euler mirror formula.

The roots of the Heine Ω function (School 6) come from the Jacobi-Heine treatment of elliptic functions [270].

Johannes Thomae (1840–1921) [497, p. 262] claims that his teacher Heine was the first to find the q-analogue of the Euler mirror formula [270, p. 310].

The main difference between the two functions is that Ω has zeros, in contrast to the Γ_q function which has no zeros, and therefore $\frac{1}{\Gamma_q}$ is entire. In his thesis Ashton [45], supervised by Lindemann, showed its connection to elliptic functions. Daum [148] tried to find all the basic analogues of Thomae's ${}_3F_2$ transformation formula [501, Eq. 11], using a notation analogous to that used by Whipple [541] and by essentially replacing the Γ_q function by the Heine Ω function.

Daum [148] concludes his thesis by saying *It is hoped, however that the use of the modified Heine Ω function, will serve to emphasize the analogy between hypergeometric series and q-hypergeometric series and simplify the notation generally.*

Sonine wrote a book [469] on the Heine Ω function in Russian, which also treats the Γ function.

School 7 is in the area of the Mellin and Gamma functions and the nineteenth century work in this area further influenced the Danish School, featuring names such as Thiele, Nielsen and Nørlund.

School 8 started with the numerous Italian books about elliptic functions written by Bellacchi in 1894, E. Pascal (II) in 1895, Bianchi (1856–1928) in 1901, and Giulio Vivanti (1859–1949) in 1900 [524].

This influenced Pia Nalli to write her paper [391], where Theta functions and a q-integral formula for a q-Riemann zeta function were given.

One can note here that the Gudermann's notation was very quickly accepted in Japan, see [378, p. 87].

2.6 Trigonometry, prosthaphaeresis, logarithms

There has always been a strong connection between mathematics and physics.

The 1648 book *Mathematical Magick* by John Wilkins (1614–1672) contains basic mechanics, but no mathematics, by today's definition.

Textbooks on mathematics in the late eighteenth century contained a variety of subjects like mechanics, optics and astronomy. One example is the book *Die Elemente der Mathematik* by Johann Friedrich Lorenz (1737–1807) from 1797, which treated such diverse subjects as refraction, parallax, geography, the atmosphere of the moon. Elementary trigonometric formulas were given, and these trigonometric functions were used to treat the physics involved. This was natural since many earlier mathematicians, like the Bernoullis, were both physicians and physicists. John Bernoulli (1) has in fact even written a memoir about mathematical medicine.

We will start with a brief history about trigonometry and its relationship to spherical triangles and astronomy.

Probably Aristarchus of Samos, Greece (-260) [392, p. 108] used ratios similar to tangens. Menelaus of Alexandria ($+100$) in his treatise on spherical trigonometry introduced the concept of sine [392, p. 108].

Aryabhata ($+510$) has for the first time used special names for sine and computed tables for every angle [392, p. 108]. His contemporary Vara-Mihira has in the year 505 given formulas that are equivalent to sine and cosine [392, p. 108]. These Indian works were then taken over by the Arabs and transmitted to Europe [392, p. 108]. The Egyptian mathematician and astronomer Ibn Yunus (950–1009) demonstrated the product formula for cos and made many astronomical observations. During the doldrums of the dark ages not much happened in trigonometry until the renaissance with the rediscovery of the old Arabic culture. Prosthaphaeresis (the Greek word for addition and subtraction) is a technique for computing products quickly using trigonometric identities, which predated logarithms. Myriads of books have been written on trigonometry in Latin before the modern notations sin and cos were introduced by Euler. The Greek and Alexandrian mathematicians were prominent in proof theory and geometry, including conic sections. Certainly these ancient scientists had some notation for trigonometric functions and some of this great research has survived through Latin translations during the renaissance.

Georg Purbach (1423–1461), who was born near Linz, has enriched the trigonometry and astronomy with new tables and theorems. He actually combined the chords of Ptolemaios with the sines of the arabs and in that way introduced the first tables of sines with decimals. His student in Vienna Johannes Müller (1436–1476) continued Purbach's research and inter alia introduced the tangent in astronomy; Müller called it *foecundus* (Latin: fruitful) because of its great advantages. His work on flat and spherical trigonometry in five volumes was printed in Nuremberg in 1533.

The trigonometry of Regiomontanus (Müller) looked about the same as now, the biggest difference being that logarithms were not used. Many other mathematicians have continued his development of trigonometry, e.g. John Werner (1468–1528), who wrote a work in five volumes on triangles. Werner used the product formula for sin and also contributed to the development of instruments. In the footsteps of Werner, Georg Rheticus (1514–1574) used Prosthaphaeresis computations for instruments. Through his friendship with Duke Albert of Prussia, Rheticus agreed in 1541 to the printing of Copernicus work *De Revolutionibus*. Two years earlier, Rheticus visited Nicholaus Copernicus (1473–1543), whose great book could not yet be printed. Copernicus *De Revolutionibus* was finally printed in 1543 in Nuremberg; in an annex Rheticus adds tables for sin and cos. In 1596, Rheticus' student Valentin Otto (1550–1603) published several books on plane and spherical trigonometry, based on Rheticus' computations.

Erasmus Reinhold (1511–1553) and Franciscus Maurolycus (1494–1575) have also published tables of tangents or secants.

In 1591, Philippus Lansbergen (1561–1632) computed tables for sin, sec and tan by hand in *Geometria Triangularium*. This is a short, concise work.

The Danish physicist and mathematician Thomas Fincke (1561–1656) was born in Flensburg and later worked in Copenhagen. Fincke has introduced the concept of sec.

Probably the first western European work dealing with systematic computations in plane and spherical triangles was written 1579 by François Viète (1540–1603), called 'the father of modern algebraic notation'.

Viète used a certain notation for multiplication and found [230] formulas almost equivalent to multiplication and division for complex numbers as well as the de Moivre's theorem.

Joost Bürgi (1552–1632) was also an advocate for Prosthaphaeresis. At about the same time as Napier, he invented the logarithms, but unfortunately he did not dare to publish his invention until 1620 [110, p. 165]. Bürgi, who was a Swiss clockmaker was also a member of the so-called *Rosenkreuzer* society, to which we will come back later.

James VI of Scotland, who was on a journey to Norway in the year 1590 together with his entourage, including Dr. John Craig, visited the island Ven on his return to Scotland. There Tycho Brahe (1546–1601) had constructed a big machine for prosthaphaeresis computations to ease the burden of calculation. Craig told his friend John Napier (1550–1617) about the visit to the famous astronomer and that inspired Napier to develop his logarithms and the generation of his tables, a work to which he dedicated his remaining 25 years.

Johannes Kepler (1571–1630), who was collecting Tycho Brahe's immense data, read Napier's book on logarithms in 1616; he found that he could describe his laws for orbital periods and semimajor axes for planetary ellipses as a straight line in a log-log diagram.

Thanks to these laws Isaac Newton (1642–1727) was able to discover the gravitation law. At the request of Kepler, Bürgi finally brought himself to publish his important book on logarithms in 1620. Logarithms have been in great use ever since; even physicians and nurses have employed these tables for various long computations.

In Ulm (1627), Kepler then published his Rudolphine tables with tables for $\log \sin x$ together with so-called Antilogarithms, a forerunner of the exponential. In 1801, James Wilson in *scriptores Logarithmici* published a work on the use of logarithms in navigation. Wilson has also written a book on finite differences in 1820.

2.7 The development of calculus

The first calculating machine was built by the German astronomer Wilhelm Schickard (1592–1635) in 1623 [361, p. 48] and was designed for Kepler. The Schickard calculator could add, subtract, multiply and divide, but remained unknown for 300 years. In 1642, Blaise Pascal (1623–1662) constructed a mechanical calculator, capable of addition and subtraction, called Pascal's calculator or the *Pascaline*, in order to help his father with his calculations of taxes [361, p. 16]. Some of his calculators were also exhibited in museums both in Paris and Dresden, but they failed to be a commercial success. Although Pascal made further improvements and built fifty machines, the *Pascaline* became little more than a toy and status symbol for the very rich families in Europe, since it was extremely expensive. Also, people feared it might create unemployment, since it could do the work of six accountants.

Gottfried Wilhelm Leibniz made two attempts to build a calculating machine before he succeeded in 1673; his machine could do addition, subtraction, multiplication and possibly division [283, p. 79].

Since the q-difference operator is fundamental for our treatment, we will go through the historical development of the calculus in some detail. We will outline the development of the infinitesimal calculus and find that the q-integral occurs in geometric disguise. Euclid computed the volume of a pyramid by a geometric series in Elements XII 3/5 [65, p. 48], [459, p. 98]. Archimedes used a geometric series to do the quadrature of the parabola. In the 17th century each mathematician did his own proofs in calculus. Pierre de Fermat (1601–1665), Gilles Roberval (1602–75) and Evangelista Torricelli (1608–47) had great success in the theory of integration. All three, independent of each other, found the integral and derivative of power functions, but in a geometric way. Roberval kept his post as professor in Paris by winning every contest that was set up. Thus he could not publish his discoveries since then he would reveal the secret's of his methods [527, p. 21]. Roberval's win of the competition in 1634 was probably due to his knowledge of indivisibles [527, p. 21].

Between 1628 and 1634 Roberval invented his method of infinitesimals [527, p. 59]. Roberval plotted graphs of trigonometric functions before 1637 in connection with a volume calculation [527, p. 67]. He was also the first to compute certain trigonometric integrals [527, p. 72].

In 1635 Bonaventura Cavalieri (1598–1647) was the first to publish integrals of power functions x^n in his book *Geometria indivisibilis continuorum*; but he proved the result explicitly only for the first few cases, including $n = 4$, while, as he stated, the general proof which he published was communicated to him by a French mathematician Jean Beaugrand (1590–1640), who quite probably had got it from Fermat. Beaugrand made a trip to Italy in 1635 to tell Cavalieri about Fermat's achievements [366, p. 51]. Cavalieri's method was much like Roberval's, but mathematically inferior [527, p. 21].

Fermat was a famous mathematician who founded modern number theory, analytic geometry (together with Descartes), and introduced the precursor of the q-integral.

In that time period, father Marin Mersenne (1588–1648) kept track on science in France and knew about all important discoveries, a kind of human internet. Mersenne had made the work of Fermat, René Descartes (1596–1650) and Roberval known in Italy, both through correspondence with Galilei (1564–1642) dating from 1635 and during a pilgrimage to Rome in 1644.

Fermat's contribution became known through a translation of *Diophantus Arithmetica* by Claude Gaspard de Bachet (1591–1639) in the year 1621. Fermat adhered to the algebraic notation of Viète and relied heavily on Pappus in his development of calculus. Like Kepler, Fermat uses the fact that extreme values of polynomials are characterized by multiple roots of the function put equal to zero [284, p. 63].

Originally Fermat put $f(x+h) = f(x-h)$ for extreme values [292], then developed the expression in terms of powers of h, and finally decided the type of the extreme value from the sign. Later in 1643–44 he even talked of letting $h \to 0$ [284, p. 63]. The leading mathematician of the first part of the seventeenth century was Fermat, who was very talented in languages and handwritings [284, p. 62]. Few results in the history of science have been so closely examined as Fermat's method of maxima and minima [245, p. 24].

Laplace acclaimed Fermat as the discoverer of the differential calculus, as well as a codiscoverer of analytic geometry. Fermat was the first to consider analytic geometry in $\mathbb{R}^3$.

Descartes contended himself with $\mathbb{R}^2$ [66, p. 83]. According to Moritz Cantor (1829–1920) [96, p. 800], Descartes and Fermat were the greatest mathematicians mentioned here.

At that time Fermat and then also Torricelli, had already generalized the power formula to rational exponents $n \neq -1$. Fermat determined areas under curves which he called general parabolas and general hyperbolas. This was equivalent to calculating integrals for fractional powers of x.

Vincenzo Viviani (1622–1703), another prominent pupil of Galilei, determined the tangent to the cycloid.

In 1665, the first two scientific journals were published, *Philosophical Transactions of the Royal Society* in England and *Journal des Savants* in France. The idea of private, for-profit, journal publishing was already established during this time. Fermat however, chose not to publish, probably for political reasons. It is also possible that Fermat was influenced by the old Greek habit of not publishing one's own proofs [366, p. 31].

Roberval was the first to (q-)integrate certain trigonometric functions. At about the same time important contributions were made by the English mathematician John Wallis (1616–1703) in Oxford, who in his book *the Arithmetic of Infinitesimals* (1655) stressed the notion of the limit, see Section 3.7.8.

It was in mathematics that Wallis became an outstanding scientist in his country, although he engaged himself in a wide range of interests.

Blaise Pascal associated himself with his contemporaries in Paris, like Roberval and Mersenne. He learned a method similar to q-integration from his masters and also corresponded with Fermat. Their short correspondence in 1654 [66, p. 107] founded the theory of probability. During several months from 1658 to 1659, Pascal summed infinite series, calculated derivatives of the trigonometric functions geometrically and found power series for sine and cosine. However, in Pascal's time there were no signs for sine and cosine. During the last years of his life Pascal also published the philosophical work *Lettres provinciales* under the pseudonym Louis de Montalte.

Trigonometric tables were published in great numbers in the early seventeenth century; e.g. Mathias Bernegger (1582–1640) published such tables 1612 and 1619 in Strasbourg.

During his life, Jacob Heinlein (1588–1660) had mostly worked as a priest; when he was inaugurated by Kepler in the mathematical world at the time, Heinlein was allowed to lecture on mathematics in Thübingen for a time after Schickard's death in 1635.

Heinlein published *Synopsis mathematica universalis* posthumously in Thübingen in 1663 and 1679, where some trigonometric functions appear. This book was then translated into English and issued in three editions in London, in 1702, 1709 and 1729. In the English expanded edition you will also find trigonometric tables.

The power series for sin and arcsin were communicated by Henry Oldenburg (1626–1678) to Leibniz in 1675 [10]. These series were also known to Georg Mohr (1640–1697) and John Collins (1625–1683). Isaac Barrow (1630–77), who was Isaac Newton's teacher in Cambridge [66, p. 117], published his geometrical lectures in 1664. Barrow was familiar to the concept of drawing tangents and curves, probably from the works of Cavalieri and Pascal. Barrow developed a kind of calculus in a geometrical way, but did not have a suitable algebraic notation for it.

J. M. Child claims that Barrow was the first to give a rigorous demonstration of the derivative for fractional powers of x. But probably Fermat forestalled him. Barrow also touched upon logarithmic differentiation.

Leibniz was a child prodigy in Leipzig, where he learned Latin and Greek by studying books in his father's library. At the age of 15 Leibniz explained the theorems of Euclidean geometry to his fellow students at the university. Leibniz also studied philosophy and was fascinated by Descartes's ideas. Leibniz had found remarks on combinations of letters in a book by Christopher Clavius (1537–1612) [283, p. 3]. Leibniz could decipfer both letter- and number codes and in 1666 published a thesis [357], where the mathematical foundations of combinations were given.

He was also interested in alchemy and became a member of an alchemical society in Leipzig [3]. It was a secret society named after Christian Rosenkreuz (1378–1484), and founded in the seventeenth century. Its aim was to further knowledge, in particular mathematics and alchemy. Despite his outstanding qualifications in law, Leibniz was not given his doctor's degree in Leipzig, so he turned to another city... He stayed in Nuremberg for several months 1667 to learn the secrets of the *Rosenkreuzer* and their scientific books. Because of his outstanding knowledge in alchemy, he was elected as secretary of the society [346, p. 109]. He remembers that he had Cavalieri's book about indivisibles in his hands during this stay [283, p. 5]. During this time, Leibniz was completely under the spell of the concept of indivisibles and had no clear idea of the real nature of infinitesimal calculus [283, p. 8].

Bored by the trivialities of the alchemists and realizing that the world's scientific centre was in France, Leibniz then entered the diplomatic service for several German royal families. As a diplomat, he first went to Paris in 1672, where he mingled in scientific and mathematical circles for four years. He was advised by the Dutch mathematician Huygens to read Pascal's work of 1659 *A Treatise on the Sines of a Quadrant of a Circle*.

In January and February 1673 he visited the Royal Society and was elected to membership [540, p. 260]. He wished to display one of the first models of his calculating machine [283, p. 24]. An English calculating machine was also shown to him, which used Napier's bones [283, p. 25]. On 6 April 1673 Oldenburg mailed Leibniz a long report, which Collins had drawn up for him, on the status of British mathematics. James Gregory's (1638–1675) and Newton's work dominated the report, which included a number of series expansions, although no suggestion of the method of proof was given [540, p. 260]. At the end of 1675, Leibniz had received only some of Newton's results, all confined to infinite series [540, p. 262]. At Leibniz' second visit to London, it appeared that Leibniz' method proved to be more general, but that left Collins wholly unmoved. Even before Leibniz' visit, Collins had been impressed enough to urge Newton anew to publish his method. But since Newton at the time was engrossed in other interests, he did not respond to Collins' suggestion and Oldenburg, who was engaged in Newton's research, unfortunately died two years later [540, p. 264].

Leibniz rewrote Pascal's proof of $\sin' x = \cos x$ in terms of increment in y/increment in x, using finite differences.

The idea of a limit in the definition of the derivative was introduced by Jean d'Alembert (1717–83) and Augustin Louis Cauchy in 1821.

Earlier, there was a suspicion that Leibniz got many of his ideas from the unpublished works of Newton, but nowadays there is agreement that Leibniz and Newton have arrived at their results independently. They have both contributed successfully to the development of calculus; Leibniz was the one who started with integration while Newton started with differentiation.

It was Leibniz, however, who named the new discipline. In fact, it was his symbolism that built up European mathematics. Leibniz kept an important correspondence with Newton, where he introduced and forcefully emphasized his own ideas on the subject of tangents and curves. Newton says explicitly that he got the hint of the method of the differential calculus from Fermat's method of drawing tangents [66, p. 83].

Leibniz also continued the use of infinity that had been used by Kepler and Fermat. In fact Kepler started with an early calculus for the purpose of calculating the perimeter of an ellipse and the optimal shape of wine boxes. When we, according to Leibniz, speak of infinitely great or infinitely small quantities, we mean quantities that are infinitely large or infinitely small, i.e. as large or small as you please. Leibniz said: It will be sufficient if, when we speak of infinitely large (or more strictly unlimited), or of infinitely small quantities, it is understood that we mean quantities that are indefinitely large or indefinitely small, i.e. as large as you please, or as small as you please... These notions were then further used in the works of Euler, Gauß, Heine, and the present author.

Newton called his calculus the "the science of fluxions". Newton used the notation $\dot{x}$ for his fluxions; this symbol is much used in mechanics today. He wrote his *Methodus fluxionum et serium infinitorum* in 1670–1671, but this work was not published until 1736, nine years after his death. It was the astronomer Edmond Halley (1656–1742) who paid for the printing of the masterpiece *Principia* in 1687, where the first rigorous theory of mechanics and gravitation was given. At that time there was only one journal in England where Newton could publish, *Philosophical Transactions of the Royal Society*, dating back to 1665. Newton's theory of light was published there 1672. The *Transactions of the Cambridge Philosophical Society* did not appear until 1821–1928.

On the other hand, the infinity symbol was often used in the circle around Leibniz. In 1682 the first scientific journal of the German lands, *Acta Eruditorum*, was founded in Leipzig, with Otto Mencke (1644–1707) as editor. This journal was very broad in scope and at that time mathematics and astronomy were considered to be one subject. Leibniz published his first paper on calculus in this journal in 1684. One other important author was Ehrenfried Walter von Tschirnhaus (1651–1708), who published on tangents under the pseudonym D. T. The Bernoulli brothers, Jakob and John, soon also started to publish in *Acta Eruditorum*. The result was that almost all of the elementary calculus that we now know was published here before the end of the eighteenth century. For instance, the well-known Taylor formula was published in a different form by Johann Bernoulli in 1694 [210].

We will stop from time to time and note some connections in order to motivate the q-umbral calculus introduced here. Our first concern is of computational nature. Today there are several programs used by scientists for computations. The q-calculus computations are especially hard because of their symbolic nature; there are only two programs which can do the job well. We are only going to grade these two programs on a relative scale, since an absolute evaluation is not possible; we will take the ability of drawing colour graphs into consideration. To describe the relative grades we write a table which compares Maple and Mathematica on the one hand, and prosthaphaeresis and logarithms on the other hand. The astute reader will immediately observe that logarithms have the higher grade, since it shortens down the calculations considerably. The present paper continues the logarithmic method for q-calculus which enables additions instead of multiplications in computations. The resemblance to hypergeometric formulas is also appealing. The q-addition corresponds to the Viète formula for cos and to the de Moivre theorem. We try to use a uniform notation for q-special functions, like the Eulerian notations sin and cos which are now generally accepted. All this is called renaissance [170], [168], [165] in the table.

For each row, the relative merit is higher to the right.

Viète	Descartes
prosthaphaeresis	logarithms
congruences	index calculus
Pascaline	Babbage computer
fluxions	Arbogast notation
Basic	Fortran
Maple	Mathematica
Gasper/Rahman	Renaissance

2.8 The Faulhaber mathematics

In this section we refer to Schneider's biography [455] of Faulhaber. Johannes Faulhaber (1580–1635) was an outstanding Ulmian mathematician, who early was apprenticed as reckoner. He spoke, however, with a few exceptions, neither Latin nor French [455, p. 180]. He could, therefore, not access the large Latin literature, which was available at the time. The complex numbers of Rafael Bombelli (1526–1572) [76] were also inaccessible to him. Faulhaber was a representative of discrete mathematics and we will briefly summarize his contributions.

1. The first 16 Bernoulli numbers (1631) [161, p. 128].
2. A formula equivalent to (2.4).
3. Pascal's triangle.
4. Power sums similar to Section 5.3.
5. The formula (compare (5.89))

$$\sum_{s=k}^{n-1} \binom{s}{k} = \binom{n}{k+1}. \tag{2.10}$$

2.9 Descartes, Leibniz, Hindenburg, Arbogast

Descartes had travelled a lot in his youth and perhaps met Faulhaber in the year 1620 in Ulm [11, p. 223]. Faulhaber was a member of the *Rosenkreuzer*. The *Rosenkreuzer* liked to use unusual signs in their texts; e.g. Faulhaber showed the astrological sign of Jupiter to Descartes. The *Rosenkreuzer* also liked to collect scientific books; Faulhaber had collected several books about algebra and geometry in his home. According to [373, p. 50], Descartes attempted in vain to contact the brotherhood during his travels in Germany. Around that time mathematics in France was not prosperous, due to the dominance of the Catholic Church (Richelieu, Mazarin). As we have seen, the most talented scientists had to keep a secret correspondence: they knew what had happened to Galilei. In 1666, the French Academy of Sciences was founded at the suggestion of Colbert. The situation further improved after the translation of *Principia* into French in 1759 by Émilie du Châtelet (1706–1749). In Germany the situation was different, some states had changed to Protestantism. The *Rosenkreuzer* were opposed to the Catholic Church and preferred a reform of the religious system in continental Europe. They disliked the opposition of the Catholic Church to scientific ideas and wanted a change. This may have been the most important reason why the *Rosenkreuzer* brotherhood was a secret society. There were also English mathematicians, like John Dee (1527–1609), who were *Rosenkreuzer*. Dee had written a long preface to the English translation of *Elements*. He was in the service of Queen Elizabeth and had visited many people on his long trips, including Tycho Brahe. The famous scientist Robert Hooke (1635–1703), contemporary of Newton, wrote in codes and Newton was a dedicated alchemist who virtually gave up science during the last third of his life [64, p. 196]. When Descartes returned to Paris in October 1628, there were rumours that he had become a *Rosenkreuzer*. According to [3], Descartes started to keep a secret notebook with signs used by the *Rosenkreuzer*. Then at the end of 1628, Descartes definitely moved to Holland, where he stayed for 21 years.

The infinite series were introduced by Newton. The formal power series were conceptually introduced by Euler [190]. We will see that this concept prevailed for quite a while until the introduction of modern analysis by Cauchy in 1821.

Leibniz did not have many pupils in Germany, since he worked a lot as a librarian and travelled frequently. While in France mathematical physics was flourishing (Laplace, Lagrange, Legendre (1752–1833), Biot (1774–1862)), German mathematics had a weak scientific position in the time between Frederick the great (1712–1786) and the Humboldt education reform [312, p. 173], [95, p. 256]. One exception was Georg Simon Klügel (1739–1812), who introduced a relatively modern concept of trigonometric functions in [339] (1770) and [338] (1805).

Combinatorial notions such as permutation and combination had been introduced by Pascal and by Jacob Bernoulli in [70].

In the footsteps of Leibniz, Hindenburg, a professor of physics and philosophy in Leipzig founded the first modern School of combinatorics with the intention of promoting this subject to a major position in mathematics. One of the ways to achieve this aim was through journals. The *Acta Eruditorum* continued until 1782, then Hindenburg and John Bernoulli (III) (1744–1807) edited *Leipziger Magazin für reine und angewandte Mathematik* (1786–1789). This was followed by *Archiv der reinen und angewandten mathematik* (1794–1799) with Hindenburg as sole editor. Hindenburg used certain complicated notations for binomial coefficients and powers; one can feel the influence of the *Rosenkreuzer* secret codes here. The main advantages of the Hindenburg combinatorial School was the use of combinatorics in power series and the partial transition from the Latin language of Euler. The disadvantages were the limitation to formal computations and the old-fashioned notation.

A partial improvement was made by Bernhard Friedrich Thibaut (1775–1832) [312, p. 193], [496], who expressed the multinomial theorem, a central formula in the Hindenburg School, in a slightly more modern way. Thibaut also made a strict distinction between equality of formal power series and equality in finite formulas [312, p. 196], a central theme in q-calculus.

A similar discussion about formal power series was made by Gudermann in 1825 [253].

The Hindenburg combinatorial School can be divided into two phases [312, p. 171].

1. 1780–1808.
2. 1808–1840.

The first important book of Hindenburg [280] began with a long quotation from Leibniz [357]; this quotation would be normal in Hindenburg combinatorial School publications until 1801 [312, p. 178].

One of the main reasons for this is the so-called multinomial expansion theorem, a central formula in the Hindenburg combinatorial School, which was first mentioned in a letter 1695 from Leibniz to John Bernoulli, who proved it.

Hindenburg thought that a mechanical calculation would be an important objective for his School [312, p. 222] and that is exactly what we do in this book.

The bricks are the basic formulas for the *q-factorials*. The general formulas and developments can often be expressed through the q-binomial coefficients as (4.74) and (4.75).

Heinrich August Rothe (1773–1842) introduced a sign for sums, which was used by Gudermann. In 1793 Rothe found a formula for the inversion of a formal power series, improving on a formula found, with no proof provided, by Hieronymus Eschenbach (1764–1797) in 1789 [312, p. 200]. This invention gave the combinatorial School a rise in Germany, as can be seen from the list of its ensuing publications [312, p. 201].

Rothe's presentation was marked by clarity, order and completeness, according to a report in the newspaper *Jenaische Allgemeine Literaturzeitung* in 1804. Rothe was the teacher of Martin Ohm, of whom we shall hear much later.

In the year 1800, Louis Arbogast suggested [42] to substitute a capital D for the little $\frac{dy}{dx}$ of Leibniz to simplify the computations. Arbogast [42, p. v] writes: *The Leibniz theory of combinatorics has been improved by Hindenburg, who has studied the development of functions of one variable and given the multinomial theorem. The procedure and notation of Hindenburg is not familiar. I don't know his writings* *except for the title; I have followed my proper ideas. The procedure I will give is very analytical.*

In [42, p. 127] Arbogast gives Taylor's formula for a function of two variables. The use of signs for sums would significantly increase the clarity of this long formula… At any rate, Arbogast's formula had a long-lasting influence on the development of calculus in Germany and in England.

This can be seen from the different notations in two publications by Hindenburg. In 1795, the Journal *Archiv der Reinen und Angewandten Mathematik* received some papers on the Taylor formula, in which it was expressed through a difference operator. In 1803, however, Hindenburg [279, p. 180] also used the symbol D and apparently noticed the difference between the two. The above-mentioned journal also received some military reports, among others from Johann Heinrich Lambert (1728–1777), mathematician and physicist. Hindenburg, who was also a physicist, read Lambert's articles in physics with interest, which in turn had a strong influence on Gudermann.

From about this time onwards, the theory of special functions according to Euler can be said to have started. This development was parallel to the theory of the Bernoulli numbers and the Stirling numbers.

2.10 The Fakultäten

In 1730 James Stirling (1692–1770), by a remarkable numerical analysis, confirmed a result which would now be written in the form [159, p. 18]:

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

The Frenchman Alexandre-Théophile Vandermonde (1735–1796) found the same formula [516]. This article was translated into German and the formula appeared in [515, p. 77].

In 1772 Vandermonde introduced the following notation [515]:

Definition 14 The falling factorial is defined by

$$(x)_{n-} \equiv x(x-1)(x-2)\cdots(x-n+1). \tag{2.11}$$

We will now explain how the development of the Fakultäten occurred in parallel with the development of Newton's binomial theorem and the fluxions. To avoid the metaphysical difficulties of the fluxions, John Landen (1719–1790) suggested to use a purely algebraic method, which could be compared with Lagrange's operational method. That's why Landen was called the English d'Alembert.

The Italian mathematician A. M. Lorgna (1730–1796) had similar ideas. In the period 1791–1806 a series of books entitled *Scriptores Logarithmici* [371] were published in London. In the second volume Landen gives a proof of Newton's binomial theorem with positive quotient-exponent. In volume five he gives the proof for exponent $-\frac{1}{n}$. The name and the method has many similarities with the so-called *Fakultäten*. As we shall see, the fluxions disappeared in the English teaching about twenty years later.

In 1798 Christian Kramp (1760–1826) introduced the *Fakultäten*, a function similar to the Γ function. In 1800 Arbogast used the word Faktoriell and in 1808 Kramp introduced the notation $n!$.

Vandermonde [515] used the function

$$f(u,x,y) \equiv \prod_{m=0}^{y-1}(u+mx). \tag{2.12}$$

When $y \in \mathbb{N}$, the following relations hold:

$$f\big(u,x,y+y'\big) = f(u,x,y)f\big(u+yx,x,y'\big), \tag{2.13}$$

$$f(u,x,1) = u, \tag{2.14}$$

$$f(ku,kx,y) = k^y f(u,x,y), \tag{2.15}$$

$$f(u,x,y) = f(u+yx-x,-x,y), \tag{2.16}$$

$$f(u,0,y) = u^y. \tag{2.17}$$

This combinatorial School of Vandermonde and Kramp enjoyed a certain popularity in the period 1772–1856. The aim of this School was to divide the Fakultäten into four classes: positive, negative, whole and fractional exponents.

Each class had its own laws, similar as for the q-factorial. The *Fakultäten* were also considered by Ettingshausen [525, p. 190]. Arbogast also writes a lot about the *Fakultäten* in his book [42, p. 364]. Vandermonde and Kramp [347] tried to extend this function to all $y \in \mathbb{R}_+$ by the definitions (2.13)–(2.17). This however did not turn out well, as was shown by Friedrich Wilhelm Bessel (1784–1846) in 1812 [71, p. 241]. Bessel tried to mend this theory by defining the factorial $f(u,x,y)$ in another way. In this connection, Bessel [71, p. 348] published a formula very similar to the Euler reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2.18}$$

This formula was expressed in terms of *Fakultäten*. Two other similar formulas were published in the same paper. In 1824, M. Ohm and Ludwig Öttinger (1797–1869) made further attempts in *Crelle Journal* to rescue the theory of the *Fakultäten* without the introduction of complex numbers and of the principal branch of the logarithm. In 1843 and 1856 Weierstraß [539] published a long paper, which put the theory of analytic factorials on a solid mathematical foundation by connecting the formulas to the Γ function. Weierstraß' investigations also contained Bessel's formula (2.18). Thereafter, the Γ function was mainly used. Nevertheless, the *Fakultäten* have continued to live in the Russian textbook by Guelfond, *Calcul des Différences Finies* [257, p. 25]; this book was translated from Russian into many languages.

2.11 Königsberg School

The Königsberg School was able to develop quickly, as there were not so many visitors and one could therefore concentrate on research. It included many celebrities, such as Immanuel Kant (1724–1804) and Jacobi. The proximity of St. Petersburg was also noticeable, e.g. for the Bernoulli numbers. The Bessel function, which was introduced in 1824 by Bessel, was investigated earlier by Jacob Bernoulli, Daniel Bernoulli (1700–1782), Euler and Lagrange. There is also a link to differential equations: the Bessel differential equation is related to the Riccati differential equation, which was introduced in 1724 by Francesco Riccati (1676–1754).

This paved the way for the development of the hypergeometric function, the basis of special functions and the prerequisite for q-hypergeometric functions.

Jacobi discovered the theta functions by a brilliant formal derivation. After Jacobi's death (1851) his School was continued by his favourite student Friedrich Julius Richelot (1808–1875).

Franz Ernst Neumann (1798–1895) and his doctoral student Louis Saalschütz (1835–1913) have both contributed to the theory of special functions. Neumann, who formulated the law of electric induction in 1845 and 1847, but with very different notation, has also contributed to the theory of Bessel functions (Neumann function).

In his book [446] Saalschütz summarized the present knowledge in the field of Bernoulli numbers, and has [446, p. 54–116], according to Gould, given 38 explicit formulas for Bernoulli numbers.

Saalschütz also republished the Euler-Pfaff-Saalschütz summation formula for hypergeometric functions.

Reinhold Hoppe (1816–1900), mathematical physicist and professor in Berlin, also belonged to the Königsberg School.

2.12 Viennese School

This is a School without q-calculus.

Ettingshausen introduced the notation $\binom{m}{n}$ for the binomial coefficients in his book about combinatorics [525, p. 195] published in 1826 and used the so-called Stirling numbers. At that time, Euler's *Vollstaendige Anleitung zur Integralrechnung* was also translated from Latin into German and published in Vienna.

Jozef Petzval (1807–1891) [227] was a world famous optician and good mathematics teacher, who wrote an excellent textbook on differential equations.

Leopold Gegenbauer (1849–1903) wrote excellent articles about orthogonal polynomials; the Gegenbauer polynomials are named after him.

Lothar Koschmieder (1890–1974) worked in Wroclaw, Brno, Aleppo, Graz and Tübingen. In the forties Koschmieder published, among other journals, in Austrian journals. He worked a great deal with differential operator computations on polynomials and (multiple) hypergeometric series.

2.13 Göttingen School

In 1808 Gauß wrote his only article on q-analysis [442], where he introduced the q-binomial coefficients and computed some sums of these. This has evidently influenced Rothe and Johann Philipp von Grüson (1768–1857) to find the following important theorem. According to Ward [531, p. 255] and Kupershmidt [349, p. 244], the identity (2.19) was already known to Euler. Gauß in 1876 [222] proved this formula.

Theorem 2.13.1 *Rothe* (1811) [441], [526, p. 36] (1814) *von Grüson*. *Fundamental Theorem of q-calculus*:

$$\sum_{n=0}^{m}(-1)^n\binom{m}{n}_q q^{\binom{n}{2}}u^n = (u;q)_m. \tag{2.19}$$

We note that the names of two students of Gauß, Grünert and Gudermann, can be found in the next section, and this is no coincidence.

Johannes Friedrich Posselt (1794–1823) used *Fakultäten* in his Göttingen thesis *Dissertatio analytica de functionibus quibusdam symmetricis* (1818); he spent his last years at the observatory in Jena.

Without ever having passed through a baccalaureate degree, Moritz Stern (1807–1894), born in Frankfurt, received in 1829 the Doctorate for a treatise on the theory of continued fractions. Stern then became interested in the field of number theory, to which he would devote by far the greater part of his life. In the period until the appointment to professor in 1859, Stern also pursued the teaching, which Gauß did not like: the two were good friends.

In 1847 Stern wrote the book *Zur Theorie der Eulerschen Integrale*. Stern had a considerable role in the massive reform of university mathematics teaching that took place in this period.

2.14 The combinatorial School: Gudermann, Grünert

The goal of the combinatorial School was to develop functions in power series by Taylor's formula. Taylor's formula was originally formulated with finite difference-quotients, so-called fluxions. Earlier, there were two names for finite differences, after Brook Taylor (1685–1731) and Jasques Cousin (1739–1800). Instead, in 1795, Hindenburg introduced the notation [281, p. 94] $\overset{k}{y}$ in *Archiv der Reinen and Angewandten Mathematik.*

Christoph Gudermann, encouraged by his close friendship with August Crelle (1780–1855), was first school teacher in Cleve, later professor in Münster. He first wrote in German and later alternately turned to Latin, the common scientific language of that time and could therefore reach international recognition.

Gudermann wrote an excellent Latin in a time when the Latin in Europe had declined.

Crelle was very interested in the contemporary mathematical questions and was able to find publishers for Gudermann's textbooks.

Gudermann even had a crucial role as the teacher of Weierstraß. It is reported that thirteen students came to the first lecture of Gudermann about elliptic functions. At the end of the semester only one remained, namely Weierstraß. Gudermann was one of the first who discovered Weierstraß' extraordinary talent for mathematics. Weierstraß was inspired by Gudermann's theories on series expansions and often expressed his great gratitude for his old teacher. Weierstraß, for his part, developed and modified the Gudermann combinatorial School.

Bernard Riemann was probably also influenced by Gudermann. In his textbook on elliptic functions [432] one finds the series (7.32)–(7.34) with the Gudermann notation. This book contains the Riemann lectures of 1855–56 and 1861–62.

Influenced by Lambert, who introduced the hyperbolic functions, Gudermann, among other things, developed the function $\frac{1}{\cosh(x)}$ in powers of x, using the work of Scherk on the so-called Euler numbers. The Gudermann names for the trigonometric functions had many followers until 1908 [94, p. 173].

Gudermann used the sign for sums of Rothe and also a product symbol for $\sin x$ and $\cos x$ in the form of infinite products [255, p. 68]. Gudermann has often developed his functions by Taylor's formula; he then used a precursor of the Pochhammer symbol − in the disguise of Kramp's notation.

One could say that the circle around Gudermann formed a School of its own. This School consisted, among others, of Johann August Grünert (1797–1872), editor of *Archiv der Mathematik und Physik*, which started in 1841, and Oscar Schlömilch, editor of *Zeitschrift für Mathematik and Physik*, which started in 1856. These two journals differed from Crelle's Journal, which had a more purely mathematical character. Grünert, mathematician and physicist, also took over the completion of the dictionary of Klügel and the preparation of two supplementary volumes. He was a student of Pfaff and Gauß, wrote, among other things, early on *Fakultäten*, and thus composed tables of Stirling numbers [250, p. 71], [252, p. 279]. Stirling numbers were later used in series expansions for Bernoulli functions [545, p. 210]. Grünert worked almost 39 years as a full professor in Greifswald, where he founded a mathematical seminar in 1825, and also made his private library available to his students.

In Sweden, Malmsten and Björling both contributed to the *Grünert Archiv*. This magazine also included publications about hyperbolic functions and spherical trigonometry. This last name is a modern term for analytical Sphaerics, which was treated in [254]. We will later return to contributions about elliptic functions in Grünert Archiv.

You could say that this was an earlier start of the Mathematical Reviews and Centralblatt Mathclass section 33 (Special functions with applications) in Europe.

Grünert had a conflict with Grassmann in 1862, and perhaps this is why his name is not mentioned in Klein's eminent book [336]; Klein also treats Gudermann unfairly.

In 1853, when Grünert was 56, *Archiv der Mathematik und Physik* began its decline. After the death of Grünert in 1872, Hoppe took over the editorial.

2.15 Heidelberg School

Franz Ferdinand Schweins (1780–1856) taught mathematics 46 years at the University of Heidelberg. Heidelberg became the centre of the combinatorial School under Schweins. Öttinger was director of the Pädagogicum in Durlach (near Karlsruhe) in 1820, professor at the Gymnasium in Heidelberg in 1822 and next lecturer at the University of Heidelberg from 1831 to 1836.

One might therefore speak of a Heidelberg School. In the textbook by Öttinger of 1831 [408, p. 26] one finds Euler's special case of the formula (2.20) below; the proof has been carried out by using partitions, just like Euler did. The textbook by Schweins [458, p. 613] contains the *Fakultäten*. He often referred to Jozef Maria Hoene Wronski (1778–1853). Schweins then proves the following theorem (2.20) with the help of the *Fakultäten*, that together with (2.19) forms the basis of q-analysis.

Theorem 2.15.1 *The Schweinsian q-binomial theorem* [457, p. 294, (497)]:

$$\sum_{n=0}^{\infty} \frac{\langle a; q\rangle_n}{\langle 1; q\rangle_n} z^n = \frac{(zq^a; q)_\infty}{(z; q)_\infty},$$

$$|z| < 1,\ 0 < |q| < 1. \tag{2.20}$$

In the following, we call this relation simply the q-binomial theorem; it is more general than the two Euler formulas (6.188), (6.189).

Remark 2 The notation of Schweins leaves much to be desired, yet it is better than Hindenburg's notation.

Remark 3 In his book [458, p. 317–] Schweins gives a theory of "Producte mit Versetzungen", which according to Muir [389] is equivalent to a deep theory of determinants. His colleague Öttinger has written a similar book *Differential und Differenzenrechnung* in 1831. It is clear that the two have worked together.

2.16 Weierstraß, formal power series and the Γ function

Karl Weierstraß was a colleague of M. Ohm, Kummer and Hoppe in Berlin. In the footsteps of Gudermann, Weierstraß introduced the modern Analysis.

A famous student of Weierstraß, Gösta Mittag-Leffler (1846–1927) tells [491, p. 214] that formal power series were always used. The Euler and Gaussian formulas for the Γ function were here brought together in modern notation.

Earlier Adrien-Marie Legendre had denoted the Γ function by Γ and computed the Euler integrals for $\Gamma(x)$. As we have already mentioned, in 1856 Weierstraß replaced the *Fakultäten* by the complex Γ function. In the same year Alfred Enneper's dissertation *Über die Funktion* Γ *von Gauß mit komplexem Argument* appeared. Based on the Legendre integral representation of the Gamma function, Enneper proceeds purely formal and arrives through long logarithmic computations to a series of profound results.

These two works by Weierstraß and Enneper (1830–1885) thus have virtually extended the Γ function to the complex plane. In the present book the Γ_q function is extended to the complex plane, compare [180]. The elliptic functions in a more general form than Gudermann's also played an important role in Weierstraß work. The number of Weierstraß students was high, and we list here only those who mainly dealt with special functions. Nicolai Bugaev (1837–1903), doctorate in 1866, had a gifted student Sonine, who worked with Laguerre polynomials and Bessel functions. Mathias Lerch (1860–1922) has written some interesting articles on Theta functions, which are similar to q-analysis. The works of Lerch in q-analysis have many similarities with those of Leopold Schendel [450], who has also written a book on this subject [451]. Schendel published work on the q-Gaussian Taylor series and pointed out an expansion of the logarithmic integral.

2.17 Halle q-analysis School

In 1844 Gudermann published his famous book on elliptic functions [256]. Two years later, in 1846, E. Heine, a docent in Bonn, who had studied with Gauß, Lejeune Dirichlet (1805–1859) and Jacobi, wrote the following letter to his professor Dirichlet. This letter, which is a natural continuation of the work of Gudermann, was published in the same year [269] in the Crelle Journal.

> Sehr viele Reihen, darunter auch solche, auf welche die elliptischen Funktionen führen, sind in der allgemeinen Reihe
>
> $$1+\sum_{k=1}^{\infty}\frac{\prod_{m=0}^{k-1}(q^{a+m}-1)\prod_{m=0}^{k-1}(q^{b+m}-1)}{\prod_{m=0}^{k-1}(q^{1+m}-1)\prod_{m=0}^{k-1}(q^{c+m}-1)}z^k$$
>
> enthalten, die ich zur Abkürzung mit
>
> $${}_2\phi_1(a,b;c;q,z)$$
>
> bezeichne, gerade so wie es bei der hypergeometrischen Reihe zu geschehen pflegt, in welche unser ϕ für $q=1$ übergeht. Es scheint mir nicht uninteressant, die ϕ ganz ähnlich zu behandeln, wie **Gauß** the ${}_2\mathrm{F}_1$ in den 'Disquisitiones generales' untersucht hat. Ich

> will hier nur flüchtige **Andeutungen** zu einer solchen Übertragung geben. Es entspricht jedem F im §5 der Disquisitiones genau eine Reihe ϕ. Einigen Formeln entsprechen zwei oder mehr verschiedene ϕ, nämlich denen, in welchen a oder b unendlich werden. So hat die Reihe für e^t zwei Analoga.

Heine worked as a professor in Halle, he often went to Berlin, as his sister, who was married to Felix Mendelssohn, lived there.

Heine introduced the q-hypergeometric functions and proved their transformation formulas formally by continued fractions [270]. This was the first time that q-equations had to be corrected; certainly it would have been better for Heine (already in 1847) to use the improved notation (1.57).

Heine translated the work of the Swedish mathematician Göran Dillner (1832–1906) about quaternions into German. Heine published his famous book on spherical harmonics [272] in 1861. In the same year, Thomae began his studies at the Universität Halle, near his home. Heine had the greatest influence on Thomae, who thus developed his great liking for function theory.

The astronomer Ernst Schering (1833–1897) was tutor of Thomae in Göttingen 1864, and wrote a treatise called *Allgemeine Transformation der Thetafunctionen*. In 1867 Thomae became a docent in Halle, where he was a colleague of Heine and Georg Cantor (1845–1918). Together with the Reverend Jackson, Thomae has developed the so-called q-integral, the inverse of the q-derivative. Thomae also wrote important works about hypergeometric series and in fact many years these two subjects have developed together.

The cooperation with Heine in q-analysis lasted till 1879.

Karl Heun (1859–1929), a student of Schering and Enneper in the period 1878–80 in Göttingen, did not stay long in Halle between April and October 1880. Heun went back to Göttingen and began his doctoral work, which was inspired by Heine; his supervisor was again Schering. In 1881 Heun defended his doctoral dissertation *Kugelfunctions and Lamésche functions als Determinanten*. The Heun equation is a linear differential equation of second order of Fuchsian type with four singular points.

2.18 Jakob Friedrich Fries, Martin Ohm, Babbage, Peacock and Herschel

In the following we frequently quote Elaine Koppelman [345] and E. P. Ozhigova [409].

Robert Woodhouse (1773–1827), George Peacock (1791–1858), Charles Babbage (1791–1871) and John Herschel (1792–1871) were all from Cambridge.

Woodhouse discussed at length the importance of good notation [345, p. 177], since the development of calculus in Cambridge had been slow until 1820 [345, p. 155]. His conclusion was that the notation of Arbogast is by far superior.

Cajori [93] has drawn a similar conclusion.

Already in 1803 Woodhouse had tried to put calculus on a rigorous algebraic basis by a formal power series development, similar to Lagrange, in his important work *Principles of Analytic Calculation*, which had a lasting influence on Babbage [345, p. 178]. Another attempt by Woodhouse to bring mathematics at Cambridge up-to-date was in 1804, when he published a paper on elliptic integrals in the *Philosophical Transactions of the Royal Society*. He fully realised the significance of the topic, which earlier had received little attention in Cambridge.

The first immediate reaction to Woodhouse's book from 1803 came when Rev. John Brinkley (1763–1835), in a paper in *Phil. Trans. Royal Soc. London* in 1807, started the first symbolic calculus in England. Brinkley's paper contained some abbreviations for expressions like $\frac{x^n}{n!}$ and $\frac{\dot{x}}{n!}$. Here $\dot{x}$ is the fluxion of x. Brinkley also calculated with expressions for "differences of nothing", the precursor of Stirling numbers. Of course the Stirling numbers had been known already to Thomas Harriot (1560–1621), but Brinkley probably was not aware of this. Although Brinkley knew about Arbogast, he writes: *My publication has hitherto been delayed by my unwillingness to offer a fluxional notation different from either that of Newton or Leibniz, each of which is very inconvenient as far as regards the application of the theorems for finding fluxions*.

Brinkley's work became widely known in Russia, partly due to his fame as an astronomer. His work was also published in France by the mathematician and astronomer Dominique François Jean Arago (1786–1853) in 1827 [409, 138]. After receiving the chair of astronomy at Trinity College, Dublin, in 1790, Brinkley had to wait 18 years until the new telescope was erected, and still stands. He had eighteen years more in which to use it. During the first of these periods Brinkley devoted himself to mathematical research; during the second he became a celebrated astronomer.

Jakob Friedrich Fries (1773–1843) was a German philosopher, who valued mathematics very highly. In 1822 Fries's work on the mathematical philosophy of nature appeared.

Fries says: Every philosophy that matches the exact sciences may be true, any that contradicts them must necessarily be false.

Among others, Gauß highly valued the philosophy of Fries, and Schlömilch was a student of Fries.

The fluxion concept was dominant in England until 1820, when the four people from Cambridge managed to recognize the notation of Leibniz and Arbogast in England [345, p. 156]. This led shortly to the introduction of operator calculus [345, p. 156] or umbral calculus. The fluxion notation was cumbersome, an expression could have many meanings [409, p. 139]. Other mathematicians could not understand the fluxions [409, p. 139].

The astronomical tables had a great importance for the navigation.

Herschel and Babbage have pointed to the many errors in the astronomical tables and insisted that an automatic calculating machine is needed [409, p. 139]. Many computations were then made using logarithms, see the book by Wilson from 1820 about calculus of finite differences.

In the translation of Lacroix's book by Babbage, Peacock and Herschel (1816) it is claimed in the preface that calculus was discovered by Fermat, made analytical by Newton and enriched with a powerful and comprehensive notation by Leibniz [345, p. 181]. Before Babbage dropped this subject he once again stressed the importance of a good notation for calculus [345, p. 184]. E. T. Bell (adviser of Morgan Ward) wrote that operational mathematics, which was developed in England during the period 1835–1860, despite its obvious utility, was scarcely reputable mathematics, because no validity condition or validity region accompanied the formulas obtained [345, p. 188]. In his book *Treatise on Algebra* (1830) Peacock studied the relationship between algebra and natural numbers; he called it symbolic algebra.

One of Peacock's students was De Morgan [409, p. 143]. England thus became the centre for symbolic calculations. Combinatorial analysis (Germany) and symbolic computations (England, France, Italy) developed in two different directions [409, p. 121]. The symbolic calculus was introduced in 1880 in Italy [115]; in the footsteps of Koschmieder, Johann Cigler reintroduced the symbolic method in Austria 1979 [133].

In 1837 the *Cambridge Mathematical Journal* was founded by, among others, Duncan Gregory (1813–1844), to provide a place for publication of short mathematical research papers and thus encourage young researchers. In an 1845 letter to John Herschel [345, p. 189], De Morgan described the journal and Gregory's contributions as full of very original communications, very full of symbols. In Gregory's first paper on the separation of symbols, the linear differential equation with constant coefficients was treated. Similar studies had already been published by Cauchy in France; Gregory was familiar with Cauchy's and Brisson's works on this subject [345, p. 190]. As pointed out by De Morgan in 1840 [345, p. 234], the symbolic algebra method gives a strong presumption of truth, not a method of proof. Gregory correctly claimed that the operations of multiplication and function differentiation obey the same laws [345, p. 192]. Gregory's methods only applied to differential operators with constant coefficients.

A generalization to non-commutative operators was given in 1837 [345, p. 195] by Robert Murphy. The studies of non-commutative operators were continued by, among others, George Boole, William Donkin (1814–1869) and Charles Graves (1812–1899). More general functional operators were studied by W. H. L. Russell [345, p. 204], William Spottiswoode (1825–1883), William Hamilton (1805–1865) and William Clifford (1845–1879).

Let's summarize: the calculus of operations, imported from France and extended by Babbage and Herschel, was an important mathematical research area in England between 1835 and 1865 [345, p. 213]. Most of these articles were published in the *Cambridge Mathematical Journal* and its successor, the *Cambridge and Dublin Mathematical Journal* (CDMJ) (1845–1854).

The CDMJ handled all kinds of subjects such as physics and astronomy. In CDMJ 1 De Morgan summarized the work of Arbogast.

In CDMJ 3, Rev. Brice Bronwin wrote an article about umbral calculus. This description is typical of the many so-called mathematicians in England who were clergymen and were not familiar with the modern Analysis of Cauchy. Formal power series were introduced in England in the 1880s by Oliver Heaviside. Soon after England took over the lead in umbral calculus.

The *Quarterly Journal of Pure and Applied Mathematics* (QJPAM) rose from the ashes of the CDMJ in April 1855 [144]. The first two British editors were Ferrers and J. J. Sylvester (1814–1897), who was editor until 1878. It was here that the first works on umbral calculus were published by Horner in 1861, John-Charles Blissard in the years 1861–68 [144] and Glaisher. It was Sylvester who coined the name umbral calculus.

F. H. Jackson, the first master of q-calculus in the twentieth century, who was also a priest like Blissard, published many of his papers in QJPAM.

From about 1860 the calculus of operations split into different areas, some of which are:

1. umbral calculus.
2. q-calculus.
3. theory of linear operators [345, p. 214].
4. algebra.

These different subjects are however far from disjoint.

The theory of formal power series within the Hindenburg combinatorial School continued under Martin Ohm, who under the influence of Cauchy's *Cours d'analyse*, obtained convergence criteria for the known elementary transcendental functions in certain regions. Ohm defined mathematical analysis using seven basic operations and built a rather involved theory, which clearly was a forerunner of later attempts to base all of mathematics on the integers [345, p. 226].

Some students of M. Ohm were

1. Eduard Heine, who introduced the q-hypergeometric series.
2. Leo Pochhammer (1841–1920), known for the Pochhammer symbol and the Pochhammer integral.
3. Friedrich Prym (1841–1915), who made many investigations about special functions and even founded a mathematical School in Würzburg.

Martin Ohm made mathematics into a clean, accurate subject, without physics. He also wrote physics books. In a textbook of 1862 (short guide) Ohm writes on page 62 about *Fakultäten*. In 1848 Ohm unsuccessfully tried to merge the Γ function, which after 1856 was universal and the *Fakultäten*.

In 1833 Hamilton read a paper expressing complex numbers as algebraic couples, and in 1837 he presented an article on the arithmetization of analysis [372]. This was a careful, detailed and logical criticism of the foundations of algebra, and it represents an important step in the development of modern abstract algebra (C. C. Macduffee, 1945) [345, p. 222].

In 1846 Hamilton published a series of papers with the title *On symbolic geometry*. Again he cited Peacock and Martin Ohm as the authors who had inspired him to a deeper appreciation of the new School of algebra [345, p. 222]. Like De Morgan, Hamilton wanted, at first, a system which would form an associative and commutative division algebra over the reals [345, p. 228]. Out of this finally grew the famous quaternions.

2.19 Different styles in q-analysis

In mathematics there are different styles. There is the so-called real analysis, in modern literature often connected with Sobolev spaces, where one the main purposes is to find different norms and inequalities for integrals. Then there is the field of formal computations with special functions, which is closely connected with number theory and induction.

In q-analysis, there is a similar layout. Here in the real analysis case, often the Γ_q function is used to formulate inequalities. Also q-integrals occur, but here far from all possibilities are exhausted.

The formal computations within q-analysis play, as we will see, a big role. In most cases, a real analysis article contains few formal calculations and vice versa. But this need not always be the case, both styles can learn from each other.

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