Chapter 2

Slice monogenic functions

2.1 Clifford algebras

Clifford algebras will be the setting in which we will work throughout this book. They were introduced under the name of geometric algebras by Clifford in 1878. Since then, several people have extensively studied them and nowadays there are, in the literature, several possible ways to introduce Clifford algebras: for example one can use exterior algebras, or present them as a quotient of a tensor algebra or by means of a universal property (see [23], [31], [34], or [75] for a survey on the various possible definitions). In this book, we will adopt an equivalent but more direct approach, using generators and relations.

Definition 2.1.1. *Given n elements $e_1, \dots, e_n$, $n = p + q$, $p, q \geq 0$, which will be called* imaginary units*, together with the* defining relations

$$\begin{gathered} e_i^2 = +1, \quad \textit{for} \quad i = 1, \dots, p, \\ e_i^2 = -1, \quad \textit{for} \quad i = p+1, \dots, n, \\ e_i e_j + e_j e_i = 0, \quad i \neq j. \end{gathered}$$

Assume that

$$e_1 e_2 \dots e_n \neq \pm 1 \quad \text{if } p - q \equiv 1 (\operatorname{mod} 4). \tag{2.1}$$

We will call (universal) Clifford algebra *the algebra over $\mathbb{R}$ generated by $e_1, \dots, e_n$ and we will denote it by $\mathbb{R}_{p,q}$.*

Remark 2.1.2. It is immediate that $\mathbb{R}_{p,q}$, as a real vector space and has dimension 2^n, $n = p + q$.

An element in $\mathbb{R}_{p,q}$, called a *Clifford number*, can be written as

$$a = a_0 + a_1 e_1 + \dots + a_n e_n + a_{12} e_1 e_2 + \dots + a_{123} e_1 e_2 e_3 + \dots + a_{12\dots n} e_1 e_2 \dots e_n.$$

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Denote by A an element in the power set $\wp(1, \dots, n)$. If $A = i_1 \dots i_r$, then the element $e_{i_1} \dots e_{i_r}$ can be written as $e_{i_1 \dots i_r}$ or, in short, e_A. Thus, in a more compact form, we can write a Clifford number as

$$a = \sum_A a_A e_A.$$

Possibly using the defining relations, we will order the indices in A as $i_1 < \dots < i_r$. When $A = \emptyset$ we set $e_\emptyset = 1$.

We now give some examples of real Clifford algebras $\mathbb{R}_n$ of low dimension.

Example 2.1.3. First of all, we point out that the index $n = 0$ is allowed in the definition, and in this case we obtain the real numbers. For $n = 1$ we have that $\mathbb{R}_{0,1}$ is the algebra generated by e_1 over $\mathbb{R}$ with the relation $e_1^2 = -1$. Hence there is an $\mathbb{R}$-algebra isomorphism $\mathbb{R}_{0,1} \cong \mathbb{C}$ where $\mathbb{C}$ denotes, as customary, the algebra of complex numbers.

Example 2.1.4. For $n = 2$, the Clifford algebra $\mathbb{R}_{0,2}$ is generated by e_1 and e_2. This real algebra is the so-called algebra of quaternions and it is usually denoted by the symbol $\mathbb{H}$. A quaternion q is traditionally written as $q = x_0 + ix_1 + jx_2 + kx_3$ where the imaginary units i, j, k anti-commute among them and satisfy $i^2 = j^2 = k^2 = -1$. With the identification

$$e_1 \to i, \quad e_2 \to j,$$

(and the consequent $e_1 e_2 \to k$), it is immediate to identify $\mathbb{R}_{0,2}$ with $\mathbb{H}$.

Example 2.1.5. We now compare the two Clifford algebras $\mathbb{R}_{1,1}$ generated by the elements e_1 and ϵ_1 such that $e_1^2 = -1$ and $e_2^2 = +1$, and $\mathbb{R}_{2,0}$ generated by the elements ε_1 and ε_2 both having square $+1$. These two Clifford algebras are isomorphic. In fact, let us consider the matrices

$$\eta_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \eta_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\eta_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \eta_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They form a basis for the vector space $M(2, \mathbb{R})$ of 2×2 real matrices. The map

$$\varphi : \ \mathbb{R}_{1,1} \mapsto M(2, \mathbb{R})$$

defined by $\varphi(e_1) = \eta_2$, $\varphi(e_2) = \eta_1$ can be extended to an isomorphism for which $\varphi(1) = \eta_0$, and $\varphi(e_2 e_1) = \eta_3$. The map

$$\psi : \ \mathbb{R}_{2,0} \mapsto M(2, \mathbb{R})$$

defined by $\psi(\varepsilon_1) = \eta_1$, $\psi(\varepsilon_2) = \eta_3$ can be extended to an isomorphism for which $\psi(1) = \eta_0$, $\psi(\varepsilon_1 \varepsilon_2) = \eta_2$. Thus the Clifford algebras $\mathbb{R}_{1,1}$ and $\mathbb{R}_{2,0}$ are isomorphic but, as the reader can verify, they are not isomorphic to $\mathbb{R}_{0,2}$.

The case of $\mathbb{R}_{0,n}$ will be the only case we will use in this book. For this reason, we will write $\mathbb{R}_n$ instead of $\mathbb{R}_{0,n}$.

Definition 2.1.6. *Let $k \in \mathbb{N}$ and $0 \leq k \leq n$. The linear subspace of $\mathbb{R}_n$ generated by the $\binom{n}{k}$ elements of the form $e_A = e_{i_1} \dots e_{i_k}$, $i_\ell \in \{1, \dots, n\}$, $i_1 < \dots < i_k$, will be denoted by $\mathbb{R}_n^k$. The elements in $\mathbb{R}_n^k$ are called k-vectors.*

For $k = 0$, the subspace $\mathbb{R}_n^0$ is identified with the space of scalars $\mathbb{R}$; for $k = 1$ we have the subspace $\mathbb{R}_n^1$ of 1-vectors, also called vectors for short and denoted by $\underline{x}$, with basis $\{e_1, \dots, e_n\}$; an element $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ can be identified with a vector $\underline{x} \in \mathbb{R}_n^1$ in the Clifford algebra using the map:

$$(x_1, x_2, \dots, x_n) \mapsto \underline{x} = x_1 e_1 + \dots + x_n e_n.$$

The subspace $\mathbb{R}_n^2$ consists of 2-vectors or bivectors, and has basis $\{e_{ij} = e_i e_j,\ i < j\}$. In general, for any subset $A = \{i_1, \dots, i_k\}$ of $N = \{1, \dots, n\}$ of cardinality $|A| = k$, the elements $e_A = e_{i_1} \dots e_{i_k}$, $i_1 < \dots < i_k$, form a basis for the $\binom{n}{k}$-dimensional vector space $\mathbb{R}_n^k$ of the k-vectors. Every element belonging to $\mathbb{R}_n^0 \oplus \mathbb{R}_n^1$ is a sum of a scalar and a vector. It is called paravector. An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ can be identified with a paravector $\mathbf{x} \in \mathbb{R}_n^0 \oplus \mathbb{R}_n^1$ by the map:

$$(x_0, x_1, \dots, x_n) \mapsto \mathbf{x} = x_0 + x_1 e_1 + \dots + x_n e_n.$$

Note also that every element $a \in \mathbb{R}_n$ may also be uniquely written as

$$a = [a]_0 + [a]_1 + \dots + [a]_k + \dots + [a]_n$$

where $[\cdot]_k : \mathbb{R}_n \to \mathbb{R}_n^k$ denotes the projection of $\mathbb{R}_n$ onto the space of k-vectors. Finally, a can be written in the form

$$a = a_+ + a_-$$

where $[a]_+ = [a]_0 + [a]_2 + \dots$, and $[a]_- = [a]_1 + [a]_3 + \dots$. We hence have a direct sum decomposition

$$\mathbb{R}_n = \mathbb{R}_{n,+} \oplus \mathbb{R}_{n,-}$$

where $\mathbb{R}_{n,+}$ is the even subalgebra generated by the bivectors e_{ij}, while $\mathbb{R}_{n,-}$ contains all the elements a that may be written in the form $a = -e_1(e_1 a)$, $e_1 a \in \mathbb{R}_{n,+}$. Note that $\mathbb{R}_{n,-}$ is not an algebra while $\mathbb{R}_{n,+}$ is an algebra isomorphic to $\mathbb{R}_{n-1}$.

Among the elements in the Clifford algebra $\mathbb{R}_n$, we can consider the product of all the imaginary units e_i:

Definition 2.1.7. *The product $e_N := e_1 \dots e_n$ is called pseudoscalar.*

Remark 2.1.8. If n is odd the pseudoscalar commutes with any element of the Clifford algebra $\mathbb{R}_n$ since it can be verified that

$$e_j e_N = e_N e_j,$$

while when n is even e_N anticommutes with any imaginary unit in the Clifford algebra:

$$e_j e_N = -e_N e_j.$$

As a consequence of the remark, we immediately have the following result:

Proposition 2.1.9. *The center of a Clifford algebra* $\mathbb{R}_n$ *is* $\mathbb{R}$ *for* n *even, while it is* $\mathbb{R} \oplus e_N\mathbb{R} = \{x + e_N y \mid x, y \in \mathbb{R}\}$ *for* n *odd.*

Proposition 2.1.10. *The Clifford algebra* $\mathbb{R}_n$, $n \geq 3$, *contains zero divisors.*

Proof. Since $n \geq 3$, $\mathbb{R}_n$ contains the element e_{123}. We have

$$(1 - e_{123})(1 + e_{123}) = 1 - e_{123} + e_{123} - e_{123}e_{123} = 1 - e_{123}^2 = 0. \qquad \square$$

In a Clifford algebra it is possible to introduce several involutions, but for our purposes we will simply consider the so-called conjugation:

Definition 2.1.11. *Let* $a, b \in \mathbb{R}_n$. *The conjugation is defined by*

$$\bar{e}_j = -e_j, \quad j = 1, \dots, n, \qquad \overline{ab} = \bar{b}\bar{a}.$$

As a consequence of the definition, for any $a \in \mathbb{R}_n$, $a = \sum a_A e_A$, we have

$$\bar{a} = \sum a_A \bar{e}_A = [a]_0 - [a]_1 - [a]_2 + [a]_3 + [a]_4 - \dots$$

i.e., for any $a \in \mathbb{R}_n^k$ we have the 4-periodicity

$$\bar{a} = a \quad \text{for} \quad k \equiv 0, 3 \bmod 4,$$
$$\bar{a} = -a \quad \text{for} \quad k \equiv 1, 2 \bmod 4.$$

The following properties of the conjugation can be easily verified by direct computation:

Proposition 2.1.12. *The conjugation of Clifford numbers satisfies:*

(1) $\bar{\bar{a}} = a$ *for all* $a \in \mathbb{R}_n$;

(2) $\overline{a + b} = \bar{a} + \bar{b}$ *for all* $a, b \in \mathbb{R}_n$;

(3) $a + \bar{a} = 2[a]_0$ *for all paravectors* a.

The conjugation allows us to introduce an inner product defined on the real linear space of Clifford numbers:

Proposition 2.1.13. *Let* $a, b \in \mathbb{R}_n$. *Then*

$$\langle a, b \rangle := [\bar{a}b]_0 = [b\bar{a}]_0 = [\bar{b}a]_0,$$

is a positive definite inner product on $\mathbb{R}_n$.

Proof. Let $a = \sum_A a_A e_A$, $b = \sum_B b_B e_B$. We have

$$\bar{a}b = (\overline{\sum_A a_A e_A})(\sum_B b_B e_B) = \sum_{A,B} a_A b_B \bar{e}_A e_B$$

and since $\bar{e}_A e_A = (-1)^{|A|(|A|+1)/2} e_A e_A = 1$ we obtain

$$\bar{a}b = \sum_A a_A b_A + \sum_{A \neq B} a_A b_B \bar{e}_A e_B$$

and so $[\bar{a}b]_0 = \sum_A a_A b_A$. Thus $[\bar{a}b]_0$ coincides with the scalar product of the vectors in $\mathbb{R}^{2^n}$ corresponding to the real components of a and b and it defines a scalar product. The fact that it coincides with $[b\bar{a}]_0$ and $[\bar{b}a]_0$ can be proved by similar computations. □

We note that the inner product defined by Proposition 2.1.13 behaves like a scalar product on the space of vectors and, if $\underline{x}$ and $\underline{y}$ are two vectors we have

$$\langle \underline{x}, \underline{y} \rangle = \frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}).$$

The wedge product of two vectors $\underline{x}$ and $\underline{y}$ is defined by

$$\underline{x} \wedge \underline{y} = \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}).$$

Note that the wedge product represents the directed and oriented surface measure of the parallelogram individuated by $\underline{x}$ and $\underline{y}$. It is also immediate that the product of two vectors can be written as

$$\underline{x}\ \underline{y} = \frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}) + \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}) = \langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y}. \tag{2.2}$$

Note also that, in the case of vectors, the scalar product can be written as

$$\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^{n} x_i y_i,$$

and, if by $|\underline{x}|$ we denote the Euclidean norm of a vector $\underline{x}$, we have

$$|\underline{x}| = \sqrt{\langle \underline{x}, \underline{x} \rangle} \tag{2.3}$$

which is the length of the vector $\underline{x}$.

We will say that two nonzero vectors $\underline{x}$, $\underline{y}$ are orthogonal if $\langle \underline{x}, \underline{y} \rangle = 0$. As customary, a basis $\{u_1, \ldots u_s\}$ of a subspace U of the Euclidean space $\mathbb{R}^n$ is said to be orthonormal if $|u_i| = 1$ and $\langle u_i, u_j \rangle = 0$ for every u_i, u_j, such that $u_i \neq u_j$.

In general, given an element $a = \sum_A a_A e_A \in \mathbb{R}_n$ we can define its modulus as

$$|a| = (\sum_A a_A^2)^{\frac{1}{2}}.$$

The proof of Proposition 2.1.13 shows that

$$|a|^2 = [a\bar{a}]_0 = \langle a, a \rangle,$$

thus generalizing formula (2.3) to the case of a general Clifford number. We have the following properties:

Proposition 2.1.14. *The modulus of Clifford numbers satisfies:*

(1) $|\lambda a| = |\lambda|\,|a|$ *for all* $\lambda \in \mathbb{R}$, $a \in \mathbb{R}_n$;

(2) $||x| - |y|| \leq |x - y| \leq |x| + |y|$;

However, the modulus is not multiplicative, as shown in the next result.

Proposition 2.1.15. *For any two elements* $a, b \in \mathbb{R}_n$ *we have*

$$|ab| \leq C_n |a|\,|b|$$

where C_n *is a constant depending only on the dimension of the Clifford algebra* $\mathbb{R}_n$. *Moreover, we have* $C_n \leq 2^{n/2}$.

Remark 2.1.16. The modulus is multiplicative in the case of complex numbers and quaternions. To have a multiplicative modulus when enlarging the field of real numbers one has to abandon the notion of order to get $\mathbb{C}$ and then the notion of commutativity to get $\mathbb{H}$. There is another possibility to enlarge further the dimension: by abandoning associativity one obtains the (division) algebra of octonions. In fact, Hurwitz' theorem shows that the only algebras over the real field with multiplicative modulus are the field of real numbers, the field of complex numbers, the quaternion skew field and the alternative algebra of octonions.

Inside a Clifford algebra there is the possibility, in some special cases, to have that the modulus is multiplicative. These cases are described in the following result:

Proposition 2.1.17. *Let* $b \in \mathbb{R}_n$ *be such that* $b\bar{b} = |b|^2$. *Then*

$$|ab| = |a||b|.$$

Proof. Consider $|ab|$. We have:

$$|ab|^2 = [ab\overline{ab}]_0 = [ab\bar{b}\bar{a}]_0 = [a|b|^2\bar{a}]_0 = [a\bar{a}]_0|b|^2 = |a|^2|b|^2. \qquad \square$$

Note that the result holds, in particular, when a is paravector $\mathbf{x}$. Moreover any nonzero paravector $\mathbf{x}$ admits an inverse, the so-called Kelvin inverse, defined by

$$\mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{|\mathbf{x}|^2}.$$

2.2 Slice monogenic functions: definition and properties

As mentioned in Section 2.1, an element $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be identified with a vector $\underline{x} = x_1e_1 + \ldots + x_ne_n \in \mathbb{R}^1_n$ while an element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ can be identified with the paravector

$$\mathbf{x} = x_0 + x_1e_1 + \ldots + x_ne_n = x_0 + \underline{x} \in \mathbb{R}^0_n \oplus \mathbb{R}^1_n.$$

In the sequel, with an abuse of notation, we will write $\underline{x} \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can be interpreted as a function of the paravector $\mathbf{x}$. Note also that an element $\mathbf{x}$ will be often denoted as

$$\mathbf{x} = \mathrm{Re}[\mathbf{x}] + \underline{x},$$

to emphasize its real and vector part, respectively.

The theory of slice monogenic functions was first developed in [26] where the authors study a new notion of monogenicity for functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}_n$. It is worth noting, however, that the exposition we propose here offers a significantly improved theory, and reorganizes the ideas of [26] in a new more powerful fashion as in [15], [18], [27], [28], [29], [53].

To introduce the theory of slice monogenic functions, we need some definitions and notation.

Definition 2.2.1. *We will denote by* $\mathbb{S}$ *the set of unit vectors:*

$$\mathbb{S} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_n^2 = 1\}.$$

From a geometric point of view, $\mathbb{S}$ is an $(n-1)$-sphere in the Euclidean space of vectors $\mathbb{R}^n$ and if $I \in \mathbb{S}$, then $I^2 = -1$.

The two-dimensional real subspace of $\mathbb{R}^{n+1}$ generated by 1 and I is the plane $\mathbb{R} + I\mathbb{R}$. It will be denoted by $\mathbb{C}_I$, in fact it is isomorphic to the complex plane. Note that the isomorphism between the vector space $\mathbb{C}_I$ and $\mathbb{C}$ is also an algebra isomorphism, thus $\mathbb{C}_I$ will be referred to as a "complex plane".

An element in $\mathbb{C}_I$ will be denoted by $u + Iv$. Conversely, given a paravector $\mathbf{x}$, it will be possible to write it as an element in a suitable complex plane $\mathbb{C}_I$. In fact, either $\mathbf{x}$ is a real number, or we can write it as $\mathbf{x} = \mathrm{Re}[\mathbf{x}] + \frac{\underline{x}}{|\underline{x}|}|\underline{x}|$. Since $\mathrm{Re}[\mathbf{x}]$, $|\underline{x}|$ are real numbers and $\frac{\underline{x}}{|\underline{x}|}$ is a unit vector, we have written the given paravector as $\mathbf{x} = u + I_{\mathbf{x}}v$, with $u = \mathrm{Re}[\mathbf{x}]$, $v = |\underline{x}|$ and $I_{\mathbf{x}} = \frac{\underline{x}}{|\underline{x}|}$.

Definition 2.2.2. *Let* $U \subseteq \mathbb{R}^{n+1}$ *be an open set and let* $f : U \to \mathbb{R}_n$ *be a real differentiable function. Let* $I \in \mathbb{S}$ *and let* f_I *be the restriction of* f *to the complex plane* $\mathbb{C}_I$ *and denote by* $u + Iv$ *an element on* $\mathbb{C}_I$*. We say that* f *is a left slice monogenic (for short* s-monogenic*) function if, for every* $I \in \mathbb{S}$*, we have*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = \frac{1}{2}\left(\frac{\partial}{\partial u} f_I(u + Iv) + I\frac{\partial}{\partial v} f_I(u + Iv)\right) = 0$$

on $U \cap \mathbb{C}_I$. We will denote by $\mathcal{M}(U)$ the set of left s-monogenic functions on the open set U or by $\mathcal{M}^L(U)$ when confusion may arise. We say that f is a right slice monogenic (for short right s-monogenic*) function if, for every $I \in \mathbb{S}$, we have*

$$\frac{1}{2}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}I\right) f_I(u+Iv) = \frac{1}{2}\left(\frac{\partial}{\partial u}f_I(u+Iv)+\frac{\partial}{\partial v}f_I(u+Iv)I\right) = 0,$$

on $U \cap \mathbb{C}_I$. We will denote by $\mathcal{M}^R(U)$ the set of right s-monogenic functions on the open set U.

Remark 2.2.3. The theory of right s-monogenic functions is equivalent to the theory of (left) s-monogenic functions. In the sequel, we will mainly consider s-monogenicity on the left, but we will introduce some basic tools for right s-monogenic functions in order to treat the functional calculus for n-tuples of noncommuting operators.

Definition 2.2.4. *We define the notion of I-derivative by means of the operator:*

$$\partial_I := \frac{1}{2}\left(\frac{\partial}{\partial u} - I\frac{\partial}{\partial v}\right).$$

For consistency, we will denote by $\overline{\partial}_I$ the operator $\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right)$.

Using the notation we have just introduced, the condition of left s-monogenicity will be expressed, in short, by

$$\overline{\partial}_I f = 0.$$

Right s-monogenicity will be expressed, with an abuse of notation, by

$$f\overline{\partial}_I = 0.$$

Remark 2.2.5. It is easy to verify that the (left) s-monogenic functions on $U \subseteq \mathbb{R}^{n+1}$ form a right $\mathbb{R}_n$-module. In fact it is trivial that if $f, g \in \mathcal{M}(U)$, then for every $I \in \mathbb{S}$ one has $\overline{\partial}_I f_I = \overline{\partial}_I g_I = 0$, thus $\overline{\partial}_I (f+g)_I = 0$. Moreover, for any $a \in \mathbb{R}_n$ we have $\overline{\partial}_I(f_I a) = (\overline{\partial}_I f)a = 0$. Analogously, the right s-monogenic functions on $U \subseteq \mathbb{R}^{n+1}$ form a left $\mathbb{R}_n$-module.

Definition 2.2.6. *Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Its s-derivative ∂_s is defined as*

$$\partial_s(f) = \begin{cases} \partial_I(f)(\mathbf{x}) & \mathbf{x} = u + Iv,\ v \neq 0, \\ \partial_u f(u) & u \in \mathbb{R}. \end{cases} \tag{2.4}$$

Note that the definition of s-derivative is well posed because it is applied only to s-monogenic functions. Moreover, for such functions, it coincides with the partial derivative with respect to the scalar component u, in fact we have:

$$\partial_s(f)(u+Iv) = \partial_I(f_I)(u+Iv) = \partial_u(f_I)(u+Iv). \tag{2.5}$$

Note incidentally that

$$\partial_u(f_I)(u+Iv) = \partial_u(f)(u+Iv).$$

Proposition 2.2.7. *Let U be an open set in $\mathbb{R}^{n+1}$ and let $f:\ U \to \mathbb{R}_n$ be an s-monogenic function. The s-derivative $\partial_s f$ of f is an s-monogenic function, moreover*

$$\partial_s^m f(u+Iv) = \frac{\partial^m f}{\partial u^m}(u+Iv).$$

Proof. The first part of the statement follows from

$$\overline{\partial}_I(\partial_s f(u+Iv)) = \partial_s(\overline{\partial}_I f(u+Iv)) = 0. \tag{2.6}$$

The second part follows from (2.5). □

We now provide some examples of s-monogenic functions. It is interesting to note that in the classical theory of monogenic functions (see [7], [34]) the monomials, and thus the polynomials, in the paravector variable are not monogenic functions. However polynomials (and also converging power series) in the paravector variable turn out to be s-monogenic functions, provided that the coefficients are written on the right.

Example 2.2.8. The monomials $\mathbf{x}^n a_n$, $a_n \in \mathbb{R}_n$ are s-monogenic, thus also the polynomials $\sum_{n=0}^N \mathbf{x}^n a_n$ are s-monogenic. Note that these polynomials have coefficients written on the right: indeed, polynomials with left coefficients are not, in general, s-monogenic. To avoid confusion, we will call polynomials of the form $\sum_{n=0}^N \mathbf{x}^n a_n$ s-monogenic polynomials. Moreover, as we will see in the sequel, any power series $\sum_{n\geq 0} \mathbf{x}^n a_n$ is s-monogenic in its domain of convergence.

Remark 2.2.9. Note that the complex plane $\mathbb{C} = \mathbb{R}_1$ can be seen both as $\mathbb{R}^2$ and as $\mathbb{R}_1$. It is immediate, from Definition 2.2.2, that the space of holomorphic functions $f:\ \mathbb{C} \to \mathbb{C}$ coincides with the space of s-monogenic functions from $\mathbb{R}^2$ to $\mathbb{R}_1$. For this reason we will consider the case $n > 1$ (obviously, all the results that we will prove are valid also in the case $n = 1$).

Proposition 2.2.10. *Let $I = I_1 \in \mathbb{S}$. It is possible to choose $I_2, \dots, I_n \in \mathbb{S}$ such that $I_1, \dots, I_n$ form an orthonormal basis for the Clifford algebra $\mathbb{R}_n$ i.e., they satisfy the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$.*

Proof. First of all, note that since $\underline{x} \wedge \underline{y} = -\underline{y} \wedge \underline{x}$, formula (2.2) gives

$$\underline{x}\,\underline{y} + \underline{y}\,\underline{x} = 2\langle \underline{x}, \underline{y}\rangle.$$

Then it sufficient to select the vectors I_r in a way such that $\langle I_r, I_r\rangle = -1$ and $\langle I_s, I_r\rangle = 0$, for $s = 1, \dots, n$, $r = 2, \dots, n$. Since $I_r = \sum_{\ell=1}^n x_{r\ell} e_\ell$ the two conditions translate into

$$\langle I_r, I_r\rangle = -\sum_{\ell=1}^n x_{r\ell}^2$$

and

$$\langle I_s, I_r \rangle = -\sum_{\ell=1}^{n} x_{s\ell} x_{r\ell}.$$

By identifying each vector I_r with its components $(x_1, \ldots, x_n) \in \mathbb{R}^n$ we conclude using the Gram–Schmidt algorithm. □

A simple and yet extremely important feature of s-monogenic functions is that their restrictions to a complex plane $\mathbb{C}_I$ can be written as a suitable linear combination of 2^{n-1} holomorphic functions, as proved in the following:

Lemma 2.2.11 (Splitting Lemma). *Let $U \subseteq \mathbb{R}^{n+1}$ be an open set. Let $f : U \to \mathbb{R}_n$ be an s-monogenic function. For every $I = I_1 \in \mathbb{S}$ let $I_2, \ldots, I_n$ be a completion to a basis of $\mathbb{R}_n$ satisfying the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$. Then there exist 2^{n-1} holomorphic functions $F_A : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for every $z = u + Iv$,*

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A, \quad I_A = I_{i_1} \ldots I_{i_s},$$

where $A = i_1 \ldots i_s$ is a subset of $\{2, \ldots, n\}$, with $i_1 < \ldots < i_s$, or, when $|A| = 0$, $I_\emptyset = 1$.

Proof. Let $z = u + Iv$. Since f is $\mathbb{R}_n$-valued, there are functions $F_A : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A = \sum_{|A|=0}^{n-1} (f_A + g_A I) I_A.$$

We now need to show that the functions F_A are holomorphic. Since f is s-monogenic we have that its restriction to $\mathbb{C}_I$ satisfies

$$\left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0$$

and so

$$\begin{aligned} &\sum \left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) (f_A + g_A I) I_A \\ &= \frac{\partial}{\partial u} f_A + I \frac{\partial}{\partial v} f_A + \frac{\partial}{\partial u} g_A I - \frac{\partial}{\partial v} g_A = 0. \end{aligned}$$

Since the imaginary units commute with any real-valued function, we obtain the system:

$$\begin{cases} \dfrac{\partial}{\partial u} f_A - \dfrac{\partial}{\partial v} g_A = 0, \\ \dfrac{\partial}{\partial v} f_A + \dfrac{\partial}{\partial u} g_A = 0 \end{cases}$$

for all multi-indices A. Therefore all the functions $F_A = f_A + g_A I$ satisfy the standard Cauchy–Riemann system and so they are holomorphic. □

Example 2.2.12. To clarify our result, we consider explicitly the case of $\mathbb{R}_4$-valued functions. A function $f : U \subseteq \mathbb{R}^5 \to \mathbb{R}_4$ can be written as

$$\begin{aligned} f =& f_0 + f_1 I_1 + f_2 I_2 + f_3 I_3 + f_4 I_4 + f_{12} I_{12} + f_{13} I_{13} + f_{14} I_{14} + f_{23} I_{23} \\ &+ f_{24} I_{24} + f_{34} I_{34} + f_{123} I_{123} + f_{124} I_{124} + f_{134} I_{134} + f_{234} I_{234} + f_{1234} I_{1234} \end{aligned}$$

and grouping as prescribed in the statement of the Lemma, we obtain

$$\begin{aligned} f = &(f_0 + f_1 I_1) + (f_2 + f_{12} I_1) I_2 + (f_3 + f_{13} I_1) I_3 + (f_4 + f_{14} I_1) I_4 \\ &+ (f_{23} + f_{123} I_1) I_{23} + (f_{24} + f_{124} I_1) I_{24} + (f_{34} + f_{134} I_1) I_{34} \\ &+ (f_{234} + f_{1234} I_1) I_{234}. \end{aligned}$$

To develop a meaningful theory of s-monogenic functions we need some additional hypotheses on the open sets on which they are defined. For example, the natural class of open sets in which we can prove the Identity Principle is given by the domains whose intersection with any complex plane $\mathbb{C}_I$ is connected. We introduce these domains in the following definition:

Definition 2.2.13. *Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a* slice domain (s-domain for short) *if $U \cap \mathbb{R}$ is nonempty and if $U \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.*

In this class of domains it is possible to prove the following Identity Principle:

Theorem 2.2.14 (Identity Principle). *Let U be an s-domain in $\mathbb{R}^{n+1}$. Let $f : U \to \mathbb{R}_n$ be an s-monogenic function, and let Z be its zero set. If there is an imaginary unit I such that $\mathbb{C}_I \cap Z$ has an accumulation point, then $f \equiv 0$ on U.*

Proof. Let us consider the restriction f_I of f to the plane $\mathbb{C}_I$, for $I \in \mathbb{S}$. By the Splitting Lemma we have

$$f_I(z) = \sum_{|A|=0}^{n-1} F_A(z) I_A$$

with $F_A : U \cap \mathbb{C}_I \to \mathbb{C}_I$ holomorphic for every multi-index A and $z = u + Iv$. Since $\mathbb{C}_I \cap Z$ has an accumulation point, we deduce that all the functions F_A vanish identically on $U \cap \mathbb{C}_I$ and thus $f_I = 0$ on $U \cap \mathbb{C}_I$. In particular f_I vanishes in the points of U on the real axis. Any other plane $\mathbb{C}_{I'}$ is such that $f_{I'}$ vanishes on $U \cap \mathbb{R}$ which has an accumulation point. If we apply the Splitting Lemma to $f_{I'}$, we can write $f_{I'} = \sum_{A'} F_{A'} I_{A'}$ and thus its components $F_{A'}$ vanish on $U \cap \mathbb{R}$ and thus they vanish identically on $U \cap \mathbb{C}_{I'}$. This fact implies that also $f_{I'}$ vanish on $\mathbb{C}_{I'}$, thus $f \equiv 0$ on U. □

Analogously to what happens in the complex case, we can prove the following consequence of the Identity Principle.

Corollary 2.2.15. *Let U be an s-domain in $\mathbb{R}^{n+1}$. Let $f, g : U \to \mathbb{R}_n$ be s-monogenic functions. If there is an imaginary unit I such that $f = g$ on a subset of $\mathbb{C}_I$ having an accumulation point, then $f \equiv g$ on U.*

Among the domains in $\mathbb{R}^{n+1}$ there is a special subclass which is useful to provide a Representation Formula for s-monogenic functions. In order to define them, it is useful to suitably denote the $(n-1)$-sphere associated to a paravector.

Let $\mathbf{s} = s_0 + \underline{s} = s_0 + I_{\mathbf{s}}|\underline{s}| \in \mathbb{R}^{n+1}$ be a paravector; we denote by $[\mathbf{s}\,]$ the set

$$[\mathbf{s}\,] = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \mathbf{x} = s_0 + I|\underline{s}|,\ I \in \mathbb{S}\}.$$

The set $[\mathbf{s}\,]$ is either reduced to a real point or it is the $(n-1)$-sphere defined by $\mathbf{s}$, i.e., the $(n-1)$-dimensional sphere with center at the real point s_0 and radius $|\underline{s}|$.

Remark 2.2.16. Observe that the relation: "$\mathbf{x} \sim \mathbf{s}$ if and only if $x_0 = s_0$ and $|\mathbf{x}| = |\mathbf{s}|$" is an equivalence relation. Given a paravector $\mathbf{s}$, its equivalence class contains only the element $\mathbf{s}$ when $\mathbf{s}$ is a real number, while it contains infinitely many elements when $\mathbf{s}$ is not real and corresponds to the $(n-1)$-dimensional sphere $[\mathbf{s}\,]$.

Definition 2.2.17. *Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is* axially symmetric *if, for all $\mathbf{s} = u + Iv \in U$, the whole $(n-1)$-sphere $[\mathbf{s}\,]$ is contained in U.*

Observe that axially symmetric sets are invariant under rotations that fix the real axis.

In order to state the next result we need some notation. Given an element $\mathbf{x} = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$I_{\mathbf{x}} = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} \text{ otherwise.} \end{cases}$$

We have the following:

Theorem 2.2.18 (Representation Formula). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let f be an s-monogenic function on U.*

(1) *For any vector $\mathbf{x} = u + I_{\mathbf{x}}v \in U$ the following formulas hold:*

$$f(\mathbf{x}) = \frac{1}{2}\Big[1 - I_{\mathbf{x}}I\Big]f(u + Iv) + \frac{1}{2}\Big[1 + I_{\mathbf{x}}I\Big]f(u - Iv) \tag{2.7}$$

and

$$f(\mathbf{x}) = \frac{1}{2}\Big[f(u + Iv) + f(u - Iv) + I_{\mathbf{x}}I[f(u - Iv) - f(u + Iv)]\Big]. \tag{2.8}$$

(2) *Moreover, the two quantities*

$$\alpha(u,v) := \frac{1}{2}[f(u+Iv)+f(u-Iv)] \tag{2.9}$$

and

$$\beta(u,v) := I\frac{1}{2}[f(u-Iv)-f(u+Iv)] \tag{2.10}$$

do not depend on $I \in \mathbb{S}$.

Proof. The result is trivial for real paravectors, in fact we have the identity

$$f(u) = \frac{1}{2}\Big[1 - I_{\mathbf{x}} I\Big] f(u) + \frac{1}{2}\Big[1 + I_{\mathbf{x}} I\Big] f(u)$$

for any $I_{\mathbf{x}} \in \mathbb{S}$. If $\mathbf{x} \notin \mathbb{R}$ and if we write $\mathbf{x} = u + I_{\mathbf{x}} v$ we can set

$$\phi(u+I_{\mathbf{x}}v) := \frac{1}{2}\Big[f(u+Iv)+f(u-Iv)+I_{\mathbf{x}}I[f(u-Iv)-f(u+Iv)]\Big],$$

and observe that if $I = I_{\mathbf{x}}$ we have

$$\phi(u+I_{\mathbf{x}}v) = f(\mathbf{x}).$$

Let us show that $\left(\frac{\partial}{\partial u} + I_{\mathbf{x}}\frac{\partial}{\partial v}\right)\phi(u+I_{\mathbf{x}}v) = 0$ for all $\mathbf{x} \in U \cap \mathbb{C}_I$. Indeed we have:

$$\begin{aligned}
&\Big(\frac{\partial}{\partial u} + I_{\mathbf{x}}\frac{\partial}{\partial v}\Big)\phi(u+I_{\mathbf{x}}v) \\
&= \frac{1}{2}\Big[1 - I_{\mathbf{x}}I\Big]\frac{\partial}{\partial u}f(u+Iv) + \frac{1}{2}\Big[1 + I_{\mathbf{x}}I\Big]\frac{\partial}{\partial u}f(u-Iv) \\
&\quad + \frac{1}{2}I_{\mathbf{x}}\Big[1 - I_{\mathbf{x}}I\Big]\frac{\partial}{\partial v}f(u+Iv) + \frac{1}{2}I_{\mathbf{x}}\Big[1 + I_{\mathbf{x}}I\Big]\frac{\partial}{\partial v}f(u-Iv).
\end{aligned}$$

Using the fact that f is s-monogenic, we can write

$$\begin{aligned}
&\Big(\frac{\partial}{\partial u} + I_{\mathbf{x}}\frac{\partial}{\partial v}\Big)\phi(u+I_{\mathbf{x}}v) \\
&= \frac{1}{2}\Big[1 - I_{\mathbf{x}}I\Big](-I)\frac{\partial}{\partial v}f(u+Iv) + \frac{1}{2}\Big[1 + I_{\mathbf{x}}I\Big]I\frac{\partial}{\partial v}f(u-Iv) \\
&\quad + \frac{1}{2}I_{\mathbf{x}}\Big[1 - I_{\mathbf{x}}I\Big]\frac{\partial}{\partial v}f(u+Iv) + \frac{1}{2}I_{\mathbf{x}}\Big[1 + I_{\mathbf{x}}I\Big]\frac{\partial}{\partial v}f(u-Iv) = 0.
\end{aligned}$$

Since the function ϕ is s-monogenic and $\phi \equiv f$ on $\mathbb{C}_I$, then ϕ coincides with f on U by the Identity Principle. The second part of the proof follows directly from

(2.8). In fact we have

$$\begin{aligned}
&\frac{1}{2}[f(u+Iv)+f(u-Iv)]\\
&=\frac{1}{2}\Big\{\frac{1}{2}\Big[f(u+Jv)+f(u-Jv)\Big]+I\frac{1}{2}\Big[J[f(u-Jv)-f(u+Jv)]\Big]\\
&\quad+\frac{1}{2}\Big[f(u+Jv)+f(u-Jv)\Big]-I\frac{1}{2}\Big[J[f(u-Jv)-f(u+Jv)]\Big]\Big\}\\
&=\frac{1}{2}\Big[f(u+Jv)+f(u-Jv)\Big]
\end{aligned}$$

and so α, and similarly β, depend on u, v only. □

Remark 2.2.19. Note that the operator $\overline{\partial}_I$ is not a constant coefficients differential operator since the imaginary unit I changes with the point $u+Iv$. This shows that f per se does not satisfy a system of constant coefficients differential equations; however, as the next corollary shows, its components α and β do, and they give an s-monogenic function if they satisfy some additional conditions, see [87].

Corollary 2.2.20. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain, and $D \subseteq \mathbb{R}^2$ be such that $u + Iv \in U$ whenever $(u, v) \in D$ and let $f : U \to \mathbb{R}_n$. The function f is an s-monogenic function if and only if there exist two differentiable functions $\alpha, \beta : D \subseteq \mathbb{R}^2 \to \mathbb{R}_n$ satisfying $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = -\beta(u, -v)$ and the Cauchy–Riemann system*

$$\begin{cases} \partial_u\alpha - \partial_v\beta = 0, \\ \partial_u\beta + \partial_v\alpha = 0, \end{cases} \tag{2.11}$$

and such that

$$f(u + Iv) = \alpha(u, v) + I\beta(u, v). \tag{2.12}$$

Proof. If f is s-monogenic, then we can apply Theorem 2.2.18 and we can set $\alpha(u, v)$ and $\beta(u, v)$ as in (2.9) and (2.10). Then $f(u+Iv) = \alpha(u, v)+I\beta(u, v)$, and $\alpha(u, v) = \alpha(u, -v)$, $\beta(u, v) = \beta(u, -v)$ by their definitions. The proof of Theorem 2.2.18 shows that the pair α, β satisfies the Cauchy–Riemann system. The converse is immediate: any function of the form $f(u+Iv) = \alpha(u, v)+I\beta(u, v)$ is well defined on an axially symmetric open set. In fact,

$$f(u - Iv) = \alpha(u, -v) + I\beta(u, -v) = \alpha(u, v) - I\beta(u, v).$$

The fact that α and β satisfy the Cauchy–Riemann system guarantees that f is an s-monogenic function. □

The Representation Formula has several interesting consequences.

Corollary 2.2.21. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}^{n+1}$ be an s-monogenic function. For any choice of $u, v \in \mathbb{R}$ such that $u+Iv \in U$ there exist $a, b \in \mathbb{R}_n$ such that*

$$f(u + Iv) = a + Ib, \tag{2.13}$$

for all $I \in \mathbb{S}$. In particular, the image of the $(n-1)$-sphere $[u+Iv]$ is the set $\{a+Ib : \ I \in \mathbb{S}\}$.

Proof. It is a direct application of Theorem 2.2.18. □

Another consequence of the Representation Formula is the fact that any holomorphic map defined on a suitable domain can be uniquely extended to an s-monogenic function:

Lemma 2.2.22 (Extension Lemma). *Let $J \in \mathbb{S}$ and let D be a domain in $\mathbb{C}_J$, symmetric with respect to the real axis and such that $D \cap \mathbb{R} \neq \emptyset$. Let U_D be the axially symmetric s-domain defined by*

$$U_D = \bigcup_{u+Jv \in D, \ I \in \mathbb{S}} (u+Iv).$$

If $f : \ D \to \mathbb{C}_J$ is holomorphic, then the function $\mathrm{ext}(f) : \ U_D \to \mathbb{R}_n$ defined by

$$\mathrm{ext}(f)(u+Iv) := \frac{1}{2}\Big[f(u+Jv)+f(u-Jv)\Big]+I\frac{1}{2}\Big[J[f(u-Jv)-f(u+Jv)]\Big] \quad (2.14)$$

is the unique s-monogenic extension of f to U_D.

Similarly, let $J_2, \dots, J_n$ be a completion of J to an orthonormal basis of $\mathbb{R}_n$ and let

$$f : \ D \to \mathbb{R}_n$$

defined by $f = \sum_{|A|=0}^{n-1} F_A J_A$, $A \subseteq \{2, \dots, n\}$, $F_A : \ D \to \mathbb{C}_J$ holomorphic. Then, $\overline{\partial}_J f(u+Jv) = 0$ and the function obtained by extending each of its holomorphic components F_A is the unique s-monogenic extension of f to U_D.

Proof. The fact that $\mathrm{ext}(f)$ is s-monogenic follows by the proof of Theorem 2.2.18. When $I = J$ in (2.14) we have that $\mathrm{ext}(f)(u+Jv) = f(u+Jv)$, and hence $\mathrm{ext}(f)$ is the unique extension of f by the Identity Principle. The second part is immediate. □

The second part of Theorem 2.2.18 shows that for every $I, K \in \mathbb{S}$ we have

$$f(u+Iv) = \alpha(u,v) + I\beta(u,v) \qquad \text{and} \qquad f(u+Kv) = \alpha(u,v) + K\beta(u,v).$$

By subtracting the two expressions and assuming that $I \neq K$, we have

$$\alpha(u,v) = (I-K)^{-1}[If(u+Iv) - Kf(u+Kv)]$$

and

$$\beta(u,v) = (I-K)^{-1}[f(u+Iv) - f(u+Kv)].$$

Thus the Representation Formula admits the following generalization:

Theorem 2.2.23 (Representation Formula, II). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let f be an s-monogenic function on U. For any vector $u + Jv \in U$ the following formula holds:*

$$f(u + Jv) = (I - K)^{-1}[If(u + Iv) - Kf(u + Kv)] \\ + J(I - K)^{-1}[f(u + Iv) - f(u + Kv)]. \tag{2.15}$$

As a consequence we have that the values of an s-monogenic function f on an axially symmetric set U are uniquely determined by its values on the two half-planes $U \cap \mathbb{C}^+_J$, $U \cap \mathbb{C}^+_K$ through formula (2.15). Moreover we have the following generalization of the extension lemma:

Lemma 2.2.24 (Extension Lemma, II). *Let U be an s-domain in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Let $\tilde{U}$ be the axially symmetric s-domain defined by*

$$\tilde{U} = \bigcup_{u+Jv\in U,\ I\in\mathbb{S}} (u + Iv)$$

There exists a unique s-monogenic extension of f to the whole $\tilde{U}$.

Proof. By construction, it is immediate that $\tilde{U}$ is an axially symmetric s-domain. Observing that U is an open set, we consider another axially symmetric s-domain W obtained as the union of all the open balls $B(x, r_x) \subset U$ with center at a point on the real axis $x \in U$, i.e.,

$$W = \cup_{x\in U\cap\mathbb{R}} B(x, r_x).$$

The restriction of f to W is an s-monogenic function which can be uniquely extended to a function $\tilde{f}$ defined on a maximal, axially symmetric, s-domain set $U_{\max}$ such that $W \subseteq U_{\max} \subseteq \tilde{U}$. Our goal is now to show that $U_{\max}$ coincides with $\tilde{U}$. Assume the contrary, and suppose that there exists $\mathbf{y} = y_0 + Iy_1 \in \tilde{U} \cap \partial U_{\max}$. Since $\mathbf{y} \in \tilde{U}$, there exists $J \in \mathbb{S}$ such that $y_0 + Jy_1 \in U$ and since U is open, there is an open ball with center at $\mathbf{y}$ contained in U. So there exist $K \in \mathbb{S}$ and $\tilde{\mathbf{y}} = y_0 + Ky_1$ such that the two discs Δ_J and Δ_K of radius ε with center at $\mathbf{y}$ and $\tilde{\mathbf{y}}$ on the plane $\mathbb{C}_J$, $\mathbb{C}_K$, respectively, are contained in U. Let us define

$$\tilde{g}(u + Jv) := (I - K)^{-1}[If(u + Iv) - Kf(u + Kv)] \\ + J(I - K)^{-1}[f(u + Iv) - f(u + Kv)]$$

on the set $D = \{\mathbf{x} = u + Jv \mid (u - y_0)^2 + (v - y_1)^2 < \varepsilon\}$. Then the function $\tilde{g}$ coincides with $\tilde{f}$ on $D \cap U_{\max}$. The function h defined by $h(\mathbf{x}) = \tilde{f}(\mathbf{x})$ for $\mathbf{x} \in U_{\max}$ and $h(\mathbf{x}) = \tilde{g}(\mathbf{x})$ for $\mathbf{x} \in D$ is the s-monogenic extension of f to the axially symmetric open set $D \cup U_{\max}$ contradicting the maximality of $U_{\max}$. This completes the proof. □

2.3 Power series

As we have already observed in the previous section, polynomials in the paravector variable $\mathbf{x}$ are s-monogenic. However, it is no longer true that a polynomial $f(\mathbf{x})$ of the form $f(\mathbf{x}) = (\mathbf{x} - a)^n$, where $a \in \mathbb{R}_n$ is s-monogenic, in general. If $a \in \mathbb{R}$, however, then $f(\mathbf{x})$ is s-monogenic and so are power series centered at a point on the real axis, where they converge. In this section we will provide a detailed study of s-monogenic functions which can be expanded into power series.

Proposition 2.3.1. *If $B = B(0, R) \subseteq \mathbb{R}^{n+1}$ is a ball centered in 0 with radius $R > 0$, then $f : B \to \mathbb{R}_n$ is an s-monogenic function if and only if f has a series expansion of the form*

$$f(\mathbf{x}) = \sum_{m \geq 0} \mathbf{x}^m \frac{1}{m!} \frac{\partial^m f}{\partial u^m}(0) \tag{2.16}$$

converging on B.

Proof. If a function admits a series expansion as in (2.16) it is obviously s-monogenic where the series converges. The converse requires the Splitting Lemma. Consider an element $I = I_1 \in \mathbb{S}$ and the corresponding plane $\mathbb{C}_I$. Let $\Delta \subset \mathbb{C}_I$ be a disc with center in the origin and radius $r < R$ and let us set $z = u + Iv$. The restriction of f to the plane $\mathbb{C}_I$ can be written as $f_I(z) = \sum F_A(z) I_A$. Since every function $F_A(z)$ is holomorphic, it admits an integral representation via the Cauchy formula, i.e.,

$$F_A(z) = \frac{1}{2\pi I} \int_{\partial\Delta(0,r)} \frac{F_A(\zeta)}{\zeta - z}\, d\zeta,$$

for any $z \in \Delta$ and therefore

$$f_I(z) = \sum_{|A|=0}^{n-1} \left(\frac{1}{2\pi I} \int_{\partial\Delta(0,r)} \frac{F_A(\zeta)}{\zeta - z}\, d\zeta \right) I_A.$$

Now observe that ζ and z commute because they lie on the same plane $\mathbb{C}_I$, so we can expand the denominator in each integral in power series, as in the classical case:

$$\begin{aligned} F_A(z) &= \frac{1}{2\pi I} \int_{\partial\Delta(0,r)} \sum_{m \geq 0} \left(\frac{z}{\zeta}\right)^m \frac{F_A(\zeta)}{\zeta}\, d\zeta \\ &= \sum_{m \geq 0} z^m \int_{\partial\Delta(0,r)} \sum_{m \geq 0} \frac{F_A(\zeta)}{\zeta^{m+1}}\, d\zeta \\ &= \sum_{m \geq 0} z^m \frac{1}{m!} \frac{\partial^m F_A}{\partial z^m}(0). \end{aligned}$$

Plugging this expression into $f_I(z) = \sum F_A(z)I_A$ we obtain

$$f_I(z) = \sum_{|A|=0}^{n-1} \sum_{m\geq 0} z^m \frac{1}{m!} \frac{\partial^m F_A}{\partial z^m}(0) I_A = \sum_{m\geq 0} z^m \frac{1}{m!} \frac{\partial^m f}{\partial z^m}(0),$$

and using the definition of s-derivative together with Proposition 2.2.7, we get

$$\sum_{m\geq 0} z^m \frac{1}{m!} \frac{1}{2} \left(\frac{\partial}{\partial u} - I \frac{\partial}{\partial v} \right)^m f(0) = \sum_{m\geq 0} z^m \frac{1}{m!} \frac{\partial^m}{\partial u^m} f(0).$$

Finally observe that the coefficients of the power series do not depend on the choice of the unit I, thus $f_I(z)$ is the restriction to $\mathbb{C}_I$ of the function defined in (2.16) and the statement follows. □

The following two results can be proved as in the complex case:

Proposition 2.3.2. *The s-derivative of a power series*

$$\sum_{n\geq 0} \mathbf{x}^n a_n, \quad a_n \in \mathbb{R}_n$$

equals

$$\sum_{n\geq 0} n\mathbf{x}^{n-1} a_n$$

and has the same radius of convergence of the original series.

Corollary 2.3.3. *Let $f: B \to \mathbb{R}_n$ be an s-monogenic function. Then $f \equiv 0$ on B if and only if $\partial_s^n f(0) = 0$ for all $n \in \mathbb{N}$.*

The next proposition shows that s-monogenic functions whose power series expansion have real coefficients play a privileged role.

Proposition 2.3.4. *The product of two functions $f, g : B(0,R) \to \mathbb{R}_n$ such that the series expansion of f has real coefficients is an s-monogenic function. Moreover, the composition of f with an s-monogenic function $h : B(0,R') \to \mathbb{R}_n$ is an s-monogenic function whenever the composition is defined.*

Proof. Let

$$f(\mathbf{x}) = \sum_{m\geq 0} \mathbf{x}^m a_m,$$

$$g(\mathbf{x}) = \sum_{m\geq 0} \mathbf{x}^m b_m,$$

$$h(\mathbf{x}) = \sum_{m\geq 0} \mathbf{x}^m c_m,$$

be s-monogenic functions with $a_m \in \mathbb{R}$, $b_m, c_m \in \mathbb{R}_n$. Since real coefficients commute with the variable $\mathbf{x}$ we have

$$(fg)(\mathbf{x}) = \sum_{s\geq 0} \mathbf{x}^s (a_0 b_s + a_1 b_{s-1} + \ldots + a_s b_0).$$

Now consider $(h \circ f)(\mathbf{x}) = h(f(\mathbf{x}))$; we have

$$h(f(\mathbf{x})) = \sum_{m\geq 0} (\sum_{r\geq 0} \mathbf{x}^r a_r)^m c_m.$$

Since the coefficients a_r commute with the variables we can group them on the right and the statement follows. □

Corollary 2.3.5. *Let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Then the function $f(\mathbf{x} - y_0)$, $y_0 \in \mathbb{R}$, is an s-monogenic function in the open set $U' = \{\mathbf{x}' = \mathbf{x} - y_0,\ \mathbf{x} \in U\}$.*

Proposition 2.3.6. *Let $B = B(y_0, R) \subseteq \mathbb{R}^{n+1}$ be the ball centered in $y_0 \in \mathbb{R}$ with radius $R > 0$, then $f : B \to \mathbb{R}_n$ is an s-monogenic function if and only if it has a series expansion of the form*

$$f(\mathbf{x}) = \sum_{m\geq 0} (\mathbf{x} - y_0)^m \frac{1}{m!} \frac{\partial^m f}{\partial u^m}(y_0). \tag{2.17}$$

Proof. Consider the transformation of coordinates $\mathbf{z} = \mathbf{x} - y_0$. Since the function $f(\mathbf{z})$ is s-monogenic in a ball centered in the origin with radius $R > 0$, we can apply Proposition 2.3.1. Using the inverse transformation $\mathbf{x} = \mathbf{z} + y_0$, we obtain the statement. □

The result extends to s-domains as follows:

Corollary 2.3.7. *Let f be an s-monogenic function on an s-domain $U \subseteq \mathbb{R}^{n+1}$. Then for any point on the real axis y_0 in U, the function f can be represented in power series*

$$f(\mathbf{x}) = \sum_{n\geq 0} (\mathbf{x} - y_0)^n \frac{1}{n!} \frac{\partial^n f}{\partial u^n}(y_0)$$

on the ball $B(y_0, R)$, where $R = R_{y_0}$ is the largest positive real number such that $B(y_0, R)$ is contained in U.

Proof. Since f is s-monogenic in y_0, then, for every $I \in \mathbb{S}$, f can be expanded in power series on the disc $\Delta_I = B(y_0, R_I)$ of radius R_I on the plane $\mathbb{C}_I$. The radius R turns out to be $\min_{I\in\mathbb{S}} R_I$ which is nonzero because y_0 is an internal point in U. □

Corollary 2.3.8. *Let $f : B(y_0, R) \to \mathbb{R}_n$ be an s-monogenic function. If there exists $I \in \mathbb{S}$ such that $f(\mathbb{C}_I) \subseteq \mathbb{C}_I$, then the series expansion of f,*

$$f(\mathbf{x}) = \sum_{n\geq 0} (\mathbf{x} - y_0)^n \frac{1}{n!} \frac{\partial^n f}{\partial u^n}(y_0),$$

has all its coefficients in $\mathbb{C}_I$. Consequently, if there are two different units $I, J \in \mathbb{S}$ such that $f(\mathbb{C}_I) \subseteq \mathbb{C}_I$ and $f(\mathbb{C}_J) \subseteq \mathbb{C}_J$, then the coefficients are real.

Proof. If $I \in \mathbb{S}$ is such that $f(\mathbb{C}_I) \subseteq \mathbb{C}_I$, then for any real number y_0 we have $f(y_0) = f_I(y_0) \in \mathbb{C}_I$. Therefore $\dfrac{\partial^n f}{\partial u^n}(y_0) \in \mathbb{C}_I$ for any $n \in \mathbb{N}$, $y_0 \in \mathbb{R}$, and the conclusion follows. The second part is immediate. □

We now introduce a product among s-monogenic polynomials which preserves the s-monogenicity:

Definition 2.3.9. *Let $f(\mathbf{x}) = \sum_{i=0}^{n} \mathbf{x}^i a_i$ and $g(\mathbf{x}) = \sum_{i=0}^{m} \mathbf{x}^i b_i$, for $a_i, b_i \in \mathbb{R}_n$. We define the s-monogenic product of f and g as*

$$f * g(\mathbf{x}) := \sum_{j=0}^{n+m} \mathbf{x}^j c_j$$

with $c_j = \sum_{i+k=j} a_i b_k$. We will denote by f^{*n} the product $f * \ldots * f$, n-times.

This product is computed by taking the coefficients of the polynomials on the right, as in the case in which the variables and the coefficients commute and coincides with the standard product of polynomials with coefficients in a division algebra (see [71]). We adopt this definition also in this setting and we extend it to the case of the product of series. If $f(\mathbf{x}) = \sum_{i\geq 0} \mathbf{x}^i a_i$ and $g(\mathbf{x}) = \sum_{i\geq 0} \mathbf{x}^i b_i$ are s-monogenic series, we define their s-monogenic product as

$$f * g(\mathbf{x}) := \sum_{j\geq 0} \mathbf{x}^j c_j$$

with $c_j = \sum_{i+k=j} a_i b_k$. Note that when the coefficients of a polynomial or a series f are real numbers, the s-monogenic product coincides with the usual product, i.e., $f * g = fg$ (see Proposition 2.3.4). This product will be generalized in the sequel to s-monogenic functions which are not necessarily power series.

We conclude this section by showing that s-monogenic functions are infinitely differentiable:

Proposition 2.3.10. *An s-monogenic function $f : U \to \mathbb{R}_n$ on an axially symmetric s-domain $U \subseteq \mathbb{R}^{n+1}$ is infinitely differentiable on U.*

Proof. The differentiability of f on the real axis follows from Corollary 2.3.7 since for any point of the real axis there is a ball in which the function f can be expressed in power series. To prove differentiability outside the real axis consider formula (2.7) and write $\mathbf{x}$ as $\mathbf{x} = x_0 + \underline{x} = x_0 + \frac{\underline{x}}{|\underline{x}|}\underline{x}$:

$$f(\mathbf{x}) = \frac{1}{2}\Big[f(x_0 + I|\underline{x}|) + f(x_0 - I|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}I[f(x_0 - I|\underline{x}|) - f(x_0 + I|\underline{x}|]\Big].$$

The function f is s-monogenic and hence, by definition, its restriction f_I to $\mathbb{C}_I$ is infinitely differentiable on $U \cap \mathbb{C}_I$ for any $I \in \mathbb{S}$. It is therefore obvious that f can be obtained as a composition of the functions f_I, x_0, $\underline{x} = \sum_\ell e_\ell x_\ell$, and $|\underline{x}|$, which are all infinitely differentiable outside the real axis with respect to the variables x_ℓ, $\ell = 0, \dots, n$. This concludes the proof. $\square$

2.4 Cauchy integral formula, I

A main result in the theory of s-monogenic functions is an analog of the Cauchy integral formula. We will present two versions of such a Cauchy formula: the one discussed in this section is less general than the second version, but it is enough to prove several properties of s-monogenic functions.

Theorem 2.4.1. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. If $\mathbf{x} \in U$, then*

$$f(\mathbf{x}) = \frac{1}{2\pi}\int_{\partial\Delta_{\mathbf{x}}(a,r)} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_{\mathbf{x}}} f(\zeta)$$

where $d\zeta_{I_{\mathbf{x}}} := -d\zeta I_{\mathbf{x}}$ and $a \in \mathbb{R}$, $r > 0$ are such that

$$\overline{\Delta_{\mathbf{x}}(a,r)} = \{u + I_{\mathbf{x}} v \mid (u-a)^2 + v^2 \leq r^2\} \subset \mathbb{C}_{I_{\mathbf{x}}}$$

contains $\mathbf{x}$ and is contained in U.

Proof. With no loss of generality, we will assume $a = 0$. Consider the integral

$$\frac{1}{2\pi}\int_{\partial\Delta_{\mathbf{x}}(0,r)} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f(\xi).$$

Set $I_{\mathbf{x}} := I_1$, complete to a basis $I_1, \dots, I_n$ of the Clifford algebra $\mathbb{R}_n$, satisfying the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$. Using the Splitting Lemma, we can write the restriction of f to $\mathbb{C}_{I_{\mathbf{x}}}$ as $f_{I_{\mathbf{x}}} = \sum_A F_A I_A$. We have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial\Delta_{\mathbf{x}}(0,r)} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f_{I_{\mathbf{x}}}(\xi)\\
&= \frac{1}{2\pi}\int_{\partial\Delta_{\mathbf{x}}(0,r)} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} \sum_A F_A(\xi) I_A
\end{aligned}$$

$$
\begin{aligned}
&= \sum_A \frac{1}{2\pi} \int_{\partial \Delta_{\mathbf{x}}(0,r)} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} F_A(\xi) I_A \\
&= \sum_A F_A(\mathbf{x}) I_A \\
&= f(\mathbf{x}).
\end{aligned}
$$

□

Remark 2.4.2. Let $B_1 = B(0, R_1)$, $B_2 = B(0, R_2)$ be two balls centered at the origin and with radii $0 < R_1 < R_2$. The same argument used in the previous proof shows that if a function f is s-monogenic in a neighborhood of the annular domain $B_2 \setminus B_1$, then for any $\mathbf{x} \in B_2 \setminus B_1$, it satisfies

$$
\begin{aligned}
f(\mathbf{x}) = \frac{1}{2\pi} &\int_{\partial(B_2 \cap \mathbb{C}_{I_{\mathbf{x}}})} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_{\mathbf{x}}} f(\zeta) \\
&- \frac{1}{2\pi} \int_{\partial(B_1 \cap \mathbb{C}_{I_{\mathbf{x}}})} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_{\mathbf{x}}} f(\zeta).
\end{aligned}
$$

Remark 2.4.3. The function $\mathcal{I}_{\mathbf{y}}(\mathbf{x}) := (\mathbf{x} - \mathbf{y})^{-1}$ corresponding to the Cauchy kernel in Theorem 2.4.1 is not s-monogenic on $\mathbb{R}^{n+1} \setminus \{\mathbf{y}\}$, unless $\mathbf{y} = y_0 \in \mathbb{R}$. In particular, the function

$$
\mathcal{I}_0(\mathbf{x}) = \mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{|\mathbf{x}|^2} \tag{2.18}
$$

is s-monogenic in $\mathbb{R}^{n+1} \setminus \{0\}$.

Theorem 2.4.4 (Cauchy formula outside a ball). *Let $B = B(0, R)$ and let $B^c = \mathbb{R}^{n+1} \setminus \overline{B}$. Let $f : B^c \to \mathbb{R}_n$ be an s-monogenic function with $\lim_{\mathbf{x} \to \infty} f(\mathbf{x}) = a$. If $\mathbf{x} \in B^c$, then*

$$
f(\mathbf{x}) = a - \frac{1}{2\pi} \int_{\partial \Delta_{\mathbf{x}}(0,r)} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_{\mathbf{x}}} f(\zeta)
$$

where $0 < R < r < |\mathbf{x}|$ and the complement of the set $\overline{\Delta_{\mathbf{x}}(0, r)}$ is contained in B^c and contains $\mathbf{x}$.

Proof. The proof is based on the Splitting Lemma and on the analogous result for holomorphic functions of a complex variable. Let $\mathbf{x} \in \mathbb{R}^{n+1} \setminus \overline{B}$ and let $I_{\mathbf{x}}$ be the corresponding imaginary unit. Consider $r' > r > R$, and the discs $\Delta = \Delta_{\mathbf{x}}(0, r)$, $\Delta' = \Delta_{\mathbf{x}}(0, r')$ on the plane $\mathbb{C}_{I_{\mathbf{x}}}$ having radius r and r' respectively and such that $\mathbf{x} \in \Delta'$. Since f is s-monogenic on $\Delta' \setminus \Delta$ we can apply the Cauchy formula to the set $\overline{\Delta'} \setminus \Delta$ to compute $f(\mathbf{x})$. We obtain

$$
\begin{aligned}
f(\mathbf{x}) &= \frac{1}{2\pi} \int_{\partial \Delta' \setminus \partial \Delta} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f(\xi) \\
&= \frac{1}{2\pi} \int_{\partial \Delta'} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f(\xi) \\
&\quad - \frac{1}{2\pi} \int_{\partial \Delta} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f(\xi).
\end{aligned}
$$

Let us set $I_1 := I_{\mathbf{x}}$ and complete to an orthonormal basis $I_1, \dots, I_n$ of the Clifford algebra $\mathbb{R}_n$. The Splitting Lemma gives $f_{I_{\mathbf{x}}} = \sum_A F_A I_A$ and we can write

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2\pi} \int_{\partial\Delta'} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f(\xi) \\ &\quad - \frac{1}{2\pi} \int_{\partial\Delta} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} f(\xi) \\ &= \sum_A \frac{1}{2\pi} \int_{\partial\Delta'} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} F_A(\xi) I_A \\ &\quad - \sum_A \frac{1}{2\pi} \int_{\partial\Delta} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} F_A(\xi) I_A. \end{aligned}$$

Let us now consider a single component F_A at a time. By computing the integral on $\partial\Delta'$ in spherical coordinates, and by letting $r' \to \infty$, we obtain that the integral equals $a_A = \lim_{r' \to \infty} F_A$, and therefore:

$$F_A(\mathbf{x}) = a_A - \frac{1}{2\pi} \int_{\partial\Delta} (\xi - \mathbf{x})^{-1} d\xi_{I_{\mathbf{x}}} F_A(\xi).$$

Taking the sum of the various components multiplied with the corresponding units I_A we get the statement with $a = \sum_A a_A I_A$. □

Theorem 2.4.5 (Cauchy estimates). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Let $y_0 \in U \cap \mathbb{R}$, $I \in \mathbb{S}$, and $r > 0$ be such that $\overline{\Delta_I(y_0, r)} = \{(u + Iv) : (u - y_0)^2 + v^2 \leq r^2\}$ is contained in $U \cap \mathbb{C}_I$. If $M_I = \max\{|f(\mathbf{x})| : \mathbf{x} \in \partial\Delta_I(y_0, r)\}$ and if $M = \inf\{M_I : I \in \mathbb{S}\}$, then*

$$\frac{1}{n!} \left| \frac{\partial^n f}{\partial u^n}(y_0) \right| \leq \frac{M}{r^n}, \quad n \geq 0.$$

Proof. For any $I \in \mathbb{S}$, it is possible to write

$$\frac{1}{n!} \frac{\partial^n f}{\partial u^n}(y_0) = \frac{1}{2\pi I} \int_{\partial\Delta_I(y_0, r)} \frac{d\zeta}{(\zeta - y_0)^{n+1}} f(\zeta).$$

Therefore, for any $I \in \mathbb{S}$ we can write the following sequence of inequalities:

$$\begin{aligned} \frac{1}{n!} \left| \frac{\partial^n f}{\partial u^n}(y_0) \right| &\leq \frac{1}{2\pi} \int_{\partial\Delta_I(y_0, r)} \frac{|f(\zeta)|}{r^{n+1}} d\zeta \\ &\leq \frac{1}{2\pi} \int_{\partial\Delta_I(y_0, r)} \frac{M_I}{r^{n+1}} d\zeta = \frac{M_I}{r^n}. \end{aligned}$$

By taking the infimum, for $I \in \mathbb{S}$, of the right-hand side of the inequality we prove the assertion. □

Using the previous result it is immediate to show the following

Theorem 2.4.6 (Liouville). *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be an entire s-monogenic function. If f is bounded, then f is constant on $\mathbb{R}^{n+1}$.*

Proof. Suppose that $|f| \leq M$ on $\mathbb{R}^{n+1}$. By the previous theorem we have:

$$\frac{1}{n!}\left|\frac{\partial^n f}{\partial u^n}(0)\right| \leq \frac{M}{r^n}, \quad n \geq 0,$$

and by letting $r \to +\infty$ we obtain

$$\frac{\partial^n f}{\partial u^n}(0) = 0$$

for any $n > 0$, which implies $f(\mathbf{x}) = c$, with $c \in \mathbb{R}_n$. □

Corollary 2.4.7. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ be an entire s-monogenic function. If $\lim_{\mathbf{x}\to\infty} f$ exists, then f is constant on $\mathbb{R}^{n+1}$.*

Theorem 2.4.8. *Let U be an open set in $\mathbb{R}^{n+1}$. If $f : U \to \mathbb{R}_n$ is an s-monogenic function, then*

$$\int_{\partial\Delta} d\mathbf{x} f(\mathbf{x}) = 0$$

for any disc $\Delta \subset U \cap \mathbb{C}_I$ with center in a point on the real axis.

Proof. This result is an easy consequence of the analogous result for holomorphic functions of one complex variable and of the Splitting Lemma. □

Conversely, we have the following result:

Theorem 2.4.9. *Let U be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Assume that*

$$\int_{\gamma_I} d\mathbf{x} f(\mathbf{x}) = 0$$

for any closed, piecewise $\mathcal{C}^1$ curve γ_I contained in $U \cap \mathbb{C}_I$ and homotopic to a point. Then f is an s-monogenic function.

Proof. This is a consequence of the classical Morera's theorem and of the definition of s-monogenic function. □

Proposition 2.4.10. *Let $f : B(0, R) \to \mathbb{R}_n$ be the s-monogenic function expressed by the series $\sum \mathbf{x}^m a_m$ converging on B. Then the composition of the functions f and $\mathcal{I}_0 = \mathbf{x}^{-1}$ is s-monogenic on $\mathbb{R}^{n+1} \setminus B(0, 1/R)$ and it can be expressed by the series $\sum \mathbf{x}^{-m} a_m$ converging on $\mathbb{R}^{n+1} \setminus B(0, 1/R)$.*

Proof. Proposition 2.3.4 implies that $f \circ \mathcal{I}_0$ is an s-monogenic function on $\mathbb{R}^{n+1} \setminus B(0, 1/R)$. The statement follows from the analogous result for holomorphic functions in one complex variable. □

Theorem 2.4.11 (Laurent series). *Let f be an s-monogenic function in a spherical shell $A = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid R_1 < |\mathbf{x}| < R_2\}$, $0 < R_1 < R_2$. Then f admits the unique Laurent expansion*

$$f(\mathbf{x}) = \sum_{m\geq 0} \mathbf{x}^m a_m + \sum_{m\geq 1} \mathbf{x}^{-m} b_m \tag{2.19}$$

where

$$a_m = \frac{1}{m!}\partial_s^m f(0), \qquad b_m = \frac{1}{2\pi}\int_{\partial(B(0,R_1')\cap\mathbb{C}_{I_\mathbf{x}})} \zeta^{m-1} d\zeta_{I_\mathbf{x}} f(\zeta).$$

The two series in (2.19) *converge in the open ball $B(0, R_2)$ and in $\mathbb{R}^{n+1}\setminus\overline{B(0, R_1)}$, respectively.*

Proof. Let $\mathbf{x} \in A$, then there exist two positive real numbers R_1', R_2' such that $A' = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid R_1' < |\mathbf{x}| < R_2'\} \subset A$, and $\mathbf{x} \in A'$. Using the Cauchy integral formula, we can write

$$f(\mathbf{x}) = \frac{1}{2\pi}\int_{\partial(A'\cap\mathbb{C}_{I_\mathbf{x}})} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_\mathbf{x}} f(\zeta) = f_1(\mathbf{x}) + f_2(\mathbf{x})$$

where

$$f_1(\mathbf{x}) = \frac{1}{2\pi}\int_{\partial(B(0,R_2')\cap\mathbb{C}_{I_\mathbf{x}})} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_\mathbf{x}} f(\zeta)$$

and

$$f_2(\mathbf{x}) = -\frac{1}{2\pi}\int_{\partial(B(0,R_1')\cap\mathbb{C}_{I_\mathbf{x}})} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_\mathbf{x}} f(\zeta).$$

The first integral is associated to the first series in the Laurent expansion, by Proposition 2.3.1. Let us consider the second integral, set $I_1 = I_\mathbf{x}$ and let us use the Splitting Lemma and write f_2 as $\sum_A F_A I_A$. Now we can reason as in the case of functions in one complex variable, and consider the single components of $f_2(\mathbf{x})$. In $\mathbb{R}^{n+1} \setminus \overline{B(0, R_1')}$, we have

$$\begin{aligned} F_A(\mathbf{x}) &= -\frac{1}{2\pi}\int_{\partial(B(0,R_1')\cap\mathbb{C}_{I_\mathbf{x}})} (\zeta - \mathbf{x})^{-1}\, d\zeta_{I_\mathbf{x}} F_A(\zeta) \\ &= \frac{1}{2\pi}\int_{\partial(B(0,R_1')\cap\mathbb{C}_{I_\mathbf{x}})} \sum_{m\geq 0} \mathbf{x}^{-m-1}\zeta^m d\zeta_{I_\mathbf{x}} F_A(\zeta) \end{aligned}$$

where we have used the fact that on the plane $\mathbb{C}_{I_\mathbf{x}}$ the variables ζ and $\mathbf{x}$ commute. Now, using the uniform convergence of the series we can write

$$F_A(\mathbf{x}) = \sum_{m\geq 0} \mathbf{x}^{-m-1}\frac{1}{2\pi}\int_{\partial(B(0,R_1')\cap\mathbb{C}_{I_\mathbf{x}})} \zeta^m d\zeta_{I_\mathbf{x}} F_A(\zeta) = \sum_{m\geq 0} \mathbf{x}^{-m-1} b_{m+1,A}$$

where

$$b_{m+1,A} := b_{m+1,I_{\mathbf{x}};A} = \frac{1}{2\pi}\int_{\partial(B(0,R_1')\cap\mathbb{C}_{I_{\mathbf{x}}})} \zeta^m d\zeta_{I_{\mathbf{x}}} F_A(\zeta).$$

Finally, we obtain:

$$\tilde{f}_2(\mathbf{x}) = \sum_A F_A(\mathbf{x}) I_A = \sum_{m\geq 0}\sum_A \mathbf{x}^{-m-1} b_{m+1,A} I_A.$$

Note that $\tilde{f}_2(\mathbf{x})$ coincides with $f_2(\mathbf{x})$ on the plane $\mathbb{C}_{I_{\mathbf{x}}}$, thus they coincide everywhere and the coefficients $b_{m+1,A}$ do not depend on the choice of the imaginary unit $I_{\mathbf{x}}$. The statement follows. □

2.5 Zeros of slice monogenic functions

As it is well known, the Fundamental Theorem of Algebra does not hold in $\mathbb{R}_n$ for $n \geq 3$, thus we cannot guarantee that a polynomial in the paravector variable $\mathbf{x}$ has a zero, not even if it is a degree-one polynomial. The following examples are instructive to show what can happen in a Clifford algebra.

Example 2.5.1. Consider the Clifford algebra $\mathbb{R}_n$, $n \geq 2$ and the polynomial $p(\mathbf{x}) = \mathbf{x}e_1 - e_2 \in \mathbb{R}_n[\mathbf{x}]$. The only zero of p is e_1e_2 which does not belong to $\mathbb{R}^{n+1}$.

Example 2.5.2. Consider the Clifford algebra $\mathbb{R}_n$, $n \geq 2$ and the polynomial $p(\mathbf{x}) = \mathbf{x}^2 - \mathbf{x}(e_1e_2 - 2e_1) + 2e_2 \in \mathbb{R}_n[\mathbf{x}]$. It can be easily verified that p vanishes for $\mathbf{x} = -2e_1$ and $\mathbf{x} = -\frac{1}{5}(4e_1 + 3e_1e_2)$. However, only $\mathbf{x} = -2e_1$ is a zero of p in $\mathbb{R}^{n+1}$.

Example 2.5.3. Consider the Clifford algebra $\mathbb{R}_n$, $n \geq 2$ and the polynomial $p(\mathbf{x}) = \mathbf{x}^2 - \mathbf{x}(e_1 + 2e_2) + 2e_1e_2 \in \mathbb{R}_n[\mathbf{x}]$. It can be easily verified that both $\mathbf{x} = e_1$ and $\mathbf{x} = \frac{1}{5}(8e_1 + 6e_2)$ are zeros of p in $\mathbb{R}^{n+1}$.

It is nevertheless interesting to attempt to characterize the set of zeros for those polynomials for which such a set is not empty. Let us start by showing that each $(n-1)$-sphere $[\mathbf{s}]$ is characterized by a second degree equation.

Proposition 2.5.4. *Let $\mathbf{s} = s_0 + \underline{s} \in \mathbb{R}^{n+1}$. Consider the equation*

$$\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2 = 0. \tag{2.20}$$

Then, $\mathbf{x} = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ is a solution if and only if $\mathbf{x} \in [\mathbf{s}]$.

Proof. The result is immediate when $\mathbf{s} = s_0 \in \mathbb{R}$. Let us suppose that $\mathbf{s} \notin \mathbb{R}$. It is immediate that $\mathbf{x} \in [\mathbf{s}]$ is a solution. Conversely, let $\mathbf{x}$ be a solution, i.e., $(x_0 + \underline{x})^2 - 2Re[s](x_0 + \underline{x}) + |\mathbf{s}|^2 = 0$. A direct computation shows that this is possible if and only if $\underline{x} = 0$ or $x_0 = s_0$. The first possibility does not give any solution, while the second gives $|\mathbf{x}| = |\mathbf{s}|$, i.e., the equivalence class of $\mathbf{s}$. □

An obvious consequence of the proposition, which will be useful in the sequel, is that any paravector $\mathbf{s}$ satisfies the identity

$$\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{s} + |\mathbf{s}|^2 = 0. \tag{2.21}$$

As a consequence of the Representation Formula II, we obtain the following immediate result on the zeros of an s-monogenic function:

Proposition 2.5.5. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}^{n+1}$ be an s-monogenic function. If $f(u + Iv) = f(u + Kv) = 0$ for some $I, K \in \mathbb{S}$, $I \neq K$, then f vanishes on the entire $(n-1)$-sphere $[u + Iv]$.*

In particular, we have

Corollary 2.5.6. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}^{n+1}$ be an s-monogenic function. If $f(u + Iv) = f(u - Iv) = 0$ for some $I \in \mathbb{S}$, then f vanishes on the entire $(n-1)$-sphere $[u + Iv]$.*

In other words, the zero set of an s-monogenic function having two zeros on a certain $(n-1)$-sphere contains the entire sphere. There are s-monogenic functions whose zero set is made only by the union of isolated $(n-1)$-spheres (in particular, points on the real axis). Among these functions there are power series with real coefficient, as proved in the following:

Proposition 2.5.7. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. If f has a series representation*

$$f(\mathbf{x}) = \sum_{m \geq 0} (\mathbf{x} - y_0)^m a_m$$

with real coefficients a_m, at some point on the real axis $y_0 \in U$, then every real zero is isolated. If $u_0 + v_0 I_0$, for some $I_0 \in \mathbb{S}$, is a nonreal zero, then $u_0 + v_0 I$ is a zero for any $I \in \mathbb{S}$. In particular, if $f \not\equiv 0$, the zero set of f is either empty or it is the union of isolated points (belonging to $\mathbb{R}$) and isolated $(n-1)$-spheres.

Proof. We will first prove that for all $I \in \mathbb{S}$ we have $f(U \cap \mathbb{C}_I) \subseteq U \cap \mathbb{C}_I$. This fact is true in a suitable disc $B \cap \mathbb{C}_I \subset \mathbb{C}_I$ containing y_0, since the series $f(\mathbf{x}) = \sum_{m \geq 0} (\mathbf{x} - y_0)^m a_m$ converging on B, has real coefficients by hypothesis. The Splitting Lemma on the plane $\mathbb{C}_I$ implies that $f_I(u + Iv) = F(u + Iv)$ in that disc on $\mathbb{C}_I$. Therefore, $F_A = 0$ for $A \neq \emptyset$ on $B \cap \mathbb{C}_I$ and by the Identity Principle for holomorphic functions we obtain that all the holomorphic functions F_A are identically zero on $U \cap \mathbb{C}_I$ for $A \neq \emptyset$. Hence $f(U \cap \mathbb{C}_I) \subseteq U \cap \mathbb{C}_I$ for all $I \in \mathbb{S}$ from which it follows that $f(u) \in \mathbb{R}$ for all $u \in U \cap \mathbb{R}$. By the Identity Principle we get that $f(u + I_0 v) \equiv F(u + I_0 v)$ on $U \cap \mathbb{C}_{I_0}$ and, being $F(u)$ real-valued for all $u \in U \cap \mathbb{R}$, we have that $F(u + I_0 v) = \overline{F(u - I_0 v)}$ on $U \cap \mathbb{C}_{I_0}$. Since

$$0 = f(u_0 + I_0 v_0) = F(u_0 + I_0 v_0) = \overline{F(u_0 - I_0 v_0)}$$

it turns out that

$$F(u_0 - I_0 v_0) = f(u_0 - I_0 v_0) = 0.$$

The statement follows from Corollary 2.5.6. The fact that the real zeros and the spheres are isolated follows from the Identity Principle. □

As a consequence, we get a description of the zero set of a polynomial with real coefficients in the paravector variable:

Corollary 2.5.8. *Let p be a polynomial in the paravector variable $\mathbf{x}$ with real coefficients. Then the zero set of p is the union of isolated points (belonging to $\mathbb{R}$) and isolated $(n-1)$-spheres.*

Remark 2.5.9. As we have already pointed out, in the case $n = 1$ the set of s-monogenic functions coincide with the set of holomorphic functions in one complex variable (by identifying $\mathbb{R}^2$ with $\mathbb{C}$). Proposition 2.5.7 corresponds to the well-known result saying that the zeros of a holomorphic function whose series expansion has real coefficients has isolated zeros which are either real or complex conjugates.

To show that any s-monogenic function has zero set consisting of a union of isolated $(n-1)$-spheres (which might be reduced to a point on the real axis) and isolated points, we associate to each s-monogenic function defined on an axially symmetric s-domain U, an auxiliary function defined on U and denoted by f^σ. The function f^σ has two main properties: on one hand it vanishes on the zero set of f, on the other hand, it defines a holomorphic function which takes elements from $U \cap \mathbb{C}_I$ to $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

The idea used to construct the function f^σ is based on the observation that, given a vector with 2^{n-1} complex components w_A, the vector with components $w_A \bar{w}_A$ is zero if and only if $w_A = 0$ for all A. Now note that the Splitting Lemma allows to write the restriction f_I of an s-monogenic function f in terms of a vector of 2^{n-1} holomorphic functions $F_A : U \cap \mathbb{C}_I \to \mathbb{C}_I$ as

$$f_I(z) = \sum_A F_A(z) I_A.$$

Consider the vector with components $F_A(z)\overline{F_A(\bar{z})}$. The components are obviously holomorphic and if $F_A(z_0) = 0$ also $F_A(z_0)\overline{F_A(\bar{z}_0)} = 0$. We then define the function $f_I^\sigma : U \cap \mathbb{C}_I \to \mathbb{C}_I$ by

$$f_I^\sigma(z) = \sum_A F_A(z)\overline{F_A(\bar{z})}.$$

Using the Extension Lemma 2.2.22, we can extend the function f_I^σ to an s-monogenic function defined on U:

Definition 2.5.10. *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Let $I \in \mathbb{S}$ and let*

$$f_I^\sigma(z) = \sum_A F_A(z)\overline{F_A(\bar{z})}.$$

We define $f^\sigma : U \to \mathbb{R}_n$ by:

$$f^\sigma(\mathbf{x}) := \mathrm{ext}(f_I^\sigma)(\mathbf{x}).$$

We have the following property:

Lemma 2.5.11. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Then f vanishes identically on U if and only if f^σ vanishes identically on U.*

Proof. When $f \equiv 0$, it is immediate that $f^\sigma \equiv 0$. Conversely, consider the restriction f_I of f to a plane $\mathbb{C}_I$, which, by the Splitting Lemma, can be written as $f_I(z) = \sum_A F_A(z) I_A$ where $F_A : U \cap \mathbb{C}_I \to \mathbb{C}_I$ are holomorphic functions. Then the functions F_A admit series expansion at any point of $U \cap \mathbb{C}_I$. Consider $y_0 \in U \cap \mathbb{C}_I$ belonging to the real axis and the series expansion of F_A at a point y_0:

$$F_A(z) = \sum_{m\geq 0} (z - y_0)^m a_{Am}, \qquad a_{Am} \in \mathbb{C}_I$$

which holds in a suitable disc $\Delta(y_0, R) \subseteq U \cap \mathbb{C}_I$ of radius R and centered in $y_0 \in \mathbb{R}$. Then, on $\Delta(y_0, R)$, we have

$$\overline{F_A(\bar{z})} = \sum_{m\geq 0} (z - y_0)^m \bar{a}_{Am}.$$

Moreover on $\Delta(y_0, R)$ we can write

$$\begin{aligned} f_I^\sigma(z) &= \sum_A F_A(z)\overline{F_A(\bar{z})} \\ &= \sum_A \sum_{m\geq 0} (z - y_0)^m c_{Am} = \sum_{m\geq 0} (z - y_0)^m (\sum_A c_{Am}), \end{aligned}$$

where

$$c_{Am} = \sum_{i=0}^{m} a_{Ai}\bar{a}_{A\ m-i}.$$

Now, if $f^\sigma \equiv 0$, then $f_I^\sigma \equiv 0$. So, in the disc $\Delta(y_0, R)$ we have that $\sum_A c_{A0} = \sum_A |a_{A0}|^2 = 0$ so $a_{A0} = 0$ for all multi-indices A. Now, by induction, assume that $a_{Ai} = 0$ for $i = 0, 1, \ldots, k-1$, $k \geq 1$ for all multi-indices A. Consider the coefficient

$$\sum_A c_{A\,2k} = \sum_A \sum_{i=0}^{2k} a_{Ai}\bar{a}_{A\,2k-i}$$

which is zero because $f_I^\sigma \equiv 0$. By assumption we have $a_{Ai}\bar{a}_{A\ 2k-i} = 0$ when $i = 0, \ldots, k-1$ since $a_{Ai} = 0$ and $a_{Ai}\bar{a}_{A\,2k-i} = 0$ when $i = k+1, \ldots, 2k$ since $\bar{a}_{A\ 2k-i} = 0$. Thus, $\sum_A c_{A\,2k} = \sum_A |a_{Ak}|^2$ is zero if and only if $a_{Ak} = 0$ for all multi-indices A. We conclude that $f_I^\sigma \equiv 0$ in the disc $\Delta(y_0, R) \cap \mathbb{C}_I$ implies that all the coefficients a_{Ai} vanish, thus also f_I vanishes identically on the same disc.

By the Identity Principle f vanishes identically. □

The zero set of f^σ is described in the following result:

Lemma 2.5.12. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain, let $f: U \to \mathbb{R}_n$ be an s-monogenic function, and let $f \not\equiv 0$. If there exists $I \in \mathbb{S}$ for which $f^\sigma(u_0 + Iv_0) = 0$, then $f^\sigma(u_0 + Jv_0) = 0$ for all $J \in \mathbb{S}$. Moreover, the zero set of f^σ consists of isolated $(n-1)$-spheres (which might reduce to points on the real axis).*

Proof. Consider the restriction f_I^σ of f^σ to the plane $\mathbb{C}_I$. We have:

$$\overline{f_I^\sigma(\bar{z})} = \overline{\sum_A F_A(\bar{z})\overline{F_A(z)}} = \sum_A \overline{F_A(\bar{z})}F_A(z) = f_I^\sigma(z),$$

thus $f_I^\sigma(u_0 + Iu_0) = 0$ if and only if $f_I^\sigma(u_0 - Iu_0) = 0$. So, if $f^\sigma(u_0 + Iv_0) = 0$, then, by the Representation Formula, $f^\sigma(u_0 + Ju_0) = 0$ for all $J \in \mathbb{S}$. The second part of the statement follows by the Identity Principle: if the $(n-1)$-spheres of zeros were not isolated, on each plane we would get accumulation points of zeros and thus f^σ would be identically zero by the Identity Principle which contradicts the fact that $f^\sigma \not\equiv 0$ by Lemma 2.5.11. □

Lemma 2.5.13. *Let $U \subseteq \mathbb{R}_n$ be an axially symmetric s-domain and let $f: U \to \mathbb{R}_n$ be an s-monogenic function. If $u + Iv$ is a zero of f, then it is also a zero of f^σ.*

Proof. The restriction of f to the plane $\mathbb{C}_I$ can be written, by the Splitting Lemma, as $f_I(z) = \sum_A F_A(z)I_A$. The condition $f(u + Iv) = 0$ implies that, on the plane $\mathbb{C}_I$ it is also $F_A(u + Iv) = 0$ for all A. Thus $f_I^\sigma(u + Iv) = 0$ and the statement follows. □

We are now in a position to prove the following theorem which describes the zero set of an s-monogenic function defined on an axially symmetric s-domain.

Theorem 2.5.14 (Structure of the Zero Set). *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f: U \to \mathbb{R}_n$ be an s-monogenic function. Suppose that f does not vanish identically. Then if the zero set of f is nonempty, it consists of the union of isolated $(n-1)$-spheres and/or isolated points.*

Proof. Suppose that the zero set of f is nonempty and that f does not vanish identically, thus also f^σ does not vanish identically by Lemma 2.5.11. By Lemma 2.5.13 any zero of f is a zero of f^σ, i.e., denoting by Z_{f^σ} and Z_f the zero set of f^σ and f respectively, we have $Z_f \subseteq Z_{f^\sigma}$. If Z_f contains two points on an $(n-1)$-sphere $[\mathbf{s}]$, then Z_f contains the whole sphere. Indeed, suppose that $u_0 + Iv_0$, $u_0 + Jv_0 \in [\mathbf{s}]$, $I \neq J$, and $f(u_0 + Iv_0) = f(u_0 + Jv_0) = 0$. Then by the Representation Formula we get

$$f(u_0 + Jv_0) = \frac{1}{2}[1 + JI]f(u_0 - Iv_0) = 0.$$

The element $1 + JI = (-I + J)I$ is invertible since it is product of two invertible elements, thus $f(u_0 - Iv_0) = 0$ and the statement follows from Proposition 2.5.6.

When a sphere belongs to Z_f, then it is isolated. Indeed, let $\mathbf{x}_0$ be a point on this sphere. If there were a sequence $\{\mathbf{x}_n\}$ of zeros, $\mathbf{x}_n \notin [u_0 + I_0 v_0]$, such that $\mathbf{x}_n \to \mathbf{x}_0$, then the corresponding spheres $[\mathbf{x}_n]$ would belong to Z_{f^σ}, which is absurd by Lemma 2.5.12.

Similarly, suppose that Z_f contains a point $\mathbf{x}_0 = u_0 + Jv_0$, without containing the sphere $u_0 + Iv_0$, $I \in \mathbb{S}$ generated by it. Then we have to show that the point $u_0 + I_0 v_0$ is isolated. Indeed, if there were a sequence $\{\mathbf{x}_n\}$ of zeros, $\mathbf{x}_n \notin [u_0 + Jv_0]$ (otherwise the whole sphere $[u_0 + Jv_0]$ would belong to Z_f), such that $\mathbf{x}_n \to \mathbf{x}_0$, then the corresponding spheres would belong to Z_{f^σ} which is absurd by Lemma 2.5.12. □

Remark 2.5.15. The result already obtained in Proposition 2.5.5 can be obtained also as a consequence of the previous theorem. In fact, given a converging power series $\sum_{m\geq 0} \mathbf{x}^m a_m$, $a_m \in \mathbb{R}_n$, if there are two different elements in a given equivalence class $[\mathbf{s}]$, which are solutions to the equation

$$\sum_{m\geq 0} \mathbf{x}^m a_m = 0,$$

then all the elements in the equivalence class are solutions.

We close this section with an immediate corollary of the previous theorem, which yields a nice description of the zero set of a polynomial:

Corollary 2.5.16. *Let $p(\mathbf{x})$ be a polynomial in $\mathbb{R}_n[\mathbf{x}]$, with right coefficients, which does not vanish identically. Then, if the zero set of p is nonempty, it consists of isolated points or isolated $(n-1)$-spheres.*

2.6 The slice monogenic product

It is immediate to see that the product of two s-monogenic functions is not, in general, s-monogenic. Nevertheless, as we indicated in Section 2.3, it is possible to define a product among s-monogenic power series by mimicking the process used to define a product of polynomials in skew fields. We can extend this idea to the case of s-monogenic functions defined on axially symmetric s-domains, to define an s-monogenic product. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f, g :\ U \to \mathbb{R}_n$ be s-monogenic functions. For any $I \in \mathbb{S}$ set $I = I_1$ and consider a completion to a basis $\{I_1, \ldots, I_n\}$ of $\mathbb{R}_n$ such that $I_i I_j + I_j I_i = -2\delta_{ij}$. The Splitting

Lemma guarantees the existence of holomorphic functions $F_A, G_A : U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that, for all $z = u + Iv \in U \cap \mathbb{C}_I$,

$$f_I(z) = \sum_A F_A(z) I_A, \qquad g_I(z) = \sum_B G_B(z) I_B,$$

where A, B are subsets of $\{2, \ldots, n\}$ and, by definition, $I_\emptyset = 1$. We define the function $f_I * g_I : U \cap \mathbb{C}_I \to \mathbb{R}_n$ as

$$\begin{aligned} f_I * g_I(z) = &\sum_{|A| \text{even}} (-1)^{\frac{|A|}{2}} F_A(z) G_A(z) + \sum_{|A| \text{odd}} (-1)^{\frac{|A|+1}{2}} F_A(z) \overline{G_A(\bar{z})} \\ &+ \sum_{|A| \text{even}, B \neq A} F_A(z) G_B(z) I_A I_B + \sum_{|A| \text{odd}, B \neq A} F_A(z) \overline{G_B(\bar{z})} I_A I_B. \end{aligned} \tag{2.22}$$

Then $f_I * g_I(z)$ is obviously a holomorphic map on $\mathbb{C}_I$, i.e., $\overline{\partial}_I (f_I * g_I)(z) = 0$, and hence its unique s-monogenic extension to U, according to the Extension Lemma 2.2.22, is given by

$$f * g(\mathbf{x}) := \operatorname{ext}(f_I * g_I)(\mathbf{x}).$$

Definition 2.6.1. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f, g : U \to \mathbb{R}_n$ be s-monogenic functions. The function*

$$f * g(\mathbf{x}) = \operatorname{ext}(f_I * g_I)(\mathbf{x})$$

defined as the extension of (2.22) is called the s-monogenic product of f and g. This product is also called the $*$-product of f and g.

Remark 2.6.2. It is immediate to verify that the $*$-product is associative, distributive but, in general, not commutative.

The following example shows the dramatic difference between polynomials in a division algebra and polynomials in a Clifford algebra. Even a simple result such as $\deg(p_1 * p_2) = \deg(p_1) + \deg(p_2)$ fails (we can only conclude that $\deg(p_1 * p_2) \leq \deg(p_1) + \deg(p_2)$) and it is impossible to deduce the zeros of the product from the zeros of the factors. This is in stark contrast with the case of polynomials in division algebras, where it is possible to obtain explicit formulas to deduce the zeros of $f * g$ from the zeros of f and g (see, e.g., [71]).

Example 2.6.3. Consider the two polynomials $p_1(\mathbf{x}) = 1 + \mathbf{x}(1 - e_1 e_2 e_3)$ and $p_2(\mathbf{x}) = 1 + \mathbf{x}(1 + e_1 e_2 e_3) \in \mathbb{R}_3[\mathbf{x}]$. None of them has roots in $\mathbb{R}^4$ because $(1 \pm e_1 e_2 e_3)$ are zero divisors. Their product $p_1 * p_2(\mathbf{x}) = (1 + \mathbf{x}(1 - e_1 e_2 e_3)) * (1 + \mathbf{x}(1 + e_1 e_2 e_3)) = 1 + 2\mathbf{x}$ is a degree-one polynomial and has the real number $-1/2$ as its root.

The s-monogenic product is however an important tool to obtain s-monogenic functions. In particular, it allows us to define the inverse of an s-monogenic function with respect to the $*$-product. As we have already mentioned, not all the Clifford numbers admit an inverse with respect to the product in the Clifford algebra

$\mathbb{R}_n$. Those Clifford numbers $a \in \mathbb{R}_n$ for which $a\bar{a}$ is a real nonzero number admit inverse $a^{-1} = \bar{a}(a\bar{a})^{-1}$. In particular the existence of the inverse can be guaranteed for all nonzero vectors. Similarly, for s-monogenic functions we can guarantee the existence of an inverse with respect to the $*$-product, if we suitably restrict their codomains. To introduce the notion of inverse we need some preliminary definitions.

Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. Let us consider the restriction $f_I(z)$ of f to the plane $\mathbb{C}_I$ and it usual representation (given by the Splitting Lemma)

$$f_I(z) = \sum_A F_A(z) I_A.$$

Let us define the function $f_I^c : U \cap \mathbb{C}_I \to \mathbb{C}_I$ as

$$\begin{aligned} f_I^c(z) &:= \sum_A F_A^c(z) I_A \qquad (2.23) \\ &= \sum_{|A|\equiv 0} \overline{F_A(\bar{z})} I_A - \sum_{|A|\equiv 1} F_A(z) I_A - \sum_{|A|\equiv 2} \overline{F_A(\bar{z})} I_A + \sum_{|A|\equiv 3} F_A(z) I_A, \end{aligned}$$

where the equivalence $\equiv$ is intended as $\equiv$ (mod4), i.e., the congruence modulo 4. Since any function F_A is obviously holomorphic it can be uniquely extended to an s-monogenic function on U, according to the Extension Lemma 2.2.22. Thus we can give the following definition:

Definition 2.6.4. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. The function*

$$f^c(\mathbf{x}) = \mathrm{ext}(f_I^c)(\mathbf{x})$$

is called the s-monogenic conjugate of f.

This definition of conjugate behaves, for power series and thus for polynomials, as the conjugation on the coefficients as proven in the next result:

Proposition 2.6.5. *Let $f : B(y_0, R) \to \mathbb{R}_n$ be an s-monogenic function on an open ball in $\mathbb{R}^{n+1}$ centered at a point on the real axis y_0. If*

$$f(\mathbf{x}) = \sum_{m\geq 0} (\mathbf{x} - y_0)^m a_m,$$

then, for $a_m \in \mathbb{R}_n$, we have

$$f^c(\mathbf{x}) = \sum_{m\geq 0} (\mathbf{x} - y_0)^m \bar{a}_m.$$

Proof. We will suppose without loss of generality that $y_0 = 0$. By Corollary 2.3.7, given any $I \in \mathbb{S}$, the coefficients of the power series expansion of f can be obtained as the coefficients of the power series of f_I. By the Splitting Lemma with respect to an orthonormal completion of I to a basis of $\mathbb{R}_n$, for all $z = u + Iv \in B(0,R) \cap \mathbb{C}_I$ we have

$$f_I(z) = \sum_A F_A(z) I_A = \sum_A \sum_{m\geq 0} z^m \frac{1}{m!}\Big(\frac{\partial^m F_A}{\partial u^m}(0)\Big) I_A = \sum_{m\geq 0} z^m \frac{1}{m!} \partial_s^m f(0)$$

and hence the relation

$$f_I^c(z) = \sum_{|A|\equiv 0} \overline{F_A(\bar{z})} I_A - \sum_{|A|\equiv 1} F_A(z) I_A - \sum_{|A|\equiv 2} \overline{F_A(\bar{z})} I_A + \sum_{|A|\equiv 3} F_A(z) I_A \tag{2.24}$$

$$= \sum_{m\geq 0} \frac{z^m}{m!}\Big(\sum_{|A|\equiv 0} \frac{\partial^m \overline{F_A}}{\partial u^m}(0) - \sum_{|A|\equiv 1} \frac{\partial^m F_A}{\partial u^m}(0) - \sum_{|A|\equiv 2} \frac{\partial^m \overline{F_A}}{\partial u^m}(0) + \sum_{|A|\equiv 3} \frac{\partial^m F_A}{\partial u^m}(0)\Big) I_A \tag{2.25}$$

$$= \sum_{m\geq 0} z^m \frac{1}{m!} \overline{\partial_s^m f(0)}, \tag{2.26}$$

where the equivalence $\equiv$ is intended as the congruence modulo 4, proves the assertion. □

Using the notion of $*$-multiplication of s-monogenic functions, it is possible to associate to any s-monogenic function f its "symmetrization" or "normal form", denoted by f^s. We will show that all the zeros of f^s are $(n-1)$-spheres (possibly reduced to a point on the real axis) and that if $\mathbf{x}$ is a zero of f (isolated or not), then the $(n-1)$-sphere $[\mathbf{x}]$ is a zero of f^s.

Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f:\ U \to \mathbb{R}_n$ be an s-monogenic function. As usual, using the Splitting Lemma we can write

$$f_I(z) = \sum_A F_A(z) I_A;$$

here we will use the notation $[f_I]_0$ to denote the "scalar" part of the function f_I, i.e., the part whose coefficient in the Splitting Lemma is $I_\emptyset = 1$. With this notation, we define the function $f^s:\ U \cap \mathbb{C}_I \to \mathbb{C}_I$ as

$$\begin{aligned} f_I^s &:= [f_I * f_I^c]_0 \qquad (2.27)\\ &= \Big[\big(\sum_B F_B(z) I_B\big)\big(\sum_{|A|\equiv 0} \overline{F_A(\bar{z})} I_A - \sum_{|A|\equiv 1} F_A(z) I_A \\ &\quad - \sum_{|A|\equiv 2} \overline{F_A(\bar{z})} I_A + \sum_{|A|\equiv 3} F_A(z) I_A\big)\Big]_0. \end{aligned}$$

We have

$$\begin{aligned} f_I * f_I^c = &\sum_{|B|\text{even},|A|\equiv 0} F_B(z)\overline{F_A(\bar z)} I_B I_A - \sum_{|B|\text{even},|A|\equiv 1} F_B(z)F_A(z) I_B I_A \\ &- \sum_{|B|\text{even},|A|\equiv 2} F_B(z)\overline{F_A(\bar z)} I_B I_A + \sum_{|B|\text{even},|A|\equiv 3} F_B(z)F_A(z) I_B I_A \\ &+ \sum_{|B|\text{odd},|A|\equiv 0} F_B(z)F_A(z) I_B I_A - \sum_{|B|\text{odd},|A|\equiv 1} F_B(z)\overline{F_A(\bar z)} I_B I_A \\ &- \sum_{|B|\text{odd},|A|\equiv 2} F_B(z)F_A(z) I_B I_A + \sum_{|B|\text{odd},|A|\equiv 3} F_B(z)\overline{F_A(\bar z)} I_B I_A. \end{aligned}$$

The terms from which the scalar part arises are the ones with $A = B$, i.e.,

$$\begin{aligned} [f_I * f_I^c]_0 = &\sum_{|A|\equiv 0} F_A(z)\overline{F_A(\bar z)} I_A^2 - \sum_{|A|\equiv 2} F_A(z)\overline{F_A(\bar z)} I_A^2 \\ &- \sum_{|A|\equiv 1} F_A(z)\overline{F_A(\bar z)} I_A^2 + \sum_{|A|\equiv 3} F_A(z)\overline{F_A(\bar z)} I_A^2 = \sum_A F_A(z)\overline{F_A(\bar z)}. \end{aligned}$$

Then f_I^s is obviously holomorphic and hence its unique s-monogenic extension to U defined by

$$f^s(\mathbf{x}) := \mathrm{ext}(f_I^s)(\mathbf{x})$$

is s-monogenic.

Definition 2.6.6. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f : U \to \mathbb{R}_n$ be an s-monogenic function. The function*

$$f^s(\mathbf{x}) = \mathrm{ext}(f_I^s)(\mathbf{x})$$

defined by the extension of $f_I^s = [f_I * f_I^c]_0$ from $U \cap \mathbb{C}_I$ to the whole U is called the symmetrization of f.

Remark 2.6.7. Notice that formula (2.27) yields that, for all $I \in \mathbb{S}$, $f^s(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I$.

Remark 2.6.8. Note that the function f^σ introduced in Definition 2.5.10 to study the zero set of an s-monogenic function coincides with f^s for all s-monogenic functions f.

It is now easy to verify the following facts.

Proposition 2.6.9. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain and let $f, g \in \mathcal{M}(U)$. Then*

$$f^s g = f^s * g = g * f^s.$$

Moreover, if Z_{f^s} is the zero set of f^s, then

$$(f^s)^{-1} g = (f^s)^{-1} * g = g * (f^s)^{-1} \quad \text{on } U \setminus Z_{f^s}.$$

Proof. Since $f^s(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I$, the series expansion of f_I^s in a small ball with center at a real point has real coefficients so, in that ball, we have $f_I^s g_I = f_I^s * g_I = g_I * f_I^s$. By the Identity Principle $f_I^s * g_I = f_I^s g_I = g_I * f_I^s$ on $U \cap \mathbb{C}_I$ and so, by the Extension Lemma, $f^s * g = g * f^s$. Reasoning in the same way with the function $(f^s)^{-1}$, whose restriction to $\mathbb{C}_I$ takes $(U \setminus Z_{f^s}) \cap \mathbb{C}_I$ to $\mathbb{C}_I$, we get the final part of the statement. □

Definition 2.6.10. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Let $f : U \to \mathbb{R}_n$ be an s-monogenic function such, that for some $I \in \mathbb{S}$ its restriction f_I to the complex plane $\mathbb{C}_I$ satisfies the condition*

$$f_I * f_I^c \ \ \textit{has values in } \mathbb{C}_I.$$

We define the function:

$$f^{-*} := \mathrm{ext}((f_I^s)^{-1} f_I^c)$$

where $f_I^s = [f_I * f_I^c]_0 = f_I * f_I^c$, and we will call it s-monogenic inverse of the function f.

The next proposition shows that the function f^{-*} is the inverse of f with respect to the $*$-product:

Proposition 2.6.11. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Let $f : U \to \mathbb{R}_n$ be an s-monogenic function such that for some $I \in \mathbb{S}$ we have $f_I * f_I^c$ has values in $\mathbb{C}_I$. Then on $U \setminus Z_{f^s}$ we have:*

$$f^{-*} * f = f * f^{-*} = 1.$$

Proof. To prove the statement it is sufficient to show that on the plane $\mathbb{C}_I$ we have:

$$f_I * (f_I^s)^{-1} f_I^c = (f_I^s)^{-1} f_I^c * f_I = 1.$$

Using associativity and Proposition 2.6.9, we easily compute:

$$f_I * ((f_I^s)^{-1} * f_I^c) = (f_I^s)^{-1} * f_I * f_I^c = f_I^{s\,-1} * (f_I^s) = 1,$$

and

$$((f_I^s)^{-1} * f_I^c) * f_I = (f_I^s)^{-1} * f_I^c * f_I = (f_I^s)^{-1} * f_I^s = 1.$$

The result now follows from the Extension Lemma 2.2.22. □

Example 2.6.12. Consider the function $f(\mathbf{x}) = \mathbf{x} - \mathbf{s}$ defined on $\mathbb{R}^{n+1}$. As it is well known, the inverse $(\mathbf{x} - \mathbf{s})^{-1}$ is not an s-monogenic function, unless $\mathbf{s} \in \mathbb{R}$. Since the function

$$(f_I * f_I^c)(z) = (z - \mathbf{s})(z - \overline{\mathbf{s}}) = z^2 - 2\mathrm{Re}[\mathbf{s}]z + |\mathbf{s}|^2$$

has real coefficients and thus has values in $\mathbb{C}_I$, we can consider the s-monogenic inverse of f. According to Definition 2.6.10, f^{-*} is defined for $\mathbf{x} \notin [\mathbf{s}]$, and it is the function

$$f^{-*}(\mathbf{x}) = (\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\mathbf{x} - \overline{\mathbf{s}}).$$

As we will see in the next section, the expression $(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\mathbf{x} - \overline{\mathbf{s}})$ cannot be simplified, unless $\mathbf{s} \in \mathbb{R}$ and in this case it coincides with $(\mathbf{x}-\mathbf{s})^{-1}$, i.e., the standard inverse of f.

Example 2.6.13. The notions of s-monogenic inverse and s-monogenic multiplication allow us to introduce s-monogenic quotients (left and right) of s-monogenic functions. Let $f, g : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ be two s-monogenic functions. On $U \setminus Z_{g^s}$ we can define the functions

$$g^{-*} * f \qquad \text{and} \qquad f * g^{-*}.$$

Let us consider the function $g^{-*} * f$ (the other case can be treated in a similar way): by definition it is the extension of

$$g_I^{-*} * f_I = (g_I^s)^{-1} g_I^c * f_I,$$

which is an $\mathbb{R}_n$-valued function satisfying

$$\overline{\partial}_I((g_I^s)^{-1} g_I^c * f_I) = 0$$

and such that $Z_{g^s} \cap \mathbb{C}_I$ consists of isolated points.

2.7 Slice monogenic Cauchy kernel

We begin this section with the following crucial definition, which is the starting point to find a Cauchy formula with s-monogenic kernel.

Definition 2.7.1. *Let* $\mathbf{x}$, $\mathbf{s} \in \mathbb{R}^{n+1}$. *We call*

$$S^{-1}(\mathbf{s}, \mathbf{x}) := \sum_{n \geq 0} \mathbf{x}^n \mathbf{s}^{-1-n}$$

the noncommutative Cauchy kernel series.

Remark 2.7.2. The noncommutative Cauchy kernel series is convergent for $|\mathbf{x}| < |\mathbf{s}|$.

Theorem 2.7.3. *Let* $\mathbf{x}$, $\mathbf{s} \in \mathbb{R}^{n+1}$ *be such that* $\mathbf{x}\mathbf{s} \neq \mathbf{s}\mathbf{x}$. *Then, the function*

$$S(\mathbf{s}, \mathbf{x}) = -(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2),$$

is the inverse of the noncommutative Cauchy kernel series.

Proof. Let us verify that

$$-(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2) \sum_{n \geq 0} \mathbf{x}^n \mathbf{s}^{-1-n} = 1.$$

We therefore obtain

$$(-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x}) \sum_{n\geq 0} \mathbf{x}^n \mathbf{s}^{-1-n} = \mathbf{s} + \mathbf{x} - 2\ \mathrm{Re}[\mathbf{s}]. \tag{2.28}$$

Observing that $-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x}$ commutes with $\mathbf{x}^n$ we can rewrite this last equation as

$$\sum_{n\geq 0} \mathbf{x}^n (-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x})\mathbf{s}^{-1-n} = \mathbf{s} + \mathbf{x} - 2\ \mathrm{Re}[\mathbf{s}].$$

Now the left-hand side can be written as

$$\begin{aligned}
&\sum_{n\geq 0} \mathbf{x}^n (-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x})s^{-1-n} \\
&= (-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x})\mathbf{s}^{-1} + \mathbf{x}^1(-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x})\mathbf{s}^{-2} \\
&\quad + \mathbf{x}^2(-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x})\mathbf{s}^{-3} + \dots \\
&= -\Big(|\mathbf{s}|^2\mathbf{s}^{-1} + \mathbf{x}(-2\mathrm{Re}[\mathbf{s}]\mathbf{s} + |\mathbf{s}|^2)\mathbf{s}^{-2} + \mathbf{x}^2(\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{s} + |\mathbf{s}|^2)\mathbf{s}^{-3} \\
&\quad + \mathbf{x}^3(\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{s} + |\mathbf{s}|^2)\mathbf{s}^{-4} + \dots\Big).
\end{aligned}$$

Using the identity (2.20)

$$\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{s} + |\mathbf{s}|^2 = 0$$

we get

$$\begin{aligned}
&\sum_{n\geq 0} \mathbf{x}^n (-|\mathbf{s}|^2 - \mathbf{x}^2 + 2\mathrm{Re}[\mathbf{s}]\mathbf{x})\mathbf{s}^{-1-n} = -|\mathbf{s}|^2\mathbf{s}^{-1} + \mathbf{x}\mathbf{s}^2\mathbf{s}^{-2} \\
&= -|\mathbf{s}|^2\mathbf{s}^{-1} + \mathbf{x} = -\overline{\mathbf{s}}\mathbf{s}\mathbf{s}^{-1} + \mathbf{x} = -\overline{\mathbf{s}} + \mathbf{x} = \mathbf{s} - 2\ \mathrm{Re}[\mathbf{s}] + \mathbf{x}
\end{aligned}$$

which equals the right-hand side of (2.28). □

When $\mathbf{x}$, $\mathbf{s}$ commute, the function $S(\mathbf{s},\mathbf{x})$ becomes

$$S(\mathbf{s},\mathbf{x}) = -(\mathbf{x}-\overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathbf{x}\mathrm{Re}[\mathbf{s}] + |\mathbf{s}|^2) = -(\mathbf{x}-\overline{\mathbf{s}})^{-1}(\mathbf{x}-\overline{\mathbf{s}})(\mathbf{x}-\mathbf{s}) = \mathbf{s}-\mathbf{x}$$

which is, trivially, the inverse of the standard sum of the Cauchy kernel series $S^{-1}(\mathbf{s},\mathbf{x}) = \sum_{n\geq 0} \mathbf{x}^n\mathbf{s}^{-1-n} = (\mathbf{s}-\mathbf{x})^{-1}$.

As a direct consequence of this observation and of the previous result, we can explicitly write the sum of the noncommutative Cauchy kernel series:

Theorem 2.7.4. *Let* $\mathbf{x}$, $\mathbf{s} \in \mathbb{R}^{n+1}$ *be such that* $\mathbf{x}\mathbf{s} \neq \mathbf{s}\mathbf{x}$. *Then*

$$\sum_{n\geq 0} \mathbf{x}^n\mathbf{s}^{-1-n} = -(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\mathbf{x}-\overline{\mathbf{s}}),$$

for $|\mathbf{x}| < |\mathbf{s}|$. *If* $\mathbf{x}\mathbf{s} = \mathbf{s}\mathbf{x}$, *then*

$$\sum_{n\geq 0} \mathbf{x}^n \mathbf{s}^{-1-n} = (\mathbf{s} - \mathbf{x})^{-1},$$

for $|\mathbf{x}| < |\mathbf{s}|$.

Definition 2.7.5. *We will call the expression*

$$S^{-1}(\mathbf{s}, \mathbf{x}) = -(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\mathbf{x} - \overline{\mathbf{s}}), \tag{2.29}$$

defined for $\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2 \neq 0$, *the noncommutative Cauchy kernel.*

Remark 2.7.6. With an abuse of notation we have used the same symbol $S^{-1}(\mathbf{s}, \mathbf{x})$ to denote the noncommutative Cauchy kernel series and the noncommutative Cauchy kernel. This notation will not create confusion in the following since from the context it will be clear which object we are considering.

Note that the noncommutative Cauchy kernel is defined on a set which is larger than the set $\{(\mathbf{x}, \mathbf{s}) \ : \ |\mathbf{x}| < |\mathbf{s}|\}$ where the noncommutative Cauchy kernel series is convergent.

Remark 2.7.7. We now observe that the expression

$$(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\mathbf{x} - \overline{\mathbf{s}})$$

involves an inverse which does not exist if we set $\mathbf{x} = \overline{\mathbf{s}}$; indeed, in this case we have

$$\overline{\mathbf{s}}^2 - 2\mathrm{Re}[\mathbf{s}]\overline{\mathbf{s}} + |\mathbf{s}|^2 = 0.$$

One may wonder if the factor $(\mathbf{x} - \overline{\mathbf{s}})$ can be simplified. The next theorem shows that this is not possible and the function

$$(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\mathbf{x} - \overline{\mathbf{s}})$$

cannot be extended to a continuous function in $\mathbf{x} = \overline{\mathbf{s}}$.

Theorem 2.7.8. *Let* $S^{-1}(\mathbf{s}, \mathbf{x})$ *be the noncommutative Cauchy kernel and let* $\mathbf{x}\mathbf{s} \neq \mathbf{s}\mathbf{x}$. *Then* $S^{-1}(\mathbf{s}, \mathbf{x})$ *is irreducible and* $\lim_{\mathbf{x}\to\overline{\mathbf{s}}} S^{-1}(\mathbf{s}, \mathbf{x})$ *does not exist.*

Proof. We prove that we cannot find a degree-one polynomial $Q(\mathbf{x})$ such that

$$\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2 = (\mathbf{s} + \mathbf{x} - 2\mathrm{Re}[\mathbf{s}])Q(\mathbf{x}).$$

The existence of $Q(\mathbf{x})$ would allow the simplification

$$S^{-1}(\mathbf{s}, \mathbf{x}) = Q^{-1}(\mathbf{x})(\mathbf{s} + \mathbf{x} - 2\mathrm{Re}[\mathbf{s}])^{-1}(\mathbf{s} + \mathbf{x} - 2\mathrm{Re}[\mathbf{s}]) = Q^{-1}(\mathbf{x}).$$

We proceed as follows: first of all note that $Q(\mathbf{x})$ has to be a monic polynomial of degree one, so we set

$$Q(\mathbf{x}) = \mathbf{x} - \mathbf{r}$$

where $\mathbf{r} = r_0 + \sum_{j=1}^n r_j e_j$. The equality

$$(\mathbf{s} + \mathbf{x} - 2\ \mathrm{Re}[\mathbf{s}])(\mathbf{x} - \mathbf{r}) = \mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2$$

gives

$$\mathbf{s}\mathbf{x} - \mathbf{s}\mathbf{r} - \mathbf{x}\mathbf{r} + 2\mathrm{Re}[\mathbf{s}]\mathbf{r} - |\mathbf{s}|^2 = 0.$$

Solving for $\mathbf{r}$, we get

$$\mathbf{r} = (\mathbf{s} + \mathbf{x} - 2\ \mathrm{Re}[\mathbf{s}])^{-1}(\mathbf{s}\mathbf{x} - |\mathbf{s}|^2),$$

which depends on $\mathbf{x}$. Let us now prove that the limit does not exist. Let $\mathfrak{e} = \varepsilon_0 + \sum_{j=1}^n \varepsilon_j e_j$, and consider

$$\begin{aligned}
&S^{-1}(\mathbf{s}, \overline{\mathbf{s}} + \mathfrak{e}) = ((\overline{\mathbf{s}} + \mathfrak{e})^2 - 2(\overline{\mathbf{s}} + \mathfrak{e})\mathrm{Re}[\mathbf{s}] + |\mathbf{s}|^2)^{-1}\mathfrak{e}\\
&= ((\overline{\mathbf{s}} + \mathfrak{e})^2 - 2(\overline{\mathbf{s}} + \mathfrak{e})\mathrm{Re}[\mathbf{s}] + |\mathbf{s}|^2)^{-1}\mathfrak{e}\\
&= (\overline{\mathbf{s}}\mathfrak{e} + \mathfrak{e}\overline{\mathbf{s}} + \mathfrak{e}^2 - 2\mathfrak{e}\mathrm{Re}[\mathbf{s}])^{-1}\mathfrak{e}\\
&= (\mathfrak{e}^{-1}(\overline{\mathbf{s}}\mathfrak{e} + \mathfrak{e}\overline{\mathbf{s}} + \mathfrak{e}^2 - 2\mathfrak{e}\mathrm{Re}[\mathbf{s}]))^{-1}\\
&= (\mathfrak{e}^{-1}\overline{\mathbf{s}}\mathfrak{e} + \overline{\mathbf{s}} + \mathfrak{e} - 2\mathrm{Re}[\mathbf{s}]))^{-1}.
\end{aligned}$$

If we now let $\mathfrak{e} \to \mathbf{0}$, we obtain that the term $\mathfrak{e}^{-1}\overline{\mathbf{s}}\mathfrak{e}$ does not have a limit because the element

$$\mathfrak{e}^{-1}\overline{\mathbf{s}}\mathfrak{e} = \frac{\overline{\mathfrak{e}}}{|\mathfrak{e}|^2}\overline{\mathbf{s}}\mathfrak{e}$$

has scalar components of the type $\dfrac{\varepsilon_i \varepsilon_j s_\ell}{|\mathfrak{e}|^2}$ with $i, j, \ell \in \{0, 1, 2, 3\}$, which do not have limit. □

Proposition 2.7.9. *The function $S^{-1}(\mathbf{s}, \mathbf{x})$ is left s-monogenic in the variable $\mathbf{x}$ and right s-monogenic in the variable $\mathbf{s}$ in its domain of definition.*

Proof. The proof follows by direct computations. Consider any $I \in \mathbb{S}$ and set $\mathbf{x} = u + Iv$. We have:

$$\begin{aligned}
&\frac{\partial}{\partial u} S^{-1}(\mathbf{s}, u + Iv)\\
&\quad = ((u + Iv)^2 - 2\mathrm{Re}[\mathbf{s}](u + Iv) + |\mathbf{s}|^2)^{-2}(2u + 2Iv - 2\mathrm{Re}[\mathbf{s}])(u + Iv - \overline{\mathbf{s}})\\
&\qquad\qquad - ((u + Iv)^2 - 2\mathrm{Re}[\mathbf{s}](u + Iv) + |\mathbf{s}|^2)^{-1},
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial v} S^{-1}(\mathbf{s}, u + Iv)\\
&\quad = ((u + Iv)^2 - 2\mathrm{Re}[\mathbf{s}](u + Iv) + |\mathbf{s}|^2)^{-2}(2uI - 2v - 2\mathrm{Re}[\mathbf{s}]I)(u + Iv - \overline{\mathbf{s}})\\
&\qquad\qquad - ((u + Iv)^2 - 2\mathrm{Re}[\mathbf{s}](u + Iv) + |\mathbf{s}|^2)^{-1} I,
\end{aligned}$$

so we obtain:

$$\begin{aligned}
&\frac{\partial}{\partial u}S^{-1}(\mathbf{s},u+Iv)+I\frac{\partial}{\partial v}S^{-1}(\mathbf{s},u+Iv)\\
&=((u+Iv)^2-2\mathrm{Re}[\mathbf{s}](u+Iv)+|\mathbf{s}|^2)^{-2}(2u+2Iv-2\mathrm{Re}[\mathbf{s}])(u+Iv-\bar{\mathbf{s}})\\
&\quad-((u+Iv)^2-2\mathrm{Re}[\mathbf{s}](u+Iv)+|\mathbf{s}|^2)^{-1}\\
&\quad+((u+Iv)^2-2\mathrm{Re}[\mathbf{s}](u+Iv)+|\mathbf{s}|^2)^{-2}(-2u-2vI+2\mathrm{Re}[\mathbf{s}])(u+Iv-\bar{\mathbf{s}})\\
&\quad+((u+Iv)^2-2\mathrm{Re}[\mathbf{s}](u+Iv)+|\mathbf{s}|^2)^{-1}=0.
\end{aligned}$$

Let us now set $\mathbf{s}=u+Iv$. Then $S^{-1}(u+Iv,\mathbf{x})=F(u,v,\mathbf{x})(\mathbf{x}-u+Iv)$ where $F(u,v,\mathbf{x})$ is a function involving $\mathbf{x}$, the real variables u,v but not the imaginary unit I. Then we have:

$$\begin{aligned}
\frac{\partial}{\partial u}S^{-1}(u+Iv,\mathbf{x})&=(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-2}(-2\mathbf{x}+2u)(\mathbf{x}-u+Iv)\\
&\qquad+(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-1},
\end{aligned}$$

$$\frac{\partial}{\partial v}S^{-1}(u+Iv,\mathbf{x})=(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-2}2v(\mathbf{x}-u+Iv)-(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-1}I.$$

It follows that

$$\begin{aligned}
&\frac{\partial}{\partial u}S^{-1}(u+Iv,\mathbf{x})+\frac{\partial}{\partial y}S^{-1}(u+Iv,\mathbf{x})I\\
&=(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-2}(-2\mathbf{x}+2u)(\mathbf{x}-u+Iv)-(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-1}\\
&\quad+(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-2}2v(\mathbf{x}-u+Iv)I-(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-1}\\
&=2(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-2}(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)-2(\mathbf{x}^2-2\mathbf{x}u+u^2+v^2)^{-1}\\
&=0.
\end{aligned}$$

□

This result is obviously trivial when $S^{-1}(\mathbf{s},\mathbf{x})$ coincides with the Cauchy kernel series. However, as we have pointed out after Definition 2.29, the function $S^{-1}(\mathbf{s},\mathbf{x})$ is defined on a set which is larger than the domain of convergence of the series and therefore the direct argument in the preceding proof is necessary.

We now state some equalities which are important to prove further properties of the Cauchy kernel function.

Proposition 2.7.10. *Let* $\mathbf{x}$, $\mathbf{s}\in\mathbb{R}^{n+1}$ *be such that* $\mathbf{x}\neq\bar{\mathbf{s}}$. *Then the following identity holds:*

$$(\mathbf{x}-\bar{\mathbf{s}})^{-1}\mathbf{s}(\mathbf{x}-\bar{\mathbf{s}})-\mathbf{x}=-(\mathbf{s}-\bar{\mathbf{x}})\mathbf{x}(\mathbf{s}-\bar{\mathbf{x}})^{-1}+\mathbf{s},$$

or, equivalently,

$$-(\mathbf{x}-\bar{\mathbf{s}})^{-1}(\mathbf{x}^2-2\mathbf{x}\mathrm{Re}[\mathbf{s}]+|\mathbf{s}|^2)=(\mathbf{s}^2-2\mathrm{Re}[\mathbf{x}]\mathbf{s}+|\mathbf{x}|^2)(\mathbf{s}-\bar{\mathbf{x}})^{-1}; \tag{2.30}$$

finally, if $\mathbf{x}\notin[\mathbf{s}]$ *we have*

$$-(\mathbf{x}^2-2\mathrm{Re}[\mathbf{s}]\mathbf{x}+|\mathbf{s}|^2)^{-1}(\mathbf{x}-\bar{\mathbf{s}})=(\mathbf{s}-\overline{\mathbf{x}})(\mathbf{s}^2-2\mathrm{Re}[\mathbf{x}]\mathbf{s}+|\mathbf{x}|^2)^{-1}. \tag{2.31}$$

Proof. One may prove the identities by direct computations. Let us prove (2.31). To show that the formula is an identity, we multiply by $(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)$ on the left and by $(\mathbf{s}^2 - \mathrm{Re}[\mathbf{x}]\mathbf{s} + |\mathbf{x}|^2)$ on the right. We obtain:

$$\mathbf{x}^2\mathbf{s} - 2\mathrm{Re}[\mathbf{s}]\mathbf{x}\mathbf{s} + 2\mathrm{Re}[\mathbf{s}]|\mathbf{x}|^2 - \overline{\mathbf{x}}|\mathbf{s}|^2 = -\mathbf{x}\mathbf{s}^2 + 2\mathrm{Re}[\mathbf{x}]\mathbf{x}\mathbf{s} - 2\mathrm{Re}[\mathbf{x}]|s|^2 + \overline{\mathbf{s}}|\mathbf{x}|^2$$

which becomes

$$(\mathbf{x}^2 - \mathrm{Re}[\mathbf{x}] + |\mathbf{x}|^2)\mathbf{s} = -\mathbf{x}(\mathbf{s}^2 - \mathrm{Re}[\mathbf{s}] + |\mathbf{s}|^2)$$

that is an identity by (2.21). Note that (2.31) holds for $\mathbf{x} \notin [\mathbf{s}]$, which is equivalent to $\mathbf{s} \notin [\mathbf{x}]$. The identity (2.30) can be proven by taking the inverse of (2.31) and it holds for $\mathbf{x} \neq \overline{\mathbf{s}}$. Easy computations show the validity of the remaining identity. □

We now consider the function $S^{-1}(\mathbf{s}, \mathbf{x}) = S_{\mathbf{s}}^{-1}(\mathbf{x})$ as a function of $\mathbf{x}$. Clearly, its singularities are the entire $(n-1)$-sphere $[\mathbf{s}]$ which reduces to the point $\{\mathbf{s}\}$ when $\mathbf{s}$ is real. The next result analyzes in detail the singularities of $S_{\mathbf{s}}^{-1}(\mathbf{x})$ on each plane $\mathbb{C}_I$ when $\mathbf{s} \notin \mathbb{R}$.

Proposition 2.7.11. *Let* $\mathbf{s} \in \mathbb{R}^{n+1} \setminus \mathbb{R}$. *If* $I \neq I_{\mathbf{s}}$, *then the function* $S^{-1}(\mathbf{s}, \mathbf{x}) = S_{\mathbf{s}}^{-1}(\mathbf{x})$ *has two singularities* $\mathrm{Re}[\mathbf{s}] \pm I|\underline{s}|$ *on the plane* $\mathbb{C}_I$. *On the plane* $\mathbb{C}_{I_{\mathbf{s}}}$, *the restriction of* $S_{\mathbf{s}}^{-1}(\mathbf{x})$, *i.e.,* $(\mathbf{x} - \mathbf{s})^{-1}$, *has only one singularity at the point* $\mathbf{s}$.

Proof. Suppose $\mathbf{s} \in \mathbb{R}^{n+1} \setminus \mathbb{R}$ and consider $S_{\mathbf{s}}^{-1}(\mathbf{x}) = (\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{x}]\mathbf{s} + |\mathbf{x}|^2)^{-1}(\mathbf{s} - \overline{\mathbf{x}})$. The singularities of $S_{\mathbf{s}}^{-1}(\mathbf{x})$ corresponds to the roots of $\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{x}]\mathbf{s} + |\mathbf{x}|^2 = 0$. This equation can be written by splitting real and imaginary parts as

$$\mathrm{Re}[\mathbf{s}]^2 - |\underline{s}|^2 - 2\mathrm{Re}[\mathbf{s}]\mathrm{Re}[\mathbf{x}] + |\mathbf{x}|^2 = 0,$$
$$(\mathrm{Re}[\mathbf{s}] - \mathrm{Re}[\mathbf{x}])\underline{s} = 0.$$

The assumption $\underline{s} \neq 0$ implies $\mathrm{Re}[\mathbf{x}] = \mathrm{Re}[\mathbf{s}]$ and so $|\mathbf{x}| = |\mathbf{s}|$, i.e., the roots correspond to the $(n-1)$-sphere $[\mathbf{s}]$. Consider now the plane $\mathbb{C}_I$. When $I \neq I_{\mathbf{s}}$, $\mathbb{C}_I$ intersect the $(n-1)$-sphere $[\mathbf{s}]$ in $\mathrm{Re}[\mathbf{s}] \pm I|\underline{s}|$ while, when $I = I_{\mathbf{s}}$, $\mathbf{x}$ and $\mathbf{s}$ commute, so

$$S_{\mathbf{s}}^{-1}(\mathbf{x}) = -(\mathbf{x} - \mathbf{s})^{-1}(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x} - \overline{\mathbf{s}}) = -(\mathbf{x} - \mathbf{s})^{-1}$$

and $\mathbf{x}$ is the only singularity of the restriction of $S_{\mathbf{s}}^{-1}(\mathbf{x})$ to the plane $\mathbb{C}_{I_{\mathbf{s}}}$. □

Remark 2.7.12. The previous proposition states that the restriction of $S^{-1}(\mathbf{s}, \mathbf{x})$ to the plane $\mathbb{C}_{I_{\mathbf{s}}}$, has a removable singularity at the point $\mathbf{x} = \overline{\mathbf{s}}$. However, equality (2.31) and the proof of Theorem 2.7.8 show that the function $S^{-1}(\mathbf{s}, \mathbf{x})$ still has a singularity at the point $\mathbf{x} = \overline{\mathbf{s}}$.

The kernel $S^{-1}(\mathbf{s}, \mathbf{x})$ is a left s-monogenic function in $\mathbf{x}$ and a right s-monogenic function in $\mathbf{s}$ so, in principle, it cannot be used in both the Cauchy formulas for left and for right s-monogenic functions. Thus one has to establish which kernel has to be used for a Cauchy formula for right s-monogenic functions. Note that the series expansion of a kernel which is right (resp. left) s-monogenic in the variable $\mathbf{x}$ (resp. $\mathbf{s}$) is of the following form

Definition 2.7.13. *Let* $\mathbf{x}$, $\mathbf{s} \in \mathbb{R}^{n+1}$. *We call*

$$S_R^{-1}(\mathbf{s},\mathbf{x}) := \sum_{n\geq 0} \mathbf{s}^{-n-1}\mathbf{x}^n, \tag{2.32}$$

the right noncommutative Cauchy kernel series.

Remark 2.7.14. *The right noncommutative Cauchy kernel series is convergent for* $|\mathbf{x}| < |\mathbf{s}|$.

We have the following:

Proposition 2.7.15. *The sum of the series* (2.32) *is given by the function*

$$S_R^{-1}(\mathbf{s},\mathbf{x}) = -(\mathbf{x}-\bar{\mathbf{s}})(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}, \tag{2.33}$$

which is defined for $\mathbf{x} \notin [\mathbf{s}]$. *Moreover,* $S_R^{-1}(\mathbf{s},\mathbf{x})$ *is right (resp. left) s-monogenic in the variable* $\mathbf{x}$ *(resp.* $\mathbf{s}$*).*

Proof. It follows the same lines of the proof of Theorem 2.7.4. We just sketch some of the computations. The statement is proved if we show that, for $|\mathbf{x}| < |\mathbf{s}|$, we have

$$\Big(\sum_{n\geq 0} \mathbf{s}^{-n-1}\mathbf{x}^n\Big)(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2) = -(\mathbf{x}-\bar{\mathbf{s}}). \tag{2.34}$$

By computing the product at the left-hand side of (2.34), we obtain:

$$\begin{aligned}
&\mathbf{s}^{-1}\mathbf{x}^2 - 2\mathbf{s}^{-1}\mathrm{Re}[\mathbf{s}]\mathbf{x} + \mathbf{s}^{-1}|\mathbf{s}|^2 + \mathbf{s}^{-2}\mathbf{x}^3 - 2\mathbf{s}^{-2}\mathrm{Re}[\mathbf{s}]\mathbf{x}^2 + \mathbf{s}^{-2}\mathbf{x}|\mathbf{s}|^2 + \dots\\
&= -2\mathbf{s}^{-1}\mathrm{Re}[\mathbf{s}]\mathbf{x} + \mathbf{s}^{-1}|\mathbf{s}|^2 + \mathbf{s}^{-2}\mathbf{x}|\mathbf{s}|^2 + \sum_{n\geq 2} \mathbf{s}^{-(n+1)}(\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{s} + |\mathbf{s}|^2)\mathbf{x}^n\\
&= -2\mathbf{s}^{-1}\mathrm{Re}[\mathbf{s}]\mathbf{x} + \mathbf{s}^{-1}|\mathbf{s}|^2 + \mathbf{s}^{-2}\mathbf{x}|\mathbf{s}|^2\\
&= \mathbf{s}^{-2}(-2\mathrm{Re}[\mathbf{s}] + \bar{\mathbf{s}})\mathbf{s}\mathbf{x} + \mathbf{s}^{-1}\mathbf{s}\bar{\mathbf{s}} = -\mathbf{x} + \bar{\mathbf{s}}.
\end{aligned}$$

The fact that function $S_R^{-1}(\mathbf{s},\mathbf{x})$, which is defined for $\mathbf{x} \notin [\mathbf{s}]$, is left s-monogenic in the variable $\mathbf{s}$ and right s-monogenic in the variable $\mathbf{x}$ can be proved by a direct computation. This concludes the proof. □

Definition 2.7.16. *We will call the expression*

$$S_R^{-1}(\mathbf{s},\mathbf{x}) = -(\mathbf{x}-\bar{\mathbf{s}})(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}, \tag{2.35}$$

defined for $\mathbf{x}^2 - 2\mathbf{x}\mathrm{Re}[\mathbf{s}] + |\mathbf{s}|^2 \neq 0$, *the right noncommutative Cauchy kernel.*

Remark 2.7.17. Analogous considerations as in Remarks 2.7.6 and 2.7.7 and in Theorem 2.7.8 can be done for the right noncommutative Cauchy kernel $S_R^{-1}(\mathbf{s},\mathbf{x})$.

Proposition 2.7.18. *Suppose that* $\mathbf{x}$ *and* $\mathbf{s} \in \mathbb{R}^{n+1}$ *are such that* $\mathbf{x} \notin [\mathbf{s}]$. *The following identity holds:*

$$S_R^{-1}(\mathbf{s},\mathbf{x}) = (\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{x}]\mathbf{s} + |\mathbf{x}|^2)^{-1}(\mathbf{s}-\bar{\mathbf{x}}) = -(\mathbf{x}-\bar{\mathbf{s}})(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}. \tag{2.36}$$

Proof. One may prove the identity by direct computations (compare with the proof of Proposition 2.7.10). □

Remark 2.7.19. The identities (2.31) and (2.36) can be proved not only by direct computation but also in a longer way which can be of some interest. We sketch the lines of this alternative proof. Consider the function $f(\mathbf{x}) = \mathbf{s} - \mathbf{x}$. It is such that $f_I * f_I^c$ has values in $\mathbb{C}_I$ thus it admits an s-monogenic inverse (see Example 2.6.12). One may construct its s-monogenic inverse with respect to the two variables $\mathbf{x}$ and $\mathbf{s}$ on the left and on the right. If one constructs, e.g., the left inverse with respect to $\mathbf{x}$, see Definition 2.6.10, one gets

$$(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)^{-1}(\bar{\mathbf{s}} - \mathbf{x}).$$

By direct computation it follows that this function is right s-monogenic with respect to $\mathbf{s}$, thus it must coincide, by the Identity Principle, with the right s-monogenic inverse of $(\mathbf{s} - \mathbf{x})$ with respect to $\mathbf{s}$, i.e.,

$$(\mathbf{s} - \bar{\mathbf{x}})(\mathbf{s}^2 - 2\mathrm{Re}[\mathbf{x}]\mathbf{s} + |\mathbf{x}|^2)^{-1}$$

thus relation (2.31) holds. Note that we have not provided the construction of the right s-monogenic inverse of a function f, but it is not difficult to check that, when it exists, it coincides with the extension of the function $f_I^c(f_I * f_I^c)^{-1}$. Similarly, one can construct the left s-monogenic inverse of $\mathbf{s} - \mathbf{x}$ with respect to $\mathbf{s}$, then one shows that it is right s-monogenic with respect to $\mathbf{x}$ and so it follows that it must coincide with the right s-monogenic inverse with respect to $\mathbf{x}$, thus equality (2.36) holds.

By comparing the Cauchy kernel functions $S^{-1}(\mathbf{s}, \mathbf{x})$ and $S_R^{-1}(\mathbf{s}, \mathbf{x})$, we conclude that the two functions are different, thus the kernel to be used for the Cauchy formula for right s-monogenic functions is not the kernel $S^{-1}(\mathbf{s}, \mathbf{x})$ used for left s-monogenic functions. However we have the following relation.

Proposition 2.7.20. *Let $x, s \in \mathbb{R}^{n+1}$. The following identity holds:*

$$S^{-1}(\mathbf{x}, \mathbf{s}) = -S_R^{-1}(\mathbf{s}, \mathbf{x}), \qquad \textit{for} \qquad \mathbf{x} \notin [\mathbf{s}].$$

Proof. The identities (2.31) and (2.36) show that by exchanging the role of the variables $\mathbf{x}$ and $\mathbf{s}$ we get $S^{-1}(\mathbf{x}, \mathbf{s}) = -S_R^{-1}(\mathbf{s}, \mathbf{x})$. □

2.8 Cauchy integral formula, II

In this section we prove a Cauchy formula for an s-monogenic function with s-monogenic kernel which is more general than the one proved in Section 2.4. In fact, the formula does not depend on the plane in which the integration path is chosen.

Let us recall the well-known Stokes' theorem in the complex plane (see for example [2]).

Theorem 2.8.1. *Let C be a bounded open set in $\mathbb{C}$ such that its boundary ∂C is a finite union of continuously differentiable Jordan curves. If $f \in \mathcal{C}^1(\overline{C})$, then*

$$\int_{\partial C} f dz = \int_C df \wedge dz = 2i \int_C \frac{\partial f}{\partial \bar{z}} dx \wedge dy.$$

When considering $\mathbb{R}_n$-valued functions, the Stokes' theorem can be rephrased as follows:

Lemma 2.8.2. *Let D_I be a bounded open set on a plane $\mathbb{C}_I$ such that its boundary ∂D_I is a finite union of continuously differentiable Jordan curves. Let f, $g \in \mathcal{C}^1(\overline{D}_I)$ be $\mathbb{R}_n$-valued functions. Then*

$$\int_{\partial D_I} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) = 2 \int_{D_I} ((g(\mathbf{s})\bar{\partial}_I) f(\mathbf{s}) + g(\mathbf{s})(\bar{\partial}_I f(\mathbf{s}))) d\sigma$$

where $\mathbf{s} = u + Iv$ is the variable on $\mathbb{C}_I$, $d\mathbf{s}_I = -Id\mathbf{s}$, $d\sigma = du \wedge dv$.

Proof. Let us choose $n-1$ imaginary units $I_2, \ldots, I_n$ such that $I, I_2, \ldots, I_n$ form an orthonormal basis of $\mathbb{R}_n$ satisfying the defining relations $I_r I_s + I_s I_r = -2\delta_{rs}$. Then it is possible to write

$$f(\mathbf{s}) = \sum_{|A|=0}^{n-1} F_A(\mathbf{s}) I_A,$$
$$g(\mathbf{s}) = \sum_{|A|=0}^{n-1} I_A G_A(\mathbf{s}),$$

where $\mathbf{s} \in \mathbb{C}_I$, $I_A = I_{i_1} \ldots I_{i_s}$, $A = i_1 \ldots i_s$ is a subset of $\{2, \ldots, n\}$ and $F_A(\mathbf{s})$, $G_A(\mathbf{s})$ have values in the complex plane $\mathbb{C}_I$. We have

$$\int_{\partial D_I} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) = \int_{\partial D_I} \Big(\sum_{|A|=0}^{n-1} I_A G_A(\mathbf{s}) \Big) d\mathbf{s}_I \Big(\sum_{|B|=0}^{n-1} F_B(\mathbf{s}) I_B \Big)$$
$$= \sum_{|A|=0,|B|=0}^{n-1} I_A \Big(\int_{\partial D_I} G_A(\mathbf{s}) d\mathbf{s}_I F_B(\mathbf{s}) \Big) I_B.$$

We now use the usual Stokes' theorem in the complex plane $\mathbb{C}_I$ and we write

$$\int_{\partial D_I} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) = \sum_{|A|=0,|B|=0}^{n-1} I_A \Big(\int_{D_I} \frac{\partial}{\partial \bar{\mathbf{s}}} (G_A(\mathbf{s}) F_B(\mathbf{s})) d\bar{\mathbf{s}} \wedge d\mathbf{s}_I \Big) I_B$$
$$= 2 \sum_{|A|=0,|B|=0}^{n-1} I_A \Big(\int_{D_I} (\partial_u + I\partial_v)(G_A(\mathbf{s}) F_B(\mathbf{s})) d\sigma \Big) I_B;$$

we recall that I commutes with F_A and G_B which have values in $\mathbb{C}_I$, and that $d\sigma$ is real, thus we obtain

$$\begin{aligned}\int_{\partial D_I} g(\mathbf{s})d\mathbf{s}_I f(\mathbf{s}) &= 2\sum_{|A|,|B|=0}^{n-1}\Big(\int_{D_I} I_A(\partial_u(G_A)+\partial_v(G_A)I)F_B I_B d\sigma \\ &\quad + \int_{D_I} I_A G_A(\partial_u F_B + I\partial_v F_B)I_B d\sigma\Big) \\ &= 2\int_{D_I}\sum_{|A|,|B|=0}^{n-1} I_A(G_A\overline{\partial}_I)F_B I_B d\sigma \\ &\quad + 2\int_{D_I}\sum_{|A|,|B|=0}^{n-1} I_A G_A(\overline{\partial}_I F_B)I_B d\sigma \\ &= 2\int_{D_I}((g(\mathbf{s})\bar{\partial}_I)f(\mathbf{s}) + g(\mathbf{s})(\bar{\partial}_I f(\mathbf{s})))d\sigma\end{aligned}$$

and we get the statement. □

An immediate consequence of the above lemma is the following:

Corollary 2.8.3. *Let f and g be left s-monogenic and right s-monogenic functions, respectively, defined on an open set U. For any $I \in \mathbb{S}$ and any open bounded set D_I in $U \cap \mathbb{C}_I$ whose boundary is a finite union of continuously differentiable Jordan curves, we have*

$$\int_{\partial D_I} g(\mathbf{s})d\mathbf{s}_I f(\mathbf{s}) = 0.$$

Theorem 2.8.4 (The Cauchy formula with s-monogenic kernel). *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Suppose that $\partial(U \cap \mathbb{C}_I)$ is a finite union of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Set $d\mathbf{s}_I = -d\mathbf{s}I$ for $I \in \mathbb{S}$. If f is a (left) s-monogenic function on a set that contains $\overline{U}$, then*

$$f(\mathbf{x}) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}) \tag{2.37}$$

where $S^{-1}(\mathbf{s},\mathbf{x})$ is defined in (2.29) and the integral does not depend on U and on the imaginary unit $I \in \mathbb{S}$.

If f is a right s-monogenic function on a set that contains $\overline{U}$, then

$$f(\mathbf{x}) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(\mathbf{s})d\mathbf{s}_I S_R^{-1}(\mathbf{s},\mathbf{x}) \tag{2.38}$$

$$= -\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} f(\mathbf{s})d\mathbf{s}_I S^{-1}(\mathbf{x},\mathbf{s}) \tag{2.39}$$

where $S_R^{-1}(\mathbf{s},\mathbf{x})$ is defined in (2.35) and the integral (2.38) does not depend on the choice of the imaginary unit $I \in \mathbb{S}$ and on U.

Proof. First of all, the integral at the right-hand side of (2.37) does not depend on the open set U: this follows from the fact that $S^{-1}(\mathbf{s},\mathbf{x})$ is right s-monogenic in s, and Corollary 2.8.3. Let us show that the integral (2.37) does not depend on the choice of the imaginary unit $I \in \mathbb{S}$. The zeros of the function $\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2 = 0$ consist either of a real point $\mathbf{x}$ or a 2-sphere $[\mathbf{x}]$. On $\mathbb{C}_{I_\mathbf{x}}$ we find only the point $\mathbf{x}$ as a singularity and the result follows from the Cauchy formula on the plane $\mathbb{C}_{I_\mathbf{x}}$. When the singularity is a real number, the integral reduces again to the Cauchy integral of complex analysis. If the zero is not real, on any complex plane $\mathbb{C}_I$ we find the two zeros $s_{1,2} = x_0 \pm I|\underline{x}|$. In this case, we calculate the residues in the points s_1 and s_2 on the plane $\mathbb{C}_I$ for $I \neq I_\mathbf{x}$. Let us start with s_1 by setting the positions

$$\begin{aligned}
\mathbf{s} &= x_0 + I|\underline{x}| + \varepsilon e^{I\theta},\\
s_0 &= x_0 + \varepsilon\cos\theta,\\
\bar{\mathbf{s}} &= x_0 - I|\underline{x}| + \varepsilon e^{-I\theta},\\
d\mathbf{s}_I &= -[\varepsilon I e^{I\theta}]I d\theta = \varepsilon e^{I\theta} d\theta,
\end{aligned}$$

and

$$|\mathbf{s}|^2 = x_0^2 + 2x_0\varepsilon\cos\theta + \varepsilon^2 + |\underline{x}|^2 + 2\varepsilon\sin\theta|\underline{x}|.$$

We have

$$\begin{aligned}
2\pi I_1^\varepsilon = \int_0^{2\pi} &-(-2\mathbf{x}\varepsilon\cos\theta + 2x_0\varepsilon\cos\theta + \varepsilon^2 + 2\varepsilon\sin\theta|\underline{x}|)^{-1}\\
&\cdot (\mathbf{x} - [x_0 - I|\underline{x}| + \varepsilon e^{-I\theta}])\varepsilon e^{I\theta} d\theta f(x_0 + I|\underline{x}| + \varepsilon e^{I\theta}),
\end{aligned}$$

and for $\varepsilon \to 0$ we get

$$\begin{aligned}
&2\pi I_1^0\\
&= \int_0^{2\pi} (2\mathbf{x}\cos\theta - 2x_0\cos\theta - 2\sin\theta|\underline{x}|)^{-1}(\underline{x} + I|\underline{x}|)e^{I\theta} d\theta f(x_0 + I|\underline{x}|)\\
&= \frac{1}{2}\int_0^{2\pi} (\underline{x}\cos\theta - \sin\theta|\underline{x}|)^{-1}(\underline{x} + I|\underline{x}|)e^{I\theta} d\theta f(x_0 + I|\underline{x}|)\\
&= -\frac{1}{2|\underline{x}|^2}\int_0^{2\pi} (\underline{x}\cos\theta + \sin\theta|\underline{x}|)(\underline{x} + I|\underline{x}|)[\cos\theta + I\sin\theta] d\theta f(x_0 + I|\underline{x}|)\\
&= -\frac{1}{2|\underline{x}|^2}\int_0^{2\pi} [(\underline{x})^2\cos\theta + \sin\theta|\underline{x}|\underline{x} + \underline{x}I|\underline{x}|\cos\theta\\
&\quad + \sin\theta|\underline{x}|^2 I][\cos\theta + I\sin\theta] d\theta f(x_0 + I|\underline{x}|).
\end{aligned}$$

With some calculations we obtain

$$\begin{aligned}
2\pi I_1^0 &= -\frac{1}{2|\underline{x}|^2}\int_0^{2\pi} \Big[(\underline{x})^2 + \underline{x}I|\underline{x}|\cos^2\theta + \sin^2\theta|\underline{x}|\underline{x}I\Big] d\theta f(x_0 + I|\underline{x}|)\\
&= -\frac{1}{2|\underline{x}|^2}\Big[2\pi(\underline{x})^2 + \pi\underline{x}I|\underline{x}| + \pi|\underline{x}|\underline{x}I\Big] f(x_0 + I|\underline{x}|)
\end{aligned}$$

$$= \frac{\pi}{|\underline{x}|}\Big[|\underline{x}| - \underline{x}I\Big] f(x_0 + I|\underline{x}|).$$

Recalling that $\underline{x}/|\underline{x}| = I_{\mathbf{x}}$ we get the first residue

$$I_1^0 = \frac{1}{2}\Big[1 - I_{\mathbf{x}}I\Big] f(x_0 + I|\underline{x}|).$$

With analogous calculations we prove that the residue in s_2 is

$$I_2^0 = \frac{1}{2}\Big[1 + I_{\mathbf{x}}I\Big] f(x_0 - I|\underline{x}|).$$

So by the residues theorem we get

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}) = I_1^0 + I_2^0.$$

The statement now follows from the Representation Formula. Formula (2.38) can be deduced with similar arguments while formula (2.39) is a consequence of Proposition 2.7.20. □

We conclude this section with the formula for the derivatives of an s-monogenic function using the s-monogenic Cauchy kernel.

Theorem 2.8.5 (Derivatives using the s-monogenic Cauchy kernel). *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain. Suppose that $\partial(U\cap\mathbb{C}_I)$ is a finite union of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Set $d\mathbf{s}_I = -d\mathbf{s}I$ for $I \in \mathbb{S}$. Let f be an s-monogenic function on an open set that contains $\overline{U}$ and set $\mathbf{x} = x_0 + \underline{x}$, $\mathbf{s} = s_0 + \underline{s}$. Then*

$$\partial_{x_0}^n f(\mathbf{x}) = \frac{n!}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-n-1}(\mathbf{x}-\overline{\mathbf{s}})^{*(n+1)} d\mathbf{s}_I f(\mathbf{s})$$
$$= \frac{n!}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} [S^{-1}(\mathbf{s},\mathbf{x})(\mathbf{x}-\overline{\mathbf{s}})^{-1}]^{n+1}(\mathbf{x}-\overline{\mathbf{s}})^{*(n+1)} d\mathbf{s}_I f(\mathbf{s}) \tag{2.40}$$

where

$$(\mathbf{x}-\overline{\mathbf{s}})^{*n} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}\mathbf{x}^{n-k}\overline{\mathbf{s}}^k, \tag{2.41}$$

and $S^{-1}(\mathbf{s},\mathbf{x})$ is defined in (2.29). Moreover, the integral does not depend on U and on the imaginary unit $I \in \mathbb{S}$.

Proof. First of all, we recall that the s-derivative defined in (2.4) coincides, for s-monogenic functions, with the partial derivative with respect to the scalar coordinate x_0. To compute $\partial_{x_0}^n f(\mathbf{x})$, we can compute the derivative of the integrand, since f and its derivatives with respect to x_0 are continuous functions on $\partial(U\cap\mathbb{C}_I)$. Thus we get

$$\partial_{x_0}^n f(\mathbf{x}) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \partial_{x_0}^n[S^{-1}(\mathbf{s},\mathbf{x})]d\mathbf{s}_I f(\mathbf{s}).$$

To prove the statement, it is sufficient to compute $\partial_{x_0}^n [S^{-1}(\mathbf{s},\mathbf{x})]$ by recurrence. Consider the derivative of $\partial_{x_0} S^{-1}(\mathbf{s},\mathbf{x})$:

$$\begin{aligned}
&\partial_{x_0} S^{-1}(\mathbf{s},\mathbf{x}) \\
&= -(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-2}(2\mathbf{x} - 2s_0)(\mathbf{x} - \overline{\mathbf{s}}) - (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-1} \\
&= (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-2}[2\mathbf{x}^2 - 2\mathbf{x}\overline{\mathbf{s}} - 2s_0\mathbf{x} + 2s_0\overline{\mathbf{s}} - \mathbf{x}^2 + 2s_0\mathbf{x} - |\mathbf{s}|^2] \\
&= (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-2}[\mathbf{x}^2 - 2\mathbf{x}\overline{\mathbf{s}} + \overline{\mathbf{s}}^2] = (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-2}(\mathbf{x} - \overline{\mathbf{s}})^{*2}.
\end{aligned}$$

We now assume

$$\partial_{x_0}^n S^{-1}(\mathbf{s},\mathbf{x}) = (-1)^{n+1} n!(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-(n+1)}(\mathbf{x} - \overline{\mathbf{s}})^{*(n+1)},$$

and we compute $\partial_{x_0}^{n+1} S^{-1}(\mathbf{s},\mathbf{x})$. We have

$$\begin{aligned}
&\partial_{x_0}^{n+1} S^{-1}(\mathbf{s},\mathbf{x}) = \partial_{x_0}[(-1)^{n+1} n!(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-(n+1)}(\mathbf{x} - \overline{\mathbf{s}})^{*(n+1)}] \\
&= (-1)^{n+2}(n+1)!(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-(n+2)}(2\mathbf{x} - 2s_0)(\mathbf{x} - \overline{\mathbf{s}})^{*(n+1)} \\
&\quad + (-1)^{n+1}(n+1)!(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-(n+1)}(\mathbf{x} - \overline{\mathbf{s}})^{*n} \\
&= (-1)^{n+2}(n+1)!(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-(n+2)}[(2\mathbf{x} - 2s_0)(\mathbf{x} - \overline{\mathbf{s}}) \\
&\quad - (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)] * (\mathbf{x} - \overline{\mathbf{s}})^{*n};
\end{aligned}$$

here we have used the fact that the s-monogenic product coincides with the usual one when the coefficients are real numbers, so

$$\begin{aligned}
&\partial_{x_0}^{n+1} S^{-1}(\mathbf{s},\mathbf{x}) \\
&= (-1)^{n+2}(n+1)!(\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-(n+2)}[\mathbf{x}^2 - 2\mathbf{x}\overline{\mathbf{s}} + \overline{\mathbf{s}}^2] * (\mathbf{x} - \overline{\mathbf{s}})^{*n}.
\end{aligned}$$

We get the last equality in (2.40) by recalling that

$$S^{-1}(\mathbf{s},\mathbf{x})(\mathbf{x} - \overline{\mathbf{s}})^{-1} = (\mathbf{x}^2 - 2s_0\mathbf{x} + |\mathbf{s}|^2)^{-1}.$$ □

Theorem 2.8.6 (Cauchy formula II outside an axially symmetric s-domain). *Let $U \subset \mathbb{R}^{n+1}$ be a bounded axially symmetric s-domain and assume that $U^c = \mathbb{R}^{n+1} \setminus \overline{U}$ is connected. Let $f : U^c \to \mathbb{R}_n$ be a left s-monogenic function with $\lim_{\mathbf{x}\to\infty} f(x) = a$. If $\mathbf{x} \in U^c$, then*

$$f(\mathbf{x}) = a - \frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})\, d\mathbf{s}_I f(\mathbf{s}),$$

where V is an axially symmetric s-domain containing U such that $\partial(V \cap \mathbb{C}_I)$ is a union of a finite number of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Moreover, the integral does not depend on V and on the imaginary unit $I \in \mathbb{S}$.

Proof. Let $\mathbf{x} \in U^c$. Then there exists $r > 0$ and a real point α such that the ball $B = B(\alpha, r)$ satisfies $B \supset U$ and $\mathbf{x} \in B$. Let V be an axially symmetric s-domain containing U such that $\partial(V \cap \mathbb{C}_I)$ is a union of a finite number of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Then f is s-monogenic on $B \setminus V$ and we can apply the Cauchy formula to compute $f(\mathbf{x})$. We obtain

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2\pi}\int_{\partial((B\setminus V)\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}) \\ &= \frac{1}{2\pi}\int_{\partial(B\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}) - \frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}). \end{aligned}$$

By setting the positions

$$\mathbf{s} = \alpha + re^{I\theta}$$

we can compute the integral on $\partial(B\cap\mathbb{C}_I)$ in the standard way, and letting $r \to \infty$ we obtain that the integral equals $a = \lim_{r\to\infty} f$, therefore,

$$f(\mathbf{x}) = a - \frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}).$$

The integral does not depend on V and on the imaginary unit $I \in \mathbb{S}$, thanks to the Cauchy formula on bounded axially symmetric s-domains. □

We finally obtain a version of the Borel-Pompeiu formula.

Theorem 2.8.7 (Borel-Pompeiu formula). *Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric open bounded set such that $\partial(U\cap\mathbb{C}_I)$ is a union of a finite number of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. Let $f : \overline{U} \to \mathbb{R}^{n+1}$ be a function of class $\mathcal{C}^1$ and set $d\mathbf{s}_I = -Id\mathbf{s}$. For every $\mathbf{x} \in U$, $\mathbf{x} = u + I_{\mathbf{x}}v$ and $I \in \mathbb{S}$, we have*

$$\begin{aligned} &\frac{1}{2}\Big[1 - I_{\mathbf{x}}I\Big]f(u+Iv) + \frac{1}{2}\Big[1 + I_{\mathbf{x}}I\Big]f(u-Iv) \\ &= \frac{1}{2\pi}\Big(\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}) + \int_{U\cap\mathbb{C}_I} S^{-1}(\mathbf{s},\mathbf{x})\overline{\partial}_I f(\mathbf{s})d\mathbf{s}_I \wedge d\bar{\mathbf{s}}\Big). \end{aligned} \tag{2.42}$$

In particular, when $I = I_{\mathbf{x}}$ we have

$$\begin{aligned} f(\mathbf{x}) = \frac{1}{2\pi}\Big(&\int_{\partial(U\cap\mathbb{C}_{I_{\mathbf{x}}})} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_{I_{\mathbf{x}}} f(\mathbf{s}) \\ &+ \int_{U\cap\mathbb{C}_{I_{\mathbf{x}}}} S^{-1}(\mathbf{s},\mathbf{x})\overline{\partial}_{I_{\mathbf{x}}} f(s)d\mathbf{s}_{I_{\mathbf{x}}} \wedge d\bar{\mathbf{s}}\Big). \end{aligned} \tag{2.43}$$

Proof. Let us set $\mathbf{x} = u + I_{\mathbf{x}}v$ and let us define

$$U_\varepsilon = \{\mathbf{s} = u' + I_s v' \in U \mid |(u+Iv) - (u'+Iv')| > \varepsilon \ \ \forall I \in \mathbb{S}\}$$

where ε is a positive number less than the distance from the $(n-1)$-sphere $u + \mathbb{S}v$

defined by $\mathbf{x}$ to the complement of U. The zeros of the function $\mathbf{x}^2-2\mathrm{Re}[\mathbf{s}]\mathbf{x}+|\mathbf{s}|^2=0$ consist either of a point on the real axis, or an $(n-1)$-sphere $u+Iv$. On $\mathbb{C}_{I_\mathbf{x}}$ we find only the point $\mathbf{x}$ as a singularity and the result follows from the Pompeiu formula on the complex plane $\mathbb{C}_{I_\mathbf{x}}$. When the singularity is a real number, S^{-1} is the standard Cauchy kernel and again the statement follows from the Pompeiu formula on the complex plane $\mathbb{C}_I$ for every $I\in\mathbb{S}$. If the zero is not real, on any complex plane $\mathbb{C}_I$ we find two zeros $\mathbf{s}_{1,2}=x_0\pm I|\underline{x}|$. Thus $\partial U_\varepsilon=\partial U-\partial B_1-\partial B_2$ where ∂B_i is the boundary of ball B_i with center $\mathbf{s}_i$ and radius ε.

From Lemma 2.8.2 applied to the functions $S^{-1}(\mathbf{s},\mathbf{x})$, $f(\mathbf{s})$ and since $S^{-1}(\mathbf{s},\mathbf{x})$ is right s-monogenic in the variable $\mathbf{s}$, we obtain

$$\frac{1}{2\pi}\int_{U_\varepsilon\cap\mathbb{C}_I} S^{-1}(\mathbf{s},\mathbf{x})\overline{\partial}_I f(\mathbf{s})d\mathbf{s}_I\wedge d\bar{\mathbf{s}}+\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s})$$
$$=\mathfrak{I}_1^\varepsilon(\mathbf{x})+\mathfrak{I}_2^\varepsilon(\mathbf{x})$$

where

$$\mathfrak{I}_1^\varepsilon(\mathbf{x}):=\frac{1}{2\pi}\int_{\partial(B_1\cap\mathbb{C}_I)} S^{-1}(s,\mathbf{x})d\mathbf{s}_I f(\mathbf{s}),$$
$$\mathfrak{I}_2^\varepsilon(\mathbf{x}):=\frac{1}{2\pi}\int_{\partial(B_2\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}).$$

With similar computations as in the proof of Theorem 2.8.4, by letting $\varepsilon\to 0$ and after some computations we get

$$\mathfrak{I}_1^0(\mathbf{x})=\frac{1}{2}\Big[1-I_\mathbf{x}I\Big]f(x_0+I|\underline{x}|).$$

Similarly, the integral related to $\mathbf{s}_2$ turns out to be

$$\mathfrak{I}_2^0(\mathbf{x})=\frac{1}{2}\Big[1+I_\mathbf{x}I\Big]f(x_0-I|\underline{x}|).$$

So we get

$$\mathfrak{I}_1^0(\mathbf{x})+\mathfrak{I}_2^0(\mathbf{x})=\frac{1}{2}\Big[1-I_qI\Big]f(x_0+I|\underline{x}|)+\frac{1}{2}\Big[1+I_\mathbf{x}I\Big]f(x_0-I|\underline{x}|),$$

and this concludes the proof. □

Remark 2.8.8. Note that formula (2.43) is not surprising and in fact is the exact analog of the Borel-Pompeiu formula in the complex case. Formula (2.42) on the other hand, highlights a new phenomenon: given a point $\mathbf{x}$ and an imaginary unit $I\in\mathbb{S}$ there are exactly two points in $\mathbb{C}_I$ on the same sphere of $\mathbf{x}$ and formula (2.42) shows how to obtain an integral representation of f at those points.

Remark 2.8.9. The Cauchy formula in Theorem 2.8.4 follows as an immediate consequence of the Borel-Pompeiu formula and of the Representation Formula.

2.9 Duality Theorems

In this section we prove the algebraic isomorphism between the $\mathbb{R}_n$-module of functionals acting on $\mathcal{G}(K) := \operatorname{ind\,lim}_U \mathcal{M}^R(U)$ where K is a connected axially symmetric compact set such that its intersection with every complex plane $\mathbb{C}_I$ remains connected, and the $\mathbb{R}_n$-module of s-monogenic functions defined in the complement of K and vanishing at infinity. The results we obtain are the analogs, in this setting, of those obtained by Köthe in [67] and generalized by Grothendieck, see [57].

Consider the set $\mathcal{C}^\infty(U, \mathbb{R}_n)$ of infinitely differentiable functions defined on an open set $U \subseteq \mathbb{R}^{n+1}$ with values in $\mathbb{R}_n$. This set is an $\mathbb{R}_n$-bimodule with respect to the standard sum of functions and multiplication of a function by a Clifford number. To endow $\mathcal{C}^\infty(U, \mathbb{R}_n)$ with a locally convex topology, we follow [7] and consider an increasing sequence of compact sets $\{K_j\}_{j\in\mathbb{N}}$, $K_j \subset \mathbb{R}^{n+1}$, such that

$$K_0 \Subset K_1 \Subset \dots, \qquad U = \cup_{j=0}^{\infty} K_j,$$

and we introduce the family of seminorms $\{p_{j,r},\ j, r \in \mathbb{N}\}$ defined by

$$p_{j,r}(f) := \sup_{|\alpha|\le r} \sup_{\mathbf{x}\in K_j} |\partial^\alpha f(\mathbf{x})|, \quad f \in \mathcal{C}^\infty(U, \mathbb{R}_n),$$

where

$$\partial^\alpha = \frac{\partial^{\alpha_0}}{\partial x_0^{\alpha_0}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \qquad |\alpha| = \sum_{i=0}^{n} \alpha_i.$$

This topology coincides with the product topology $\prod_A \mathcal{C}^\infty(U, \mathbb{R})$ where A is a multi-index which can be identified with an element in the power set of $\{1, \dots, n\}$. Thus we have the following result:

Theorem 2.9.1. *The set $\mathcal{C}^\infty(U, \mathbb{R}_n)$ is a Fréchet $\mathbb{R}_n$-bimodule.*

Proposition 2.9.2. *Let U be an open set in $\mathbb{R}^{n+1}$. The sets $\mathcal{M}^R(U)$ (resp. $\mathcal{M}^L(U)$) are Fréchet left (resp. right) $\mathbb{R}_n$-modules with respect to the topology of uniform convergence over compact sets.*

Proof. The set $\mathcal{C}^\infty(U)$ with the topology of uniform convergence on compact sets is a Fréchet bimodule. The sets $\mathcal{M}^R(U)$ and $\mathcal{M}^L(U)$ are closed submodules of $\mathcal{C}^\infty(U)$. Indeed, if we choose a sequence $\{f_m\}_{m\in\mathbb{N}} \subset \mathcal{M}^R(U)$, then, by definition, for every $I \in \mathbb{S}$ we have that the function f_m satisfies $\overline{\partial}_I f_{m,I}(u+Iv) = 0$ on $U \cap \mathbb{C}_I$. Let f be the limit function of $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{C}^\infty(U)$. The restriction of $f \in \mathcal{C}^\infty(U)$ to a plane $\mathbb{C}_I$ is the limit of the restrictions $f_{m,I}$ thus, by the uniform convergence of the derivatives of $\{f_{m,I}\}$, it satisfies $\overline{\partial}_I f_I(u + Iv) = 0$ on $U \cap \mathbb{C}_I$. This proves that $\mathcal{M}^R(U)$ is a Fréchet module with the topology induced by the topology of $\mathcal{C}^\infty(U)$. The same argument applies to $\mathcal{M}^L(U)$. □

Remark 2.9.3. The same argument used in the proof shows also that $\mathcal{M}^R(U)$ and $\mathcal{M}^L(U)$ are Montel modules, since they are closed submodules of $\mathcal{C}^\infty(U)$ which is a Montel $\mathbb{R}_n$-bimodule.

Definition 2.9.4. *Let $K \subset \mathbb{R}^{n+1}$ be a compact set. We define a set of germs of functions defined by*

$$\mathcal{G}(K) := \operatorname{ind} \lim_{U \text{open} \supset K} \mathcal{M}^R(U).$$

In the sequel, we will use the same letter φ both to denote an element $\varphi \in \mathcal{G}(K)$ and an s-monogenic extension of φ to some neighborhood $U \subseteq \mathbb{R}^{n+1}$ of K. Because of Proposition 2.9.2, $\mathcal{G}(K)$ is a limit of Fréchet $\mathbb{R}_n$-modules, and it is naturally endowed with an LF–topology: a seminorm on $\mathcal{G}(K)$ is every seminorm that is continuous on every $\mathcal{M}^R(U)$. Even though $\mathcal{G}(K)$ is not a Fréchet $\mathbb{R}_n$-module itself, it is possible to characterize its topology in terms of convergence of sequences as in the following result:

Proposition 2.9.5. *Let $K \subset \mathbb{R}^{n+1}$ be a compact set. A sequence $\{\varphi_j\}$ of germs in $\mathcal{G}(K)$ converges to a germ $\varphi \in \mathcal{G}(K)$, if $\varphi_j(\mathbf{x})$ converges uniformly to $\varphi(\mathbf{x})$ in a neighborhood $U \subset \mathbb{R}^{n+1}$ of K.*

Proof. It is a consequence of the definition of inductive limit topology of $\mathcal{G}(K)$. □

Definition 2.9.6. *We call a connected compact set K such that $K \cap \mathbb{R} \neq \emptyset$ and its intersection $K \cap \mathbb{C}_I$ is connected for all $I \in \mathbb{S}$ an s-compact set.*

Let K be an s-compact set in $\overline{\mathbb{R}}^{n+1} := \mathbb{R}^{n+1} \cup \{\infty\}$.

We denote by $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \setminus K)$ the right $\mathbb{R}_n$-module of left s-monogenic functions on $\overline{\mathbb{R}}^{n+1} \setminus K$ which vanish at infinity.

Theorem 2.9.7. *Let K be an axially symmetric s-compact set in $\mathbb{R}^{n+1}$. There is an $\mathbb{R}_n$-module isomorphism*

$$(\mathcal{G}(K))' \cong \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \setminus K)$$

where $(\mathcal{G}(K))'$ is the set of left $\mathbb{R}_n$–linear continuous functionals on $\mathcal{G}(K)$.

Proof. Let us define a map $T : \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \setminus K) \to (\mathcal{G}(K))'$. For any function $f \in \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \setminus K)$ we construct a functional $\mu = \mu_f$. Let $g \in \mathcal{G}(K)$ and let us denote by the same symbol g also its s-monogenic extension to an axially symmetric s-domain $U \supset K$. Let us fix an element $I \in \mathbb{S}$ and define

$$\langle \mu_f, g \rangle := \int_{\partial(U \cap \mathbb{C}_I)} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}). \tag{2.44}$$

We have to show that the definition does not depend on the choice of U and on the extension g. If we replace U by another axially symmetric s-domain V containing K we have

$$\begin{aligned} &\int_{\partial(U \cap \mathbb{C}_I)} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) - \int_{\partial(V \cap \mathbb{C}_I)} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) \\ &= \int_{\partial((U \setminus V) \cap \mathbb{C}_I)} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) = 0 \end{aligned}$$

by Lemma 2.8.2; indeed f, g are s-monogenic functions on the left and on the right, respectively, on $U \setminus V$. If we replace g by another extension, the value of integral (2.44) is not affected since all the extensions of g coincide on small open sets containing K. The map μ_f is left $\mathbb{R}_n$-linear and continuous on $\mathcal{G}(K)$ by its definition. Thus the map T defined by

$$T(f) = \mu_f,$$

is well defined and right $\mathbb{R}_n$-linear. Let us now show that there is a map T' which is the inverse of T. Let us consider any $\mu \in (\mathcal{G}(K))'$, and define the function

$$\mathfrak{F}(\mathbf{x}) := -\frac{1}{2\pi} \langle \mu, S^{-1}(\mathbf{s}, \mathbf{x}) \rangle. \tag{2.45}$$

Note that μ acts on the variable s and $S^{-1}(\mathbf{s}, \mathbf{x})$ is right s-monogenic with respect to it. Since μ is a linear functional, we have

$$\begin{aligned} \bar{\partial}_I \mathfrak{F}_I(\mathbf{x}) &= -\frac{1}{2\pi} \bar{\partial}_I \langle \mu, S^{-1}(\mathbf{s}, u + Iv) \rangle \\ &= -\frac{1}{2\pi} \langle \mu, \bar{\partial}_I S^{-1}(\mathbf{s}, u + Iv) \rangle = 0, \qquad \forall I \in \mathbb{S}. \end{aligned}$$

Thus the function $\mathfrak{F}(\mathbf{x})$ is left s-monogenic for $\mathbf{x} \notin [\mathbf{s}]$, $\mathbf{s} \in K$ so, by the hypothesis on K, it is s-monogenic on the complement of K and vanishes at infinity, i.e., $\mathfrak{F} \in \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \setminus K)$. Define now

$$T' : (\mathcal{G}(K))' \to \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \setminus K), \qquad T'(\mu) = \mathfrak{F}.$$

The map T' is well defined and right $\mathbb{R}_n$-linear. Let us show that T' is a right inverse of T, i.e., that $T \cdot T' = \mathrm{id}_{(\mathcal{G}(K))'}$. Let $\mu \in (\mathcal{G}(K))'$ and consider $T'(\mu) = \mathfrak{F}$.

The functional $T(T'(\mu))$ acts on right s-monogenic functions as follows:

$$\begin{aligned}\langle T(T'(\mu)), g\rangle &= \langle T(\mathfrak{F}), g\rangle = \int_{\partial(U\cap\mathbb{C}_I)} g(\mathbf{x})d\mathbf{x}_I\mathfrak{F}(\mathbf{x})\\ &= -\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g(\mathbf{x})d\mathbf{x}_I\langle\mu, S^{-1}(\mathbf{s},\mathbf{x})\rangle\\ &= \langle\mu, -\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g(\mathbf{x})d\mathbf{x}_I S^{-1}(\mathbf{s},\mathbf{x})\rangle\\ &= \langle\mu, \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g(\mathbf{x})d\mathbf{x}_I S_R^{-1}(\mathbf{x},\mathbf{s})\rangle = \langle\mu, g\rangle,\end{aligned}$$

so we get $T(T'(\mu)) = \mu$. Let us now show that T' is a left inverse of T, i.e., that $T'\cdot T = \mathrm{id}_{\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)}$. Consider $f\in\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)$, the functional $T(f)=\mu_f$ defined in (2.44) and $T'(\mu_f)$. By Theorem 2.8.6 and the fact that f vanishes at infinity, we have

$$\begin{aligned}T'(T(f)) = T'(\mu_f) &= -\frac{1}{2\pi}\langle\mu_f, S^{-1}(\mathbf{s},\mathbf{x})\rangle\\ &= -\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{s},\mathbf{x})d\mathbf{s}_I f(\mathbf{s}) = f(\mathbf{x}),\end{aligned}$$

that is $T'(T(f)) = f$. This concludes the proof. □

In analogy with the complex case, we give the following definition.

Definition 2.9.8. *The function*

$$\mathfrak{F}(\mathbf{x}) := -\frac{1}{2\pi}\langle\mu, S^{-1}(\mathbf{s},\mathbf{x})\rangle$$

is called the Fantappié indicatrix of the functional $\mu\in(\mathcal{G}(K))'$.

One could be tempted to dualize Theorem 2.9.7 by simply taking the dual of the sets in its statement. Since

$$(\mathcal{G}(K))' \cong \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K),$$

one could take the dual on both sides and obtain

$$(\mathcal{G}(K))'' \cong (\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))',$$

and attempt to conclude that

$$\mathcal{G}(K) \cong (\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))'$$

by using some reflexivity property of $\mathcal{G}$. This approach, however, is premature, and at this stage we need to give a direct proof of such an isomorphism. In the next section, we will show how to make such an attempt rigorous.

Theorem 2.9.9. *Let K be an axially symmetric s-compact set in $\mathbb{R}^{n+1}$. Then there is an $\mathbb{R}_n$-module isomorphism*

$$(\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))' \cong \mathcal{G}(K).$$

Proof. Fix any $g \in \mathcal{G}(K)$ and consider, for every $f \in \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)$, the integral

$$\langle \phi_g, f\rangle := \int_{\partial(U\cap\mathbb{C}_I)} g(\mathbf{s}) d\mathbf{s}_I f(\mathbf{s}) \tag{2.46}$$

where U denotes an axially symmetric s-domain containing K and where we have fixed $I \in \mathbb{S}$. For any $g \in \mathcal{G}(K)$, the integral (2.46) defines a continuous right linear $\mathbb{R}_n$–functional ϕ_g on $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)$. Therefore we have a map

$$\mathcal{T} : \mathcal{G}(K) \longrightarrow (\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))',$$

defined by setting $\mathcal{T}(g) = \phi_g$ for any fixed $g \in \mathcal{G}(K)$. The map is injective: if $g_1 \neq g_2$, then the functionals ϕ_{g_1}, ϕ_{g_2} (defined by g_1 and g_2) are different. Indeed, let $\mathbf{x} \in K$ and consider the action of the two functionals on the function $S^{-1}_R(\mathbf{s},\mathbf{x}) = S^{-1}_{R,\mathbf{x}}(\mathbf{s}) \in \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)$; then we have

$$\frac{1}{2\pi}\langle \phi_{g_1}, S^{-1}_R(\mathbf{s},\mathbf{x})\rangle = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g_1(\mathbf{s}) d\mathbf{s}_I S^{-1}_R(\mathbf{s},\mathbf{x}) = g_1(\mathbf{x}),$$

and

$$\frac{1}{2\pi}\langle \phi_{g_2}, S^{-1}_R(\mathbf{s},\mathbf{x})\rangle = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} g_2(\mathbf{s}) d\mathbf{s}_I S^{-1}_R(\mathbf{s},\mathbf{x}) = g_2(\mathbf{x}),$$

hence we have a one-to-one mapping:

$$\mathcal{T} : \mathcal{G}(K) \longrightarrow (\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))'.$$

To conclude the proof it is sufficient to show that $\mathcal{T}$ admits a right inverse. Let

$$\phi : \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K) \to \mathbb{R}_n$$

be a continuous right $\mathbb{R}_n$–linear map, acting continuously on $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)$ with its natural topology. It allows us to define

$$\psi(\mathbf{x}) = \frac{1}{2\pi}\langle \phi, S^{-1}_R(\mathbf{s},\mathbf{x})\rangle, \tag{2.47}$$

where the functional ϕ acts on the variable $\mathbf{s}$. The function $\psi(\mathbf{x})$ is right s-monogenic, as one can check directly, hence $\psi \in \mathcal{G}(K)$. Let

$$\mathcal{T}' : \ (\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))' \to \mathcal{G}(K)$$

be the map defined by

$$\mathcal{T}'(\phi) = \frac{1}{2\pi}\langle \phi, S_R^{-1}(\mathbf{s},\mathbf{x})\rangle = \psi(\mathbf{x}).$$

Now we have to show that $\mathcal{T}\cdot\mathcal{T}' = \mathrm{id}_{(\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))'}$. Since

$$\begin{aligned}\langle \mathcal{T}(\mathcal{T}'(\phi)), f\rangle = \langle \mathcal{T}(\psi), f\rangle &= \frac{1}{2\pi}\langle \mathcal{T}\Big(\langle \phi, S_R^{-1}(\mathbf{s},\mathbf{x})\rangle\Big), f\rangle \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \langle \phi, S_R^{-1}(\mathbf{s},\mathbf{x})\rangle d\mathbf{x}_I f(\mathbf{x}) \\ &= \langle \phi, -\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{x},\mathbf{s}) d\mathbf{x}_I f(\mathbf{x})\rangle,\end{aligned}$$

by Theorem 2.8.6 we get

$$\langle \phi, -\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} S^{-1}(\mathbf{x},\mathbf{s}) d\mathbf{x}_I f(\mathbf{x})\rangle = \langle \phi, f\rangle,$$

which concludes the proof. □

Corollary 2.9.10. *Let $\overline{B} = \overline{B(0,r)}$ be the closed ball in $\mathbb{R}^{n+1}$ centered at the origin and with radius $r > 0$. The dual of $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus B)$ is the set of all right s-monogenic functions defined in a neighborhood of $\overline{B}$.*

2.10 Topological Duality Theorems

In the previous sections we have proved the two $\mathbb{R}_n$-modules isomorphisms:

$$(\mathcal{G}(K))' \cong \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K)$$

and

$$\mathcal{G}(K) \cong (\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1}\setminus K))'.$$

We now want to show that those isomorphisms are actually topological isomorphisms. To this end we need to introduce a special class of infinite-order differential operators which is of independent interest. We recall that for s-monogenic functions, the s-derivatives coincide with the partial derivative ∂_u with respect to the scalar part u of a paravector, so

$$F(\partial_s) = F(\partial_u) = \sum_{m\geq 0} \partial_u^m a_m.$$

Proposition 2.10.1. *Let $F(\partial_s)$ be defined as above and let f be a left s-monogenic function in an axially symmetric s-domain $U \subseteq \mathbb{R}^{n+1}$. The function $F(\partial_s)f$ is a left s-monogenic function in U if and only if*

$$\lim_{m\to+\infty} \sqrt[m]{|a_m| m!} = 0. \tag{2.48}$$

Proof. Suppose that condition (2.48) holds, choose $I \in \mathbb{S}$, and consider the restriction of f to the plane $\mathbb{C}_I$. By the Splitting Lemma, f_I can be written as $f_I(z) = \sum_A F_A(z)I_A$, $z = u + Iv$ and each holomorphic function F_A can be expanded into a power series at a point $z_0 \in \mathbb{C}_I$. Thus $f_I(z)$ can be expanded into a power series with center at z_0 and, by the usual Cauchy estimates on the plane $\mathbb{C}_I$, we also deduce that

$$\frac{1}{m!}\left|\frac{\partial^m f}{\partial u^m}(z_0)\right| \leq \frac{M}{\delta^m}, \quad m \geq 0, \quad \text{for } |\mathbf{x} - z_0| \leq \delta.$$

Since $|a_m m!| < \varepsilon$ for all $m \in \mathbb{N}$, we deduce that the series $\sum_m \partial_s^m f(\mathbf{x})a_m$ converges locally uniformly on $\mathbb{C}_I$. It is immediate to verify that

$$\overline{\partial}_I[F(\partial_s)f_I(z)] = \overline{\partial}_I[\sum_{m\geq 0} \partial_u^m f_I(z)a_m] = \sum_{m\geq 0} \partial_u^m \overline{\partial}_I f_I(z)a_m = 0,$$

and since the choice of I is arbitrary we get that $\sum_{m\geq 0} \partial_u^m f(\mathbf{x})a_m$ is an s-monogenic function.

Conversely, suppose by an absurdity that $\sum_{m\geq 0} \partial_s^m f(\mathbf{x})a_m$ is s-monogenic but (2.48) does not hold. The result follows as in the complex case, see [63], Lemma 1.8.1. Indeed, suppose we negate (2.48). Then for some $\varepsilon > 0$ there is a subsequence a_{k_j} such that

$$\sqrt[k_j]{|a_{k_j}|k_j!} \geq 2\varepsilon \qquad \text{for all } k_j, \quad k_j \to +\infty.$$

We now apply $F(\partial_s)$ to the s-monogenic function $(\mathbf{x} - y_0)^{-1}$, with $y_0 \in U \cap \mathbb{R}$, and we obtain

$$F(\partial_s)(\mathbf{x} - y_0)^{-1} = \sum_{k\geq 0} \frac{a_k(-1)^k k!}{(\mathbf{x} - y_0 - \varepsilon)^{k+1}} = \sum_{k\geq 0} F_k(\mathbf{x}). \tag{2.49}$$

Consider $|\mathbf{x} - y_0| \leq \varepsilon$, and assume, by taking if necessary a subsequence $\mathbf{x}_j$, that $\mathbf{x}_j \to y_0$. Then we get

$$|F_k(\mathbf{x}_j)| \geq \frac{(2\varepsilon)^{k_j}}{|\mathbf{x}_j - y_0 - \varepsilon|^{k_j+1}} \geq \frac{1}{2\varepsilon},$$

thus for $|\mathbf{x} - y_0| \leq \varepsilon$ the series (2.49) does not converge locally uniformly which contradicts the hypothesis. □

Proposition 2.10.2. *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-domain. An operator of the type $F(\partial_{\mathbf{s}})$ acts continuously on $\mathcal{M}^L(U)$, for any U.*

Proof. If $f \in \mathcal{M}^L(U)$ we know that the estimate for $F(\partial_{\mathbf{s}})f$ depends only on the maximum norm of f, so continuity follows. □

Theorem 2.10.3. *Let $K \subseteq \mathbb{R}^{n+1}$ be an axially symmetric s-compact set. The sequence $\{g_k\}$ converges to $g \in \mathcal{G}(K)$ if and only if the sequence $\{F(\partial_{\mathbf{s}})g_k(\mathbf{x})\}$ converges pointwise on K for all $F(\partial_{\mathbf{s}})$.*

Proof. Let g_k be defined in an axially symmetric s-domain U containing K and let $g_{k,I} = \sum_A F_{k,A}(z)I_A$ be the restriction of g_k to a plane $\mathbb{C}_I$ obtained using the Splitting Lemma 2.2.11. The convergence of g_k to g in the topology of $\mathcal{M}^R(U)$ is equivalent to the convergence, for every multi-index A, of $\{F_{k,A}\}$ to some function F_A which is holomorphic in $U \cap \mathbb{C}_I$. Theorem 4.1.10 in [63] shows that the convergence of $F_{k,A}$ is equivalent, for every A, to the pointwise convergence of $\{F(\partial_s)F_{k,A}\}$ for every $F(\partial_s)$. This in turn is equivalent to the convergence of $g_{k,I}$ on the plane $\mathbb{C}_I$. We conclude the proof by applying the Representation Formula (2.7). □

Theorem 2.10.4. *Let $K \subset \mathbb{R}^{n+1}$ be an axially symmetric s-compact set. The isomorphism*

$$(\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \backslash K))' \cong \mathcal{G}(K).$$

is topological.

Proof. If $g_k \to g$ in $\mathcal{G}(K)$ it means that $g_k \to g$ uniformly in a neighborhood of K. With respect to the duality defined by (2.46), we have $\langle \phi_{g_k}, f\rangle \to \langle \phi_g, f\rangle$ uniformly when f varies in a bounded subset of $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \backslash K)$, thus $\phi_{g_k} \to \phi_g$.

Conversely, suppose that $\phi_k \to \phi$ in $(\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \backslash K))'$, with its natural topology. Then the functions

$$g_k(\mathbf{x}) = \frac{1}{2\pi}\langle \phi_k, S_R^{-1}(\mathbf{s}, \mathbf{x})\rangle$$

defined by (2.47) are right s-monogenic in a neighborhood U of K which can be chosen to be an axially symmetric s-domain. Now we have to show that the sequence $\{g_k\}$ converges uniformly in some suitable neighborhood of K. By Theorem 2.10.3 it is enough to prove that $\{F(\partial_u)g_k\}$ converges pointwise for all infinite-order differential operators $F(\partial_u)$ satisfying condition (2.48). From the continuity of ϕ_k, fixing any $\mathbf{x} \in K$, we have

$$\begin{aligned} F(\partial_u)g_k(\mathbf{x}) = \frac{1}{2\pi}\langle \phi_k, F(\partial_u)S_R^{-1}(\mathbf{s}, \mathbf{x})\rangle &\to \frac{1}{2\pi}\langle \phi, F(\partial_u)S_R^{-1}(\mathbf{s}, \mathbf{x})\rangle \\ &= F(\partial_u)\langle \phi, \frac{1}{2\pi}S_R^{-1}(\mathbf{s}, \mathbf{x})\rangle \end{aligned}$$

and the statement follows by setting $g(\mathbf{x}) = \frac{1}{2\pi}\langle \phi, S_R^{-1}(\mathbf{s}, \mathbf{x})\rangle$. □

Corollary 2.10.5. *Let $K \subset \mathbb{R}^{n+1}$ be an axially symmetric s-compact set. The isomorphism*

$$(\mathcal{G}(K))' \cong \mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \backslash K)$$

is topological.

Proof. We have pointed out that $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \backslash K)$ is a Montel module thus it is reflexive. So, by Theorem 2.10.4, the dual of $\mathcal{G}(K)$ is $\mathcal{M}^L_\infty(\overline{\mathbb{R}}^{n+1} \backslash K)$ itself. □

We conclude this section by looking at a very special case of a compact set, namely $K = \{0\}$. Recall that s-monogenic functions outside the origin can be represented by a Laurent-type series of the form

$$f(\mathbf{x}) = \sum_{m \geq 0} \mathbf{x}^m a_m + \sum_{m \geq 1} \mathbf{x}^{-m} b_m \tag{2.50}$$

converging in a spherical shell

$$A = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid R_1 < |\mathbf{x}| < R_2\}, \quad 0 < R_1 < R_2.$$

This formula contains two series: one with positive powers of the variable, and one with negative powers of the variable. It is clear that, in order for the Laurent series to give a function which vanishes at infinity, the portion with positive powers must vanish. Thus, we can say that s-monogenic functions outside the origin, which vanish at infinity, are represented by Laurent series where only negative powers of the variable appear. An additional condition is the consequence of the fact that we are requiring the Laurent series to converge everywhere. For this to be true, we need to ask that the series has radius of convergence equal to infinity, and this yields, once again, condition (2.48). We can therefore state the following result:

Corollary 2.10.6. *The $\mathbb{R}_n$-module $(\mathcal{G}(\{0\}))'$ is isomorphic to the $\mathbb{R}_n$-module of infinite-order differential operators acting on s-monogenic functions.*

2.11 Notes

Note 2.11.1. On the kernel $S^{-1}(\mathbf{s}, \mathbf{x})$. Unlike the case of regular or monogenic functions which are defined as the elements of the kernel of first-order differential operators (the Cauchy Fueter operator for the case of regular functions and the Dirac operator for the case of monogenic functions), it is not possible to consider s-regular and s-monogenic functions as solutions of a globally defined operator. Specifically, these functions are defined as those functions whose restrictions to a family of planes satisfy a family of first-order operators on those planes. The Cauchy kernel that we have constructed is, on each of those planes, the fundamental solution for the relevant operator; this justifies our choice of nomenclature, even though strictly speaking this is somewhat of an abuse of notation because the kernel is not the solution on $\mathbb{R}^4$ or $\mathbb{R}^{n+1}$ of a globally-defined operator.

In fact, the fundamental solution to the equation

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = \delta(u + Iv), \quad I \in \mathbb{S}, \tag{2.51}$$

on the plane $\mathbb{C}_I$, where $\delta(u + Iv)$ is the Dirac delta distribution, is (see [59])

$$f_I(u + Iv) = \frac{1}{\pi} \frac{1}{u + Iv}, \quad I \in \mathbb{S}. \tag{2.52}$$

By the Extension Lemma, we uniquely extend the function in (2.52) to the entire space $\mathbb{R}^{n+1} \setminus \{0\}$ to get

$$f(x) = \frac{1}{\pi}\,\frac{1}{\mathbf{x}} = -\frac{1}{\pi}\, S^{-1}(0, \mathbf{x}).$$

If the delta distribution is not centered at the origin but at a point α on the real axis, the solution becomes

$$f(\mathbf{x} - \alpha) = \frac{1}{\pi}\,\frac{1}{\mathbf{x} - \alpha} = -\frac{1}{\pi}\, S^{-1}(\alpha, \mathbf{x}).$$

If α is not real, then the function $(\mathbf{x} - \alpha)^{-1}$ is not s-monogenic, thus we have to consider its s-monogenic inverse $(\mathbf{x} - \alpha)^{-*}$ which is precisely

$$(\mathbf{x} - \alpha)^{-*} = -S^{-1}(\alpha, \mathbf{x}).$$

Let us now consider another feature of the Cauchy kernel series. Let $\mathbf{x}, \mathbf{s} \in \mathbb{R}^{n+1}$ such that $\mathbf{xs} \neq \mathbf{sx}$ and denote by $S(\mathbf{s}, \mathbf{x})$ the inverse of the noncommutative Cauchy kernel series $S^{-1}(\mathbf{s}, \mathbf{x})$. Our next goal is to show that the function $S(\mathbf{s}, \mathbf{x})$, satisfying the equation

$$S^2(\mathbf{s}, \mathbf{x}) + S(\mathbf{s}, \mathbf{x})\mathbf{x} - \mathbf{s}\, S(\mathbf{s}, \mathbf{x}) = 0 \tag{2.53}$$

is the inverse of the noncommutative Cauchy kernel series.

Lemma 2.11.2. *Let* $\mathbf{x}, \mathbf{s} \in \mathbb{R}^{n+1}$*. Then* $S(\mathbf{s}, \mathbf{x}) := \mathbf{s} - \mathbf{x}$ *is a solution of equation (2.53) if and only if* $\mathbf{s}\,\mathbf{x} = \mathbf{x}\,\mathbf{s}$*.*

In general, when $\mathbf{s}$, $\mathbf{x}$ do not commute, the equation (2.53) has another nontrivial solution:

Theorem 2.11.3. *Let* $\mathbf{x}, \mathbf{s} \in \mathbb{R}^{n+1}$ *be such that* $\mathbf{xs} \neq \mathbf{sx}$*. The equation* (2.53) *has the nontrivial solution*

$$S(\mathbf{s}, \mathbf{x}) = -(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2).$$

Proof. Let us plug $-(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)$ into (2.53) and show that

$$\begin{aligned}
&(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)\\
&- (\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)\mathbf{x}\\
&+ \mathbf{s}(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2) = 0
\end{aligned}$$

is an identity. We multiply on the left by $(\mathbf{x} - \overline{\mathbf{s}})$ and we get

$$\begin{aligned}
&(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)\\
&- (\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)\mathbf{x}\\
&+ (\mathbf{x} - \overline{\mathbf{s}})\mathbf{s}(\mathbf{x} - \overline{\mathbf{s}})^{-1}(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2) = 0.
\end{aligned} \tag{2.54}$$

We observe that $\mathbf{x}$ and $(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)$ commute and that the element

$$\mathbf{u} := (\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)$$

is invertible where it is nonzero. Indeed

$$\begin{aligned}\mathbf{u}\bar{\mathbf{u}} &= (\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)(\bar{\mathbf{x}}^2 - 2\mathrm{Re}[\mathbf{s}]\bar{\mathbf{x}} + |\mathbf{s}|^2)\\ &= |\mathbf{x}|^4 - 2\mathbf{x}|\mathbf{x}|^2\mathrm{Re}[\mathbf{s}] + \mathbf{x}^2|\mathbf{s}|^2 - 2\bar{\mathbf{x}}|\mathbf{x}|^2\mathrm{Re}[\mathbf{s}] + 4|\mathbf{x}|^2\mathrm{Re}[\mathbf{s}]^2\\ &\quad - 2\mathbf{x}\mathrm{Re}[\mathbf{s}]|\mathbf{s}|^2 + \bar{\mathbf{x}}^2|\mathbf{s}|^2 - 2\bar{\mathbf{x}}\mathrm{Re}[\mathbf{s}]|\mathbf{s}|^2 + |\mathbf{s}|^4\\ &= |\mathbf{x}|^4 - 2\mathrm{Re}[\mathbf{x}]|\mathbf{x}|^2\mathrm{Re}[\mathbf{s}]\\ &\quad + (\mathrm{Re}[\mathbf{s}]^2 - |\underline{s}|^2)|\mathbf{s}|^2 + 4|\mathbf{x}|^2\mathrm{Re}[\mathbf{s}]^2 - 2\mathrm{Re}[\mathbf{x}]\mathrm{Re}[\mathbf{s}]|\mathbf{s}|^2 + |\mathbf{s}|^4\end{aligned}$$

therefore $\mathbf{u}\bar{\mathbf{u}} \in \mathbb{R}$, thus the inverse of $\mathbf{u}$ is $\bar{\mathbf{u}}/|\mathbf{u}|^2$. By multiplying equality (2.54) by $\mathbf{u}^{-1}$ on the right, we obtain:

$$(\mathbf{x}^2 - 2\mathrm{Re}[\mathbf{s}]\mathbf{x} + |\mathbf{s}|^2)(\mathbf{x} - \bar{\mathbf{s}})^{-1} - \mathbf{x} + (\mathbf{x} - \bar{\mathbf{s}})\mathbf{s}(\mathbf{x} - \bar{\mathbf{s}})^{-1} = 0.$$

We multiply by $\mathbf{x} - \bar{\mathbf{s}}$ on the right and we get the identity

$$-2\mathrm{Re}[\mathbf{s}]\mathbf{x} + \mathbf{x}\bar{\mathbf{s}} + \mathbf{x}\mathbf{s} = 0. \qquad \square$$

Note 2.11.4. Historical notes and further readings. The study of s-monogenic functions is a relatively new field of research: they were introduced in 2007 in [26] (published two years later), in an effort to generalize the notion of slice regularity (see [48], [49]) to the setting of Clifford algebras. Further properties of s-monogenic functions which are collected in this book are treated in [18], [27], [28], [29]. The Runge theorem is proved in [30] for a slightly different class of functions that, however, coincide with the class of s-monogenic functions over axially symmetric s-domains.

The most studied and well-known generalization of holomorphic functions to the Clifford algebras setting is Clifford analysis, intended as the study of functions in the kernel of the Dirac operator. It is nowadays a widely developed topic which the reader can approach in the classical references [7] and [34]. More recent books, which address in a less detailed way the topic of monogenic functions but give some insights to further developments of the theory, are [23] and [31]. Finally, a very friendly introduction to classical complex analysis and its higher-dimensional generalizations containing also historical remarks is given in the textbook [58]. Clifford analysis is a very rich and well-developed theory which, however, does not allow one to treat power series in the paravector variable and for this reason other theories have been introduced. With no claim of completeness, we mention for example the hyperholomorphic functions studied by Eriksson and Leutwiler in [74], [38], [39], [40] and Cliffordian holomorphic functions introduced by Laville and Ramadanoff [72], [73]. Slice monogenic functions admit power series expansion in terms of the paravector variable, at least on discs centered at points on the real axis, and this property will allow us to deal with a functional calculus for n-tuples of linear operators (see the next chapter).

It is worth noticing, however, that the theory of s-monogenic functions is not, strictly speaking, a generalization of the theory of holomorphic functions of a complex variable: holomorphic functions, can be obtained as s-monogenic functions for $n = 1$, see Remark 2.2.9, but given an s-monogenic function there is no possibility to restrict its domain or codomain in order to obtain a holomorphic function.

The s-monogenic functions, as well as s-regular functions in one quaternionic variable, have several forerunners in the literature. Fueter in his paper [43], but see also [33], [98], considered the problem of constructing regular functions (in the sense of Cauchy–Fueter) starting from holomorphic functions. Thus he introduced functions of the form

$$f(q) = \alpha(q_0, |\mathrm{Im}(q)|) + \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}\beta(q_0, |\mathrm{Im}(q)|) \tag{2.55}$$

where α, β are defined on the upper complex plane $\mathbb{C}^+$, have real values and α, β satisfy the Cauchy–Riemann system. The function Δf, now called the Fueter transform of f, is Cauchy–Fueter regular. Note that, in light of this result, the function $\sum \Delta q^n a_n$ is (Cauchy–Fueter) regular in q where it converges. This approach was generalized to functions of a paravector variable: it is sufficient to rewrite (2.55) by replacing the quaternion q by a paravector $\mathbf{x} \in \mathbb{R}^{n+1}$ and $\mathrm{Im}(q)$ by the vector $\underline{x}$. If n is odd, it is possible to show that $\Delta^{(n-1)/2} f$ is a monogenic function in the sense of [7]. This result, known as Fueter's mapping theorem, has been proved by Sce in [94] and then generalized by Qian, see [87], when n is an even number. Later on, Fueter's theorem was generalized to the case in which a function f as above is multiplied by a monogenic homogeneous polynomial of degree k, see [68], [83], [96] and to the case in which the function f is defined on an open set U, not necessarily chosen in the upper complex plane, see [88]. This last result is important because in this case a function of the form (2.55), with q replaced by $\mathbf{x}$, is s-monogenic in the sense of our definition, even though we are allowed to consider α and β with values in the Clifford algebra $\mathbb{R}_n$. Fueter's mapping theorem allows us to construct monogenic functions starting from s-monogenic functions, moreover it allows us to show that the class of monogenic functions which comes from s-monogenic ones corresponds to the axially monogenic functions (see [24]).

The class of functions (2.55), whose importance for Fueter's mapping theorem is clear, is also known in the literature as the class of radially holomorphic functions, see for example [58]. They are also related to the so-called standard intrinsic functions studied by Rinehart and then by Cullen, see [89], [32] respectively. These studies were the starting point for a deep generalization carried out by Ghiloni and Perotti in their paper [53]. In this paper, the authors study functions with values in a real alternative algebra A which are slice functions, i.e., they are of the form $f(u, v) = \alpha(u, v) + I\beta(u, v)$ where $\alpha(u, -v) = \alpha(u, v)$ and $\beta(u, -v) = -\beta(u, v)$, I is an element chosen in a suitable subset of the algebra such that $I^2 = -1$, (u, v)

are real numbers which correspond to the "real part" and to the modulus of the "imaginary part" of a variable chosen in a suitable subset of the algebra A. Then by requiring that the pair of functions (α, β) satisfy the Cauchy–Riemann system, one obtains the so-called slice regular functions according to [53] (compare with Corollary 2.2.20). We do not enter into the details of this interesting construction: it is sufficient to observe that the treatment is general enough to include, when we consider open sets that are axially symmetric and which properly intersect the real axis, the case of s-monogenic and s-regular functions treated in this book.

Finally, we point out that the study of zeros of polynomials of a paravector variable, which we started in our work as a byproduct of the study of s-monogenic functions, has been the topic of the researches of Qian and Yang, see [104]. Moreover, polynomials with coefficients in a Clifford algebra can also be treated with the techniques developed by Ghiloni and Perotti, see the aforementioned papers and [54].

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