2

Basic Representation Theory of the Jacobi Group

Depending on whether we look at the archimedean, a $\mathfrak{p}$-adic or the adelic case, the methods for studying representations are sometimes very different. In this chapter we will collect some general material, mainly going back to Mackey, which will be useful in all three cases. We start by explaining the induction procedure, and apply it to describe the representations of the Heisenberg group. We treat the representations of the Jacobi group G^J with trivial central character and set the way for all further discussions of the cases with non-trivial central character by introducing a certain projective representation of G^J, the **Schrödinger-Weil representation** (others would perhaps call it the oscillator representation). This fundamental representation will later on be elaborated thoroughly in the different cases, and will allow to reduce, in a sense to be made precise later, the G^J-theory to the metaplectic theory.

2.1 Induced representations

There is a general method (studied in detail by Mackey) to construct representations of a locally compact group G by an induction process starting from representations of a subgroup B. As we will apply this method later on at several occasions, we sketch here this procedure following essentially Kirillov [Ki] pp. 183-184.

There are two natural realizations of an **induced representation**:

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1.) in a space of vector valued functions ϕ on the group G that transform according to a given representation σ of B under left translations by elements of the group B,

2.) in a space of vector valued functions F on the coset space $\mathbf{X} = B\backslash G$.

The transition from one model to the other is sometimes a difficult task, as we will see later on.

The first realization

To describe the first realization, we will consider a closed subgroup B of G and a representation σ of B in a Hilbert space $V = V_\sigma$. We denote by $d_r g$ and $d_r b$ right Haar measure on G resp. B and by $\Delta_G(g)$ and $\Delta_B(b)$ the modular function with

$$d_r(g_0 g) = \Delta_G(g_0) d_r g$$

resp. correspondingly for $\Delta_B(b)$. Then we induce from σ a representation

$$\pi = \operatorname{ind}_B^G \sigma$$

of G given by right translation ζ on the space $\mathcal{H} = \mathcal{H}_\pi$ of measurable V_σ-valued functions ϕ on G with the two properties

i) $$\phi(bg) = \left(\frac{\Delta_B(b)}{\Delta_G(b)}\right)^{1/2} \sigma(b)\phi(g) \qquad \text{for all } b \in B \text{ and } g \in G.$$

ii) $$\int_{\mathbf{X}} \|\phi(s(x))\|_v^2 \, d\mu_s(x) < \infty.$$

Here

$$s : \mathbf{X} = B\backslash G \to G$$

is a **Borel section** of the projection $p : G \to B\backslash G$ given by $g \mapsto Bg$. Then every $g \in G$ can uniquely be written in the form

$$g = b \cdot s(x), \qquad b \in B, \quad x \in \mathbf{X},$$

and G (as a set) can be identified with $B \times \mathbf{X}$. Under this identification, the Haar measure on G goes over into a measure equivalent to the product of a quasi-invariant measure on $\mathbf{X}$ and the Haar measure on B. More precisely, if a quasi-invariant measure μ_s on $\mathbf{X}$ is appropriately chosen, then the following equalities are valid.

$$d_r g = \frac{\Delta_G(b)}{\Delta_B(b)} d\mu_s(x) d_r b \qquad \text{and} \qquad \frac{d\mu_s(xg)}{d\mu_s(x)} = \frac{\Delta_B(b(x,g))}{\Delta_G(b(x,g))},$$

where $b(x,g) \in B$ is defined by the relation

$$s(x)g = b(x,g)s(xg).$$

If G is unimodular, i.e. $\Delta_G \equiv 1$, and if it is possible to select a subgroup K that is complementary to B in the sense that almost every element of G can uniquely be written in the form

$$g = b \cdot k, \quad b \in B, \quad k \in K,$$

then it is natural to identify $\mathbf{X} = B \backslash G$ with K and to chose s as the embedding of K in G. In this case, we have

$$dg = \Delta_B(b)^{-1} d_r b \, d_r k = d_l b \, d_r k.$$

If both G and B are unimodular (or more generally, if $\Delta_G(b)$ and $\Delta_B(b)$ coincide for $b \in B$), then there exists a G–invariant measure on $\mathbf{X} = B \backslash G$. If it is possible to extend Δ_B to a multiplicative function on G, then there exists a relatively invariant measure on $\mathbf{X}$ which is multiplied by the factor $\Delta_B(g)\Delta_G(g)^{-1}$ under translation by g.

It is a fundamental fact that $\pi = \mathrm{ind}_B^G \, \sigma$ is **unitary** if σ is. In this case $\mathcal{B} = \mathcal{B}_\pi$ is a Hilbert space with a G-invariant scalar product of the form

$$\langle \phi_1, \phi_2 \rangle = \int_G \langle \phi_1(g), \phi_2(g) \rangle_V \, d\mu(g),$$

where the measure μ on G is such that

$$\int_G \|\phi(g)\|_V^2 \, d\mu(g) = \int_{\mathbf{X}} \|\phi(s(x))\|_V^2 \, d\mu_s(x)$$

holds for all $\phi \in X$.

The second realization

Using the section $s : \mathbf{X} \to G$, we associate to each $\phi \in X$ a function f on $\mathbf{X}$ defined by

$$f(x) := \phi(s(x)).$$

Obviously ϕ is uniquely determined by f and we have an isomorphism of $\mathcal{B}_\pi$ onto the space $\mathcal{B}^\pi = L^2(\mathbf{X}, \mu_s, V)$ of V-valued functions on $\mathbf{X}$ having summable square norm with respect to the measure μ_s. The problem now is to exhibit the representation operator corresponding to the right translation ρ on $\mathcal{H}_\pi$. It can be shown that we have

$$\pi(g)f(x) = A(g,x)f(xg) \qquad \text{for} \quad f \in \mathcal{H}^\pi$$

where the operator valued function $A(g,x)$ is defined by the equality

$$A(g,x) = \left(\frac{\Delta_B(g)}{\Delta_G(b)}\right)^{1/2} \sigma(b),$$

in which the element $b \in B$ is defined from the relation

$$s(x)g = bs(xg).$$

2.2 The Schrödinger representation

As an example, we will discuss the **Heisenberg group** and its **Schrödinger representation**. From now on, almost everything depends on the choice of some additive character of the underlying field. Thus we will now introduce the so-called **additive standard characters**, following [Tate], 2.2. For every prime p (including $p = \infty$) we can define a homomorphism of additive groups

$$\lambda : \ \mathbb{Q}_p \longrightarrow \mathbb{R}/\mathbb{Z}$$

as follows. If $\mathbb{Q}_p = \mathbb{R}$, then $\lambda(x) = -x \bmod 1$. If p is finite, then we map a Laurent series in p to its main part:

$$\lambda\Big(\sum_{i>>-\infty}^{\infty} a_i p^i \Big) = \sum_{i>>-\infty}^{-1} a_i p^i.$$

If F is a finite extension of $\mathbb{Q}_p$, then the additive standard character

$$\psi : \ F \longmapsto S^1$$

is defined by

$$\psi(x) = e^{-2\pi i \lambda(\mathrm{Tr}(x))},$$

where Tr is the trace mapping $F \to \mathbb{Q}_p$. Hence if $F = \mathbb{R}$, then

$$\psi(x) = e^{2\pi i x},$$

and if $F = \mathbb{C}$, then

$$\psi(x) = e^{4\pi i \mathrm{Re}(x)}.$$

Caution: Our character is precisely the *inverse* of the character defined in [Tate]. We have made our choice of characters analogous to that in the papers [Be1]-[Be6] in the real case.

For $m \in F$, the notation

$$\psi^m(x) = \psi(mx)$$

will be used throughout. From [Tate] 2.2 it is known that the map $m \mapsto \psi^m$ identifies F with its own character group. It is also important to know that if F is discrete and $\mathfrak{d}$ denotes the absolute different of F then $\mathfrak{d}^{-1}$ is the greatest ideal of F, on which ψ is trivial. In particular, if $F = \mathbb{Q}_p$, then ψ is trivial on $\mathbb{Z}_p$ and on no bigger ideal.

Now, let F be a number field, $\{\mathfrak{p}\}$ the set of places of F, and $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. We can define a global additive character ψ of the adele ring $\mathbb{A}$ of F by

$$\psi(x) = \prod_{\mathfrak{p}} \psi_\mathfrak{p}(x_\mathfrak{p}) \qquad \text{for all } x = (x_\mathfrak{p})_\mathfrak{p} \in \mathbb{A},$$

where ψ_p are the local standard characters defined above. The adele ring is also self-dual via the identification $\mathbb{A} \ni m \mapsto \psi^m$ (cf. [Tate] Theorem 4.1.1). The global character thus defined has the special property that $\psi(x) = 1$ for all $x \in F$, i.e., it is a character of $\mathbb{A}/F$. Every other such character is then of the form ψ^m with $m \in F$ ([Tate] Theorem 4.1.4). We will always consider these characters in the global theory.

Returning now to local considerations, we let F be a local field of characteristic 0, and consider

$$\begin{array}{ccccl} G & = & H & = & \{h = (\lambda, \mu, \kappa) : \ \lambda, \mu, \kappa \in F\} \\ B & = & B_H & = & \{b = (0, \mu, \kappa) : \ \mu, \kappa \in F\}. \end{array}$$

For ψ the additive standard character of F as explained above and $m \in F^*$, let

$$\sigma(b) = \sigma(0, \mu, \kappa) = \psi^m(\kappa) = \psi(m\kappa).$$

Here we have the simplest situation, i.e. G and B are unimodular and we have the decomposition

$$H = B_H A_H \qquad \text{with} \quad A_H = \{a = (\lambda, 0, 0) : \ \lambda \in F\}$$

and

$$h = (\lambda, \mu, \kappa) = (0, \mu, \kappa')(\lambda, 0, 0) \qquad \text{with} \quad \kappa' = \kappa + \lambda\mu.$$

This already shows that the **first realization** of $\pi = \mathrm{ind}_B^G \sigma$ is given by right translation ρ on the space $\mathcal{H}_\pi$ of measurable $\mathbb{C}$-valued functions ϕ on H with

$$\phi(bh) = \psi(m\kappa)\phi(h) \qquad \text{for all} \quad b \in B_H \quad \text{and} \quad h \in H$$

and

$$\int_F |\phi(\lambda, 0, 0)|^2 \, d\lambda \, < \, \infty.$$

This realization is sometimes called the **Heisenberg representation**.

The restriction map $\phi \mapsto f$ given by

$$f(x) = \phi(x, 0, 0)$$

intertwines this model with the usual **Schrödinger representation** π_S^m on the space $\mathcal{H}^\pi = L^2(F)$. The prescription given above for the representation operator $A(g, x)$ here means to solve the equation

$$s(x)h = bs(xh)$$

for given x, i. e. $s(x) = (x, 0, 0)$, and $h = (\lambda, \mu, \kappa)$ by

$$b = (0, \mu, \kappa + 2x\mu + \lambda\mu).$$

This means we have for $f \in L^2(F)$ the well known formula

$$(\pi_S^m(\lambda, \mu, \kappa)f)(x) = \psi^m(\kappa + (2x + \lambda)\mu)f(x + \lambda). \tag{2.1}$$

One can see directly that π_S^m is a unitary representation. In case F is non-archimedean, it is customary to regard π_S^m as a representation on the space of smooth vectors of π_S^m, which is just the Schwartz space $\mathcal{S}(F)$.

The representation theory of the Heisenberg group is very simple, due to the following theorem which we give in both the real and the $\mathfrak{p}$-adic cases. Proofs can for instance be found in [LV], 1.3 (for the real case) and [MVW], 2.I.2, 2.I.8 (for the $\mathfrak{p}$-adic case). The notion of **smooth representation** appearing in Theorem 2.2.2 will be explained in Section 5.1.

2.2.1 Theorem. (Archimedean Stone–von Neumann theorem)

i) π_S^m *is an irreducible unitary representation of* $H(\mathbb{R})$ *with central character* ψ^m*, and every such is isomorphic to* π_S^m*.*

ii) A unitary representation of $H(\mathbb{R})$ *with central character* ψ^m *decomposes into a direct sum of Schrödinger representations* π_S^m*.*

2.2.2 Theorem. (Non-archimedean Stone–von Neumann theorem)
Let F *be a* $\mathfrak{p}$*-adic field.*

i) The representation π_S^m *on* $\mathcal{S}(F)$ *is an irreducible, smooth representation of* $H(F)$ *with central character* ψ^m*, and every such is isomorphic to* π_S^m*.*

ii) A smooth representation of $H(\mathbb{R})$ *with non-trivial central character* ψ^m *decomposes into a direct sum of Schrödinger representations* π_S^m*.*

It is indeed the Stone–von Neumann theorem which enables much of our treatment of the representation theory of the Jacobi group.

2.3 Mackey's method for semidirect products

The aim of the present and the following sections is to compute the unitary dual of the Jacobi group over a local field or over the adeles of a number field. We will make a distinction between the representations which have trivial central character and those which do not. In the first case we impose a general method of Mackey for determining the unitary dual of certain semidirect products. The second case can be treated more directly by using only the Stone–von Neumann theorem. In this section we begin with presenting Mackey's method in a degree of generality that suffices for our purposes.

Let G' be a locally compact topological group and H' a commutative closed normal subgroup, such that the exact sequence

$$1 \longrightarrow H' \longrightarrow G' \longrightarrow G'/H' \longrightarrow 1 \tag{2.2}$$

splits, i.e., G' is a semidirect product of $G := G'/H'$ with H':

$$G' = G \ltimes H'.$$

We wish to determine the unitary representations of G' in terms of those of G and H'. The method to be described goes back to Mackey [Ma1] and is repeated, for instance, in [Ma2], p. 77.

Assume the unitary dual $\widehat{H'}$ is known and has been given the topology of uniform convergence on compact subsets. G' operates on H' by conjugation, and this induces an operation of G' on $\widehat{H'}$:

$$\begin{array}{rcl} G' \times \widehat{H'} & \longrightarrow & \widehat{H'}, \\ (g, \sigma) & \longmapsto & \sigma^g, \end{array}$$

where the representation σ^g is given by

$$\sigma^g(h) = \sigma(ghg^{-1}) \qquad \text{for all } h \in H'.$$

Of course, if $g \in H$, then σ^g is equivalent to σ. Hence H operates trivially on $\widehat{H'}$, and only the action of G has to be considered.

Mackey's theory does not work for arbitrary semidirect products. One has to impose a certain smoothness condition on the orbits of G' in $\widehat{H'}$. Namely it is demanded that for every G'-orbit Ω in $\widehat{H'}$ and for every $\sigma \in \Omega$ with stabilizer $G'_\sigma \subset G'$ the canonical bijection

$$G'_\sigma \backslash G' \longrightarrow \Omega$$

be a homeomorphism. If this condition is fulfilled then H' is called regularly embedded, and $G' = G \ltimes H'$ is called a regular semidirect product.

The result of Mackey is now as follows.

2.3.1 Theorem. *Let G' be a locally compact topological group and H' a closed commutative normal subgroup such that the sequence (2.2) splits. Assume that H' is of type I and regularly embedded. For every $\sigma \in \widehat{H'}$ let G'_σ the stabilizer of σ under the above action of G' on $\widehat{H'}$, and*

$$\check{G}'_\sigma = \{\tau \in \widehat{G'_\sigma} :\ \tau\Big|_{H'} \text{ is a multiple of } \sigma\}.$$

Then the induced representation

$$\operatorname{Ind}_{G'_\sigma}^{G'} \tau$$

is irreducible for every $\tau \in \check{G}'_\sigma$, and $\widehat{G'}$ is a disjoint union

$$\widehat{G'} = \bigcup_{\widehat{H'}/G} \{\operatorname{Ind}_{G'_\sigma}^{G'} \tau :\ \tau \in \check{G}'_\sigma\}.$$

2.4 Representations of G^J with trivial central character

Let R be a local field of characteristic 0 ($\mathbb{R}$ and $\mathbb{C}$ included) or the ring of adeles of a number field, and let G^J be the Jacobi group over R. In this section we determine the irreducible unitary representations of G^J which have trivial central character. These representations are obviously in 1-1 correspondence with the irreducible unitary representations of the group

$$G' := G^J/Z \simeq G \ltimes H', \qquad \text{where } H' := R^2.$$

Now G' contains H' as an abelian normal subgroup which allows determination of its unitary dual by means of the method described in the last section.

The first step is to determine the irreducible unitary representations of H'. This is very easy in our case because R is self-dual. Hence the unitary dual $\widehat{H'}$ identifies with R^2 itself by associating with $(m_1, m_2) \in R^2$ the unitary character

$$\begin{array}{rcl} R^2 & \longrightarrow & \mathbb{C}^*, \\ (\lambda, \mu) & \longmapsto & \psi^{m_1}(\lambda)\psi^{m_2}(\mu). \end{array}$$

G operates on H' by conjugation and thus also on $\widehat{H'}$:

$$\begin{array}{rcl} G \times \widehat{H'} & \longrightarrow & \widehat{H'}, \\ (M, \sigma) & \longmapsto & (X \mapsto \sigma(XM)) \end{array}$$

(XM means matrix multiplication). A small calculation shows that under the above identification $\widehat{H'} = R^2$ this operation goes over to the natural action

$$\begin{array}{rcl} G \times R^2 & \longrightarrow & R^2, \\ (M, Y) & \longmapsto & MY \end{array}$$

(now think of $Y \in R^2$ as a column vector). This makes it obvious that $\widehat{H'}$ decomposes into two G-orbits, one of them consisting only of the trivial representation. As a representative for the non-trivial characters we choose

$$\begin{aligned} \Psi:\ H' &\longrightarrow \mathbb{C}, \\ (\lambda,\mu) &\longmapsto \psi(\lambda) \end{aligned}$$

(corresponding to the point $(1,0) \in R^2$). The stabilizer of the trivial representation is certainly G itself, and the stabilizer of Ψ is

$$G'_\Psi = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} (\lambda,\mu) :\ c,\lambda,\mu \in R \right\}.$$

Theorem 2.3.1 gives the following result, where we leave it as an exercise to check the hypotheses in this theorem.

2.4.1 Proposition. *The irreducible unitary representations of G' are exactly the following:*

i) The representations σ where $\sigma\big|_{H'}$ is trivial and $\sigma\big|_{G}$ is an irreducible unitary representation of G.

ii) The representations $\mathrm{Ind}_{G'_\Psi}^{G'}\,\tau$, where τ runs through the irreducible unitary representation of G'_Ψ whose restriction to H' is a multiple of Ψ.

It remains to describe more closely the representations appearing in ii). Suppose τ is an irreducible unitary representation of G'_Ψ whose restriction to H' is a multiple of Ψ. Then an element $(\lambda,\mu) \in H'$ operates by multiplication with $\psi(\lambda)$. Thus every subspace which is invariant under the matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is yet invariant under G'_Ψ. Hence the restriction of τ to the matrix group must be irreducible. This group being isomorphic to R itself we see that our representation is one-dimensional and the matrices act through a unitary character of R. Conversely, given such a unitary character ψ^r with $r \in R$ it is immediately checked that

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} (\lambda,\mu) \longmapsto \psi^r(c)\psi(\mu)$$

defines a homomorphism $G'_\Psi \to \mathbb{C}^*$. So the representations τ from which we start our induction constitute a one-parameter family indexed by $r \in R$. Putting everything together we have the following result.

2.4.2 Theorem. *The irreducible unitary representations of G^J with trivial central character are exactly the following.*

i) The representations σ where $\sigma\Big|_H$ is trivial and $\sigma\Big|_G$ is an irreducible unitary representation of G.

ii) The representations $\mathrm{Ind}_{G'_\Psi}^{G^J}\,\tau_r$, where

$$\begin{array}{rcl} \tau_r:\ G'_\Psi & \longrightarrow & \mathbb{C}^*, \\ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}(\lambda,\mu) & \longmapsto & \psi(rc+\mu). \end{array}$$

2.5 The Schrödinger-Weil representation

It will turn out in the following section that every irreducible unitary (respectively smooth) representation π of G^J with non-trivial central character can be written as a tensor product of two representations, where one factor is a certain standard representation independent of π. The present section is devoted to introducing this so-called **Schrödinger-Weil representation**, which is not really a representation of G^J but a projective one. The construction is standard and carried out in much greater generality in [We].

Let R be the real or complex numbers, a $\mathfrak{p}$-adic field, or the adele ring of a number field, and consider $G^J = G \ltimes H$ over R. The starting point is the Schrödinger representation

$$\pi_S^m:\ H \longrightarrow \mathrm{GL}(V)$$

with central character ψ^m, $m \in R^*$, which was discussed in Section 2.2. Now G operates on H by conjugation inside G^J in the following way:

$$\begin{array}{rcl} G \times H & \longrightarrow & H, \\ (M,h) & \longmapsto & MhM^{-1} = (XM^{-1},\kappa), \qquad h = (X,\kappa),\ X \in R^2,\ \kappa \in R. \end{array}$$

In particular, M leaves the central part of h untouched. Hence the irreducible unitary representation

$$\begin{array}{rcl} H & \longrightarrow & \mathrm{GL}(V) \\ h & \longmapsto & \pi_S^m(MhM^{-1}) \end{array}$$

has central character ψ^m, just like π_S^m. By the Stone–von Neumann theorem, this conjugated representation must be equivalent to π_S^m itself, i.e., there is a unitary operator

$$\pi_W^m(M):\ V \longrightarrow V$$

such that

$$\pi_S^m(MhM^{-1}) = \pi_W^m(M)\pi_S^m(h)\pi_W^m(M)^{-1} \qquad \text{for all } h \in H. \tag{2.3}$$

By Schur's lemma, $\pi_W^m(M)$ is determined up to nonzero scalars. We fix one $\pi_W^m(M)$ for each $M \in G$ arbitrarily. Now for $M_1, M_2 \in G$ we have

$$\begin{aligned}&\pi_W^m(M_1)\pi_W^m(M_2)\pi_S^m(h)\pi_W^m(M_2)^{-1}\pi_W^m(M_1)^{-1}\\ &\quad = \quad \pi_W^m(M_1M_2)\pi_S^m(h)\pi_W^m(M_1M_2)^{-1},\end{aligned}$$

and again by Schur's lemma there must exist a scalar $\lambda(M_1, M_2)$ of absolute value 1 such that

$$\pi_W^m(M_1M_2) = \lambda(M_1, M_2)\pi_W^m(M_1)\pi_W^m(M_2). \tag{2.4}$$

From the associativity law in G it follows that

$$\lambda(M_1M_2, M_3)\lambda(M_1, M_2) = \lambda(M_1, M_2M_3)\lambda(M_2, M_3),$$

which just says that λ is a 2-cocycle for the trivial G-modul S^1. The freedom in multiplying the operators $\pi_W^m(M)$ with scalars of absolute value 1 amounts to changing λ by a coboundary. Hence the representation π_S^m we started with determines in a unique way an element

$$\lambda \in H^2(G, S^1).$$

From [We] or [Ku1] it is known that

- $H^2(G(R), S^1)$ is trivial if $R = \mathbb{C}$.
- $H^2(G(R), S^1)$ consists of exactly two elements if $R = \mathbb{R}$ or $R = F$ a $\mathfrak{p}$-adic field.

It is further known that λ represents the non-trivial element of $H^2(G(R), S^1)$ if R is real or $\mathfrak{p}$-adic. In [Ge2] a version of this cocycle can be found which has the property that

$$\lambda\Big|_{\mathcal{O}^* \times \mathcal{O}^*} = 1 \qquad\qquad \text{if } R \text{ is } \mathfrak{p}\text{-adic and not an extension of } \mathbb{Q}_2.$$

We will use in all that follows this cocycle in the real or $\mathfrak{p}$-adic case, $\lambda = 1$ in the complex case, and the product of the corresponding local cocycles in the adelic case. Coming back to the above notations we see that

$$M \longmapsto \pi_W^m(M)$$

is a projective representation of G on V with multiplier λ. It is called the **Weil representation** with character ψ^m. Note that π_W^m is an ordinary representation exactly in the complex case. Otherwise we can make π_W^m into an ordinary representation by going over to the **metaplectic group** Mp (also denoted $\widetilde{G}$, or $\mathrm{Mp}(R)$), which is by definition the topological group extension of G by $\{\pm 1\}$ determined by the cocycle λ. In other words, as a set we have

$$\mathrm{Mp} = G \times \{\pm 1\},$$

the multiplication is defined by

$$(M,\varepsilon)(M',\varepsilon') = (MM', \lambda(M,M')\varepsilon\varepsilon'),$$

and there is an exact sequence of topological groups

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp} \longrightarrow G \longrightarrow 1.$$

Now the map

$$(M,\varepsilon) \longmapsto \pi_W^m(M)\varepsilon$$

obviously defines a representation of Mp in the ordinary sense. It is also called the **Weil representation**.

We put the Schrödinger and the Weil representation together and define

$$\begin{aligned} \pi_{SW}^m :\ G^J &\longrightarrow \mathrm{GL}(V), \\ hM &\longmapsto \pi_S^m(h)\pi_W^m(M) \qquad\qquad \text{for all } h \in H,\ M \in G. \end{aligned}$$

The defining property (2.3) of π_W^m immediately shows that π_{SW}^m is a projective representation of G^J with multiplier λ, the latter extended canonically to G^J. It is called the **Schrödinger-Weil representation** of G^J with central character ψ^m. We give the same name to the corresponding ordinary representation of the two-fold cover $\widetilde{G^J}$ of G^J which is defined analogously to G^J. Note that there is a commutative diagram

$$\begin{array}{ccc} \widetilde{G^J} & \longrightarrow & \widetilde{G} \\ \downarrow & & \downarrow \\ G^J & \longrightarrow & G \end{array}$$

and that $\widetilde{G^J}$ identifies with the semidirect product of $\widetilde{G}$ with H.

Finally we give some explicit formulas for the Weil representation. There will be the appearence of the so-called **Weil constant**. This is a function

$$\gamma :\ R^* \longrightarrow S^1$$

which depends on the different cases and on the character ψ^m.

- If $R = \mathbb{C}$ then γ is the constant function 1.
- If $R = \mathbb{R}$ then
$$\gamma(a) = e^{\pi i\, \mathrm{sgn}(m)\mathrm{sgn}(a)/4}.$$
- If $R = F$ is a $\mathfrak{p}$-adic field, then
$$\gamma(a) = \lim_{n\to\infty} \int\limits_{\omega^{-n}\mathcal{O}} \psi^m(ax^2)\,dx \Big/ |\ldots|.$$
- If $R = \mathbb{A}$ then γ is the (well-defined) product of local Weil constants.

If the dependence on the character ψ^m is to be emphasized, we write γ_m instead of γ. Though not obvious in the non-archimedean case, the Weil constant is always an eighth root of unity (see [We] or [Sch1]).

As a further ingredient to the explicit formulas below there is the (second) **Hilbert symbol**

$$(\cdot,\cdot):\ R^* \times R^* \longrightarrow \{\pm 1\}.$$

If R is a local field then it is defined as

$$(a,b) = 1 \quad \Longleftrightarrow \quad b \text{ is a norm from } R(\sqrt{a}).$$

In particular the Hilbert symbol is constantly 1 in the complex case. The global Hilbert symbol is defined to be the product of the local symbols. More about Hilbert symbols can be found in texts on algebraic number theory.

Now we are ready to state the explicit formulas for the Weil representation. As a model for π_S^m the Schwartz space $\mathcal{S}(R)$ is used. Then the associated Weil representation acts on the same space as follows.

$$\left(\pi_W^m\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}f\right)(x) = \psi^m(bx^2)f(x). \tag{2.5}$$

$$\left(\pi_W^m\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}f\right)(x) = (a,-1)\gamma(a)\gamma(1)^{-1}|a|^{1/2}f(ax). \tag{2.6}$$

$$\pi_W^m\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}f = \gamma(1)\hat{f}. \tag{2.7}$$

Here $\hat{f}$ denotes the Fourier transformation of $f \in \mathcal{S}(R)$:

$$\hat{f}(x) = |2m|^{1/2}\int_R f(y)\psi^m(2xy)\,dy.$$

The factor $|2m|^{1/2}$ normalizes the measure on R to make Fourier inversion hold:

$$\hat{\hat{f}}(x) = f(-x).$$

It is not easy to deduce the formulas (2.5)–(2.7), but it is easy to prove them. It just has to be checked that (2.3) holds with these operators, but we will not carry this out. For the real case, see [Mum] Lemma 8.2 or [LV] Section 2.5.

Assume now R to be a local field. Since the Schrödinger representation is irreducible, the Schrödinger-Weil representation is also. But if we restrict π_{SW}^m to $\mathrm{SL}(2,R)$, i.e., we consider the Weil representation π_W^m, then from the formulas (2.5)–(2.7) we immediately find the invariant subspaces $\mathcal{S}(F)^+$ and $\mathcal{S}(F)^-$ consisting of even resp. odd Schwartz functions. Let $\pi_W^{m\pm}$ denote the subrepresentations on these spaces. They are called the **positive** (resp. **negative**) or **even** (resp. **odd**) **Weil representations**.

2.5.1 Proposition. *The positive and negative Weil representations are irreducible, and we have*

$$\pi_W^m = \pi_W^{m+} \oplus \pi_W^{m-}.$$

Between the irreducible Weil representations there are exactly the following equivalences:

$$\pi_W^{m\pm} \simeq \pi_W^{m'\pm} \qquad \Longleftrightarrow \qquad mF^{*2} = m'F^{*2}.$$

Proof: It is easy to see from (2.5) and (2.7) that the isomorphism

$$\begin{array}{ccc} \mathcal{S}(F) & \longrightarrow & \mathcal{S}(F), \\ f & \longmapsto & (x \mapsto f(ax)), \end{array}$$

intertwines π_W^m with $\pi_W^{a^2m}$, for any $a \in F^*$. So if $R = \mathbb{C}$, we are done. The case $R = \mathbb{R}$ will follow from our considerations in the first part of Section 3.2. For the $\mathfrak{p}$-adic case, see [MVW] 2.II.1. $\square$

2.6 Representations of G^J with non-trivial central character

Let again G^J be the real or $\mathfrak{p}$-adic Jacobi group. In principle Mackey's method could also be used to determine the unitary representations of G^J with non-trivial central character. Since the Heisenberg group is not commutative, one would have to check carefully the hypotheses made in [Ma1]. However, we prefer a direct method similar to the construction in [We]. The procedure is also described in Kirillov [Ki] pp. 218–219.

When dealing with the real Jacobi group, we are interested in unitary representations, while for the $\mathfrak{p}$-adic Jacobi group, we consider smooth representations. Both cases can be treated in a very similar way. The decisive point is to have the Stone–von Neumann theorem at hand. We treat the unitary case and leave the minor changes for the $\mathfrak{p}$-adic case to the reader.

So let π be a unitary representation of the real Jacobi group G^J on a Hilbert space V with central character ψ^m, $m \neq 0$. The restriction of π to the Heisenberg group decomposes into unitary representations, each of which must be equivalent to the Schrödinger representation π_S^m with central character ψ^m, by the Stone–von Neumann theorem 2.2.1. So this restriction is isotypical, and consequently we may assume that V is a Hilbert tensor product

$$V = V_1 \otimes V_2,$$

where H acts trivially on V_1 and where V_2 is a representation space for π_S^m.

From the defining property (2.3) of π_W^m, which also acts on V_2, it follows easily that

$$\pi(M^{-1})(\mathbf{1}_{V_1}\otimes\pi_W^m(M))\pi(h)=\pi(h)\pi(M^{-1})(\mathbf{1}_{V_1}\otimes\pi_W^m(M)),$$

i.e., the operator $\pi(M^{-1})(\mathbf{1}_{V_1}\otimes\pi_W^m(M))$ commutes with the action of the Heisenberg group. Hence it must be of the form

$$\pi(M^{-1})(\mathbf{1}_{V_1}\otimes\pi_W^m(M))=\tilde{\pi}(M)\otimes\mathbf{1}_{V_2}\qquad\text{with } \tilde{\pi}(M)\in\mathrm{Aut}(V_1).$$

As a result we were able to separate the action of G in one on V_1 and one on V_2:

$$\pi(M)=\tilde{\pi}(M)\otimes\pi_W^m(M).\tag{2.8}$$

More generally, for every element $g=hM$ of the Jacobi group with $M\in G$ and $h\in H$ we have

$$\pi(hM)=\tilde{\pi}(M)\otimes\pi_{SW}^m(hM),$$

where π_{SW}^m is the Schrödinger-Weil representation introduced in the last chapter. From (2.4) it follows that for M_1, $M_2\in G$

$$\tilde{\pi}(M_1M_2)=\lambda(M_1,M_2)^{-1}\tilde{\pi}(M_1)\tilde{\pi}(M_2).$$

In other words, $\tilde{\pi}$ and π_{SW}^m are both projective representations of G resp. G^J with multiplier λ^{-1} resp. λ. After tensorizing the cocycles cancel and the result is an ordinary representation of G^J. Summarizing we obtain the following result.

2.6.1 Theorem. *The above construction gives a 1-1 correspondence*

$$\tilde{\pi}\longmapsto\tilde{\pi}\otimes\pi_{SW}^m$$

between the irreducible unitary projective representations of $\mathrm{SL}(2,\mathbb{R})$ *with multiplier* λ *and the irreducible unitary representations of* $G^J(\mathbb{R})$ *with non-trivial central character* ψ^m*.*

The corresponding non-archimedean result is as follows.

2.6.2 Theorem. *Let* F *be a* $\mathfrak{p}$*-adic field. There is a 1-1 correspondence*

$$\tilde{\pi}\longmapsto\tilde{\pi}\otimes\pi_{SW}^m$$

between the irreducible smooth projective representations of $\mathrm{SL}(2,F)$ *with multiplier* λ *and the irreducible smooth representations of* $G^J(F)$ *with non-trivial central character* ψ^m*.*

The only difference in the complex case is that π_{SW}^m is a representation, not a projective one. Then $\tilde{\pi}$ will also turn out to be a representation of G, and we get the following result:

2.6.3 Theorem. *The map*

$$\tilde{\pi} \longmapsto \tilde{\pi} \otimes \pi_{SW}^m$$

establishes a 1-1 correspondence between irreducible, unitary representation of $\mathrm{SL}(2,\mathbb{C})$ *and irreducible, unitary representations of* $G^J(\mathbb{C})$ *with central character* ψ^m *(*$m \in \mathbb{C}^*$*).*

We refer the reader to Knapp [Kn] II, §4, for a classification of the irreducible, unitary representations of $\mathrm{SL}(2,\mathbb{C})$, and thus for a classification of irreducible, unitary representations of $G^J(\mathbb{C})$.

Much more will be said in the following chapters about the correspondence $\tilde{\pi} \mapsto \tilde{\pi} \otimes \pi_{SW}^m$, with specific reference to the underlying field.

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