# CHAPTER II

# $\mathbb{Z}^d$-actions on compact abelian groups

## 5. The dual module

According to Theorem 4.2, $\mathbb{Z}^d$ is of Markov type for every $d \geq 1$, and $\mathbb{Z}^d$-actions by automorphisms of compact groups enjoy the properties described in (4.10), Propositions 4.9–4.10, Remark 4.15, and Theorem 4.11. Just as compact, abelian groups like $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ have automorphisms with very intricate dynamical properties, there is an abundance of examples of interesting $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups. In this section we introduce a general formalism for the investigation of such actions which will also give us a systematic approach to constructing actions with specified properties.

Let $d \geq 1$, and let $\alpha\colon \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ be an action of $\mathbb{Z}^d$ by automorphisms of $X$. For every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ we denote by $\hat{\alpha}_{\mathbf{n}}$ the automorphism of $\hat{X}$ dual to $\alpha_{\mathbf{n}}$ and write $\hat{\alpha}\colon \mathbb{Z}^d \longmapsto \operatorname{Aut}(\hat{X})$ for the resulting $\mathbb{Z}^d$-action dual to $\alpha$. Under the action $\hat{\alpha}$ the group $\hat{X}$ becomes a $\mathbb{Z}^d$-module, and hence a module over the group ring $\mathbb{Z}[\mathbb{Z}^d]$. In order to make this explicit we denote by

$$\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \tag{5.1}$$

the ring of Laurent polynomials in the (commuting) variables $u_1, \dots, u_d$ with coefficients in $\mathbb{Z}$. A typical element $f \in \mathfrak{R}_d$ will be written as

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}}, \tag{5.2}$$

where $c_f(\mathbf{n}) \in \mathbb{Z}$ and $u^{\mathbf{n}} = u_1^{n_1} \cdot \ldots \cdot u_d^{n_d}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, and where $c_f(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n} \in \mathbb{Z}^d$. Then $\mathfrak{R}_d \cong \mathbb{Z}[\mathbb{Z}^d]$, $\mathfrak{R}_d$ acts on $\hat{X}$ by

$$(f, a) \mapsto f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) \hat{\alpha}_{\mathbf{n}}(a) \tag{5.3}$$

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for every $f \in \mathfrak{R}_d$, $a \in \hat{X}$, and $\hat{X}$ is an $\mathfrak{R}_d$-module. Note that

$$\hat{\alpha}_{\mathbf{n}}(a) = \hat{\alpha}_{\mathbf{n}}(a) = u^{\mathbf{n}} \cdot a \tag{5.4}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \hat{X}$. Conversely, if $\mathfrak{M}$ is an $\mathfrak{R}_d$-module (always assumed to be countable), then $\mathbb{Z}^d$ has an obvious action $\hat{\alpha}^{\mathfrak{M}} \colon \mathbf{n} \mapsto \hat{\alpha}^{\mathfrak{M}}_{\mathbf{n}}$ on $\mathfrak{M}$ given by

$$\hat{\alpha}^{\mathfrak{M}}_{\mathbf{n}}(a) = u^{\mathbf{n}} \cdot a \tag{5.5}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. We write $X = \widehat{\mathfrak{M}}$ for the dual group of $\mathfrak{M}$ and obtain a dual action

$$\alpha^{\mathfrak{M}} \colon \mathbf{n} \mapsto \alpha^{\mathfrak{M}}_{\mathbf{n}} \in \mathrm{Aut}(X) \tag{5.6}$$

of $\mathbb{Z}^d$ on $X$. For future reference we collect these observations in a lemma.

LEMMA 5.1. *Let $\alpha \colon \mathbf{n} \mapsto \alpha_{\mathbf{n}}$ be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group $X$, and let $\hat{\alpha} \colon \mathbf{n} \mapsto \hat{\alpha}_{\mathbf{n}}$ be the dual action of $\mathbb{Z}^d$ on the dual group $\hat{X}$ of $X$. If $\mathfrak{R}_d$ is the ring defined in* (5.1) *then $\hat{X}$ is an $\mathfrak{R}_d$-module under the $\mathfrak{R}_d$-action* (5.3). *Conversely, if $\mathfrak{M}$ is an $\mathfrak{R}_d$-module, then* (5.5) *and* (5.6) *define $\mathbb{Z}^d$-actions $\hat{\alpha}^{\mathfrak{M}} = \hat{\alpha}$ and $\alpha^{\mathfrak{M}} = \alpha$ by automorphisms of $\mathfrak{M}$ and $X^{\mathfrak{M}} = \widehat{\mathfrak{M}}$, respectively.*

EXAMPLES 5.2. Let $d \geq 1$.

(1) Let $\mathfrak{M} = \mathfrak{R}_d$. Since $\mathfrak{R}_d$ is isomorphic to the direct sum $\sum_{\mathbb{Z}^d} \mathbb{Z}$ of copies of $\mathbb{Z}$ indexed by $\mathbb{Z}^d$, the dual group $X = \widehat{\mathfrak{R}_d}$ is isomorphic to the cartesian product $\mathbb{T}^{\mathbb{Z}^d}$ of copies of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in \mathbb{T}^{\mathbb{Z}^d}$ as $x = (x_{\mathbf{n}}) = (x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d)$ with $x_{\mathbf{n}} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and choose the following identification of $X^{\mathfrak{R}_d} = \widehat{\mathfrak{R}_d}$ and $\mathbb{T}^{\mathbb{Z}^d}$: for every $x = (x_{\mathbf{n}})$ in $\mathbb{T}^{\mathbb{Z}^d}$ and $f \in \mathfrak{R}_d$,

$$\langle x, f \rangle = e^{2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{n}}}, \tag{5.7}$$

where $f$ is given by (5.2). Under this identification the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{R}_d}$ on $X^{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$ becomes the shift-action

$$\alpha^{\mathfrak{R}_d}_{\mathbf{n}}(x)_{\mathbf{m}} = (\sigma_{\mathbf{n}}(x))_{\mathbf{m}} = x_{\mathbf{m}+\mathbf{n}}, \tag{5.8}$$

with $\mathbf{n} \in \mathbb{Z}^d$ and $x = (x_{\mathbf{m}}) \in X^{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$.

(2) Let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal, and let $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$. Since $\mathfrak{M}$ is a quotient of the additive group $\mathfrak{R}_d$ by a $\hat{\alpha}^{\mathfrak{R}_d}$-invariant subgroup, the dual group $X^{\mathfrak{M}}$ is the $\alpha^{\mathfrak{R}_d}$-invariant subgroup

$$\begin{aligned} X^{\mathfrak{R}_d/\mathfrak{a}} &= \{x \in X^{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d} : \langle x, f \rangle = 1 \text{ for every } f \in \mathfrak{a}\} \\ &= \Big\{x \in \mathbb{T}^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = 0 \pmod 1 \\ &\qquad \text{for every } f \in \mathfrak{a} \text{ and } \mathbf{m} \in \mathbb{Z}^d\Big\}, \end{aligned} \tag{5.9}$$

and $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is the restriction of $\alpha^{\mathfrak{R}_d}$ to $X^{\mathfrak{M}} \subset \mathbb{T}^{\mathbb{Z}^d}$, i.e.

$$\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{a}} = \sigma_{\mathbf{n}}^{X^{\mathfrak{R}_d/\mathfrak{a}}} \tag{5.10}$$

for every $\mathbf{n} \in \mathbb{Z}^d$.

(3) Let $X \subset \mathbb{T}^{\mathbb{Z}^d} = \widehat{\mathfrak{R}_d}$ be a closed subgroup, and let $X^\perp = \{f \in \mathfrak{R}_d : \langle x, f\rangle = 1 \text{ for every } x \in X\}$ be the annihilator of $X$ in $\widehat{\mathfrak{R}_d}$. Then $X$ is shift-invariant if and only if $X^\perp$ is an ideal in $\mathfrak{R}_d$: indeed, if $X^\perp$ is an ideal, it is obviously invariant under multiplication by the group of units $\{u^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \mathfrak{R}_d$, i.e. $X^\perp$ is $\hat{\alpha}^{\mathfrak{R}_d}$-invariant; conversely, if $X^\perp$ is $\hat{\alpha}^{\mathfrak{R}_d}$-invariant, then (5.3) shows that $f \cdot a \in X^\perp$ for every $f \in \mathfrak{R}_d$ and $a \in X^\perp$. In other words, $X^\perp$ is an ideal.

(4) Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module, and let $\{a_1, \dots, a_k\}$ be a set of generators for $\mathfrak{M}$, i.e. $\mathfrak{M} = \mathfrak{R}_d \cdot a_1 + \dots + \mathfrak{R}_d \cdot a_k$. The surjective homomorphism $(f_1, \dots, f_k) \mapsto f_1 \cdot a_1 + \dots + f_k \cdot a_k$ from $\mathfrak{R}_d^k$ to $\mathfrak{M}$ induces a dual injective homomorphism $\phi\colon X^{\mathfrak{M}} \longmapsto X^{\mathfrak{R}_d^k} \cong (\mathbb{T}^k)^{\mathbb{Z}^d} = Y$ such that $\alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \phi = \sigma_{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $\sigma_{\mathbf{n}}$ is the shift on $(\mathbb{T}^k)^{\mathbb{Z}^d}$ defined in (5.8). In particular, $\phi$ embeds $X^{\mathfrak{M}}$ as a closed, shift-invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$. Conversely, if $X \subset (\mathbb{T}^k)^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup, then $\hat{X} = \mathfrak{R}_d^k/X^\perp$, and $X^\perp$ is a submodule of $\mathfrak{R}_d^k$. $\boxdot$

EXAMPLES 5.3. (1) Let $\alpha$ be the automorphism of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ determined by the matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. In Example 2.18 (2) we have seen that $\alpha$ (or, more precisely, the $\mathbb{Z}$-action on $\mathbb{T}^2$ defined by $\alpha$) is conjugate to $(X^{\mathfrak{R}_1/(f)}, \alpha^{\mathfrak{R}_1/(f)})$, where $(f) \subset \mathfrak{R}_1$ is the principal ideal generated by the characteristic polynomial $f(u_1) = 1 + u_1 - u_1^2$ of $A$. Indeed, an element $x \in X = \widehat{\mathfrak{R}_1} = \mathbb{T}^{\mathbb{Z}}$ satisfies that $\langle x, u_1^n f\rangle = 1$ if and only if $x_n + x_{n+1} - x_{n+2} = 0 \pmod 1$, and hence

$$X^{\mathfrak{R}_1/(f)} = \{x \in \mathbb{T}^{\mathbb{Z}} : x_n + x_{n+1} - x_{n+2} = 0 \pmod 1 \text{ for all } n \in \mathbb{Z}\}$$

(cf. (5.7) and (5.9)). The continuous group isomorphism $\phi = \pi_{\{0,1\}}\colon X^{\mathfrak{R}_1/(f)} \longmapsto \mathbb{T}^2$ makes the diagram

$$\begin{array}{ccc} X^{\mathfrak{R}_1/(f)} & \xrightarrow{\alpha^{\mathfrak{R}_1/(f)}} & X^{\mathfrak{R}_1/(f)} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{T}^2 & \xrightarrow[\alpha]{} & \mathbb{T}^2 \end{array} \tag{5.11}$$

commute, and the automorphism $\alpha^{\mathfrak{R}_1/(f)}$ is equal to the shift on $X^{\mathfrak{R}_1/(f)}$.

(2) Example (1) depends on the fact that the matrix $A$ is conjugate (over $\mathbb{Z}$) to the companion matrix of its characteristic polynomial. If $\alpha$ is the automorphism of $\mathbb{T}^2$ defined by $A = \left(\begin{smallmatrix} 3 & 4 \\ 1 & 1 \end{smallmatrix}\right)$, then the characteristic polynomial of $A$ is $f(u_1) = -1 - 4u_1 + u_1^2$, and $AM = MB$, where $B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 4 \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$. The

map $\phi\colon X^{\mathfrak{R}_1/(f)} \longmapsto \mathbb{T}^2$ given by $\phi(x) = (x_0+3x_1, x_1)$ for all $x \in X^{\mathfrak{R}_1/(f)} \subset \mathbb{T}^{\mathbb{Z}}$ is a group isomorphism, and the diagram (5.11) commutes.

If $A' = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, then the characteristic polynomial of $A'$ is again equal to $f(u_1) = -1 - 4u_1 + u_1^2$, $A'M = MB$ with $M = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right)$, but there is no matrix $M'$ with integer entries and determinant 1 such that $A'M' = M'B$. The homomorphism $\phi'\colon X^{\mathfrak{R}_1/(f)} \longmapsto \mathbb{T}^2$ with $\phi'(x) = (x_0 + 3x_1, 2x_1)$ for all $x \in X^{\mathfrak{R}_1/(f)} \subset \mathbb{T}^{\mathbb{Z}}$ is surjective, and we write $\psi' = \hat{\phi}\colon \mathbb{Z}^2 \longmapsto \mathfrak{R}_1/(f)$ for the dual homomorphism, which is injective, but not bijective. The $\mathfrak{R}_1$-module $\mathfrak{M} = \hat{X}$ arising from the $\mathbb{Z}$-action $n \mapsto (A')^n$ via Lemma 5.1 is (isomorphic to) the submodule $\psi'(\mathbb{Z}^2)$ of $\mathfrak{R}_1/(f)$. We claim that $\mathfrak{M}$ is not isomorphic to $\mathfrak{R}_1/(f)$—in fact, $\mathfrak{M}$ is not even *cyclic*, i.e. not of the form $\mathfrak{M} = \mathfrak{R}_1 \cdot a$ for some $a \in \mathfrak{M}$. Indeed, if $\mathfrak{M}$ were cyclic, there would exist an element $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ such that $\{(A')^n\mathbf{m} : n \in \mathbb{Z}\}$ generates $\mathbb{Z}^2$, which is equivalent to the condition that

$$\{\mathbf{m}, A'\mathbf{m}\} = \{(m_1, m_2), (3m_1 + 2m_2, 2m_1 + m_2)\}$$

generates $\mathbb{Z}^2$. Hence

$$\det \left(\begin{smallmatrix} m_1 & 3m_1+2m_2 \\ m_2 & 2m_1+m_2 \end{smallmatrix}\right) = 2m_1^2 - 2m_1m_2 - 2m_2^2 = 1,$$

which is obviously impossible.

(3) Let $f = 2 - u_1 \in \mathfrak{R}_1$, and let $(f)$ be the principal ideal generated by $f$. According to (5.7) and (5.9),

$$X = X^{\mathfrak{R}_1/(f)} = \{x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : 2x_n = x_{n+1} \pmod 1 \text{ for all } n \in \mathbb{Z}\},$$

and $\alpha^{\mathfrak{R}_1/(f)}$ is equal to the shift-action $\sigma$ of $\mathbb{Z}$ on $X$. The zero coordinate projection $\phi = \pi_{\{0\}}\colon X \longmapsto \mathbb{T}$ is surjective and satisfies that $\phi \cdot \sigma_1 = T \cdot \phi$, where $T\colon \mathbb{T} \longmapsto \mathbb{T}$ is the surjective homomorphism consisting of multiplication by 2 modulo 1.

(4) Let $f_1 = 2-u_1$, $f_2 = 3-u_2$, and let $\mathfrak{a} = (f_1, f_2) = f_1\mathfrak{R}_2 + f_2\mathfrak{R}_2 \subset \mathfrak{R}_2$. Then

$$\begin{aligned} X = X^{\mathfrak{R}_2/\mathfrak{a}} = \{x = (x_{m,n}) \in \mathbb{T}^{\mathbb{Z}^2} : 2x_{(m,n)} &= x_{(m+1,n)} \pmod 1 \text{ and} \\ 3x_{(m,n)} &= x_{(m,n+1)} \pmod 1 \text{ for every } (m,n) \in \mathbb{Z}^2\}, \end{aligned}$$

and $\alpha^{\mathfrak{R}_2/\mathfrak{a}} = \sigma$ is the shift-action of $\mathbb{Z}^2$ on $X^{\mathfrak{R}_2/\mathfrak{a}}$. The zero coordinate projection $\phi = \pi_{\{(0,0)\}}\colon X \longmapsto \mathbb{T}$ is again surjective and satisfies that $\phi \cdot \sigma_{\mathbf{n}} = T_{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^2$, where $T$ is the $\mathbb{N}^2$-action on $\mathbb{T}$ defined by $T_{(m,n)}(t) = 2^m 3^n t \pmod 1$ for every $(m,n) \in \mathbb{Z}^2$ and $t \in \mathbb{T}$.

(5) Let

$$\begin{aligned} X = \{x = (x_{\mathbf{n}}) \in \mathbb{Z}_{/2}^{\mathbb{Z}^2} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = 0 \pmod 2 \\ \text{for all } \mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2\}. \end{aligned}$$

From (5.7) and (5.9) we see that the shift-action $\sigma$ of $\mathbb{Z}^2$ on the full, shift-invariant subgroup $X \subset \mathbb{Z}_{/2}^{\mathbb{Z}^2}$ is conjugate to $(X^{\mathfrak{R}_2/\mathfrak{a}}, \alpha^{\mathfrak{R}_2/\mathfrak{a}})$, where $\mathfrak{a} = (2, 1 + u_1 + u_2) \subset \mathfrak{R}_2$ is the ideal generated by 2 and $1 + u_1 + u_2$.

(6) Let $d \geq 1$. A Laurent polynomial $f \in \mathfrak{R}_d$ is *primitive* if the highest common factor of its coefficients is equal to 1. Suppose that $f$ is primitive and $m > 1$ an integer, and let $(f)$ and $(mf)$ be the principal ideals in $\mathfrak{R}_d$ generated by $f$ and $mf$, respectively. The map $h \mapsto mh$ from $\mathfrak{R}_d$ to $\mathfrak{R}_d$ induces an injective homomorphism $\xi\colon \mathfrak{R}_d/(f) \longmapsto \mathfrak{R}_d/(mf)$, the dual homomorphism $\phi\colon X^{\mathfrak{R}_d/(mf)} \longmapsto X^{\mathfrak{R}_d/(f)}$ is surjective, and $\ker(\phi) \cong \mathbb{Z}_{/m}^{\mathbb{Z}^d}$. The group $X^{\mathfrak{R}_d/(f)}$ is connected, and the connected component of the identity in $X^{\mathfrak{R}_d/(mf)}$ is isomorphic to $X^{\mathfrak{R}_d/(f)}$.

More generally, if $\mathfrak{a} \subset \mathfrak{R}_d$ is an arbitrary ideal such that the additive group $\mathfrak{R}_d/\mathfrak{a}$ is torsion-free (or, equivalently, such that $X^{\mathfrak{R}_d/\mathfrak{a}}$ is connected), and if $m \geq 1$ is an integer, then we obtain an exact sequence

$$0 \longrightarrow \mathbb{Z}_{/m}^{\mathbb{Z}^d} \xrightarrow{\psi} X^{\mathfrak{R}_d/m\mathfrak{a}} \xrightarrow{\phi} X^{\mathfrak{R}_d/\mathfrak{a}} \longrightarrow 0,$$

where $\phi\colon X^{\mathfrak{R}_d/m\mathfrak{a}} \longmapsto X^{\mathfrak{R}_d/\mathfrak{a}}$ is the surjection dual to the injective homomorphism $\xi\colon \mathfrak{R}_d/\mathfrak{a} \longmapsto \mathfrak{R}_d/m\mathfrak{a}$ consisting of multiplication by $m$, and where $\psi$ is the inclusion map. Note that $\psi \cdot \sigma_{\mathbf{n}}(x) = \alpha_{\mathbf{n}}^{\mathfrak{R}_d/m\mathfrak{a}} \cdot \psi(x)$ and $\phi \cdot \alpha_{\mathbf{n}}^{\mathfrak{R}_d/m\mathfrak{a}}(y) = \alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{a}} \cdot \phi(y)$ for all $\mathbf{n} \in \mathbb{Z}^d$, $x \in \mathbb{Z}_{/m}^{\mathbb{Z}^d}$, and $y \in X^{\mathfrak{R}_d/m\mathfrak{a}}$, where $\sigma$ is the shift-action of $\mathbb{Z}^d$ on $\mathbb{Z}_{/m}^{\mathbb{Z}^d}$, and that the map $\phi$ induces an isomorphism of the connected component of the identity in $X^{\mathfrak{R}_d/m\mathfrak{a}}$ with $X^{\mathfrak{R}_d/\mathfrak{a}}$. ⊡

The next proposition is a straightforward consequence of Theorem 4.2 and Pontryagin duality (cf. also Example 5.2 (4)).

PROPOSITION 5.4. *Let $X$ be a compact, abelian group, $\alpha$ a $\mathbb{Z}^d$-action by automorphisms of $X$. The following conditions are equivalent.*

1. *The $\mathfrak{R}_d$-module $\mathfrak{M} = \hat{X}$ obtained via* Lemma 5.1 *is Noetherian;*
2. *$(X, \alpha)$ satisfies the d.c.c.;*
3. *$(X, \alpha)$ is conjugate to a subshift of $(\mathbb{T}^n)^{\mathbb{Z}^d}$ for some $n \geq 1$.*

The Noetherian $\mathfrak{R}_d$-modules form a particularly well-behaved class of $\mathfrak{R}_d$-modules, and it is therefore not surprising that $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups satisfying the d.c.c. have many exceptional properties. As a first illustration of the rôle played by the descending chain condition, let us consider the set of periodic points for a $\mathbb{Z}^d$-action $\alpha$ on a compact, abelian group $X$.

DEFINITION 5.5. Let $\Gamma$ be a countable group and let $\alpha$ be a $\Gamma$-action by automorphisms of a compact group $X$. A point $x \in X$ is *periodic* under $\alpha$ (or *$\alpha$-periodic*) if its orbit $\alpha_\Gamma(x) = \{\alpha_\gamma(x) : \gamma \in \Gamma\}$ is finite. If $\beta \in \mathrm{Aut}(X)$ then a point $x \in X$ is *periodic* under $\beta$ if $\beta^n(x) = x$ for some $n \geq 1$.

The following examples show that a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group need not have *any* periodic points other than the fixed point $\mathbf{0}_X$, but in Theorem 5.7 we shall see that the set of $\alpha$-periodic points is *dense* if $(X,\alpha)$ satisfies the d.c.c.

EXAMPLES 5.6. (1) Let $X = \widehat{\mathbb{Q}}$ be the dual group of the additive group $\mathbb{Q}$, and consider the automorphism $\alpha$ of $X$ dual to multiplication by $\frac{3}{2}$ on $\mathbb{Q}$. If $x \in X$ is a periodic point of $\alpha$, i.e. if $\alpha^n(x) = x$ for some $n \geq 1$, then $\langle \alpha^n(x) - x, a\rangle = \langle x, (\frac{3^n}{2^n} - 1)a\rangle = 1$ for every $a \in \mathbb{Q}$. However, $(\frac{3^n}{2^n} - 1) \neq 0$, so that $\langle x, a\rangle = 1$ for every $a \in \mathbb{Q}$. This shows that $x = \mathbf{0}_X$.

(2) Let $Y = \mathbb{Z}_{/2}^{\mathbb{Z}}$. For every $n \geq 2$ we define a continuous, shift commuting, surjective homomorphism $\phi_n \colon Y \longmapsto Y$ by setting $(\phi_n(y))_m = \sum_{k=m}^{m+n-1} y_k$ for every $m \in \mathbb{Z}$ and $y = (y_k,\, k \in \mathbb{Z}) \in Y$. We put $\psi_n = \phi_n$ for every $n \geq 2$ and denote by $X$ the projective limit

$$Y \xleftarrow{\psi_2} Y \xleftarrow{\psi_3} \dots \xleftarrow{\psi_n} Y \xleftarrow{\psi_{n+1}} \dots \tag{5.12}$$

The shift $\sigma$ on $Y$ commutes with the maps $\psi_n$ and induces an automorphism $\alpha$ of the projective limit $X$ in (5.12). Suppose that $\alpha$ has a periodic point $x \in X$ with period $n$, say. We can write $x$ as $(x^{(k)},\, k \geq 1)$ with $x^{(k)} \in Y$ and $\psi_k(x^{(k)}) = x^{(k-1)}$ for every $k \geq 2$. Since $x$ has period $n$, $\sigma^n(x^{(k)}) = x^{(k)}$ for every $k \geq 1$. However, $\psi_{nk}(x^{(nk)}) = \phi_{nk}(x^{(nk)}) = x^{(nk-1)} \in \{\mathbf{0}, \mathbf{1}\}$ for every $k \geq 1$, where $\mathbf{0} = (\dots, 0, 0, 0, \dots)$ and $\mathbf{1} = (\dots, 1, 1, 1, \dots)$ are the fixed points of $\sigma$ in $Y$. As $k$ can be arbitrarily large we see that $x^{(k)} \in \{\mathbf{0}, \mathbf{1}\}$ for every $k \geq 0$. Finally we observe that, if $k \geq 2$ is even, then $x^{(k-1)} = \psi_k(x^{(k)}) = \mathbf{0}$. This shows that $x^{(k)} = \mathbf{0}$ for every $k \geq 1$, i.e. that $x = \mathbf{0}_X$.

(3) We stay with the notation of Example (2) and set $\psi_n = \phi_2$ for every $n \geq 2$ in (5.12). The projective limit $X$ in (5.12) can be written as $X = \{x = (x_{(m,n)}) \in \mathbb{Z}_{/2}^{\mathbb{Z}\times\mathbb{N}} : x_{(m,n)} = x_{(m,n+1)} + x_{(m+1,n+1)} \pmod 2$ for every $m \in \mathbb{Z}$ and $n \geq 1\}$, and $\alpha$ is the horizontal shift on $X$ defined by $(\alpha(x))_{(m,n)} = x_{(m+1,n)}$ for all $x \in X$ and $(m,n) \in \mathbb{Z} \times \mathbb{N}^*$. The same argument as in Example (2) shows that every point $x \in X$ with period $2^k$, $k \geq 0$ is equal to the identity element $\mathbf{0}_X$, but that there exist $2^{k-1}$ points of period $k$ if $k \geq 1$ is odd (for every sequence $y = (y_m) \in Y$ with $y_{(m+k)} = y_m$ and $\sum_{j=0}^{k-1} x_{m+j} = 0 \pmod 2$ for all $m \in \mathbb{Z}$ there exists a unique point $x \in X$ with $\alpha^k(x) = x$ and $x_{(m,1)} = y_m$ for all $m \in \mathbb{Z}$).

If $\mathfrak{a} \subset \mathfrak{R}_2$ is the ideal $(2, 1 + u_2 + u_1u_2) = 2\mathfrak{R}_2 + (1 + u_2 + u_1u_2)\mathfrak{R}_2$, then (5.7) and (5.9) show that $(X^{\mathfrak{R}_2/\mathfrak{a}}, \alpha^{\mathfrak{R}_2/\mathfrak{a}})$ is (conjugate to) the shift-action of $\mathbb{Z}^2$ on

$$\begin{aligned} X' = \{x = (x_{(m,n)}) \in \mathbb{Z}_{/2}^{\mathbb{Z}^2} : x_{(m,n)} &+ x_{(m,n+1)} + x_{(m+1,n+1)} \\ &= 0 \pmod 2 \ \text{ for every } (m,n) \in \mathbb{Z}^2\}, \end{aligned}$$

and a comparison of $X'$ with the definition of $X$ in the preceding paragraph reveals that $X$ is equal to the projection of $X'$ onto its coordinates in the upper half plane of $\mathbb{Z}^2$, and that this projection sends the horizontal shift $\sigma_{(1,0)}$ of $X'$ to the automorphism $\alpha$ of $X$. In particular we see that the shift-action $\sigma$ of $\mathbb{Z}^2$ on $X'$ has only one point with horizontal period $2^k$ for every $k \geq 0$ (the identity element). We also refer to Example 5.3 (5): the $\mathbb{Z}^2$-action $\alpha^{\mathfrak{R}_2/\mathfrak{a}}$ appearing there obviously has the same property.

(4) Let $\psi_n = \phi_3$ for every $n \geq$ in (5.12). Then the resulting automorphism $\alpha$ of the projective limit $X$ in (5.12) has only one point with period $3^k$, $k \geq 0$, but there exist $2^k$ points with period $k$ for every $k$ which is not divisible by 3.

(5) Let $(p_n,\, n \geq 2)$ be a sequence of rational primes in which every prime occurs infinitely often, and let $(q_n,\, n \geq 2)$ be a sequence of odd primes in which every odd prime occurs infinitely often. If $\psi_n = \phi_{p_n}$ for every $n \geq 2$, then the automorphism $\alpha$ of the projective limit $X$ in (5.12) has no periodic points other than the fixed point $\mathbf{0}_X$. However, if $\psi_n = \phi_{q_n}$, $n \geq 2$, then the resulting automorphism $\alpha$ will have $2^{2^k}$ periodic points with period $2^k$ for every $k \geq 0$, but only one point with period $2l+1$ for every $l \geq 0$ (the fixed point $\mathbf{0}_X$).

None of the automorphisms $\alpha$ in Examples (1)–(5) satisfies the d.c.c. $\boxdot$

THEOREM 5.7. *Let $X$ be a compact, abelian group, and let $\alpha$ be a $\mathbb{Z}^d$-action by automorphisms of $X$. If $(X, \alpha)$ satisfies the d.c.c. then the set of $\alpha$-periodic points is dense in $X$.*

PROOF. Let $\mathfrak{M} = \hat{X}$ be the $\mathfrak{R}_d$-module arising from Lemma 5.1. Fix a non-zero element $a \in \mathfrak{M}$ and choose a submodule $\mathfrak{M}_a \subset \mathfrak{M}$ which is maximal with respect to the property that $a \notin \mathfrak{M}_a$. Then the $\mathfrak{R}_d$-module $\mathfrak{M}' = \mathfrak{M}/\mathfrak{M}_a$ has the minimal non-zero submodule $\mathfrak{M}'_1 = (\mathfrak{R}_d \cdot a + \mathfrak{M}_a)/\mathfrak{M}_a$. Consider the ideal $\mathfrak{a} = \{f \in \mathfrak{R}_d : f \cdot \mathfrak{M}'_1 = 0\}$, and let $\mathfrak{b}$ be an ideal with $\mathfrak{a} \subsetneq \mathfrak{b} \subsetneq \mathfrak{R}_d$. The minimality of $\mathfrak{M}'_1$ implies that $\mathfrak{b} \cdot \mathfrak{M}'_1 = \mathfrak{M}'_1$, and Corollary 2.5 in [5] shows that there exists an element $x \in 1 + \mathfrak{b}$ such that $x \cdot \mathfrak{M}'_1 = \{0\}$. This contradicts our definition of $\mathfrak{a}$, and we conclude that the ideal $\mathfrak{a} \subset \mathfrak{R}_d$ is maximal, and that $\mathfrak{k} = \mathfrak{R}_d/\mathfrak{a}$ is a (necessarily finite) field.

For every $m \geq 1$ we write $\mathfrak{a}^m \subset \mathfrak{R}_d$ for the ideal generated by $\{f_1 \cdot \ldots \cdot f_m : f_i \in \mathfrak{a} \text{ for } i = 1, \ldots, m\}$. If $a' = a + \mathfrak{M}_a \in \mathfrak{a}^m \cdot \mathfrak{M}'$ for every $m \geq 1$, then $a \in \mathfrak{M}'' = \bigcap_{m \geq 1} \mathfrak{a}^m \cdot \mathfrak{M}''$, and $\mathfrak{a} \cdot \mathfrak{M}''/\mathfrak{M}''$. The argument in the preceding paragraph shows that there exists an element $y \in 1 + \mathfrak{a}$ with $y \cdot \mathfrak{M}'' = \{0\}$, and the maximality of $\mathfrak{a}$ implies that $\mathfrak{M}'' = \{0\}$, which is absurd. Hence there exists an integer $m \geq 1$ with $a' \notin \mathfrak{a}^m \cdot \mathfrak{M}'$, and the maximality of $\mathfrak{M}_a$ implies that $\mathfrak{a}^m \cdot \mathfrak{M}' = \{0\}$.

Each of the successive quotients $\mathfrak{a}^r \cdot \mathfrak{M}'/\mathfrak{a}^{r+1} \cdot \mathfrak{M}'$ in the decreasing sequence of $\mathfrak{R}_d$-modules $\mathfrak{M}' \supset \mathfrak{a} \cdot \mathfrak{M}' \supset \cdots \supset \mathfrak{a}^m \cdot \mathfrak{M}' = \{0\}$ is a Noetherian module over $\mathfrak{k}$. Since $\mathfrak{k}$ is finite we conclude that $\mathfrak{M}'$ is finite.

We have found, for every non-zero $a \in \mathfrak{M} = \hat{X}$, a submodule $\mathfrak{M}_a \subset \mathfrak{M}$ such that $a \notin \mathfrak{M}_a$ and $\mathfrak{M}/\mathfrak{M}_a$ is finite. The subgroup $X_a = \mathfrak{M}_a^\perp \subset X$ is finite, $\alpha$-invariant, and is not annihilated by (the character corresponding to) $a$. Since every point in $X_a$ must be $\alpha$-periodic, and since the $\alpha$-periodic points form a subgroup of $X$, this shows that the set of $\alpha$-periodic points is dense in $X$. □

Before turning to the problem of relating the algebraic properties of a Noetherian $\mathfrak{R}_d$-module $\mathfrak{M}$ to the dynamical properties of $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ we should discuss the extent to which $\mathfrak{M}$ and $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ determine each other. Let $d \geq 1$, and let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module which is torsion-free when regarded as an additive group or, equivalently, as a $\mathbb{Z}$-module (this is equivalent to the assumption that $X^{\mathfrak{M}} = \widehat{\mathfrak{M}}$ is connected). We define the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{M}}$ on $X^{\mathfrak{M}}$ by (5.5) and (5.6) and consider the action induced by $\alpha^{\mathfrak{M}}$ on the Čech homology group $H_1(X^{\mathfrak{M}}, \mathbb{T})$ (cf. [20]).

Lemma 5.8. *The group $H_1(X^{\mathfrak{M}}, \mathbb{T})$ is isomorphic to $X^{\mathfrak{M}}$, and the automorphism induced by $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ on $H_1(X^{\mathfrak{M}}, \mathbb{T})$ is equal to $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ for every $\mathbf{n} \in \mathbb{Z}^d$.*

Proof. In view of Example 5.2 (4) we may assume that $X = X^{\mathfrak{M}}$ is a closed, shift-invariant subgroup of $(\mathbb{T}^k)^{\mathbb{Z}^d}$, and the connectedness of $X$ allows us to assume that $X$ is full. If $F(n) = \{-n, \dots, n\}^d \subset \mathbb{Z}^d$ then $\pi_{F(n)}(X) \subset (\mathbb{T}^k)^{F(n)}$ is a finite-dimensional torus, and $X$ is equal to the projective limit

$$\pi_{F(1)}(X) \xleftarrow{\pi_{F(1)}} \pi_{F(2)}(X) \xleftarrow{\pi_{F(2)}} \pi_{F(3)}(X) \xleftarrow{\pi_{F(3)}} \dots. \tag{5.13}$$

Since $H_1(\pi_{F(k)}(X), \mathbb{T}) \cong \pi_{F(k)}(X)$ ([20]), we see from (5.13) that $H_1(X, \mathbb{T}) \cong X$, and that the automorphism induced by $\alpha_{\mathbf{n}}^{\mathfrak{M}} = \sigma_{\mathbf{n}}$ on $H_1(X, \mathbb{T})$ is equal to $\sigma_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. □

Theorem 5.9. *Let $X$ and $X'$ be compact, connected, abelian groups, and let $\alpha$ and $\alpha'$ be $\mathbb{Z}^d$-actions by automorphisms of $X$ and $X'$ which satisfy the d.c.c. The following statements are equivalent.*

(1) *The $\mathbb{Z}^d$-actions $\alpha$ and $\alpha'$ are topologically conjugate, i.e. there exists a homeomorphism $\phi\colon X \longmapsto X'$ with $\phi \cdot \alpha_{\mathbf{n}} = \alpha'_{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$;*
(2) *The $\mathbb{Z}^d$-actions $\alpha$ and $\alpha'$ are algebraically conjugate, i.e. there exists a continuous group isomorphism $\psi\colon X \longmapsto X'$ such that $\psi \cdot \alpha_{\mathbf{n}} = \alpha'_{\mathbf{n}} \cdot \psi$ for every $\mathbf{n} \in \mathbb{Z}^d$.*

Proof. The implication (2)⇒(1) is obvious. If (1) is satisfied we use Lemma 5.1 and Proposition 5.4 to find Noetherian $\mathfrak{R}_d$-modules $\mathfrak{M}$ and $\mathfrak{M}'$ such that $(X, \alpha)$ and $(X', \alpha')$ are conjugate to $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ and $(X^{\mathfrak{M}'}, \alpha^{\mathfrak{M}'})$, respectively. By Lemma 5.8, $H_1(X^{\mathfrak{M}}, \mathbb{T}) \cong X^{\mathfrak{M}}$, $H_1(X^{\mathfrak{M}'}, \mathbb{T}) \cong X^{\mathfrak{M}'}$, and for every $\mathbf{n} \in \mathbb{Z}^d$ the isomorphisms of $H_1(X^{\mathfrak{M}}, \mathbb{T})$ and $H_1(X^{\mathfrak{M}'}, \mathbb{T})$ defined by $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ and $\alpha_{\mathbf{n}}^{\mathfrak{M}'}$ are equal to $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ and $\alpha_{\mathbf{n}}^{\mathfrak{M}'}$, respectively. The continuous group isomorphism $\psi'\colon H_1(X^{\mathfrak{M}}, \mathbb{T}) \longmapsto H_1(X^{\mathfrak{M}'}, \mathbb{T})$ induced by $\phi\colon X \longmapsto X'$ satisfies that $\psi' \cdot \alpha_{\mathbf{n}}^{\mathfrak{M}} = \alpha_{\mathbf{n}}^{\mathfrak{M}'} \cdot \psi'$ for every $\mathbf{n} \in \mathbb{Z}^d$, and this implies (2). □

Corollary 5.10. *Let $d \geq 1$, and let $\mathfrak{M}$ and $\mathfrak{M}'$ be finitely generated $\mathfrak{R}_d$-modules which are torsion-free* (*as additive groups*). *The following statements are equivalent.*

(1) *The $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{M}}$ and $\alpha^{\mathfrak{M}'}$ are topologically conjugate;*
(2) *The $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{M}}$ and $\alpha^{\mathfrak{M}'}$ are algebraically conjugate;*
(3) *There exists an $\mathfrak{R}_d$-module isomorphism $\chi\colon \mathfrak{M} \longmapsto \mathfrak{M}'$.*

Proof. The equivalence of (1) and (2) is stated in Theorem 5.9. If (2) is satisfied, then any group isomorphism $\psi\colon X^{\mathfrak{M}} \longmapsto X^{\mathfrak{M}'}$ with $\psi \cdot \alpha_{\mathbf{n}}^{\mathfrak{M}} = \alpha_{\mathbf{n}}^{\mathfrak{M}'} \cdot \psi$ for all $\mathbf{n} \in \mathbb{Z}^d$ induces a dual isomorphism $\hat{\psi}\colon \mathfrak{M}' \longmapsto \mathfrak{M}$ which is easily seen to be an $\mathfrak{R}_d$-module isomorphism. The implication (3)⇒(2) is obvious. □

Concluding Remarks 5.11. (1) Most of the material of this section comes from [45], except for Lemma 5.8, Theorem 5.9, and Corollary 5.10, which come from [94]. Example 5.3 (2) is taken from [110], Example 5.3 (4) features in [23] and [89], Example 5.3 (5) comes from [56] (cf. (0.1)), and Example 5.6 (1) appears to be oral tradition attributed to Furstenberg. For $\mathbb{Z}$-actions Theorem 5.7 was first proved in [55], and the general proof presented here is due to Hartley. A more general version of Theorem 5.7 will be proved in Section 10 (Theorem 10.2).

(2) If $X$ and $X'$ are not connected, Theorem 5.9 (or the equivalence of (1) and (2) in Corollary 5.10) is not true in general. The shifts on the groups $\mathbb{Z}_{/4}^{\mathbb{Z}}$ and $(\mathbb{Z}_{/2}^2)^{\mathbb{Z}}$ are topologically, but not algebraically conjugate. However, the equivalence of (2) and (3) in Corollary 5.10 holds for *any* pair of $\mathfrak{R}_d$-modules $\mathfrak{M}$ and $\mathfrak{M}'$, whether they are torsion-free (as additive groups) or not.

## 6. The dynamical system defined by a Noetherian module

We begin with a little bit of algebra. Let $d \geq 1$, and let $\mathcal{R}$ be a commutative ring. We denote by $\mathcal{R}^\times$ the set of invertible elements (or units) in $\mathcal{R}$, write $\mathcal{R}[u_1, \dots, u_d]$ and $\mathcal{R}[u_1^{\pm 1}, \dots, u_d^{\pm d}]$ for the rings of polynomials and Laurent polynomials in the commuting variables $u_1, \dots, u_d$ with coefficients in $\mathcal{R}$, and we define $\mathfrak{R}_d$ by (5.1). For every rational prime $p$ we denote by $\overline{\mathbb{F}}_p$ the algebraic closure of the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_{/p}$ and define a homomorphism $f \mapsto f_{/p}$ from $\mathfrak{R}_d$ to

$$\mathfrak{R}_d^{(p)} = \mathbb{F}_p[u_1^{\pm 1}, \dots, u_d^{\pm d}] \tag{6.1}$$

by reducing the coefficients of $f \in \mathfrak{R}_d$ modulo $p$. An element $f \in \mathfrak{R}_d^{(p)}$ will again be written in the form (5.1) with $c_f(\mathbf{n}) \in \mathbb{F}_p$ for all $\mathbf{n} \in \mathbb{Z}^d$, where $c_f(\mathbf{n}) \neq 0$ for only finitely many $\mathbf{n} \in \mathbb{Z}^d$. For notational consistency we set $\overline{\mathbb{F}}_0$ equal to the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and put $\mathfrak{R}_d^{(0)} = \mathfrak{R}_d$ and $f_{/0} = f$ for every $f \in \mathfrak{R}_d$.

Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. We identify $\mathbb{Z}$ with the set of constant polynomials in $\mathfrak{R}_d$, denote by $p(\mathfrak{p})$ the *characteristic* $\operatorname{char}(\mathfrak{R}_d/\mathfrak{p})$ of $\mathfrak{R}_d/\mathfrak{p}$, i.e.

the unique non-negative integer such that $\mathfrak{p}\cap\mathbb{Z}=p(\mathfrak{p})\mathbb{Z}$, and define the *variety* of $\mathfrak{p}$ by

$$V(\mathfrak{p})=\{c\in(\overline{\mathbb{F}}_{p(\mathfrak{p})}^{\times})^d : f_{/p(\mathfrak{p})}(c)=0 \text{ for every } f\in\mathfrak{p}\}. \tag{6.2}$$

If $\mathfrak{a}\subset\mathfrak{R}_d$ is an arbitrary ideal we set

$$V_{\mathbb{C}}(\mathfrak{a})=\{c\in(\mathbb{C}^{\times})^d : f(c)=0 \text{ for every } f\in\mathfrak{a}\}. \tag{6.3}$$

Suppose that $\mathfrak{M}$ is an $\mathfrak{R}_d$-module. For every $f\in\mathfrak{R}_d$ we write $f_{\mathfrak{M}}\colon\mathfrak{M}\longmapsto\mathfrak{M}$ for the map $a\mapsto f\cdot a$, $a\in\mathfrak{M}$, and we denote by $\operatorname{ann}(a)=\{f\in\mathfrak{R}_d : f\cdot a=0\}$ the annihilator of an element $a\in\mathfrak{M}$. A prime ideal $\mathfrak{p}\subset\mathfrak{R}_d$ is *associated with* $\mathfrak{M}$ if $\mathfrak{p}=\operatorname{ann}(a)$ for some $a\in\mathfrak{M}$, and the module $\mathfrak{M}$ is *associated with* $\mathfrak{p}$ if $\mathfrak{p}$ is the only prime ideal in $\mathfrak{R}_d$ associated with $\mathfrak{M}$. If $\mathfrak{M}$ is Noetherian then it is associated with $\mathfrak{p}$ if and only if

$$\mathfrak{p}=\{f\in\mathfrak{R}_d : f_{\mathfrak{M}} \text{ is not injective}\}=\{f\in\mathfrak{R}_d : f_{\mathfrak{M}} \text{ is nilpotent}\} \tag{6.4}$$

(cf. Corollary VI.4.11 in [51]). If $\mathfrak{M}$ is associated with $\mathfrak{p}$ and $\mathfrak{N}\subset\mathfrak{M}$ is a non-zero submodule, then $\mathfrak{N}$ is again associated with $\mathfrak{p}$. The module $\mathfrak{M}$ is a *torsion module* if the prime ideal $\{0\}$ is not associated with $\mathfrak{M}$. We shall have to be careful to distinguish between $\mathfrak{R}_d$-modules $\mathfrak{M}$ which are not *torsion* and those which are *torsion-free* as additive groups (or $\mathbb{Z}$-modules): $\mathfrak{M}$ is a torsion module if every associated prime ideal is non-zero, $\mathfrak{M}$ is a torsion group if each of its associated primes contains a non-zero constant, and $\mathfrak{M}$ is torsion-free (as an additive group) if none of its associated primes contains a non-zero constant.

A submodule $\mathfrak{W}\subset\mathfrak{M}$ is $\mathfrak{p}$*-primary* (or $\mathfrak{p}$ *belongs to* $\mathfrak{W}$) if $\mathfrak{M}/\mathfrak{W}$ is associated with $\mathfrak{p}$. *From now on we assume that $\mathfrak{M}$ is Noetherian.* By Theorem VI.5.3 in [51] there exist primary submodules $\mathfrak{W}_1,\dots,\mathfrak{W}_m$ of $\mathfrak{M}$ with the following properties:

$$\begin{gathered}\text{the primes } \mathfrak{p}_i \text{ belonging to the submodules } \mathfrak{W}_i \text{ are all distinct;}\\ \mathfrak{W}_1\cap\dots\cap\mathfrak{W}_m=\{0\};\\ \text{for every subset } S\subsetneq\{1,\dots,m\},\ \bigcap_{i\in S}\mathfrak{W}_i\neq\{0\}.\end{gathered} \tag{6.5}$$

A family $\{\mathfrak{W}_1,\dots,\mathfrak{W}_m\}$ of primary submodules satisfying (6.5) is called a *reduced primary decomposition* of $\mathfrak{M}$, and $\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$ is the *set of associated primes* of $\mathfrak{M}$. According to the Theorems VI.5.2 and VI.5.5 in [51] the set of associated primes of $\mathfrak{M}$ is independent of the specific decomposition (6.5), and

$$\{f\in\mathfrak{R}_d : f_{\mathfrak{M}} \text{ is not injective}\}=\bigcup_{i=1,\dots,m}\mathfrak{p}_i. \tag{6.6}$$

PROPOSITION 6.1. *Let $d\geq 1$, $\mathfrak{q}\subset\mathfrak{R}_d$ a prime ideal, and let $\mathfrak{W}$ be a Noetherian $\mathfrak{R}_d$-module associated with $\mathfrak{q}$. Then there exist integers $1\leq t\leq s$ and submodules $\{0\}=\mathfrak{N}_0\subset\dots\subset\mathfrak{N}_s=\mathfrak{W}$ such that, for every $i=1,\dots,s$,*

$\mathfrak{N}_i/\mathfrak{N}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}_i$ *for some prime ideal* $\mathfrak{q} \subset \mathfrak{q}_i \subset \mathfrak{R}_d$, $\mathfrak{q}_i = \mathfrak{q}$ *for* $i = 1, \dots, t$, *and* $\mathfrak{q}_i \supsetneq \mathfrak{q}$ *for* $i = t+1, \dots, s$.

PROOF. Note that, if $\mathfrak{N} \subset \mathfrak{W}$ is a submodule, and if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal associated with $\mathfrak{W}/\mathfrak{N}$, then $\mathfrak{p} \supset \mathfrak{q}$. Indeed, if $\mathfrak{p} = \operatorname{ann}(a)$ for some $a \in \mathfrak{W}/\mathfrak{N}$, choose $b \in \mathfrak{W}$ such that $a = b + \mathfrak{N}$, and set $\mathfrak{N}' = \mathfrak{p} \cdot b = \{f \cdot b : f \in \mathfrak{p}\} \subset \mathfrak{N}$. If $\mathfrak{N}' \neq \{0\}$ then $\mathfrak{N}'$ is associated with $\mathfrak{q}$, and (6.4) shows that $g^n \in \mathfrak{p}$ for every $g \in \mathfrak{q}$ and every sufficiently large $n \geq 1$. Since $\mathfrak{p}$ is prime we conclude that $\mathfrak{q} \subset \mathfrak{p}$.

Let $\Omega_1$ be the set of submodules $\mathfrak{N} \subset \mathfrak{W}$ with the following property: there exists an integer $r \geq 1$ and submodules $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_r = \mathfrak{N}$ such that $\mathfrak{N}_i/\mathfrak{N}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}$ for every $i = 1, \dots, r$. It is clear that $\Omega_1 \neq \varnothing$, since we can find an $a \in \mathfrak{W}$ with $\operatorname{ann}(a) = \mathfrak{q}$ and $\mathfrak{N} = \mathfrak{R}_d a \cong \mathfrak{R}_d/\mathfrak{q}$. Since $\mathfrak{W}$ is Noetherian, $\Omega_1$ contains a maximal element $\mathfrak{W}'$, and we set $\mathfrak{V} = \mathfrak{W}/\mathfrak{W}'$ and consider the set of prime ideals $\{\mathfrak{q}_1, \dots, \mathfrak{q}_l\}$ associated with the $\mathfrak{R}_d$-module $\mathfrak{V}$. If $\mathfrak{q}_i = \mathfrak{q}$ for some $i \in \{1, \dots, l\}$, then there exists an element $b \in \mathfrak{W}$ with $b \notin \mathfrak{W}'$ and $\{f \in \mathfrak{R}_d : fb \in \mathfrak{W}'\} = \mathfrak{q}$, and this violates the maximality of $\mathfrak{W}'$.

Let $\Omega_2$ be the set of submodules $\mathfrak{N}$ with $\mathfrak{W}' \subset \mathfrak{N} \subset \mathfrak{W}$, for which there exist submodules $\mathfrak{W}' = \mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_t = \mathfrak{N}$ such that, for every $i = 1, \dots, t$, $\mathfrak{L}_i/\mathfrak{L}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i \supsetneq \mathfrak{q}$. Then $\Omega_2$ again has a maximal element $\mathfrak{W}''$. If $\mathfrak{W}'' \neq \mathfrak{W}$ we set $\mathfrak{V}' = \mathfrak{W}/\mathfrak{W}''$, consider the set of prime ideals associated with $\mathfrak{V}'$, all of which are strictly greater than $\mathfrak{q}$ by the argument in the first paragraph of this proof, and obtain a contradiction to the maximality of $\mathfrak{W}''$ exactly as before, where we were dealing with $\mathfrak{W}'$. Hence $\mathfrak{W}'' = \mathfrak{W}$, and the proposition is proved by setting $\mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s$ equal to $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s = \mathfrak{L}_0 \subset \cdots \subset \mathfrak{L}_t = \mathfrak{N}$. □

COROLLARY 6.2. *Let* $d \geq 1$, $\mathfrak{M}$ *a Noetherian* $\mathfrak{R}_d$*-module with associated primes* $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ *and a corresponding reduced primary decomposition* $\{\mathfrak{W}_1, \dots, \mathfrak{W}_m\}$. *Then there exist submodules* $\mathfrak{M} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$ *such that, for every* $i = 1, \dots, s$, $\mathfrak{N}_i/\mathfrak{N}_{i-1} \cong \mathfrak{R}_d/\mathfrak{q}_i$ *for some prime ideal* $\mathfrak{q}_i \subset \mathfrak{R}_d$, *and* $\mathfrak{q}_i \supset \mathfrak{p}_j$ *for some* $j \in \{1, \dots, m\}$ (*such a sequence* $\mathfrak{M} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$ *is called a prime filtration of* $\mathfrak{M}$).

PROOF. Apply Proposition 6.1 to the successive quotients of the sequence

$$\mathfrak{M} \supset \mathfrak{W}_1 \supset (\mathfrak{W}_1 \cap \mathfrak{W}_2) \supset \cdots \supset (\mathfrak{W}_1 \cap \cdots \cap \mathfrak{W}_m) = \{0\},$$

bearing in mind that

$$(\mathfrak{W}_1 \cap \cdots \cap \mathfrak{W}_i)/(\mathfrak{W}_1 \cap \cdots \cap \mathfrak{W}_{i+1}) \cong (\mathfrak{W}_1 \cap \cdots \cap \mathfrak{W}_i)/\mathfrak{W}_{i+1} \subset \mathfrak{M}/\mathfrak{W}_{i+1}$$

is associated with $\mathfrak{p}_{i+1}$ for every $i = 1, \dots, m-1$ (if $B, C$ are subgroups of an abelian group $A$ we use the symbol $B/C$ to denote $(B + C)/C$). □

Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module with a prime filtration $\mathfrak{M} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$, and define the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{M}}$ on $X = X^{\mathfrak{M}}$ by (5.5) and (5.6). For every $j = 0, \dots, s$, $Y_j = \mathfrak{N}_j^{\perp}$ is a closed, $\alpha$-invariant subgroup of $X$, and the dual group of $Y_{j-1}/Y_j$ is isomorphic to $\mathfrak{R}_d/\mathfrak{q}_j$, where $\mathfrak{q}_j \subset \mathfrak{R}_d$ is a prime ideal containing one of the associated primes of $\mathfrak{M}$. This allows one to build up $(X, \alpha)$ from the successive quotients $(Y_{j-1}/Y_j, \alpha^{Y_{j-1}/Y_j})$, which have the explicit realization (5.9)–(5.10) with $\mathfrak{a} = \mathfrak{q}_j$. However, although the prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ are canonically associated with $\mathfrak{M}$, the ideals $\mathfrak{q}_j$ appearing in Proposition 6.1 and Corollary 6.2 need no longer be canonical, and may depend on a specific prime filtration of $\mathfrak{M}$. The next corollary can help to overcome this problem.

COROLLARY 6.3. *Let $d \geq 1$, $\mathfrak{M}$ a Noetherian $\mathfrak{R}_d$-module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Then there exists a Noetherian $\mathfrak{R}_d$-module $\mathfrak{N} = \mathfrak{N}^{(1)} \oplus \cdots \oplus \mathfrak{N}^{(m)}$ and an injective $\mathfrak{R}_d$-module homomorphism $\phi\colon \mathfrak{M} \longmapsto \mathfrak{N}$ such that each of the modules $\mathfrak{N}^{(j)}$ has a prime filtration $\mathfrak{N}^{(j)} = \mathfrak{N}_{r_j}^{(j)} \supset \cdots \supset \mathfrak{N}_0^{(j)} = \{0\}$ with $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{p}_j$ for $k = 1, \dots, r_j$.*

*If $X = X^{\mathfrak{M}}$ and $Y = X^{\mathfrak{N}} = X^{\mathfrak{N}^{(1)}} \times \cdots \times X^{\mathfrak{N}^{(m)}}$, then the homomorphism $\psi\colon Y \longmapsto X$ dual to $\phi$ is surjective and satisfies that*

$$\psi \cdot \alpha_{\mathbf{n}}^{\mathfrak{N}} = \psi \cdot (\alpha_{\mathbf{n}}^{\mathfrak{N}^{(1)}} \times \cdots \times \alpha_{\mathbf{n}}^{\mathfrak{N}^{(m)}}) = \alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \psi \tag{6.7}$$

*for every $\mathbf{n} \in \mathbb{Z}^d$.*

PROOF. Choose a reduced primary decomposition $\mathfrak{W}_1, \dots, \mathfrak{W}_m$ of $\mathfrak{M}$ as in (6.5). Then the map $\phi'\colon a \mapsto (a + \mathfrak{W}_1, \dots, a + \mathfrak{W}_m)$ from $\mathfrak{M}$ into $\mathfrak{K} = \bigoplus_{i=1}^{m} \mathfrak{M}/\mathfrak{W}_i$ is injective. We fix $j \in \{1, \dots, m\}$ for the moment and apply Proposition 6.1 to find a prime filtration $\{0\} = \mathfrak{N}_0 \subset \cdots \subset \mathfrak{N}_s = \mathfrak{M}/\mathfrak{W}_j$ such that $\mathfrak{N}_k^{(j)}/\mathfrak{N}_{k-1}^{(j)} \cong \mathfrak{R}_d/\mathfrak{q}_k^{(j)}$ for every $k = 1, \dots, s_j$, where $\mathfrak{q}_k^{(j)} \subset \mathfrak{R}_d$ is a prime ideal containing $\mathfrak{p}_j$, and where there exists an $r_j \in \{1, \dots, s_j\}$ such that $\mathfrak{q}_k^{(j)} = \mathfrak{p}_j$ for $k = 1, \dots, r_j$, and $\mathfrak{q}_k^{(j)} \supsetneq \mathfrak{p}_j$ for $k = r_j + 1, \dots, s_j$. If $r_j < s_j$ we choose Laurent polynomials $g_k^{(j)} \in \mathfrak{q}_k^{(j)} \smallsetminus \mathfrak{p}_j$ for $k = r_j + 1, \dots, s_j$, set $g^{(j)} = g_{r_j+1}^{(j)} \cdot \dots \cdot g_{s_j}^{(j)}$, and note that the map $\psi^{(j)}\colon \mathfrak{M}/\mathfrak{W}_j \longmapsto \mathfrak{N}_{r_j}^{(j)}$ consisting of multiplication by $g^{(j)}$ is injective. Since $\mathfrak{N}_{r_j}^{(j)}$ has the prime filtration $\{0\} = \mathfrak{N}_0^{(j)} \subset \cdots \subset \mathfrak{N}_{r_j}^{(j)}$ whose successive quotients are all isomorphic to $\mathfrak{R}_d/\mathfrak{p}_j$, the module $\mathfrak{N} = \mathfrak{N}_{r_1}^{(1)} \oplus \cdots \oplus \mathfrak{N}_{r_m}^{(m)}$ has the required properties. The last assertion follows from duality. $\square$

EXAMPLE 6.4. In Example 5.3 (2) we considered the automorphism of $\mathbb{T}^2$ given by the matrix $A' = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and obtained that the $\mathbb{Z}$-action on $\mathbb{T}^2$ defined by $A'$ is conjugate to $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$, where $\mathfrak{M}$ is the $\mathfrak{R}_1$-module $\psi'(\mathbb{Z}^2) \subset \mathfrak{R}_1/(f)$ with $f(u_1) = -1 - 4u_1 + u_1^2$ and $\psi'(m_1, m_2) = m_1 + (3m_1 + 2m_2)u_1 \in \mathfrak{R}_1/(f)$ for every $(m_1, m_2) \in \mathbb{Z}^2$. As a submodule of $\mathfrak{R}_1/(f)$, $\mathfrak{M}$ is associated with

$(f)$. Let $a = \psi'(0,1) = 2u_1 \in \mathfrak{R}_1/(f)$, and let $\mathfrak{N} = \mathfrak{R}_1 \cdot a = 2\mathfrak{R}_1/(f)$. Then $\mathfrak{M}/\mathfrak{N} = \mathfrak{R}_1/\mathfrak{a}$, where $\mathfrak{a}$ is the prime ideal $(2, 1+u_1) = 2\mathfrak{R}_1 + \mathfrak{R}_1(1+u_1) \subset \mathfrak{R}_1$, and $\{0\} \subset \mathfrak{N} \subset \mathfrak{M}$ is a prime filtration of $\mathfrak{M}$ with $\mathfrak{M}/\mathfrak{N} \cong \mathfrak{R}_1/\mathfrak{a}$ and $\mathfrak{N}/\{0\} \cong \mathfrak{R}_1/(f)$. ⊡

Our next result shows that certain dynamical properties of the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{M}}$ on $X^{\mathfrak{M}}$ can be expressed purely in terms of the primes associated with $\mathfrak{M}$ and do not require the much more difficult analysis of the primes which may occur in a prime filtration of $\mathfrak{M}$. Recall that an element $g \in \mathfrak{R}_d$ is a *generalized cyclotomic polynomial* if it is of the form $g(u_1, \dots, u_d) = u^{\mathbf{m}} c(u^{\mathbf{n}})$, where $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$, $\mathbf{n} \neq \mathbf{0}$, and $c$ is a cyclotomic polynomial in a single variable.

THEOREM 6.5. *Let $d \geq 1$, let $\mathfrak{M}$ a Noetherian $\mathfrak{R}_d$-module with associated primes $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, and let $(X, \alpha) = (X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ be defined by* (5.5)–(5.6). *For every $i = 1, \dots, m$ we denote by $p(\mathfrak{p}_i) \geq 0$ the characteristic of $\mathfrak{R}_d/\mathfrak{p}_i$.*

(1) *The following conditions are equivalent.*
 (a) *$\alpha$ is ergodic;*
 (b) *$\alpha_{\mathbf{n}}$ is ergodic for some $\mathbf{n} \in \mathbb{Z}^d$;*
 (c) *$\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$ is ergodic for every $i \in \{1, \dots, m\}$;*
 (d) *There do not exist integers $i \in \{1, \dots, m\}$ and $l \geq 1$ with*
$$\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \subset \mathfrak{p}_i;$$
 (e) *There do not exist integers $i \in \{1, \dots, m\}$ and $l \geq 1$ with*
$$V(\mathfrak{p}_i) \subset \{c = (c_1, \dots, c_d) \in (\overline{\mathbb{F}}_{p(\mathfrak{p}_i)}^{\times})^d : c_1^l = \dots = c_d^l = 1\}.$$
(2) *The following conditions are equivalent.*
 (a) *$\alpha$ is mixing;*
 (b) *For every $i = 1, \dots, m$, $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$ is mixing;*
 (c) *None of the prime ideals associated with $\mathfrak{M}$ contains a generalized cyclotomic polynomial, i.e. $\{u^{\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\} \cap \mathfrak{p}_i = \{0\}$ for $i = 1, \dots, m$.*
(3) *Let $\Lambda \subset \mathbb{Z}^d$ be a subgroup with finite index. The following conditions are equivalent.*
 (a) *The set*
$$\mathrm{Fix}_\Lambda(\alpha) = \{x \in X : \alpha_{\mathbf{n}}(x) = x \ \text{ for every } \ \mathbf{n} \in \Lambda\}$$
 *is finite;*
 (b) *For every $i = 1, \dots, m$, the set $\mathrm{Fix}_\Lambda(\alpha^{\mathfrak{R}_d/\mathfrak{p}_i})$ is finite;*
 (c) *For every $i = 1, \dots, m$, $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \Omega(\Lambda) = \varnothing$, where*
$$\Omega(\Lambda) = \{c \in \mathbb{C}^d : c^{\mathbf{n}} = 1 \ \text{ for every } \ \mathbf{n} \in \Lambda\}$$
 *with $c = (c_1, \dots, c_d)$, $\mathbf{n} = (n_1, \dots, n_d)$, and $c^{\mathbf{n}} = c_1^{n_1} \cdot \dots \cdot c_d^{n_d}$.*
(4) *The following conditions are equivalent.*
 (a) *$\alpha$ is expansive;*

(b) *For every* $i=1,\dots,m$, $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$ *is expansive;*
(c) *For every* $i=1,\dots,m$, $V_{\mathbb{C}}(\mathfrak{p}_i)\cap\mathbb{S}^d=\varnothing$*;*
(d) *For every* $i=1,\dots,m$ *with* $p(\mathfrak{p}_i)=0$, $V(\mathfrak{p}_i)\cap\mathbb{S}^d=\varnothing$*.*

We begin the proof of Theorem 6.5 with a general proposition.

PROPOSITION 6.6. *Let* $\mathfrak{M}$ *a countable* $\mathfrak{R}_d$*-module.*

(1) *For any* $\mathbf{n}\in\mathbb{Z}^d$ *the following conditions are equivalent.*
   (a) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ *is ergodic;*
   (b) $\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$ *is ergodic for every prime ideal* $\mathfrak{p}$ *associated with* $\mathfrak{M}$*;*
   (c) *No prime ideal* $\mathfrak{p}$ *associated with* $\mathfrak{M}$ *contains a polynomial of the form* $u^{l\mathbf{n}}-1$ *with* $l\geq 1$*.*
(2) *The following conditions are equivalent.*
   (a) $\alpha^{\mathfrak{M}}$ *is ergodic;*
   (b) $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ *is ergodic for every prime ideal* $\mathfrak{p}$ *associated with* $\mathfrak{M}$*;*
   (c) *No prime ideal* $\mathfrak{p}$ *associated with* $\mathfrak{M}$ *contains a set of the form* $\{u^{l\mathbf{n}}-1:\mathbf{n}\in\mathbb{Z}^d\}$ *with* $l\geq 1$*.*
(3) *The following conditions are equivalent.*
   (a) $\alpha^{\mathfrak{M}}$ *is mixing;*
   (b) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ *is ergodic for every non-zero element* $\mathbf{n}\in\mathbb{Z}^d$*;*
   (c) $\alpha_{\mathbf{n}}^{\mathfrak{M}}$ *is mixing for every non-zero element* $\mathbf{n}\in\mathbb{Z}^d$*;*
   (d) $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ *is mixing for every prime ideal* $\mathfrak{p}$ *associated with* $\mathfrak{M}$*;*
   (e) *None of the prime ideals associated with* $\mathfrak{M}$ *contains a generalized cyclotomic polynomial.*

PROOF. From Lemma 1.2 and (5.5)–(5.6) it is clear that the $\mathbb{Z}$-action $k\mapsto\alpha_{k\mathbf{n}}^{\mathfrak{M}}$ is non-ergodic if and only if there exists a non-zero element $a\in\mathfrak{M}$ such that $(u^{l\mathbf{n}}-1)a=0$ for some $l\geq 1$. Let $\mathfrak{N}=\mathfrak{R}_d\cdot a$, and let $b\in\mathfrak{N}$ be a non-zero element such that $\mathfrak{p}=\operatorname{ann}(b)$ is maximal in the set of annihilators of elements in $\mathfrak{N}$. Then $\mathfrak{p}$ is a prime ideal associated with $\mathfrak{M}$ which contains $u^{l\mathbf{n}}-1$. This shows that (1.c)⇒(1.a). Conversely, if there exists a prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$ which contains $u^{l\mathbf{n}}-1\in\mathfrak{p}$ for some $l\geq 1$, we choose $a\in\mathfrak{M}$ with $\operatorname{ann}(a)=\mathfrak{p}$, note that $(u^{l\mathbf{n}}-1)a=0$, and obtain that (1.a)⇒(1.c).

If we apply the equivalence (1.a)⟺(1.c) to the $\mathfrak{R}_d$-module $\mathfrak{R}_d/\mathfrak{p}$, whose only associated prime is $\mathfrak{p}$, we see that $\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$ is non-ergodic if and only if $u^{l\mathbf{n}}-1\in\mathfrak{p}$ for some $l\geq 1$, which completes the proof of the first part of this lemma.

If $\alpha^{\mathfrak{M}}$ is non-ergodic, then Lemma 1.2 implies that there exists a non-zero element $a\in\mathfrak{M}$ such that the orbit $\{u^{\mathbf{m}}\cdot a:\mathbf{m}\in\mathbb{Z}^d\}$ of the $\mathbb{Z}^d$-action $\hat{\alpha}^{\mathfrak{M}}$ in (5.5) is finite. As in the proof of (1) we set $\mathfrak{N}=\mathfrak{R}_d\cdot a$, choose $0\neq b\in\mathfrak{N}$ such that $\mathfrak{p}=\operatorname{ann}(b)$ is maximal, and note that $\mathfrak{p}$ is a prime ideal which contains $\{u^{l\mathbf{m}}-1:\mathbf{m}\in\mathbb{Z}^d\}$ for some $l\geq 1$. Conversely, if there exists a prime ideal $\mathfrak{p}\subset\mathfrak{R}_d$ associated with $\mathfrak{M}$ which contains $\{u^{l\mathbf{m}}-1:\mathbf{m}\in\mathbb{Z}^d\}$ for some $l\geq 1$, and Lemma 1.2 shows that the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{M}}$ cannot be ergodic. This shows that (2.c)$\Longleftrightarrow$(2.a), and the equivalence of (2.b) and (2.c) is obtained by applying the equivalence of (2.a) and (2.c) to the $\mathfrak{R}_d$-module $\mathfrak{R}_d/\mathfrak{p}$.

In order to prove (3) we note that the equivalence (3.a)$\Longleftrightarrow$(3.b)$\Longleftrightarrow$(3.c) follows from Theorem 1.6 (2), and the proof is completed by applying the part (1) of this lemma both to $\alpha^{\mathfrak{M}}$ and to $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p}$ ranges over the set of prime ideals associated with $\mathfrak{M}$. $\square$

PROOF OF THEOREM 6.5 (1). The implication (b)$\Rightarrow$(a) is obvious. If (b) does not hold there exists, for every $\mathbf{n} \in \mathbb{Z}^d$, an $l \geq 1$ with $u^{l\mathbf{n}} - 1 \in \bigcup_{1\leq i\leq m} \mathfrak{p}_i$ (Proposition 6.6). For every $i = 1, \dots, m$, the set $\Gamma_i = \{\mathbf{n} \in \mathbb{Z}^d : u^{\mathbf{n}} - 1 \in \mathfrak{p}_i\}$ is a subgroup of $\mathbb{Z}^d$. As we have just observed, the set $\Gamma = \bigcup_{i=1}^m \Gamma_i$ contains some multiple of every element of $\mathbb{Z}^d$; if every $\Gamma_i$ has infinite index in $\mathbb{Z}^d$, then $\Gamma$ is contained in the intersection with $\mathbb{Z}^d$ of a union of $m$ at most $d-1$-dimensional subspaces of $\mathbb{R}^d$, which is obviously impossible. Hence $\Gamma_i$ must have finite index in $\mathbb{Z}^d$ for some $i \in \{1, \dots, m\}$, and we can find an integer $l \geq 1$ such that $u^{l\mathbf{n}} - 1 \in \mathfrak{p}_i$ for every $\mathbf{n} \in \mathbb{Z}^d$. This proves the implication (d)$\Rightarrow$(b). The implications (a)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d) were proved in Proposition 6.6, and the equivalence of (d) and (e) follows from Hilbert's Nullstellensatz. $\square$

PROOF OF THEOREM 6.5 (2). Use Proposition 6.6. $\square$

LEMMA 6.7. *Let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal. Then $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$ if and only if $V_{\mathbb{C}}(\mathfrak{a}) = \varnothing$.*

PROOF. If $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$ then $V_{\mathbb{C}}(\mathfrak{a}) = \varnothing$. Conversely, if $V_{\mathbb{C}}(\mathfrak{a}) = \varnothing$, then the Nullstellensatz implies that $\overline{\mathbb{Q}}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \cdot \mathfrak{a} = \overline{\mathbb{Q}}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, and there exist polynomials $f_i \in \mathfrak{a}$, $g_i \in \overline{\mathbb{Q}}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$, $i = 1, \dots, n$, with $1 = \sum_{i=1}^n f_i g_i$. The coefficients of the $g_i$ generate a finite extension field $\mathbb{K} \supset \mathbb{Q}$, and $\mathfrak{R}_d^{(\mathbb{K})} = \mathbb{K}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] = \sum_{j=1}^l v_j \mathfrak{R}_d^{(\mathbb{Q})}$ for suitably chosen elements $\{v_1, \dots, v_l\} \in \mathfrak{R}_d^{(\mathbb{K})}$, where $\mathfrak{R}_d^{(\mathbb{Q})} = \mathbb{Q}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. Since $\mathfrak{a}^{(\mathbb{Q})} = \mathfrak{R}_d^{(\mathbb{Q})} \cdot \mathfrak{a}$ is an ideal in $\mathfrak{R}_d^{(\mathbb{Q})}$ and $\mathfrak{R}_d^{(\mathbb{K})} \cdot \mathfrak{a}^{(\mathbb{Q})} = \mathfrak{R}_d^{(\mathbb{K})}$, there exist elements $\{h_{j,k} : 1 \leq j, k \leq l\} \subset \mathfrak{a}^{(\mathbb{Q})}$ such that, for every $j = 1, \dots, l$, $v_j = \sum_{k=1}^l h_{j,k} v_k$. Hence $\det(\delta_{j,k} - h_{j,k}) = 0$, where $\delta_{j,k} = 1$ for $j = k$ and $\delta_{j,k} = 0$ otherwise, and we conclude that $1 \in \mathfrak{a}^{(\mathbb{Q})}$. This proves that $\mathfrak{a} \cap \mathbb{Z} \neq \{0\}$. $\square$

PROOF OF THEOREM 6.5 (3). If $\mathfrak{b}(\Lambda) \subset \mathfrak{R}_d$ is the ideal generated by $\{u^{\mathbf{n}} - 1 : \mathbf{n} \in \Lambda\}$, then

$$V_{\mathbb{C}}(\mathfrak{b}(\Lambda)) = \{c \in \mathbb{C}^d : c^{\mathbf{n}} = 1 \text{ for every } \mathbf{n} \in \Lambda\} = \Omega(\Lambda),$$

$\mathrm{Fix}_\Lambda(\alpha)^\perp = \mathfrak{b}(\Lambda) \cdot \mathfrak{M}$, and $\widehat{\mathrm{Fix}_\Lambda(\alpha)} = \mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$ (cf. (5.5)–(5.6)). In particular, $\mathrm{Fix}_\Lambda(\alpha)$ is finite if and only if $\mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$ is finite.

Suppose that $\mathrm{Fix}_\Lambda(\alpha)$ is finite. For every $i = 1, \dots, m$ we choose $a_i \in \mathfrak{M}$ such that $\mathfrak{p}_i = \mathrm{ann}(a_i)$ and hence $\mathfrak{L}_i = \mathfrak{R}_d \cdot a_i \cong \mathfrak{R}_d/\mathfrak{p}_i$. The Artin-Rees Lemma

(Corollary 10.10 in [5]) implies that

$$\mathfrak{b}(\Lambda)^{(t)} \cdot \mathfrak{M} \cap \mathfrak{L}_i = \mathfrak{b}(\Lambda) \cdot (\mathfrak{b}(\Lambda)^{(t-1)} \cdot \mathfrak{M} \cap \mathfrak{L}_i) \subset \mathfrak{b}(\Lambda) \cdot \mathfrak{L}_i$$

for some $t \geq 1$, where $\mathfrak{b}(\Lambda)^{(t)} \subset \mathfrak{R}_d$ is the ideal generated by $\{f_1 \cdot \dots \cdot f_t : f_i \in \mathfrak{b}(\Lambda) \text{ for } i = 1, \dots, t\}$. By assumption,

$$\widehat{\mathrm{Fix}_\Lambda(\alpha)} = \mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$$

is finite. Since $\mathfrak{b}(\Lambda)$ is finitely generated we can choose $f_1, \dots f_r$ such that $\mathfrak{b}(\Lambda) = f_1\mathfrak{R}_d + \dots + f_r\mathfrak{R}_d$, and we conclude that

$$\begin{aligned} |\mathfrak{b}(\Lambda) \cdot \mathfrak{M}/\mathfrak{b}(\Lambda)^{(2)} \cdot \mathfrak{M}| &\leq \sum_{j=1}^{r} \left| f_j \cdot \mathfrak{M} \Big/ \Big( \sum_{j,j'=1}^{r} f_j f_{j'} \cdot \mathfrak{M} \Big) \right| \\ &\leq \sum_{j=1}^{r} \left| f_j \cdot \mathfrak{M} \Big/ \Big( \sum_{j'=1}^{r} f_j f_{j'} \cdot \mathfrak{M} \Big) \right| \\ &\leq r \left| \mathfrak{M} \Big/ \Big( \sum_{j'=1}^{r} f_{j'} \cdot \mathfrak{M} \Big) \right| = r|\mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}| < \infty. \end{aligned}$$

An induction argument shows that $\mathfrak{b}(\Lambda)^{(k)}\mathfrak{M}/\mathfrak{b}(\Lambda)^{(k+1)} \cdot \mathfrak{M}$ is finite for every $k \geq 1$, and we conclude that $\mathfrak{M}/\mathfrak{b}(\Lambda)^{(k)} \cdot \mathfrak{M}$ is finite for every $k \geq 1$. In particular, the modules $\mathfrak{L}_i/\mathfrak{b}(\Lambda)^{(t)} \cdot \mathfrak{M} \cong \mathfrak{L}_i/(\mathfrak{b}(\Lambda)^{(t)} \cdot \mathfrak{M} \cap \mathfrak{L}_i)$ and $\mathfrak{L}_i/\mathfrak{b}(\Lambda) \cdot \mathfrak{L}_i \cong \mathfrak{R}_d/(\mathfrak{p}_i + \mathfrak{b}(\Lambda))$ are finite. From Lemma 6.7 we conclude that $V_{\mathbb{C}}(\mathfrak{p}_i + \mathfrak{b}(\Lambda)) = V_{\mathbb{C}}(\mathfrak{p}_i) \cap \Omega(\Lambda) = \varnothing$ for every $i = 1, \dots, m$, which proves (c).

Conversely, if (c) is satisfied, we choose a prime filtration $\mathfrak{M} = \mathfrak{N}_s \supset \dots \supset \mathfrak{N}_0 = \{0\}$ of $\mathfrak{M}$ such that, for every $j = 1, \dots, s$, $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{q}_j$ for some prime ideal $\mathfrak{q}_j$ which contains one of the associated primes $\mathfrak{p}_i$ of $\mathfrak{M}$ (cf. Corollary 6.2). Since

$$V_{\mathbb{C}}(\mathfrak{q}_j + \mathfrak{b}(\Lambda)) = V_{\mathbb{C}}(\mathfrak{q}_j) \cap V_{\mathbb{C}}(\mathfrak{b}(\Lambda)) \subset V_{\mathbb{C}}(\mathfrak{p}_i) \cap V_{\mathbb{C}}(\mathfrak{b}(\Lambda)) = \varnothing$$

for every $j = 1, \dots, s$, the module $\mathfrak{R}_d/(\mathfrak{q}_j + \mathfrak{b}(\Lambda))$ is finite for every $j$ by Lemma 6.7. Hence $\mathfrak{N}_j/(\mathfrak{N}_{j-1} + \mathfrak{b}(\Lambda) \cdot \mathfrak{M})$ is finite for $j = 1, \dots, s$, since it is (isomorphic to) a quotient of $\mathfrak{R}_d/(\mathfrak{q}_j + \mathfrak{b}(\Lambda))$, and $\mathfrak{M}/\mathfrak{b}(\Lambda) \cdot \mathfrak{M}$ is finite. This implies the finiteness of $\mathrm{Fix}_\Lambda(\alpha)$ and completes the proof of the implication (c)⇒(a). The equivalence of (b) and (c) is obtained by applying what we have just proved to the $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$, $i = 1, \dots, m$. □

LEMMA 6.8. *Let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal with $V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d = \varnothing$. Then $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is expansive.*

PROOF. We assume that $X^{\mathfrak{R}_d/\mathfrak{a}} = \widehat{\mathfrak{R}_d/\mathfrak{a}}$ and $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ are given by (5.9)–(5.10). For every $f \in \mathfrak{R}_d$ of the form (5.2) we set $\|f\| = \sum_{\mathbf{n} \in \mathbb{Z}^d} |c_f(\mathbf{n})|$. Let $\{f_1, \dots, f_k\}$ be a set of generators for $\mathfrak{a}$, $\varepsilon = (10 \sum_{j=1}^{k} \|f_j\|)^{-1}$, and $N = \{x \in$

$X^{\mathfrak{R}_d/\mathfrak{a}} : \|x_{\mathbf{0}}\| < \varepsilon\}$, where $\|t\| = \min\{|t-n| : n \in \mathbb{Z}\}$ for every $t \in \mathbb{T}$. We claim that $N$ is an expansive neighbourhood of the identity $\mathbf{0}$ in $X^{\mathfrak{R}_d/\mathfrak{a}}$.

If $N$ is not expansive, there exists a point $\mathbf{0} \neq x \in \bigcap_{\mathbf{n}\in\mathbb{Z}^d} \sigma_{\mathbf{n}}(N)$. Let $\mathbf{B} = \ell^\infty(\mathbb{Z}^d)$ be the Banach space of all bounded, complex valued functions $(z_{\mathbf{n}}) = (z_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d)$ on $\mathbb{Z}^d$ in the supremum norm. Since $\|x_{\mathbf{n}}\| < \varepsilon$ for every $\mathbf{n} \in \mathbb{Z}^d$, there exists a unique non-zero point $y \in \mathbf{B}$ with $|y_{\mathbf{n}}| < \varepsilon$ and $y_{\mathbf{n}} \pmod 1 = x_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. From (5.7) and (5.9) we know that

$$\langle x, f_j \rangle = e^{2\pi i \sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) x_{\mathbf{n}}} = 1$$

and hence

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) y_{\mathbf{n}} \in \mathbb{Z}$$

for $j = 1, \dots, k$, and our choice of $\varepsilon$ implies that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) y_{\mathbf{n}} = 0 \tag{6.8}$$

for all $j$. Consider the group of isometries $\{U_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ of $\mathbf{B}$ defined by $(U_{\mathbf{n}} z)_{\mathbf{m}} = z_{\mathbf{m}+\mathbf{n}}$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $z \in \mathbf{B}$, and put

$$\begin{aligned}\mathbf{S} &= \Bigl\{ z \in \mathbf{B} : \sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) z_{\mathbf{m}+\mathbf{n}} = 0 \text{ for all } \mathbf{m} \in \mathbb{Z}^d \text{ and } j = 1, \dots, k \Bigr\} \\ &= \Bigl\{ z \in \mathbf{B} : \Bigl( \sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) U_{\mathbf{n}} \Bigr) z = 0 \text{ for } j = 1, \dots, k \Bigr\}.\end{aligned} \tag{6.9}$$

From (6.8) we know that the closed linear subspace $\mathbf{S} \subset \mathbf{B}$ is non-zero. Let $\mathcal{B}(\mathbf{S})$ be the Banach algebra of all bounded, linear operators on $\mathbf{S}$, denote by $V_{\mathbf{n}}$ the restriction of $U_{\mathbf{n}}$ to $\mathbf{S}$, and let $\mathcal{A} \subset \mathcal{B}(\mathbf{S})$ be the Banach subalgebra generated by $\{V_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$. We write $\mathcal{M}(\mathcal{A})$ for the space of maximal ideals of $\mathcal{A}$ in its usual topology. The Gelfand transform $A \mapsto \hat{A}$ from $\mathcal{A}$ to the Banach algebra $\mathcal{C}(\mathcal{M}(\mathcal{A}), \mathbb{C})$ of continuous, complex valued functions on $\mathcal{M}(\mathcal{A})$ is a norm-non-increasing Banach algebra homomorphism (cf. §11 in [75]). For every $\mathbf{n} \in \mathbb{Z}^d$, both $V_{\mathbf{n}}$ and $V_{-\mathbf{n}} = V_{\mathbf{n}}^{-1}$ are isometries of $\mathbf{S}$, and hence $|\widehat{V_{\mathbf{n}}}(\omega)| = 1$ for every $\omega \in \mathcal{M}(\mathcal{A})$. Since $\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) V_{\mathbf{n}} = 0$ (cf. (6.9)) we obtain that $\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) \widehat{V_{\mathbf{n}}}(\omega) = 0$ for every $j = 1, \dots, k$ and $\omega \in \mathcal{M}(\mathcal{A})$. Fix $\omega \in \mathcal{M}(\mathcal{A})$ and put $c_i = \widehat{V_{\mathbf{e}^{(i)}}}(\omega)$ for every $i = 1, \dots, d$, where $\mathbf{e}^{(i)}$ is the $i$-th unit vector in $\mathbb{Z}^d$. Then $\sum_{\mathbf{n}\in\mathbb{Z}^d} c_{f_j}(\mathbf{n}) c^{\mathbf{n}} = f_j(c) = 0$ for $j = 1, \dots, k$ with $c = (c_1, \dots, c_d) \in \mathbb{S}^d$. It follows that $c \in V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$, contrary to our initial assumption. This proves that $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is expansive. $\square$

PROOF OF THEOREM 6.5 (4). We begin by proving the equivalence of (a) and (c). Suppose that (c) is satisfied, but that $\alpha$ is non-expansive. We apply Corollary 6.2 and choose a prime filtration $\mathfrak{M} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$ such that, for every $j = 1, \dots, s$, $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{q}_j$ for some prime ideal $\mathfrak{q}_j \subset \mathfrak{R}_d$ which contains one of the associated primes $\mathfrak{p}_i$. Put $X_j = \mathfrak{N}_j^\perp \subset X$ and observe that $X = X_0 \supset \dots \supset X_s = \{\mathbf{1}\}$, that $X_j$ is a closed, $\alpha$-invariant subgroup of $X$, and that $X_{j-1}/X_j \cong \widehat{\mathfrak{R}_d/\mathfrak{q}_j}$ for $j = 1, \dots, s$. Then $V_{\mathbb{C}}(\mathfrak{q}_1) \subset \bigcup_{i=1}^m V_{\mathbb{C}}(\mathfrak{p}_i)$, hence $V_{\mathbb{C}}(\mathfrak{q}_1) \cap \mathbb{S}^d = \varnothing$, and Lemma 6.8 shows that $\alpha^{\mathfrak{R}_d/\mathfrak{q}_1}$ is expansive. Since $\alpha^{\mathfrak{R}_d/\mathfrak{q}_1}$ is conjugate to $\alpha^{X/X_1} = \alpha^{X_0/X_1}$ we see that $\alpha^{X_0/X_1}$ is expansive. The non-expansiveness of $\alpha$ implies that $\alpha^{X_1}$ cannot be expansive, and by repeating this argument we eventually obtain that $\alpha^{X_s}$ is non-expansive, which is absurd. This contradiction proves the expansiveness of $\alpha$.

In order to explain the idea behind the proof of the reverse implication we assume for the moment that $\mathfrak{M}$ is of the form $\mathfrak{R}_d/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathfrak{R}_d$. If $c = (c_1, \dots, c_d) \in V_{\mathbb{C}}(\mathfrak{a})$ then the evaluation map $f \mapsto f(c)$ defines an $\mathfrak{R}_d$-module homomorphism $\eta_c \colon \mathfrak{R}_d/\mathfrak{a} \longmapsto \mathbb{C}$, where $\mathbb{C}$ is an $\mathfrak{R}_d$-module under the action $(f, z) \mapsto f(c)z$, $f \in \mathfrak{R}_d, z \in \mathbb{C}$. If $W$ is the closure of $\eta_c(\mathfrak{R}_d/\mathfrak{a}) \subset \mathbb{C}$, then $\eta_c$ conjugates the $\mathbb{Z}^d$-action $\hat{\alpha}$ on $\mathfrak{M}$ to the action $\theta$ on $W$, where $\theta_{\mathbf{n}}$ is multiplication by $c^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. If $c \in V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^d$ then $\theta$ is isometric (with respect to the usual metric on $\mathbb{C}$), and the homomorphism $\eta_c$ induces an inclusion of $V = \hat{W}$ in $X^{\mathfrak{R}_d/\mathfrak{a}} = \widehat{\mathfrak{R}_d/\mathfrak{a}}$. Since $\theta$ is isometric on $W$, the dual action $\hat{\theta}$ on $V$ is also equicontinuous, and coincides with the restriction of $\alpha$ to $V$. This shows that $\alpha$ cannot be expansive.

We return to our given module $\mathfrak{M}$ with its associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and a corresponding reduced primary decomposition $\mathfrak{W}_1, \dots, \mathfrak{W}_m$. If $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d \neq \varnothing$ for some $i \in \{1, \dots, m\}$ we set $\mathfrak{M}' = \mathfrak{M}/\mathfrak{W}_i$, choose $a_1, \dots, a_k \in \mathfrak{M}'$ such that $\mathfrak{M}' = \mathfrak{R}_d a_1 + \dots + \mathfrak{R}_d a_k$, and define a surjective homomorphism $\zeta \colon \mathfrak{R}_d^k \longmapsto \mathfrak{M}'$ by $\zeta(f_1, \dots, f_k) = f_1 a_1 + \dots + f_k a_k$.

Choose a point $c = (c_1, \dots, c_d) \in V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d$, denote by $\eta_c \colon \mathfrak{R}_d \longmapsto \mathbb{C}$ the evaluation map at $c$, and observe that $\mathfrak{a} = \ker(\eta_c) \supset \mathfrak{p}_i$. Let $\mathfrak{L} = \ker(\zeta) + \mathfrak{a}^k \subset \mathfrak{R}_d^k$, and let $\mathfrak{N} = \{(0, \dots, 0, f) : f \in \mathfrak{R}_d\} \subset \mathfrak{R}_d^k$. From (6.6) (with $\mathfrak{M}$ replaced by $\mathfrak{M}'$) we see that $\operatorname{ann}(a_k) \subset \mathfrak{p}_i$, so that

$$\mathfrak{L} \cap \mathfrak{N} \subset \{(0, \dots, 0, f) : f \in \mathfrak{p}_i\} \subset \{(0, \dots, 0, f) : f \in \mathfrak{a}\}.$$

This allows us to define an additive group homomorphism $\xi \colon \mathfrak{L} + \mathfrak{N} \longmapsto \mathbb{C}$ by $\xi(a + b) = \eta_c(f)$ for all $a \in \mathfrak{L}$ and $b = (0, \dots, 0, f) \in \mathfrak{N}$. Then

$$\xi(a) = 0 \text{ for } a \in \mathfrak{L}, \tag{6.10}$$

and

$$\xi \cdot \hat{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d^k}(a) = c^{\mathbf{n}} \xi(a) \text{ for all } a \in \mathfrak{L} + \mathfrak{N},\ \mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d, \tag{6.11}$$

where $c^{\mathbf{n}} = c_1^{n_1} \cdot \dots \cdot c_d^{n_d}$. We claim that $\xi$ can be extended to a homomorphism $\bar{\xi} \colon \mathfrak{R}_d^k \longmapsto \mathbb{C}$ which still satisfies (6.10) and (6.11). Indeed, there exists a maximal extension $\xi'$ of $\xi$ to a submodule $\mathfrak{N}' \subset \mathfrak{R}_d^k$ satisfying (6.11) for every $a \in \mathfrak{N}'$. If $b \in \mathfrak{R}_d^k \smallsetminus \mathfrak{N}'$ and $\xi'(b') = 0$ for every $b' \in \mathfrak{R}_d b \cap \mathfrak{N}'$, then we put $\rho = 0$. If there exists an element $f \in \mathfrak{R}_d$ with $fb \in \mathfrak{R}_d b \cap \mathfrak{N}'$ and $\xi'(fb) \neq 0$, then $f(c) = \eta_c(f) \neq 0$: otherwise $f \in \mathfrak{a}$, $fb \in \mathfrak{a}^k \subset \mathfrak{L}$, and $\xi'(fb) = \xi(fb) = 0$ by (6.10), which is impossible. Hence we can set $\rho = \xi'(fb)/f(c)$. The map $\xi'' \colon \mathfrak{N}'' = \mathfrak{R}_d b + \mathfrak{N}' \longmapsto \mathbb{C}$, defined by $\xi''(fb + a) = f(c)\rho + \xi'(a)$ for $f \in \mathfrak{R}_d$ and $a \in \mathfrak{N}'$, is a homomorphism which extends $\xi'$ and satisfies (6.11) for all $a \in \mathfrak{N}''$. This contradiction to the maximality of $\mathfrak{N}'$ proves our claim.

We have obtained an extension $\bar{\xi} \colon \mathfrak{R}_d^k \longmapsto \mathbb{C}$ of $\xi$ satisfying (6.11) for all $a \in \mathfrak{R}_d^k$; this implies that $\ker(\bar{\xi})$ is a submodule of $\mathfrak{R}_d^k$ which contains $\ker(\zeta)$, and that $\bar{\xi}$ induces an $\mathfrak{R}_d$-module homomorphism $\Xi \colon \mathfrak{M}' \cong \mathfrak{R}_d^k/\ker(\zeta) \longmapsto \mathbb{C}$ with $\Xi(\mathfrak{M}') \supset \eta_c(\mathfrak{R}_d)$ and

$$\Xi \cdot \hat{\alpha}_{\mathbf{n}}^{\mathfrak{M}'} = \theta_{\mathbf{n}} \cdot \Xi \tag{6.12}$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where $\theta_{\mathbf{n}}$ is multiplication by $c^{\mathbf{n}}$. We denote by $W$ the closure of $\Xi(\mathfrak{M}')$ in $\mathbb{C}$ and write $V = \hat{W}$ for the dual group of $W$. Since $\Xi$ sends $\mathfrak{M}'$ to a dense subgroup of $W$, there is a dual inclusion $V \subset \widehat{\mathfrak{M}'/\ker(\Xi)} \subset \widehat{\mathfrak{M}'} \subset X$, and (6.12) shows that, for every $v \in V$ and $\mathbf{n} \in \mathbb{Z}^d$,

$$\hat{\theta}_{\mathbf{n}}(v) = \alpha_{\mathbf{n}}(v). \tag{6.13}$$

If the closed subgroup $W \subset \mathbb{C}$ is countable, then the group $\Theta = \{\theta_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \operatorname{Aut}(W)$ is finite, since it consists of isometries of $W$, and hence $\hat{\Theta} = \{\hat{\theta}_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\} \subset \operatorname{Aut}(V)$ is finite. From (6.13) it is clear that the restriction of $\alpha$ to the infinite subgroup $V \subset X$ cannot be expansive.

If $W$ is uncountable, but disconnected, we replace $W$ by its infinite, discrete quotient group $W' = W/W^\circ$, and obtain an $\alpha$-invariant subgroup $V' = \widehat{W/W^\circ} \subset V \subset X$ on which $\alpha$ is not expansive.

If $W$ is connected, it is either equal to $\mathbb{C}$ or isomorphic to $\mathbb{R}$, and the definition of $\Theta$ implies that $W$ has a basis of $\Theta$-invariant neighbourhoods of the identity. The dual group $V$ is isomorphic to $W$, and again possesses a basis of $\hat{\Theta}$-invariant neighbourhoods of the identity. Since the inclusion $V \hookrightarrow X$ is continuous, the $\mathbb{Z}^d$-action $\mathbf{n} \mapsto \hat{\theta}_{\mathbf{n}}$ on $V \subset X$ must also be non-expansive in the subspace topology, i.e. $\alpha$ is not expansive on $V$.

We have proved that there always exists an infinite, $\alpha$-invariant, but not necessarily closed, subgroup $V \subset X$ on which $\alpha$ is non-expansive in the induced topology. This shows that $\alpha$ is not expansive and completes the proof that (a)$\iff$(c).

The equivalence of (b) and (c) is seen by applying the implications (a)$\iff$(c) already proved to the $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}$, $i = 1, \dots, m$.

It is clear that (c)$\Rightarrow$(d). Conversely, if $V_{\mathbb{C}}(\mathfrak{p}_i) \cap \mathbb{S}^d \neq \varnothing$ for some $i \in \{1, \dots, m\}$, choose $f_1, \dots, f_k$ in $\mathfrak{R}_d$ with $\mathfrak{p}_i = f_1\mathfrak{R}_d + \dots + f_k\mathfrak{R}_d$, and define polynomials $g_j, h_j$, $j = 1, \dots, k$, in

$$\mathcal{R}_d = \mathbb{Q}[x_1, \dots, x_d, y_1, \dots, y_d]$$

by

$$g_j(a_1,\dots,a_d,b_1,\dots,b_d)=\mathrm{Re}(f_j(a_1+b_1\sqrt{-1},\dots,a_d+b_d\sqrt{-1}))$$

and

$$h_j(a_1,\dots,a_d,b_1,\dots,b_d)=\mathrm{Im}(f_j(a_1+b_1\sqrt{-1},\dots,a_d+b_d\sqrt{-1}))$$

for all $j=1,\dots,k$ and $(a_1,\dots,a_d,b_1,\dots,b_d)\in\mathbb{R}^{2d}$, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of $z\in\mathbb{C}$. For $l=1,\dots,d$ we put

$$\chi_l(x_1,\dots,x_d,y_1,\dots,y_d)=x_l^2+y_l^2-1\in\mathcal{R}_d.$$

The ideal $\mathfrak{J}\subset\mathcal{R}_d$ generated by $\{g_1,\dots,g_k,h_1,\dots,h_k,\chi_1,\dots,\chi_k\}$ satisfies that $V_{\mathbb{C}}(\mathfrak{J})\cap\mathbb{R}^{2d}\neq\varnothing$. Hence $\mathfrak{J}$ does not contain a polynomial of the form $1+\sum_{j=1}^r\psi_j^2$ with $r\geq 1$ and $\psi_j\in\mathcal{R}_d$, and the real version of Hilbert's Nullstellensatz implies that $V_{\mathbb{C}}(\mathfrak{J})\cap\mathbb{R}^{2d}\cap\overline{\mathbb{Q}}^{2d}\neq\varnothing$ (proposition 4.1.7 and corollaire 4.1.8 in [11]). In particular we see that (d) cannot be satisfied, and this shows that (d)⇒(c) and completes the proof of Theorem 6.5 (4). □

Before we start listing some useful corollaries of Theorem 6.5 we give an elementary characterization of the connectedness of a group $X$ carrying a $\mathbb{Z}^d$-action by automorphisms in terms of the prime ideals associated with the $\mathfrak{R}_d$-module $\hat{X}$.

PROPOSITION 6.9. *Let $\alpha$ be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group $X$, and let $\mathfrak{M}=\hat{X}$ be the $\mathfrak{R}_d$-module defined by* Lemma 5.1. *The following conditions are equivalent.*

(1) *$X$ is connected;*
(2) *$V_{\mathbb{C}}(\mathfrak{p})\neq\varnothing$ for every prime ideal $\mathfrak{p}\subset\mathfrak{R}_d$ associated with $\mathfrak{M}$.*

PROOF. Suppose that $X$ is connected, and let $\mathfrak{p}\subset\mathfrak{R}_d$ be a prime ideal associated with $\mathfrak{M}$. Then there exists an element $a\in\mathfrak{M}$ with $\mathfrak{R}_d\cdot a\cong\mathfrak{R}_d/\mathfrak{p}$, which implies that $X^{\mathfrak{R}_d/\mathfrak{p}}$ is a quotient group of $X$. In particular, $X^{\mathfrak{R}_d/\mathfrak{p}}$ is connected, so that $\mathfrak{R}_d/\mathfrak{p}$ is a torsion-free, abelian group, and Lemma 6.7 implies that $V_{\mathbb{C}}(\mathfrak{p})\neq\varnothing$. Conversely, if $X$ is disconnected, then there exists—by duality theory—a non-zero element $a\in\mathfrak{M}$ and a positive integer $m$ with $ma=0$, and we set $\mathfrak{N}=\mathfrak{R}_d\cdot a$ and observe that $\mathfrak{N}$ (and hence $\mathfrak{M}$) has an associated prime ideal $\mathfrak{p}$ containing a non-zero constant (cf. (6.6)). In particular, $V_{\mathbb{C}}(\mathfrak{p})=\varnothing$. □

COROLLARY 6.10 (OF THEOREM 6.5). *If $\alpha$ is an ergodic $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group $X$ satisfying the d.c.c., then $\alpha_{\mathbf{n}}$ is ergodic for some $\mathbf{n}\in\mathbb{Z}^d$.*

PROOF. Lemma 5.1, Proposition 5.4, and Theorem 6.5 (1). □

COROLLARY 6.11. *Let $d\geq 2$, and let $(f)\subset\mathfrak{R}_d$ be a principal ideal. Then $\alpha^{\mathfrak{R}_d/(f)}$ is ergodic.*

PROOF. By Theorem 6.5 (1), the non-ergodicity of $\alpha$ implies that $V(\mathfrak{p}_i)$ is finite for at least one of the associated primes of $\mathfrak{M} = \mathfrak{R}_d/(f)$. However, the associate primes of $\mathfrak{M}$ are are all principal (they are given by the prime factors of $f$ in $\mathfrak{R}_d$), and have infinite varieties. $\square$

COROLLARY 6.12. *Let $d \geq 1$ and $f \in \mathfrak{R}_d$. If $f$ is not divisible by any generalized cyclotomic polynomial then $\alpha^{\mathfrak{R}_d/(f)}$ is mixing.*

PROOF. If $\mathfrak{p}$ is one of the associated primes of $\mathfrak{R}_d/(f)$ then $\mathfrak{p} = (h)$ for a prime factor $h$ of $f$ in $\mathfrak{R}_d$, and $\mathfrak{p}$ contains a polynomial of the form $u^{\mathbf{n}} - 1$ for some (non-zero) $\mathbf{n} \in \mathbb{Z}^d$ if and only if $h = c(u^{\mathbf{n}})$ for some cyclotomic polynomial $c$ (cf. Theorem 6.5 (2)). $\square$

COROLLARY 6.13. *Let $X$ be a compact, abelian group, and let $\alpha$ be an expansive $\mathbb{Z}^d$-action by automorphisms of $X$. Then the $\mathfrak{R}_d$-module $\mathfrak{M} = \hat{X}$ is a Noetherian torsion module.*

PROOF. According to (4.10) and Proposition 5.4, $\mathfrak{M}$ is Noetherian, and by Theorem 6.5 (4), $\{0\}$ cannot be an associated prime ideal of $\mathfrak{M}$. $\square$

COROLLARY 6.14. *Let $X$ be a compact, connected group, and let $\alpha$ be an expansive $\mathbb{Z}^d$-action by automorphisms of $X$. Then $X$ is abelian and $\alpha$ is ergodic.*

PROOF. Theorem 2.4 shows that $X$ is abelian, and (4.10) and Proposition 5.4 allow us to assume that $(X, \alpha) = (X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ for some Noetherian $\mathfrak{R}_d$-module $\mathfrak{M}$. By recalling Proposition 6.9 and comparing the conditions (1.e) and (4.c) in Theorem 6.5 we see that $\alpha$ is ergodic. $\square$

COROLLARY 6.15. *Let $X$ be a compact group, and let $\alpha$ be an expansive $\mathbb{Z}^d$-action by automorphisms of $X$. If $Y \subset X$ is a closed, normal, $\alpha$-invariant subgroup, then $\alpha^Y$ and $\alpha^{X/Y}$ are both expansive.*

PROOF. The expansiveness of $\alpha^Y$ is obvious. In order to see that $\alpha^{X/Y}$ is expansive we note that the connected component of the identity $X^\circ \subset X$ is abelian by Corollary 2.5. The group $X/X^\circ$ is zero-dimensional, and $X/(Y + X^\circ)$ is a quotient of a zero-dimensional group and hence again zero dimensional. Since the $\mathbb{Z}^d$-action $\alpha^{X/(Y+X^\circ)}$ satisfies the d.c.c., Corollary 3.4 implies that $\alpha^{X/(Y+X^\circ)}$ is expansive.

The group $(Y + X^\circ)/Y$ is isomorphic to $X^\circ/(Y \cap X^\circ)$, and this isomorphism carries $\alpha^{(Y+X^\circ)/Y}$ to $\alpha^{X^\circ/(Y\cap X^\circ)}$. We apply Lemma 5.1 to the abelian groups $X^\circ$ and $X^\circ/(Y \cap X^\circ)$, and obtain $\mathfrak{R}_d$-modules $\hat{X}^\circ = \mathfrak{M}$ and $\widehat{X^\circ/(Y \cap X^\circ)} = \mathfrak{N} \subset \mathfrak{M}$ satisfying (5.3)–(5.4). Since $\alpha^{X^\circ}$ is expansive, Theorem 6.5 (4) implies that $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$ for every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$. Every prime ideal associated with $\mathfrak{N}$ is also associated with $\mathfrak{M}$, and Theorem 6.5 (4) implies that $\alpha^{\mathfrak{N}}$ is expansive. This implies the expansiveness of both $\alpha^{X^\circ/(Y\cap X^\circ)}$ and $\alpha^{(Y+X^\circ)/Y}$.

Suppose that $x \in X \smallsetminus Y$. If $x \notin Y + X^\circ$ then the expansiveness of $\alpha^{X/(Y+X^\circ)}$ guarantees the existence of an open neighbourhood $N'(\mathbf{1}_X)$ of the identity in $X$ such that $\alpha_{\mathbf{m}}(x) \notin N'(\mathbf{1}_X) + Y + X^\circ \supset N'(\mathbf{1}_X) + Y$ for some $\mathbf{m} \in \mathbb{Z}^d$. If $x \in Y + X^\circ$ then the expansiveness of $\alpha^{(Y+X^\circ)/Y}$ allows us to choose a neighbourhood $N''(\mathbf{1}_X)$ of the identity in $X$ with $\alpha_{\mathbf{m}}(x) \notin N''(\mathbf{1}_X) + Y$ for some $\mathbf{m} \in \mathbb{Z}^d$. Put $N(\mathbf{1}_X) = N'(\mathbf{1}_X) \cap N''(\mathbf{1}_X)$. Then there exists, for every $x \in X \smallsetminus Y$, an $\mathbf{m} \in \mathbb{Z}^d$ with $\alpha_{\mathbf{m}}(x) \notin N(\mathbf{1}_X) + Y$, which shows that $\alpha^{X/Y}$ is expansive. □

In view of Theorem 6.5 we introduce the following definition, which will help to simplify terminology.

DEFINITION 6.16. Let $d \geq 1$, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. The ideal $\mathfrak{p}$ will be called *ergodic*, *mixing*, or *expansive* if the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic, mixing, or expansive.

EXAMPLES 6.17. (1) Let $n \geq 1$, $\alpha = A \in \mathrm{GL}(n, \mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^n)$, and let $\beta = \hat{A} = A^\top \in \mathrm{Aut}(\mathbb{Z}^n)$. The $\mathfrak{R}_1$-module $\mathfrak{M} = \mathbb{Z}^n$ arising from $\alpha$ via Lemma 5.1 is Noetherian, and $\mathrm{ann}(\mathbf{m}) = \{f \in \mathfrak{R}_1 : f(A^\top)\mathbf{m} = 0\}$ for every $\mathbf{m} \in \mathbb{Z}^n$. In particular, the associated primes of $\mathfrak{M}$ are the principal ideals $(h)$, where $h$ runs through the prime factors of the characteristic polynomial $\chi_A = \chi_{A^\top}$ of $A$ (or $A^\top$) in $\mathfrak{R}_1$. In this setting Theorem 6.5 (1) reduces to the following well known facts about toral automorphisms: (i) $\alpha$ is ergodic if and only if no root of $\chi_A$ is a root of unity; (ii) $\alpha$ is expansive if and only if no root of $\chi_A$ has modulus 1.

(2) The automorphism $\alpha$ in Example 5.6 (1) does not satisfy the d.c.c. (cf. Theorem 5.7), and is therefore non-expansive by (6.10). However, if we replace $\mathbb{Q}$ by $\mathbb{Z}[\frac{1}{6}] = \{k/6^l : k \in \mathbb{Z}, l \geq 0\} \cong \mathfrak{R}_1/(2u_1 - 3) = \mathfrak{M}$, where the isomorphism between $\mathfrak{R}_1/(2u_1 - 3)$ and $\mathbb{Z}[\frac{1}{6}]$ is the evaluation $f \mapsto f(\frac{3}{2})$, then the automorphism $\beta'$ of $\mathbb{Z}[\frac{1}{6}]$ consisting of multiplication by $\frac{3}{2}$ is conjugate to multiplication by $u_1$ on $\mathfrak{M}$. Since $\mathfrak{p} = (2u_1 - 3) \subset \mathfrak{R}_1$ is a prime ideal, $\mathfrak{M}$ is associated with $\mathfrak{p}$, $V_{\mathbb{C}}(\mathfrak{p}) = \{\frac{3}{2}\}$, and the automorphism $\alpha'$ on $X = \widehat{\mathbb{Z}[\frac{1}{6}]}$ dual to $\beta'$ is expansive by Theorem 6.5 (4). An explicit realization of $\alpha'$ can be obtained from Example 5.2 (2) by setting $\alpha'$ equal to the shift $\sigma$ on $X' = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : 3x_k = 2x_{k+1} \text{ for every } k \in \mathbb{Z}\}$.

(3) Let $\mathfrak{p} \subset \mathfrak{R}_1$ be a prime ideal. Since the ring $\mathfrak{R}_1^{(\mathbb{Q})} = \mathbb{Q}[u_1^{\pm 1}]$ of Laurent polynomials with rational coefficients is a principal ideal domain, $\mathfrak{R}_1/\mathfrak{p}$ must be finite if $\mathfrak{p}$ is non-principal. In order to see this, assume that $\mathfrak{p} \subsetneq \mathfrak{R}_1$ is a non-principal prime ideal, and choose two irreducible elements $g, h \in \mathfrak{p}$ with $g\mathfrak{R}_1 \neq h\mathfrak{R}_1$. We assume without loss in generality that $g\mathfrak{R}_1 \neq m\mathfrak{R}_1$ for any $m \in \mathbb{Z}$. Then $\mathfrak{q} = \{\frac{1}{n} f : n \geq 1, f \in \mathfrak{p}\} \subset \mathfrak{R}_1^{(\mathbb{Q})}$ is an ideal strictly containing the maximal ideal $g\mathfrak{R}_1^{(\mathbb{Q})}$, and therefore equal to $\mathfrak{R}_1^{(\mathbb{Q})}$. We conclude that $\mathfrak{p}$ contains a prime constant $p$, and hence the ideal $(p, g) = p\mathfrak{R}_1 + g\mathfrak{R}_1$. It follows that $\mathfrak{R}_1/\mathfrak{p}$ is a quotient of the finite ring $\mathfrak{R}_1/(p, g) \cong \mathfrak{R}_1^{(p)}/g_{/p}\mathfrak{R}_1^{(p)}$ (cf. (6.1)). In

particular, if $\mathfrak{p} \subset \mathfrak{R}_1$ is a non-principal prime ideal, then $X^{\mathfrak{R}_1/\mathfrak{p}} = \widehat{\mathfrak{R}_1/\mathfrak{p}}$ is finite, and $\alpha^{\mathfrak{R}_1/\mathfrak{p}}$ is non-ergodic.

If $\mathfrak{p} = (f)$ for some $f \in \mathfrak{R}_1$, the automorphism $\alpha = \alpha^{\mathfrak{R}_1/\mathfrak{p}}$ is non-ergodic if and only if $f$ divides $u_1^n - 1$ for some $n \geq 1$ (Theorem 6.5 (1)) (as $f$ is irreducible this means that $\pm u_1^n f$ is cyclotomic for some $n \in \mathbb{Z}$), and $\alpha$ is expansive if and only if $f$ is non-zero and has no roots of modulus 1 (Theorem 6.5 (4)). Since we can write $X = X^{\mathfrak{R}_1/\mathfrak{p}}$ in the form (5.9) we see that $X$ is (isomorphic to) a finite-dimensional torus if and only if there exists $n \in \mathbb{Z}$ and $s \geq 1$ such that $u_1^n f(u_1) = c_0 + c_1 u_1 + \dots + c_s u_1^s$ with $|c_0 c_s| = 1$. If $|c_0 c_s| > 1$, then $X$ is a *finite-dimensional solenoid*, i.e. $\hat{X}$ is isomorphic to a subgroup of $\mathbb{Q}^s$ (Example (2) and Example 5.3 (3)).

(4) Let $\alpha$ be an ergodic automorphism of a compact, abelian group $X$, and let $\mathfrak{M} = \hat{X}$ be the $\mathfrak{R}_1$-module arising from $\alpha$ via Lemma 5.1. Then every prime ideal $\mathfrak{p} \subset \mathfrak{R}_1$ associated with $\mathfrak{M}$ is principal, and $\mathfrak{p} \neq (f)$ for any cyclotomic polynomial $f \subset \mathfrak{R}_1$ (Proposition 6.6 and Example (3)). ⊡

Further examples of expansive automorphisms of compact, abelian groups will appear in Chapter 3.

Examples 6.18. In the following illustrations of Theorem 6.5 we consider $\mathfrak{R}_2$-modules of the form $\mathfrak{M} = \mathfrak{R}_2/\mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{R}_2$ is an ideal, realize $X = X^{\mathfrak{M}} \subset \mathbb{T}^{\mathbb{Z}^2}$ as in Example 5.2 (2), and denote by $\alpha = \alpha^{\mathfrak{M}}$ the shift-action of $\mathbb{Z}^2$ on $X$.

(1) Let $\mathfrak{a} = (1+u_1+u_2)$. Since $\mathfrak{a}$ is prime, $\mathfrak{M}$ is associated with $\mathfrak{a}$. Corollary 6.11 shows that $\alpha$ is ergodic, and Corollary 6.12 implies that $\alpha$ is mixing. Since $((-1+i\sqrt{-3})/2, (-1-i\sqrt{-3})/2) \in V_{\mathbb{C}}(\mathfrak{a}) \cap \mathbb{S}^2$, $\alpha$ is not expansive by Theorem 6.5 (4). Moreover, $V_{\mathbb{C}}(\mathfrak{a}) \cap \Omega(3\mathbb{Z}^2) \neq \varnothing$, so that $\mathrm{Fix}_{3\mathbb{Z}^2}(\alpha)$ is infinite by Theorem 6.5 (3). note that $\mathrm{Fix}_{3\mathbb{Z}^2}(\alpha)$ consists of all points

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & c & a & \cdot \\ \cdot & a+2b+c & a+b+2c & 2a+b+c & a+2b+c & \cdot \\ \cdot & -a-b & -b-c & -a-c & -a-b & \cdot \\ \cdot & a & b & c & a & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

with $a, b, c \in \mathbb{T}$ and $3a + 3b + 3c = 0 \pmod 1$. In particular, the connected component of the identity $\mathrm{Fix}_{3\mathbb{Z}^2}(\alpha)^\circ \subset \mathrm{Fix}_{3\mathbb{Z}^2}(\alpha)$ is isomorphic to $\mathbb{T}^2$.

(2) Let $\mathfrak{a} = (2 + u_1 + u_2) \subset \mathfrak{R}_2$. The action $\alpha$ is ergodic, mixing, non-expansive, and $(-1,-1) \in V_{\mathbb{C}}(\mathfrak{a}) \cap \Omega(2\mathbb{Z}^2) \neq \varnothing$. The points in $\mathrm{Fix}_{2\mathbb{Z}^2}(\alpha)$ are of the form

$$\begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & a & \cdot \\ \cdot & -2a-b & -a-2b & -2a-b & \cdot \\ \cdot & a & b & a & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

with $4a + 4b = 1 \pmod 1$, and $\mathrm{Fix}_{2\mathbb{Z}^2}(\alpha)^\circ$ is isomorphic to $\mathbb{T}$.

(3) Let $\mathfrak{a} = (2 - u_1 - u_2) \subset \mathfrak{R}_2$. Then $\alpha$ is again ergodic, mixing, and non-expansive. Since $(1,1) \in V_{\mathbb{C}}(\mathfrak{a})$, $\alpha$ has uncountably many fixed points, and hence $\mathrm{Fix}_\Lambda(\alpha)$ is uncountable for every subgroup $\Lambda \subset \mathbb{Z}^d$.

(4) If $\mathfrak{a} = (3 + u_1 + u_2) \subset \mathfrak{R}_2$, then $\alpha$ is ergodic, mixing, expansive, and the expansiveness of $\alpha$ implies directly that $\mathrm{Fix}_\Lambda(\alpha)$ is finite for every subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index.

(5) In Example 5.3 (5) we considered the ideal $\mathfrak{a} = (2, 1 + u_1 + u_2) \subset \mathfrak{R}_2$. Then $V_{\mathbb{C}}(\mathfrak{a}) = \varnothing$, and Theorem 6.5 (4) re-establishes the fact that $\alpha$ is expansive. Since the polynomial $1 + u_1 + u_2$ is prime in $\mathfrak{R}_2^{(2)} = \mathbb{Z}_{/2}[u_1^{\pm 1}, u_2^{\pm 2}]$, the ideal $\mathfrak{a}$ is prime, and as in Corollary 6.12 we see that $\alpha$ is mixing (since every prime polynomial in $\mathbb{Z}_{/2}[u]$ divides a polynomial of the form $u^l - 1$ for some $l \geq 1$, (the analogue of) Corollary 6.12 reduces to checking that $1 + u_1 + u_2 \in \mathfrak{R}_2^{(2)}$ is not a polynomial in the single variable $u^{\mathbf{n}}$ for some $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^2$).

(6) Let $\mathfrak{a} = (4, 1 + u_1 - u_2 + 2u_2^2 + u_1 u_2) \subset \mathfrak{R}_2$. Since every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$ must contain both the polynomial $1 + u_1 - u_2 + 2u_2^2 + u_1 u_2$ and the constant 2, the prime ideals associated with $\mathfrak{M}$ are given by $\mathfrak{p}_1 = (2, 1 - u_1)$ and $\mathfrak{p}_2 = (2, 1 - u_2)$. In particular, $\alpha$ is ergodic and expansive, but not mixing: the automorphisms $\alpha_{(1,0)}$ and $\alpha_{(0,1)}$ are non-ergodic, whereas $\alpha_{(1,1)}$ is ergodic.

(7) Let $\mathfrak{a} = (6 - 2u_1, 2 - 3u_1 - 5u_2^2)$. The prime ideals associated with $\mathfrak{M} = \mathfrak{R}_2/\mathfrak{a}$ are given by $\mathfrak{p}_1 = (3 - u_1, 7 + 5u_2^2)$, $\mathfrak{p}_2 = (3, 1 + u_2)$, $\mathfrak{p}_3 = (3, 1 - u_2)$, and the $\mathbb{Z}^2$-action $\alpha$ is ergodic and expansive, but non-mixing. In this example $\alpha_{(0,1)}$ is non-ergodic (because of $\mathfrak{p}_3$), but $\alpha_{(1,0)}$ is ergodic.

(8) If $\mathfrak{a} = (1 + u_1 + u_1^2, 1 - u_2)$ then $\alpha$ is non-ergodic, since $\mathfrak{a}$ is prime and contains $\{u^{3\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^2\}$. ⊡

Concluding Remarks 6.19. (1) Most of the material in this section is taken from [94]. For Example 6.17 (2) we refer to [71].

(2) If $d \geq 2$, Corollary 6.10 is incorrect without the assumption that $(X, \alpha)$ satisfies the d.c.c.: indeed, let, for every $\mathbf{n} \in \mathbb{Z}^d$, $\mathfrak{N}_{\mathbf{n}} = \mathfrak{R}_d/(u^{\mathbf{n}} - 1)$. Then $\mathfrak{N}_{\mathbf{n}}$ is an $\mathfrak{R}_d$-module, and the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{N}_{\mathbf{n}}}$ is ergodic by Corollary 6.11. We denote by $\mathfrak{M} = \sum_{\mathbf{n} \in \mathbb{Z}^d} \mathfrak{N}_{\mathbf{n}}$ the direct sum of the modules $\mathfrak{N}_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, and write a typical element $a \in \mathfrak{M}$ as $a = (a_{\mathbf{n}})$ with $a_{\mathbf{n}} \in \mathfrak{N}_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. The $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{M}}$ arising from the $\mathfrak{R}_d$-module $\mathfrak{M}$ via Lemma 5.1 is ergodic by Lemma 1.2. However, $\alpha_{\mathbf{n}}$ is non-ergodic for *every* $\mathbf{n} \in \mathbb{Z}^d$: if $\mathbf{n} = \mathbf{0}$, this assertion is obvious, and if $\mathbf{n} \neq \mathbf{0}$, then the non-zero element $a(\mathbf{n}) \in \mathfrak{M}$ defined by

$$a(\mathbf{n})_{\mathbf{m}} = \begin{cases} 1 & \text{for } \mathbf{m} = \mathbf{n} \\ 0 & \text{for } \mathbf{m} \neq \mathbf{n} \end{cases}$$

satisfies that $u^{\mathbf{n}} a(\mathbf{n}) = a(\mathbf{n})$, and hence $\alpha_{\mathbf{n}}$ is non-ergodic by Lemma 1.2 (applied to the $\mathbb{Z}$-action $k \mapsto \alpha_{k\mathbf{n}}$).

(3) Let $\mathfrak{M}$ be a countable $\mathfrak{R}_d$-module, and define $(X^{\mathfrak{M}}, \alpha^{\mathfrak{M}})$ by Lemma 5.1. For every $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) \in \mathfrak{R}_d$ we define a group homomorphism

$$\alpha_f^{\mathfrak{M}} = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})\alpha_{\mathbf{n}}^{\mathfrak{M}} \colon X^{\mathfrak{M}} \longmapsto X^{\mathfrak{M}} \tag{6.14}$$

by setting

$$\alpha_f^{\mathfrak{M}}(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})\alpha_{\mathbf{n}}^{\mathfrak{M}}(x)$$

for every $x \in X^{\mathfrak{M}}$, and note that $\alpha_f^{\mathfrak{M}}$ commutes with $\alpha^{\mathfrak{M}}$ (i.e. $\alpha_f^{\mathfrak{M}} \cdot \alpha_{\mathbf{n}}^{\mathfrak{M}} = \alpha_{\mathbf{n}}^{\mathfrak{M}} \cdot \alpha_f^{\mathfrak{M}}$ for every $\mathbf{n} \in \mathbb{Z}^d$), and that $\alpha_f^{\mathfrak{M}}$ is dual to the homomorphism

$$f_{\mathfrak{M}} \colon \mathfrak{M} \longmapsto \mathfrak{M} \tag{6.15}$$

consisting of multiplication by $f$. In particular, $\alpha_f^{\mathfrak{M}}$ is surjective if and only if $f_{\mathfrak{M}}$ is injective, i.e. if and only if $f$ does not lie in any prime ideal associated with $\mathfrak{M}$ (cf. (6.4)). If $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \mathfrak{R}_d$, then (5.9) shows that

$$X^{\mathfrak{R}_d/\mathfrak{a}} = \{x \in \mathbb{T}^{\mathbb{Z}^d} = X^{\mathfrak{R}_d} : \alpha_f^{\mathfrak{R}_d}(x) = \mathbf{0}_X \text{ for every } f \in \mathfrak{a}\}, \tag{6.16}$$

and *every* $\alpha$-commuting homomorphism $\psi \colon X^{\mathfrak{M}} \longmapsto X^{\mathfrak{M}}$ is of the form $\psi = \alpha_f^{\mathfrak{M}}$ for some $f \in \mathfrak{R}_d$: indeed, if $\hat{\psi} \colon \mathfrak{R}_d/\mathfrak{a} \longmapsto \mathfrak{R}_d/\mathfrak{a}$ is the homomorphism dual to $\psi$, then $\hat{\psi}(1) = f + \mathfrak{a}$ for some $f \in \mathfrak{R}_d$, and $\psi = \alpha_f^{\mathfrak{R}_d/\mathfrak{a}}$. For every ideal $\mathfrak{a} \subset \mathfrak{R}_d$ we set $\mathfrak{a}^\perp = X^{\mathfrak{R}_d/\mathfrak{a}} = \widehat{\mathfrak{R}_d/\mathfrak{a}} \subset \widehat{\mathfrak{R}_d} = \mathbb{T}^{\mathbb{Z}^d}$, and observe that $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is the restriction of the shift-action $\sigma$ of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ to $\mathfrak{a}^\perp$. For every $f \in \mathfrak{R}_d$ the sequence

$$0 \longrightarrow (\mathfrak{a} + (f))^\perp \longrightarrow \mathfrak{a}^\perp \xrightarrow{\alpha_f^{\mathfrak{R}_d}} \mathfrak{b}^\perp \longrightarrow 0, \tag{6.17}$$

is exact, where

$$\mathfrak{b} = \{g \in \mathfrak{R}_d : fg \in \mathfrak{a}\}. \tag{6.18}$$

In particular, $\alpha_f^{\mathfrak{R}_d/\mathfrak{a}} \colon \mathfrak{a}^\perp \longmapsto \mathfrak{a}^\perp$ is surjective if and only if $\mathfrak{a} = \mathfrak{b}$.

(4) Let $p > 1$ be a rational prime, and let $\alpha$ be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group $X$ with the property that $px = 0$ for every $x \in X$. If $\mathfrak{M} = \hat{X}$ is the $\mathfrak{R}_d$-module arising from lemma 5.1, then $pa = 0$ for every $a \in \mathfrak{M}$, so that $\mathfrak{M}$ may be viewed as an $\mathfrak{R}_d^{(p)}$-module. Conversely, suppose that $\mathfrak{N}$ is a countable $\mathfrak{R}_d^{(p)}$-module. Exactly as in (5.1)–(5.6) we can define a $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{N}}$ on the dual group $X = X^{\mathfrak{N}} = \widehat{\mathfrak{N}}$ of $\mathfrak{N}$. Since $pa = 0$ for every $a \in \mathfrak{N}$, the group $X$ is totally disconnected, and $x^p = \mathbf{1}_X$ for every $x \in X$. Since $\mathfrak{R}_d^{(p)}$ is a quotient ring of $\mathfrak{R}_d$, $\mathfrak{N}$ is also an $\mathfrak{R}_d$-module, and we write $\mathfrak{N}'$ instead of $\mathfrak{N}$ if we wish to emphasize that $\mathfrak{N}$ is viewed as an $\mathfrak{R}_d$-module. If $\mathfrak{N}$ is Noetherian (either as an $\mathfrak{R}_d$-module or as an $\mathfrak{R}_d^{(p)}$-module—the two conditions

are obviously equivalent), then we can realize $(X^{\mathfrak{N}}, \alpha^{\mathfrak{N}}) = (X^{\mathfrak{N}'}, \alpha^{\mathfrak{N}'})$ as the shift-action $\sigma$ of $\mathbb{Z}^d$ on a closed, shift-invariant subgroup $X \subset (\mathbb{T}^k)^{\mathbb{Z}^d}$ for some $k \geq 1$ (Example 5.2 (3)–(4)). Since $px = \mathbf{0}_X$ for every $x \in X$, we know that $x_{\mathbf{n}} \in (F_p)^k$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $F_p = \{\frac{k}{p} : k = 0, \dots, p-1\} \subset \mathbb{T}$, and the obvious identification of $F_p$ with the prime field $\mathbb{F}_p$ allows us to regard $X$ (and hence $X^{\mathfrak{N}}$) as a closed, shift-invariant subgroup of $(\mathbb{F}_p^k)^{\mathbb{Z}^d}$, and $\alpha^{\mathfrak{N}}$ as the shift-action on $X$.

In particular, if $\mathfrak{a} \subset \mathfrak{R}_d^{(p)}$ is an ideal, and if $\mathfrak{N} = \mathfrak{R}_d^{(p)}/\mathfrak{a}$, then we may regard $\alpha^{\mathfrak{N}} = \alpha^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$ as the shift-action of $\mathbb{Z}^d$ on the subgroup

$$X^{\mathfrak{R}_d^{(p)}/\mathfrak{a}} = \left\{ x = (x_{\mathbf{m}}) \in \mathbb{F}_p^{\mathbb{Z}^d} : \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = \mathbf{0}_{\mathbb{F}_p} \text{ for all } f \in \mathfrak{a},\, \mathbf{m} \in \mathbb{Z}^d \right\} \tag{6.19}$$

of $\mathbb{F}_p^{\mathbb{Z}^d}$. Conversely, if $X \subset \mathbb{F}_p^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup, then

$$X^{\perp} = \mathfrak{a} \subset \mathfrak{R}_d^{(p)} \cong \widehat{\mathbb{F}_p^{\mathbb{Z}^d}} \tag{6.20}$$

is an ideal, $X \cong X^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$, and the isomorphism between $X$ and $X^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$ carries the shift-action $\sigma$ of $\mathbb{Z}^d$ on $X$ to $\alpha^{\mathfrak{R}_d^{(p)}/\mathfrak{a}}$.

Every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ associated with an $\mathfrak{R}_d^{(p)}$-module $\mathfrak{N}$ defines a prime ideal $\mathfrak{p}' = \{f \in \mathfrak{R}_d : f_{/p} \in \mathfrak{p}\} \subset \mathfrak{R}_d$, and $\mathfrak{p}'$ varies over the set of prime ideals in $\mathfrak{R}_d$ associated with $\mathfrak{N}'$ as $\mathfrak{p}$ varies over the prime ideals in $\mathfrak{R}_d^{(p)}$ associated with $\mathfrak{N}$. As we have seen in Example 6.18 (5), the dynamical properties of $\alpha^{\mathfrak{N}'}$ expressed in terms of the associated primes $\mathfrak{p}' \subset \mathfrak{R}_d$ of $\mathfrak{N}'$ have an analogous expression in terms of the prime ideals $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ associated with $\mathfrak{N}$. In particular, $\alpha = \alpha^{\mathfrak{N}} = \alpha^{\mathfrak{N}'}$ *is non-ergodic if and only if* $V(\mathfrak{p})$ *is finite for some prime ideal* $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ *associated with* $\mathfrak{N}$, and $\alpha$ *is mixing if and only if no prime ideal* $\mathfrak{p} \subset \mathfrak{R}_d^{(p)}$ *associated with* $\mathfrak{N}$ *contains a polynomial in a single variable* $u^{\mathbf{n}}$, $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$. Furthermore, *if* $\mathfrak{N}$ *is Noetherian, then* $\mathrm{Fix}_{\Lambda}(\alpha)$ *is finite for every subgroup* $\Lambda \subset \mathbb{Z}^d$ *of finite index, and* $\alpha$ *is expansive.*

The algebraic advantage in viewing an $\mathfrak{R}_d$-module $\mathfrak{M}$ with $pa = 0$ for all $a \in \mathfrak{M}$ as an $\mathfrak{R}_d^{(p)}$-module is that $\mathfrak{R}_d^{(p)}$ is a ring of polynomials with coefficients in the *field* $\mathbb{F}_p$, which simplifies the ideal structure of $\mathfrak{R}_d^{(p)}$ when compared with that of $\mathfrak{R}_d$. As far as the dynamics are concerned there is, of course, no difference between viewing $\mathfrak{M}$ as a module over either of the rings $\mathfrak{R}_d$ or $\mathfrak{R}_d^{(p)}$.

## 7. The dynamical system defined by a point

The results in Section 6 show that many questions about $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups can be reduced to questions about $\mathbb{Z}^d$-actions of the form $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal. In this section we consider prime ideals of the form $\mathfrak{p} = \mathfrak{j}_c = \{f \in \mathfrak{R}_d : f(c) = 0\}$ with $c =$

$(c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$. The groups $X^{\mathfrak{R}_d/\mathfrak{j}_c}$ arising from these ideals via Lemma 5.1 turn out to be connected and finite-dimensional (i.e. finite-dimensional tori or solenoids); conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal such that $X^{\mathfrak{R}_d/\mathfrak{p}}$ is connected and finite-dimensional, then $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^\times)^d$ (Corollary 7.4).

Let $\mathbb{K}$ be an algebraic number field, i.e. a finite extension of $\mathbb{Q}$. A *valuation* of $\mathbb{K}$ is a homomorphism $\phi\colon \mathbb{K} \longmapsto \mathbb{R}^+$ with the property that $\phi(a) = 0$ if and only if $a = 0$, $\phi(ab) = \phi(a)\phi(b)$, and $\phi(a+b) \leq c \cdot \max\{\phi(a), \phi(b)\}$ for all $a, b \in \mathbb{K}$ and some $c \in \mathbb{R}$ with $c \geq 1$. The valuation $\phi$ is *non-trivial* if $\phi(\mathbb{K}) \supsetneqq \{0, 1\}$, *non-archimedean* if $\phi$ is non-trivial and we can set $c = 1$, and *archimedean* otherwise. Two valuations $\phi, \psi$ of $\mathbb{K}$ are *equivalent* if there exists an $s > 0$ with $\phi(a) = \psi(a)^s$ for all $a \in \mathbb{K}$. An equivalence class $v$ of non-trivial valuations of $\mathbb{K}$ is called a *place* of $\mathbb{K}$, and $v$ is *finite* if $v$ contains a non-archimedean valuation, and *infinite* otherwise. If $v$ is finite, all valuations $\phi \in v$ are non-archimedean.

Let $v$ be a place of $\mathbb{K}$, and let $\phi \in v$ be a valuation. A sequence $(a_n,\ n \geq 1)$ is *Cauchy* with respect to $\phi$ if there exists, for every $\varepsilon > 0$, an integer $N \geq 1$ such that $\phi(a_m - a_n) < \varepsilon$ whenever $m, n \geq N$. It is clear that this definition does not depend on the valuation $\phi \in v$, so that we may call $(a_n)$ a Cauchy sequence for $v$. Two Cauchy sequences $(a_n)$ and $(b_n)$ for $v$ are *equivalent* if $\lim_{n\to\infty} \phi(a_n - b_n) = 0$, and this notion of equivalence again only depends on $v$ and not on $\phi$. With respect to the obvious operations the set of equivalence classes of Cauchy sequences for $v$ is a field, denoted by $\mathbb{K}_v$, which contains $\mathbb{K}$ as a dense subfield (every $a \in \mathbb{K}$ is identified with the equivalence class of the constant Cauchy sequence $(a, a, a, \dots)$ in $\mathbb{K}_v$). The field $\mathbb{K}_v$ is the *completion of* $\mathbb{K}$ *in the $v$-adic topology.*

Ostrowski's Theorem (Theorem 2.2.1 in [16]) states that every non-trivial valuation $\phi$ of $\mathbb{Q}$ is either equivalent to the absolute value (i.e. there exists a $t > 0$ with $\phi(a)^t = |a|$ for every $a \in \mathbb{Q}$), or to the $p$-adic valuation for some rational prime $p \geq 2$ (i.e. there exists a $t > 0$ such that $\phi(\frac{m}{n})^t = p^{(n'-m')} = |\frac{m}{n}|_p$ for all $\frac{m}{n} \in \mathbb{Q}$, where $m = p^{m'}m''$, $n = p^{n'}n''$, and neither $m''$ nor $n''$ are divisible by $p$). It is easy to see that the valuations $|\cdot|_\infty$, $|\cdot|_p$, $|\cdot|_q$ are mutually inequivalent whenever $p, q$ are distinct rational primes, i.e. that the places of $\mathbb{Q}$ are indexed by the set $\Pi \cup \{\infty\}$, where $\Pi \subset \mathbb{N}$ denotes the set of rational primes. The completion $\mathbb{Q}_\infty$ of $\mathbb{Q}$ is equal to $\mathbb{R}$, and for every rational prime $p$ the completion $\mathbb{Q}_p$ of $\mathbb{Q}$ is the field of $p$-adic rationals.

For every valuation $\phi$ of $\mathbb{K}$, the restriction of $\phi$ to $\mathbb{Q} \subset \mathbb{K}$ is a valuation of $\mathbb{Q}$ and is equivalent either to $|\cdot|_\infty$ or to $|\cdot|_p$ for some rational prime $p$. In the first case the place $v \ni \phi$ is infinite (or *lies above* $\infty$), and in the second case $v$ *lies above* $p$ (or $p$ *lies below* $v$). We denote by $w$ the place of $\mathbb{Q}$ below $v$ and observe that $\mathbb{K}_v$ is a finite-dimensional vector space over the locally compact, metrizable field $\mathbb{Q}_w$ and hence locally compact and metrizable in its own right. Choose a Haar measure $\lambda_v$ on $\mathbb{K}_v$ (with respect to addition), fix a compact set $C \subset \mathbb{K}_v$ with non-empty interior, and write $\mathrm{mod}_{\mathbb{K}_v}(a) = \lambda_v(aC)/\lambda_v(C)$ for the *module* of an element $a \in \mathbb{K}_v$. The map $\mathrm{mod}_{\mathbb{K}_v} \colon \mathbb{K} \longmapsto \mathbb{R}^+$ is continuous, independent of the choice of $\lambda_v$, and its restriction to $\mathbb{K}$ is a valuation in $v$ which is denoted by $|\cdot|_v$.

Above every place $v$ of $\mathbb{Q}$ there are at least one and at most finitely many places of $\mathbb{K}$. Indeed, if $\mathbb{K} = \mathbb{Q}(a_1, \dots, a_n)$ with $\{a_1, \dots, a_n\} \subset \overline{\mathbb{Q}}$, and if $f$ is the minimal polynomial of $a_1$ over $\mathbb{Q}$, then $f$ is irreducible over $\mathbb{Q}$, but $f$ may be reducible over $\mathbb{Q}_v$; we write $f = f_1 \cdot \dots \cdot f_k$ for the decomposition of $f$ into irreducible factors over $\mathbb{Q}_v$ and consider the field $\mathbb{Q}_v[x]/(f_i)$, where $(f_i)$ denotes the principal ideal in the ring $\mathbb{Q}_v[x]$ generated by $f_i$. We define an injective field homomorphism $\zeta\colon \mathbb{K}^{(1)} = \mathbb{Q}_v(a_1) \longmapsto \mathbb{Q}_v[x]/(f_i)$ by setting $\zeta(a_1) = x$ and $\zeta(b) = b$ for every $b \in \mathbb{Q}_v$ and put $\phi_i(a) = \mathrm{mod}_{Q_v[x]/(f_i)}(\zeta(a))$ for every $a \in \mathbb{K}^{(1)}$. Then $\phi_i$ is a valuation of $\mathbb{K}^{(1)}$ whose place $w_i$ lies above $v$. The places $w_1, \dots, w_k$ are all distinct, and they are the only places of $\mathbb{K}^{(1)}$ above $v$ (Theorem III.1 in [109]). In exactly the same way we find finitely many places of $\mathbb{K}^{(2)} = \mathbb{K}^{(1)}(a_2) = \mathbb{Q}(a_1, a_2)$ above each place of $\mathbb{K}^{(1)}$, and after $n$ steps we obtain that there are at least one and at most finitely many places of $\mathbb{K}$ above each place of $\mathbb{Q}$. A place $v$ of $\mathbb{K}$ is infinite if and only it lies above $\infty$; in this case $v$ is either *real* (if $\mathbb{K}_v = \mathbb{R}$) or *complex* (if $\mathbb{K}_v = \mathbb{C}$).

We write $P^{\mathbb{K}}$, $P^{\mathbb{K}}_{\mathfrak{f}}$, and $P^{\mathbb{K}}_{\infty}$, for the sets of places, finite places, and infinite places of $\mathbb{K}$. For every $v \in P^{\mathbb{K}}$, $\mathcal{R}_v = \{r \in \mathbb{K}_v : |r|_v \le 1\}$ is a compact subset of $\mathbb{K}_v$. If $v \in P^{\mathbb{K}}_{\mathfrak{f}}$, then $\mathcal{R}_v$ is, in addition, open, and is the unique maximal compact subring of $\mathbb{K}_v$; furthermore there exists a *prime element* $\pi_v \in \mathcal{R}_v$ such that $\pi_v \mathcal{R}_v$ is the unique maximal ideal of $\mathcal{R}_v$. For every $v \in P^{\mathbb{K}}_{\mathfrak{f}}$ we set $\mathfrak{o}_v = \mathbb{K} \cap \mathcal{R}_v$, and we note that $\mathfrak{o}_{\mathbb{K}} = \bigcap_{v \in P^{\mathbb{K}}_{\mathfrak{f}}} \mathfrak{o}_v$ is the ring of integral elements in $\mathbb{K}$ (Theorem V.1 in [109]). The set

$$\begin{aligned} \mathbb{K}_{\mathbb{A}} = \Bigl\{ \omega = (\omega_v,\, v \in P^{\mathbb{K}}) \in \prod_{v \in P^{\mathbb{K}}} \mathbb{K}_v : \\ |\omega_v|_v \le 1 \text{ for all but finitely many } v \in P^{\mathbb{K}} \Bigr\}, \end{aligned} \tag{7.1}$$

furnished with that topology in which the subgroup

$$\begin{aligned} \bigl\{ \omega = (\omega_v,\, v \in P^{\mathbb{K}}) \in \mathbb{K}_{\mathbb{A}} : |\omega_v|_v \le 1 \text{ for every } v \in P^{\mathbb{K}}_{\mathfrak{f}} \bigr\} \\ \cong \prod_{v \in P^{\mathbb{K}}_{\infty}} \mathbb{K}_v \times \prod_{v \in P^{\mathbb{K}}_{\mathfrak{f}}} \mathcal{R}_v \end{aligned}$$

carries the product topology and is open in $\mathbb{K}_{\mathbb{A}}$, is the locally compact *adele ring* of $\mathbb{K}$. The diagonal embedding $i\colon \xi \mapsto (\xi, \xi, \dots)$ of $\mathbb{K}$ in $\mathbb{A}_{\mathbb{K}}$ maps $\mathbb{K}$ to a discrete, co-compact subring of $\mathbb{K}_{\mathbb{A}}$ (cf. [16], [109]).

We fix a non-trivial character $\chi \in i(\mathbb{K})^{\perp} \subset \widehat{\mathbb{K}_{\mathbb{A}}}$ and define, for every $a \in \mathbb{K}$, a character $\chi_a \in i(\mathbb{K})^{\perp} \subset \widehat{\mathbb{K}_{\mathbb{A}}}$ by setting

$$\chi_a(\omega) = \chi(i(a)\omega)$$

for every $\omega \in \mathbb{K}_{\mathbb{A}}$. By [16] or [109], the map $a \mapsto \chi_a$ is an isomorphism of the discrete, additive group $\mathbb{K}$ onto $i(\mathbb{K})^{\perp} \subset \widehat{\mathbb{K}_{\mathbb{A}}}$. The resulting identification

$$\widehat{\mathbb{K}} \cong \mathbb{K}_{\mathbb{A}}/i(\mathbb{K}) \tag{7.2}$$

depends, of course, on the chosen character $\chi$. In order to make the isomorphism (7.2) a little more canonical we consider, for every $w \in P^{\mathbb{K}}$, the subgroup

$$\Omega(\{w\})' = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : \omega_v = 0 \ \text{ for every } \ v \neq w\} \cong \mathbb{K}_w$$

of $\mathbb{K}_{\mathbb{A}}$ and denote by $\chi^{(w)} \in \widehat{\mathbb{K}_w}$ the character induced by the restriction of $\chi$ to $\Omega(\{w\})'$. After replacing $\chi$ by a suitable $\chi_a$, $a \in \mathbb{K}$, if necessary, we may assume that the induced characters $\chi^{(w)} \in \widehat{\mathbb{K}_w}$, $w \in P_{\mathfrak{f}}^{\mathbb{K}}$, satisfy that

$$\begin{gathered} \mathcal{R}_w \subset \ker(\chi^{(w)}) = \{\omega \in \mathbb{A}_w : \chi^{(w)}(\omega) = 1\}, \\ \pi_w^{-1}\mathcal{R}_w \not\subset \ker(\chi^{(w)}) \end{gathered} \tag{7.3}$$

for every $w \in P_{\mathfrak{f}}^{\mathbb{K}}$, where $\pi_w \in \mathcal{R}_w$ is the prime element appearing in the preceding paragraph (cf. [109]). With this choice of $\chi$ we have that

$$\chi \in \bigl(i(\mathbb{K}) + \Omega(P_{\mathfrak{f}}^{\mathbb{K}})'\bigr)^{\perp},$$

where

$$\Omega(P_{\mathfrak{f}}^{\mathbb{K}})' = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : \omega_v = 0 \ \text{ for every } \ v \in P_{\infty}^{\mathbb{K}} = P^{\mathbb{K}} \smallsetminus P_{\mathfrak{f}}^{\mathbb{K}}\}.$$

Now consider a finite subset $F \subset P^{\mathbb{K}}$ which contains $P_{\infty}^{\mathbb{K}}$, denote by

$$i_F \colon \mathbb{K} \longmapsto \prod_{v \in F} \mathbb{K}_v \tag{7.4}$$

the diagonal embedding $r \mapsto (r, \dots, r)$, $r \in \mathbb{K}$, put

$$R_F = \{a \in \mathbb{K} : |a|_v \leq 1 \ \text{ for every } \ v \notin F\}, \tag{7.5}$$

and observe that $i_F(R_F)$ is a discrete, additive subgroup of $\prod_{v \in F} \mathbb{K}_v$. If

$$\begin{gathered} \Omega = \Omega(F) = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : |\omega_v|_v \leq 1 \ \text{ for every } \ v \in P^{\mathbb{K}} \smallsetminus F\}, \\ \Omega' = \Omega(P^{\mathbb{K}} \smallsetminus F)' = \{\omega = (\omega_v) \in \mathbb{K}_{\mathbb{A}} : \omega_v = 0 \ \text{ for every } \ v \in F\}, \\ \Omega'' = \Omega \cap \Omega', \end{gathered}$$

then $i(\mathbb{K}) + \Omega'' = i(\mathbb{K}) + \Omega'$, and (7.3) implies that $\chi \in (i(\mathbb{K}) + \Omega'')^{\perp} = (i(\mathbb{K}) + \Omega')^{\perp}$ and

$$R_F = \{a \in \mathbb{K} : \chi_a \in (i(\mathbb{K}) + \Omega')^{\perp}\}.$$

Hence

$$\widehat{R_F} = \mathbb{K}_{\mathbb{A}}/(i(\mathbb{K}) + \Omega') \cong \Bigl(\prod_{v \in F} \mathbb{K}_v\Bigr) \Big/ i_F(R_F). \tag{7.6}$$

Let $d \geq 1$, $c = (c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$, and $\mathfrak{j}_c = \{f \in \mathfrak{R}_d : f(c) = 0\}$. We wish to investigate the dynamical system $(X, \alpha) = (X^{\mathfrak{R}_d/\mathfrak{j}_c}, \alpha^{\mathfrak{R}_d/\mathfrak{j}_c})$ determined by $c$. Denote by $\mathbb{K} = \mathbb{Q}(c)$ the algebraic number field generated by $\{c_1, \dots, c_d\}$ and put

$$F(c) = \{v \in P_{\mathrm{f}}^{\mathbb{K}} : |c_i|_v \neq 1 \text{ for some } i \in \{1, \dots, d\}\}, \tag{7.7}$$

which is finite by Theorem III.3 in [109], and

$$R_c = R_{P(c)}, \tag{7.8}$$

where $P(c) = P_\infty^{\mathbb{K}} \cup F(c)$. Then $R_c$ is an $\mathfrak{R}_d$-module under the action $(f, a) \mapsto f(c)a$, and we define the $\mathbb{Z}^d$-action

$$\alpha^{(c)} = \alpha^{R_c} \tag{7.9}$$

on the compact group

$$Y^{(c)} = \widehat{R_c} = \Bigl(\prod_{v \in P(c)} \mathbb{K}_v\Bigr) \Bigm/ i_F(R_c) \tag{7.10}$$

by (5.5)–(5.6), where we use (7.6) to identify $\widehat{R_c}$ and $\bigl(\prod_{v\in P(c)} \mathbb{K}_v\bigr) \bigm/ i_F(R_c)$.

THEOREM 7.1. *There exists a continuous, surjective, finite-to-one homomorphism* $\phi\colon Y^{(c)} \longmapsto X^{\mathfrak{R}_d/\mathfrak{j}_c}$ *such that the diagram*

$$\begin{array}{ccc} Y^{(c)} & \xrightarrow{\alpha_{\mathbf{m}}^{(c)}} & Y^{(c)} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ X^{\mathfrak{R}_d/\mathfrak{j}_c} & \xrightarrow[\alpha_{\mathbf{m}}^{\mathfrak{R}_d/\mathfrak{j}_c}]{} & X^{\mathfrak{R}_d/\mathfrak{j}_c} \end{array} \tag{7.11}$$

*commutes for every* $\mathbf{m} \in \mathbb{Z}^d$.

PROOF. The evaluation map $\eta_c\colon f \mapsto f(c)$ induces an isomorphism $\boldsymbol{\eta}$ of the $\mathfrak{R}_d$-module $\mathfrak{R}_d/\mathfrak{j}_c$ with the submodule $\eta_c(\mathfrak{R}_d) \subset R_c \subset \mathbb{K}$; in particular

$$\boldsymbol{\eta}(\hat{\alpha}_{\mathbf{m}}^{\mathfrak{R}_d/\mathfrak{j}_c}(a)) = \hat{\alpha}_{\mathbf{m}}^{\eta_c(\mathfrak{R}_d)}(\boldsymbol{\eta}(a)) = \hat{\alpha}_{\mathbf{m}}^{R_c}(\boldsymbol{\eta}(a)) \tag{7.12}$$

for every $a \in \mathfrak{R}_d/\mathfrak{j}_c$ and $\mathbf{m} \in \mathbb{Z}^d$.

We claim that $R_c/\eta_c(\mathfrak{R}_d)$ is finite. Indeed, since $\mathbb{K} = \mathbb{Q}(c)$ is algebraic, every $a \in \mathbb{K}$ can be written as $a = b/m$ with $b \in \mathbb{Z}[c] = \mathbb{Z}[c_1, \dots, c_d]$ and $m \geq 1$. In particular, since the ring of integers $\mathfrak{o}(c) = \mathfrak{o}_{\mathbb{K}} \subset \mathbb{K}$ is a finitely generated $\mathbb{Z}$-module, there exist positive integers $m_0, M_0$ with $m_0\mathfrak{o}(c) \subset \mathbb{Z}[c] \subset \eta_c(\mathfrak{R}_d)$ and $|\mathfrak{I}_c/\eta_c(\mathfrak{R}_d)| \leq |\mathfrak{o}(c)/m_0\mathfrak{o}(c)| = M_0 < \infty$.

According to the definition of $F(c)$ there exists, for every $v \in F(c)$, an element $a_v \in \eta_c(\mathfrak{R}_d)$ such that $|a_v|_v > 1$ and $|a_v|_w = 1$ for all $w \in P_{\mathrm{f}}^{\mathbb{K}} \smallsetminus F(c)$. Then $|a_v^n\mathfrak{o}(c)/\eta_c(\mathfrak{R}_d)| \leq M_0$ and $|(\sum_{v\in F(c)} a_v^n\mathfrak{o}(c))/\eta_c(\mathfrak{R}_d)| \leq M_0^{|F(c)|}$ for all

$n > 0$. As $n \to \infty$, $\sum_{v \in F(c)} a_v^n \mathfrak{o}(c)$ increases to $R_c$, and we conclude that $|R_c/\eta_c(\mathfrak{R}_d)| \le M_0^{|F(c)|} < \infty$.

The inclusion map $\mathfrak{R}_d/\mathfrak{j}_c \cong \eta_c(\mathfrak{R}_d) \hookrightarrow R_c$ induces a dual, surjective, finite-to-one homomorphism $\phi\colon Y^{(c)} \longmapsto X = \widehat{\mathfrak{R}_d/\mathfrak{j}_c}$, and the diagram (7.11) commutes by (7.12). □

This comparison between $R_c$ and $\eta_c(\mathfrak{R}_d)$ shows that the $\mathbb{Z}^d$-actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ are closely related. The group $R_c$ can be determined much more easily than $\eta_c(\mathfrak{R}_d)$ and has other advantages, e.g. for the computation of entropy in Section 7; on the other hand $R_c$ may not be a cyclic $\mathfrak{R}_d$-module, in contrast to $\eta_c(\mathfrak{R}_d) \cong \mathfrak{R}_d/\mathfrak{j}_c$. Since $R_c$ is torsion-free (as an additive group), $Y^{(c)}$ and $X^{\mathfrak{R}_d/\mathfrak{j}_c}$ are both connected.

Proposition 7.2. *Let $d \ge 1$, $c = (c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$, and let $(X^{\mathfrak{R}_d/\mathfrak{j}_c}, \alpha^{\mathfrak{R}_d/\mathfrak{j}_c})$ and $(Y^{(c)}, \alpha^{(c)})$ be defined as in* Theorem 7.1.

(1) *For every $\mathbf{m} \in \mathbb{Z}^d$, the following conditions are equivalent.*
  (a) *$\alpha^{(c)}_{\mathbf{m}}$ is ergodic;*
  (b) *$\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}_{\mathbf{m}}$ is ergodic;*
  (c) *$c^{\mathbf{m}}$ is not a root of unity.*

(2) *The following conditions are equivalent.*
  (a) *$\alpha^{(c)}$ is ergodic;*
  (b) *$\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ is ergodic;*
  (c) *At least one coordinate of $c$ is not a root of unity.*

(3) *The following conditions are equivalent.*
  (a) *$\alpha^{(c)}$ is mixing;*
  (b) *$\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ is mixing;*
  (c) *$c^{\mathbf{m}} \ne 1$ for all non-zero $\mathbf{m} \in \mathbb{Z}^d$.*

(4) *If $\alpha^{(c)}$ is ergodic then the groups $\mathrm{Fix}_\Lambda(\alpha^{(c)})$ and $\mathrm{Fix}_\Lambda(\alpha^{\mathfrak{R}_d/\mathfrak{j}_c})$ are finite for every subgroup $\Lambda \subset \mathbb{Z}^d$ with finite index.*

(5) *The following conditions are equivalent.*
  (a) *$\alpha^{(c)}$ is expansive;*
  (b) *$\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ is expansive;*
  (c) *The orbit of $c$ under the diagonal action of the Galois group $\mathrm{Gal}[\overline{\mathbb{Q}} : \mathbb{Q}]$ on $(\overline{\mathbb{Q}}^\times)^d$ does not intersect $\mathbb{S}^d$.*

Proof. The $\mathfrak{R}_d$-modules $R_c$ and $\mathfrak{R}_d/\mathfrak{j}_c$ are both associated with the prime ideal $\mathfrak{j}_c$, $V_{\mathbb{C}}(\mathfrak{j}_c) = \mathrm{Gal}[\overline{\mathbb{Q}} : \mathbb{Q}](c)$, and all assertions follow from Theorem 6.5. □

Proposition 7.3. *Let $N(c)$ be the cardinality of the orbit $\mathrm{Gal}[\overline{\mathbb{Q}} : \mathbb{Q}](c)$ of $c$ under the Galois group. Then $Y^{(c)} \cong \mathbb{T}^{N(c)}$ if and only if $c_i$ is an algebraic unit for every $i = 1, \dots, d$ (i.e. $c_i$ and $c_i^{-1}$ are integral in $\mathbb{Q}(c)$ for $i = 1, \dots, d$). If at least one of the coordinates of $c$ is not a unit, then $Y^{(c)}$ is a projective limit of copies of $\mathbb{T}^{N(c)}$.*

PROOF. We use the notation established in (7.1)–(7.8). The number $N(c)$ is equal to the degree $[\mathbb{Q}(c):\mathbb{Q}]$. If $N_{\mathbb{R}}(c)$ and $N_{\mathbb{C}}(c)$ are the numbers of real and complex (infinite) places of $\mathbb{Q}(c)$ then $N(c) = N_{\mathbb{R}}(c) + 2N_{\mathbb{C}}(c)$, and the connected component of the identity in $\prod_{v\in P(c)} \mathbb{K}_v$ is isomorphic to $\mathbb{R}^{N(c)}$. The condition that every coordinate of $c$ be a unit is equivalent to the assumption that $F(c) = \varnothing$; in this case $Y^{(c)}$ is isomorphic to the quotient of $\mathbb{R}^{N(c)}$ by the discrete, co-compact subgroup $i_{P(c)}(R_c)$, i.e. $Y^{(c)} \cong \mathbb{T}^{N(c)}$. If $F(c) \neq \varnothing$ then $Y^{(c)}$ is isomorphic to the quotient of $\mathbb{R}^{N(c)} \times \prod_{v\in F(c)} \mathbb{K}_v$ by $i_{P(c)}(R_c)$. In order to prove the assertion about the projective limit we choose, for every $v \in F(c)$, a prime element $p_v \in \mathbb{K}_v$ (i.e. an element with $p_v\mathcal{R}_v = \{a \in \mathbb{K}_v : |a|_v < 1\}$), and set $\Delta_n = i_{P(c)}(R_c) + \prod_{v\in F(c)} p_v^n \bar{\mathcal{R}}_v$ for every $n \geq 1$. Then $\bigcap_{n\geq 1} \Delta_n = i_{P(c)}(R_c)$, and $Y^{(c)}$ is the projective limit of the groups $Y_n = Y^{(c)}/\Delta_n \cong \mathbb{T}^{N(c)}$, $n \geq 1$, where the last isomorphism is established by meditation. $\square$

If $X$ is a compact, connected, abelian group with dual group $\hat{X}$, then $\hat{X}$ is torsion-free, and the map $a \mapsto 1 \otimes a$ from $\hat{X}$ into the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{X}$ is therefore injective. We denote by $\dim X$ the dimension of the vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{X}$ over $\mathbb{Q}$ and note that this definition of $\dim X$ is consistent with the usual topological dimension of $X$: in particular, $0 < \dim Y^{(c)} = N(c) < \infty$ in Proposition 7.3. With this terminology we obtain the following corollary of Theorem 7.1 and Proposition 7.3.

COROLLARY 7.4. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal, and let $(X^{\mathfrak{R}_d/\mathfrak{p}}, \alpha^{\mathfrak{R}_d/\mathfrak{p}})$ be defined as in* Lemma 5.1. *The following conditions are equivalent.*

(1) *$X^{\mathfrak{R}_d/\mathfrak{p}}$ is a connected, finite-dimensional, abelian group;*

(2) *$\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^{\times})^d$.*

*Furthermore, if $\alpha$ is an ergodic $\mathbb{Z}^d$-action by automorphisms of a compact, connected, finite-dimensional abelian group $X$, then the $\mathfrak{R}_d$-module $\mathfrak{M} = \hat{X}$ has only finitely many associated prime ideals, each of which is of the form $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^{\times})^d$.*

PROOF. The implication (2)$\Rightarrow$(1) is clear from Theorem 7.1, Proposition 7.3, and the definition of $\dim X$. Conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal such that $X^{\mathfrak{R}_d/\mathfrak{p}} = \widehat{\mathfrak{R}_d/\mathfrak{p}}$ is connected, then $\mathfrak{p}$ does not contain any non-zero constants, and the map $a \mapsto 1 \otimes a$ from $\mathfrak{R}_d/\mathfrak{p}$ into the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$ is injective. This allows us to regard $\mathfrak{R}_d/\mathfrak{p}$ as a subring of $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$. The variety $V(\mathfrak{p})$ is non-empty by Proposition 6.9, and is finite if and only if each of the elements $u_i + \mathfrak{p} \in \mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$, $i = 1, \dots, d$, is algebraic over the subring $\mathbb{Q} \subset \mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$. In particular, if $V(\mathfrak{p})$ is finite, then $\mathfrak{p} = \mathfrak{j}_c$ for every $c \in V(\mathfrak{p})$, which implies (2). If $V(\mathfrak{p})$ is infinite, then at least one of the elements $u_j + \mathfrak{p}$ is transcendental over $\mathbb{Q} \subset \mathbb{Q} \otimes_{\mathbb{Z}} (\mathfrak{R}_d/\mathfrak{p})$, and the powers $u_j^k + \mathfrak{p}$, $k \in \mathbb{Z}$, are rationally independent. This is easily seen to imply that $\dim X^{\mathfrak{R}_d/\mathfrak{p}} = \infty$.

In order to prove the last assertion we assume that $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal associated with $\mathfrak{M}$. Then $X^{\mathfrak{R}_d/\mathfrak{p}}$ is (isomorphic to) a quotient group of

$X$, hence connected and finite-dimensional, and Proposition 6.9 and the first part of this corollary together imply that $\mathfrak{p} = \mathfrak{j}_c$ for some $c \in (\overline{\mathbb{Q}}^{\times})^d$. If $\mathfrak{M}$ has infinitely many distinct associated prime ideals $\{\mathfrak{j}_{c^{(1)}}, \mathfrak{j}_{c^{(2)}}, \dots\}$, then we can find, for every $i \geq 1$, an element $a_i \in \mathfrak{M}$ with $\mathfrak{R}_d \cdot a_i \cong \mathfrak{R}_d/\mathfrak{j}_{c^{(i)}}$. If $b \in \bigl(\sum_{i=1}^{j-1} \mathfrak{R}_d \cdot a_i\bigr) \cap \mathfrak{R}_d \cdot a_j \neq \{0\}$ for some $j > 1$, then the submodule $\mathfrak{R}_d \cdot b \subset \mathfrak{M}$ has an associated prime ideal $\mathfrak{j}$ which strictly contains $\mathfrak{j}_{c^{(j)}}$; in particular, $\mathfrak{j}$ must contain a non-zero constant, in violation of the fact that every prime ideal $\mathfrak{p}$ associated with $\mathfrak{R}_d \cdot b$ (and hence with $\mathfrak{M}$) must satisfy that $V_{\mathbb{C}}(\mathfrak{p}) \neq \varnothing$. It follows that $\mathfrak{M}$ has a submodule isomorphic to $\mathfrak{R}_d/\mathfrak{j}_{c^{(1)}} \oplus \mathfrak{R}_d/\mathfrak{j}_{c^{(2)}} \oplus \cdots$, and hence that $\dim X = \infty$. This contradiction proves that there are only finitely many distinct prime ideals associated with $\mathfrak{M}$. □

Example 7.5. If $\alpha$ is a $\mathbb{Z}^d$-action by automorphisms of a compact, connected, finite-dimensional, abelian group, then the $\mathfrak{R}_d$-module $\mathfrak{M} = \hat{X}$ need not be Noetherian (cf. Corollary 7.4): if $\alpha$ is the automorphism of $X = \widehat{\mathbb{Q}}$ in Example 5.6 (1) consisting of multiplication by $\frac{3}{2}$, then $\dim(X) = 1$, but $\mathfrak{M} = \hat{X} = \mathbb{Q}$ is not Noetherian (cf. Example 6.17 (2). ⊡

The following Examples 7.6 show that the $\mathbb{Z}^d$-actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}/\mathfrak{j}_c}$ may be, but need not be, topologically conjugate.

Examples 7.6. (1) If $c = 2$ then $F(c) = \{2\}$, $R_c = \mathbb{Z}[\frac{1}{2}]$, and we claim that the automorphism $\alpha_1^{(c)}$ on $Y^{(c)} = \widehat{R_c} = (\mathbb{R} \times \mathbb{Q}_2)/i_{F(c)}(\mathbb{Z}[\frac{1}{2}])$, which is multiplication by 2, is conjugate to the shift $\alpha_1^{\mathfrak{R}_1/(2-u_1)}$ on the group $X^{\mathfrak{R}_1/(2-u_1)}$ described in Example 5.3 (3). In order to verify this we note that there exists, for every $(s,t) \in \mathbb{R} \times \mathbb{Q}_2$, a unique element $r \in \mathbb{Z}[\frac{1}{2}]$ with $r + s \in [0,1)$ and $r + t \in \mathbb{Z}_2$. This allows us to identify $Y^{(c)} = \widehat{\mathbb{Z}[\frac{1}{2}]}$ with $(\mathbb{R} \times \mathbb{Z}_2)/i_{F(c)}(\mathbb{Z})$. An element $a = \frac{k}{2^l} \in \mathbb{Z}[\frac{1}{2}]$ defines a character on $Y^{(c)} = (\mathbb{R} \times \mathbb{Z}_2)/i_{F(c)}(\mathbb{Z})$ by $\langle a, (s,t) + i_{F(c)}(\mathbb{Z})\rangle = e^{2\pi i(\mathrm{Int}(as) + \mathrm{Frac}(at))}$ for every $s \in \mathbb{R}$ and $t \in \mathbb{Z}_2$, where $\mathrm{Int}(as)$ is the integral part of $as \in \mathbb{R}$ and $\mathrm{Frac}(at) \in [0,1)$ is the (well-defined) fractional part of $at \in \mathbb{Q}_2$. Consider the homomorphism $\phi\colon Y^{(c)} \longmapsto \mathbb{T}^{\mathbb{Z}}$ defined by $e^{2\pi i(\phi(y))_m} = \langle 2^m, y\rangle$ for every $y \in Y^{(c)}$ and $m \in \mathbb{Z}$. Then $\phi$ is injective, $\phi(Y^{(c)}) \subset X^{\mathfrak{R}_1/(2-u_1)}$, and it is not difficult to see that $\phi\colon Y^{(c)} \longmapsto X^{\mathfrak{R}_1/(2-u_1)}$ is a continuous group isomorphism which makes the diagram (7.11) commute. In particular, if we write a typical element $y \in Y^{(c)}$ as $y = (s,t) + i_{F(c)}(\mathbb{Z})$ with $s \in \mathbb{R}$ and $t \in \mathbb{Z}_2$, then

$$(\phi((0,t) + i_{F(c)}(\mathbb{Z})))_m = 0 \quad \text{and} \quad (\phi((s,0) + i_{F(c)}(\mathbb{Z})))_m = 2^m s \pmod 1$$

for every $m \geq 0$.

Proposition 7.2 shows that the automorphism $\alpha^{(c)} = \alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ is expansive and hence ergodic.

(2) If $c = 3/2$ then $F(c) = \{2, 3\}$, $R_c = \mathbb{Z}[\frac{1}{6}]$, and we see as in Example (1) that multiplication by $\frac{3}{2}$ on

$$Y^{(c)} = \widehat{R_c} = (\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3)/i_{F(c)}(\mathbb{Z}[\tfrac{1}{6}]) \cong (\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/i_{F(c)}(\mathbb{Z})$$

is conjugate to the shift $\alpha^{\mathfrak{R}_1/\mathfrak{I}_c}$ on $X^{\mathfrak{R}_1/\mathfrak{j}_c}$ in Example 6.18 (2). The $\mathbb{Z}$-action $\alpha^{(c)} = X^{\mathfrak{R}_1/\mathfrak{j}_c}$ is expansive and ergodic by Proposition 7.2.

(3) Let $c = 2 + \sqrt{5}$. Then $\eta_c(\mathfrak{R}_1) = \{k + l\sqrt{5} : k, l \in \mathbb{Z}\} \cong \mathbb{Z}^2$, $F(c) = \varnothing$, and $R_c$ is equal to the set $\mathfrak{o}(c) = \mathfrak{o}_{\mathbb{Q}(c)}$ of integral elements in $\mathbb{Q}(c)$. Since $\mathfrak{o}_{\mathbb{Q}(c)} = \{k\frac{1+\sqrt{5}}{2} + l\frac{1-\sqrt{5}}{2} : k, l \in \mathbb{Z}\}$ (cf. Lemma 10.3.3 in [16]), $R_c \neq \eta_c(\mathfrak{R}_1)$. By Proposition 7.2, the $\mathbb{Z}$-actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ are both expansive (and hence ergodic), but we claim that they are not topologically conjugate. According to Corollary 5.10 this amounts to showing that $R_c$ and $\mathfrak{R}_1/\mathfrak{j}_c$ are not isomorphic as $\mathfrak{R}_1$-modules, and we establish this by showing that $R_c$ is not cyclic. In terms of the $\mathbb{Z}$-basis $\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$ for $R_c$, multiplication by $c$ is represented by the matrix $A = \left(\begin{smallmatrix} 5 & -2 \\ 2 & -1 \end{smallmatrix}\right)$. If the module $R_c$ is cyclic, then there exists a vector $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ such that $\{\mathbf{m}, A\mathbf{m}\} = \{(m_1, m_2), (5m_1 - 2m_2, 2m_1 - m_2)$ generates $\mathbb{Z}^2$, and as in Example 5.3 (2) we see that this impossible.

In this example $X^{\mathfrak{R}_1/\mathfrak{j}_c} \cong Y^{(c)} \cong \mathbb{T}^2$. The matrix $A' = \left(\begin{smallmatrix} 2 & 5 \\ 1 & 2 \end{smallmatrix}\right)$ represents multiplication by $c$ in terms of the $\mathbb{Z}$-basis $\{1, \sqrt{5}\}$ of $\eta_c(\mathfrak{R}_1)$, and the matrices $A$ and $A'$ define non-conjugate automorphisms of $\mathbb{T}^2$ with identical characteristic polynomials (cf. Example 5.3 (2)).

(4) Let $c = \frac{1+\sqrt{5}}{2}$. Then $\eta_c(\mathfrak{R}_1) = \mathfrak{o}(\mathbb{Q}(c)) = R_c$, and the $\mathbb{Z}$-actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ are algebraically conjugate. However, a little care is needed in identifying $\widehat{R_c}$ with $Y^{(c)}$ in (7.10). The set $P(c) = P_\infty^{\mathbb{Q}(c)}$ consists of the two real places determined by the embeddings $\sqrt{5} \mapsto \sqrt{5}$ and $\sqrt{5} \mapsto -\sqrt{5}$ of $\mathbb{Q}(c) = \mathbb{Q}(\sqrt{5})$ in $\mathbb{R}$, so that $Y^{(c)} = \mathbb{R}^2/i_{P(c)}(R_c)$ with $i_{P(c)}(R_c) = \{(k + l\frac{1+\sqrt{5}}{2}, k + l\frac{1-\sqrt{5}}{2}) : (k, l) \in \mathbb{Z}^2\} \subset \mathbb{R}^2$. Under the usual identification of $\widehat{\mathbb{R}^2}$ with $\mathbb{R}^2$ given by $\langle (t_1, t_2), (s_1, s_2) \rangle = e^{2\pi i(s_1 t_1 + s_2 t_2)}$ for every $(s_1, s_2), (t_1, t_2) \in \mathbb{R}^2$, the annihilator $i_{P(c)}(R_c)^\perp \subset \widehat{\mathbb{R}^2} = \mathbb{R}^2$ is of the form $i_{P(c)}(R_c)^\perp = \frac{1}{\sqrt{5}} \cdot i_{P(c)}(R_c)$, and

$$\widehat{Y^{(c)}} = i_{P(c)}(R_c)^\perp = \frac{1}{\sqrt{5}} \cdot i_{P(c)}(R_c) = i_{P(c)}\left(\frac{1}{\sqrt{5}} \cdot R_c\right) \cong \frac{1}{\sqrt{5}} \cdot R_c \cong R_c.$$

(5) Let $\omega = (-1 + \sqrt{-3})/2$ and $c = 1 + 3\omega \in \overline{\mathbb{Q}}$. Then $\mathbb{K} = \mathbb{Q}(\omega)$ and $F(c) = \{7\}$. We claim that $R_c \neq \eta_c(\mathfrak{R}_1)$. Indeed, since the minimal polynomial $f(u) = u^2 + u + 1$ of $\omega$ is irreducible over the field $\mathbb{Q}_3$ of triadic rationals, there exists a unique place $v$ of $\mathbb{K}$ above 3, and $\mathbb{K}_v = \mathbb{Q}_3(\omega)$. Let $\mathcal{R}_v = \{a \in \mathbb{K}_v : |a|_v \leq 1\}$ and $\mathfrak{o}_v = \mathbb{K} \cap \mathcal{R}_v$. As $|3|_v = 1/9$, every $a \in S = \mathbb{Z} + 3\mathfrak{o}_v \subset \mathfrak{o}_v$ with $|a|_v < 1$ satisfies that $|a|_v \leq 3^{-2}$. In particular, $\zeta = 1 - \omega \in \mathfrak{o}_v \smallsetminus S$, since $\zeta^2 = (1 - \omega)^2 = -3\omega$ and hence $|\zeta|_v = 1/3$ (cf. p.139 in [16]). Since $\eta_c(\mathfrak{R}_1) \subset S$ and $\zeta \in \mathfrak{o}(c) \subset R_c$ we conclude that $\zeta \in R_c \smallsetminus \eta_c(\mathfrak{R}_1) \neq \varnothing$.

In order to verify that $\eta_c(\mathfrak{R}_1) \cong \mathfrak{R}_1/\mathfrak{j}_c$ and $R_c$ are non-isomorphic we take an arbitrary, non-zero element $a \in R_c$ and note that

$$\begin{aligned}\{|b|_v : b \in \eta_c(\mathfrak{R}_1) \cdot a\} &= \{|f(c)|_v |a|_v : f \in \mathfrak{R}_1\} \subset \{|a|_v |b|_v : b \in S\} \\ &\subsetneq \{3^{-n} : n \geq 0\} = \{|b|_v : b \in R_c\}.\end{aligned}$$

Hence $R_c$ is not cyclic, in contrast to $\mathfrak{R}_1/\mathfrak{j}_c$. Corollary 5.10 shows that the $\mathbb{Z}^d$-actions $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ are not topologically conjugate. In this example the isomorphic groups $Y^{(c)}$ and $X^{\mathfrak{R}_1/\mathfrak{j}_c}$ are projective limits of two-dimensional tori, and the automorphisms $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_1/\mathfrak{j}_c}$ are expansive (and ergodic) by Proposition 7.2. ⊡

Examples 7.7. (1) Let $c = (2,3) \subset (\overline{\mathbb{Q}}^{\times})^2$. Then $\mathfrak{j}_c = (u_1 - 2, u_2 - 3) \subset \mathfrak{R}_2$, $F(c) = \{2,3\}$, $R_c = \mathbb{Z}[\frac{1}{6}]$, and as in Example 7.6 (1) one sees that the $\mathbb{Z}^2$-action $\alpha^{(c)}$ on $Y^{(c)}$ is conjugate to shift-action $\alpha^{\mathfrak{R}_2/\mathfrak{j}_c}$ on the group $X^{\mathfrak{R}_2/\mathfrak{j}_c}$ appearing in in Example 5.3 (4). Note that $\alpha^{\mathfrak{R}_2/\mathfrak{j}_c}$ is expansive and mixing; in fact, $\alpha_{\mathbf{n}}^{\mathfrak{R}_2/\mathfrak{j}_c}$ is expansive for every non-zero $\mathbf{n} \in \mathbb{Z}^2$ (Proposition 7.2). The group $Y^{(c)} = (\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3)/i_{F(c)}(\mathbb{Z}[\frac{1}{6}]) \cong (\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/i_{F(c)}(\mathbb{Z})$ is the same as in Example 7.6 (2), but $X^{\mathfrak{R}_2/\mathfrak{j}_c}$ is now a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^2}$. In order to describe an explicit isomorphism $\phi\colon Y^{(c)} \longmapsto X^{\mathfrak{R}_2/\mathfrak{j}_c}$ we proceed as in Example 7.6 (1): identify $Y^{(c)}$ with $(\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3)/i_{F(c)}(\mathbb{Z})$, and write the character of $Y^{(c)}$ defined by an element $a = \frac{j}{2^k 3^l} \in \mathbb{Z}[\frac{1}{6}]$ as $\langle a, (r,s,t) + i_{F(c)}(\mathbb{Z})\rangle = e^{2\pi i(\mathrm{Int}(ar) + \mathrm{Frac}(as) + \mathrm{Frac}(at))}$ for every $r \in \mathbb{R}$, $s \in \mathbb{Z}_2$ and $t \in \mathbb{Z}_3$. If $\phi\colon Y^{(c)} \longmapsto \mathbb{T}^{\mathbb{Z}^2}$ is the map given by $e^{2\pi i(\phi(y))_{(n_1,n_2)}} = \langle 2^{n_1} 3^{n_2}, y\rangle$ for every $y \in Y$and $(n_1, n_2) \in \mathbb{Z}^2$, then $\phi$ is injective, $\phi(Y^{(c)}) = X^{\mathfrak{R}_2/\mathfrak{j}_c}$, and $\phi$ makes the diagram (7.11) commute.

(2) Let $\mathbb{K} \supset \mathbb{Q}$ be an algebraic number field. We denote by $\mathfrak{o}_{\mathbb{K}} \subset \mathbb{K}$ the ring of integers and write $\mathfrak{U}_{\mathbb{K}} \subset \mathfrak{o}_{\mathbb{K}}$ for the group of units (i.e. $\mathfrak{U}_{\mathbb{K}} = \{a \in \mathfrak{o}_{\mathbb{K}} : a^{-1} \in \mathfrak{o}_{\mathbb{K}}\}$). By Theorem 10.8.1 in [16], $\mathfrak{U}_{\mathbb{K}}$ is isomorphic to the cartesian product $F \times \mathbb{Z}^{r+s-1}$, where $F$ is a finite, cyclic group consisting of all roots of unity in $\mathbb{K}$ and $r$ and $s$ are the numbers of real and complex places of $\mathbb{K}$. We set $d = r + s - 1$, choose generators $c_1, \dots, c_d \in \mathfrak{U}_{\mathbb{K}}$ such that every $a \in \mathfrak{U}_{\mathbb{K}}$ can be written as $a = uc_1^{k_1} \cdot \dots \cdot c_d^{k_d}$ with $u \in F$ and $k_1, \dots, k_d \in \mathbb{Z}$, and set $c = (c_1, \dots, c_d)$. Then $X^{\mathfrak{R}_d/\mathfrak{j}_c} \cong Y^{(c)} \cong \mathbb{T}^{r+2s}$, and the $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ and $\alpha^{(c)}$ are mixing by Proposition 7.2.

(3) Let $d \geq 1$, and let $\mathfrak{a} \subset \mathfrak{R}_d$ be an ideal with $V(\mathfrak{a}) \neq \varnothing$ (or, equivalently, with $V_{\mathbb{C}}(\mathfrak{a}) \neq \varnothing$). For every $c \in V(\mathfrak{a})$ the evaluation map $\eta_c\colon f \mapsto f(c)$ from $\mathfrak{R}_d/\mathfrak{a}$ to $\mathbb{Q}(c)$ induces a dual, injective embedding of $X^{\mathfrak{R}_d/\mathfrak{j}_c}$ in $X^{\mathfrak{R}_d/\mathfrak{a}}$, so that we may regard $X^{\mathfrak{R}_d/\mathfrak{j}_c}$ as a subgroup of $X^{\mathfrak{R}_d/\mathfrak{a}}$; in this picture $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ is the restriction of $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ to $X^{\mathfrak{R}_d/\mathfrak{j}_c}$. In fact, if $\mathfrak{a}$ is *radical*, i.e. if $\mathfrak{a} = \sqrt{\mathfrak{a}} = \{f \in \mathfrak{R}_d : f^k \in \mathfrak{a}$ for some $k \geq 1\}$, then $\mathfrak{a} = \{f \in \mathfrak{R}_d : f(c) = 0$ for every $c \in V(\mathfrak{a})\}$, and the group generated by $X^{\mathfrak{R}_d/\mathfrak{j}_c}$, $c \in V_{\mathbb{C}}(\mathfrak{a})$, is dense in $X^{\mathfrak{R}_d/\mathfrak{a}}$. In general, $\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ is expansive if and only if $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ is expansive for every $c \in V(\mathfrak{a})$, but

$\alpha^{\mathfrak{R}_d/\mathfrak{a}}$ may be mixing in spite of $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ being non-ergodic for some $c \in V(\mathfrak{a})$: take, for example, $d = 2$, $\mathfrak{a} = (1+u_1+u_2) \subset \mathfrak{R}_2$, and $c = ((-1+i\sqrt{-3})/2, (-1-i\sqrt{-3})/2) \in V(\mathfrak{a})$ (Theorem 6.5, Proposition 7.2, and Example 6.18 (1)). ⊡

CONCLUDING REMARK 7.8. Theorem 7.1, Proposition 7.2, and Example 7.6 (5) are taken from [94], and Example 7.6 (4) was pointed out to me by Jenkner. The possible difference between $\alpha^{(c)}$ and $\alpha^{\mathfrak{R}_d/\mathfrak{j}_c}$ for $c \in (\overline{\mathbb{Q}}^{\times})^d$ allows the construction of analogues to Williams' Example 5.3 (2) for $\mathbb{Z}^d$-actions.

## 8. The dynamical system defined by a prime ideal

In this section we continue our investigation of the structure of the $\mathbb{Z}^d$-actions $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal. For prime ideals of the form $\mathfrak{j}_c$, $c \in (\overline{\mathbb{Q}}^{\times})^d$, the work was done in Section 7, and for $\mathfrak{p} = \{0\}$ we already know that $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is the shift-action of $\mathbb{Z}^d$ on $X^{\mathfrak{R}_d/\mathfrak{p}} = \mathbb{T}^{\mathbb{Z}^d}$. Another case which can be dealt with easily are the non-ergodic prime ideals (Definition 6.16).

PROPOSITION 8.1. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. Then $\mathfrak{p}$ is non-ergodic if and only if $\mathfrak{p}$ is either maximal, or of the form $\mathfrak{j}_c$ for a point $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d$ with $c_1^l = \cdots = c_d^l = 1$ for some $l \geq 1$. Furthermore, if $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is non-ergodic, then $X^{\mathfrak{R}_d/\mathfrak{p}}$ is either finite or a finite-dimensional torus, and there exists an integer $L \geq 1$ such that $\alpha_{L\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}} = id_{X^{\mathfrak{R}_d/\mathfrak{p}}}$ for every $\mathbf{n} \in \mathbb{Z}^d$.*

PROOF. This is just a re-wording of Theorem 6.5 (1). An ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is maximal if and only if $\mathfrak{R}_d/\mathfrak{p}$ is a finite field; in particular, the characteristic $p(\mathfrak{p})$ is positive for any maximal ideal $\mathfrak{p}$.

Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal such that $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is non-ergodic. If $p = p(\mathfrak{p}) > 0$, then Theorem 6.5 (1.e) implies that $V(\mathfrak{p}) \subset (\overline{\mathbb{F}}_{p(\mathfrak{p})}^{\times})^d$ is finite and that $\mathfrak{p}$ is therefore maximal. In particular, $\mathfrak{R}_d/\mathfrak{p} \cong \mathbb{F}_{p^l}$ for some $l \geq 1$, where $\mathbb{F}_{p^l}$ is the finite field with $p^l$ elements, and $\alpha_{(p^l-1)\mathbf{n}}$ is the identity map on $X^{\mathfrak{R}_d/\mathfrak{p}} \cong \mathbb{F}_{p^l}$ for every $\mathbf{n} \in \mathbb{Z}^d$. Conversely, if $\mathfrak{p}$ is maximal, then $|X^{\mathfrak{R}_d/\mathfrak{p}}| = |\mathfrak{R}_d/\mathfrak{p}|$ is finite, and $\alpha$ is therefore non-ergodic.

If $p(\mathfrak{p}) = 0$, then Theorem 6.5 (1.e) guarantees the existence of an integer $l \geq 1$ with $c_1^l = \cdots = c_d^l = 1$ for every $c = (c_1, \dots, c_l) \in V(\mathfrak{p}) = V_{\mathbb{C}}(\mathfrak{p})$, so that $V(\mathfrak{p})$ is finite, and the primality of $\mathfrak{p}$ allows us to conclude that $\mathfrak{p} = \mathfrak{j}_c$ for some $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d$ with $c_1^l = \cdots = c_d^l = 1$. From the definition of $\alpha^{(c)}$ in (7.9)–(7.10), Theorem 7.1, and Proposition 7.3, it is clear that $X^{\mathfrak{R}_d/\mathfrak{p}}$ is a finite-dimensional torus, and that $\alpha_{l\mathbf{n}}$ is the identity map on $X^{\mathfrak{R}_d/\mathfrak{p}}$ for every $\mathbf{n} \in \mathbb{Z}^d$. Conversely, if $\mathfrak{p} = \mathfrak{j}_c$ for some $c = (c_1, \dots, c_d) \in \overline{\mathbb{Q}}^d$ with $c_1^l = \cdots = c_d^l = 1$, then Theorem 6.5 (1.e) shows that $\alpha$ is non-ergodic. □

Next we consider ergodic prime ideals $\mathfrak{p} \subset \mathfrak{R}_d$ with $p(\mathfrak{p}) > 0$. We call a subgroup $\Gamma \subset \mathbb{Z}^d$ *primitive* if $\mathbb{Z}^d/\Gamma$ is torsion-free; a non-zero element $\mathbf{n} \in \mathbb{Z}^d$ is *primitive* if the subgroup $\{k\mathbf{n} : k \in \mathbb{Z}\} \subset \mathbb{Z}^d$ is primitive. The following proposition shows that there exists, for every ergodic prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ with

$p(\mathfrak{p}) > 0$, a maximal primitive subgroup $\Gamma \subset \mathbb{Z}^d$ and a finite, abelian group $G$ such that the restriction $\alpha^\Gamma$ of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ to $\Gamma$ is topologically and algebraically conjugate to the shift-action of $\Gamma$ on $G^\Gamma$.

PROPOSITION 8.2. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be an ergodic prime ideal with $p = p(\mathfrak{p}) > 0$, and assume that $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is the shift-action of $\mathbb{Z}^d$ on the closed, shift-invariant subgroup $X = X^{\mathfrak{R}_d/\mathfrak{p}} \subset \mathbb{F}_p^{\mathbb{Z}^d}$ defined by* (6.19)*. Then there exists an integer $r = r(\mathfrak{p}) \in \{1, \dots, d\}$, a primitive subgroup $\Gamma = \Gamma(\mathfrak{p}) \subset \mathbb{Z}^d$, and a finite set $Q = Q(\mathfrak{p}) \subset \mathbb{Z}^d$ with the following properties.*

(1) $\Gamma \cong \mathbb{Z}^r$;
(2) $\mathbf{0} \in Q$, *and* $Q \cap (Q + \mathbf{m}) = \varnothing$ *whenever* $\mathbf{0} \neq \mathbf{m} \in \Gamma$;
(3) *If $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$, then the coordinate projection $\pi_{\bar{\Gamma}} \colon X \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$, which restricts any point $x \in X \subset \mathbb{F}_p^{\mathbb{Z}^d}$ to its coordinates in $\bar{\Gamma}$, is a continuous group isomorphism; in particular, the $\Gamma$-action $\alpha^\Gamma : \mathbf{n} \mapsto \alpha_{\mathbf{n}}$, $\mathbf{n} \in \Gamma$, is* (*isomorphic to*) *the shift-action of $\Gamma$ on $(\mathbb{F}_p^Q)^\Gamma$.*

PROOF. This is Noether's normalization lemma in disguise. Consider the prime ideal $\mathfrak{p}' = \{f_{/p} : f \in \mathfrak{p}\} \subset \mathfrak{R}_d^{(p)}$ defined in Remark 6.19 (4), and write $\mathbf{e}^{(i)}$ for the $i$-th unit vector in $\mathbb{Z}^d$. We claim that there exists a matrix $A \in \mathrm{GL}(d, \mathbb{Z})$ and an integer $r$, $1 \leq r \leq d$, such that the elements $v_i = u^{A\mathbf{e}^{(i)}} + \mathfrak{p}'$ are algebraically independent in the ring $\mathcal{R} = \mathfrak{R}_d^{(p)}/\mathfrak{p}'$ for $i = 1, \dots, r$, and $v_j = u^{A\mathbf{e}^{(j)}} + \mathfrak{p}'$ is an algebraic unit over the subring $\mathbb{F}_p[v_1^{\pm1}, \dots, v_{j-1}^{\pm1}] \subset \mathcal{R}$ for $j = r + 1, \dots, d$. Indeed, if $u_1' = u_1 + \mathfrak{p}', \dots, u_d' = u_d + \mathfrak{p}'$ are algebraically independent elements of $\mathcal{R}$, then $\mathfrak{p}' = \{0\}$, and the assertion holds with $r = d$, and with $A$ equal to the $d \times d$ identity matrix. Assume therefore (after renumbering the variables, if necessary) that there exists an irreducible Laurent polynomial $f \in \mathfrak{p}'$ of the form $f = g_0 + g_1 u_d + \dots + g_l u_d^l$, where $g_i \in \mathbb{F}_p[u_1^{\pm1}, \dots, u_{d-1}^{\pm1}]$ and $g_0 g_l \neq 0$. If the supports of $g_0$ and $g_l$ are both singletons, then $u_d$ and $u_d^{-1}$ are both integral over the subring $\mathbb{F}_p[{u_1'}^{\pm1}, \dots, {u_{d-1}'}^{\pm1}] \subset \mathcal{R}$. If the support of either $g_0$ or $g_l$ is not a singleton one can find integers $k_1, \dots, k_d$ such that substitution of the variables $w_i = u_i u_d^{k_i}$, $i = 1, \dots, d-1$, in $f$ leads to a Laurent polynomial $g(w_1, \dots, w_{d-1}, u_d) = u_d^{k_d} f(u_1, \dots, u_d)$ of the form $g = g_0' + g_1' u_d + \dots + g_{l'}' u_d^{l'}$, where $g_i' \in \mathbb{F}_p[w_1^{\pm1}, \dots, w_{d-1}^{\pm1}]$, and where the supports of $g_0'$ and $g_{l'}'$ are both singletons. We set

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 & k_1 \\ 0 & 1 & \dots & 0 & k_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & k_{d-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$w_i' = w_i + \mathfrak{p}' = u^{B\mathbf{e}^{(i)}} + \mathfrak{p}'$, $i = 1, \dots, d-1$, and note that $w_d'$ and ${w_d'}^{-1}$ are integral over $\mathbb{F}_p[{w_1'}^{\pm1}, \dots, {w_{d-1}'}^{\pm1}] \subset \mathcal{R}$. If the elements $w_1', \dots, w_{d-1}'$ are algebraically independent in $\mathcal{R}$, then our claim is proved; if not, then we can apply the same argument to $w_1, \dots, w_{d-1}$ instead of $u_1, \dots, u_d$, and iteration of this procedure leads to a matrix $A \in \mathrm{GL}(d,\mathbb{Z})$ and an integer $r \geq 0$ such that the elements $v_j' = u^{A\mathbf{e}^{(j)}} + \mathfrak{p}' \in \mathcal{R}$ satisfy that $v_1',\dots,v_r'$ are algebraically independent, and $v_j'$ and ${v_j'}^{-1}$ are integral over $\mathcal{R}^{(j-1)} = \mathbb{F}_p[{v_1'}^{\pm 1},\dots,{v_{j-1}'}^{\pm 1}] \subset \mathcal{R}$ for $j > r$, where $\mathcal{R}^{(0)} = \mathbb{F}_p$ if $r = 0$ (in which case $\mathcal{R}$ must be finite). From Theorem 3.2 it is clear that the ergodicity of $\alpha$ implies that $r \geq 1$, and this completes the proof of our claim.

For the remainder of this proof we assume for simplicity that $A$ is the $d\times d$ identity matrix, so that $v_i = u_i$ for $i = 1,\dots,d$ (this is—in effect—equivalent to replacing $\alpha$ by the $\mathbb{Z}^d$-action $\alpha'\colon \mathbf{n} \mapsto \alpha'_{\mathbf{n}} = \alpha_{A\mathbf{n}}$). The argument in the preceding paragraph gives us, for each $j = r+1,\dots,d$, an irreducible polynomial $f_j(x) = \sum_{k=0}^{l_j} g_k^{(j)} x^k$ with coefficients in the ring $\mathbb{F}_p[u_1^{\pm1},\dots,u_{j-1}^{\pm1}] \subset \mathfrak{R}_d$ such that $h_j(u_j) = h_j(u_1,\dots,u_{j-1},u_j) \in \mathfrak{p}'$ and the supports of $g_0^{(j)}$ and $g_{l_j}^{(j)}$ are singletons. Let $\Gamma \subset \mathbb{Z}^d$ be the group generated by $\{\mathbf{e}^{(1)},\dots,\mathbf{e}^{(r)}\}$, $Q = \{0\}\times\dots\times\{0\}\times\{0,\dots,l_{r+1}-1\}\times\{0,\dots,l_d-1\} \subset \mathbb{Z}^d$, and let $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m}+\mathbf{n} : \mathbf{m}\in\Gamma,\ \mathbf{n}\in Q\}$. We write $\pi_{\bar{\Gamma}}\colon X \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$ for the coordinate projection which restricts every $x\in X$ to its coordinates in $\bar{\Gamma}$ and note that $\pi_{\bar{\Gamma}}\colon X \longmapsto \mathbb{F}_p^{\bar{\Gamma}}$ is a continuous group isomorphism. In other words, the restriction of $\alpha$ to the group $\Gamma \cong \mathbb{Z}^r$ is conjugate to the shift-action of $\Gamma$ on $(\mathbb{F}_p^Q)^\Gamma$. $\square$

If the prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ satisfies that $p(\mathfrak{p}) = 0$, then the analysis of the action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ becomes somewhat more complicated. We denote by $\kappa\colon \widehat{\mathbb{Q}} \longmapsto \mathbb{T}$ the surjective group homomorphism dual to the inclusion $\hat{\kappa}\colon \mathbb{Z} \longmapsto \mathbb{Q}$. If $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal with $p(\mathfrak{p}) = 0$ we regard $X^{\mathfrak{R}_d/\mathfrak{p}}$ as the subgroup (5.9) of $\mathbb{T}^{\mathbb{Z}^d}$, and define a closed, shift-invariant subgroup $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \subset \widehat{\mathbb{Q}}^{\mathbb{Z}^d}$ by

$$\bar{X}^{\mathfrak{R}_d/\mathfrak{p}} = \Bigl\{ x = (x_{\mathbf{n}}) \in \widehat{\mathbb{Q}}^{\mathbb{Z}^d} : \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) x_{\mathbf{m}+\mathbf{n}} = \mathbf{0}_{\widehat{\mathbb{Q}}^{\mathbb{Z}^d}} \text{ for every } f \in \mathfrak{p} \Bigr\}. \tag{8.1}$$

The restriction of the shift-action $\sigma$ of $\mathbb{Z}^d$ on $\widehat{\mathbb{Q}}^{\mathbb{Z}^d}$ to $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$ will be denoted by $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ (cf. (2.1)). Define a continuous, surjective homomorphism $\boldsymbol{\kappa}\colon \widehat{\mathbb{Q}}^{\mathbb{Z}^d} \longmapsto \mathbb{T}^{\mathbb{Z}^d}$ by $(\boldsymbol{\kappa}(x))_{\mathbf{n}} = \kappa(x_{\mathbf{n}})$ for every $x = (x_{\mathbf{m}}) \in \widehat{\mathbb{Q}}^{\mathbb{Z}^d}$ and $\mathbf{n}\in\mathbb{Z}^d$, and write

$$\boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}\colon \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto X^{\mathfrak{R}_d/\mathfrak{p}} \tag{8.2}$$

for the restriction of $\boldsymbol{\kappa}$ to $\bar{X}^{\mathfrak{R}_d/\mathfrak{p}}$. The map $\boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}$ is surjective, and the diagram

$$\begin{array}{ccc} \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} & \xrightarrow{\bar{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}} & \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \\ {\scriptstyle\boldsymbol{\kappa}}\downarrow & & \downarrow{\scriptstyle\boldsymbol{\kappa}} \\ X^{\mathfrak{R}_d/\mathfrak{p}} & \xrightarrow[\alpha_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}]{} & X^{\mathfrak{R}_d/\mathfrak{p}} \end{array} \tag{8.3}$$

commutes for every $\mathbf{n} \in \mathbb{Z}^d$.

In order to explain this construction in terms of the dual modules we consider the ring $\mathfrak{R}_d^{(\mathbb{Q})} = \mathbb{Q}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{R}_d$, regard $\mathfrak{R}_d$ as the subring of $\mathfrak{R}_d^{(\mathbb{Q})}$ consisting of all polynomials with integral coefficients, and denote by $\mathfrak{p}^{(\mathbb{Q})} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{p} \subset \mathfrak{R}_d^{(\mathbb{Q})}$ the prime ideal in $\mathfrak{R}_d^{(\mathbb{Q})}$ corresponding to $\mathfrak{p}$. Since $p(\mathfrak{p}) = 0$, every $\mathfrak{R}_d$-module $\mathfrak{N}$ associated with $\mathfrak{p}$ is embedded injectively in the $\mathfrak{R}_d^{(\mathbb{Q})}$-module $\mathfrak{N}^{(\mathbb{Q})} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{N}$ by

$$\hat{\imath}^{\mathfrak{N}} \colon a \mapsto 1 \otimes_{\mathbb{Z}} a,\ a \in \mathfrak{N}, \tag{8.4}$$

and $\mathfrak{N}^{(\mathbb{Q})}$ is associated with $\mathfrak{p}^{(\mathbb{Q})}$. Since $\mathfrak{R}_d \subset \mathfrak{R}_d^{(\mathbb{Q})}$, $\mathfrak{N}^{(\mathbb{Q})}$ is an $\mathfrak{R}_d$-module, and we can define the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{N}^{(\mathbb{Q})}}$ on $X^{\mathfrak{N}^{(\mathbb{Q})}}$ as in Lemma 5.1. Note that the set of prime ideals associated with the $\mathfrak{R}_d$-module $\mathfrak{N}^{(\mathbb{Q})}$ is the same as that of $\mathfrak{N}$; in particular, $\alpha^{\mathfrak{N}^{(\mathbb{Q})}}$ is ergodic if and only if $\alpha^{\mathfrak{N}}$ is ergodic and, for every $\mathbf{n} \in \mathbb{Z}^d$, $\alpha_{\mathbf{n}}^{\mathfrak{N}^{(\mathbb{Q})}}$ is ergodic if and only if $\alpha_{\mathbf{n}}^{\mathfrak{N}}$ is ergodic. The homomorphism

$$\imath^{\mathfrak{N}} \colon X^{\mathfrak{N}^{(\mathbb{Q})}} \longmapsto X^{\mathfrak{N}} \tag{8.5}$$

dual to

$$\hat{\imath} \colon \mathfrak{N} \longmapsto \mathfrak{N}^{(\mathbb{Q})} \tag{8.6}$$

is surjective, and the diagram

$$\begin{array}{ccc} X^{\mathfrak{N}^{(\mathbb{Q})}} & \xrightarrow{\alpha_{\mathbf{n}}^{\mathfrak{N}^{(\mathbb{Q})}}} & \bar{X}^{\mathfrak{N}^{(\mathbb{Q})}} \\ \boldsymbol{\kappa}\big\downarrow & & \big\downarrow\boldsymbol{\kappa} \\ X^{\mathfrak{N}} & \xrightarrow[\alpha_{\mathbf{n}}^{\mathfrak{N}}]{} & X^{\mathfrak{N}} \end{array} \tag{8.7}$$

commutes for every $\mathbf{n} \in \mathbb{Z}^d$. For $\mathfrak{N} = \mathfrak{R}_d/\mathfrak{p}$ we obtain that

$$\begin{aligned} X^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}} &= \bar{X}^{\mathfrak{R}_d/\mathfrak{p}}, \\ \alpha^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}} &= \bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}, \\ \imath^{\mathfrak{R}_d/\mathfrak{p}} &= \boldsymbol{\kappa}^{\mathfrak{R}_d/\mathfrak{p}}. \end{aligned} \tag{8.8}$$

PROPOSITION 8.3. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal with $p(\mathfrak{p}) = 0$ which is not of the form $\mathfrak{p} = \mathfrak{j}_c$ for any $c \in \overline{\mathbb{Q}}^d$. Then the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ on $X = X^{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic, and there exists an integer $r = r(\mathfrak{p}) \in \{1, \dots, d\}$, a primitive subgroup $\Gamma = \Gamma(\mathfrak{p}) \subset \mathbb{Z}^d$, and a finite set $Q = Q(\mathfrak{p}) \subset \mathbb{Z}^d$ with the following properties.*

(1) $\Gamma \cong \mathbb{Z}^r$;
(2) $\mathbf{0} \in Q$, *and* $Q \cap (Q + \mathbf{m}) = \varnothing$ *whenever* $\mathbf{0} \neq \mathbf{m} \in \Gamma$;

(3) *If* $\bar{\Gamma} = \Gamma + Q = \{\mathbf{m}+\mathbf{n} : \mathbf{m} \in \Gamma, \mathbf{n} \in Q\}$, *then the coordinate projection* $\pi_{\bar{\Gamma}} \colon \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \widehat{\mathbb{Q}}^{\bar{\Gamma}}$, *which restricts any point* $x \in \bar{X}^{\mathfrak{R}_d/\mathfrak{p}} \subset \widehat{\mathbb{Q}}^{\mathbb{Z}^d}$ *to its coordinates in* $\bar{\Gamma}$, *is a continuous group isomorphism; in particular, the* $\Gamma$-*action* $\mathbf{n} \mapsto \bar{\alpha}_{\mathbf{n}}^{\mathfrak{R}_d/\mathfrak{p}}$, $\mathbf{n} \in \Gamma$, *is (isomorphic to) the shift-action of* $\Gamma$ *on* $(\widehat{\mathbb{Q}}^Q)^\Gamma$.

Proof. The proof is completely analogous to that of Proposition 8.2. We find a matrix $A \in \mathrm{GL}(d, \mathbb{Z})$ and an integer $r \in \{1, \dots, d\}$ with the following properties: if $v_j = u^{A\mathbf{e}^{(j)}}$ and $v_j' = v_j + \mathfrak{p}$ for $j = 1, \dots, d$, then $v_1', \dots, v_r'$ are algebraically independent elements of $\mathcal{R} = \mathfrak{R}_d/\mathfrak{p}$, and there exists, for each $j = r+1, \dots, d$, an irreducible polynomial $f_j(x) = \sum_{k=0}^{l_j} g_k^{(j)}(x^k)$ with coefficients in the ring $\mathbb{Z}[v_1^{\pm 1}, \dots, v_{j-1}^{\pm 1}] \subset \mathfrak{R}_d$ such that $f_j(v_1, \dots, v_{j-1}, v_j) \in \mathfrak{q}$ and the supports of $g_0^{(j)}$ and $g_{l_j}^{(j)}$ are singletons.

We assume again that $A$ is the $d \times d$ identity matrix, so that $v_j = u_j$ for $j = 1, \dots, d$ and $\Gamma \cong \mathbb{Z}^r$ is generated by $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(r)}$, set $Q = \{0\} \times \dots \times \{0\} \times \{0, \dots, l_{r+1} - 1\} \times \dots \times \{0, \dots, l_d - 1\} \subset \mathbb{Z}^d$, and complete the proof in the same way as that of Proposition 3.4, using (8.1) instead of (6.19). The ergodicity of $\bar{\alpha}^{\mathfrak{R}_d/\mathfrak{p}}$ is obvious from the conditions (1)–(3), and from (8.3) we conclude the ergodicity of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$. □

Remarks 8.4. (1) We can extend the definition of $r(\mathfrak{p})$ in Proposition 8.2 and 8.3 to ergodic prime ideals of the form $\mathfrak{p} = \mathfrak{j}_c, c \in (\overline{\mathbb{Q}}^\times)^d$, by setting $r(\mathfrak{j}_c) = 0$. Then the integer $r(\mathfrak{p})$ is a well-defined property of the prime ideal $\mathfrak{p}$, and is in particular independent of the choice of the primitive subgroup $\Gamma \subset \mathbb{Z}^d$ in Proposition 8.2 or 8.3 (it is easy to see that there is considerable freedom in the choice of $\Gamma$): if $r', \Gamma', Q'$ are a positive integer, a primitive subgroup of $\mathbb{Z}^d$, and a finite subset of $\mathbb{Z}^d$, satisfying the conditions (1)–(3) in either of the Propositions 8.2 or 8.3, then $r' = r(\mathfrak{p})$. This follows from Noether's normalization theorem; a dynamical proof using entropy will be given in Section 24.

(2) If $\mathfrak{p} \subset \mathfrak{R}_d$ is an ergodic prime ideal with $p(\mathfrak{p}) > 0$, then the subgroup $\Gamma \subset \mathbb{Z}^d$ in Proposition 8.2 is a maximal subgroup of $\mathbb{Z}^d$ for which the restriction $\alpha^\Gamma$ of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ to $\Gamma$ is expansive. In particular, $r(\mathfrak{p})$ is the smallest integer for which there exists a subgroup $\Gamma \cong \mathbb{Z}^r$ in $\mathbb{Z}^d$ such that $\alpha^\Gamma$ is expansive.

(3) Even if the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ in Proposition 8.3 is expansive, the action $\alpha^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}}$ is non-expansive. By proving a more intricate version of Proposition 8.3 one can analyze the structure of the group $X^{\mathfrak{R}_d/\mathfrak{p}}$ directly, without passing to $X^{(\mathfrak{R}_d/\mathfrak{p})^{(\mathbb{Q})}}$: if $X^{\mathfrak{R}_d/\mathfrak{p}}$ is written as a shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}^d}$ (cf. (5.9)), and if $r = r(\mathfrak{p}), \Gamma, Q$ are given as in Proposition 8.3, then the projection $\pi_{\bar{\Gamma}} \colon X^{\mathfrak{R}_d/\mathfrak{p}} \longmapsto \mathbb{T}^{\bar{\Gamma}}$ is still surjective, but need no longer be injective; the kernel of $\pi_{\bar{\Gamma}}$ is of the form $Y^\Gamma$ for some compact, zero-dimensional group $Y$ (cf. Example 8.5 (2)).

EXAMPLES 8.5. (1) Let $\mathfrak{p} = (2, 1 + u_1 + u_2) \subset \mathfrak{R}_2$ (cf. Example 5.3 (5)). Then $p(\mathfrak{p}) = 2$, $r(\mathfrak{p}) = 1$, and we may set $\Gamma = \{(k,k) : k \in \mathbb{Z}\} \cong \mathbb{Z}$ and $Q = \{(0,0),(1,0)\} \subset \mathbb{Z}^2$ in Proposition 8.2. If $X = X^{\mathfrak{R}_2/\mathfrak{p}}$ is written in the form (6.19) as

$$X = \bigl\{x = (x_{\mathbf{m}}) \subset \mathbb{F}_2^{\mathbb{Z}^d} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = \mathbf{0}_{\mathbb{F}_2} \text{ for all } (m_1,m_2) \in \mathbb{Z}^2\bigr\},$$

then the projection $\pi_{\bar{\Gamma}} \colon X \longmapsto \mathbb{F}_2^{\bar{\Gamma}}$ sends the shift $\alpha_{(1,1)}^{\mathfrak{R}_2/\mathfrak{p}} = \alpha_{(1,1)}$ on $X$ to the shift on $\mathbb{F}_2^{\bar{\Gamma}} \cong (\mathbb{Z}_{/2} \times \mathbb{Z}_{/2})^{\mathbb{Z}}$. Note that, although $\alpha_{(1,1)}$ acts expansively on $X$, other elements of $\mathbb{Z}^2$ may not be expansive; for example, $\alpha_{(1,0)}$ is non-expansive.

(2) Let $\mathfrak{p} = (3 + u_1 + 2u_2) \subset \mathfrak{R}_2$. Then $p(\mathfrak{p}) = 0$, $r(\mathfrak{p}) = 1$, and $\Gamma$ and $Q$ may be chosen as in Example (1). Note that $X^{\mathfrak{R}_2/\mathfrak{p}} = X = \{x = (x_{\mathbf{m}}) \subset \mathbb{T}^{\mathbb{Z}^d} : x_{(m_1,m_2)} + x_{(m_1+1,m_2)} + x_{(m_1,m_2+1)} = \mathbf{0}_{\mathbb{T}} \text{ for all } (m_1,m_2) \in \mathbb{Z}^2\}$; the coordinate projection $\pi_{\bar{\Gamma}} \colon X \longmapsto \mathbb{T}^{\bar{\Gamma}}$ in Proposition 8.3 is not injective; for every $x \in X$, the coordinates $x_{(m_1,m_2)}$ with $m_1 \geq m_2$ are completely determined by $\pi_{\bar{\Gamma}}(x)$, but each of the coordinates $x_{(k,k+1)}$, $k \in \mathbb{Z}$, has two possible values. Similarly, if we know the coordinates $x_{(m_1,m_2)}$, $m_1 \geq m_2 - r$ of a point $x = (x_{\mathbf{m}}) \in X$ for any $r \geq 0$, then there are exactly two (independent) choices for each of the coordinates $x_{(k,k+r+1)}$, $k \in \mathbb{Z}$. This shows that the kernel of the surjective homomorphism $\pi_{\bar{\Gamma}} \colon X \longmapsto \mathbb{T}^{\bar{\Gamma}} \cong (\mathbb{T}^2)^{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}_2^{\Gamma}$, where $Y = \mathbb{Z}_2$ denotes the group of dyadic integers.

If $\mathfrak{p}$ is replaced by the prime ideal $\mathfrak{p}' = (1 + 3u_1 + 2u_2) \subset \mathfrak{R}_2$, then $\Gamma$ and $Q$ remain unchanged, but the kernel of $\pi_{\bar{\Gamma}}$ becomes isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_3)^{\Gamma}$, where $\mathbb{Z}_3$ is the group of tri-adic integers. Finally, if $\mathfrak{p}'' = (1 + u_1 + u_2) \subset \mathfrak{R}_2$, and if $\Gamma$ and $Q$ are as in Example (1), then $\pi_{\bar{\Gamma}} \colon X^{\mathfrak{R}_2/\mathfrak{p}''} \longmapsto (\mathbb{T}^Q)^{\mathbb{Z}}$ is a group isomorphism. $\boxdot$

CONCLUDING REMARK 8.6. The material in this section (with the exception of Proposition 8.1) is taken from [38].

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