1
Prologue

1.1
Tides in Newton's Gravity

A brief review of Newtonian gravity is useful not only as a limit of weak-field relativistic gravity, but also as a reminder of the principles upon which general relativity was formulated. Newtonian gravity is conveniently formulated in a fixed rectilinear coordinate system in terms of an absolute time coordinate. In such coordinates as these, Newton's laws of motion and gravitation describe the motion of a body of mass m falling freely about another body of mass M by the force

$$\boldsymbol{F} = m\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{GMm}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3}(\boldsymbol{x}-\boldsymbol{x}') , \tag{1.1}$$

where $\boldsymbol{x}$ is the position of the body with mass m, $\boldsymbol{x}'$ is the position of the body with mass M, t is the absolute time coordinate, and $G \simeq 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is Newton's gravitational constant. Famously, the quantity m cancels and

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\frac{GM}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3}(\boldsymbol{x}-\boldsymbol{x}') . \tag{1.2}$$

If there is a continuous distribution of matter then we can sum up all contributions to the acceleration from all pieces of the distribution to obtain

$$\frac{d^2\boldsymbol{x}}{dt^2} = -G\int\limits_{\text{body}} \frac{\boldsymbol{x}-\boldsymbol{x}'}{\|\boldsymbol{x}-\boldsymbol{x}'\|^3}\rho(\boldsymbol{x}')d^3\boldsymbol{x}' = \boldsymbol{\nabla}\left[G\int\limits_{\text{body}}\frac{\rho(\boldsymbol{x}')}{\|\boldsymbol{x}-\boldsymbol{x}'\|}d^3\boldsymbol{x}'\right] , \tag{1.3}$$

where ρ is the mass distribution (density) and $\boldsymbol{\nabla}$ is the gradient operator in $\boldsymbol{x}$. Therefore, the *acceleration* of the body (with respect to the Newtonian system of rectilinear coordinates) is

$$\boldsymbol{a} = \frac{d^2\boldsymbol{x}}{dt^2} = -\boldsymbol{\nabla}\Phi(\boldsymbol{x}) , \tag{1.4}$$

where

$$\Phi(\boldsymbol{x}) := -G\int\limits_{\text{body}}\frac{\rho(\boldsymbol{x}')}{\|\boldsymbol{x}-\boldsymbol{x}'\|}d^3\boldsymbol{x}' \tag{1.5}$$

Gravitational-Wave Physics and Astronomy, First Edition. Jolien D. E. Creighton, Warren G. Anderson.

is the *Newtonian potential*. The Newtonian potential satisfies the *Poisson equation*

$$\nabla^2 \Phi(\boldsymbol{x}) = -G \int \rho(\boldsymbol{x}') \nabla^2 \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}'\|} d^3 \boldsymbol{x}' = 4\pi G \rho(\boldsymbol{x}) \,, \tag{1.6}$$

where we have used

$$\nabla^2 \frac{1}{\|\boldsymbol{x} - \boldsymbol{x}'\|} = -4\pi \delta(\boldsymbol{x} - \boldsymbol{x}') \,. \tag{1.7}$$

Because the mass of the falling body does not enter into the equations of motion, any two bodies will fall the same way. If you can only see nearby free-falling bodies, you cannot tell whether you're falling or not. You feel the same if you are freely falling toward some massive object as you would if you were in no gravitational field whatsoever. The gravitational acceleration describes the motion of the falling body with respect to the absolute Newtonian coordinates – but is there any way for a freely falling observer to know if they are accelerating or not?

Einstein codified the observation that freely falling objects fall together as a principle known as the *equivalence principle*: a freely falling observer could always set up a local (freely falling) frame in which all the laws of physics are the same as they would be if that observer were not in a gravitational field. The coordinate acceleration $\boldsymbol{a}$ does not have any physical importance (as it does in Newtonian gravity) because one can always choose a frame of reference – freely falling with the observer – in which the observer is at rest.

Example 1.1 Coordinate acceleration in non-inertial frames of reference

An inertial frame of reference in Newtonian mechanics is any frame of reference that can be related to the absolute Newtonian frame of reference by a uniform velocity and a constant translation of position. That is, if $\boldsymbol{x}$ is the location of a particle in one inertial frame of reference, then another inertial frame of reference will have $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x}_0 - \boldsymbol{v}t$ for some constant vectors $\boldsymbol{x}_0$ and $\boldsymbol{v}$. Inertial frames preserve the form of Newton's second law since $\boldsymbol{a}' = d^2\boldsymbol{x}'/dt^2 = d^2\boldsymbol{x}/dt^2 = \boldsymbol{a}$.

In non-Cartesian coordinates, however, the form of the coordinate acceleration is different. For example, for a two-dimensional system we could express the location of a particle in polar coordinates $r = (x^2 + y^2)^{1/2}$ and $\phi = \arctan(y/x)$. In these coordinates, the coordinate velocity of a particle is given by $dr/dt = \boldsymbol{v} \cdot \boldsymbol{e}_r$ and $d\phi/dt = r^{-1}\boldsymbol{v} \cdot \boldsymbol{e}_\phi$ where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\phi$ are unit vectors in the r- and ϕ-directions, and the equations of motion for the particle are $F_r = m[d^2r/dt^2 + r(d\phi/dt)^2]$ and $F_\phi = m[d^2\phi/dt^2 + 2r^{-1}(dr/dt)(d\phi/dt)]$. Even when there is no force on the particle, $\boldsymbol{F} = 0$, there is still a coordinate acceleration in that d^2r/dt^2 and $d^2\phi/dt^2$ do not vanish except for purely radial motion. This merely arises because of the choice of non-Cartesian coordinates – the geometrical form of Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$ still holds.

A non-inertial frame is a frame that is accelerating relative to an inertial frame. A common example is a uniformly rotating reference frame with angular velocity vector $\boldsymbol{\omega}$. In such a reference frame, Newton's second law has the form $\boldsymbol{F} = m\boldsymbol{a} +$

$m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2m\boldsymbol{\omega} \times \boldsymbol{v}$ where the two additional terms, the *centrifugal force*, $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ and the *Coriolis force*, $2m\boldsymbol{\omega} \times \boldsymbol{v}$, arise because the frame of reference is non-inertial. These are known as fictitious forces.

A *freely falling frame of reference* in Newtonian theory is a non-inertial frame of reference because it is accelerating relative to the absolute set of Newtonian coordinates. The following coordinate transformation relates a freely falling frame of reference (primed coordinates) at point $\boldsymbol{x}_0$ with the absolute Newtonian coordinates (unprimed): $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x}_0 - \frac{1}{2}\boldsymbol{g}t^2$, where $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi(\boldsymbol{x}_0)$ is a constant. It is straightforward to see that $\boldsymbol{a}' = d^2\boldsymbol{x}'/dt^2 = -\boldsymbol{\nabla}[\Phi(\boldsymbol{x}) - \Phi(\boldsymbol{x}_0)]$ which vanishes at point $\boldsymbol{x}_0$.

In fact, there *is* a way to tell if you are falling. If there is another object that is some small distance away from you then its acceleration will be slightly different. Suppose $\boldsymbol{\zeta}$ is the vector pointing from you to the other object. The acceleration of that object is

$$\boldsymbol{a}(\boldsymbol{x} + \boldsymbol{\zeta}) = \boldsymbol{a}(\boldsymbol{x}) + (\boldsymbol{\zeta} \cdot \boldsymbol{\nabla})\boldsymbol{a}(\boldsymbol{x}) + O(\zeta^2) \tag{1.8}$$

and so the relative acceleration or *tidal acceleration* is

$$\Delta a_i = -\zeta^j \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = -\mathcal{E}_{ij}\zeta^j \, , \tag{1.9}$$

where

$$\mathcal{E}_{ij} := \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \tag{1.10}$$

is known as the *tidal tensor field*. The tidal acceleration is not really local since it depends on the separation $\boldsymbol{\zeta}$ between falling bodies. The tidal field, however, is a local quantity, and it encodes the presence of the gravitational field. We will see later that in General Relativity, the tidal field is a measure of the spacetime curvature.

In the above expressions, the indices i and j run over the three spatial coordinates $\{x^1, x^2, x^3\}$ or equivalently $\{x, y, z\}$ and ζ^i is the ith component of the vector $\boldsymbol{\zeta}$. (The three components of the vector are ζ^1, ζ^2 and ζ^3 so we would write $\zeta^i = [\zeta^1, \zeta^2, \zeta^3]$.) The tidal field is a rank-2 tensor having nine components: $\mathcal{E}_{11}, \mathcal{E}_{12}, \mathcal{E}_{13}, \mathcal{E}_{21}, \mathcal{E}_{22}, \mathcal{E}_{23}, \mathcal{E}_{31}, \mathcal{E}_{32}$ and $\mathcal{E}_{33}$. It is symmetric: $\mathcal{E}_{12} = \mathcal{E}_{21}$, $\mathcal{E}_{13} = \mathcal{E}_{31}$ and $\mathcal{E}_{23} = \mathcal{E}_{32}$, or, more concisely, $\mathcal{E}_{ij} = \mathcal{E}_{ji}$. Einstein's summation convention is being used here: there is an implicit summation over repeated indices. That is, the expression

$$\mathcal{E}_{ij}\zeta^j$$

is short-hand for

$$\sum_{j=1}^{3} \mathcal{E}_{ij}\zeta^j = \mathcal{E}_{i1}\zeta^1 + \mathcal{E}_{i2}\zeta^2 + \mathcal{E}_{i3}\zeta^3 \, .$$

For example, if two objects are separated in the x^3- or z-direction, so that ζ^1 and ζ^2 both vanish, then the three components of the tidal acceleration are

$$\Delta a_1 = -\mathcal{E}_{13}\zeta^3 \, , \quad \Delta a_2 = -\mathcal{E}_{23}\zeta^3 \, , \quad \text{and} \quad \Delta a_3 = -\mathcal{E}_{33}\zeta^3 \, .$$

Example 1.2 Tidal acceleration

Consider a body falling toward the Earth. The Newtonian potential is

$$\Phi = -\frac{G M_\oplus}{(x^2 + y^2 + z^2)^{1/2}} \, . \tag{1.11}$$

The tidal field component $\mathcal{E}_{11}$ is

$$\mathcal{E}_{11} = \frac{\partial^2 \Phi}{\partial x^2} = -G M_\oplus \left[3 \frac{x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \right] , \tag{1.12}$$

the tidal field component $\mathcal{E}_{12}$ is

$$\mathcal{E}_{12} = \frac{\partial^2 \Phi}{\partial x \partial y} = -G M_\oplus \left[3 \frac{x y}{(x^2 + y^2 + z^2)^{5/2}} \right] , \tag{1.13}$$

and so forth. The components can be written concisely as

$$\mathcal{E}_{ij} = -\frac{G M_\oplus}{r^5} \left[3 x_i x_j - \delta_{ij} r^2 \right] , \tag{1.14}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ and δ_{ij} is the *Kronecker delta*,

$$\delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \, , \end{cases} \tag{1.15}$$

and so $x_i = \delta_{ij} x^j$.

Suppose that a reference body is on the z-axis at a distance $r = z$ from the centre of the Earth. Then the tidal tensor is

$$\mathcal{E}_{ij} = \frac{G M_\oplus}{r^3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \, . \tag{1.16}$$

Consider a nearby second body that is also on the z-axis, a distance Δz farther from the centre of the Earth. The relative tidal acceleration of this body is

$$\Delta a_i = -\mathcal{E}_{ij} \zeta^j = -\mathcal{E}_{i3} \Delta z \, . \tag{1.17}$$

The only non-vanishing component is the z-component:

$$\Delta a_3 = 2 \frac{G M_\oplus}{r^3} \Delta z \, . \tag{1.18}$$

A third body is next to the reference body, lying a small distance Δx away on the x-axis. The relative tidal acceleration of this body is

$$\Delta a_i = -\mathcal{E}_{ij} \zeta^j = -\mathcal{E}_{i1} \Delta x \tag{1.19}$$

and the only non-vanishing component is the x-component:

$$\Delta a_1 = -\frac{G M_\oplus}{r^3} \Delta x \, . \tag{1.20}$$

Notice that a collection of freely falling objects will be pulled apart along the direction in which they are falling while being squeezed together in the orthogonal directions.

Unlike the coordinate acceleration, the tidal acceleration has intrinsic physical meaning. We witness ocean tides caused by the Moon and the Sun. These tides dissipate energy on the Earth. That is, tidal forces can do work. To compute the work, consider an extended body (say, the Earth) moving within a tidal field produced by another body (say, the Moon). An element of the extended body, located at a position $\boldsymbol{x}$ and having mass $\rho(\boldsymbol{x})d^3\boldsymbol{x}$, experiences a tidal force

$$F_i = -\mathcal{E}_{ij}x^j\rho(\boldsymbol{x})d^3\boldsymbol{x} \,. \tag{1.21}$$

If the element is moving through the tidal field with velocity $\boldsymbol{v}$ then there is an amount $F_i v^i$ of work per unit time done on that element. Summing over all elements that comprise the body yields the total amount of tidal work:

$$\begin{aligned}\frac{dW}{dt} &= -\int_{\text{body}} \mathcal{E}_{ij}v^i x^j \rho(\boldsymbol{x})d^3\boldsymbol{x} \\ &= -\frac{1}{2}\mathcal{E}_{ij}\frac{d}{dt}\int_{\text{body}} x^i x^j \rho(\boldsymbol{x})d^3\boldsymbol{x} \\ &= -\frac{1}{2}\mathcal{E}_{ij}\frac{dI^{ij}}{dt} \,,\end{aligned} \tag{1.22}$$

where

$$I^{ij} := \int_{\text{body}} x^i x^j \rho(\boldsymbol{x})d^3\boldsymbol{x} \tag{1.23}$$

is the *quadrupole tensor*. Note that this tensor is closely related to the *moment of inertia tensor*

$$\mathcal{I}_{ij} := \left(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}\right) I^{kl} = \int_{\text{body}} \left(r^2\delta_{ij} - x_i x_j\right)\rho(\boldsymbol{x})d^3\boldsymbol{x} \tag{1.24}$$

and also to the (traceless) *reduced quadrupole tensor*

$$I\!\!\!-_{ij} := \left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) I^{kl} = \int_{\text{body}} \left(x_i x_j - \frac{1}{3}r^2\delta_{ij}\right)\rho(\boldsymbol{x})d^3\boldsymbol{x} \,. \tag{1.25}$$

Here $r^2 = \|\boldsymbol{x}\|^2 = \delta_{ij}x^i x^j$.

Tidal work can also be performed by a dynamical system with a time-changing tidal field $\mathcal{E}_{ij}(t)$. The work performed by such a system on another body with a quadrupole tensor I^{ij} is found by integrating Eq. (1.22) by parts:

$$W = -\frac{1}{2}\,\mathcal{E}_{ij}I^{ij}\Big|_0^T + \frac{1}{2}\int_0^T \frac{d\mathcal{E}_{ij}}{dt} I^{ij}\,dt \,. \tag{1.26}$$

The first term is bounded, while the second term secularly increases with time and represents a transfer of energy from the dynamical system that is producing the time-changing tidal field to the other body. For example, the source of the time-changing tidal field might be a rotating dumbbell or a binary system of two stars in orbit about each other. Over a long time (large T) the secularly growing term will dominate, and we can write the work done *by* the dynamical source *on* the body with moment of inertia tensor I^{ij} as

$$\frac{dW}{dt} \approx \frac{1}{2}\frac{d\mathcal{E}_{ij}}{dt} I^{ij} . \tag{1.27}$$

1.2 Relativity

The *special theory of relativity* postulates that there is no preferred inertial frame: local measurements of physical quantities are the same no matter which inertial frame the measurement is made in. This is the *principle of relativity*. In particular, measurements of the speed of light in any inertial frame will always yield the same value, $c := 299\,792\,458\,\mathrm{m\,s^{-1}}$. The consequence of this is that the Newtonian separation of space and time must be abandoned. Consider a spaceship travelling at a constant speed v in the x-direction relative to the Earth (see Figure 1.1). Within the spaceship, an experimental determination of the speed of light is made in which a photon is emitted from a source in the y-direction, reflected by a mirror a distance $\frac{1}{2}\Delta y$ away from the source, and received back at the source. The time-of-flight $\Delta\tau$ is measured and the speed of light $c = \Delta y/\Delta\tau$ is computed. For an observer on the Earth, however, the distance travelled by the photon is $[(\Delta x)^2 + (\Delta y)^2]^{1/2}$, where $\Delta x = v\Delta t$ and Δt is the amount of time the observer on the Earth determines it takes the photon to travel from the emitter to the receiver. Since the observer on Earth must measure the same speed of light, $c = [(\Delta x)^2 + (\Delta y)^2]^{1/2}/\Delta t$, we see that

$$c^2 = \frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2} = \frac{(\Delta x)^2 + (c\Delta\tau)^2}{(\Delta t)^2} , \tag{1.28}$$

where we have used $\Delta y = c\Delta\tau$, and so

$$c^2(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta x)^2 . \tag{1.29}$$

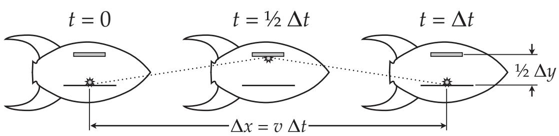

Figure 1.1 A measurement of the speed of light, performed in a rocket moving at speed v relative to the Earth, as seen by an observer on the Earth. A flash of light is produced at $t = 0$. The light travels a vertical distance $\frac{1}{2}\Delta y$, reflects off of the mirror and returns to the source after a time Δt (as measured by the observer on the Earth). The rocket has moved a horizontal distance $\Delta x = v\Delta t$ in this time.

The usual time dilation formula $\Delta t = \gamma \Delta \tau$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor, follows by setting $\Delta x = v \Delta t$. This relationship between how time is measured within the moving frame of the spaceship to how time is measured on Earth is not particular to the experiment with the photon: time really does move differently in the different inertial frames of reference.

Equation (1.29) relates the amount of time $\Delta \tau$ between two events, as recorded in an inertial frame in which the two events occur at the same spatial position (which is known as the *proper time* between the two events), to the amount of time Δt between the same two events as seen in an inertial frame in which the two events are separated by a spatial distance Δx. Since the notion of an absolute time is lost in special relativity, we understand time to simply be a new coordinate which, along with the three spatial coordinates, depends on the frame of reference. Together, the time and space coordinates are used to identify points (or events) on a four-dimensional *spacetime*. For rectilinear coordinates in an inertial frame, we define an invariant interval $(\Delta s)^2$ between two points in spacetime, (t, x, y, z) and $(t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z)$, by

$$(\Delta s)^2 := -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \,, \tag{1.30}$$

which has the same form as the Pythagorean theorem except for the factor of $-c^2$ in front of the square of the time interval. This equation is just a generalization of Eq. (1.29) with $(\Delta s)^2 := -c^2(\Delta \tau)^2$.

Special relativity is incompatible with Newtonian gravity because Newton's law of gravitation defines a force between two distant bodies in terms of their separation at a given instant in time. However, in special relativity, there is no unique notion of simultaneity. In addition, different frames of reference will make different measurements of the Newtonian gravitational force, a result that is at odds with the principle of relativity.

The *general theory of relativity* provides a description of gravity in terms of a curved spacetime. This is discussed in Chapter 2. In general relativity, the inertial frames of reference are freely falling frames, and the principle of relativity is then taken to hold in such frames of reference. Tidal acceleration is the physical manifestation of gravitation, but measurement of a tidal field requires a somewhat extended apparatus.

Of course, Newtonian gravity must be recovered in some limit of general relativity: this limit is when $GM/(c^2 R) \ll 1$ and $v/c \ll 1$ where M is the characteristic mass of the system, R is the characteristic size of the system, and v is the characteristic speed of bodies in the system. And since in Newtonian gravity a changing tidal field is capable of producing work on distant bodies, this must be true in general relativity as well. This means that in order to ensure that energy is conserved, energy must be radiated from the gravitating system that is producing the changing tidal field to the rest of the universe, because there is no way that the bodies on which the work is done can create an instantaneous reactive force on the gravitating system – this would be incompatible with relativity. The radiation is called gravitational radiation.