Chapter 2

Evolution of Geometric Quantities

In studying the long term behavior of solutions of parabolic equations and systems, in particular in the analysis of singularities, a basic step is always to obtain a priori estimates. These can be integral or pointwise; the main tool in order to get these latter is the *maximum principle*, in particular in the context of mean curvature flow.

2.1 Maximum Principle

Theorem 2.1.1. *Assume that $g(t)$, for $t \in [0,T)$, is a family of Riemannian metrics on a manifold M, with a possible boundary ∂M, such that the dependence on t is smooth.*
Let $u : M \times [0,T) \to \mathbb{R}$ be a smooth function satisfying

$$\partial_t u \leq \Delta_{g(t)} u + \langle X(p,u,\nabla u,t) \,|\, \nabla u \rangle_{g(t)} + b(u)$$

where X and b are respectively a continuous vector field and a locally Lipschitz function in their arguments.
Then, suppose that for every $t \in [0,T)$ there exists a value $\delta > 0$ and a compact subset $K \subset M \setminus \partial M$ such that at every time $t' \in (t-\delta, t+\delta) \cap [0,T)$ the maximum of $u(\,\cdot\,, t')$ is attained at least at one point of K (this is clearly true if M is compact without boundary).
Setting $u_{\max}(t) = \max_{p \in M} u(p,t)$ we have that the function $u_{\max}$ is locally Lipschitz, hence differentiable at almost every time $t \in [0,T)$ and at every differentiability time,

$$\frac{du_{\max}(t)}{dt} \leq b(u_{\max}(t))\,.$$

As a consequence, if $h : [0,T') \to \mathbb{R}$ is a solution of the ODE

$$\begin{cases} h'(t) = & b(h(t)), \\ h(0) = & u_{\max}(0), \end{cases}$$

for $T' \leq T$, then $u \leq h$ in $M \times [0,T')$.

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Moreover, if M *is connected and at some time* $\tau \in (0, T')$ *we have* $u_{\max}(\tau) = h(\tau)$*, then* $u = h$ *in* $M \times [0, \tau]$*, that is,* $u(\,\cdot\,, t)$ *is constant in space.*

Corollary 2.1.2. *Under the same hypotheses, when* M *is connected and the function* b *is nonpositive (in particular if it is identically zero), if the maximum of* u *is nondecreasing in a time interval* I*, the function* u *is constant in* $M \times I$*.*

The first part of the theorem is a consequence of the following lemma. The last claim, the *strong* maximum principle, is more involved, see the book of Landis [82] for a proof and the extensive discussion in [27, Chapter 12].

Lemma 2.1.3 (Hamilton's Trick [56]). *Let* $u : M \times (0, T) \to \mathbb{R}$ *be a* C^1 *function such that for every time* t*, there exists a value* $\delta > 0$ *and a compact subset* $K \subset M \setminus \partial M$ *such that at every time* $t' \in (t - \delta, t + \delta)$ *the maximum* $u_{\max}(t') = \max_{p \in M} u(p, t')$ *is attained at least at one point of* K*.*

Then, $u_{\max}$ *is a locally Lipschitz function in* $(0, T)$ *and at every differentiability time* $t \in (0, T)$ *we have*

$$\frac{du_{\max}(t)}{dt} = \frac{\partial u(p, t)}{\partial t}$$

where $p \in M \setminus \partial M$ *is any interior point where* $u(\,\cdot\,, t)$ *gets its maximum.*

Proof. Fixing $t \in (0, T)$, we have $\delta > 0$ and K as in the hypotheses, hence on $K \times (t - \delta, t + \delta)$ the function u is Lipschitz with some Lipschitz constant C. Consider a value $0 < \varepsilon < \delta$, then we have

$$u_{\max}(t + \varepsilon) = u(q, t + \varepsilon) \leq u(q, t) + \varepsilon C \leq u_{\max}(t) + \varepsilon C\,,$$

for some $q \in K$, hence,

$$\frac{u_{\max}(t + \varepsilon) - u_{\max}(t)}{\varepsilon} \leq C\,.$$

Analogously,

$$u_{\max}(t) = u(p, t) \leq u(p, t + \varepsilon) + \varepsilon C \leq u_{\max}(t + \varepsilon) + \varepsilon C\,,$$

for some $p \in K$, hence,

$$\frac{u_{\max}(t) - u_{\max}(t + \varepsilon)}{\varepsilon} \leq C\,.$$

With the same argument, considering $-\delta < \varepsilon < 0$, we conclude that $u_{\max}$ is a locally Lipschitz function in $(0, T)$, hence differentiable at almost every time.

Suppose that t is one of such times; let p be a point in the nonempty set $\{p \in M \setminus \partial M \,|\, u(p, t) = u_{\max}(t)\}$.

By Lagrange's theorem, for every $0 < \varepsilon < \delta$, $u(p, t+\varepsilon) = u(p,t) + \varepsilon \frac{\partial u(p,\xi)}{\partial t}$ for some ξ, hence

$$u_{\max}(t+\varepsilon) \geq u(p, t+\varepsilon) = u_{\max}(t) + \varepsilon \frac{\partial u(p,\xi)}{\partial t}\,,$$

which implies, as $\varepsilon > 0$,

$$\frac{u_{\max}(t+\varepsilon) - u_{\max}(t)}{\varepsilon} \geq \frac{\partial u(p,\xi)}{\partial t}\,.$$

Sending ε to zero, we get $u'_{\max}(t) \geq \frac{\partial u(p,t)}{\partial t}$.
If instead we choose $-\delta < \varepsilon < 0$ we get

$$\frac{u_{\max}(t+\varepsilon) - u_{\max}(t)}{\varepsilon} \leq \frac{\partial u(p,\xi)}{\partial t}$$

and when $\varepsilon \to 0$, we have $u'_{\max}(t) \leq \frac{\partial u(p,t)}{\partial t}$. Thus, we are done. ☐

Exercise 2.1.4. Prove that the conclusion of the lemma holds also if the function u is merely locally Lipschitz, provided that all the derivatives involved in the computations there exist.

Proof of Theorem 2.1.1 – *First Part.* By the previous lemma, the function $u_{\max}$ is locally Lipschitz and letting t be a differentiability time of $u_{\max}$, we have, choosing any $p \in M \setminus \partial M$ such that $u(p,t) = u_{\max}(t)$,

$$\begin{aligned} u'_{\max}(t) = \frac{\partial u(p,t)}{\partial t} &\leq \Delta_{g(t)} u + \langle X(p,u,\nabla u,t) \,|\, \nabla u \rangle_{g(t)} + b(u(p,t)) \\ &\leq b(u(p,t)) \\ &= b(u_{\max}(t))\,. \end{aligned}$$

Let now $h : [0, T') \to \mathbb{R}$ be as in the hypothesis. We define, for $\varepsilon > 0$, the approximating functions $h_\varepsilon : [0, T'') \to \mathbb{R}$ to be the maximal solutions of the family of ODE's

$$\begin{cases} h'_\varepsilon(t) = & b(h_\varepsilon(t)), \\ h_\varepsilon(0) = & u_{\max}(0) + \varepsilon\,. \end{cases}$$

It is easy to see that, as the function b is locally Lipschitz, then $\lim_{\varepsilon\to 0} h_\varepsilon = h$ uniformly on $[0, T' - \delta]$ for any $\delta > 0$. Suppose that at some positive time $u_{\max} > h_\varepsilon$ and set $\overline{t} > 0$ to be the positive infimum of such times (at time zero $u_{\max}(0) = h_\varepsilon(0) - \varepsilon$). Then, $u_{\max}(\overline{t}) = h_\varepsilon(\overline{t})$ and, setting $H_\varepsilon = h_\varepsilon - u_{\max}$, at every differentiability point of $u_{\max}$ in the interval $[0, \overline{t})$ we have $H_\varepsilon(0) = \varepsilon > 0$ and

$$H'_\varepsilon(t) \geq b(h_\varepsilon(t)) - b(u_{\max}(t)) \geq -C(h_\varepsilon(t) - u_{\max}(t)) = -CH_\varepsilon(t)$$

where $C > 0$ is a local Lipschitz constant for b.

Then, $(\log H_\varepsilon)'(t) \geq -C$ and integrating, $\log H_\varepsilon|_0^t \geq -Ct$, that is, $H_\varepsilon(t) \geq H_\varepsilon(0)e^{-Ct} = \varepsilon e^{-Ct}$. In particular, if $t \to \overline{t}$, we conclude $H_\varepsilon(\overline{t}) \geq \varepsilon e^{-C\overline{t}} > 0$ which is in contradiction with $H_\varepsilon(\overline{t}) = 0$. Hence, $u_{\max}(t) \leq h_\varepsilon(t)$ for every $t \in [0, T' - \delta)$ and sending ε to zero, $u_{\max}(t) \leq h(t)$ for every $t \in [0, T' - \delta)$. As $\delta > 0$ was arbitrary, we conclude the proof of the first part of the theorem. □

Exercise 2.1.5. When the function $u_{\max}$ is not differentiable at t, one can still actually say something using the upper derivative, that is the lim sup of the incremental ratios; we call this operator d^+. Prove that

$$\frac{d^+ u_{\max}(t)}{dt} = \sup_{\{p \in M \,|\, u(p,t) = u_{\max}(t)\}} \frac{\partial u(p,t)}{\partial t}\,.$$

Roughly speaking, the sup and the upper derivative operators can be interchanged.

The same holds for the inf and the lower derivative defined analogously.

What can be said about the left/right derivatives of $u_{\max}$?

Remark 2.1.6. Clearly, there hold analogous results for the minimum of the solution of the opposite partial differential inequality. Moreover, the maximum principle for elliptic equations easily follows as the special case where all the quantities around do not depend on the time variable t.

2.2 Comparison Principle

Theorem 2.2.1 (Comparison Principle for Mean Curvature Flow). *Let* $\varphi : M_1 \times [0, T) \to \mathbb{R}^{n+1}$ *and* $\psi : M_2 \times [0, T) \to \mathbb{R}^{n+1}$ *be two hypersurfaces moving by mean curvature, with* M_1 *compact. Then the distance between them is nondecreasing in time.*

Proof. The distance between the two hypersurfaces $\varphi_t : M_1 \to \mathbb{R}^{n+1}$ and $\psi_t : M_2 \to \mathbb{R}^{n+1}$ at time t, is given by $d_\psi^\varphi(t) = \inf_{p \in M_1, q \in M_2} |\varphi(p,t) - \psi(q,t)|$. This function is locally Lipschitz in time, as the curvature is locally bounded and the two hypersurfaces move by mean curvature, so it is differentiable almost everywhere and we assume that t is a differentiability point.

This infimum is actually a minimum as M_1 is compact, suppose then that it is positive and let (p_t, q_t) be any pair realizing such a minimum.

It is easy to see that, by minimality, the respective tangent spaces at p_t and q_t of the two hypersurfaces have to be parallel. Then we can write locally $\varphi(p,t)$ and $\psi(p,t)$ as graphs of two functions $f(p,t)$ and $h(p,t)$ over one of these tangent spaces for a small interval of time $(t - \varepsilon, t + \varepsilon)$. We can assume that $\langle e_1, \dots, e_n \rangle \subset \mathbb{R}^{n+1}$ is such a tangent space with $\varphi(p_t, t) = (0, f(0,t))$ and $\psi(q_t, t) = (0, h(0,t))$ at time t; moreover $f(0,t) > h(0,t)$.

We know, by Exercise 1.3.8 that

$$f_t = \Delta f - \frac{\mathrm{Hess} f(\nabla f, \nabla f)}{1 + |\nabla f|^2} \quad \text{and} \quad h_t = \Delta h - \frac{\mathrm{Hess} h(\nabla h, \nabla h)}{1 + |\nabla h|^2}.$$

Again, by minimality, the function $f(x,t)-h(x,t)$ has a minimum at $x=0$, hence, $\Delta f(0,t)-\Delta h(0,t)\geq 0$ and $\nabla f(0,t)=\nabla h(0,t)=0$, but we saw that for graphs, $\Delta f(0,t) = \mathrm{H}^{\varphi}(p_t,t)\langle \nu^{\varphi}(p_t,t)\,|\,e_{n+1}\rangle$ and $\Delta h(0,t) = \mathrm{H}^{\psi}(q_t,t)\langle \nu^{\psi}(q_t,t)\,|\,e_{n+1}\rangle$, thus,

$$\langle \mathrm{H}^{\varphi}(p_t,t)\nu^{\varphi}(p_t,t)-\mathrm{H}^{\psi}(q_t,t)\nu^{\psi}(q_t,t)\,|\,e_{n+1}\rangle=\Delta f(0,t)-\Delta h(0,t)\geq 0\,.$$

Now we have $\frac{p_t-q_t}{|p_t-q_t|}=e_{n+1}$ by construction and, by Lemma 2.1.3, we can conclude, as this analysis holds for all the pairs of points realizing the minimum, that

$$\begin{aligned}\frac{d}{dt}d^{\varphi}_{\psi}(t) &= \inf_{(p_t,q_t)\in M_1\times M_2 \ \text{with}\ |\varphi(p_t,t)-\psi(q_t,t)|=d^{\varphi}_{\psi}(t)} \frac{\partial}{\partial t}|\varphi(p_t,t)-\psi(q_t,t)|\\ &= \inf_{(p_t,q_t)\in M_1\times M_2 \ \text{with}\ |\varphi(p_t,t)-\psi(q_t,t)|=d^{\varphi}_{\psi}(t)} \frac{\langle p_t-q_t\,|\,\mathrm{H}^{\varphi}\nu^{\varphi}-\mathrm{H}^{\psi}\nu^{\psi}\rangle}{|p_t-q_t|}\\ &= \inf_{(p_t,q_t)\in M_1\times M_2 \ \text{with}\ |\varphi(p_t,t)-\psi(q_t,t)|=d^{\varphi}_{\psi}(t)} \langle \mathrm{H}^{\varphi}\nu^{\varphi}-\mathrm{H}^{\psi}\nu^{\psi}\,|\,e_{n+1}\rangle\\ &\geq 0\,.\end{aligned}$$

If the minimum is zero, there is nothing to show; obviously the derivative, if it exists, cannot be negative. ☐

Exercise 2.2.2. Show the following facts for a compact hypersurface moving by mean curvature.

- The diameter of the hypersurface decreases during the flow.
- The circumradius of the hypersurface (the radius of the smallest sphere enclosing the hypersurface) decreases.

Corollary 2.2.3. *Let $\varphi : M_1\times[0,T)\to\mathbb{R}^{n+1}$ and $\psi : M_2\times[0,T)\to\mathbb{R}^{n+1}$ be two hypersurfaces moving by mean curvature such that M_1 is compact, M_2 is embedded and $\varphi(M_1,0)$ is strictly "inside" $\psi(M_2,0)$. Then $\varphi(M_1,t)$ remains strictly "inside" $\psi(M_2,t)$ for every time $t\in[0,T)$.*

Proof. This is an easy consequence of the fact that the distance between the two hypersurfaces is nondecreasing, so it cannot get to zero, as it starts positive. Hence, the hypersurface "inside" cannot "touch" the other during the flow. ☐

Remark 2.2.4. By means of the continuous dependence result in Theorem 1.5.1 one has a slight improvement of the previous corollary, allowing the two hypersurfaces, one "inside" the other, to have common points at the initial time. To prove this fact one can "push" a little inside the initial hypersurface φ_0 along the gradient of the distance function from $\psi(M_2,0)$ in a local small tubular neighborhood (M_1 is compact), then conclude by the above corollary and the continuous dependence of the flow on the initial hypersurface.

By means of the strong maximum principle we can actually show something more, that is, evolving by mean curvature, the distance between two connected hypersurfaces (with at least one compact) with possibly only tangent intersections and such that they "do not cross each other", is always increasing, otherwise they must coincide.

This can be seen by using again the idea of the proof of Theorem 1.5.1, writing the two hypersurfaces as graphs over the initial "external" hypersurface in a small regular tubular neighborhood of this latter and applying the strong maximum principle to the "height" functions representing them. As a preliminary step, one has to consider an "intermediate" hypersurface close enough to the "external" one which stays in its tubular neighborhood for some positive time. We leave the technical details to the reader as an exercise.

In other words, if two connected hypersurfaces (one compact "inside" the other) touch each other at time zero but they are not the same, immediately they become disjoint, at every positive time.

Even more, in the special case of curves in the plane the number of intersections (or of self-intersections) is nonincreasing in time, see [14, 16].

Applying Corollary 2.2.3 to the case that $\varphi(M_2, 0)$ is a sphere of radius R, we have the following estimate for the maximal time of smooth existence.

Corollary 2.2.5. *Let* $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ *be the mean curvature flow of a compact hypersurface. If* $\varphi(M, 0) \subset B_R(x_0)$ *then the flow is contained in* $B_R(x_0)$ *at every time and* $T \leq R^2/(2n)$.

Hence, the mean curvature flow of every compact immersed hypersurface develops a singularity in finite time.

In particular, if $T_{\max}$ *is the maximal time of smooth existence of the flow, then* $T_{\max} \leq \operatorname{diam}^2_{\mathbb{R}^{n+1}}[\varphi(M, 0)]/2n$.

Proof. We have already seen that a sphere of radius R shrinks to a point with the rule $R(t) = \sqrt{R^2 - 2nt}$, hence at time $t = R^2/(2n)$ its radius gets to zero. As $\varphi(M, t) \subset B_{\sqrt{R^2-2nt}}(x_0)$, at most at time $t = R^2/(2n)$ the evolving hypersurface φ_t must develop a singularity, since at such time it cannot be an immersion.

The last claim is trivial. □

Another consequence of the maximum principle is the following characterization of the points of $\mathbb{R}^{n+1}$ "reached" by the flow at time T, that is, an estimate on the rate of convergence to a limit hypersurface as $t \to T$ (this will be particularly interesting when T is a singular time). Roughly speaking, if a hypersurface moving by mean curvature is "reaching" a point of the Euclidean space at some time, then it cannot stay "too far" from such a point in the past.

Proposition 2.2.6. *Let* $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ *be a mean curvature flow and define* $\mathcal{S}$ *to be the set of points* $x \in \mathbb{R}^{n+1}$ *such that there exists a sequence of pairs* $(p_i, t_i) \in M \times [0, T)$ *with* $t_i \nearrow T$ *and* $\varphi(p_i, t_i) \to x$.

Then, $\mathcal{S}$ is closed (and bounded if M is compact), moreover $x \in \mathcal{S}$ if and only if for every $t \in [0,T)$ the closed ball of radius $\sqrt{2n(T-t)}$ and center x intersects $\varphi(M,t)$.

Proof. One implication is obvious.
Suppose that $x \in \mathcal{S}$ and let $d_t(x) = \min_{p\in M} |\varphi(p,t) - x|$, that is, the Euclidean distance from x to the hypersurface at time t.
The function $d_t : [0,T) \to \mathbb{R}$ is obviously locally Lipschitz and at a differentiability time with $d_t(x) > 0$, by Hamilton's trick, Lemma 2.1.3, we have

$$d_t'(x) = \frac{\partial}{\partial t}|\varphi(q,t) - x| \geq \frac{\mathrm{H}(q,t)\langle \nu(q,t)\,|\,\varphi(q,t)-x\rangle}{|\varphi(q,t)-x|}$$

for any point $q \in M$ such that $d_t(x) = |\varphi(q,t) - x|$.
As the closed ball $\overline{B}_{d_t(x)}(x)$ intersects the hypersurface φ_t only on its boundary and the vector $\frac{\varphi(q,t)-x}{|\varphi(q,t)-x|}$ is parallel to the normal $\nu(q,t)$ by minimality, an easy geometric argument on the principal eigenvalues of the second fundamental form shows that

$$\frac{\mathrm{H}(q,t)\langle \nu(q,t)\,|\,\varphi(q,t)-x\rangle}{|\varphi(q,t)-x|} \geq -n/d_t(x)\,.$$

Hence, we conclude that for almost every time $t \in [0,T)$,

$$d_t'(x) \geq -n/d_t(x)$$

if $d_t(x) \neq 0$.
Integrating this differential inequality on $[t,s]$ we get $d_t^2(x) - d_s^2(x) \leq 2n(s-t)$ and by the hypothesis on x we have $d_{t_i}^2(x) \to 0$, hence

$$d_t^2(x) = \lim_{i\to\infty} d_t^2(x) - d_{t_i}^2(x) \leq \lim_{i\to\infty} 2n(t_i - t) = 2n(T-t)$$

which is the thesis of the proposition.
The closure of $\mathcal{S}$ is obvious, if M is compact $\mathcal{S}$ is clearly also bounded by Corollary 2.2.5. □

A very important fact about hypersurfaces moving by mean curvature is the following.

Proposition 2.2.7. *If the initial hypersurface is compact and embedded, then it remains embedded during the flow.*

Proof. Given the mean curvature flow φ_t, if the hypersurface φ_0 is embedded it remains so for a small positive time, otherwise we will have a sequence of points and times, with $\varphi(p_i,t_i) = \varphi(q_i,t_i)$ and $t_i \to 0$, then, extracting a subsequence (not relabeled) such that $p_i \to p$ and $q_i \to q$, either $p \neq q$ so $\varphi(p,0) = \varphi(q,0)$, which is a contradiction, or $p = q$. By the smooth existence of the flow, in particular by the nonsingularity of the differential of $\partial_x\varphi(p,t)$ there exists a ball $B \subset M$ around p such that for $t \in [0,\varepsilon)$ the map $\varphi_t|_B$ is one-to-one, which is in contradiction with the hypotheses.

This short time embeddedness property is also immediate by revisiting the proof of the short time existence theorem, representing the moving hypersurfaces as graphs on the initial one.

This argument also implies that the embeddedness holds in an open time interval, then we assume that $T > 0$ is the first time such that the hypersurface φ_t is no more embedded. The set S of pairs (p,q) with $p \neq q$ and $\varphi(p,T) = \varphi(q,T)$ is a nonempty closed set disjoint from the diagonal in $M \times M$, otherwise φ_T fails to be an immersion at some point in M. Then, we can find a smooth open neighborhood Ω of the diagonal with $\overline{\Omega} \cap S = \emptyset$.

We consider the quantity

$$C = \inf_{t\in[0,T]} \inf_{(p,q)\in\partial\Omega} |\varphi(p,t) - \varphi(q,t)|\,,$$

then C is positive, as $\overline{\Omega} \cap S = \emptyset$ and $\partial\Omega$ is compact. We claim that the function

$$L(t) = \min_{(p,q)\in M\times M\setminus\Omega} |\varphi(p,t) - \varphi(q,t)|\,,$$

is bounded from below by $\min\{L(0), C\} > 0$ on $[0,T]$, this is clearly in contradiction with the fact that S is nonempty and contained in $M \times M \setminus \Omega$.

If at some time $L(t) < C$ it follows that $L(t)$ is achieved by some pairs (p,q) not belonging to $\partial\Omega$, then (p,q) are inner points of $M\times M\setminus\Omega$ and a geometric argument analogous to the one of the comparison Theorem 2.2.1 shows that $\frac{dL(t)}{dt} \geq 0$, hence $L(t)$ is nondecreasing in time. This last fact clearly implies the claim. $\square$

Remark 2.2.8. Theorem 2.2.1 and Proposition 2.2.7 also hold if the involved hypersurfaces are not compact, with some additional assumptions on the behavior at infinity (for instance, uniform bounds on the curvature), the analysis is anyway more complicated.

2.3 Evolution of Curvature

Now we derive the evolution equations for g, ν, Γ^i_{jk}, A and H. We already know that

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{H}h_{ij}\,.$$

Differentiating the formula $g_{is}g^{sj} = \delta^j_i$ we get

$$\frac{\partial}{\partial t} g^{ij} = -g^{is}\frac{\partial}{\partial t} g_{sl}g^{lj} = 2\mathrm{H}g^{is}h_{sl}g^{lj} = 2\mathrm{H}h^{ij}\,.$$

The derivative of the normal ν is given by

$$\left\langle \frac{\partial \nu}{\partial t} \,\middle|\, \frac{\partial \varphi}{\partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial t \partial x_i} \right\rangle = -\left\langle \nu \,\middle|\, \frac{\partial (\mathrm{H}\nu)}{\partial x_i} \right\rangle = -\frac{\partial \mathrm{H}}{\partial x_i}\,.$$

Finally the derivative of the Christoffel symbols is

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^i_{jk} &= \frac{1}{2} g^{il} \left\{ \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t} g_{kl}\right) + \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial t} g_{jl}\right) - \frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial t} g_{jk}\right) \right\} \\
&\quad + \frac{1}{2}\frac{\partial}{\partial t} g^{il} \left\{ \frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \right\} \\
&= \frac{1}{2} g^{il} \left\{ \nabla_j \left(\frac{\partial}{\partial t} g_{kl}\right) + \nabla_k \left(\frac{\partial}{\partial t} g_{jl}\right) - \nabla_l \left(\frac{\partial}{\partial t} g_{jk}\right) \right\} \\
&\quad + \frac{1}{2} g^{il} \Big\{ \frac{\partial}{\partial t} g_{kz} \Gamma^z_{jl} + \frac{\partial}{\partial t} g_{lz} \Gamma^z_{jk} + \frac{\partial}{\partial t} g_{jz} \Gamma^z_{kl} \\
&\qquad\qquad + \frac{\partial}{\partial t} g_{lz} \Gamma^z_{jk} - \frac{\partial}{\partial t} g_{jz} \Gamma^z_{kl} - \frac{\partial}{\partial t} g_{kz} \Gamma^z_{jl} \Big\} \\
&\quad - \frac{1}{2} g^{is} \frac{\partial}{\partial t} g_{sz} g^{zl} \left\{ \frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \right\} \\
&= \frac{1}{2} g^{il} \left\{ \nabla_j \left(\frac{\partial}{\partial t} g_{kl}\right) + \nabla_k \left(\frac{\partial}{\partial t} g_{jl}\right) - \nabla_l \left(\frac{\partial}{\partial t} g_{jk}\right) \right\} \\
&\quad + g^{il} \frac{\partial}{\partial t} g_{lz} \Gamma^z_{jk} - g^{is} \frac{\partial}{\partial t} g_{sz} \Gamma^z_{jk} \\
&= \frac{1}{2} g^{il} \left\{ \nabla_j \left(\frac{\partial}{\partial t} g_{kl}\right) + \nabla_k \left(\frac{\partial}{\partial t} g_{jl}\right) - \nabla_l \left(\frac{\partial}{\partial t} g_{jk}\right) \right\} \\
&= -g^{il} \left\{ \nabla_j (\mathrm{H} h_{kl}) + \nabla_k (\mathrm{H} h_{jl}) - \nabla_l (\mathrm{H} h_{jk}) \right\} \\
&= -h^i_k \nabla_j \mathrm{H} - h^i_j \nabla_k \mathrm{H} + h_{jk} \nabla^i \mathrm{H} - \mathrm{H}(\nabla_j h^i_k + \nabla_k h^i_j - \nabla^i h_{jk})\,.
\end{aligned}$$

Summarizing, we have

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= -2\mathrm{H} h_{ij} \\
\frac{\partial}{\partial t} g^{ij} &= 2\mathrm{H} h^{ij} \\
\frac{\partial}{\partial t} \nu &= -\nabla \mathrm{H} \\
\frac{\partial}{\partial t} \Gamma^i_{jk} &= \nabla \mathrm{H} * \mathrm{A} + \mathrm{H} * \nabla \mathrm{A} = \nabla \mathrm{A} * \mathrm{A}\,.
\end{aligned}$$

Proposition 2.3.1. *The second fundamental form satisfies the evolution equation*

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2\mathrm{H} h_{il} g^{ls} h_{sj} + |\mathrm{A}|^2 h_{ij}\,. \tag{2.3.1}$$

It follows that

$$\frac{\partial}{\partial t}h_i^j = \Delta h_i^j + |\mathrm{A}|^2 h_i^j\,, \tag{2.3.2}$$

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,. \tag{2.3.3}$$

Proof. Keeping in mind the Gauss–Weingarten relations (1.1.1) and the previous evolution equations, we compute

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}\left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle\\
&= \left\langle \nu \,\middle|\, \frac{\partial^2(\mathrm{H}\nu)}{\partial x_i\partial x_j}\right\rangle - \left\langle \nabla\mathrm{H} \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_j}\right\rangle\\
&= \frac{\partial^2\mathrm{H}}{\partial x_i\partial x_j} - \mathrm{H}\left\langle \nu \,\middle|\, \frac{\partial}{\partial x_i}\left(h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}\right)\right\rangle\\
&\quad - \left\langle \frac{\partial\mathrm{H}}{\partial x_l}\cdot\frac{\partial\varphi}{\partial x_s}g^{ls} \,\middle|\, \Gamma_{ij}^k\frac{\partial\varphi}{\partial x_k} + h_{ij}\nu\right\rangle\\
&= \frac{\partial^2\mathrm{H}}{\partial x_i\partial x_j} - \mathrm{H}h_{jl}g^{ls}\left\langle \nu \,\middle|\, \frac{\partial^2\varphi}{\partial x_i\partial x_s}\right\rangle - \Gamma_{ij}^k\frac{\partial\mathrm{H}}{\partial x_k}\\
&= \nabla_i\nabla_j\mathrm{H} - \mathrm{H}h_{il}g^{ls}h_{sj}\,.
\end{aligned}$$

Then using Simons' identity (1.1.4) we conclude that

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2\mathrm{H}h_{il}g^{ls}h_{sj} + |\mathrm{A}|^2 h_{ij}\,.$$

The other equations follow from straightforward computations, as $\frac{\partial}{\partial t}g^{ij} = 2\mathrm{H}h^{ij}$. □

Remark 2.3.2. Since it will be useful in the sequel, we see in detail the evolution equations in the special one-dimensional case of the flow by curvature $\gamma : \mathbb{S}^1\times[0,T)$ of a closed curve in the plane.

We denote by θ the parameter on $\mathbb{S}^1$ and by $s = s(\theta,t) = \int_0^\theta |\partial_\theta\gamma(\theta,t)|\,d\theta$ the arclength, $\tau = \partial_s\gamma$ is the tangent unit vector and $\nu = \mathrm{R}\tau$ is the unit normal, where $\mathrm{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of an angle of $\pi/2$, finally $k = \langle\partial_s\tau\,|\,\nu\rangle$ is the curvature.

Notice that $\partial_s = |\gamma_\theta|^{-1}\partial_\theta$ and that the evolution equation reads $\partial_t\gamma = k\nu = \partial_{ss}^2\gamma$. Then, we easily get the commutation rule $\partial_t\partial_s = \partial_s\partial_t + k^2\partial_s$ which implies

$$\begin{aligned}
\partial_t\tau &= \partial_t\partial_s\gamma = \partial_s\partial_t\gamma + k^2\partial_s\gamma = \partial_s(k\nu) + k^2\tau = k_s\nu,\\
\partial_t\nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t\tau = -k_s\tau,\\
\partial_t k &= k_{ss} + k^3\,.
\end{aligned}$$

Now we deal with the covariant derivatives of A.

Lemma 2.3.3. *The following formula for the interchange of time and covariant derivative of a tensor T holds:*

$$\frac{\partial}{\partial t}\nabla T = \nabla\frac{\partial}{\partial t}T + T * \mathrm{A} * \nabla\mathrm{A}\,.$$

Proof. We suppose that $T = T_{i_1\dots i_k}$ is a covariant tensor, the general case is analogous, as it will be clear by the following computation:

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_j T_{i_1\dots i_k} &= \frac{\partial}{\partial t}\Big(\frac{\partial T_{i_1\dots i_k}}{\partial x_j} - \sum_{s=1}^{k}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\Big)\\ &= \frac{\partial}{\partial x_j}\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}\Gamma^l_{ji_s}\frac{\partial T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}}{\partial t}\\ &\quad - \sum_{s=1}^{k}\frac{\partial}{\partial t}\Gamma^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\\ &= \nabla_j\frac{\partial T_{i_1\dots i_k}}{\partial t} - \sum_{s=1}^{k}(\mathrm{A} * \nabla\mathrm{A})^l_{ji_s}T_{i_1\dots i_{s-1}li_{s+1}\dots i_k}\,,\end{aligned}$$

which is the formula we wanted. □

Lemma 2.3.4. *We have, for $k > 0$, denoting by ∇^k the kth iterated covariant derivative,*

$$\frac{\partial}{\partial t}\nabla^k h_{ij} = \Delta\nabla^k h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}.$$

Proof. We work by induction on $k \in \mathbb{N}$. The case $k = 0$ is given by equation (2.3.1); we then suppose that the formula holds for $k-1$. We have, by the previous lemma,

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k h_{ij} &= \nabla\frac{\partial}{\partial t}\nabla^{k-1}h_{ij} + \nabla^{k-1}\mathrm{A} * \nabla\mathrm{A} * \mathrm{A}\\ &= \nabla\Big(\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k-1\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}\Big)\\ &\quad + \nabla^{k-1}\mathrm{A} * \nabla\mathrm{A} * \mathrm{A}\\ &= \nabla\Delta\nabla^{k-1}h_{ij} + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}}\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A}.\end{aligned}$$

Interchanging now the Laplacian and the covariant derivative and recalling that $\mathrm{Riem} = \mathrm{A} * \mathrm{A}$, we have the conclusion, as all the extra terms we get are of the form $\mathrm{A} * \mathrm{A} * \nabla^k\mathrm{A}$ and $\mathrm{A} * \nabla\mathrm{A} * \nabla^{k-1}\mathrm{A}$. □

Proposition 2.3.5. *The following formula holds:*

$$\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 = \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}. \tag{2.3.4}$$

Proof. We compute

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 &= 2g\Big(\nabla^k \mathrm{A}, \frac{\partial}{\partial t}\nabla^k \mathrm{A}\Big) + \nabla^k \mathrm{A} * \nabla^k \mathrm{A} * \mathrm{A} * \mathrm{A} \\
&= 2g\Big(\nabla^k \mathrm{A}, \Delta\nabla^k \mathrm{A} + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A}\Big) \\
&\quad + \nabla^k \mathrm{A} * \nabla^k \mathrm{A} * \mathrm{A} * \mathrm{A} \\
&= 2g\left(\nabla^k \mathrm{A}, \Delta\nabla^k \mathrm{A}\right) + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A} \\
&= \Delta|\nabla^k \mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k \,|\, p,q,r\in\mathbb{N}} \nabla^p \mathrm{A} * \nabla^q \mathrm{A} * \nabla^r \mathrm{A} * \nabla^k \mathrm{A}\,.
\end{aligned}$$

□

2.4 Consequences of Evolution Equations

Let us see some consequences of application of the maximum principle to evolution equations for curvature.

Suppose that we have a mean curvature flow of a compact hypersurface M in the time interval $[0, T)$; we have seen that

$$\frac{\partial}{\partial t}|\mathrm{A}|^2 = \Delta|\mathrm{A}|^2 - 2|\nabla \mathrm{A}|^2 + 2|\mathrm{A}|^4 \leq \Delta|\mathrm{A}|^2 + 2|\mathrm{A}|^4$$

and

$$\frac{\partial}{\partial t}\mathrm{H} = \Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2\,.$$

First we deal with the so-called *mean convex* hypersurfaces that play a major role in the subject.

A hypersurface is mean convex if $\mathrm{H} \geq 0$ everywhere. We will see in the next proposition that this property is preserved by the mean curvature flow. Mean convexity is a significant generalization of convexity; for instance, it is general enough to allow the neckpinch behavior described in Section 1.4, in particular, mean convex hypersurfaces do not necessarily shrink to a point at the singular time.

Proposition 2.4.1. *Assume that the initial, compact hypersurface satisfies* $\mathrm{H} \geq 0$. *Then, under the mean curvature flow, the minimum of* H *is increasing, hence* H *is positive for every positive time.*

Proof. Arguing by contradiction, suppose that in an interval $(t_0, t_1) \subset \mathbb{R}^+$ we have $\mathrm{H}_{\min}(t) < 0$ and $\mathrm{H}_{\min}(t_0) = 0$ ($\mathrm{H}_{\min}$ is obviously continuous in time and $\mathrm{H}_{\min}(0) \geq 0$).

Let $|\mathrm{A}|^2 \leq C$ in such an interval. Then

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \qquad \text{implies} \qquad \frac{\partial \mathrm{H}_{\min}}{\partial t} \geq C\mathrm{H}_{\min}$$

for almost every $t \in (t_0, t_1)$.

Integrating this differential inequality in $[s, t] \subset (t_0, t_1)$ we get $\mathrm{H}_{\min}(t) \geq e^{C(t-s)}\mathrm{H}_{\min}(s)$, then sending $s \to t_0^+$ we conclude $\mathrm{H}_{\min}(t) \geq 0$ for every $t \in (t_0, t_1)$ which is a contradiction.

Since then $\mathrm{H} \geq 0$ we get

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta \mathrm{H} + \mathrm{H}|\mathrm{A}|^2 \geq \Delta \mathrm{H} + \mathrm{H}^3/n\,.$$

With the notation of Theorem 2.1.1, we let $u = -\mathrm{H}$, $X = 0$ and $b(x) = x^3/n$, then, if $\mathrm{H}_{\min}(0) = 0$ the ODE solution $h(t)$ is always zero; so if at some positive time $\mathrm{H}_{\min}(\tau) = 0$, we have that $\mathrm{H}(\,\cdot\,, \tau)$ is constant equal to zero on M, but there are no compact hypersurfaces with zero mean curvature. Hence, $\mathrm{H}_{\min}$ is always increasing during the flow and H is positive on all M at every positive time. ☐

Actually, this proposition can be slightly improved as follows.

Proposition 2.4.2. *If the initial, compact hypersurface satisfies* $|\mathrm{A}| \leq \alpha\mathrm{H}$ *for some constant* α, *then* $|\mathrm{A}| \leq \alpha\mathrm{H}$ *for every positive time.*

Proof. We know that $\mathrm{H} > 0$ for every positive time, hence also $|\mathrm{A}| > 0$ for every positive time which implies that it is smooth as $|\mathrm{A}|^2$.

Let $[0, T)$ be the interval of smooth existence of the flow. Computing the evolution equation of the function $f = |\mathrm{A}| - \alpha\mathrm{H}$, we get

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{1}{2|\mathrm{A}|}(\Delta|\mathrm{A}|^2 - 2|\nabla\mathrm{A}|^2 + 2|\mathrm{A}|^4) - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)\\
&= \Delta|\mathrm{A}| + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla\mathrm{A}|^2) + |\mathrm{A}|^3 - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)\\
&= \Delta f + |\mathrm{A}|^2 f + \frac{1}{2|\mathrm{A}|}(2|\nabla|\mathrm{A}||^2 - 2|\nabla\mathrm{A}|^2)\\
&\leq \Delta f + |\mathrm{A}|^2|f|\,,
\end{aligned}$$

as the term $|\nabla|\mathrm{A}||^2 - |\nabla\mathrm{A}|^2$ is nonpositive.

Hence, choosing any $T' < T$, if C is the maximum of $|\mathrm{A}|^2$ on $M \times [0, T']$, we have $\partial_t f \leq \Delta f + C|f|$ on $M \times [0, T']$. By the maximum principle Theorem 2.1.1, as $f_{\max}(0) \leq 0$, we conclude $f \leq 0$ on $M \times [0, T']$. By the arbitrariness of $T' < T$, the thesis follows. ☐

Corollary 2.4.3. *If $H > 0$ for the initial, compact, n-dimensional hypersurface, then there exists $\alpha_0 > 0$ such that $\alpha_0 |\mathrm{A}|^2 \leq \mathrm{H}^2 \leq n|\mathrm{A}|^2$ everywhere on M for every time.*

If the initial hypersurface has positive scalar curvature, then the same holds for every positive time.

Proof. The first claim is immediate by the compactness of M and the previous proposition (the second inequality is algebraic).

Recalling that the scalar curvature is equal to $\mathrm{H}^2 - |\mathrm{A}|^2$, positive scalar curvature implies that $\mathrm{H} > 0$ (H cannot change sign on M and there is always a point where it is positive, as M is compact) and $\mathrm{H}^2/|\mathrm{A}|^2 > 1$, the second part of this corollary is also a consequence of Proposition 2.4.2. ☐

Corollary 2.4.4. *Assume that the initial, compact hypersurface has* $\mathrm{H} \geq 0$*, then, if* A *is not bounded as $t \to T$ then* H *is also not bounded.*

Proof. Immediate consequence of Proposition 2.4.1 and the estimate of the previous corollary. ☐

Now we consider the evolution equation of $|\mathrm{A}|^2$ which implies

$$\frac{\partial}{\partial t} |\mathrm{A}|^2_{\max} \leq 2|\mathrm{A}|^4_{\max} .$$

Notice that $|\mathrm{A}|^2_{\max}$ is always positive, otherwise at some time t we would have $\mathrm{A} = 0$ identically on M, which would imply that M is a hyperplane in $\mathbb{R}^{n+1}$ in contradiction with the compactness hypothesis of M. Hence, we can divide both members by $|\mathrm{A}|^2_{\max}$ obtaining the following differential inequality for the locally Lipschitz function $1/|\mathrm{A}|^2_{\max}$, holding at almost every time $t \in [0, T)$,

$$-\frac{d}{dt} \frac{1}{|\mathrm{A}|^2_{\max}} \leq 2 .$$

Integrating in time in any interval $[t, s] \subset [0, T)$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} - \frac{1}{|\mathrm{A}(\,\cdot\,, s)|^2_{\max}} \leq 2(s - t) .$$

Suppose now that A is not bounded in $[0, T)$, that is, there exists a sequence of times $s_i \nearrow T$ such that $|\mathrm{A}(\,\cdot\,, s_i)|^2_{\max} \to +\infty$. Substituting these times s_i in the previous inequality and sending $i \to \infty$, we get

$$\frac{1}{|\mathrm{A}(\,\cdot\,, t)|^2_{\max}} \leq 2(T - t) .$$

Exercise 2.4.5. Show that the only compact hypersurfaces in $\mathbb{R}^{n+1}$ with constant mean curvature are the spheres. What can be said about a compact hypersurface in $\mathbb{R}^{n+1}$ with constant $|\mathrm{A}|$?

In other words, we have proved the following.

Proposition 2.4.6. *If the second fundamental form* A *during the mean curvature flow of a compact hypersurface is not bounded as* $t \to T < +\infty$*, then it must satisfy the following lower bound for its blow-up rate:*

$$\max_{p\in M} |\mathrm{A}(p,t)| \geq \frac{1}{\sqrt{2(T-t)}}$$

for every $t \in [0,T)$*.*
Hence,

$$\lim_{t\to T} \max_{p\in M} |\mathrm{A}(p,t)| = +\infty\,.$$

Exercise 2.4.7. Assume that the initial, compact hypersurface has $\mathrm{H} > 0$, then the maximal time of smooth existence of the flow can be estimated as $T_{\max} \leq \frac{n}{2\mathrm{H}^2_{\min}(0)}$.

Proposition 2.4.8. *If the second fundamental form is bounded in the interval* $[0,T)$ *with* $T < +\infty$*, then all its covariant derivatives are also bounded.*

Proof. By Proposition 2.3.5 we have

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k \mathrm{A}|^2 &= \Delta|\nabla^k\mathrm{A}|^2 - 2|\nabla^{k+1}\mathrm{A}|^2 + \sum_{p+q+r=k\,|\,p,q,r\in\mathbb{N}} \nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}\\ &\leq \Delta|\nabla^k\mathrm{A}|^2 + P(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)|\nabla^k\mathrm{A}|^2 + Q(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)\,,\end{aligned}$$

where P and Q are smooth functions independent of time (actually they are polynomials in their arguments). Notice that in the arguments of P, Q there is not $\nabla^k\mathrm{A}$; indeed, in the terms $\nabla^p\mathrm{A} * \nabla^q\mathrm{A} * \nabla^r\mathrm{A} * \nabla^k\mathrm{A}$ there can be only one or two occurrences of $\nabla^k\mathrm{A}$, since $p+q+r=k$ and $p,q,r \in \mathbb{N}$. If there are two, suppose that $r = k$, then necessarily $p = q = 0$ and we estimate $|\mathrm{A}*\mathrm{A}*\nabla^k\mathrm{A}*\nabla^k\mathrm{A}| \leq |\mathrm{A}|^2|\nabla^k\mathrm{A}|^2$; if there is only one this means that $p,q,r<k$ and we again estimate $|\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}*\nabla^k\mathrm{A}| \leq |\nabla^p\mathrm{A}*\nabla^q\mathrm{A}*\nabla^r\mathrm{A}|^2/2 + |\nabla^k\mathrm{A}|^2/2$.

Reasoning by induction on k, being the case $k = 0$ in the hypotheses, we assume that all the covariant derivatives of A up to order $k-1$ are bounded, hence also $P(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)$ and $Q(|\mathrm{A}|,\dots,|\nabla^{k-1}\mathrm{A}|)$ are bounded, thus

$$\frac{\partial}{\partial t}|\nabla^k\mathrm{A}|^2 \leq \Delta|\nabla^k\mathrm{A}|^2 + C|\nabla^k\mathrm{A}|^2 + D\,.$$

By the maximum principle, this implies

$$\frac{d}{dt}|\nabla^k\mathrm{A}|^2_{\max} \leq C|\nabla^k\mathrm{A}|^2_{\max} + D\,,$$

and since the interval $[0,T)$ is bounded, the quantity $|\nabla^k \mathrm{A}|^2_{\max}$ is also bounded, as one can obtain an easy exponential estimate for the function $u(t) = |\nabla^k \mathrm{A}|^2_{\max}$, integrating the ordinary differential inequality $u' \leq Cu + D$, holding for almost every time $t \in [0,T)$. □

Proposition 2.4.9. *If the second fundamental form is bounded in the interval* $[0,T)$ *with* $T < +\infty$*, then* T *cannot be a singular time for the mean curvature flow of a compact hypersurface* $\varphi : M \times [0,T) \to \mathbb{R}^{n+1}$.

Proof. By the previous proposition we know that all the covariant derivatives of A are bounded by constants depending on T and the geometry of the initial hypersurface. As H is bounded, we have

$$|\varphi(p,t) - \varphi(p,s)| \leq \int_s^t |\mathrm{H}(p,\xi)|\, d\xi \leq C(t-s)$$

for every $0 \leq s \leq t < T$, then the maps $\varphi_t = \varphi(\,\cdot\,,t)$ uniformly converge to a continuous limit map $\varphi_T : M \to \mathbb{R}^{n+1}$ as $t \to T$.
We fix now a vector $v = \{v^i\} \in T_pM$,

$$\frac{d}{dt}\log|v|^2_g = \frac{\frac{\partial g_{ij}}{\partial t} v^i v^j}{|v|^2_g} = \frac{-2\mathrm{H}h_{ij}v^iv^j}{|v|^2_g} \leq C\frac{|\mathrm{A}|^2|v|^2_g}{|v|^2_g} \leq C;$$

then, for every $0 \leq s \leq t < T$,

$$\left|\log\frac{|v|^2_{g(t)}}{|v|^2_{g(s)}}\right| \leq \int_s^t \left|\frac{d}{d\xi}\log|v|^2_{g(\xi)}\right| d\xi \leq C(t-s)$$

which implies that the metrics $g(t)$ are all equivalent and the norms $|\cdot|_{g(t)}$ uniformly converge, as $t \to T$, to an equivalent norm $|\cdot|_T$ which is continuous. As the parallelogram identity passes to the limit, it must hold also for $|\cdot|_T$, hence this latter comes from a metric tensor g_T which can be obtained by polarization. Moreover, since g_T is equivalent to all the other metrics, it is also positive definite. Another consequence of such equivalence is that we are free to use any of these metrics in doing our estimates.

By the evolution equation for the Christoffel symbols, we see that

$$\left|\Gamma^k_{ij}(t)\right| \leq \left|\Gamma^k_{ij}(0)\right| + \int_0^t \left|\frac{\partial}{\partial\xi}\Gamma^k_{ij}(\xi)\right| d\xi \leq C + \int_0^T |\mathrm{A} * \nabla\mathrm{A}|\, d\xi \leq C + DT\,,$$

for some constants depending only on the initial hypersurface. Thus, the Christoffel symbols are equibounded in time, after fixing a local chart. This implies for every tensor S,

$$\left|\,\left|\frac{\partial S}{\partial x_i}\right| - |\nabla_i S|\,\right| \leq C|S|\,,$$

that is, the derivatives in coordinates differ by the relative covariant ones by equibounded terms.

In the rest of the proof, by simplicity, we will denote by ∂ the coordinate derivatives and by ∇ the covariant ones.

As the time derivative of the Christoffel symbols is a tensor of the form $\mathrm{A} * \nabla\mathrm{A}$, we have

$$|\partial_t\partial^s_{l_1\dots l_s}\Gamma^k_{ij}| = |\partial^s_{l_1\dots l_s}\partial_t\Gamma^k_{ij}| = |\partial^s_{l_1\dots l_s}\mathrm{A} * \nabla\mathrm{A}|\,,$$

hence, by an induction argument on the order s and integration as above, one can show that $\left|\partial^s_{l_1\dots l_s}\Gamma^k_{ij}\right| \leq C$ for every $s \in \mathbb{N}$.
Then, again by induction, the following formula (where we avoid indicating the indices) relating the iterated covariant and coordinate derivatives of a tensor S, holds:

$$|\,|\nabla^s S| - |\partial^s S|\,| \leq \sum_{i=1}^{s}\sum_{j_1+\dots+j_i+k\leq s-1}\left|\partial^{j_1}\Gamma\dots\partial^{j_i}\Gamma\partial^k S\right| \leq C\sum_{k=1}^{s-1}\left|\partial^k S\right|\,.$$

This implies that if a tensor has all its covariant derivatives bounded, also all the coordinate derivatives are bounded. In particular this holds for the tensor A, that is, $\left|\partial^k\mathrm{A}\right| \leq C_k$. Moreover, by induction, as $\nabla^k g = 0$ all the coordinate derivatives of the metric tensor g are equibounded.

We already know that $|\varphi|$ is bounded and $|\partial\varphi| = 1$, then by the Gauss–Weingarten relations (1.1.1),

$$\partial^2\varphi = \Gamma\partial\varphi + \mathrm{A}\nu\,, \qquad \partial\nu = \mathrm{A} * \partial\varphi\,,$$

we get

$$\begin{aligned}
|\partial^k\varphi| &= \left|\sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\Gamma\partial^{i+1}\varphi + \sum_{i=0}^{k-2}\binom{k-2}{i}\partial^{k-2-i}\mathrm{A}\partial^i\nu\right| \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}|\partial^{i-1}(\mathrm{A} * \partial\varphi)| + C \\
&= C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\Bigg|\sum_{p+q+r=i-1}\partial^p\mathrm{A} * \partial^q g * \partial^{r+1}\varphi\Bigg| + C \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}\sum_{r=0}^{i-1}|\partial^{r+1}\varphi| + C \\
&\leq C\sum_{i=0}^{k-2}|\partial^{i+1}\varphi| + C\sum_{i=1}^{k-2}|\partial^i\varphi| + C \\
&\leq C\sum_{i=0}^{k-1}|\partial^i\varphi|
\end{aligned}$$

where we estimated with a constant all the occurrences of $\partial^k \mathrm{A}$ and $\partial^k g$. Hence, we obtain by induction that $|\partial^k \varphi| < C_k$ for constants C_k independent of time $t \in [0, T)$. By the Ascoli–Arzelà theorem we can conclude that $\varphi_T : M \to \mathbb{R}^{n+1}$ is a smooth immersion and the convergence $\varphi(\,\cdot\,, t) \to \varphi_T$ is in C^∞.

Moreover, with the same argument, repeatedly differentiating the evolution equation $\partial_t \varphi = \mathrm{H}\nu$ one gets also uniform boundedness of the time derivatives of the map φ, that is $|\partial_t^s \partial_x^k \varphi| \leq C_{s,k}$. Hence the map $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ can be extended smoothly to the boundary of the domain of φ with the map φ_T.

By means of the short time existence Theorem 1.5.1 we can now “restart” the flow with the immersion φ_T, obtaining a smooth extension of the map φ which is in contradiction with the fact that T was the maximal time of smooth existence. ☐

Open Problem 2.4.10. Recently the condition of bounded second fundamental form was weakened by Le and Sesum [84] to a lower bound on A and an integral bound on H.

An interesting open problem is whether actually a uniform bound only on the mean curvature H is sufficient to exclude singularities during the flow (see [85]).

Thus, we conclude this section stating the following slightly improved version of Theorem 1.5.1.

Theorem 2.4.11. *For any initial, compact, smooth hypersurface immersed in* $\mathbb{R}^{n+1}$ *there exists a unique mean curvature flow which is smooth in a maximal time interval* $[0, T_{\max})$.

Moreover, $T_{\max}$ *is finite and*

$$\max_{p \in M} |\mathrm{A}(p, t)| \geq \frac{1}{\sqrt{2(T_{\max} - t)}}$$

for every $t \in [0, T_{\max})$.

Notice that it follows that the maximal time of smooth existence of the flow can be estimated from below as $T_{\max} \geq \frac{1}{2|\mathrm{A}(\,\cdot\,,0)|^2_{\max}}$.

2.5 Convexity Invariance

Corollary 2.4.3 is a consequence of a more general invariance property of the elementary symmetric polynomials of the curvatures, as we are going to show.

We recall that the *elementary symmetric polynomial* of degree k of $\lambda_1, \ldots, \lambda_n$ is defined as

$$\mathrm{S}_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

for $k = 1, \ldots, n$. In particular, if λ_i are the eigenvalues of the second fundamental form A we have $\mathrm{S}_1 = \mathrm{H}$, S_2 is the scalar curvature and $|\mathrm{A}|^2 = \mathrm{S}_1^2 + 2\mathrm{S}_2$.

It is not difficult to show that

$$\begin{aligned} \lambda_1 \geq 0, \dots, \lambda_n \geq 0 \quad &\iff \quad \mathrm{S}_1 \geq 0, \dots, \mathrm{S}_n \geq 0\,, \\ \lambda_1 > 0, \dots, \lambda_n > 0 \quad &\iff \quad \mathrm{S}_1 > 0, \dots, \mathrm{S}_n > 0\,. \end{aligned} \tag{2.5.1}$$

These polynomials enjoy various concavity properties, see [73, 91].

Proposition 2.5.1. *Let* $\Gamma_k \subset \mathbb{R}^n$ *denote the connected component of* $\{\mathrm{S}_k > 0\}$ *containing the positive cone. Then* $\mathrm{S}_l > 0$ *in* Γ_k *for all* $l = 1, \dots, k$ *and the quotient* $\mathrm{S}_{k+1}/\mathrm{S}_k$ *is concave on* Γ_k*.*

The above properties remain unchanged if we regard the polynomials S_k as functions of the Weingarten operator h^i_j instead of the principal curvatures, as we have the following algebraic result, see [9, Lemma 2.22] or [73, Lemma 2.11].

Proposition 2.5.2. *Let* $f(\lambda_1, \dots, \lambda_n)$ *be a symmetric convex* (*concave*) *function of its arguments and let* $F(A) = f$ (*eigenvalues of* A) *for any* $n \times n$ *symmetric matrix* A *whose eigenvalues belong to the domain of* f*. Then* F *is convex* (*concave*)*.*

We are now ready to derive the evolution equations of some relevant quantities and to apply the maximum principle to obtain some invariance properties.

Proposition 2.5.3. *Let* $F(h^i_j)$ *be a homogeneous function of degree one. Let* φ *be a mean curvature flow of a compact, n-dimensional hypersurface with* $\mathrm{H} > 0$ *and such that* h^i_j *belongs everywhere to the domain of* F*. Then,*

$$\frac{\partial}{\partial t}\frac{F}{\mathrm{H}} - \Delta\frac{F}{\mathrm{H}} = \frac{2}{\mathrm{H}}\Big\langle \nabla\mathrm{H} \,\Big|\, \nabla\frac{F}{\mathrm{H}}\Big\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\, g^{pq}\nabla_p h^i_j \nabla_q h^k_l\,.$$

As a consequence, if F *is concave* (*convex*)*, any pinching of the form* $F \geq \alpha\mathrm{H}$ ($F \leq \alpha\mathrm{H}$) *is preserved during the flow by the maximum principle, as the last term is then nonnegative* (*nonpositive*)*.*

Proof. A straightforward computation using formula (2.3.2) in Proposition 2.3.1 and Euler's theorem on homogeneous functions yields

$$\begin{aligned} \frac{\partial}{\partial t}\frac{F}{\mathrm{H}} &= \frac{1}{H}\frac{\partial F}{\partial h^i_j}(\Delta h^i_j + |\mathrm{A}|^2 h^i_j) - \frac{F}{\mathrm{H}^2}(\Delta\mathrm{H} + |\mathrm{A}|^2\mathrm{H}) \\ &= \Delta\frac{F}{\mathrm{H}} + \frac{2}{\mathrm{H}}\Big\langle \nabla\mathrm{H} \,\Big|\, \nabla\frac{F}{\mathrm{H}}\Big\rangle - \frac{1}{\mathrm{H}}\frac{\partial^2 F}{\partial h^i_j \partial h^k_l}\, g^{pq}\nabla_p h^i_j \nabla_q h^k_l\,. \qquad \square \end{aligned}$$

In particular, the previous proposition can be applied to the quantity $F = \mathrm{S}_{k+1}/\mathrm{S}_k$, provided $\mathrm{S}_k \neq 0$. This leads to the following result, which generalizes Corollary 2.4.3.

Proposition 2.5.4. *Let the initial, compact hypersurface satisfy* $\mathrm{S}_k > 0$ *everywhere for a given* $k \in \{1, \dots, n\}$ *and let* $\varphi : M \times [0, T) \to \mathbb{R}^{n+1}$ *be its evolution by mean curvature. Then, for any* $i = 2, \dots, k$ *there exists* α_i *such that* $\mathrm{S}_i \geq \alpha_i \mathrm{H}^i > 0$ *for every* $p \in M$ *and* $t \in [0, T)$*.*

Proof. We assume that the hypersurface M is connected, otherwise we argue component by component.

For every pair of of points p and q in M, the set of principal curvatures at p and the set of principal curvatures at q belong to the same connected component of $\{\mathrm{S}_k > 0\} \subset \mathbb{R}^n$, seeing S_k as a map from $\mathbb{R}^n$ to $\mathbb{R}$ (connect with an arc the two points). Then, as the initial hypersurface is compact, there exists a point $p \in M$ where all the principal curvatures are positive (consider a tangent sphere containing the hypersurface), hence, the set of principal curvatures at all the points of M belongs to the connected component Γ_k of the positive cone defined in Proposition 2.5.1. Hence, for every $i = 1, \dots, k$ we have $\mathrm{S}_i > 0$ everywhere on the initial hypersurface. In particular $\mathrm{H} = \mathrm{S}_1 > 0$ and, by compactness, we have $\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1}$ for suitable constants $\beta_i > 0$, for any $i = 2, \dots, k$.

We know from Proposition 2.4.2 that $\mathrm{H} > 0$ everywhere on M for every $t \in [0, T)$. Then we can consider the quotient $\mathrm{S}_2/\mathrm{H}^2 = \mathrm{S}_2/(\mathrm{H}\mathrm{S}_1)$ which is well defined for every t and it is greater than β_2 at time $t = 0$. By Proposition 2.5.3 its minimum is nondecreasing, hence $\mathrm{S}_2 \geq \beta_2 \mathrm{H}^2$ for every $t \in [0, T)$.

We now apply the same procedure to the quotient $\mathrm{S}_3/(\mathrm{H}\mathrm{S}_2)$ to conclude that it is greater than β_3 for every $t \in [0, T)$, then in general $\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1}$ for $i = 2, \dots, k$. Multiplying together all these inequalities we get

$$\mathrm{S}_i \geq \beta_i \mathrm{H}\mathrm{S}_{i-1} \geq \beta_i \beta_{i-1} \mathrm{H}^2 \mathrm{S}_{i-2} \geq \dots \geq \beta_i \beta_{i-1} \cdots \beta_2 \mathrm{H}^i$$

and the claim follows by setting $\alpha_i = \beta_i \beta_{i-1} \cdots \beta_2$. □

Corollary 2.5.5. *If the initial, compact hypersurface is strictly convex, it remains strictly convex under the mean curvature flow.*

Proof. Strict convexity is equivalent to the set of conditions $\mathrm{S}_1, \dots, \mathrm{S}_n > 0$ on the eigenvalues of the second fundamental form, by relations (2.5.1) and these conditions are preserved under the mean curvature flow, by the previous proposition. □

Remark 2.5.6. By Hamilton's strong maximum principle for tensors in [56, Section 8] (Theorem C.1.3 in Appendix C), if an initial, compact hypersurface is only convex (not necessarily strictly convex), then it becomes immediately strictly convex. Even more precisely, in this case, the smallest eigenvalue of the second fundamental form on all M increases in time.

Indeed, the Weingarten operator is nonnegative definite for every positive time and satisfies (see Proposition 2.3.1)

$$\frac{\partial}{\partial t} h_i^j = \Delta h_i^j + |\mathrm{A}|^2 h_i^j \,,$$

then by Theorem C.1.3 its rank (hence the rank of A) is constant in some time interval $(0, \delta)$; moreover, the null space is invariant by parallel transport and time. Then, supposing that such rank m is less than the dimension n of the hypersurface,

we have an $(n-m)$-dimensional subspace $N_p \subset T_pM$ at every point $p \in M$, invariant by parallel transport, where $\mathrm{A}_p(v,v)=0$ for every $v \in N_p$.
If $v \in T_pM$ is a vector in the null space, any geodesic γ in M starting at p is also a geodesic in $\mathbb{R}^{n+1}$ as $\dot\gamma$ remains always in the null space of A and

$$\nabla^{\mathbb{R}^{n+1}}_{\dot\gamma}\dot\gamma = \nabla^{M}_{\dot\gamma}\dot\gamma + \mathrm{A}(\dot\gamma,\dot\gamma)\nu = 0\,.$$

Hence, all the $(n-m)$-dimensional null space (as an affine subspace of $\mathbb{R}^{n+1}$) is contained in M, which is in contradiction with the compactness of M.

Remark 2.5.7. If the initial hypersurface is not convex, it is not true that the smallest eigenvalue of A increases; think of Angenent's homothetically shrinking torus we mentioned in Section 1.4 (see [17]).

Notice that the results about the strict monotonicity of geometric quantities during the flow are valid when the initial hypersurface is compact and can fail otherwise. For instance, an evolving cylinder does not become immediately strictly convex.

Proposition 2.5.8. *If for a constant $\alpha \in \mathbb{R}$ there holds $\mathrm{A} \geq \alpha \mathrm{H} g$ (as forms) for the initial, compact hypersurface, this condition is preserved during the mean curvature flow.*

Proof. We consider the function $f = h_{ij}v^iv^j - \alpha\mathrm{H}g_{ij}v^iv^j$ where $v^i(p,t)$ is a time-dependent smooth vector field such that $\partial v^i/\partial t = \mathrm{H}h^i_kv^k$,

$$\begin{aligned}
\frac{\partial f}{\partial t} =& \frac{\partial h_{ij}}{\partial t}v^iv^j + 2h_{ij}v^i\frac{\partial v^j}{\partial t} - \alpha\frac{\partial \mathrm{H}}{\partial t}g_{ij}v^iv^j + 2\alpha\mathrm{H}^2h_{ij}v^iv^j - 2\alpha\mathrm{H}g_{ij}v^i\frac{\partial v^j}{\partial t}\\
=&(\Delta h_{ij} - 2\mathrm{H}h^2_{ij} + |\mathrm{A}|^2h_{ij})v^iv^j + 2\mathrm{H}h^2_{ij}v^iv^j - \alpha(\Delta\mathrm{H} + \mathrm{H}|\mathrm{A}|^2)g_{ij}v^iv^j\\
=&(\Delta h_{ij} + |\mathrm{A}|^2h_{ij})v^iv^j - \alpha\Delta\mathrm{H}g_{ij}v^iv^j - \alpha\mathrm{H}|\mathrm{A}|^2g_{ij}v^iv^j\\
=&\Delta(h_{ij}v^iv^j - \alpha\mathrm{H}g_{ij}v^iv^j) + |\mathrm{A}|^2(h_{ij} - \alpha\mathrm{H}g_{ij})v^iv^j\\
&- 4(\nabla_kh_{ij} - \alpha\nabla_k\mathrm{H}g_{ij})v^i\nabla^kv^j\\
&- 2(h_{ij} - \alpha\mathrm{H}g_{ij})\nabla_kv^i\nabla^kv^j - 2(h_{ij} - \alpha\mathrm{H}g_{ij})v^i\Delta v^j\\
=&\Delta f + |\mathrm{A}|^2f - 4(\nabla_kh_{ij} - \alpha\nabla_k\mathrm{H}g_{ij})v^i\nabla^kv^j\\
&- 2(h_{ij} - \alpha\mathrm{H}g_{ij})\nabla_kv^i\nabla^kv^j - 2(h_{ij} - \alpha\mathrm{H}g_{ij})v^i\Delta v^j\,.
\end{aligned}$$

Let $\mu(t)$ be the smallest value of $h_{ij}(q,t)v^iv^j - \alpha\mathrm{H}g_{ij}(q,t)v^iv^j$ for t fixed, $q \in M$ and $v \in T_qM$ a unit tangent vector of (M,g_t).
Since μ is a locally Lipschitz function, it is differentiable at almost every time; moreover by the hypotheses, we have $\mu(0) \geq 0$.
We suppose that there exists an open interval of time (t_0,t_1) where μ is negative and $\mu(t_0)=0$. Let $t \in (t_0,t_1)$ be a differentiability point of μ, then there exists a point $p \in M$ and a unit vector $v \in T_pM$ such that

$$\mu(t) = h_{ij}(p,t)v^iv^j - \alpha\mathrm{H}(p,t)g_{ij}(p,t)v^iv^j \leq h_{ij}(q,t)w^iw^j - \alpha\mathrm{H}(q,t)g_{ij}(q,t)w^iw^j$$

for every $q \in M$ and $w \in T_qM$ of unit norm. We extend the vector v in space to a vector field that we still call v with the following properties:

- $g_t(v(q), v(q)) \leq 1$ for every $q \in M$,
- $\nabla_{g_t} v(p) = 0$,
- $\Delta_{g_t} v(p) = 0$,

this can be done, for instance, working in local coordinates.
Now we extend v also locally in time by solving the ODE $\partial v^i/\partial t = \mathrm{H}h^i_k v^k$ and we consider the associated function f as above.
Notice that since $\mu(t)$ is negative in (t_0, t_1), the function $f(\,\cdot\,, t)$ gets a minimum in space at $p \in M$. Indeed, if $f(q,t) < 0$, we have $v(q) \neq 0$ and

$$\begin{aligned} f(p,t) = \mu(t) &\leq \frac{h_{ij}(q,t)v^i(q)v^j(q) - \alpha\mathrm{H}(q,t)g_t(v(q),v(q))}{g_t(v(q),v(q))} \\ &= \frac{f(q,t)}{g_t(v(q),v(q))} \\ &\leq f(q,t) \end{aligned}$$

as $g_t(v(q), v(q)) \leq 1$ by construction. Hence, $\Delta f(p,t) \geq 0$ and at the point (p,t) we have

$$\frac{\partial f}{\partial t} = \Delta f + |\mathrm{A}|^2 f \geq Cf$$

where $C > 0$ is a constant such that $|\mathrm{A}|^2 \leq C$ on $[0, t_1)$.
By this inequality, given $\varepsilon > 0$, there exists some $t_2 \in [t_0, t]$, such that if $\overline{t} \in [t_2, t]$ we have

$$f(p, \overline{t}) < f(p,t) - C(t - \overline{t})f(p,t) + \varepsilon(t - \overline{t})\,.$$

Since $v(p, \overline{t})$ is still a unit vector, as $\partial g(v,v)/\partial t = -2\mathrm{H}h_{ij}v^iv^j + 2g(\partial v/\partial t, v) = 0$ so the norm of v is constant, we get

$$\mu(\overline{t}) \leq f(p,\overline{t}) < f(p,t) - C(t-\overline{t})f(p,t) + \varepsilon(t-\overline{t}) = \mu(t) - C(t-\overline{t})\mu(t) + \varepsilon(t-\overline{t})\,.$$

In other words $\frac{\mu(t)-\mu(\overline{t})}{t-\overline{t}} \geq C\mu(t) - \varepsilon$ and since t is a differentiability time for μ, passing to the limit as $\overline{t} \nearrow t$, we obtain $\mu'(t) \geq C\mu(t) - \varepsilon$.
Finally, as ε is arbitrarily small, we conclude that $\mu'(t) \geq C\mu(t)$.
Since this relation holds at every differentiability time in (t_0, t_1) where $\mu < 0$, hence almost everywhere, we can integrate it in the interval (t_0, t_1). Recalling that $\mu(t_0) = 0$ by continuity, we conclude that $\mu(t)$ must be identically zero in $[t_0, t_1)$ which is in contradiction with the hypotheses.
Notice the similarities with the proofs of Lemma 2.1.3 and Proposition 2.4.1. □

Exercise 2.5.9. Show that for an initial hypersurface with $\mathrm{H} > 0$ the smallest eigenvalue of the form h_{ij}/H is nondecreasing during the flow.

Finally, further invariance properties for the mean curvature flow can be obtained again by means of Hamilton's maximum principle for tensors [56, Sections 4 and 8] (whose proof is a generalization of the argument above), see Appendix C. Let us first recall a definition; we say that an immersed hypersurface is k-convex, for some $1 \leq k \leq n$, if the sum of the k smallest principal curvatures is nonnegative at every point. In particular, one-convexity coincides with convexity, while n-convexity means nonnegativity of the mean curvature H, that is, mean convexity. Then we mention the following result generalizing Corollary 2.5.5 (see [75]).

Proposition 2.5.10. *If an initial, compact hypersurface is k-convex, then it is so for every positive time under the mean curvature flow.*

Proof. The result follows from Hamilton's maximum principle for tensors, provided we show that the inequality $\lambda_1 + \cdots + \lambda_k \geq \alpha\mathrm{H}$ describes a convex cone in the set of all matrices, and that this cone is invariant under the system of ODE's $dh^i_j/dt = |\mathrm{A}|^2 h^i_j$ for the Weingarten operator.
As

$$(\lambda_1 + \cdots + \lambda_k)(p) = \min_{\substack{e_1,\dots,e_k \in T_pM \\ g_p(e_i,e_j)=\delta_{ij}}} \left\{\mathrm{A}_p(e_1,e_1) + \cdots + \mathrm{A}_p(e_k,e_k)\right\},$$

the quantity $\lambda_1 + \cdots + \lambda_k$ is a concave function of the Weingarten operator, being the infimum of a family of linear maps. Therefore the inequality $\lambda_1 + \cdots + \lambda_k \geq \alpha\mathrm{H}$ describes a convex cone of matrices. In addition, the system $dh^i_j/dt = |\mathrm{A}|^2 h^i_j$ changes the Weingarten operator by homotheties, thus leaves any cone invariant. The conclusion follows. □

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