Chapter 2
Symplectic capacities

2.1 Definition and application to embeddings

In the following we introduce a special class of symplectic invariants discovered by I. Ekeland and H. Hofer in [68, 69] for subsets of $\mathbb{R}^{2n}$. They were led to these invariants in their search for periodic solutions on convex energy surfaces and called them symplectic capacities. The concept of a symplectic capacity was extended to general symplectic manifolds by H. Hofer and E. Zehnder in [123]. The existence proof of these invariants is based on a variational principle; it is not intuitive, and will be postponed to the next chapter. Taking their existence for granted, the aim of this chapter is rather to deduce the rigidity of some symplectic embeddings and, in addition, the rigidity of the symplectic nature of mappings under limits in the supremum norm, which will give rise to the notion of a "symplectic homeomorphism".

Definition of symplectic capacity. We consider the class of all symplectic manifolds (M,ω) possibly with boundary and of fixed dimension $2n$. A symplectic capacity is a map $(M,\omega)\mapsto c(M,\omega)$ which associates with every symplectic manifold (M,ω) a nonnegative number or ∞, satisfying the following properties A1–A3.

A1. **Monotonicity:** $\quad c(M,\omega)\leq c(N,\tau)$

if there exists a symplectic embedding $\varphi:(M,\omega)\to(N,\tau)$.

A2. **Conformality:** $\quad c(M,\alpha\omega)=|\alpha|c(M,\omega)$

for all $\alpha\in\mathbb{R},\alpha\neq 0$.

A3. **Nontriviality:** $\quad c(B(1),\omega_0)=\pi=c(Z(1),\omega_0)$

for the open unit ball $B(1)$ and the open symplectic cylinder $Z(1)$ in the standard space $(\mathbb{R}^{2n},\omega_0)$. For convenience, we recall that with the symplectic coordinates $(x,y)\in\mathbb{R}^{2n}$,

$$B(r)=\Big\{(x,y)\in\mathbb{R}^{2n}\ \Big|\ |x|^2+|y|^2<r^2\Big\}$$

and

$$Z(r)=\big\{(x,y)\in\mathbb{R}^{2n}\ \big|\ x_1^2+y_1^2<r^2\big\}$$

for $r>0$. It is often convenient to replace (A3) by the following weaker axiom (A3′):

A3′. **Weak nontriviality:** $\quad 0<c\big(B(1),\omega_0\big)$ and $c\big(Z(1),\omega_0\big)<\infty$.

It should be pointed out that the axioms (A1)–(A3) do not determine a unique capacity function. There are indeed many ways to construct capacity functions as

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we shall see later on. We first illustrate the concept and deduce some simple consequences of the axioms (A1)–(A3). In the special case of 2-dimensional symplectic manifolds, $n = 1$, the modulus of the total area

$$c(M, \omega) := \left| \int_M \omega \right|$$

is an example of a symplectic capacity function. It agrees with the Lebesgue measure in $(\mathbb{R}^2, \omega_0)$. In contrast, if $n > 1$, then the symplectic invariant $(\text{ vol })^{\frac{1}{n}}$ is excluded by axiom (A3), since the cylinder has infinite volume. If $\varphi : (M, \omega) \to (N, \sigma)$ is a symplectic diffeomorphism between the two manifolds M and N, one applies the monotonicity axiom to φ and φ^{-1} and concludes

$$c(M, \omega) = c(N, \sigma).$$

Therefore, the capacity is indeed a symplectic invariant. Observe also that, by means of the inclusion mapping, we have for open subsets of (M, ω) the monotonicity property

$$U \subset V \implies c(U) \leq c(V).$$

In order to describe some simple examples in $(\mathbb{R}^{2n}, \omega_0)$ we start with

Lemma 1. For $U \subset (\mathbb{R}^{2n}, \omega_0)$ open and $\lambda \neq 0$,

$$c(\lambda U) = \lambda^2 c(U).$$

Proof. This is a consequence of the conformality axiom. The diffeomorphism

$$\varphi : \lambda U \to U, \quad x \mapsto \frac{1}{\lambda} x$$

satisfies $\varphi^*(\lambda^2 \omega_0) = \lambda^2 \varphi^* \omega_0 = \omega_0$. Therefore, $\varphi : (\lambda U, \omega_0) \to (U, \lambda^2 \omega_0)$ is symplectic, so that $c(\lambda U, \omega_0) = c(U, \lambda^2 \omega_0) = \lambda^2 c(U, \omega_0)$ as claimed. ∎

For the open ball of radius $r > 0$ in $\mathbb{R}^{2n}$ we find, in particular

$$c(B(r)) = r^2 c\Big(B(1)\Big) = \pi r^2. \tag{2.1}$$

Since $B(r) \subset \overline{B(r)} \subset B(r + \varepsilon)$ for every $\varepsilon > 0$ we conclude by monotonicity that $c(\overline{B(r)}) = \pi r^2$. We see that in the special case of $(\mathbb{R}^2, \omega_0)$

$$c\Big(B(r)\Big) = c\Big(\overline{B(r)}\Big) = \text{area } \Big(B(r)\Big)$$

agrees with the Lebesgue measure of the disc. This can be used to show that the capacity agrees with the Lebesgue measure for a large class of sets in $\mathbb{R}^2$, as has been observed by K.F. Siburg [194].

Proposition 1. If $D \subset \mathbb{R}^2$ is a compact and connected domain with smooth boundary, then

$$c(D, \omega_0) = \text{ area } (D).$$

Proof. By removing finitely many compact curves from D, we find a simply connected domain $D_0 \subset D$ satisfying $m(D_0) = m(D)$, which in view of the uniformization theorem is diffeomorphic to the unit disc $B(1) \subset \mathbb{R}^2$. Therefore, there exists a $\rho > 0$ and a diffeomorphism $\varphi : B(\rho) \to D_0$ satisfying, in addition, $m(B(\rho)) = m(D_0)$. Given $\varepsilon > 0$ we find $r < \rho$ such that $D_1 := \varphi(\overline{B(r)}) \subset D_0$ satisfies $m(D_1) \geq m(D) - \varepsilon$. By the theorem of Dacorogna-Moser there is, therefore, a measure preserving diffeomorphism $\psi : \overline{B(r)} \to D_1$. Since this ψ is symplectic we can estimate using the monotonicity, the invariance under symplectic diffeomorphisms and the normalization

$$m(D) - \varepsilon \;\leq\; m(D_1) \;=\; m\Big(\overline{B(r)}\Big) \;=\; c\Big(\overline{B(r)}\Big) \;=\; c(D_1) \;\leq\; c(D).$$

On the other hand, there exists a diffeomorphism $\varphi : D \to \overline{B(R)} \backslash \{$ finitely many open discs of total measure $\leq \varepsilon\}$. Choosing R appropriately we can assume, again by Dacorogna and Moser's theorem, that φ is symplectic so that

$$c(D) \;\leq\; c\Big(\overline{B(R)}\Big) \;=\; \pi R^2 \;\leq\; m(D) + \varepsilon.$$

To sum up: $m(D) - \varepsilon \leq c(D) \leq m(D) + \varepsilon$ for every $\varepsilon > 0$ and the result follows. ■

Clearly,

$$0 < c(U) < \infty$$

for every open and bounded set in $(\mathbb{R}^{2n}, \omega_0)$, since U contains a small ball and is contained in a large ball. A similar argument as for $B(r)$ above shows for symplectic cylinders that

$$c(Z(r)) = \pi r^2. \tag{2.2}$$

Therefore, if an open set U satisfies

$$B(r) \subset U \subset Z(r)$$

for some $r > 0$, we find by the monotonicity that $\pi r^2 = c(B(r)) \leq c(U) \leq c(Z(r)) = \pi r^2$ and hence

$$c(U) = \pi r^2. \quad \text{(Fig. 2.1)}$$

This demonstrates that very different (in shape, size, measure and topology) open sets can have the same capacity, if $n > 1$. Recall that the ellipsoids $E \subset \mathbb{R}^{2n}$ introduced in the previous chapter are characterized by the (linear) symplectic

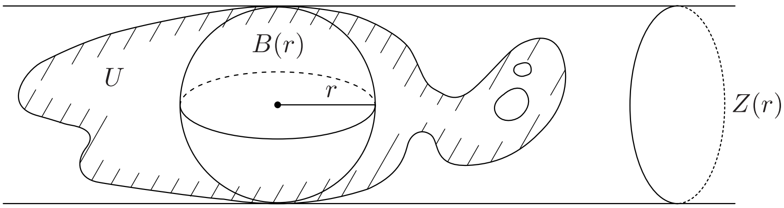

Fig. 2.1

invariants $r_1(E) \leq r_2(E) \leq \ldots \leq r_n(E)$. Applying a linear symplectic map which preserves the capacities we may assume, in view of Theorem 9 of Chapter 1, that

$$B(r_1) \subset E \subset Z(r_1)\ ,$$

where $r_1 = r_1(E)$. We conclude that the capacity of an ellipsoid E is then determined by the smallest linear symplectic invariant $r_1(E)$.

Proposition 2. The capacity of an ellipsoid E in $(\mathbb{R}^{2n}, \omega_0)$ is given by

$$c(E) = \pi r_1(E)^2 \quad .$$

We see that every capacity c extends the smallest linear invariants $\pi r_1(E)^2$ of ellipsoids E and the question arises, whether the linear invariants $\pi r_j(E)^2$ for $j > 1$ also have extensions to invariants in the nonlinear case. We shall come back to this question later on. Symplectic cylinders are "based" on symplectic 2-planes. They are very different from cylinders "based" on isotropic 2-planes on which the 2-form ω_0 vanishes, as for example $Z_1(r) = \{(x,y) \in \mathbb{R}^{2n} \mid x_1^2 + x_2^2 < r^2\}$. We claim that

$$c(Z_1(r)) = +\infty, \quad \text{for all } \ r > 0.$$

This is easily seen as follows. For every ball $B(N)$ there is a linear symplectic embedding $\varphi : B(N) \to Z_1(r)$. Therefore, $\pi N^2 = c(B(N)) \leq c(Z_1(r))$. This holds true for every N and the claim follows.

In order to generalize this example we recall some definitions. If $V \subset \mathbb{R}^{2n}$ is a linear subspace, its symplectic complement $V^{\perp}$ is defined by

$$V^{\perp} = \{x \in \mathbb{R}^{2n} \mid \ \omega_0\ (v, x) = 0 \text{ for all } v \in V\}\,.$$

Clearly $(V^{\perp})^{\perp} = V$ and $\dim V^{\perp} = \dim \mathbb{R}^{2n} - \dim V$, since ω_0 is nondegenerate. A linear subspace V is called isotropic if $V \subset V^{\perp}$, that is $\omega(v_1, v_2) = 0$ for all v_1 and $v_2 \in V$.

Proposition 3. Assume $\Omega \subset \mathbb{R}^{2n}$ is an open bounded nonempty set and assume $W \subset \mathbb{R}^{2n}$ is a linear subspace with codim $W = 2$. Consider the cylinder $\Omega + W$, then

$$c(\Omega + W) = +\infty \quad \text{if } W^{\perp} \text{ is isotropic}$$

$$0 < c(\Omega + W) < \infty \quad \text{if } W^{\perp} \text{ is not isotropic.}$$

Proof. We may assume that Ω contains the origin. Observe that $\dim W^{\perp} = 2$. Therefore, in the second case, $W^{\perp}$ is a symplectic subspace and $\mathbb{R}^{2n} = W^{\perp} \oplus W$. Choosing a symplectic basis (e_1, f_1) in $W^{\perp}$ we can, therefore, assume by a linear symplectic change of coordinates that

$$W = \{(x, y) \mid \ x_1 = y_1 = 0\}.$$

Since Ω is bounded, we have for $z \in \Omega + W$, that $x_1^2 + y_1^2 < N^2$ for some N; consequently $\Omega + W \subset B^2(N) \times \mathbb{R}^{2n-2}$, and hence $c(\Omega + W) \le c(B^2(N) \times \mathbb{R}^{2n-2}) = \pi N^2 < \infty$ by (2.2). This proves the second statement. To prove the first statement we can assume that

$$W = \{(x, y) \mid \ x_1 = x_2 = 0\}.$$

There exists $\alpha > 0$, so that the point $(x, y) \in \Omega + W$ if $x_1^2 + x_2^2 < \alpha^2$. Hence every ball $B(R)$ can be symplectically embedded in $\Omega + W$: simply define the linear symplectic map φ by $\varphi(x, y) = (\varepsilon x, \frac{1}{\varepsilon} y)$; then $\varphi(B(R)) \subset \Omega + W$ provided $\varepsilon > 0$ is sufficiently small. Consequently, by the monotonicity of a capacity, $c(\Omega \times W) \ge c(B(R)) = \pi R^2$. This holds true for every $R > 0$ so that $c(\Omega + W) = +\infty$ as claimed. ■

In view of the monotonicity property, the symplectic invariants $c(M, \omega)$ represent, in particular, obstructions of symplectic embeddings. An immediate consequence of the axioms is the celebrated squeezing theorem of Gromov [107] which gave rise to the concept of a capacity.

Theorem 1. (Gromov's squeezing theorem) There is a symplectic embedding $\varphi : B(r) \to Z(R)$ if and only if $R \ge r$.

Proof. If φ is a symplectic embedding, then using the monotonicity property of the capacity, together with (2.1) and (2.2), we have

$$\pi r^2 = c\big(B(r)\big) \le c\big(Z(R)\big) = \pi R^2,$$

and the theorem follows. ■

The next result also illustrates the difference between volume preserving and symplectic diffeomorphisms. We consider in $(\mathbb{R}^4, \omega_0)$ with symplectic coordinates (x_1, y_1, x_2, y_2) the product of symplectic open 2-balls $B(r_1) \times B(r_2)$. By a linear symplectic map we can assume that $r_1 \le r_2$.

Proposition 4. There is a symplectic diffeomorphism $\varphi : B(r_1) \times B(r_2) \to B(s_1) \times B(s_2)$ if and only if $r_1 = s_1$ and $r_2 = s_2$.

Note that, in contrast, there is a linear volume preserving diffeomorphism $\psi : B(1) \times B(1) \to B(r) \times B(\frac{1}{r})$ for every $r > 0$. As $r \to 0$, we evidently have

$$\begin{cases} c\Big(B(r) \times B\Big(\frac{1}{r}\Big)\Big) & \to \quad 0 \\ \mathrm{vol}\Big(B(r) \times B\Big(\frac{1}{r}\Big)\Big) & = \quad \text{const.} \end{cases}$$

Proof. Since $r_1 \leq r_2$ we can use the diffeomorphism φ to define the symplectic embedding $B^4(r_1) \to B(r_1) \times B(r_2) \overset{\varphi}{\to} B(s_1) \times B(s_2) \to B(s_1) \times \mathbb{R}^2 = Z(s_1)$, where the first and last mappings are the inclusion mappings. By the monotonicity of c, we conclude $s_1 \geq r_1$. Applying the same argument to φ^{-1}, we find $r_1 \geq s_1$, so that $r_1 = s_1$. Now φ is volume preserving; hence $r_1 r_2 = s_1 s_2$ and the result follows. ■

Clearly, if one assumes that φ is smooth up to the boundary, then the conditions on the radii follow simply from the invariance of the actions $|A(\partial B(r_j))| = \pi r_j^2$ under symplectic diffeomorphism. One might expect the same rigidity as in Proposition 4 to hold also in the general case of a product of n open symplectic 2-balls in $\mathbb{R}^{2n}$. This is indeed the case, but does not follow from the capacity function c alone. Actually, the proof given in [52] is rather subtle and uses the symplectic homology theory, as developed by A. Floer and H. Hofer in [90], see also K. Cieliebak, A. Floer, H. Hofer and K. Wysocki [52]. Finally the restrictions for symplectic embeddings of ellipsoids mentioned in the previous chapter follow immediately from Proposition 2.

Proposition 5. Assume E and F are two ellipsoids in $(\mathbb{R}^{2n}, \omega_0)$. If $\varphi : E \to F$ is a symplectic embedding, then

$$r_1(E) \leq r_1(F) \quad .$$

The existence of one capacity function permits the construction of many other capacity functions.

As an illustration we shall prove that the Gromov-width $D(M, \omega)$ which appears in Gromov's work [107] and which was explained in the introduction is a symplectic capacity satisfying (A1)–(A3). Recall that there is always a symplectic embedding $\varphi : (B(\varepsilon), \omega_0) \to (M, \omega)$ for ε small by Darboux's theorem and define

$$D(M, \omega) = \sup \left\{ \pi r^2 \;\middle|\; \begin{array}{l}\text{there is a}\\ \text{symplectic embedding}\end{array} \varphi : \Big(B(r), \omega_0\Big) \to (M, \omega) \right\}.$$

Theorem 2. The Gromov-width $D(M, \omega)$ is a symplectic capacity. Moreover

$$D(M, \omega) \;\leq\; c(M, \omega)$$

for every capacity function c.

Because a compact symplectic manifold (M, ω) has a finite volume we conclude $D(M, \omega) < \infty$ for compact manifolds. This is in contrast to the special capacity function c_0 constructed in the next chapter which can take on the value ∞ for certain compact manifolds.

Proof. We have already verified the monotonicity axiom (A1) in the introduction. In order to verify the conformality axiom (A2), that $D(M, \alpha\omega) = |\alpha| D(M, \omega)$ for $\alpha \neq 0$, it is sufficient to show that to every symplectic embedding

$$\varphi : \Big(B(r), \omega_0\Big) \to (M, \alpha\omega),$$

there corresponds a symplectic embedding

$$\hat{\varphi} \; : \; \Big(B\Big(\frac{r}{\sqrt{|\alpha|}}\Big), \omega_0\Big) \to (M, \omega),$$

and conversely, so that by definition of D, we conclude that $D(M, \alpha\omega) = |\alpha| D(M, \omega)$. If $\varphi \; : \; (B(r), \omega_0) \; \to \; (M, \alpha\omega)$ is a symplectic embedding, then $\varphi^*(\alpha\omega) = \omega_0$ so that

$$\varphi^* \omega \; = \; \frac{1}{\alpha} \, \omega_0.$$

Abbreviating $\rho = \frac{r}{\sqrt{|\alpha|}}$ we define the diffeomorphism $\psi : B(\rho) \to B(r)$ by setting $\psi(x) = \sqrt{|\alpha|} \cdot x$ and find

$$\psi^* \Big(\frac{1}{\alpha} \, \omega_0\Big) \; = \; \frac{|\alpha|}{\alpha} \, \omega_0.$$

Thus, if $\alpha > 0$, the map $\hat{\varphi} = \varphi \circ \psi : (B(\rho), \omega_0) \to (M, \omega)$ is the desired symplectic embedding. If $\alpha < 0$ we first introduce the symplectic diffeomorphism $\psi_0 : (B(\rho), \omega_0) \to (B(\rho), -\omega_0)$ by setting $\psi_0(u, v) = (-u, v)$ for all $(u, v) \in \mathbb{R}^{2n}$, and find the desired embedding $\hat{\varphi} = \varphi \circ \psi \circ \psi_0 : (B(\rho), \omega_0) \to (M, \omega)$.

The verification of $D(B(r), \omega_0) = \pi r^2$ is easy. If $\varphi : B(R) \to B(r)$ is a symplectic embedding, we conclude $R \leq r$ since φ is volume preserving. On the other hand the identity map induces a symplectic embedding $B(r) \to B(r)$ so that the claim follows. Since there exists a symplectic embedding $\varphi : B(R) \to Z(r)$ if and only if $r \geq R$ by Gromov's squeezing theorem, we conclude that $D(Z(r), \omega_0) = \pi r^2$, hence the Gromov-width satisfies also the nontriviality axiom (A3). In order to prove the last statement of the theorem, we assume $c(M, \omega)$ to be any capacity. If $\varphi : B(r) \to M$ is a symplectic embedding we conclude by monotonicity that $\pi r^2 = c(B(r), \omega_0) \leq c(M, \omega)$. Taking the supremum, we find $D(M, \omega) \leq c(M, \omega)$ as claimed in the theorem. ■

To a given capacity c one can associate its inner capacity $\check{c}$, defined as follows:

$$\check{c}(M, \omega) = \sup \big\{ c(U, \omega) \;\mid\; U \subset M \text{ open and } \overline{U} \subset M \backslash \partial M \big\} \; .$$

Correspondingly, we introduce the following

Definition. A capacity c has inner regularity at M if

$$\check{c}(M, \omega) = c(M, \omega).$$

Proposition 6. The function $\check{c}$ is a capacity having inner regularity and it satisfies $\check{c} \leq c$. In addition, if d is any capacity having inner regularity and satisfying $d \leq c$, then $d \leq \check{c}$.

Proof. The proof follows readily from the definitions and the axioms for capacity. Assume, for example, that d is a symplectic capacity satisfying $d \leq c$ and having inner regularity.

Then

$$\begin{aligned} \check{d}(M) &= \sup\left\{d(U)\mid\ U \subset M, \overline{U} \subset M \backslash \partial M\right\} \\ &\leq \sup\left\{c(U)\mid\ U \subset M, \overline{U} \subset M \backslash \partial M\right\} \\ &= \check{c}(M)\ , \end{aligned}$$

as claimed. ■

Because it is the smallest capacity, the Gromov-width $D(M, \omega)$ has inner regularity; another example having this property is the capacity c_0 introduced in Chapter 3. If we consider subsets of a given manifold we can also define the concept of outer regularity (relative to the manifold). The outer capacity of a set is defined as the infimum taken over the capacities of open neighborhoods of the closure of the given set. We shall return to this concept in the next section.

2.2 Rigidity of symplectic diffeomorphisms

We consider a sequence $\psi_j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of symplectic diffeomorphisms in $(\mathbb{R}^{2n}, \omega_0)$. By definition, the first derivatives satisfy the identity

$$\psi_j'(x)^T J\ \psi_j'(x)\ =\ J\ , \quad x \in \mathbb{R}^{2n}.$$

Therefore, if the sequence ψ_j converges in C^1 then the limit $\psi(x) = \lim \psi_j(x)$ is also a symplectic map. By contrast, we shall now assume that the sequence ψ_j only converges locally uniformly to a map

$$\psi(x)\ =\ \lim_{j\to\infty}\ \psi_j(x),$$

which is, therefore, a continuous map. Since $\det \psi_j'(x) = 1$ for every x, we find, in view of the transformation formula for integrals, that

$$\int f\Big(\psi(x)\Big)\, dx\ =\ \int f(x)\, dx \tag{2.3}$$

for all $f \in C_c^\infty(\mathbb{R}^{2n})$, so that ψ is measure preserving. Assume now that ψ is differentiable, then evidently $\det \psi'(x) = \pm 1$. However it is a striking phenomenon that ψ is even symplectic,

$$\psi'(x)^T J \psi'(x)\ =\ J,$$

if it is assumed to be differentiable. Hence the symplectic nature survives under topological limits.

Theorem 3. Let $\varphi_j : B(1) \to (\mathbb{R}^{2n}, \omega_0)$ be a sequence of symplectic embeddings converging locally uniformly to a map $\varphi : B(1) \to \mathbb{R}^{2n}$. If φ is differentiable at $x = 0$, then $\varphi'(0) = A$ is a symplectic map, i.e., $A^*\omega_0 = \omega_0$.

We see that, in general, a volume preserving diffeomorphism cannot be approximated by symplectic diffeomorphisms in the C^0-topology. By using, locally, Darboux charts we deduce immediately from Theorem 3

Theorem 4. (Eliashberg, Gromov) The group of symplectic diffeomorphisms of a compact symplectic manifold (M, ω) is C^0-closed in the group of all diffeomorphisms of M.

In the early seventies M. Gromov proved the alternative that the group of symplectic diffeomorphisms either is C^0-closed in the group of all diffeomorphisms or its C^0-closure is the group of volume preserving diffeomorphisms. That symplectic diffeomorphisms can be distinguished from volume preserving diffeomorphisms by global properties which are stable under C^0-limits was announced in the early eighties by Y. Eliashberg in his preprint "Rigidity of symplectic and contact structure", (1981) [78], which in full form has not been published. Proofs are partially contained in Eliashberg [71], 1987. Gromov gave a proof of Theorem 4 in [107] using the techniques of pseudoholomorphic curves. Both Eliashberg und Gromov deduced the C^0-stability from non embedding results. In his book [108] Gromov uses so-called Nash-Moser techniques of hard implicit function theorems, while Eliashberg [71], 1987, uses an analogue of Theorem 3. Following the strategy of I. Ekeland and H. Hofer in [68], we shall show next, that Theorem 3 is an easy consequence of the existence of any capacity function c.

It is convenient in the following to extend the capacity to all subsets of $\mathbb{R}^{2n}$. To do so we take a capacity function c given on the open subsets $U \subset \mathbb{R}^{2n}$ and define for an arbitrary subset $A \subset \mathbb{R}^{2n}$:

$$c(A) = \inf \left\{ c(U) \middle| A \subset U \ \text{ and } \ U \ \text{ open } \right\}.$$

Then the monotonicity property

$$A \subset B \implies c(A) \leq c(B)$$

holds true for all subsets of $\mathbb{R}^{2n}$. From the symplectic invariance of the capacity on open sets, one deduces the invariance

$$c\big(\varphi(A)\big) = c(A)$$

under every symplectic embedding φ defined on an open neighborhood of A.

Proof of Theorem 3. Without loss of generality we shall assume in the following that $\varphi(0) = 0$. We first claim that the linear map $\varphi'(0) = A$ is an isomorphism. Indeed, because φ is differentiable at 0, we have $\varphi(x) = Ax + O(|x|)$, so that for the open balls B_ε of radius $\varepsilon > 0$ and centered at 0,

$$\frac{m\Big(\varphi(B_\varepsilon)\Big)}{m(B_\varepsilon)} \longrightarrow |\det A| \ \text{ as } \ \varepsilon \to 0\,.$$

On the other hand, because the symplectic diffeomorphisms φ_j are volume preserving and $\varphi_j \to \varphi$ uniformly, we have $m(\varphi(B_\varepsilon)) = m(B_\varepsilon)$ and hence $|\det A| = 1$, so that A is an isomorphism, as claimed. We shall see later on (Lemma 3) that A is an isomorphism under weaker assumption: instead of requiring φ_j to be symplectic, we shall merely require these mappings to preserve a given capacity function.

Next we claim that to prove Theorem 3, it is sufficient to show that

$$A^*\omega_0 \;=\; \lambda\omega_0 \;\text{ for some }\; \lambda \neq 0\,. \tag{2.4}$$

Indeed, with φ_j we can also consider the symplectic embeddings $(\varphi_j, id) : B(1) \times \mathbb{R}^{2n} \to (\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_0 \oplus \omega_0)$ and hence conclude for the derivative at $(0,0)$, given by $\bar{A} = (A, 1)$, that also $\bar{A}^*(\omega_0 \oplus \omega_0) = \mu(\omega_0 \oplus \omega_0)$ for some $\mu \neq 0$. On the other hand, in view of (2.4), $\bar{A}^*(\omega_0 \oplus \omega_0) = (\lambda\omega_0) \oplus \omega_0$ and consequently $\mu = 1 = \lambda$, as required in Theorem 4, proving our claim. In order to prove (2.4) we make use of the following algebraic lemma due to Y. Eliashberg [71].

Lemma 2. Assume A is a linear isomorphism satisfying $A^*\omega_0 \neq \lambda\omega_0$. Then for every $a > 0$ there are symplectic matrices U and V such that $U^{-1}AV$ has the form

$$U^{-1}AV = \left(\begin{array}{c|c} \begin{matrix} a & 0 \\ 0 & a \end{matrix} & 0 \\ \hline * & * \end{array}\right),$$

with respect to the splitting of $\mathbb{R}^{2n} = \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$ into symplectic subspaces.

Postponing the proof of the lemma, we first show that $A^*\omega_0 = \lambda\omega_0$ for some $\lambda \neq 0$. Arguing by contradiction, we assume that $A^*\omega_0 \neq \lambda\omega_0$ and apply the lemma. Defining the symplectic maps $\psi_j := U^{-1}\varphi_j V$ in the neighborhood of the origin, we conclude that $\psi_j \to \psi := U^{-1}\varphi V$ locally uniformly, and $\psi'(0) = U^{-1}AV$. Choosing a suitable constant a in the lemma, we have $U^{-1}AV(B(1)) \subset Z(\frac{1}{8})$ and hence $U^{-1}\psi V(B(\varepsilon)) \subset Z(\frac{\varepsilon}{4})$ provided ε is sufficiently small. Because $\psi_j \to \psi$ locally uniformly, $U^{-1}\psi_j V(B(\varepsilon)) \subset Z(\frac{\varepsilon}{2})$ if j is sufficiently large and ε sufficiently small. Since $U^{-1}\psi_j V$ is symplectic, we conclude by the invariance and monotonicity property of a capacity that $c(U^{-1}\psi_j V(B(\varepsilon))) = c(B(\varepsilon)) \leq c(Z(\frac{\varepsilon}{2}))$, which contradicts the nontriviality Axiom (A3). We have proved the statement in (2.4) and it remains to prove Lemma 2.

Proof of Lemma 2. Let B be the symplectic adjoint of A satisfying

$$\omega_0(Ax, y) = \omega_0(x, By)$$

for all x, y, and abbreviate $\omega = B^*\omega_0$. Then $\omega \neq \lambda\omega_0$, as is easily verified using the fact that A is an isomorphism. We claim that there is an x such that $\omega(x,\cdot) \neq \lambda\omega_0(x,\cdot)$ for every λ. Arguing by contradiction, we assume that for every x, there exists a $\lambda(x) \in \mathbb{R}$ satisfying $\omega(x,\cdot) = \lambda(x)\omega_0(x,\cdot)$. If $x \neq 0$ there exists ξ such that $\omega_0(\xi, x) \neq 0$, since ω_0 is nondegenerate. This remains true for all y in a neighborhood $U(x)$ of x. Hence

$$\begin{aligned}\lambda(\xi)\,\omega_0(\xi, y) &= \omega(\xi, y) = -\omega(y, \xi) \\ &= -\lambda(y)\omega_0(y, \xi) \\ &= \lambda(y)\omega_0(\xi, y),\end{aligned}$$

which implies that $\lambda(\xi) = \lambda(y)$ for y in a neighborhood of x. Since $\mathbb{R}^{2n}\backslash\{0\}$ is connected and since the function $\lambda(x)$ on $\mathbb{R}^{2n}\backslash\{0\}$ is locally constant, it is constant. Therefore $\omega(x, \cdot) = \lambda\omega_0(x, \cdot)$ for $x \neq 0$ and hence for every x. This contradicts the assumption that $\omega \neq \lambda\omega_0$ and proves our claim. Consequently there exists an x such that the linear map $(\omega_0(x, \cdot), \omega(x, \cdot)) : \mathbb{R}^{2n} \to \mathbb{R}^2$ is surjective. For a given $a > 0$, we therefore find an y satisfying

$$\omega_0(x, y) = 1 \text{ and } \omega(x, y) = a^2.$$

Recalling $\omega(x, y) = \omega_0(Bx, By)$, we can choose two symplectic bases $(e_1, f_1, \ldots)$ and $(e'_1, f'_1, \ldots)$ such that $e_1 = x, f_1 = y$ and $e'_1 = \frac{1}{a}Bx, f'_1 = \frac{1}{a}By$. In these bases

$$Be_1 = ae'_1,\ Bf_1 = af'_1.$$

From $\langle JAx, y\rangle = \langle Jx, By\rangle = \langle B^T Jx, y\rangle$ we read off $A = -JB^T J$. Representing now A in the new bases as a map from $\mathbb{R}^{2n}$ with basis $(e_1, f_1, \ldots)$ onto $\mathbb{R}^{2n}$ with basis $(e'_1, f'_1, \ldots)$, we find the representation $U^{-1}AV$ of the desired form. The symplectic matrices are defined by their column vectors as $U = [e_1, f_1, \ldots]$ and $V = [e'_1, f'_1, \ldots]$. ■

We know that symplectic diffeomorphisms preserve the capacities. Theorem 3 can, therefore, be deduced from the following, even more surprising statement for continuous mappings due to I. Ekeland and H. Hofer [68].

Theorem 5. Let c be a capacity. Assume $\psi_j : B(1) \to \mathbb{R}^{2n}$ is a sequence of continuous mappings satisfying

$$c(\psi_j(E)) = c(E)$$

for all (small) ellipsoids $E \subset B(1)$ and converging locally uniformly to

$$\psi(x) = \lim \psi_j(x).$$

If ψ is differentiable at 0, then $\psi'(0) = A$ is either symplectic or antisymplectic:

$$A^*\omega_0 = \omega_0 \text{ or } A^*\omega_0 = -\omega_0.$$

Note that the mappings are not required to be invertible.

In order to prove Theorem 5, we start with

Lemma 3. Let c be a capacity. Consider a sequence φ_j of continuous mappings in $\mathbb{R}^{2n}$ converging locally uniformly to the map φ. Assume that $c(\varphi_j(E)) = c(E)$ for the open ellipsoids for all j. If $\varphi'(0)$ exists it is an isomorphism.

Proof. Arguing by contradiction, we assume that A is not surjective, so that $A(\mathbb{R}^{2n})$ is contained in a hyperplane H. Composing, if necessary, with a linear symplectic map we may assume that

$$A(\mathbb{R}^{2n}) \subset H \;=\; \{(x,y)|\; x_1 = 0\}\;. \tag{2.5}$$

Defining the linear symplectic map ψ by

$$\psi(x,y) \;=\; \Big(\frac{1}{\alpha}\,x_1, x_2, \ldots, x_n,\, \alpha\, y_1, y_2, \ldots, y_n\Big)$$

we can choose $\alpha > 0$ so small that

$$\psi A\Big(B(1)\Big) \subset B^2(\frac{1}{16}) \times \mathbb{R}^{2n-2} \;=\; Z(\frac{1}{16})\;,$$

where the open 2-disc B^2 on the right hand side is contained in the symplectic plane with the coordinates $\{x_1, y_1\}$. Be definition of a derivative, we have $|\psi\varphi(x) - \psi A(x)| \leq a(|x|)\,|x|$ where $a(s) \to 0$ as $s \to 0$. Consequently

$$\psi\varphi\Big(B(\varepsilon)\Big) \subset Z(\frac{\varepsilon}{4})$$

if ε is sufficiently small. Since $\psi\varphi_j$ converges locally uniformly to $\psi\varphi$,

$$\psi\varphi_j\Big(B(\varepsilon)\Big) \subset Z(\frac{\varepsilon}{2})\;,$$

provided j is sufficiently large. By assumption $\psi\varphi_j$ preserves the capacity and so by monotonicity

$$c\Big(B(\varepsilon)\Big) \;=\; c\Big(\psi\varphi_j(B(\varepsilon))\Big) \;\leq\; c\Big(Z(\frac{\varepsilon}{2})\Big) \;=\; \frac{1}{4}c\Big(B(\varepsilon)\Big)\;.$$

This contradiction shows that A is surjective. ■

Proof of Theorem 5. We may assume that $\psi(0) = 0$. By Lemma 3, $A = \psi'(0)$ is an isomorphism and we shall prove first that $A^*\omega_0 = \lambda\omega_0$ for some $\lambda \neq 0$. Arguing by contradiction, we assume $A^*\omega_0 \neq \lambda\omega_0$ and find (by Lemma 2) symplectic maps U and V satisfying $U^{-1}AV(B(1)) \subset Z(\frac{1}{8})$. Proceeding now as in Theorem 4, we define the sequence $\varphi_j := U^{-1}\psi_j V$. Then $\varphi_j \to \varphi := U^{-1}\psi V$ locally uniformly, and $\varphi'(0) = U^{-1}AV$. Hence $\varphi(B(\varepsilon)) \subset Z(\frac{\varepsilon}{4})$ and consequently $\varphi_j(B(\varepsilon)) \subset Z(\frac{\varepsilon}{2})$ for j sufficiently large and $\varepsilon > 0$ sufficiently small. Since, by assumption on ψ_j, we have $c(\varphi_j(B(\varepsilon))) = c(B(\varepsilon))$, we infer by the monotonicity of a capacity that $c(B(\varepsilon)) \leq c(Z(\frac{\varepsilon}{2}))$, contradicting Axiom (A3) for a capacity.

We have demonstrated that $A^*\omega_0 = \lambda\omega_0$. By conformality a linear antisymplectic map preserves the capacities. Composing the maps ψ_j and ψ with the symplectic map $B = \left(\frac{1}{\sqrt{\lambda}}A\right)^{-1}$ if $\lambda > 0$ and with the antisymplectic map $B = \left(\frac{1}{\sqrt{-\lambda}}A\right)^{-1}$ if $\lambda < 0$, we are therefore reduced to the case

$$A = \alpha\,1, \text{ with } \alpha > 0$$

and we have to show that $\alpha = 1$. If $\alpha < 1$ there is a small ball and an $\alpha < r < 1$ such that $\psi_j(B(\varepsilon)) \subset B(r\varepsilon)$ for j large. Since ψ_j preserves the capacities we find $c(B(\varepsilon)) = c(\psi_j(B(\varepsilon)) \leq c(B(r\varepsilon)) = r^2 c(B(\varepsilon))$ which is a contradiction. In the case $\alpha > 1$ we shall show that

$$\psi_j\left(B(\varepsilon)\right) \supset B(r\varepsilon) \tag{2.6}$$

for some $\alpha > r > 1$, ε small and j large, which leads to the contradiction $r^2 c(B(\varepsilon)) = c(B(r\varepsilon)) \leq c(B(\varepsilon))$. Consequently, $\alpha = 1$ as claimed in the theorem.

To prove (2.6) we have to show that for every $y \in B(r\varepsilon)$ there exists an $x \in B(\varepsilon)$ solving $\psi_j(x) = y$. We use an index argument based on the Brouwer mapping degree. Fix $1 < r < \alpha$. Then, in view of $A(B(\varepsilon)) = B(\alpha\varepsilon)$,

$$\deg(B(\varepsilon), A, y) = 1,$$

for $y \in B(r\varepsilon)$, and the proof follows if we can show that

$$\deg(B(\varepsilon), \psi_j, y) = \deg(B(\varepsilon), A, y).$$

In view of the homotopy invariance of the degree, it is sufficient to verify that the homotopy $h(t,x) = t\psi_j(x) + (1-t)Ax$ satisfies $h(t,x) \neq y$ if $x \in \partial B(\varepsilon)$ and $0 \leq t \leq 1$.

Recall $\psi(0) = 0$ and hence $|\psi(x) - Ax| \leq a(|x|)|x|$, with $a(s) \to 0$ as $s \to 0$. Since $\psi_j \to \psi$ uniformly on compact sets we have, for every $\sigma > 0$ and $j \geq j_0(\sigma)$

$$|\psi_j(x) - Ax| \leq a(|x|)|x| + \sigma. \tag{2.7}$$

Arguing by contradiction, we assume $t\psi_j(x) + (1-t)Ax = y$ for $x \in \partial B(\varepsilon)$ and $y \in B(r\varepsilon)$. Then

$$t(\psi_j(x) - Ax) = y - Ax.$$

The right hand side is larger than $|Ax| - |y| = \alpha\varepsilon - |y| \geq (\alpha - r)\varepsilon$. The left hand side, however is smaller than $a(|\varepsilon|)\varepsilon + \sigma$ according to (2.7). Therefore, choosing $\sigma = a(|\varepsilon|)\varepsilon$ and ε sufficiently small we get a contradiction, hence proving that $h(t,x) \neq y$ for all $0 \leq t \leq 1$ and all $x \in \partial B(\varepsilon)$. This finishes the proof of Theorem 5. ■

As an interesting special case we conclude from Theorem 5 the following

Corollary. A linear map A in $\mathbb{R}^{2n}$ preserving the capacities of ellipsoids, $c(A(E)) = c(E)$ is either symplectic or antisymplectic.

$$A^*\omega_0 = \omega_0 \text{ or } A^*\omega_0 = -\omega_0.$$

Let c be any capacity function and consider a homeomorphism h of $\mathbb{R}^{2n}$ satisfying

$$c(h(E)) = c(E) \text{ for all (small) ellipsoids } E.$$

Then, if h is in addition differentiable, we conclude from Theorem 5 that h is a diffeomorphism which is either symplectic or antisymplectic, $h'(x)^*\omega_0 = \pm\omega_0$. This is analogous to a measure preserving homeomorphism, i.e., a homeomorphism satisfying (2.3). Here we conclude that $\det h'(x) = \pm 1$ in case h is differentiable. We see that every capacity function c singles out the distinguished group of homeomorphisms preserving the capacity of all open sets. The elements of this group of homeomorphisms have the additional property that they are symplectic or antisymplectic in case they are differentiable. This group can, therefore, be viewed as a topological version of the group of symplectic diffeomorphisms. It is not known whether this group is closed under locally uniform limits. But the following weaker results hold true.

Theorem 6. Assume $h_j : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a sequence of homeomorphisms satisfying

$$c(h_j(E)) = c(E)$$

for all (resp. all small) ellipsoids E. Assume h_j converges locally uniformly to a homeomorphism h of $\mathbb{R}^{2n}$. Then

$$c(h(E)) = c(E)$$

for all (resp. all small) ellipsoids E.

Proof. Since $h^{-1} \circ h_j \to id$ locally uniformly, we conclude for every ellipsoid E and every $0 < \varepsilon < 1$

$$h^{-1} \circ h_j((1-\varepsilon)E) \subset E \subset h^{-1} \circ h_j((1+\varepsilon)E),$$

if only j is sufficiently large. This topological fact is easily verified using the same degree argument as above. Hence

$$h_j((1-\varepsilon)E) \subset h(E) \subset h_j((1+\varepsilon)E).$$

By monotonicity and conformality, $(1-\varepsilon)^2 c(E) = c((1-\varepsilon)E) = c(h_j((1-\varepsilon)E)) \leq c(h(E)) \leq c(h_j((1+\varepsilon)E)) = c((1+\varepsilon)E) = (1+\varepsilon)^2 c(E)$. In short

$$(1-\varepsilon)^2 c(E) \leq c(h(E)) \leq (1+\varepsilon)^2 c(E).$$

This holds true for every $\varepsilon > 0$ so that $c(h(E)) = c(E)$ as desired. ■

In order to generalize this statement, we denote by $\mathcal{O}$ the family of open and bounded sets of $\mathbb{R}^{2n}$ and associate with $\Omega \in \mathcal{O}$

$$\begin{aligned} \check{c}(\Omega) &= \sup\left\{c(U)\,|U \text{ open and } \overline{U} \subset \Omega\right\} \\ \hat{c}(\Omega) &= \inf\left\{c(U)\,|U \text{ open and } U \supset \overline{\Omega}\right\} . \end{aligned}$$

The distinguished family $\mathcal{O}_c$ of open sets is defined by the condition

$$\mathcal{O}_c = \{\Omega \in \mathcal{O}\,|\,\hat{c}\;(\Omega) = \check{c}(\Omega)\}\,.$$

Proposition 7. Assume h_j is a sequence of homeomorphisms of $\mathbb{R}^{2n}$ converging locally uniformly to a homeomorphism h of $\mathbb{R}^{2n}$. If $c(h_j(U)) = c(U)$ for all $U \in \mathcal{O}$ and all j, then $c(h(\Omega)) = c(\Omega)$ for all $\Omega \in \mathcal{O}_c$.

Proof. If $\Omega \in \mathcal{O}_c$ and $\varepsilon > 0$ we find $\hat{U}, \check{U} \in \mathcal{O}$ with $\hat{U} \supset \overline{\Omega}$ and $\overline{\check{U}} \subset \Omega$ satisfying

$$c(\hat{U}) \;\le\; c(\Omega) + \varepsilon \quad \text{and} \quad c(\check{U}) \;\ge\; c(\Omega) - \varepsilon.$$

For j large we have by the above degree argument $h_j(\hat{U}) \supset h(\Omega) \supset h_j(\check{U})$, and hence using the monotonicity property of the capacity

$$c(\hat{U}) = c(h_j(\hat{U})) \ge c(h(\Omega)) \ge c(h_j(\check{U})) = c(\check{U}),$$

so that $c(\Omega) + \varepsilon \ge c(h(\Omega)) \ge c(\Omega) - \varepsilon$. This holds true for every $\varepsilon > 0$ and the theorem is proved. ■

Definition. A capacity c is called inner regular, respectively, outer regular if

$$c(U) \;=\; \check{c}(U) \quad \text{resp. if} \quad c(U) \;=\; \hat{c}(U)$$

for all $U \in \mathcal{O}$.

Proposition 8. Assume the capacity c is inner regular or outer regular. Assume φ_j and φ are homeomorphisms of $\mathbb{R}^{2n}$ and

$$\varphi_j \to \varphi \quad \text{and} \quad \varphi_j^{-1} \to \varphi^{-1}$$

locally uniformly. If $c(\varphi_j(U)) = c(U)$ for all $U \in \mathcal{O}$ and all j, then also

$$c\Big(\varphi(U)\Big) \;=\; c(U) \quad \text{for all} \quad U \in \mathcal{O}.$$

Proof a) Assume c is inner regular and let Ω be open and bounded. Then if U is open and $\overline{U} \subset \Omega$ we have $\varphi_j(\overline{U}) \subset \varphi(\Omega)$ if j is large and thus $c(\varphi_j(\overline{U})) = c(\overline{U}) \leq c(\varphi(\Omega))$ so that $c(U) \leq c(\varphi(\Omega))$. Hence, taking the supremum we find $c(\Omega) \leq c(\varphi(\Omega))$. Similarly one shows that $c(\Omega) \leq c(\varphi^{-1}(\Omega))$ for every open and bounded set Ω. Consequently, since φ is a homeomorphism $c(\Omega) = c(\varphi^{-1} \circ \varphi(\Omega)) \geq c(\varphi(\Omega)) \geq c(\Omega)$ and thus $c(\varphi(\Omega)) = c(\Omega)$ as desired.

b) If c is outer regular and Ω is open and bounded we conclude for $\varepsilon > 0$ that $\varphi_j(\Omega) \subset U_\varepsilon(\varphi(\Omega))$ if j is sufficiently large; here $U_\varepsilon = \{x| \ \text{dist}\ (x, U) < \varepsilon\}$. Therefore, $c(\Omega) = c(\varphi_j(\Omega)) \leq c(U_\varepsilon(\varphi(\Omega)))$ and taking the infimum on the right hand side we find $c(\Omega) \leq c(\varphi(\Omega))$. Arguing as in the part a) we conclude that $c(\varphi(\Omega)) = c(\Omega)$ for every $\Omega \in \mathcal{O}$. ■

So far we have deduced from the existence of a symplectic capacity some surprising phenomena about symplectic mappings. Nevertheless, the notion itself is still rather mysterious and raises many questions. It is, for example, not known whether the knowledge of the capacities of small sets is sufficient to understand the capacity of larger sets. To be more precise, does, for example, a homeomorphism preserving the capacity of small sets preserve the capacity of large sets?

We have no example of a homeomorphism which preserves one capacity but not another. Neither do we know whether a homeomorphism preserving the capacity of open sets, also preserves the Lebesgue measure of open sets. But it is easy to see that a homeomorphism preserving all capacities of open sets is necessarily measure preserving. In order to prove this we first define a special embedding capacity γ. Introducing for $r > 0$ the open cube

$$Q(r) \ = \ (0, r)^{2n} \ \subset \ \mathbb{R}^{2n},$$

having edges parallel to the coordinate axis, we define

$$\gamma(M, \omega) := \ \sup \Big\{ r^2 \ | \ \text{there is a symplectic embedding} \ \ \varphi : Q(r) \to M \Big\}.$$

Clearly γ is a capacity satisfying the Axioms (A1), (A2) and the weak nontriviality condition (A3$'$). It is not normalized. One can prove, by using the n-th order capacity function c_n of Ekeland and Hofer in [69], that $\gamma(B(r)) = \frac{1}{n}\pi r^2$, moreover $\gamma(Z(r)) = \pi r^2$.

Proposition 9. Assume h is a homeomorphism of $\mathbb{R}^{2n}$ satisfying

$$\gamma\Big(h(\Omega)\Big) \ = \ \gamma(\Omega)$$

for all $\Omega \in \mathcal{O}$. Then h preserves the Lebesgue measure μ of open sets, i.e.,

$$\mu\Big(h(\Omega)\Big) \ = \ \mu(\Omega)$$

for all $\Omega \in \mathcal{O}$.

Proof. From the definition of γ we infer

$$\mu\Big(Q(r)\Big) \;=\; r^{2n} \;=\; \gamma\Big(Q(r)\Big)^n.$$

Since every symplectic embedding is volume preserving we find for the capacity $\gamma(\Omega)$ of the open and bounded set $\Omega \subset \mathbb{R}^{2n}$ that

$$\mu(\Omega) \;\geq\; \sup_r \mu\Big(Q(r)\Big) \;=\; \gamma(\Omega)^n,$$

where the supremum is taken over those r, for which there is a symplectic embedding $Q(r) \to \Omega$. If Q is any open cube having its edges parallel to the coordinate axes we conclude that

$$\mu(Q) \;=\; \gamma(Q)^n \;=\; \gamma\Big(h(Q)\Big)^n \;\leq\; \mu\Big(h(Q)\Big).$$

It follows that

$$\mu(\Omega) \;\leq\; \mu\Big(h(\Omega)\Big),$$

for every $\Omega \in \mathcal{O}$. Indeed assume $\Omega \in \mathcal{O}$; then given $\varepsilon > 0$ we find, in view of the regularity of the Lebesgue measure, finitely many disjoint open cubes Q_j contained in Ω such that $\mu(\Omega) - \varepsilon \leq \sum \mu(Q_j)$. Hence by the estimate above

$$\begin{aligned}
\mu(\Omega) - \varepsilon \;&\leq\; \sum \mu\Big(h(Q_j)\Big) \\
&=\; \mu\Big(h(\bigcup_j Q_j)\Big) \\
&\leq\; \mu\Big(h(\Omega)\Big).
\end{aligned}$$

This holds true for every $\varepsilon > 0$ and the claim follows. By the same argument, $\mu(\Omega) \leq \mu(h^{-1}(\Omega))$ and hence $\mu(\Omega) \leq \mu(h(\Omega)) \leq \mu(h^{-1} \circ h(\Omega)) = \mu(\Omega)$, proving the proposition. ■

Corollary. If a homeomorphism of $\mathbb{R}^{2n}$ preserves all the capacities of open sets in $\mathbb{R}^{2n}$, then it also preserves the Lebesgue measure.

All our considerations so far are based on the existence of a capacity not yet established. In the next chapter we shall construct a very special capacity function defined dynamically by means of Hamiltonian systems.

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