Chapter 2

Classical Polygons

2.1 Introduction

The notion of a *generalized polygon* arose from the classification of *trialities* of the geometry of the quadric $\mathbf{Q}(7, \mathbb{K})$ in $\mathbf{PG}(7, \mathbb{K})$, $\mathbb{K}$ any field (see Subsection 2.4.2 below) in TITS [1959]. But already in TITS [1954], [1955] generalized polygons arise as a geometrical interpretation of complex Lie groups. This generalized the well-known connection between the *classical* complex Lie groups and the complex projective spaces and quadrics. It was noticed by TITS [1959] that an axiomatization of these geometries in the relative rank 2 case gave rise to geometries (generalized polygons) corresponding to the exceptional groups of Lie type G_2 (*Dickson's groups*). Also the geometries related to *twisted versions* of groups of Lie type are covered; in particular the Ree groups of type ${}^2F_4$ produce generalized octagons (see TITS [1960]). The projective planes and generalized quadrangles corresponding to classical groups were called *classical polygons* (see KANTOR [1986a]). Soon the term *classical* became a synonym of *corresponding to a group of Lie type* for some authors, but others stuck to the corresponding group notion, i.e., classical polygons correspond to classical groups. When TITS [1974], [1976a] introduced his ideas on the *Moufang condition* and *Moufang polygons*, the term *classical* was sometimes indistinguishable from the term *Moufang*. The situation at this moment is that the class of *Moufang polygons* is a well-defined class of generalized polygons, but the proof of Tits' (and Weiss') enumeration has not yet been completely published; see Chapter 5 for more details. On the other hand, the class of *classical polygons* is not so well defined, but once a point of view is taken, all classical polygons can usually be enumerated. For instance, in PAYNE & THAS [1984], a finite classical quadrangle is one that can be fully embedded in a finite projective space. All of them are known (see Chapter 8, in particular Theorem 8.5.16).

We will take the following point of view. Roughly, the generalized polygons which will be defined in this chapter will be called *classical*. This term is motivated by the fact that these polygons are studied much more than others and hence serve

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as the *classical objects* (in the sense of *standard examples*). One exception will be the Ree–Tits octagons, which we will only explicitly introduce in Chapter 3 and which we also want to call *classical*; another exception is the projective plane related to a non-associative alternative division ring, which we introduce in this chapter, but which will not be called *classical* for historical reasons. In this way, all finite Moufang polygons will be called *classical*. In addition to these classical examples, there are other examples of *Moufang polygons*, and we will define some in Chapter 3, the notable examples being the *mixed quadrangles* and *mixed hexagons*. Finally, when discussing the Moufang property in Chapter 5, we will meet other polygons, and we will give them names, too. The main thing is that we will have a name for large classes of polygons sharing some important common group-theoretic (and geometric) properties. This enables one to state propositions in an elegant way. Note that the classes that we will define will not necessarily be disjoint. For instance, we will see that some symplectic quadrangles are both classical and mixed.

It is worth remarking that, by definition, the dual of a classical polygon is always itself a classical polygon, unlike in Payne & Thas [1984] and unlike the other names that we will assign to subclasses of classical polygons. So if it matters what the points are, then we call a classical polygon by its specific name, such as orthogonal quadrangle, twisted triality hexagon. If it doesn't matter, then we simply say "classical polygon".

Concerning notation, we have chosen to follow Payne & Thas [1984] for the finite generalized quadrangles, and one class of hexagons. This implies that we do not use the group notation used by Kantor [1986a]. A table with "translations" (Table 2.1) can be found at the end of this chapter.

2.2 Classical and alternative projective planes

Since we want to emphasize generalized n-gons for $n > 3$, we will be brief in this section. The **classical projective planes** are in fact the *Desarguesian planes*. Other, related, planes are the *alternative planes*. Among the Desarguesian planes, one has the more restricted class of *Pappian* planes.

Desarguesian projective planes

2.2.1 Construction

Let V be a three-dimensional right vector space over a skew field $\mathbb{K}$ (i.e., the scalars are written on the right of the vectors). We define $\Gamma = \mathbf{PG}(2,\mathbb{K}) = (\mathcal{P},\mathcal{L},I)$ as follows. The points of Γ are the 1-spaces of V; the lines of Γ are the 2-spaces of V; incidence is symmetrized inclusion (here, an i-space is just an i-dimensional subspace, $i = 1, 2$; we will use this terminology later for arbitrary i in n-dimensional vector spaces, $n \geq i$).

Another way of seeing $\mathbf{PG}(2,\mathbb{K})$ is as follows. The elements of $\mathcal{P}$ are the triples $(x,y,z)\in\mathbb{K}\times\mathbb{K}\times\mathbb{K}$ up to a right non-zero scalar, $(x,y,z)\neq(0,0,0)$; the elements of $\mathcal{L}$ are the triples $[u,v,w]$ up to a left non-zero scalar, $[u,v,w]\neq[0,0,0]$; the point represented by (x,y,z) is incident with the line represented by $[u,v,w]$ if and only if $ux+vy+wz=0$.

A third way of seeing this geometry is the following. The points are of three types: the pairs $(x,y)\in\mathbb{K}\times\mathbb{K}$, the elements (m), $m\in\mathbb{K}$ and a symbol (∞). The lines are dually also of three types: the pairs $[m,k]\in\mathbb{K}\times\mathbb{K}$, the elements $[x]$, $x\in\mathbb{K}$ and the symbol $[\infty]$. Incidence is defined as follows: the point (∞) is incident with $[\infty]$ and $[x]$, for all $x\in\mathbb{K}$; the point (m), $m\in\mathbb{K}$, is incident with $[\infty]$ and $[m,k]$ for all $k\in\mathbb{K}$; the point (x,y), $x,y\in\mathbb{K}$, is incident with $[x]$, for all $x\in\mathbb{K}$, and with $[m,k]$ **if and only if** $mx+y=k$.

It is an easy exercise to show that these three ways produce isomorphic projective planes (in particular, the second description arises from the first by introducing coordinates in V). We call them the **Desarguesian planes (over** $\mathbb{K}$**)**. They have the following well-known characterizing property (see HILBERT [1899], BAER [1942]):

2.2.2 Theorem. *A projective plane* Γ *is Desarguesian* **if and only if** *the following holds: for all triples* (a_1,a_2,a_3) *and* (b_1,b_2,b_3) *of pairwise distinct points, with* $a_i\neq b_i$ *and* $a_ib_i\neq a_jb_j$, $i,j\in\{1,2,3\}$, $i\neq j$, *the lines* a_1b_1, a_2b_2 *and* a_3b_3 *are concurrent* <u>*if and only if*</u> *the points* $a_1a_2\cap b_1b_2$, $a_2a_3\cap b_2b_3$ *and* $a_1a_3\cap b_1b_3$ *are collinear.* □

The configuration formed by the ten points a_i, b_i, $a_ia_j\cap b_ib_j$, $a_1b_1\cap b_1b_2$, $i,j\in\{1,2,3\}$, $i\neq j$, and the ten lines a_ib_i, a_ia_j, b_ib_j and the line containing the points $a_1a_2\cap b_1b_2$ and $a_2a_3\cap b_2b_3$, $i,j\in\{1,2,3\}$, $i\neq j$, in a Desarguesian projective plane (but also in any plane whenever the two conditions of the previous theorem are satisfied for these points and lines) is usually called a **Desargues configuration**.

If the skew field $\mathbb{K}$ is commutative, i.e., if $\mathbb{K}$ is a field, then we sometimes call $\mathbf{PG}(2,\mathbb{K})$ **a Pappian plane**. All Pappian planes share the following well-known characterizing property (see e.g. HUGHES & PIPER [1973]):

2.2.3 Theorem. *A projective plane* Γ *is Pappian* **if and only if** *for all triples* (a_1,a_2,a_3) *and* (b_1,b_2,b_3) *of distinct collinear points, with* $a_i\notin b_1b_2$ *and* $b_i\notin a_1a_2$, $i=1,2,3$, *the points* $a_1b_2\cap b_1a_2$, $a_2b_3\cap b_2a_3$ *and* $a_1b_3\cap b_1a_3$ *are collinear.* □

The configuration induced by the points and lines in the statement of the previous theorem is, similarly as above with the Desargues configuration, usually called a **Pappus configuration**.

Desarguesian planes are also called **classical planes**.

Alternative projective planes

2.2.4 Construction

Let $\mathbb{D}$ be an **alternative field**, also called an **alternative division ring**, i.e., addition in $\mathbb{D}$ defines a commutative group; multiplication in $\mathbb{D}^\times$, where $\mathbb{D}^\times$ denotes $\mathbb{D}$ without the neutral element 0 with respect to addition, has a neutral element 1; there is a two-sided inverse x^{-1} for every element $x \neq 0$; both distributive laws hold; and $(yx)x^{-1} = y$, $y = x^{-1}(xy)$ for all $x, y \in \mathbb{D}$. We define the following geometry Γ. The points of Γ are of three types: the pairs $(x, y) \in \mathbb{D} \times \mathbb{D}$, the elements (m), $m \in \mathbb{D}$ and a symbol (∞). The lines are dually also of three types: the pairs $[m, k] \in \mathbb{D} \times \mathbb{D}$, the elements $[x]$, $x \in \mathbb{D}$ and the symbol $[\infty]$. Incidence is defined as follows: the point (∞) is incident with $[\infty]$ and $[x]$, for all $x \in \mathbb{D}$; the point (m), $m \in \mathbb{D}$, is incident with $[\infty]$ and $[m, k]$ for all $k \in \mathbb{D}$; the point (x, y), $x, y \in \mathbb{D}$, is incident with $[x]$, for all $x \in \mathbb{D}$, and with $[m, k]$ **if and only if** $mx + y = k$.

This is completely similar to the third construction of a Desarguesian plane above. Hence, if $\mathbb{D}$ is a skew field, then Γ is the Desarguesian plane over $\mathbb{D}$. If $\mathbb{D}$ is not a skew field, then we call Γ an **alternative plane (over $\mathbb{D}$)**. Alternative planes and Desarguesian planes share the following characterizing property (see again e.g. Hughes & Piper [1973]):

2.2.5 Theorem. *A projective plane is alternative or classical* **if and only if** *for every point p and every line L incident with p, for all triples of pairwise distinct points (a_1, a_2, a_3) and (b_1, b_2, b_3) with $a_i \neq b_i$ and $a_ib_i \neq a_jb_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$, such that p is incident with a_kb_k, $k = 1, 2$, and $a_1a_2 \cap b_1b_2$ and $a_2a_3 \cap b_2b_3$ are incident with L, we have that p is incident with a_3b_3 if and only if L is incident with $a_1a_3 \cap b_1b_3$.* □

The configuration induced by the elements in the previous theorem is a special case of the Desargues configuration; it is often called the **little Desargues configuration** (see Hughes & Piper [1973]).

The reason why we mention these characterizations is the following. From the description of alternative planes, one might get the impression that one flag, namely $\{(\infty), [\infty]\}$, plays a special role. But that is not true. It only plays a special role in the construction. The characterization theorems make this clear because no special flag is hypothesized there.

We classify non-associative alternative fields in Appendix B.

2.3 Classical generalized quadrangles

In this section, we define the classical generalized quadrangles. Our definition is based on Chapter 10 of Bruhat & Tits [1972], modified slightly by Tits [1995]; see also Chapter 8 of Tits [1974].

2.3.1 σ-quadratic forms

Let $\mathbb{K}$ be a skew field and σ an anti-automorphism of order at most 2. This implies in particular that $\mathbb{K}$ is a field if σ is the identity because in that case $ab = (ab)^\sigma = b^\sigma a^\sigma = ba$, for all $a, b \in \mathbb{K}$. Let V be a — not necessarily finite-dimensional — right vector space over $\mathbb{K}$ and let $g : V \times V \to \mathbb{K}$ be a $(\sigma, 1)$-linear form, i.e., for all $v_1, v_2, w_1, w_2 \in V$ and all $a_1, a_2, b_1, b_2 \in \mathbb{K}$, we have

$$g(v_1a_1 + v_2a_2, w_1b_1 + w_2b_2) = \\ a_1^\sigma g(v_1, w_1)b_1 + a_1^\sigma g(v_1, w_2)b_2 + a_2^\sigma g(v_2, w_1)b_1 + a_2^\sigma g(v_2, w_2)b_2.$$

We define $f : V \times V \to \mathbb{K}$ as follows:

$$f(x, y) = g(x, y) + g(y, x)^\sigma.$$

It is clear that f is also $(\sigma, 1)$-linear and moreover f satisfies $f(x, y)^\sigma = f(y, x)$, for all $x, y \in V$. Therefore we say that f is a **(σ-)Hermitian form**. Denote $\mathbb{K}_\sigma := \{t^\sigma - t : t \in \mathbb{K}\}$. We define $q : V \to \mathbb{K}/\mathbb{K}_\sigma$ as

$$q(x) = g(x, x) + \mathbb{K}_\sigma,$$

for all $x \in V$. We call q a **σ-quadratic form (over $\mathbb{K}$)**. Let W be a subspace of V. We say that q is **anisotropic over** W if $q(w) = 0$ **if and only if** $w = 0$, for all $w \in W$ (where we have written the zero vector as 0, and the element $0 + \mathbb{K}_\sigma$ also as 0; the context always makes it clear which zero is meant by "0"). It is **non-degenerate** if it is anisotropic over the subspace $\{v \in V : f(v, w) = 0, \text{ for all } w \in V\}$. From now on we assume that q is non-degenerate.

Note that, if $q(v) = 0$, then $q(vk) = 0$, for all $k \in \mathbb{K}$. Indeed, $q(v) = 0$ is equivalent to $g(v, v) \in \mathbb{K}_\sigma$, so put $g(v, v) = t^\sigma - t$. But then $g(vk, vk) = k^\sigma g(v, v)k = (k^\sigma tk)^\sigma - (k^\sigma tk)$. Hence the inverse image $q^{-1}(0)$ is a union of 1-spaces. We say that q **has Witt index** 2, if $q^{-1}(0)$ contains 2-spaces, but no higher-dimensional subspaces.

For a non-degenerate σ-quadratic form q over $\mathbb{K}$ with Witt index 2, we define the following geometry $\Gamma = \mathsf{Q}(V, q)$. The points of Γ are the 1-spaces in $q^{-1}(0)$; the lines are the 2-spaces in $q^{-1}(0)$; and incidence is symmetrized inclusion.

Before showing that $\mathsf{Q}(V, q)$ is a weak generalized quadrangle, we start with a lemma.

2.3.2 Lemma. *With the above notation, we have* $\mathbb{K} \neq \mathbb{K}_\sigma$.

Proof. The lemma is clear if σ is the identity. If σ is not the identity, then we remark that every element x of $\mathbb{K}_\sigma$ satisfies the relation $x^\sigma = -x$. Hence if $\mathbb{K} = \mathbb{K}_\sigma$, then $x^\sigma = -x$ for all $x \in \mathbb{K}$. If the characteristic of $\mathbb{K}$ is equal to 2, then this implies that $\sigma = 1$; if the characteristic of $\mathbb{K}$ is not equal to 2, then this implies that $1^\sigma = -1 \neq 1$, a contradiction as well. $\square$

2.3.3 Theorem (Bruhat & Tits [1972]). *The geometry* $\mathsf{Q}(V,q)$ *defined above is a weak generalized quadrangle.*

Proof. We first show that, given a point p and a line L of $\mathsf{Q}(V,q)$, with p not incident with L, there exists a unique point p' incident with L and collinear with p. Since the line pp' will then be well defined (because two distinct 1-spaces in V define a unique 2-space), this implies that there is also a unique line pp' incident with p and concurrent with L. Let v be a vector on the 1-space corresponding to p, and let a, b be two non-proportional vectors in the 2-space corresponding to L. To show that there is at least one such a point p', we may assume that p is not collinear with the points of $\mathsf{Q}(V,q)$ represented by a and b.

Now we investigate what it means algebraically for two vectors v and a to represent collinear points. By definition, this means that $q(vk_1+ak_2)=0$, for all $k_1,k_2\in\mathbb{K}$. Since $q(v)=q(a)=0$, this is equivalent to

$$\begin{aligned} g(vk_1+ak_2, vk_1+ak_2) &= k_1^\sigma g(v,a)k_2 + k_2^\sigma g(a,v)k_1 \\ &= k_1^\sigma f(v,a)k_2 + (k_1^\sigma g(a,v)^\sigma k_2)^\sigma - k_1^\sigma g(a,v)^\sigma k_2 \in \mathbb{K}_\sigma, \end{aligned}$$

hence to $k_1^\sigma f(v,a)k_2\in\mathbb{K}_\sigma$, for all $k_1,k_2\in\mathbb{K}$. If $f(v,a)\neq 0$, then this implies $\mathbb{K}=\mathbb{K}_\sigma$, contradicting Lemma 2.3.2. Hence $f(v,a)=0$. Conversely, it follows from $f(v,a)=0$ that the points $v\mathbb{K}$ and $a\mathbb{K}$ of $\mathsf{Q}(V,q)$ are collinear (use the same equality above).

Hence we may assume that $f(v,a)\neq 0\neq f(v,b)$, and we know that $f(a,b)=0$. We are looking for a scalar $k\in\mathbb{K}$ such that $f(v,a+bk)=0$. Clearly

$$k=-f(v,b)^{-1}f(v,a)$$

satisfies that condition. Hence we have shown that there is at least one point p' on L collinear with p. Suppose there are two such points. Without loss of generality, we may take the points $a\mathbb{K}$ and $b\mathbb{K}$. Hence $f(v,a)=f(v,b)=0$. Note that $f(v,v)=0$; indeed, $q(v)=0$ by definition, so $g(v,v)\in\mathbb{K}_\sigma$, hence we can put $g(v,v)=t^\sigma-t$. But $f(v,v)=g(v,v)+g(v,v)^\sigma=t^\sigma-t+t-t^\sigma=0$. So by the linearity $f(vk_1+ak_2+bk_3, v\ell_1+a\ell_2+b\ell_3)=0$, for all $k_i,\ell_i\in\mathbb{K}$, $i\in\{1,2,3\}$. This readily implies that the subspace of V generated by v,a,b is contained in $q^{-1}(0)$. Since the Witt index equals 2, we necessarily have $p\,\mathbf{I}\,L$, a contradiction.

So we have shown condition (i) of Lemma 1.4.1 (page 15). We now show condition $(ii)'$ of that same lemma.

It is clear that every line in $\mathsf{Q}(V,q)$ contains $|\mathbb{K}|+1$ points, hence all lines are thick. Also, it is clear that two distinct points of $\mathsf{Q}(V,q)$ are contained in at most one line, since two different 1-spaces of V are contained in exactly one 2-space of V.

All that is left to show is that every point is incident with at least two lines. To that end, we first claim that for every point p in $\mathsf{Q}(V,q)$, there is at least one point p' of $\mathsf{Q}(V,q)$ not collinear with p in $\mathsf{Q}(V,q)$. Indeed, suppose $p=v\mathbb{K}$. Let w be any vector of V and suppose that $w\mathbb{K}$ is not a point of $\mathsf{Q}(V,q)$. Note that $w\mathbb{K}$ certainly exists since we may otherwise assume that every 1-space of V is a point of $\mathsf{Q}(V,q)$ collinear with p. This would imply $f(v,w)=0$, for every vector w, and since $q(v)=0$, this contradicts the non-degeneracy of q. Note that this argument implies also that we may assume that $f(v,w)\neq 0$. Now the vector $v+wk$, $k\in\mathbb{K}$, defines a point of $\mathsf{Q}(V,q)$ **if and only if** $q(v+wk)=0$. This condition is equivalent to

$$k^\sigma g(w,w)k+g(v,w)k+k^\sigma g(w,v)\in\mathbb{K}_\sigma.$$

Noting that $k^\sigma g(w,v)-g(w,v)^\sigma k\in\mathbb{K}_\sigma$, we deduce that, by dividing on the right by k,

$$k=-g(w,w)^{-\sigma}f(w,v)$$

is a non-zero solution, since $f(v,w)\neq 0$. Hence the claim.

Now let p be any point of $\mathsf{Q}(V,q)$. Since the Witt index is equal to 2, there is at least one line L in $\mathsf{Q}(V,q)$. By the first part of the proof, we may assume that $L\,\mathbf{I}\,p$. Let p' be a point of $\mathsf{Q}(V,q)$ not collinear with p, then there is a line M through p' meeting L in some point $p''\neq p$. There is some point p''' not collinear with p'' in $\mathsf{Q}(V,q)$. Hence there is a line M' through p''' meeting M in a point distinct from p''. But now there is a line L' through p meeting M', and clearly $L'\neq L$ (otherwise we violate condition (i) of Lemma 1.4.1 that we showed above). So we have at least two lines L,L' in $\mathsf{Q}(V,q)$ through p.

The theorem now follows from Lemma 1.4.1. □

Next, we look for a standard equation for q.

2.3.4 Proposition. *Let q be a non-degenerate σ-quadratic form over $\mathbb{K}$ on the vector space V. Then there exist four vectors e_i, $i\in\{-2,-1,1,2\}$, a direct sum decomposition*

$$V=e_{-2}\mathbb{K}\bigoplus e_{-1}\mathbb{K}\bigoplus V_0\bigoplus e_1\mathbb{K}\bigoplus e_2\mathbb{K}$$

and a non-degenerate anisotropic σ-quadratic form $q_0:V_0\to\mathbb{K}/\mathbb{K}_\sigma$ such that for all $v=e_{-2}x_{-2}+e_{-1}x_{-1}+v_0+e_1x_1+e_2x_2$, with $x_i\in\mathbb{K}$, $i\in\{-2,-1,1,2\}$ and $v_0\in V_0$,

$$q(v)=x_{-2}^\sigma x_2+x_{-1}^\sigma x_1+q_0(v_0).$$

Proof. We already know that $\mathsf{Q}(V,q)$ is a weak quadrangle. Let $e_{-2}\mathbb{K}$, $e_{-1}\mathbb{K}$, $e_1\mathbb{K}$ and $e_2\mathbb{K}$ be four points of $\mathsf{Q}(V,q)$ such that $e_i\mathbb{K}$ and $e_j\mathbb{K}$ are opposite **if and only if** $i+j=0$. It is readily seen that these four 1-spaces cannot be contained in a 3-space (otherwise there arise triangles in $\mathsf{Q}(V,q)$). Hence the sum $e_{-2}\mathbb{K}+e_{-1}\mathbb{K}+e_1\mathbb{K}+e_2\mathbb{K}$ is direct. Now note that by the previous proof, we have

$$f(e_i,e_j)=0,\qquad\qquad i+j\neq 0.$$

Upon replacing e_i by a multiple, we may assume that $f(e_i,e_{-i})=1$, for all $i\in\{-2,-1,1,2\}$. Now we define the subspace

$$V_0:=\{v\in V:f(v,e_i)=0,\ \text{for all } i=-2,-1,1,2\}.$$

For any $v \in V$, it is an elementary exercise to check that the vector

$$v - e_{-2}f(e_2, v) - e_{-1}f(e_1, v) - e_1 f(e_{-1}, v) - e_2 f(e_{-2}, v)$$

belongs to V_0. Also, no non-zero vector v^* generated by e_{-2}, e_{-1}, e_1 and e_2 belongs to V_0. Indeed, such a vector v^* would satisfy by assumption $f(v^*, e_i) = 0$, for all $i \in \{-2, -1, 1, 2\}$, which implies, putting $v^* = e_{-2}a_{-2} + e_{-1}a_{-1} + e_1 a_1 + e_2 a_2$, $a_i \in \mathbb{K}$, that all a_i are equal to 0, $i \in \{-2, -1, 1, 2\}$, a contradiction. Therefore, the following sum is indeed direct:

$$V = e_{-2}\mathbb{K} \bigoplus e_{-1}\mathbb{K} \bigoplus V_0 \bigoplus e_1\mathbb{K} \bigoplus e_2\mathbb{K}.$$

Now let $v = e_{-2}x_{-2} + e_{-1}x_{-1} + v_0 + e_1 x_1 + e_2 x_2$, with $x_i \in \mathbb{K}$ and $v_0 \in V_0$, $i \in \{-2, -1, 1, 2\}$. Then one calculates easily that, using $a + \mathbb{K}_\sigma = a^\sigma + \mathbb{K}_\sigma$,

$$\begin{aligned} q(v) &= x_{-2}^\sigma f(e_{-2}, e_2)x_2 + x_{-1}^\sigma f(e_{-1}, e_1)x_1 + q(v_0) \\ &= x_{-2}^\sigma x_2 + x_{-1}^\sigma x_1 + q(v_0). \end{aligned}$$

Now let $q_0 : V_0 \to \mathbb{K}/\mathbb{K}_\sigma$ be the restriction of q to V_0. Suppose $q_0(v) = 0$, for some non-zero vector $v \in V_0$. Since we also have $f(v, e_i) = 0$, the point $v\mathbb{K}$ of $\mathsf{Q}(V, q)$ is collinear with $e_i\mathbb{K}$, for all $i \in \{-2, -1, 1, 2\}$, a contradiction.

The proposition is proved. □

To determine the order of $\mathsf{Q}(V, q)$, we still need to know how many lines there are through any point. The next proposition gives the answer.

2.3.5 Proposition. *Let* $\mathsf{Q}(V, q)$ *be a classical weak quadrangle and suppose* q *has the standard equation of Proposition* 2.3.4*. Define the set*

$$\widehat{X} = \{(v, k) \in V_0 \times \mathbb{K} : k \in -q_0(v)\}.$$

Then $\mathsf{Q}(V, q)$ *contains exactly* $|\widehat{X}| + 1$ *lines through each point.*

Proof. Without loss of generality, we may consider the point $e_1\mathbb{K}$. We already know that all points of $\mathsf{Q}(V, q)$ collinear with $e_1\mathbb{K}$ are represented by vectors v such that $f(v, e_1) = 0$. Also, the number of lines through $e_1\mathbb{K}$ is equal to the number of points of $\mathsf{Q}(V, q)$ collinear with both $e_1\mathbb{K}$ and $e_{-1}\mathbb{K}$. Such points have representatives satisfying in addition $f(v, e_{-1}) = 0$. It is now readily seen that $v \in e_{-2}\mathbb{K} + V_0 + e_2\mathbb{K}$, so we can put (with obvious notation) $v = e_{-2}x_{-2} + v_0 + e_2 x_2$. If $x_{-2} = 0$, then $0 = q(v) = q_0(v_0)$, so $v_0 = 0$ (since q_0 is anisotropic) and $v = e_2$. This already gives one point of $\mathsf{Q}(V, q)$ collinear with both $e_1\mathbb{K}$ and $e_{-1}\mathbb{K}$.

Suppose now $x_{-2} \neq 0$, then we may take $x_{-2} = 1$. We have $0 = q(v) = x_2 + q_0(v_0)$. Hence $x_2 \in -q_0(v_0)$. Conversely, if $x_2 \in -q_0(v_0)$, then the vector $v = e_{-2} + v_0 + e_2 x_2$ defines a point of $\mathsf{Q}(V, q)$ collinear with both $e_1\mathbb{K}$ and $e_{-1}\mathbb{K}$.

The proposition now follows easily. □

There is an interesting corollary to Proposition 2.3.5.

2.3.6 Corollary. *The weak quadrangle* $\mathsf{Q}(V, q)$ *is a generalized quadrangle* **if and only if** V *has dimension at least* 5 *or* σ *is not the identity.*

Proof. According to Proposition 2.3.5, the weak quadrangle $\mathsf{Q}(V, q)$ is not thick if and only if the set $\widehat{X}$ has only one element, i.e., if $0 \in V$ is the only element of V_0 and if $0 \in \mathbb{K}$ is the only element of $\mathbb{K}_\sigma$. Hence $\mathsf{Q}(V, q)$ is non-thick **if and only if** $V_0 = \{0\}$ and $\sigma = 1$. The corollary follows. □

The points and lines of the quadrangle $\mathsf{Q}(V, q)$ can be seen as living in the projective space $\mathbf{PG}(V)$ associated to V in the standard way. Therefore we sometimes refer to that representation of $\mathsf{Q}(V, q)$ as a **standard embedding** of $\mathsf{Q}(V, q)$.

For a discussion of the regular points and lines of the classical quadrangles, we refer to Proposition 3.4.8 on page 106. Also, in the subsections following Subsection 3.4.7, we prove that some classical quadrangles with $\sigma = 1$ are anti-isomorphic to certain other classical quadrangles with $\sigma \neq 1$.

2.3.7 Definitions. Members of the special class of classical quadrangles with $\sigma = 1$ are called **orthogonal quadrangles**. The rest is called **Hermitian quadrangles**. The duals of the classical quadrangles are also called **classical**.

Sometimes we also denote the orthogonal quadrangle $\mathsf{Q}(V, q)$ by $\mathsf{Q}(d-1, \mathbb{K}, q)$, where V is d-dimensional over $\mathbb{K}$. And the Hermitian quadrangle $\mathsf{Q}(V, q)$ with anti-automorphism σ is also sometimes denoted by $\mathsf{H}(d-1, \mathbb{K}, q, \sigma)$, where V is d-dimensional over $\mathbb{K}$.

In the finite case, a σ-quadratic form q of Witt index 2 is, up to isomorphism and up to a scalar factor, determined by the dimension, the field and the kind (= orthogonal or Hermitian). Hence we delete the q and the σ from the notation. For $\mathbb{K} \cong \mathbf{GF}(s)$, we then use $\mathsf{Q}(d, s)$ for the orthogonal quadrangle in d-dimensional projective space over $\mathbf{GF}(s)$ (and $d = 4, 5$, see below), and $\mathsf{H}(d, s)$ is the Hermitian quadrangle in d-dimensional projective space over $\mathbf{GF}(s)$ with corresponding involutory field automorphism $x \mapsto x^{\sqrt{s}}$ (and here, $d = 3, 4$, see also below).

2.3.8 Quadrics as orthogonal quadrangles

Let V be a right $(d+1)$-dimensional vector space over a (commutative) field $\mathbb{K}$ and let $\mathbf{PG}(V)$ denote the corresponding d-dimensional projective space. Let Q be a quadric in $\mathbf{PG}(V)$ of **Witt index** 2, i.e., Q contains lines but no planes of $\mathbf{PG}(V)$. Then the points and lines of Q are, with the natural incidence relation, the points and lines of a generalized quadrangle Γ. We show that *there is a* 1*-quadratic form* $q : V \to \mathbb{K}$ *such that* Γ *is isomorphic to* $\mathsf{Q}(d, \mathbb{K}, q)$.

Let Q have equation

$$\sum_{i \leq j = 0}^{d} a_{ij} x_i x_j = 0,$$

with $a_{ij} \in \mathbb{K}$, with respect to a basis in $\mathbf{PG}(V)$, or equivalently, in V. Put

$$g((x_0, x_1, \dots, x_d), (y_0, y_1, \dots, y_d)) = \sum_{i \le j = 0}^{d} a_{ij} x_i y_j.$$

This is clearly a bilinear form. The corresponding 1-quadratic form

$$q((x_0, x_1, \dots, x_d)) = \sum_{i \le j = 0}^{d} a_{ij} x_i x_j$$

is zero precisely on the 1-spaces of V which correspond to points of Q (noting that $\mathbb{K}_\sigma = \{0\}$ here).

Conversely, it is easily seen that any orthogonal quadrangle $\mathsf{Q}(d, \mathbb{K}, q)$ arises from a quadric with equation $q(v) = 0$. Hence the class of orthogonal quadrangles coincides with the class of quadrics of Witt index 2 in projective space.

2.3.9 Hermitian varieties as Hermitian quadrangles

Let V be a vector space over the skew field $\mathbb{K}$ and suppose that $f : V \times V \to \mathbb{K}$ is a σ-Hermitian form, i.e., f is $(\sigma, 1)$-linear and $f(v, w) = f(w, v)^\sigma$, with σ non-trivial. The corresponding Hermitian variety $\mathcal{H}$ in $\mathbf{PG}(V)$ is a generalized quadrangle $\Gamma(\mathcal{H})$ **if and only if** $\mathcal{H}$ contains lines but no planes.

Recall that the points of $\mathcal{H}$ correspond to the 1-spaces of V with representatives v such that $f(v, v) = 0$.

Suppose that the characteristic of $\mathbb{K}$ is not equal to 2. We choose a basis $(e_i)_{i \in J}$ in V and we put an arbitrary order on J. We define

$$\begin{aligned} g(e_i, e_j) &= f(e_i, e_j) && \text{if } i < j, \\ g(e_i, e_i) &= \tfrac{1}{2} f(e_i, e_i) && \\ g(e_i, e_j) &= 0 && \text{if } i > j. \end{aligned}$$

One can check easily that the associated σ-quadratic form q of the thus defined $(\sigma, 1)$-linear form reads

$$q(v) = \frac{1}{2} f(v, v) + \mathbb{K}_\sigma.$$

It is clear that, if $f(v, v) = 0$, then also $q(v) = 0$. Suppose now $q(v) = 0$. This means that, since $2^\sigma = 2$, $f(v, v) \in \mathbb{K}_\sigma$. Write $f(v, v) = k^\sigma - k$. Since $f(x, x) = f(x, x)^\sigma$, we have $k^\sigma = k$, hence $f(v, v) = 0$. Consequently $\mathsf{Q}(V, q)$ defines a quadrangle which is isomorphic to the quadrangle $\Gamma(\mathcal{H})$ defined above.

Suppose now that the characteristic of $\mathbb{K}$ is 2. We choose a basis $(e_i)_{i \in J}$ in V such that $f(e_i, e_i) = 0$, for all $i \in J$ (this can easily be done since we may assume that the points of $\mathcal{H}$ generate $\mathbf{PG}(V)$). Now we can define $g : V \times V \to \mathbb{K}$ as above (deleting the factor $\frac{1}{2}$ of course). One checks that $f(x, y) = g(x, y) + g(y, x)^\sigma$. Define

$$q : V \to \mathbb{K}/\mathbb{K}_\sigma : v \mapsto g(v, v) + \mathbb{K}_\sigma.$$

If k is a representative of $q(v)$ in $\mathbb{K}$, then $k = g(v,v) + l^\sigma + l$. Hence

$$f(v,v) = g(v,v) + g(v,v)^\sigma = g(v,v) + l^\sigma + l + g(v,v)^\sigma + l + l^\sigma = k + k^\sigma.$$

In general, however (i.e., when $\mathbb{K}$ is not commutative; see below for the commutative case), the quadrangle $\mathsf{Q}(V,q)$ is only a subquadrangle of $\Gamma(\mathcal{H})$. Indeed, if $q(v) = 0$, then 0 is a representative of $q(v)$, and hence by the above, $f(v,v) = 0 + 0^\sigma = 0$. To obtain $\Gamma(\mathcal{H})$, we have to consider another σ-quadratic form. Define $V_* = \mathbb{K}^{(\sigma)}/\mathbb{K}_\sigma$, with $\mathbb{K}^{(\sigma)}$ the set of fixed points of σ in $\mathbb{K}$ (and note that, if $k \in \mathbb{K}_\sigma$, then $k^\sigma = k$ so that this expression makes sense). We turn V_* into a (not necessarily finite-dimensional) right vector space over $\mathbb{K}$ by defining $v \cdot k = k^\sigma v k$, for all $v \in V_*$ and $k \in \mathbb{K}$. We choose a basis $(e_i^*)_{i \in J_*}$ in V_*, choose a representative g_i in $\mathbb{K}^{(\sigma)}$ for every e_i^* and define $g_*(e_i^*, e_j^*) = 0$ for $i \neq j$, and $g_*(e_i^*, e_i^*) = g_i$. It is readily checked that this determines a $(\sigma,1)$-linear form g_* with $g_*(v_*, v_*)$ a representative of v_* in $\mathbb{K}^{(\sigma)}$. We put $V' = V \bigoplus V_*$ and define, with the obvious notation,

$$q'(v + v_*) = g_*(v_*, v_*) + q(v) = v_* + q(v).$$

So q' is by definition a σ-quadratic form. We claim that the canonical projection onto V of $q'(0)^{-1}$ is bijective and coincides with $\mathcal{H}$.

Indeed, first suppose that $q'(v+v_*) = 0$. Then any representative k of $q(v)$ belongs to $\mathbb{K}^{(\sigma)}$ and hence $f(v,v) = k^\sigma + k = k + k = 0$ (see above). Now suppose that $f(v,v) = 0$. Then putting $v_* = q(v)$, we obtain $q'(v + v_*) = q(v) + v_* = 0$. Clearly if $q'(v + v'_*) = 0$, then $q(v) = v'_*$ and hence $v_* = v'_*$. The claim is proved.

Hence we have shown that every Hermitian variety $\mathcal{H}$ containing lines but no planes gives rise to a classical quadrangle $\Gamma(\mathcal{H})$. The representation of $\Gamma(\mathcal{H})$ as the Hermitian variety $\mathcal{H}$ in a projective space will also be called a **standard embedding of** $\Gamma(\mathcal{H})$.

Finally, we would like to introduce the following notation. Since the σ-quadratic form q does not play any role in $\mathsf{H}(3,\mathbb{K},q,\sigma)$ (simply look at the standard equation for q), we may denote that quadrangle by $\mathsf{H}(3,\mathbb{K},\sigma)$. In particular, inequivalent σ-quadratic forms have inequivalent associated anti-automorphisms σ.

2.3.10 D_ℓ-quadrangles

In general, the **Witt index** of a σ-quadratic form q defined on some vector space V is the dimension of the subspaces of highest dimension in $q^{-1}(0)$. For a fixed dimension and a fixed field, a lot of cases can occur. For instance, there may be several non-equivalent σ-quadratic forms (of different Witt index, for different σ). The corresponding geometries are *(classical) polar spaces (of rank r)*, where r is the Witt index of the σ-quadratic form. So the classical generalized quadrangles are in fact (up to duality) classical polar spaces of rank 2. Polar spaces (we will not need the precise definition of such geometries) can be viewed as spherical buildings and thus they are assigned a diagram and a type (see Subsection 1.3.7 on page 8). For one particular such type and diagram (namely, D_ℓ), the polar space is completely and uniquely determined by the field and the dimension of the vector space (or alternatively, the rank of the polar space), and this gives rise to an important subclass of classical quadrangles, as we will now explain. However, we take the projective point of view.

Consider the projective space $\mathbf{PG}(2\ell-1,\mathbb{K})$ and let Q be a quadric in that space, i.e., the null set of a (homogeneous) quadratic equation in the coordinates in $\mathbf{PG}(2\ell-1,\mathbb{K})$. We say the Q is **non-degenerate** if no point of Q is collinear (on Q, i.e., the joining line has all its points on Q) with all other points of Q. The **Witt index** of Q is said to be k if the (projective) dimension of the projective subspace of highest dimension contained in Q is equal to $k-1$. For arbitrary ℓ and $\mathbb{K}$, there always exists a quadric Q_ℓ of Witt index ℓ (the "split case", or, in French, "forme déployée") in $\mathbf{PG}(2\ell-1,\mathbb{K})$ and it is projectively unique (which means that one such quadric can always be transformed into any other by a collineation of the projective space, and this collineation can be chosen to come from a linear map in the underlying vector space). The standard equation is given by $X_0X_1 + X_2X_3 + \cdots + X_{2l-2}X_{2l-1} = 0$ and one can see that for instance the subspace with equations $X_0 = X_2 = \cdots = X_{2l-2} = 0$ of projective dimension $\ell-1$ is contained in Q. The corresponding polar space is of type D_ℓ. Note that Q is never a generalized quadrangle; weak quadrangles appear for $\ell = 2$. Now let Q' be any non-degenerate quadric in $\mathbf{PG}(2\ell-1,\mathbb{K})$ with equation $F(X_1,\ldots,X_{2\ell}) = 0$. Then by extending the field $\mathbb{K}$ to its quadratic closure (or algebraic closure) $\overline{\mathbb{K}}$ and considering the equation $F(X_1,\ldots,X_{2\ell}) = 0$ over $\overline{\mathbb{K}}$, we obtain a polar space of type D_ℓ over $\overline{\mathbb{K}}$. If $\ell > 1$ and Q' has Witt index 2, then we say that the corresponding generalized quadrangle is **of type** (D_ℓ), or a D_ℓ**-quadrangle**. If the characteristic of $\mathbb{K}$ is not equal to 2, then every orthogonal quadrangle in odd-dimensional projective space over $\mathbb{K}$ is a D_ℓ-quadrangle; if $\mathbb{K}$ has characteristic 2, then the quadric over $\overline{\mathbb{K}}$ might be degenerate and if it is, we do not have a D_ℓ-quadrangle.

Some quadrangles of type (D_ℓ) will turn up as ideal subquadrangles of the so-called *exceptional Moufang quadrangles*; see Chapter 5.

Classical quadrangles over special fields

2.3.11 Commutative fields

We already know that, if $\sigma = 1$, then $\mathsf{Q}(V,q)$, with q a σ-quadratic form as above, is isomorphic to the quadrangle arising from a quadric of Witt index 2 in $\mathbf{PG}(V)$. Suppose now that $\mathbb{K}$ is commutative and $\sigma \neq 1$. Let q be a σ-quadratic form and f the associated σ-Hermitian form. If the characteristic of $\mathbb{K}$ is not equal to 2, then

$$q^{-1}(0) = \{v \in V : f(v,v) = 0\}.$$

This can be shown as in Subsection 2.3.9 above. If the characteristic of $\mathbb{K}$ is equal to 2, then clearly $\mathbb{K}_\sigma = \mathbb{K}^{(\sigma)}$. Indeed, $\mathbb{K}_\sigma \subseteq \mathbb{K}^{(\sigma)}$ because $(k^\sigma + k)^\sigma = k + k^\sigma$; $\mathbb{K}^{(\sigma)} \subseteq \mathbb{K}_\sigma$ because $l(k^\sigma + k) = (lk)^\sigma + (lk)$ for all $l \in \mathbb{K}^{(\sigma)}$, and hence, since

$\mathbb{K}_\sigma$ is non-trivial ($\sigma \neq 1$), we deduce $\mathbb{K}^{(\sigma)} \cdot \mathbb{K}_\sigma \subseteq \mathbb{K}_\sigma$. Let g be the $(\sigma, 1)$-linear form associated with q, i.e., $g(x,y) + g(y,x)^\sigma = f(x,y)$. If $f(v,v) = 0$, then $g(v,v) = g(v,v)^\sigma \in \mathbb{K}^{(\sigma)} = \mathbb{K}_\sigma$, hence $q(v) = 0$. Conversely, if $q(v) = 0$, then $g(v,v) \in \mathbb{K}_\sigma = \mathbb{K}^{(\sigma)}$, so $f(v,v) = 0$.

This shows that in the commutative case *any classical quadrangle arises from a quadric or a Hermitian variety containing lines but no planes, in some projective space.*

Also remark that in $\mathbf{PG}(4, \mathbb{K})$, all non-degenerate quadrics are projectively equivalent (this follows from Proposition 2.3.4). Hence we denote such an orthogonal quadrangle by $\mathsf{Q}(4, \mathbb{K})$, without referring to the (unique) 1-quadratic form.

2.3.12 Finite fields

Since finite skew fields are fields, all finite classical quadrangles arise from quadrics or Hermitian varieties. Let $\mathbb{K} = \mathbf{GF}(s)$. It is well known (see e.g. Artin [1957], page 144, or O'Meara [1971], page 157) that for d odd there are exactly two isomorphism classes of (non-degenerate) quadrics in $\mathbf{PG}(d, s)$: members of one class have Witt index $\frac{d-1}{2}$, members of the other class have Witt index $\frac{d+1}{2}$ (and are of type $(D_{\frac{d+1}{2}})$). This is essentially due to the fact that every element of $\mathbf{GF}(s)$ can be written as a sum of two squares. Hence the dimension determines the quadrangle and the cases are: $d = 3$ (weak non-thick quadrangle) and $d = 5$. The classical quadrangle corresponding to the latter is a D_3-quadrangle and is denoted by $\mathsf{Q}(5, s)$, since the quadratic form is — up to isomorphism and a factor — determined by the dimension. If d is even, there is a unique isomorphism class of non-degenerate quadrics in $\mathbf{PG}(d, s)$ and they contain maximal projective subspaces of dimension $\frac{d-2}{2}$. So only $d = 4$ produces generalized quadrangles, and we denote such a quadrangle by $\mathsf{Q}(4, s)$.

The situation for Hermitian varieties is even simpler: for every dimension d, there is — up to isomorphism — just one example over $\mathbf{GF}(s)$ (with s a perfect square; the involutory field automorphism $x \mapsto x^{\sqrt{s}}$ is uniquely determined) and it has maximal projective subspaces of dimension $\frac{d-1}{2}$ (for d odd) or $\frac{d-2}{2}$ (for d even), see Scharlau [1985], page 39. So only $d = 3$ and $d = 4$ give us quadrangles and we denote them, respectively, by $\mathsf{H}(3, s)$ and $\mathsf{H}(4, s)$, in conformity with previous notation.

2.3.13 Algebraically closed fields

Not surprisingly, in any algebraically closed field $\mathbb{K}$ of characteristic not equal to 2, or, more generally, in any quadratically closed field of characteristic not equal to 2, a quadric of Witt index 2 has the standard equation

$$X_0^2 + X_1^2 + \cdots + X_\ell^2 = 0,$$

with $\ell = 3, 4$ (see O'Meara [1971], Section 61B), but the case $\ell = 3$ corresponds to a weak non-thick quadrangle (see Corollary 2.3.6 above). So there is a unique orthogonal quadrangle over $\mathbb{K}$, namely $\mathsf{Q}(4, \mathbb{K})$.

2.3.14 The classical (skew) fields $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$

SYLVESTER's theorem implies that for any ordered field $\mathbb{K}$ where each positive element is a square, coordinates can be chosen in such a way that any quadric of Witt index 2 in $\mathbf{PG}(\ell, \mathbb{K})$ is given by the equation

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + \cdots + X_\ell^2 = 0,$$

with $\ell \geq 4$ (see for instance ARTIN [1957], page 149 or O'MEARA [1971], Section 61A). Hence every orthogonal quadrangle over $\mathbb{R}$ is uniquely determined by the dimension ℓ. Therefore we can denote this orthogonal quadrangle by $\mathsf{Q}(\ell, \mathbb{R})$.

Of course, there are no Hermitian quadrangles over $\mathbb{R}$ since there are no non-trivial field automorphisms in $\mathbb{R}$.

Every involutory field automorphism in $\mathbb{C}$ is conjugate (in $\mathrm{Aut}\,\mathbb{C}$) to the standard conjugation map $a + ib \mapsto (a + ib)^* = a - ib$, where $i = \sqrt{-1}$ and $a, b \in \mathbb{R}$. It follows that, unlike the finite case, for every dimension $\ell \geq 3$, there is a unique Hermitian quadrangle over $\mathbb{C}$ in $\mathbf{PG}(\ell, \mathbb{C})$ and we briefly denote it by $\mathsf{H}(\ell, \mathbb{C})$. The corresponding Hermitian form is equivalent to

$$f(x, y) = -x_0^* y_0 - x_1^* y_1 + \sum_{r=2}^{\ell} x_r^* y_r,$$

where $x = (x_0, x_1, \ldots, x_\ell)$ and similarly for y.

Since $\mathbb{C}$ is algebraically closed, it follows from the previous subsection that there is only one orthogonal quadrangle over $\mathbb{C}$, namely $\mathsf{Q}(4, \mathbb{C})$.

Every involutory anti-automorphism of $\mathbb{H}$, the standard quaternions $\mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ over $\mathbb{R}$, is conjugate (in $\mathrm{Aut}\,\mathbb{H}$) to **either** the standard conjugation $a + ib + jc + kd \mapsto (a + ib + jc + kd)^* = a - ib - jc - kd$, **or** the skew conjugation $a + ib + jc + kd \mapsto (a + ib + jc + kd)^\diamond = a - ib + jc + kd$, where $a, b, c, d \in \mathbb{R}$. If σ is the standard conjugation, then for every dimension $\ell \geq 3$, there exists a unique σ-Hermitian form

$$f(x, y) = -x_0^* y_0 - x_1^* y_1 + \sum_{r=2}^{\ell} x_r^* y_r,$$

where $x = (x_0, x_1, \ldots, x_\ell)$ and similarly for y. The corresponding quadrangle only depends on ℓ and hence can be denoted by $\mathsf{H}(\ell, \mathbb{H}, \mathbb{R})$, where the presence of $\mathbb{R}$ replaces the notation σ in that $\mathbb{R}$ is the field of fixed elements of σ. This quadrangle will sometimes be referred to as a **real quaternion Hermitian quadrangle**.

If σ is the skew conjugation, then every σ-Hermitian form in $\mathbf{PG}(\ell, \mathbb{H})$ can be written as

$$f(x, y) = \sum_{r=0}^{\ell} x_r^\diamond y_r,$$

where again $x = (x_0, x_1, \ldots, x_\ell)$ and similarly for y. It is readily checked that the Witt index is equal to 2 **if and only if** $\ell = 3$ or $\ell = 4$. In this case we obtain the

Hermitian quadrangles $\mathsf{H}(3, \mathbb{H}, \mathbb{C})$ and $\mathsf{H}(4, \mathbb{H}, \mathbb{C})$, using similar notation to that above, and they will be called the **complex quaternion Hermitian quadrangles**, because the centre $\mathbb{R}$ together with the set $\mathbb{H}_\sigma$ generate a complex subfield of $\mathbb{H}$.

Certain generalized quadrangles discussed in this subsection are dual to others; for a complete account on this matter, see Subsection 9.6.4 on page 417 in Chapter 9.

2.3.15 Local fields

Quadrics in $\mathbf{PG}(\ell, \mathbb{K})$ of Witt index 2, and with $\mathbb{K}$ a finite extension of $\mathbb{Q}_p$ (the field of p-adic numbers) or $\mathbf{GF}(q)((t))$ (the field of Laurent series over $\mathbf{GF}(q)$), exist only for $\ell \in \{3, 4, 5\}$ and they can all be classified; see O'Meara [1971], Section 63C (compare also Scharlau [1985], pages 91, 185, 217).

2.3.16 Number fields

Here, the situation is much more complicated. We simply refer to O'Meara [1971], Section 66 and Lam [1973], Chapter 6. We will not need those results in the rest of the book.

The symplectic quadrangle

We are now going to define a very important class of classical quadrangles separately and in a way different from that above. We will indicate the proof of the fact that the quadrangle is classical (and postpone a detailed proof to the next chapter; see Proposition 3.4.13 on page 109). The construction below has also some nice applications to the theory of projective spaces (for instance, to the construction of *ovoids*; see Subsection 7.6.25 on page 340), and it is similar to the construction of a class of classical hexagons.

2.3.17 Symplectic polarities and their quadrangles

Let $\mathbb{K}$ be any (commutative) field and consider in $\mathbf{PG}(3, \mathbb{K})$, with respect to some chosen basis, the symplectic polarity τ which maps the point (y_0, y_1, y_2, y_3) to the plane with equation $y_1X_0 - y_0X_1 + y_3X_2 - y_2X_3 = 0$. A line L of $\mathbf{PG}(3, \mathbb{K})$ is called **totally isotropic** if $L^\tau = L$. We define the following geometry $\mathsf{W}(\mathbb{K})$. The points of $\mathsf{W}(\mathbb{K})$ are the points of $\mathbf{PG}(3, \mathbb{K})$; the lines of $\mathsf{W}(\mathbb{K})$ are the totally isotropic lines of τ. We show that $\mathsf{W}(\mathbb{K})$ is a generalized quadrangle. Note that every point of $\mathbf{PG}(3, \mathbb{K})$ is incident with its image (that is what a symplectic polarity is all about).

If L is a line of $\mathbf{PG}(3, \mathbb{K})$ such that $L^\tau = L$, and p is a point on L, then, since $p\,\mathbf{I}\,L$ in $\mathbf{PG}(3, \mathbb{K})$, also $L\,\mathbf{I}\,p^\tau$. Conversely, if L is a line of $\mathbf{PG}(3, \mathbb{K})$ through some point p and incident with p^τ, then for any other point $x\,\mathbf{I}\,L$, x^τ is incident with x (since every point is incident with its image) and with p (since $x\,\mathbf{I}\,p^\tau$), hence $L^\tau = (xp)^\tau = xp = L$. We have shown that a line L is totally isotropic if and only if for every point p on L, it is incident with p^τ if and only if this property holds for at least one point p on L.

Now let p be a point of $\mathsf{W}(\mathbb{K})$ not incident with some line L of $\mathsf{W}(\mathbb{K})$. Then there is a unique line M of $\mathsf{W}(\mathbb{K})$ through p meeting L. Indeed, M must be incident with p, it must be contained in p^τ and it must meet L, so M joins p with $L \cap p^\tau$; the latter is a singleton since L is not contained in p^τ (otherwise $p\,\mathbf{I}\,L$, a contradiction). Furthermore, the points $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ form an ordinary 4-gon. This shows that $\mathsf{W}(\mathbb{K})$ is a weak generalized quadrangle.

Noting that each line contains $|\mathbb{K}|+1$ points and each point is incident with $|\mathbb{K}|+1$ lines, we conclude that $\mathsf{W}(\mathbb{K})$ is a (thick) generalized quadrangle, the **symplectic quadrangle (over** $\mathbb{K}$). The polarity τ defines an anti-symmetric bilinear form q as follows:

$$q((x_0,x_1,x_2,x_3),(y_0,y_1,y_2,y_3)) = x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2.$$

By previous remarks, it is clear that two points x and y of $\mathbf{PG}(3,\mathbb{K})$, and hence of $\mathsf{W}(\mathbb{K})$, are collinear in $\mathsf{W}(\mathbb{K})$ **if and only if** $q(x,y)=0$. We call the form q the bilinear form **associated with** $\mathsf{W}(\mathbb{K})$. It defines $\mathsf{W}(\mathbb{K})$ completely.

As for $\mathsf{Q}(V,q)$ above, we will sometimes refer to the representation of $\mathsf{W}(\mathbb{K})$ in $\mathbf{PG}(3,\mathbb{K})$ just described as the **standard embedding**.

2.3.18 Grassmann coordinates

We now introduce *Grassmann coordinates* (in the special case $d=3$: *Plücker coordinates*) for the lines of a projective space $\mathbf{PG}(d,\mathbb{K})$ over a field $\mathbb{K}$. Choose a basis and coordinates and let L be a line of $\mathbf{PG}(d,\mathbb{K})$, $d \geq 2$. Consider two arbitrary points x and y on L with respective coordinates $(x_0,x_1,\ldots,x_d)$ and $(y_0,y_1,\ldots,y_d)$. Then one can easily verify that the $\binom{d+1}{2}$-tuple $(p_{ij})_{0\leq i<j\leq d}$, where

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = x_iy_j - x_jy_i,$$

is, up to a non-zero scalar multiple, independent of the points x and y on L. Hence the line L defines a unique point $p_L = (p_{ij})_{0\leq i<j\leq d}$ of $\mathbf{PG}(\binom{d+1}{2}-1,\mathbb{K})$. The coordinates of the point p_L are the **Grassmann coordinates** of L. For $d=3$, all these points constitute a quadric, the so-called **Klein quadric** (see e.g. Chapter 12 of TAYLOR [1992]), and the Grassmann coordinates are then called **Plücker coordinates**.

Now, from the expression of the bilinear form associated with τ, one immediately sees that the Grassmann coordinates of a line of $\mathsf{W}(\mathbb{K})$ satisfy $p_{01}+p_{23}=0$, and conversely, every line whose Grassmann coordinates satisfy $p_{01}+p_{23}=0$ is a line of $\mathsf{W}(\mathbb{K})$. This is thus another way to describe $\mathsf{W}(\mathbb{K})$.

This description has the advantage of making apparent the isomorphism of $\mathsf{W}(\mathbb{K})$ and the dual of $\mathsf{Q}(4,\mathbb{K})$. Indeed, the quadrangle $\mathsf{Q}(4,\mathbb{K})$ is nothing other than a quadric Q of Witt index 2 in $\mathbf{PG}(4,\mathbb{K})$. The relation $p_{01} + p_{23} = 0$ determines a hyperplane in $\mathbf{PG}(5,\mathbb{K})$ which meets the Klein quadric exactly in a non-degenerate quadric containing lines (the pencils of $\mathsf{W}(\mathbb{K})$) but no planes (because of the non-degeneracy). Since there is essentially only one such quadric, it must be isomorphic to Q.

Recall from Definition 1.9.4 (see page 39) that a *projective point* in a generalized quadrangle is a regular point for which the perp-geometry is a projective plane.

2.3.19 Theorem. *All points of the symplectic quadrangle* $\mathsf{W}(\mathbb{K})$ *over any field* $\mathbb{K}$ *are projective.*

Proof. Let τ be a symplectic polarity in $\mathbf{PG}(3,\mathbb{K})$ corresponding to the symplectic quadrangle $\mathsf{W}(\mathbb{K})$. All traces are of the form $x^\tau \cap y^\tau$ for x and y two non-collinear points in $\mathsf{W}(\mathbb{K})$, viewed as points of $\mathbf{PG}(3,\mathbb{K})$. So every trace is a line of $\mathbf{PG}(3,\mathbb{K})$ and hence determined by any two of its points. Therefore, every point is regular. It is now easily seen that the perp-geometry in a point p is nothing other than the projective plane p^τ. □

2.3.20 Corollary. *No proper full or ideal subquadrangle* Γ *of a symplectic quadrangle* $\mathsf{W}(\mathbb{K})$ *can be isomorphic to a symplectic quadrangle.*

Proof. Using Proposition 1.9.18 on page 46, we see that the perp-geometry in a point x of Γ is a proper subgeometry of the corresponding perp-geometry of $\mathsf{W}(\mathbb{K})$, except that all lines through x in both perp-geometries are the same (if Γ is an ideal subquadrangle), or all points on some line through x are the same (if Γ is a full subquadrangle). So they cannot be both projective planes by Corollary 1.8.3 (see page 34). □

Later on, we will define certain subquadrangles of the symplectic quadrangles over a field of characteristic 2, the so-called *mixed quadrangles*; see Subsection 3.4.2 on page 100. Some of these will be full or ideal proper subquadrangles of $\mathsf{W}(\mathbb{K})$.

2.4 Classical generalized hexagons

From the point of view of group theory, there are no such things as *classical hexagons*, except maybe for an example of order $(2,2)$, because no classical group is naturally associated with a generalized hexagon. The exception noted is related to the group $\mathbf{PSU}_3(3)$. This group is (sporadically) isomorphic to the exceptional group of type G_2 over $\mathbf{GF}(2)$ and a construction of a generalized hexagon of order $(2,2)$ related to the group $\mathbf{PSU}_3(3)$ is given in Subsection 1.3.12. However, we will introduce a class of classical hexagons, the name "classical" being motivated by the fact that they naturally live on classical objects like quadrics (in particular, we will see that they all live on the quadric (of type D_4) in seven-dimensional space (hence containing projective 3-spaces), whereas other important examples are related to exceptional groups of type E_6 and E_8 (see Appendix C); these only exist in the infinite case). We will also see that there is a great similarity between the symplectic quadrangle and one of the classes of classical hexagons, namely, the so-called *split Cayley hexagons*. This is an extra motivation for the name *classical hexagons*.

We start with an important definition in the theory of buildings and, in particular, the theory of polar spaces.

2.4.1 Definition. Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a geometry of rank 2. Then we say that Γ satisfies the **Buekenhout–Shult one-or-all axiom** if for every point $p \in \mathcal{P}$ and every line $L \in \mathcal{L}$ not incident with p, either all points of L are collinear with p, or exactly one point on L is collinear with p.

If the gonality of Γ is at least 3, then together with some non-degeneracy conditions, the Buekenhout–Shult one-or-all axiom characterizes the class of all polar spaces and thus provides a definition for these objects. For more details, see Buekenhout & Shult [1974].

2.4.2 The quadric $\mathbf{Q}(7, \mathbb{K})$

For the definition of the classical hexagons, we will need some understanding of the geometry of the quadric $\mathbf{Q}(7, \mathbb{K})$ of type D_4 over a field $\mathbb{K}$ in $\mathbf{PG}(7, \mathbb{K})$. It is by definition the quadric containing projective 3-spaces. Recall that there is essentially only one such for any field $\mathbb{K}$. A standard equation is given by $X_0X_1 + X_2X_3 + X_4X_5 + X_6X_7 = 0$. We will give some general properties below. Our goal is to understand geometrically how *triality* produces hexagons. The classification of trialities will not be carried out, but we will explicitly describe the examples giving rise to hexagons.

The quadric $\mathbf{Q}(7, \mathbb{K})$ has as characteristic property that every plane contained in it is itself contained in exactly two projective three-dimensional subspaces of $\mathbf{Q}(7, \mathbb{K})$. The set of three-dimensional subspaces on $\mathbf{Q}(7, \mathbb{K})$ can be subdivided in two subsets in the following way. Two 3-subspaces belong to the same subset **if and only if** their intersection is a projective space of odd dimension (empty, a line or a 3-space). Each subset is called a **set of generators**. It follows that there is a unique element of each set of generators through a plane of $\mathbf{Q}(7, \mathbb{K})$. The D_4**-geometry** $\Omega(\mathbb{K})$ **attached to** $\mathbf{Q}(7, \mathbb{K})$ is defined as follows. There are four different types of elements. The 0-*points* are the points of $\mathbf{Q}(7, \mathbb{K})$; the *lines* are the lines of $\mathbf{Q}(7, \mathbb{K})$, and we denote this set by $\mathcal{L}$; the 1-*points* are the elements of one set of generators; the 2-*points* are the elements of the other set of generators. We denote the set of i-points by $\mathcal{P}^{(i)}$, $i = 0, 1, 2$. *Incidence* is symmetrized containment for i-points and lines, $i = 0, 1, 2$; also for 0-points and j-points, $j = 1, 2$; and a 1-point is incident with a 2-point if the corresponding 3-spaces meet in a plane of $\mathbf{Q}(7, \mathbb{K})$. The key property is that every permutation of the set $\{\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ defines a geometry which is isomorphic to $\Omega(\mathbb{K})$. For $i = 0, 1, 2$, we call two i-points p and q **collinear** when they are incident with a common line.

Let p be a point, and let L be a line on $\mathbf{Q}(7,\mathbb{K})$. Then, since $\mathbf{Q}(7,\mathbb{K})$ is a quadric, either there is exactly one point on L collinear on $\mathbf{Q}(7,\mathbb{K})$ with p, or p and L are contained in a plane of $\mathbf{Q}(7,\mathbb{K})$. In the latter case, all points of L are collinear on $\mathbf{Q}(7,\mathbb{K})$ with p. Thus the geometry of points and lines on $\mathbf{Q}(7,\mathbb{K})$ satisfies the **Buekenhout–Shult one-or-all axiom** (and indeed $\mathbf{Q}(7,\mathbb{K})$ is a polar space, one of type D_4). An immediate consequence is that for a point p and a projective 3-space S (plane π) either p is collinear with all points of a plane of S (line of π), or p is contained in S (p and π generate a projective 3-space of $\mathbf{Q}(7,\mathbb{K})$ or p belongs to π).

2.4.3 Definition. Let $\Omega(\mathbb{K})$ be the geometry defined from $\mathbf{Q}(7,\mathbb{K})$ as above. A **triality** of $\Omega(\mathbb{K})$ is a map

$$\theta : \mathcal{L} \to \mathcal{L}, \mathcal{P}^{(0)} \to \mathcal{P}^{(1)}, \mathcal{P}^{(1)} \to \mathcal{P}^{(2)}, \mathcal{P}^{(2)} \to \mathcal{P}^{(0)}$$

preserving incidence in $\Omega(\mathbb{K})$ and such that θ^3 is the identity.

Let a triality θ be given. An **absolute i-point** p is an element of $\mathcal{P}^{(i)}$ which is incident with p^θ, $i = 0, 1, 2$. An **absolute line** is a line which is fixed by θ. There is some similarity with polarities in three-dimensional projective space, as pointed out by TITS [1959], and we will come back to that matter in Subsection 2.4.18.

2.4.4 Theorem (Tits [1959]). *Let θ be a triality of $\Omega(\mathbb{K})$. Suppose that one of the following hypotheses is satisfied:*

- (i) *there exists at least one absolute i-point, for some $i \in \{0, 1, 2\}$, and every absolute i-point is incident with at least two absolute lines;*
- (ii) *there exists a cycle $(L_0, L_1, \ldots, L_d)$, $d > 2$, of absolute lines (with L_i concurrent with $L_{i+1} \neq L_i$; subscripts modulo d).*

Then for every $i \in \{0, 1, 2\}$, the geometry $\Gamma^{(i)}$ with point set $\mathcal{P}^{(i)}_{\mathrm{abs}}$ the set of absolute i-points, with line set $\mathcal{L}_{\mathrm{abs}}$ the set of absolute lines and with the natural incidence, is a weak generalized hexagon with thick lines (and hence with some order). Also, the isomorphism class of this geometry is independent of $i \in \{0, 1, 2\}$.

Proof. We prove this in several steps. Without loss of generality, we take $i = 0$ and we briefly talk about *points* instead of 0-points. Also, we use the symbol $\in$ to denote incidence between a point and some other element, i.e., we consider $\Omega(\mathbb{K})$ as $\mathbf{Q}(7,\mathbb{K})$. In particular, we will also talk about *planes*, and these are the planes of $\mathbf{Q}(7,\mathbb{K})$. There is no loss of generality in doing so. In fact, a plane can abstractly be viewed as a pair of incident 1- and 2-points. Note that everything we prove for θ also holds for θ^2, since θ^2 is also a triality with obviously the same absolute lines and the same absolute i-points, for all $i \in \{0, 1, 2\}$.

Lemma 1 *Every point p of any absolute line L is an absolute point.*

Proof. Indeed, $p \in L = L^\theta \subseteq p^\theta$ and similarly for θ^2. QED

Lemma 2. *Whenever two distinct absolute points p and q are such that $p \in q^\theta$, then p is collinear with q and pq is an absolute line.*

Proof. Since q is absolute, $q \in q^\theta$ and so there is a unique line pq incident with both p and q. The line pq is in q^θ, so $(pq)^\theta$ belongs to q^{θ^2}. Also, $(pq)^\theta$ is the intersection (viewed in $\mathbf{Q}(7,\mathbb{K})$) of p^θ and q^θ, both of which contain p. Therefore, p is incident with $(pq)^\theta$ and hence p also belongs to q^{θ^2}.

Also, $p \in q^\theta$ implies $q \in p^{\theta^2}$. Hence similarly as we showed $p \in q^{\theta^2}$, this implies that $q \in p^\theta$. Interchanging the roles of p and q, we infer from the previous paragraph that $q \in (pq)^\theta$. Therefore $pq = (pq)^\theta$. QED

Lemma 3. **If** *an absolute point p is collinear with an absolute point q, but $p \notin q^\theta$ and hence $q \notin p^\theta$,* **then** *there is an absolute point x such that px and qx are absolute lines.*

Proof. We first claim that the planes $\pi_p := p^\theta \cap p^{\theta^2}$ and $\pi_q := q^\theta \cap q^{\theta^2}$ meet in a unique point. Suppose first that they meet in at least two points, say x and y. Then p is collinear with x, y, q and there are two possibilities. First, q belongs to xy. In that case $q \in p^\theta$, contrary to our assumptions. Second, x, y, q forms a triangle. Then x, y, q, p are contained in a 3-space containing π_q. But there are only two 3-spaces containing x, y, q and these are q^θ and q^{θ^2}. These give, respectively, $p \in q^\theta$ and $q \in p^\theta$, a contradiction. So we have shown that π_p and π_q meet in at most one point.

Suppose now that π_p and π_q are disjoint. We consider $L := (pq)^\theta$. This line is contained in $p^\theta \cap q^\theta$ and hence it meets the planes π_p and π_q, necessarily in unique distinct points u and v, respectively, for otherwise π_p and π_q share a common point. Similarly L^θ meets π_p and π_q in unique distinct points u' and v', respectively. So q^θ is generated by π_q and u; q^{θ^2} is generated by π_q and u', with u and u' collinear (because they belong to π_p). But that implies that u' is collinear with all points of a plane π_q of q^θ plus an extra point u, in contradiction with the consequences of the Buekenhout–Shult one-or-all axiom (see above), remembering that $q^\theta \neq q^{\theta^2}$. Hence this situation cannot occur and our first claim is proved.

Our next claim is that whenever a line L is incident with p and contained in π_p, with π_p as above, then L^θ is also incident with p and contained in π_p. Indeed, as L is incident with π_p, it is incident with p^θ and with p^{θ^2}. So L^θ is incident with p^{θ^2}, with p and with p^θ. Hence with π_p as well (on $\mathbf{Q}(7,\mathbb{K})$).

Now let x be the unique point in the intersection of π_p and π_q. If we denote by $\mathbf{I}$ the incidence relation in $\Omega(\mathbb{K})$, then from $p\mathbf{I}px\mathbf{I}x\mathbf{I}xq\mathbf{I}q$ follows $p\mathbf{I}(px)^\theta\mathbf{I}x^\theta\mathbf{I}(xq)^\theta\mathbf{I}q$. Therefore x^θ contains both p and q and hence the line pq. Now $(pq)^\theta$ is the intersection of p^θ and q^θ. Since this intersection also contains x, we see that $x\,\mathbf{I}\,(pq)^\theta$. It follows that $x^{\theta^2}\,\mathbf{I}\,(pq)^\theta\,\mathbf{I}\,x$. Hence $x\,\mathbf{I}\,x^{\theta^2}$ and applying θ, we conclude $x^\theta\,\mathbf{I}\,x$. Thus, x is an absolute point and the lines px and qx are absolute lines by Lemma 2. Therefore, Lemma 3 is proved. QED

Lemma 4. *The diameter of the incidence graph* $(G, *)$ *of* $\Gamma^{(0)}$ *is less than or equal to* 6.

Proof. If δ denotes distance (as usual), then we have to prove that $\delta(v,w) \leq 6$, for all points and lines v, w of $\Gamma^{(0)}$. If v is a point and w is a line, then by the Buekenhout–Shult one-or-all axiom, there is at least one point x on w collinear in $\Omega(\mathbb{K})$ with v. By Lemma 1, x is an absolute point at distance no more than 4 from v by Lemma 2 and Lemma 3. Hence $\delta(v,w) \leq 5$. If v, w are both points or both lines, then by considering an element z incident with w, we see that $\delta(v,w) \leq \delta(v,z) + \delta(z,w) \leq 5 + 1 = 6$. Hence the diameter of $(G, *)$ is 6 or less. QED

Lemma 5. *The gonality of* $(G, *)$ *is larger than* 3.

Proof. Suppose the lines L, M, N form a triangle, i.e., L, M, N are absolute lines and $\{x\} = L \cap M \neq \{y\} = M \cap N$. The 3-space x^θ is the unique element of the set of generators corresponding to 1-points containing the lines L and M. Similarly, y^θ contains M and N, so $y^\theta = x^\theta$ implying $x = y$. The lemma is proved. QED

Lemma 6. *The gonality of* $(G, *)$ *is larger than* 4.

Proof. Suppose the lines L, M, N, P form a quadrilateral. If they are contained in a plane of $\mathbf{Q}(7, \mathbb{K})$, then any three of them form a triangle, contradicting Lemma 5. If $\{x\} = L \cap M$, then $L \cup M \subseteq x^\theta \cap x^{\theta^2}$, hence the four "vertices" of the quadrilateral are two by two collinear (in $\mathbf{Q}(7, \mathbb{K})$). Hence they are contained in a 3-space U of $\mathbf{Q}(7, \mathbb{K})$. Since U contains the plane $\langle L, M \rangle$, and since there are only two 3-spaces through that plane, we must have $U = x^\theta$ or $U = x^{\theta^2}$. We may assume without loss of generality $U = x^\theta$. But then also $y^\theta = U$ with $\{y\} = N \cap P$, a contradiction. QED

Lemma 7. *The gonality of* $(G, *)$ *is larger than* 5.

Proof. Suppose we have a pentagon L, M, N, P, Q of lines. Let x again be the intersection of L and M. Let y and z be the intersection point of, respectively, N and P, and of P and Q. As above, M and N lie in a plane of $\mathbf{Q}(7, \mathbb{K})$, hence x is collinear with y and similarly with z. So by the Buekenhout–Shult one-or-all axiom, x is collinear with all points of the space generated by N, P, Q. We conclude that the pentagon lies entirely in a 3-space U (since a 2-space is ruled out by Lemma 5). Without loss of generality, we may again assume that $x^\theta = U$; compare Lemma 6. But then also $U = y^\theta = z^\theta$, a contradiction. QED

Lemma 8. *The gonality of* $(G, *)$ *is equal to* 6.

Proof. This is obvious if we assume that there is a circuit in $\Gamma^{(0)}$ (since the diameter is at most 6, we can always reduce that circuit to one of length 12). So we may assume that each absolute point is incident with at least two absolute lines. Also, by assumption, there exists an absolute line L. By Lemma 1, each point on L is absolute. Let $x \mathbf{I} L$. By assumption, there exists an absolute line $M \mathbf{I} x$ with $M \neq L$. Let $y \mathbf{I} M$ with $y \neq x$. Then again, y is absolute. Let N be an absolute line through y distinct from M, and let $z \neq y$ be incident with N. Let P be an absolute line through z distinct from N, and let $u \neq z$ be incident with P. Finally, let $Q \neq P$ be an absolute line through u. By the Buekenhout–Shult one-or-all axiom, there is a point v on Q collinear in $\Omega(\mathbb{K})$ with x. By Lemma 3, we have $\delta(x, v) \leq 4$. Since the gonality of $(G, *)$ is at least 6, we see that $\delta(v, x) = 4$ and the sequence (x, y, z, u, v, w), where w is collinear with both x and v in $\Gamma^{(0)}$, is an ordinary hexagon. So the gonality of $(G, *)$ is equal to 6 and its diameter is also equal to 6. QED

By Lemma 1.5.10 (see page 21), $\Gamma^{(0)}$ is a weak generalized hexagon with thick lines. Applying triality, the last assertion follows and the proof of the theorem is complete. □

2.4.5 Remark. Note that the proof above implies that two points x and y of $\Gamma^{(0)}$ are opposite **if and only if** they are not collinear in $\Omega(\mathbb{K})$ (or equivalently in $\mathbf{Q}(7, \mathbb{K})$).

2.4.6 Trilinear forms

To give an explicit example of a triality, we should have a convenient description of $\mathbf{Q}(7, \mathbb{K})$, i.e., a description of $\Omega(\mathbb{K})$ in which the i-points play the same role as the j-points for $i, j \in \{0, 1, 2\}$. This is possible by introducing a *trilinear form*, see Cartan [1938]. We follow Tits [1959] for the notation.

The points of $\mathbf{Q}(7, \mathbb{K})$ can be viewed as 8-tuples $(x_0, x_1, \ldots, x_7)$, up to a scalar multiple, with elements in $\mathbb{K}$ and satisfying the relation

$$x_0x_4 + x_1x_5 + x_2x_6 + x_3x_7 = 0.$$

The philosophy of trilinear forms is that since 1-points and 2-points play the same role as 0-points, it must be possible to label the 1-points and 2-points in the same way as the 0-points and to introduce an algebraic operation that tells one when two elements are incident. In fact, it is possible to do even better: let $J = \{0, 1, \ldots, 7\}$ and let V be an eight-dimensional vector space over $\mathbb{K}$; then there exists a trilinear form $\mathcal{T} : V \times V \times V \to \mathbb{K}$ such that a pair of points (x, y) of $\mathbf{Q}(7, \mathbb{K})$ represents an incident (0-point, 1-point)-pair in $\Omega(\mathbb{K})$ **if and only if** the linear form $\mathcal{T}(x, y, z')$ is identical zero in z'; and similarly for any cyclic permutation of the letters x, y, z. This trilinear form has the following explicit description:

$$\begin{aligned}\mathcal{T}(x,y,z) \quad = \quad & \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} + \begin{vmatrix} x_4 & x_5 & x_6 \\ y_4 & y_5 & y_6 \\ z_4 & z_5 & z_6 \end{vmatrix} \\ & + x_3(z_0y_4 + z_1y_5 + z_2y_6) + x_7(y_0z_4 + y_1z_5 + y_2z_6) \\ & + y_3(x_0z_4 + x_1z_5 + x_2z_6) + y_7(z_0x_4 + z_1x_5 + z_2x_6) \\ & + z_3(y_0x_4 + y_1x_5 + y_2x_6) + z_7(x_0y_4 + x_1y_5 + x_2y_6) \\ & - x_3y_3z_3 - x_7y_7z_7.\end{aligned}$$

For example, in order to find the equation of the 3-space on $\mathbf{Q}(7,\mathbb{K})$ which corresponds to the 1-point $y = (1,0,\dots,0)$, we simply plug in the value for y in $\mathcal{T}(x,y,z)$ and require that the coefficients of all z_i, $i \in J$, vanish. This gives us $x_1 = x_2 = x_4 = x_7 = 0$.

2.4.7 Trialities that produce generalized hexagons

Now we give the formulae for all trialities which produce (thick) generalized hexagons. Let $\mathcal{T}$ be the trilinear form as introduced above. Since a line of Ω is determined by two i-points, for all $i \in \{0,1,2\}$, it is readily seen that every permutation θ of $\mathcal{P}^{(0)} \cup \mathcal{P}^{(1)} \cup \mathcal{P}^{(2)}$ preserving incidence (and well defined on the types of points) induces a not necessarily type-preserving automorphism of $\Omega(\mathbb{K})$.

Let σ be an automorphism of $\mathbb{K}$ of order 1 or 3. Then the map

$$\tau_\sigma : \mathcal{P}^{(i)} \to \mathcal{P}^{(i+1)} : (x_j)_{j\in J} \mapsto (x_j^\sigma)_{j\in J}, \qquad i = 0,1,2 \bmod 3,$$

clearly preserves incidence in $\Omega(\mathbb{K})$ (because the trilinear form $\mathcal{T}$ is preserved). Moreover, the order of τ_σ is clearly 3. Hence τ_σ is a triality. We call τ_σ a triality of **type** ($\mathbf{I}_\sigma$), closely following TITS [1959]. There are other types, but we will not need them. We review them briefly in Subsection 2.4.18.

2.4.8 Theorem (Tits [1959]). *The geometry $\Gamma^{(i)} = (\mathcal{P}^{(i)}_{\rm abs}, \mathcal{L}_{\rm abs}, \mathbf{I})$ arising from the triality τ_σ is a generalized hexagon of order $(|\mathbb{K}|, |\mathbb{K}^{(\sigma)}|)$, where $\mathbb{K}^{(\sigma)}$ is the subfield of $\mathbb{K}$ consisting of those elements fixed by σ. Replacing σ by σ^{-1} produces an isomorphic hexagon.*

Proof. According to Theorem 2.4.4, it suffices to show that there is an ordinary hexagon in $\Gamma^{(0)}$, and that there is an absolute point incident with exactly $|\mathbb{K}^{(\sigma)}|+1$ absolute lines.

Let e_i be the 0-point with coordinates x_j, $j \in J$, all zero except x_i, which can be chosen to be equal to 1. Clearly

$$\mathcal{T}(e_i, e_i^{\tau_\sigma}, z) \equiv 0$$

if and only if $i \neq 3,7$. The 0-points incident with $e_0^{\tau_\sigma}$ are those whose coordinates satisfy $x_1 = x_2 = x_4 = x_7 = 0$ (this is the example at the end of Subsection 2.4.6); so clearly the absolute points e_5 and e_6 are incident with $e_0^{\tau_\sigma}$ and hence the lines e_5e_0 and e_0e_6 are absolute lines. Similarly, the lines e_6e_1, e_1e_4, e_4e_2 and e_2e_5 are absolute lines. These six lines in total now clearly form an ordinary hexagon.

Using the trilinear form $\mathcal{T}$ again, it takes an elementary calculation to see that the 0-points $(x_j)_{j\in J}$ incident with both $e_0^{\tau_\sigma}$ and $e_0^{\tau_\sigma^2}$ are precisely the points satisfying $x_1 = x_2 = x_3 = x_4 = x_7 = 0$ (and these indeed form a plane π in $\mathbf{PG}(7,\mathbb{K})$; this plane is denoted by π_{e_0} in the proof of Theorem 2.4.4). Every absolute line incident with e_0 lies in π; moreover, by the proof of Theorem 2.4.4, every absolute point p in π, $p \neq e_0$, gives rise to an absolute line e_0p. Now consider the 0-point p with coordinates $(0,0,0,0,0,k,1,0)$, $k \in \mathbb{K}$. Its image under τ_σ is the 3-space in $\mathbf{PG}(7,\mathbb{K})$ with equations

$$\begin{cases} 0 &= x_2 + k^\sigma x_1, \\ 0 &= x_5 - k^\sigma x_6, \\ 0 &= x_3, \\ 0 &= x_4. \end{cases}$$

This space contains p **if and only if** $k = k^\sigma$. Noting that also the point with coordinates $(0,0,0,0,0,1,0,0)$ of π is absolute, we see that the order of $\Gamma^{(i)}$ is equal to $(|\mathbb{K}|, |\mathbb{K}^{(\sigma)}|)$.

If we replace σ by σ^{-1}, then we interchange $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$. The theorem now follows directly. □

2.4.9 Definitions. Taking $\sigma = 1$, we see that over every field $\mathbb{K}$ there exists a triality that produces a generalized hexagon. We call this hexagon **classical**, and, more specifically, we speak of the **split Cayley hexagon (over** $\mathbb{K}$**)**, denoted by $\mathsf{H}(\mathbb{K})$. The reason for that name is that this hexagon can also be constructed using a split Cayley algebra over $\mathbb{K}$, see for instance SCHELLEKENS [1962a], [1962b] (and moreover, the corresponding simple algebraic group is also split). The hexagon $\mathsf{H}(\mathbb{K})$ deserves the name "classical" in more than one way: on top of the reasons already mentioned (lying on a classical polar space), it is the most important hexagon, it is the main example, and in fact, the *only* example for many fields $\mathbb{K}$. The dual of $\mathsf{H}(\mathbb{K})$ is also a **classical hexagon** and denoted $\mathsf{H}(\mathbb{K})^D$. In the finite case, the split Cayley hexagon over the Galois field $\mathbf{GF}(q)$ is denoted by $\mathsf{H}(q)$.

We call a generalized hexagon arising from a triality as in Theorem 2.4.8 with $\sigma \neq 1$ also a **classical hexagon**, or more specifically, a **twisted triality hexagon**, and we denote it by $\mathsf{T}(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$. Note that $\mathbb{K}$ is a Galois extension of degree 3 of $\mathbb{K}^{(\sigma)}$. The dual of $\mathsf{T}(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$ is denoted by $\mathsf{T}(\mathbb{K}^{(\sigma)}, \mathbb{K}, \sigma)$ and is also called **classical**. In the finite case, the field automorphism σ is — up to inverse — determined by the field $\mathbf{GF}(q^3)$ and hence we can unambiguously denote the unique twisted triality hexagon over the field $\mathbf{GF}(q^3)$ by $\mathsf{T}(q^3, q)$, and its dual by $\mathsf{T}(q, q^3)$. Note that we do not follow THAS [1995] (who writes $\mathsf{H}(q^3, q)$ for $\mathsf{T}(q^3, q)$ and has no special notation for the dual) in this notation in order to avoid confusion with the Hermitian quadrangles, in particular with $\mathsf{H}(4, 64)$.

The representation of $\mathsf{T}(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$ on $\mathbf{Q}(7,\mathbb{K})$ as above is sometimes referred to as the **standard embedding** of $\mathsf{T}(\mathbb{K}, \mathbb{K}^{(\sigma)}, \sigma)$.

We now show a property of $\mathsf{H}(\mathbb{K})$ that will allow us to construct $\mathsf{H}(\mathbb{K})$ in a more direct way on a quadric in projective 6-space.

2.4.10 Theorem (Tits [1959]). *The points and lines of* $\mathsf{H}(\mathbb{K})$*, considered as the geometry* $\Gamma^{(0)}$ *of absolute points and lines of the triality* τ_σ *with* $\sigma = 1$*, all lie in* *the hyperplane of* $\mathbf{PG}(7,\mathbb{K})$ *with equation* $x_3 + x_7 = 0$*. Conversely, every point of* $\mathbf{Q}(7,\mathbb{K})$ *in that hyperplane belongs to* $\mathsf{H}(\mathbb{K})$*.*

Proof. The necessary and sufficient condition for a 0-point p with coordinates $(x_j)_{j\in J}$ to be an absolute point is that $\mathcal{T}((x_j)_{j\in J},(x_j)_{j\in J},z)$ vanishes. This is equivalent to the following condition (as is easily computed by looking at the coefficients of the z_i, $i \in J$, and taking subscripts modulo 8):

$$\begin{cases} 0 = x_{i+4}(x_3+x_7), & i \neq 3,7, \\ 0 = x_0x_4 + x_1x_5 + x_2x_6 - x_i^2 & i = 3,7. \end{cases}$$

The result now follows readily. □

2.4.11 Proposition. *The twisted triality hexagon* $\mathsf{T}(\mathbb{K},\mathbb{K}^{(\sigma)},\sigma)$ *has an ideal sub-hexagon isomorphic to* $\mathsf{H}(\mathbb{K}^{(\sigma)})$*.*

Proof. This follows by restricting coordinates in $\Omega(\mathbb{K})$ to $\mathbb{K}^{(\sigma)}$. □

2.4.12 Remark. There is another class of hexagons closely related to the twisted triality hexagons; we will define this class in Subsection 3.5.8 (see page 114).

The dual of the twisted triality hexagons are called in the literature the *hexagons related to the groups of type* ${}^3D_4$ and, in the finite case, denoted by ${}^3D_4(q)$ or ${}^3D_4(q^3)$ (cf. KANTOR [1986a]).

Split Cayley hexagons

2.4.13 Tits' description of $\mathsf{H}(\mathbb{K})$

Recall that the absolute points of a triality of type ($\mathbf{I}_{\mathrm{id}}$) are exactly the points of the intersection of a hyperplane of $\mathbf{PG}(7,\mathbb{K})$ with $\mathbf{Q}(7,\mathbb{K})$. Considering coordinates as above, this hyperplane has equation $X_3 + X_7 = 0$. Hence substituting X_7 for X_3 and deleting X_7 (which amounts to the same as deleting X_7 and substituting $-X_3$ for X_3; this substitution is for historical reasons), we can identify the point set of $\Gamma = \mathsf{H}(\mathbb{K})$ with the point set of the "parabolic" quadric $\mathbf{Q}(6,\mathbb{K})$ in $\mathbf{PG}(6,\mathbb{K})$ with equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2.$$

A tedious explicit computation (which we will not perform) shows that the Grassmann coordinates of the lines of $\mathsf{H}(\mathbb{K})$ satisfy the following six linear equations:

$$\begin{array}{lll} p_{12} = p_{34}, & p_{54} = p_{32}, & p_{20} = p_{35}, \\ p_{65} = p_{30}, & p_{01} = p_{36}, & p_{46} = p_{31}, \end{array}$$

and conversely, every line on $\mathbf{Q}(6,\mathbb{K})$ whose Grassmann coordinates satisfy these equations is a line of Γ. This gives a complete and explicit description of $\mathsf{H}(\mathbb{K})$ on the quadric $\mathbf{Q}(6,\mathbb{K})$. It is due to TITS [1959]. By the way, one can deduce all the above equations from the first one by consecutively applying the following rule: if $p_{ij} = p_{3k}$ is in the list, then so are $p_{(i\pm4)k} = p_{3j}$ and $p_{k(j\pm4)} = p_{3i}$, where in ±4 one should choose the appropriate sign in order to obtain a number between 0 and 7.

We sometimes refer to this representation of the split Cayley hexagons as a **standard embedding**.

2.4.14 (Perfect) Symplectic hexagons

Now assume that the characteristic of $\mathbb{K}$ is 2. We first recall some properties of the quadric $\mathbf{Q}(6,\mathbb{K})$. Let $\mathbf{Q}(6,\mathbb{K})$ have equation

$$X_0X_4 + X_1X_5 + X_2X_6 = X_3^2.$$

Consider the point k with coordinates $(0,0,0,1,0,0,0)$. Let L be any line of $\mathbf{PG}(6,\mathbb{K})$ through k and suppose that L contains the point with coordinates $(x_0,x_1,x_2,0,x_4,x_5,x_6)$, $x_i \in \mathbb{K}$, $i = 0,1,2,4,5,6$. A point $(x_0,x_1,x_2,\ell,x_4,x_5,x_6)$ of L (with $\ell \in \mathbb{K}$) is contained in $\mathsf{Q}(6,\mathbb{K})$ if and only if

$$\ell^2 = x_0x_4 + x_1x_5 + x_2x_6.$$

Since the characteristic of $\mathbb{K}$ is 2, L meets $\mathbf{Q}(6,\mathbb{K})$ in at most one point. Note that, if $\mathbb{K}$ is perfect (and hence every element of $\mathbb{K}$ is a square in $\mathbb{K}$), then L meets $\mathbf{Q}(6,\mathbb{K})$ always in exactly one point. In any case, k is called the **nucleus** of $\mathbf{Q}(6,\mathbb{K})$. Hence we may project $\mathbf{Q}(6,\mathbb{K})$ from k onto the hyperplane H with equation $X_3 = 0$. Let $p(x_0,x_1,x_2,x_3,x_4,x_5,x_6)$ be any point of $\mathbf{Q}(6,\mathbb{K})$. The set of points of $\mathbf{Q}(6,\mathbb{K})$ collinear with p on $\mathbf{Q}(6,\mathbb{K})$ is given by the equations:

$$\begin{cases} 0 &= x_0X_4 + x_4X_0 + x_1X_5 + x_5X_1 + x_2X_6 + x_6X_2, \\ X_3^2 &= X_0X_4 + X_1X_5 + X_2X_6. \end{cases}$$

Hence the coordinates $(X_0,X_1,X_2,0,X_4,X_5,X_6)$ of the projection of these points from k onto the hyperplane H satisfy the equation

$$0 = x_0X_4 + x_4X_0 + x_1X_5 + x_5X_1 + x_2X_6 + x_6X_2,$$

which is the equation of a hyperplane H_p of H, and clearly the correspondence $p \mapsto H_p$ uniquely defines a symplectic polarity ρ in H. This implies that the lines of $\mathbf{Q}(6,\mathbb{K})$ are projected onto totally isotropic lines for ρ. If $\mathbb{K}$ is perfect, then one can now easily calculate that all totally isotropic lines for ρ in H are obtained in this way. Hence for $\mathbb{K}$ perfect, the geometry of $\mathbf{Q}(6,\mathbb{K})$ is isomorphic to the geometry of the symplectic space $\mathbf{W}(5,\mathbb{K})$ (which is the geometry of totally isotropic subspaces for a symplectic polarity in $\mathbf{PG}(5,\mathbb{K})$).

Considering the standard embedding of $\mathsf{H}(\mathbb{K})$ in $\mathbf{Q}(6,\mathbb{K})$, we now see that, if $\mathbb{K}$ has characteristic 2, we can represent $\mathsf{H}(\mathbb{K})$ inside the symplectic space $\mathbf{W}(5,\mathbb{K})$. This means that the points of $\mathsf{H}(\mathbb{K})$ are some points of $\mathbf{PG}(5,\mathbb{K})$, and the lines of $\mathsf{H}(\mathbb{K})$ are some lines of $\mathbf{PG}(5,\mathbb{K})$ which are moreover totally isotropic with respect to some symplectic polarity. Therefore we sometimes call $\mathsf{H}(\mathbb{K})$ a **symplectic hexagon**. The above representation is also called a **standard embedding** of $\mathsf{H}(\mathbb{K})$. Hence these geometries have two standard embeddings, and we should always make it clear which one we mean. This will usually be achieved by the choice of the name *symplectic* or *split Cayley*, and it is clear which embedding we associate with each of these names.

If $\mathbb{K}$ is perfect, then the points of the standard embedding of the symplectic hexagon $\mathsf{H}(\mathbb{K})$ are *all* points of $\mathbf{PG}(5,\mathbb{K})$. In this case, we sometimes call $\mathsf{H}(\mathbb{K})$ a **perfect symplectic hexagon**.

Since the absolute lines of a triality θ incident with an absolute point p all lie in the plane $p^\theta \cap p^{\theta^2}$, we have the following property:

2.4.15 Theorem (Ronan [1980a]). *All points of any split Cayley or twisted triality hexagon are distance-2-regular.*

Proof. By the remark preceding the theorem, we know that, for two opposite points p and q (where p and q are not collinear in $\Omega(\mathbb{K})$), the set p^q is contained in the set of points of the plane $p^\theta \cap p^{\theta^2}$ collinear in $\Omega(\mathbb{K})$ with q, which forms a line. So p^q is contained in a line of $\Omega(\mathbb{K})$ and therefore it is determined by any two of its points. □

More exactly, for the split Cayley hexagons we can be more specific.

2.4.16 Theorem. *All points of the split Cayley hexagon $\mathsf{H}(\mathbb{K})$ over any field $\mathbb{K}$ are polar points.*

Proof. Let $\mathsf{H}(\mathbb{K})$ be represented on the quadric $\mathbf{Q}(6,\mathbb{K})$. As in the proof of Theorem 2.3.19, one can see easily that the perp-geometry in a point p is exactly the projective plane on $\mathbf{Q}(6,\mathbb{K})$ containing the lines of $\mathsf{H}(\mathbb{K})$ through p. It is also readily seen that for opposite points p and q, the set $\Gamma^+(p,q)$ (see Remark 1.9.11 on page 43) "is" a projective plane of $\mathbf{Q}(6,\mathbb{K})$ through p, and every plane of $\mathbf{Q}(6,\mathbb{K})$ through p and not containing lines of $\mathsf{H}(\mathbb{K})$ arises in such a way. It follows that the span-geometry at a point p is equal to the geometry of planes and lines on $\mathbf{Q}(6,\mathbb{K})$ through p and this is known to be a generalized quadrangle, namely one isomorphic to $\mathsf{W}(\mathbb{K})$ (the points of $\mathsf{W}(\mathbb{K})$ corresponding to the planes through p and the lines of $\mathsf{W}(\mathbb{K})$ to the lines through p). □

As in Corollary 2.3.20, one can now also prove:

2.4.17 Corollary. *No proper full or ideal subhexagon Γ of a split Cayley hexagon $\mathsf{H}(\mathbb{K})$ can be isomorphic to a split Cayley hexagon.* □

Later on, we will define full and ideal subhexagons of the split Cayley hexagons over a field of characteristic 3, the so-called *mixed hexagons*; see Subsection 3.5.3 on page 112.

2.4.18 Polarities of $\mathbf{PG}(3,\mathbb{K})$ versus trialities of $\Omega(\mathbb{K})$

We now come back to the similarity with polarities in projective 3-space $\mathbf{PG}(3,\mathbb{K})$. There are essentially four kinds of polarities in $\mathbf{PG}(3,\mathbb{K})$ having absolute points and lines (in $\mathbf{PG}(3,\mathbb{K})$ itself). One can distinguish the types by looking at the set of absolute points. In $\Omega(\mathbb{K})$, there are four kinds of trialities having absolute points (and they automatically have absolute lines). We give a brief survey of results due to Tits [1959].

For a given polarity θ in $\mathbf{PG}(3,\mathbb{K})$, or for a given triality θ in $\Omega(\mathbb{K})$, and for a given absolute point p, we denote the one-dimensional projective space formed by the lines incident with p and with p^θ, or incident with p, p^θ and with p^{θ^2}, by $\mathbf{PG}(1,\mathbb{K})^{(p)}$.

1. **Pseudo-polarities** in $\mathbf{PG}(3,\mathbb{K})$ are polarities for which the set of absolute points is a proper subspace π. These only exist in characteristic 2. Suppose π is a plane. Then the set of absolute lines is the pencil of lines in π through the image of π. Hence the geometry of absolute points and lines can be considered as a *degenerate generalized quadrangle*: "degenerate", because it does not contain a proper cycle (or quadrilateral in this case); "quadrangle", because the diameter is equal to 4, as for generalized quadrangles. For a given absolute point p, the collineation induced on $\mathbf{PG}(1,\mathbb{K})^{(p)}$ is either the identity or an involution with one fixed point (an *elation*), according as $p=\pi^\theta$ or $p\neq\pi^\theta$.

   In $\Omega(\mathbb{K})$, there is a similar phenomenon, namely, in characteristic 3, there are trialities which have absolute points and lines, but there is no ordinary hexagon contained in $\Gamma^{(0)}$. More exactly, if we denote such a triality by θ, then all absolute points are contained in a four-dimensional space $\mathbf{PG}(4,\mathbb{K})$ which meets $\mathbf{Q}(7,\mathbb{K})$ in a degenerate quadric Q which is the projection from some line D of $\mathbf{Q}(7,\mathbb{K})$ of a non-degenerate conic lying in a plane of $\mathbf{PG}(4,\mathbb{K})$ skew to D. All points of Q are absolute. For each point p on L, every line incident with p, with p^θ and with p^{θ^2} is absolute (the plane $p^\theta\cap p^{\theta^2}$ lies on Q and contains D); for every other point p on Q, there is a unique absolute line incident with p, namely, the line pp', where p' lies on D such that $p\in {p'}^\theta$. Hence the geometry of absolute points and lines can be considered here as a *degenerate generalized hexagon*: there are no proper cycles and the diameter is equal to 6. For a given absolute point p, the collineation induced on $\mathbf{PG}(1,\mathbb{K})^{(p)}$ is either the identity or a collineation of order 3 with one fixed point (an *elation*), according as $p\,\mathbf{I}\,D$ or $p\in Q\setminus D$.

   Note that there exist fields (necessarily of characteristic 2) such that, in $\mathbf{PG}(3,\mathbb{K})$, there are pseudo-polarities for which the set of absolute points is a line, a point or the empty set; $\mathbb{K}$ cannot be perfect in this case. For trialities in $\Omega(\mathbb{K}')$ (where $\mathbb{K}'$ has charactistic 3), this has no analogue (so the perfectness of $\mathbb{K}'$ does not change the above described situation, provided the set of absolute points is non-empty).

2. **Orthogonal polarities** in $\mathbf{PG}(3, \mathbb{K})$ are polarities for which the set of absolute points is a non-degenerate ruled quadric $\mathbf{Q}(3, \mathbb{K})$ (i.e., a quadric containing lines, hence a quadric of type D_2). The set of absolute lines is precisely the union of the two sets of generators of $\mathbf{Q}(3, \mathbb{K})$. This is clearly a weak generalized quadrangle (denoted by $\mathsf{Q}(3, \mathbb{K})$) which is the dual of the double of a generalized digon. For a given absolute point p, the collineation induced on $\mathbf{PG}(1, \mathbb{K})^{(p)}$ is an involution with two fixed points (a *homology*). These polarities do not exist in characteristic 2, but they do exist over every field of characteristic $\neq 2$.

   In $\Omega(\mathbb{K})$, there is again a similar phenomenon. Indeed, for some fields $\mathbb{K}$ (see below for examples), there exists a triality θ such that each absolute point is incident with exactly two absolute lines. The geometry $\Gamma^{(0)}$ is in this case a weak generalized hexagon which is the dual of the double of a (thick) projective plane (over $\mathbb{K}$). For a given absolute point p, the collineation induced on $\mathbf{PG}(1, \mathbb{K})^{(p)}$ has order 3 and has two fixed points (a *homology*). These trialities exist for all fields admitting such a homology in the corresponding projective 1-space; in particular, the characteristic of the field is not 3. Examples are provided by the finite fields $\mathbf{GF}(q)$ with $q \equiv 1$ modulo 3.

3. **Symplectic polarities** in $\mathbf{PG}(3, \mathbb{K})$ are polarities θ with the property that for every point p, every line incident with p and with p^θ is an absolute line. This is enough to conclude that every point is an absolute point. So the set of absolute points forms a linear subspace of $\mathbf{PG}(3, \mathbb{K})$, namely, $\mathbf{PG}(3, \mathbb{K})$ itself. The Grassmann coordinates of the absolute lines satisfy a linear equation (see Subsection 2.3.18 above). For a given absolute point p, the collineation induced on $\mathbf{PG}(1, \mathbb{K})^{(p)}$ is always the identity. These polarities exist for all fields $\mathbb{K}$.

   In $\Omega(\mathbb{K})$, there is a similar phenomenon. Indeed, the trialities of type $(\mathbf{I}_{\mathrm{id}})$ above have the property that for every absolute point p, all lines incident with p, p^θ and p^{θ^2} are absolute lines. Moreover, the set of absolute points is the set of points lying in a linear subspace of $\mathbf{PG}(7, \mathbb{K})$, namely a hyperplane (see Subsection 2.4.13 above). Also, the Grassmann coordinates of the absolute lines satisfy a system of linear equations. For a given absolute point p, the collineation induced on $\mathbf{PG}(1, \mathbb{K})^{(p)}$ is always the identity. Finally, these trialities exist for all fields $\mathbb{K}$.

More similarities between the class of symplectic quadrangles and the class of split Cayley hexagons can be found in Chapter 3 (they both contain "mixed subpolygons"), Chapter 4 (their members admit collineations with analogous properties), Chapter 5 (they behave similarly with respect to pointminimality and line-minimality), Chapter 6 (they share a number of combinatorial, geometric and algebraic characterizations), Chapter 7 (they contain self-dual and self-polar members and the "ovoids" arising this way have similar properties) and Chapter 9 (many similarities already mentioned have topological analogues).

4. **Unitary** or **Hermitian polarities** in $\mathbf{PG}(3,\mathbb{K})$ are in bijective correspondence with the involutory field automorphisms of $\mathbb{K}$. For a given such involution σ and corresponding polarity θ, the set of absolute lines through an absolute point p is parametrized by the subfield of $\mathbb{K}$ of fixed elements under σ, together with one extra element ∞. For a given absolute point p, the collineation induced on $\mathbf{PG}(1,\mathbb{K})^{(p)}$ is the involution arising from the semi-linear map with identity matrix and the non-trivial involutory field automorphism σ. These polarities exist for all fields admitting such an involution.

   In $\Omega(\mathbb{K})$, the trialities of type $(\mathbf{I}_\sigma)$, $\sigma \neq 1$, have similar properties. They are in bijective correspondence with the class of field automorphisms σ of order 3 in $\mathbb{K}$, and $\mathbb{K}^{(\sigma)} \cup \{\infty\}$ parametrizes the set of absolute lines through an absolute point. For a given absolute point p, the collineation induced on $\mathbf{PG}(1,\mathbb{K})^{(p)}$ arises from the semi-linear map with identity matrix and associated field automorphism σ. These trialities exist for all fields admitting such an automorphism.

Note that all points of the quadrangles arising from polarities in $\mathbf{PG}(3,\mathbb{K})$ as above are distance-2-regular, just like the points of all hexagons arising from trialities in $\Omega(\mathbb{K})$; see Proposition 2.4.15.

2.5 Classical generalized octagons

There is at present no elementary geometric construction known of the classical octagons. In the literature, one is usually referred to the construction of this geometry using the *Tits system* (or (B,N)*-pair*; for these notions see Section 4.7) in the Chevalley groups of type ${}^2F_4$ (the so-called Ree groups of characterictic 2), which was also the original construction by Tits [1960] (see also Tits [1983]). The construction which we would like to give uses *metasymplectic spaces*, i.e., the point–line geometries arising from buildings of type F_4. So in fact, we should first construct such spaces. Of course, this is beyond the scope of this book. Henceforth, a rigorous existence proof of the classical octagons will not be considered in this book. But for those readers who are more or less familiar with elementary properties of metasymplectic spaces — which are nevertheless more popular than the classical octagons — we include the proof of the following theorem. The result is well known, but to the best of my knowledge no proof exists in print. The result was first announced by Tits [1960]; see also Sarli [1986]. Note, however, that I will need some elementary results concerning *polarities* of Moufang quadrangles, in particular about *ovoids* (this is unavoidable since buildings of type F_4 contain Moufang quadrangles as *residues*, and a polarity of such a building induces a polarity in some of these quadrangles). I therefore advise the reader to read first the relevant parts of Chapter 7, if necessary.

2.5.1 Definition. For a field $\mathbb{K}$ of positive characteristic p, we will call an endomorphism $\sigma : \mathbb{K} \to \mathbb{K}$ such that $x^{\sigma^2} = x^p$ for all $x \in \mathbb{K}$, a **Tits endomorphism**. Note that the endomorphism $x \mapsto x^p$ itself is called the **Frobenius endomorphism**. So a Tits endomorphism is a square root of the Frobenius endomorphism (but in general not unique; see the introduction of Section 7.6, page 322). We denote the field of squares of a field $\mathbb{K}$ of characteristic 2 by $\mathbb{K}^2$.

2.5.2 Theorem (Tits (unpublished)). *Let $\mathcal{M}$ be a metasymplectic space over some field $\mathbb{K}$, i.e., the planes of $\mathcal{M}$ are planes over $\mathbb{K}$. Suppose $\mathcal{M}$ is self-polar and let θ be a polarity. Then $\mathbb{K}$ has characteristic 2, it admits a Tits endomorphism σ and the geometry $\mathsf{O}(\mathbb{K}, \sigma)$ whose points and lines are the absolute points and lines, respectively, with natural incidence relation, is a generalized octagon with $|\mathbb{K}| + 1$ points per line and $|\mathbb{K}|^2 + 1$ lines per point.* □

Proof. We first define metasymplectic spaces axiomatically, then list some properties that we will use (these properties can be read off the diagram in most cases), and then proceed to the proof of the theorem.

Definition and properties of metasymplectic spaces

A **metasymplectic space** $\mathcal{M}$ is a building (see Subsection 1.3.7) with four types of elements, usually called *points, lines, planes* and *hyperlines*, together with a binary reflexive and symmetric incidence relation satisfying the axioms (M1) to (M4) stated below. The term building already implies that the incidence graph is connected, that no element has two or more different types and that every flag (a **flag** is a set of mutual incident elements) is contained in a chamber, i.e., a flag consisting of just four elements, one of each type.

The **residue** of a flag F is the geometry of elements distinct from those belonging to F and incident with all elements of F, subject to the incidence relation inherited from $\mathcal{M}$. The **type** of a flag is the set of types of its elements.

(M1) The residue of any flag of type {point, line} or {plane, hyperline} is a projective plane.

(M2) The residue of any flag of type {point, plane}, {line, hyperline} or {line, plane} is a generalized digon.

(M3) The residue of any flag of type {point, hyperline} is a generalized quadrangle.

(M4) If we call a **shadow** the set of points incident with a given element, then the intersection of any two shadows is again a shadow or empty. Furthermore, two distinct elements have distinct shadows.

The first three axioms tell you exactly that $\mathcal{M}$ belongs to the diagram F_4. The last axiom is the **intersection property**; see TITS [1981]. In fact, we have taken this definition in terms of points, lines, planes and hyperlines from TITS [1981].

From axiom (M4) it follows immediately that every element is determined by the set of points incident with it. So we may identify every element with the set of points incident with it. This way it makes sense to talk about intersections of hyperlines, planes, etc., and to use set-theoretic symbols as $\in$, $\subseteq$, ...

A metasymplectic space $\mathcal{M}$ now has the following properties. The proofs use only standard *diagram arguments*, i.e., using the axioms (M1), (M2) and (M3) combined with incidence properties of the generalized polygons corresponding to the various residues, or standard *apartment arguments* (see the first three chapters of TITS [1974]), i.e., the mutual position of two elements can be seen in an *apartment* of the corresponding building. Note that (M6) follows directly from (M1) and (M4).

(M5) *Let x and y be two points of $\mathcal{M}$. Then one of the following situations occurs:*

- (0) $x = y$.
- (1) *There is a unique line incident with both x and y. In this case, we call x and y* **collinear** *and we denote the unique line by xy.*
- (2) *There is a unique hyperline incident with both x and y. In this case there is no line incident with both x and y, and we call x and y* **cohyperlinear**. *We denote the unique hyperline by $x\Diamond y$.*
- (3) *There is a unique point z collinear with both x and y. In this case we call x and y* **almost opposite** *and we denote z by $x \bowtie y$.*
- (4) *There is no point collinear with both x and y. In this case we call x and y* **opposite**.

(M6) *The intersection of two hyperlines is either empty, or a point, or a plane.*

(M7) *Let x be a point and h a hyperline of $\mathcal{M}$. Then one of the following situations occurs:*

- (0) $x \in h$.
- (1) *There is a unique line L in h such that x is collinear with all points of L. Every point y of h which is collinear with all points of L is cohyperco-linear with x and $x\Diamond y$ contains L. Every other point z of h (i.e., every point z of h collinear with a unique point z' of L) is almost opposite x and $x \bowtie z = z' \in L$.*
- (2) *There is a unique point u of h cohyperlinear with x. We have $h \cap (x\Diamond u) = \{u\}$. All points v of h collinear with u are almost opposite x and $x \bowtie v \notin h$. All points w of h cohyperlinear with u are opposite x.*

(M8) *Two elements are incident* **if and only if** *the shadow of one of these elements is contained in the shadow of the other.*

(M9) *The points, lines and planes incident with a hyperline form a polar space; in particular, the Buekenhout–Shult one-or-all axiom (see* Definition 2.4.1*) and its consequences hold.*

(M10) *There is a principle of duality: if we replace point, line, plane, hyperline by, respectively, hyperline, plane, line, point in the above definitions and statements, then we obtain (mostly new) true properties.*

This principle of duality is a necessary condition for the existence of polarities. A sufficient condition is that the quadrangles which appear as a residue of a flag of type {point, hyperline} are Suzuki quadrangles (see Subsection 3.4.6 on page 103). We will not prove this. A proof would require the construction of such metasymplectic spaces and this would lead us into the theory of either Chevalley — or more generally, algebraic — groups, or Tits buildings (cf. Tits [1974], Ronan & Tits [1987] or Ronan [1989]). In Lemma 2 below, we give evidence for the fact that the Suzuki quadrangles are needed in order to have a polarity.

Proof of the theorem

Suppose now $\mathcal{M}$ is a metasymplectic space and θ is a polarity of $\mathcal{M}$, i.e., θ permutes the elements of $\mathcal{M}$ in such a way that points are mapped to hyperlines and vice versa, and lines are mapped to planes and vice versa; moreover θ preserves the incidence relation and θ^2 is the identity. An element of $\mathcal{M}$ is called **absolute** if it is incident with its image under θ. We claim that there is at least one absolute element. Indeed, let x be any point of $\mathcal{M}$. We apply (M7) to $h = x^\theta$. If $x \in x^\theta$, then x is an absolute point and the claim is proved. Suppose that there is a unique line L in x^θ all points of which are collinear with x. It is not so difficult to see that L^θ is equal to the plane generated by x and L, and hence L is an absolute line. Hence, by (M7), we may assume that there is a unique point $y \in x^\theta$ cohyperlinear with x. But then y^θ is the unique hyperline through x meeting x^θ in a point and so y is an absolute point. The claim follows.

We put Γ equal to the geometry with point set the set of absolute points, with line set the set of absolute lines and with natural incidence relation. Our goal is to show that Γ is a generalized octagon. We will do so in a sequence of lemmas.

Lemma 1. *Every point of an absolute line is absolute.*

Proof. Let L be an absolute line and $p \in L$. Applying θ, we obtain $L^\theta \subseteq p^\theta$ (using (M8)). Since L is absolute, $L \subseteq L^\theta$. Hence $p \in L \subseteq L^\theta \subseteq p^\theta$ and so p is absolute. QED

It is well known that metasymplectic spaces satisfy a so-called *Moufang condition* (see Tits [1974], [1976a]). This implies that all generalized polygons appearing as residues are Moufang polygons. So we may assume that the projective planes arising as residues of flags of type {plane, hyperline} are defined over an alternative field $\mathbb{K}$, which is also known to parametrize one of the two kinds of root groups of the generalized quadrangles arising as residues of flags of type {point, hyperline}. We denote by $\mathbf{PGL}_2(\mathbb{K})$ the group induced on a line L of $\mathbf{PG}(2,\mathbb{K})$ by the stabilizer of L in the group of automorphisms of $\mathbf{PG}(2,\mathbb{K})$ generated by all *elations* (hence in the so-called *little projective group* of $\mathbf{PG}(2,\mathbb{K})$; see Definitions 4.4.4 on page 143).

Lemma 2. *There is at least one absolute point. For every absolute point p, the residue $Q(p,p^\theta)$ of the flag $\{p,p^\theta\}$ is a Suzuki quadrangle and the absolute lines through p form an ovoid in $Q(p,p^\theta)$. Hence there are $|\mathbb{K}|^2+1$ absolute lines incident with p. Also, $\mathbb{K}$ has characteristic 2 and admits a Tits endomorphism.*

Proof. Since there is at least one absolute element, there is at least one absolute point. Indeed, if there is no absolute point, then there must be an absolute line (noting that an element a is absolute **if and only if** a^θ is absolute). But Lemma 1 implies that all points on that line are absolute. Hence the claim.

So let p be an absolute point. The polarity θ clearly induces a polarity in $Q(p,p^\theta)$. By Proposition 7.2.5 (see page 308), the absolute points with respect to that polarity form an ovoid of $Q(p,p^\theta)$. But it is easily seen that these absolute points are in fact absolute lines of $\mathcal{M}$ with respect to θ (if the lines of $\mathcal{M}$ incident with p and p^θ are called points of $Q(p,p^\theta)$). Since $Q(p,p^\theta)$ is a self-polar Moufang quadrangle, it must be either a mixed quadrangle $\mathsf{Q}(\mathbb{L},\mathbb{L}';L,L')$ (for appropriate $\mathbb{L},\mathbb{L}',L$ and L') or a Moufang quadrangle of type $(BC-CB)_2$, by Theorem 7.3.2 on page 312. The group induced on a residue of a flag of type {point, plane, hyperline} contains $\mathbf{PGL}_2(\mathbb{K})$ (looking in a residue of type {plane, hyperline}). This implies that all non-trivial root elations of $Q(p,p^\theta)$ are conjugate. Hence $Q(p,p^\theta)$ cannot be a Moufang quadrangle of type $(BC-CB)_2$, because in such a quadrangle, elements of $[U_1,U_3]\subseteq U_2$ (with the notation of Subsection 5.5.5) are never conjugate to elements of $U_2\setminus[U_1,U_3]$ (and the latter is non-empty!). Hence $Q(p,p^\theta)\cong\mathsf{Q}(\mathbb{L},\mathbb{L}';L,L')$. Now the group induced on a residue of a flag of type {point, plane, hyperline} is also contained in $\mathbf{PGL}_2(\mathbb{L})$ (looking in a residue of a flag of type {point, hyperline}). Hence, looking at the stabilizer of two points in such a residue, we readily deduce that $\mathbb{K}$ is a (commutative) field and that $\mathbb{K}=\mathbb{L}=L$, and consequently also $\mathbb{L}'=L'$, with $\mathbb{L}'$ the image of $\mathbb{L}$ under a Tits endomorphism (this follows from Theorem 7.3.2 on page 312). So $Q(p,p^\theta)$ is a Suzuki quadrangle (see Subsection 3.4.6 on page 103). The rest of the lemma follows from Proposition 7.2.3 (page 307). QED

The next lemma shows that θ has properties entirely different from polarities in projective spaces.

Lemma 3. *If a line of $\mathcal{M}$ contains two absolute points, then it is an absolute line. Also, no plane contains more than one absolute line.*

Proof. Let p and q be two distinct absolute points incident with a line M. Applying θ we see that p^θ and q^θ share a plane π. If p is collinear with all points of π, then $p\in q^\theta$ by (M7). If p is collinear with all points of a line L of π, then again by (M7), $q\in L$ and hence again $q\in p^\theta$. By (M9), there are no other possibilities. So we have shown that $q \in p^\theta$. Thus, $M \subseteq p^\theta$, implying $p \in M^\theta$. Similarly $q \in M^\theta$. Axiom (M4) implies that $M \subseteq M^\theta$.

Let the plane π contain two absolute lines L and M. Let L and M meet in p. It is easily seen that $L, M \subseteq p^\theta$. But this contradicts the fact that L and M represent non-collinear points (of an ovoid) in $Q(p, p^\theta)$. QED

Lemma 4. *Let p and q be cohyperlinear absolute points. Then there exists a unique absolute point x collinear with both p and q.*

Proof. Suppose first that $p \in q^\theta$. Then $q \in p^\theta$ and hence $p^\theta = (p\Diamond q) = q^\theta$, implying $p = q$, a contradiction.

Suppose now that the hyperlines p^θ and $p\Diamond q$ have only p in common. By (M7), q is opposite all points of p^θ which are cohyperlinear with p. Since p and q are contained in a unique hyperline, the hyperlines p^θ and q^θ meet in a unique point z. By (M7), we must have $z = p$ and hence $p\Diamond q = q^\theta$. Hence $p \in q^\theta$, a contradiction again.

So we may assume that $p\Diamond q$ meets p^θ in a plane π. This plane π is a line of the quadrangle $Q(p, p^\theta)$ and it is therefore incident with a unique absolute point of $Q(p, p^\theta)$ with respect to the polarity induced by θ in $Q(p, p^\theta)$. This point represents an absolute line L of $\mathcal{M}$ incident with p. Since q is not collinear with p, the set of points of π collinear with q is a line N distinct from L. Let $\{x\} = L \cap N$. Then x is an absolute point (since it lies on the absolute line L) collinear with both p and q. Since q is absolute, the previous lemma implies that qx is an absolute line.

If y is another absolute point of $\mathcal{M}$ collinear with both p and q, then $p \in py \,\mathbf{I}\, (py)^\theta$ implies $py \,\mathbf{I}\, p^\theta$. Similarly $qy \,\mathbf{I}\, q^\theta$, hence $y \in p^\theta \cap q^\theta$ and consequently $y = x$. QED

The previous proof also shows that, if x, y, z are three absolute points with x and y collinear, and y and z collinear, then x and z are cohyperlinear and $z \notin x^\theta$. Indeed, both x and z are contained in y^θ.

This in turn now implies:

Lemma 5. *Γ does not contain a proper pentagon.*

Proof. Let x, y, z, u, v be the consecutive collinear vertices of a proper pentagon (with x and v collinear). The point x is collinear with $v \in u^\theta$; it is cohyperlinear with $z \in u^\theta$, hence by (M7) either v and z are collinear (a contradiction), or $x \in u^\theta$. The latter contradicts our previous remark. QED

Lemma 6. *Let p and q be two almost opposite absolute points. Then there exist unique collinear absolute points x and y such that x is collinear with p and y with q.*

Proof. Since p and q are not contained in a common hyperline, the point $p \bowtie q$ is not absolute and the hyperlines p^θ and q^θ are disjoint. Let $h^\theta = p \bowtie q$. Since both p and q are collinear with $p \bowtie q$, the hyperline h meets both p^θ and q^θ in a plane, say, π_p and π_q, respectively. Using the Buekenhout–Shult one-or-all axiom in p^θ, one easily sees that there is at least one plane π in p^θ containing p and sharing a line L with π_p, $p \notin L$ (L is uniquely determined if $p \notin \pi_p$). It follows that there is a unique absolute line L_p through p meeting L and hence there is some absolute point x collinear with p and lying in π_p. Similarly, there is some absolute point y collinear with q and lying in π_q. Since the planes π_p and π_q are disjoint, the points x and y do not coincide. Hence they are either collinear — in which case the lemma is proved, up to uniqueness of x and y — or cohyperlinear. In the latter case, there is a unique absolute point z collinear with both x and y by Lemma 4. Since $x, y \in z^\theta$, we see that $z \in h$. By uniqueness of the hyperline through x and y, $z^\theta = h$. Applying θ, we deduce $z = p \bowtie q$, contradicting the fact that $p \bowtie q$ is not absolute.

There remains to show that x and y are unique. Suppose x' and y' are collinear absolute points collinear with, respectively, p and q. The point $q \notin p^\theta$ is cohyperlinear with both x and x' of p^θ. If $x \neq x'$, then (M7) implies that q is collinear with all points of a line L in p^θ and that the unique hyperline through q and $x(x')$, namely $y^\theta({y'}^\theta)$, contains L. Thus y^θ would meet p^θ in a line, hence a plane and so, applying θ, p and y would be collinear, a contradiction. QED

An almost identical argument as in the last part of the previous proof can be used for the following lemma.

Lemma 7. Γ *does not contain proper heptagons.*

Proof. Let $(p_1, p_2, \ldots, p_7, p_8 = p_1)$ be a heptagon with p_i collinear with p_{i+1}, for all i modulo 7. The point p_1 is collinear with the point p_2 belonging to p_3^θ. Hence by (M7), it is not opposite $p_4 \in p_3^\theta$. Since Γ does not contain proper pentagons by Lemma 5, p_1 and p_4 are almost opposite (cohyperlinear would imply a pentagon by Lemma 4). Also, p_1 is cohyperlinear with $p_6 \in p_5^\theta$ and almost opposite $p_5 \in p_5^\theta$. Since p_4 and p_6 are not collinear, (M7) implies that the unique hyperline through p_1 and p_6, namely p_7^θ, meets p_5^θ in at least a line, hence a plane. Therefore p_5 and p_7 are collinear, a contradiction. QED

Lemma 6 says in fact that there are no proper ordinary hexagons in Γ. Indeed, if p, x, y, q are consecutively collinear absolute points, then p is collinear with $x \in y^\theta \ni q$, and so we deduce as in the beginning of the proof of Lemma 7 that p and q are almost opposite. This brings us back to the situation of Lemma 6.

Lemma 8. *The gonality of* Γ *is equal to* 8.

Proof. In view of previous lemmas stating that there are no proper j-gons for $j \leq 7$ in Γ, we only have to exhibit a proper ordinary octagon in Γ. Let p_0 be an absolute point. Let L_0 be an absolute line through p_0 (existing by Lemma 2). Let $p_1 \neq p_0$ be on L_0. Let $L_1 \neq L_0$ be an absolute line through p_1. Continuing thus, we obtain a sequence

$$p_0 \,\mathbf{I}\, L_0 \,\mathbf{I}\, p_1 \,\mathbf{I}\, L_1 \,\mathbf{I}\, \ldots \,\mathbf{I}\, p_4 \,\mathbf{I}\, L_4,$$

with p_4 opposite p_0; otherwise there arises an ordinary proper j-gon in Γ with $j \leq 8$. Let h be any hyperline containing L_4 (for instance one can take $h = p_4^\theta$). Since p_0 is opposite $p_4 \in h$, there is a unique point $x \in h$ cohyperlinear with p_0. All points in h collinear with x are almost opposite p_0 and there is a unique such point p_5 on L_4 (unique because otherwise, by the Buekenhout–Shult one-or-all axiom, all points of L_4 are almost opposite p_0, including p_4, a contradiction). By Lemma 6, there are collinear absolute points p_6 and p_7 collinear with, respectively, p_5 and p_0. We have established an octagon $p_0 \perp p_1 \perp \ldots \perp p_7 \perp p_0$. QED

To finish the proof of Theorem 2.5.2 we only have to prove that the diameter of Γ is equal to 8. Note that we have shown above that, whenever a line L of $\mathcal{M}$ contains a point opposite some point p (in $\mathcal{M}$), then there is a (unique) point x on L almost opposite p. In fact, this follows readily from (M7)(2).

Lemma 9. *The diameter of* Γ *is equal to* 8.

Proof. This will follow as soon as we have shown that a point and a line are always at distance $j \leq 7$ from each other in the incidence graph of Γ. This will follow if we show that, for every line L and every point p, there is a point x on L at distance at most 6 from p. By Lemmas 3, 4 and 6, we may assume that all points of L are opposite p. But then we remarked above that there must be a unique point on L almost opposite x, a contradiction. QED

The theorem is proved. □

2.5.3 Definitions. We call the generalized octagon $\mathsf{O}(\mathbb{K}, \sigma)$ as in the statement of the theorem the **Ree–Tits octagon**, for obvious reasons. Both the Ree–Tits octagons and their duals will be called **classical**, but recall that the term *Ree–Tits octagon* is, as a matter of convenience, reserved for $\mathsf{O}(\mathbb{K}, \sigma)$ itself. In the finite case, it follows from the previous theorem that the field $\mathbf{GF}(q)$ has even order and $q = 2^{2e+1}$. In that case, the Tits endomorphism σ is an automorphism and is determined by $\mathbf{GF}(q)$. The corresponding Ree–Tits octagon is therefore denoted by $\mathsf{O}(q)$. We will give an explicit description of $\mathsf{O}(\mathbb{K}, \sigma)$ in the next chapter, but without proof, in view of the remarks we made in this section.

A special case occurs when $\mathbb{K}$ is a perfect field. Indeed, the corresponding Ree–Tits octagons have nicer geometric properties which also characterize them; see Subsection 6.9 on page 298. In that case, we call $\mathsf{O}(\mathbb{K}, \sigma)$ a **perfect Ree–Tits octagon**.

Commenting on things to come, we note that the proof of the characterization Theorem 6.9.3 (page 300) consists of reconstructing the metasymplectic space $\mathcal{M}$ for a given octagon satisfying the given axioms. The proof of the fact that there is essentially one polarity with at least one absolute element for a given metasymplectic space over a field $\mathbb{K}$ of characteristic 2 and a given Tits endomorphism of $\mathbb{K}$ can be deduced from various results of Tits [1974], [1962b], [1964]. An explicit proof is contained in Van Maldeghem [1998].

2.6 Table of notation for some classical polygons

We summarize part of the notation and terminology that we have introduced in this chapter in Table 2.1. For a generalized n-gon, we display n, Γ, the corresponding (simple) group G, its name and a reference. We denote by $\mathbb{K}$ any field, $\mathbb{L}$ any skew field, q any prime power, ρ (or τ) an appropriate 1-quadratic (or σ-quadratic) form on a $(d+1)$-dimensional vector space over $\mathbb{K}$ (respectively $\mathbb{L}$) (and where σ is an anti-automorphism of order 2 of $\mathbb{L}$), $\mathbb{F}$ denotes a cubic Galois extension of $\mathbb{K}$ (if that exists), and σ' is a non-trivial element of the corresponding Galois group. Also, we denote by char$\mathbb{K}$ the characteristic of the field $\mathbb{K}$. In the infinite case, it is impossible to give all orthogonal and Hermitian quadrangles. Therefore, we have restricted ourselves to mentioning only the finite cases.

n	Γ	G	Name	Remarks	Subsection
3	$\mathbf{PG}(2,\mathbb{K})$	$\mathbf{PSL}_3(\mathbb{K})$	Pappian plane		2.2.3
3	$\mathbf{PG}(2,\mathbb{L})$	$\mathbf{PSL}_3(\mathbb{L})$	Desarguesian plane		2.2.1
4	$\mathsf{W}(\mathbb{K})$	$\mathbf{PSp}_4(\mathbb{K})$	Symplectic quadrangle		2.3.17
4	$\mathsf{Q}(d,\mathbb{K},\rho)$		Orthogonal quadrangle		2.3.7
4	$\mathsf{H}(d,\mathbb{L},\tau,\sigma)$		Hermitian quadrangle		2.3.7
4	$\mathsf{Q}(4,q)$	$\mathbf{PSO}_5(q)$	Orthogonal quadrangle		2.3.12
4	$\mathsf{Q}(5,q)$	$\mathbf{PSO}_6^-(q)$	Orthogonal quadrangle		2.3.12
4	$\mathsf{H}(3,q^2)$	$\mathbf{PSU}_4(q)$	Hermitian quadrangle		2.3.12
4	$\mathsf{H}(4,q^2)$	$\mathbf{PSU}_5(q)$	Hermitian quadrangle		2.3.12
6	$\mathsf{H}(\mathbb{K})$	$\mathbf{G}_2(\mathbb{K})$	Split Cayley hexagon		2.4.9
6	$\mathsf{H}(\mathbb{K})$	$\mathbf{G}_2(\mathbb{K})$	Symplectic hexagon	char$\mathbb{K}=2$	2.4.14
6	$\mathsf{H}(\mathbb{K})$	$\mathbf{G}_2(\mathbb{K})$	Perfect symplectic hexagon	char$\mathbb{K}=2$ $\mathbb{K}$ perfect	2.4.14
6	$\mathsf{T}(\mathbb{F},\mathbb{K},\sigma')$	${}^3\mathbf{D}_4(\mathbb{F},\sigma')$	Twisted triality hexagon		2.4.9
8	$\mathsf{O}(\mathbb{K},\theta)$	${}^2\mathbf{F}_4(\mathbb{K},\theta)$	Ree–Tits octagon	char$\mathbb{K}=2$; $\theta^2=2$	2.5.3

Table 2.1. Some classical generalized polygons with their simple groups and name.

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