Introduction

In the theory of quantum mechanics the Hamiltonian H is typically self-adjoint, i.e., $H = H^*$. The self-adjointness ensures that the spectrum of the Hamiltonian, representing the energy spectrum of H, is real but it is not a necessary condition. In the literature on so-called PT-symmetric quantum mechanics (see, e.g., [BB98], [BBM99], [BBJ03], [Ben04b] and [Ben07]), it is believed that self-adjointness is rather a mathematical requirement than a physically established fact. Therefore, it was considered a surprise that operators exist which are not self-adjoint in the given quantum mechanical Hilbert space, but have real spectrum and that – if e.g. complex eigenvalues were present – they occurred only in complex conjugate pairs.

From a mathematical point of view, however, this is no surprise at all – provided one is familiar with the theory of self-adjoint operators in spaces with indefinite inner product (Krein spaces). The physical structure found to be the reason for the reality of the spectrum is PT-symmetry (space-time reflection symmetry), which amounts to self-adjointness in some Krein space. A Hamiltonian H is PT-symmetric if it commutes with PT, that is $PTH = HPT$, compare, e.g., [Ben07] and [AT10]. Here P denotes the space reflection (parity) operator and T the time reflection operator. P and T satisfy the relations $P^2 = T^2 = (PT)^2 = I$ and $PT = TP$. If $p = \mathrm{i}\,\mathrm{d}/\mathrm{d}x$ and x are the momentum and position operators, then P has the effect

$$p \mapsto -p, \quad x \mapsto -x$$

and T has the effect

$$p \mapsto -p, \quad x \mapsto x, \quad \mathrm{i} \mapsto -\mathrm{i},$$

compare, e.g., [BB98], [BBM99], [BBJ03], [Ben04b] and [Ben07].

In contrast to self-adjointness in Hilbert spaces, PT-symmetry does not necessarily lead to a completely real spectrum. For example, the Hamiltonian

$$H := p^2 + \mathrm{i}x^3$$

is not symmetric in the Hilbert space $L^2(\mathbb{R})$ since the potential is not real-valued. However, the Hamiltonian H is PT-symmetric in the Hilbert space $L^2(\mathbb{R})$:

$$\begin{aligned}\big(PTH(PT)^{-1}\big)f(x) &= \big(PTHTP\big)f(x)\\ &= \big(PTHT\big)f(-x)\\ &= \big(PTH\big)\overline{f(-x)}\\ &= \big(PT\big)\big(p^2\overline{f(-x)} + \mathrm{i}x^3\overline{f(-x)}\big)\\ &= P\big(p^2 f(-x) - \mathrm{i}x^3 f(-x)\big)\\ &= p^2 f(x) - \mathrm{i}(-x)^3 f(x) = Hf(x), \quad f \in \mathscr{D}(H),\end{aligned}$$

where $\mathscr{D}(H)$ is the maximal domain of H. More generally, for the family of PT-symmetric Hamiltonians (compare, e.g., [BB98] and [AT10])

$$H_\varepsilon := p^2 + x^2(\mathrm{i}x)^\varepsilon, \quad \varepsilon \in \mathbb{R},$$

the spectrum of H_ε was found to be real and positive if $\varepsilon \geq 0$ and partly real and partly complex if $\varepsilon < 0$ (see, e.g., [DDT01a] for a proof of the reality of the spectrum for $\varepsilon \geq 0$; for $\varepsilon < 0$ numerical results indicate the appearance of complex eigenvalues, see, e.g., [BB98] and [Ben04a]). More precisely, for $-1 < \varepsilon < 0$, there is a finite number of real positive eigenvalues and an infinite number of complex conjugate pairs of eigenvalues, if $\varepsilon \leq -1$, then there are no real eigenvalues, see, e.g., [BB98] and [Ben04a].

During the last decade PT-symmetric models have been analysed intensively, see, e.g., the review paper [Ben07] and the references therein. Within the vast literature on PT-symmetric problems there are only some mathematically rigorous papers, see, e.g. [DDT01b], [Shi02], [AK04], [LT04], [Shi04], [Shi05], [Tan06], [Tan07] and [AT10]. In particular, we mention the works of E. Caliceti, F. Cannata, S. Graffi and J. Sjöstrand (see [Cal04], [CGS05], [CG05], [Cal05], [CCG06], [CG08] and [CCG08]), who use perturbation theory for linear operators. In [Mos02], [Jap02], [AK04], [LT04], [GSZ05] and [Tan06] Krein space methods were applied to PT-symmetric problems. The paper by H. Langer and C. Tretter (see [LT04] and [LT06]) was the first where Krein space methods were used to prove rigorous abstract results for PT-symmetric problems; this approach is also crucial for this thesis.

Consider the following situation. If a self-adjoint operator A_0 in a Krein space $\big(\mathscr{K},[\cdot,\cdot]\big)$, which is also self-adjoint with respect to some Hilbert space inner product $(\cdot,\cdot)$ on $\mathscr{K}$, has an isolated real eigenvalue of definite type, then this eigenvalue remains real under a "sufficiently small" PT-symmetric perturbation V (that is, V is symmetric in $(\mathcal{K},[\cdot,\cdot])$). This theorem has been proven for the case of bounded V in [LT04] and relies on the fact that a uniformly positive subspace of a Krein space is stable, which is a well-known result in the theory of Krein spaces. The theorem mentioned above can be applied to isolated eigenvalues of PT-symmetric problems. If two simple real eigenvalues of the same type meet, they remain real after crossing. This is the case of self-adjoint operators in Hilbert spaces, where all eigenvalues are of positive type. If two real eigenvalues of different type meet, they will, in general, develop into a pair of non-real complex conjugate eigenvalues.

While the case of bounded V was treated in [LT04] (see also [LT06]), a comparable result for the case of unbounded V has been missing. The case of unbounded potentials has only been considered for a few special classes or examples of operators, see, e.g., [DDT01b], [Shi02], [CG05], [Cal05] and [CG08]. The aim of this thesis is to generalize the results of [LT04] to wide classes of unbounded potentials, e.g., to relatively bounded and relatively form-bounded operators. This includes a generalization of the results obtained in [CG05] for a special class of Schrödinger operators with relatively bounded complex polynomial potentials.

The main results of this thesis are stability results for the reality of the spectrum of a family of operators A_ε of the form

$$A_\varepsilon := A_0 + \varepsilon V, \quad \varepsilon \in [0,1];$$

in particular, we consider the case where A_ε is self-adjoint in a Krein space $(\mathcal{K},[\cdot,\cdot])$ while A_0 is also self-adjoint with respect to some Hilbert space inner product $(\cdot,\cdot)$ on $\mathcal{K}$. Furthermore, we give inclusions for the perturbed spectrum of A_ε. We found different assumptions on V to prove the respective results. More precisely, we consider the following three types of assumptions on V; in any case V is assumed to be symmetric in the Krein space $(\mathcal{K},[\cdot,\cdot])$.

(a) $\mathcal{D}(A_0) \subset \mathcal{D}(V)$ and there exist constants $\alpha \ge 0$, $0 \le \beta < 1/2$ such that

$$\|Vx\| \le \alpha\|x\| + \beta\|A_0 x\|, \quad x \in \mathcal{D}(A_0); \tag{1}$$

in this case $A_\varepsilon = A_0 + V$ is defined as an operator sum.

(b) A_0 and V are bounded from below in $(\mathcal{K},[\cdot,\cdot])$, $\mathcal{D}(\mathfrak{a}_0) \subset \mathcal{D}(\mathfrak{v})$ for the quadratic forms $\mathfrak{a}_0$ and $\mathfrak{v}$ associated with A_0 and V, respectively, and there exist constants $\alpha \ge 0$, $0 \le \beta < 1/2$ such that

$$|\mathfrak{v}[\![x]\!]| \le \alpha\|x\|^2 + \beta|\mathfrak{a}_0[\![x]\!]|, \quad x \in \mathcal{D}(\mathfrak{a}_0);$$

in this case $A_\varepsilon = A_0 \dotplus V$ is defined as a form sum, which is an extension of the operator sum.

(c) $\mathscr{D}(V) \subset \mathscr{D}(A_0)$ and there exist constants $\alpha \geq 0$, $0 \leq \beta < 1/2$ such that

$$|[Vx,x]| \leq \alpha \|x\|^2 + \beta [J|A_0|x,x], \quad x \in \mathscr{D}(V),$$

where J denotes a fundamental symmetry on $\mathscr{K}$, and $\mathscr{D}(V)$ is a core of $|A_0|^{1/2}$; in this case A_ε is the pseudo-Friedrichs extension of $A_0 + \varepsilon V$.

For example, in terms of relative boundedness properties of V with respect to A_0, case (a), our main results are the following. Since, by assumption, A_0 is self-adjoint in a Hilbert space, its spectrum is real. We establish the following conditions which guarantee the spectrum of $A_0 + V$ to be real, even when $A_0 + V$ is not self-adjoint in a Hilbert space (compare Theorem 1.44 below):

(i) Suppose λ^0 is an isolated eigenvalue of A_0 of definite type with finite multiplicity m. If

$$\frac{1}{\delta}\Big(\alpha + \beta\big(\delta + |\lambda^0|\big)\Big) < \frac{1}{2}, \tag{2}$$

where $\delta := \operatorname{dist}\big(\lambda^0, \sigma(A_0)\setminus\{\lambda^0\}\big)$, then $\sigma(A_1) \cap B_{\delta/2}(\lambda^0)$ consists of a finite system of isolated and real eigenvalues with total multiplicity m which are of the same type as λ^0.

(ii) The preceding result can be extended to the case when the spectrum of A_0 is discrete and consists of an infinite sequence of eigenvalues

$$\cdots < \lambda^0_{-2} < \lambda^0_{-1} < \lambda^0_1 < \lambda^0_2 < \cdots$$

of definite type with finite multiplicities. In this case it is necessary that (2) holds for each eigenvalue λ^0_n, $n \in \mathbb{Z}^*$. Let $\delta_n := \operatorname{dist}\big(\lambda^0_n, \sigma(A_0)\setminus\{\lambda^0_n\}\big)$, $n \in \mathbb{Z}^*$, and suppose that (1) holds with $\alpha_n \geq 0$ and $\beta_n \in [0,1/2)$, $n \in \mathbb{Z}^*$, such that

$$\gamma := \sup_{n\in\mathbb{Z}^*}\left(\frac{1}{\delta_n}\Big(\alpha_n + \beta_n\big(\delta_n + |\lambda^0_n|\big)\Big)\right) < \infty. \tag{3}$$

Then the spectrum of A_ε is discrete and consists of real eigenvalues which are of definite type for all $\varepsilon \in [0,\varepsilon_0]$, where $\varepsilon_0 \in (0,1]$ has to be chosen such that $\varepsilon_0 < 1/(2\gamma)$.

The preceding result can be illustrated by the following example (see Section 3.3 below) which was first studied in [CG05]. Consider operators induced by the differential expression

$$A_\varepsilon = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + P + \varepsilon \mathrm{i} Q, \quad \varepsilon \in [0,1],$$

in $L^2(\mathbb{R})$, where P and Q are multiplication operators by real polynomials P and Q; P is an even polynomial of degree $2p$, $p \geq 1$, with $\lim_{|x|\to\infty} P(x) = \infty$, and Q is an odd polynomial of degree $2q-1$, $q \geq 1$, such that $p > 2q$. In this special case the assumptions of (ii) are satisfied for

$$A_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + P \quad \text{and} \quad V = \mathrm{i}Q;$$

the spectrum of A_0 consists of an infinite sequence of eigenvalues $\lambda_1^0 < \lambda_2^0 < \cdots$ and the constants $\alpha_n \geq 0$ and $\beta_n \in [0,1/2)$, $n \in \mathbb{N}$, in (1) can be chosen such that (3) holds.

The results (i) and (ii) can be extended to the case where isolated compact parts of the spectrum of A_0 are considered instead of isolated eigenvalues (see Theorem 1.46 below). Furthermore, the results remain valid for cases (b) of relatively form-bounded operators and (c) of pseudo-Friedrichs extensions.

The proof of the results (i) and (ii) relies on the fact that a uniformly positive subspace of a Krein space is stable (see [LT04, Theorem 3.1]). This stability theorem applies to isolated eigenvalues or isolated (compact) parts of the spectrum of the operator family A_ε. In order to ensure that isolated eigenvalues or isolated parts of the spectrum of A_0 remain isolated under the perturbation εV, it is necessary that the perturbation εV is “sufficiently small” or, equivalently, A_ε is “sufficiently close” to A_0. While the “distance” between two bounded linear operators can be defined as the norm of their difference, the “distance” between two unbounded linear operators has to be measured in a different way. To this end the notion of generalized convergence is used, which amounts to convergence between the graphs of two unbounded linear operators or, equivalently, to the convergence of the resolvent of A_ε to the resolvent of A_0 in norm. The latter is guaranteed by assuming that V is relatively bounded (or relatively form-bounded, respectively) with respect to A_0 with relative bound (relative form-bound, respectively) less than 1.

The results achieved in this thesis are new in various aspects. In cases (a) and (b), results were known only for very particular classes of differential operators (compare [CG05] and [CG08], respectively). For case (c) the results of this thesis have been shown before in [Ves72a] and [Ves72b], but the proofs were different. In comparison to our results, [Ves72b] requires further assumptions but shows in addition to the reality of the spectrum of the pseudo-Friedrichs extension A_1, A_1 is similar to a self-adjoint operator in a Hilbert space, compare Remark 2.52 below.

The thesis is organized as follows. In Chapter 1 the case (a) of relatively bounded V is considered. The first section of this chapter gives a brief introduction into the theory of linear operators in Krein spaces. Subsequently, the reader is provided with fundamental definitions as well as elementary facts for relatively bounded operators. In Section 3 we introduce the notion of generalized convergence and we present a proof of the well-known result that, for an arbitrary family of closed linear operators T_ε, $\varepsilon \in [0,1]$, in a Banach space, T_ε converges to T_0 in the generalized sense if and only if the resolvent of T_ε converges to the resolvent of T_0 in norm. Further, we recall important results from perturbation theory regarding the change of the spectrum. If a Cauchy contour Γ separates a bounded part of the spectrum $\sigma(T_0)$ of T_0 from the rest and T_ε converges to T_0 in the generalized sense, the spectrum of T_ε is likewise separated into two parts by Γ, moreover, the isolated part enclosed by Γ changes continuously with ε. If V is A_0-bounded with A_0-bound less than 1, then A_ε converges to A_0 in the generalized sense and hence the above results apply to the family of operators $A_\varepsilon = A_0 + \varepsilon V$. Consequently, isolated eigenvalues or isolated parts of the spectrum of A_0 remain isolated under the perturbation εV. This enables us to apply the Krein space methods of [LT04] to establish criteria for the operator A_ε to have real spectrum consisting of isolated eigenvalues or isolated parts if this holds for A_0.

Chapter 2 extends the results of Chapter 1 to the case (b) of relatively form-bounded perturbations. Instead of studying the usual operator sum $A_0 + \varepsilon V$, we consider the sum $A_0 \dot{+} \varepsilon V$ of A_0 and εV defined by means of quadratic forms which is an extension of the operator sum $A_0 + \varepsilon V$. While the condition of relative form-boundedness itself is less restrictive than the one of relative boundedness, relatively form-bounded operators have to be required to be bounded from below (with respect to the respective inner product); therefore, case (b) constitutes a different class of unbounded perturbations compared to case (a). Nevertheless, as in case (a), relative form-boundedness of V with respect to A_0 with relative form-bound less than 1 guarantees that A_ε converges to A_0 in the generalized sense. This enables us to extend the results of Chapter 1 to the case of relatively form-bounded operators.

At the end of the second chapter, we consider case (c) and introduce the notion of pseudo-Friedrichs extensions. A pseudo-Friedrichs extension is an extension of the usual operator sum which is different from the form-sum introduced before; in particular, the domain inclusion is $\mathscr{D}(V) \subset \mathscr{D}(A_0)$ rather than $\mathscr{D}(A_0) \subset \mathscr{D}(V)$ (case (a)) or $\mathscr{D}(\mathfrak{a}_0) \subset \mathscr{D}(\mathfrak{v})$ (case (b)). The results are not essentially related to sesquilinear forms, but the techniques used in the proofs are similar. In the context of Krein spaces, these operators have also been studied in [Ves72a], [Ves72b] and [Ves08] where similar results were obtained, but by different proofs.

In Chapter 3 we present some examples where the results of this thesis are applied to ordinary differential operators. In Sections 3.1 and 3.2 we study a second and a fourth order differential operator, respectively, on a compact interval. The class of differential operators on $\mathbb{R}$ introduced in [CG05] which is also covered by the results of this thesis is considered in Section 3.3. For all these examples the results show that the spectrum of $A_0 + V$ remains real, even though $A_0 + V$ is not self-adjoint in a Hilbert space.

Notation. For an introduction to the theory of unbounded linear operators, the following notation and basic terminology as well as for further details we refer to [Kat95], [GGK90] and [RS80, RS75, RS79, RS78]. The domain of a linear operator A in a Banach space $\mathscr{X}$ we denote by $\mathscr{D}(A)$, the range of A by $\mathscr{R}(A)$ and the graph of A by $\mathscr{G}(A)$. If A is a closed linear operator, we denote the spectrum and the resolvent set of A by $\sigma(A)$ and $\rho(A)$, respectively.