Chapter 2

The Laplace Transform

In this chapter the emphasis of the discussion shifts from Laplace integrals $\hat{f}(\lambda)$ and $\widehat{dF}(\lambda)$ to the Laplace transform $\mathcal{L} : f \mapsto \hat{f}$ and to the Laplace-Stieltjes transform $\mathcal{L}_S : F \mapsto \widehat{dF}$. The Laplace transform is considered first as an operator acting on $L^\infty(\mathbb{R}_+, X)$ and the Laplace-Stieltjes transform as an operator on

$$\mathrm{Lip}_0(\mathbb{R}_+, X) := \left\{ F : \mathbb{R}_+ \to X : F(0) = 0,\ \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} := \sup_{t,s \geq 0} \frac{\|F(t) - F(s)\|}{|t-s|} < \infty \right\}.$$

These domains of $\mathcal{L}$ and $\mathcal{L}_S$ are relatively easy to deal with and have immediate and important applications to abstract differential and integral equations.

The following observation is the key to one of the basic structures of Laplace transform theory. If $f \in L^\infty(\mathbb{R}_+, X)$, then $t \mapsto F(t) := \int_0^t f(s)\,ds$ belongs to $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and

$$\mathcal{L}(f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt = \int_0^\infty e^{-\lambda t}\,dF(t) = T_F(e_{-\lambda}),$$

where $T_F : g \mapsto \int_0^\infty g(s)dF(s)$ is a bounded linear operator from $L^1(\mathbb{R}_+)$ into X, and where $e_{-\lambda}$ denotes the exponential function $t \mapsto e^{-\lambda t}$. The operator T_F is fundamental to Laplace transform theory. In Section 2.1 it is shown that $\Phi_S : F \mapsto T_F$ is an isometric isomorphism between $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and $\mathcal{L}(L^1(\mathbb{R}_+), X)$ (Riesz-Stieltjes representation theorem). This representation is crucial for the following reason. The main purpose of Laplace transform theory is to translate properties of the generating function F into properties of the resulting function $\lambda \mapsto r(\lambda) = \int_0^\infty e^{-\lambda t}\,dF(t)$ and vice versa. Since $F(t) = T_F\chi_{[0,t]} = \int_0^\infty \chi_{[0,t]}(s)\,dF(s)$ and $r(\lambda) = T_F e_{-\lambda} = \int_0^\infty e^{-\lambda s}\,dF(s)$, the generating function F as well as the resulting function r are evaluations of the *same* bounded linear operator acting on different total subsets of $L^1(\mathbb{R}_+)$.

W. Arendt et al., *Vector-valued Laplace Transforms and Cauchy Problems: Second Edition*, Monographs in Mathematics 96, DOI 10.1007/978-3-0348-0087-7_2, © Springer Basel AG 2011

In Section 2.2, the range of the Laplace-Stieltjes transform acting on $\mathrm{Lip}_0(\mathbb{R}_+, X)$ is characterized. It is shown that a function $r : \mathbb{R}_+ \to X$ has a Laplace-Stieltjes representation $r = \mathcal{L}_S(F)$ for some $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ if and only if r is a C^∞-function whose Taylor coefficients satisfy the estimate

$$\|r\|_W := \sup_{n\in\mathbb{N}_0} \sup_{\lambda>0} \frac{\lambda^{n+1}}{n!} \|r^{(n)}(\lambda)\| < \infty. \tag{2.1}$$

This can be rephrased by saying that the Laplace-Stieltjes transform is an isometric isomorphism between the Banach spaces $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and

$$C_W^\infty((0,\infty), X) := \{r \in C^\infty((0,\infty), X) : \|r\|_W < \infty\}.$$

If the Banach space X has the Radon-Nikodym property (see Section 1.2), then (and only then) "Widder's growth conditions" (2.1) are necessary and sufficient for r to have a Laplace representation $r = \mathcal{L}(f)$ for some $f \in L^\infty(\mathbb{R}_+, X)$; i.e., Banach spaces with the Radon-Nikodym property are precisely those Banach spaces in which the Laplace transform is an isometric isomorphism between $L^\infty(\mathbb{R}_+, X)$ and $C_W^\infty((0,\infty), X)$. For $X = \mathbb{C}$, this is a classical result usually known as "Widder's Theorem".

If $r = \mathcal{L}_S(F)$ for some $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, then the inverse Laplace-Stieltjes transform has many different representations. A few of them, such as

$$\begin{aligned} F(t) &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{r(\lambda)}{\lambda}\, d\lambda = \lim_{n\to\infty} \sum_{j=1}^{\infty} (-1)^{j+1} e^{tnj} r(nj) \\ &= \lim_{k\to\infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \frac{d^k}{dt^k} \left(\frac{r(\lambda)}{\lambda}\right)\bigg|_{\lambda=k/t}, \end{aligned}$$

will be proved in Section 2.3.

In Section 2.4, the results of the previous sections are extended to functions with exponential growth at infinity; i.e., we investigate the Laplace transform acting on functions f with $\operatorname{ess\,sup}_{t\geq 0} \|e^{-\omega t} f(t)\| < \infty$.

In applications it is usually impossible to verify whether or not a given function r satisfies Widder's growth conditions (2.1). Thus, in Sections 2.5 and 2.6 some complex growth conditions are discussed which are necessary (and in a certain sense sufficient) for a holomorphic function $r : \{\mathrm{Re}\,\lambda > \omega\} \to X$ to have a Laplace representation. In Section 2.5, the growth condition considered is $\sup_{\mathrm{Re}\,\lambda>\omega} \|\lambda^{1+b} r(\lambda)\| < \infty$ for some $b > 0$.

In Section 2.6, we discuss functions r which are holomorphic in a sector $\Sigma := \{|\arg(\lambda)| < \frac{\pi}{2} + \varepsilon\}$ and satisfy $\sup_{\lambda\in\Sigma} \|\lambda r(\lambda)\| < \infty$. We will see that any such r is the Laplace transform of a function which is holomorphic in the sector $\{|\arg(\lambda)| < \varepsilon\}$. The final class of functions which we will consider are the completely monotonic ones; i.e., C^∞-functions r with values in an ordered Banach space such that $(-1)^n r^{(n)}(\lambda) \geq 0$ for all $n \in \mathbb{N}_0$ and $\lambda > \omega$. In the scalar case, Bernstein's theorem states that a function r is completely monotonic if and only if it is the Laplace-Stieltjes transform of an increasing function. In Section 2.7 we investigate for which ordered Banach spaces Bernstein's theorem holds.

2.1 Riesz-Stieltjes Representation

In the following sections the emphasis will be on the properties of the Laplace transform $\mathcal{L} : f \mapsto \hat{f}$ and the Laplace-Stieltjes transform $\mathcal{L}_S : F \mapsto \widehat{dF}$. As is the case with all linear operators, the choice of the domain is crucial. For the Laplace-Stieltjes transform $\mathcal{L}_S$ the most convenient choice of the domain space is

$$\mathrm{Lip}_0(\mathbb{R}_+, X) := \Big\{ F : \mathbb{R}_+ \to X : \; F(0) = 0, \; \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} := \sup_{t,s \geq 0} \frac{\|F(t) - F(s)\|}{|t-s|} < \infty \Big\}.$$

If $F(t) = \int_0^t f(s)\,ds$ for $f \in L^\infty(\mathbb{R}_+, X)$, then $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ and

$$\int_0^\infty e^{-\lambda t}\,dF(t) = \int_0^\infty e^{-\lambda t} f(t)\,dt \quad (\lambda > 0),$$

by Proposition 1.10.1. Thus, any result for $\mathcal{L}_S$ acting on $\mathrm{Lip}_0(\mathbb{R}_+, X)$ translates into one for $\mathcal{L}$ acting on $L^\infty(\mathbb{R}_+, X)$. However, since there are Banach spaces in which not every Lipschitz continuous function is the antiderivative of an L^∞-function (see Section 1.2), the Laplace-Stieltjes transform is a true generalization of the Laplace transform. It is the generalization needed to deal effectively with Laplace transforms of vector-valued functions.

In this section we investigate the Riesz-Stieltjes operator Φ_S which assigns to $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ a bounded linear operator $T_F : L^1(\mathbb{R}_+) \to X$ such that

$$T_F f := \int_0^\infty f(s)\,dF(s) := \lim_{\tau\to\infty} \int_0^\tau f(s)\,dF(s),$$

when $f \in L^1(\mathbb{R}_+)$ is continuous. It will be shown that Φ_S is an isometric isomorphism between $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and $\mathcal{L}(L^1(\mathbb{R}_+), X)$, the space of all bounded linear operators from the Banach space $L^1(\mathbb{R}_+)$ into X (Riesz-Stieltjes representation). This observation is fundamental for the whole chapter. To see why the Riesz-Stieltjes representation is such an important tool, observe that

$$F(t) = T_F \chi_{[0,t]} \;\; (t \geq 0)\,, \quad \text{and} \quad \widehat{dF}(\lambda) = T_F e_{-\lambda} \;\; (\lambda > 0).$$

Thus, if one knows F, then the operator T_F is specified on the set of characteristic functions $\chi_{[0,t]}$ $(t > 0)$, which is total in $L^1(\mathbb{R}_+)$. Therefore, T_F and, in particular, the Laplace integrals $T_F e_{-\lambda} = \widehat{dF}(\lambda)$ $(\lambda > 0)$ are completely determined.

Conversely, the Laplace integrals $\widehat{dF}(\lambda)$ determine T_F on the set of exponential functions $e_{-\lambda}$ ($\lambda > 0$), which is also total in $L^1(\mathbb{R}_+)$ (Lemma 1.7.1). Hence, the Laplace integrals $\widehat{dF}(\lambda)$ determine the properties of T_F and, in particular, the properties of $F(t) = T_F\chi_{[0,t]}$ ($t \geq 0$).

Theorem 2.1.1 (Riesz-Stieltjes Representation). *There exists a unique isometric isomorphism* $\Phi_S : F \mapsto T_F$ *from* $\mathrm{Lip}_0(\mathbb{R}_+, X)$ *onto* $\mathcal{L}(L^1(\mathbb{R}_+), X)$ *such that*

$$T_F\chi_{[0,t]} = F(t) \tag{2.2}$$

for all $t \geq 0$ *and* $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$. *Moreover,*

$$T_F g = \lim_{t\to\infty} \int_0^t g(s)\, dF(s) := \int_0^\infty g(s)\, dF(s) \tag{2.3}$$

for all continuous functions $g \in L^1(\mathbb{R}_+)$.

Note that it is part of the claim that the improper integral in (2.3) converges. We shall call the isomorphism Φ_S the *Riesz-Stieltjes operator*.

Proof. Let $D := \mathrm{span}\{\chi_{[0,t)} :\ t > 0\}$, the space of step functions, which is dense in $L^1(\mathbb{R}_+)$. For each $f \in D$ there exists a unique representation

$$f = \sum_{i=1}^n \alpha_i \chi_{[t_{i-1},t_i)},$$

where $0 = t_0 < t_1 < \ldots < t_n,\ \alpha_i \in \mathbb{C}$ ($i = 1, \ldots, n$). Let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$. Define $T_F : D \to X$ by

$$T_F(f) = T_F\left(\sum_{i=1}^n \alpha_i \chi_{[t_{i-1},t_i)}\right) := \sum_{i=1}^n \alpha_i (F(t_i) - F(t_{i-1})).$$

Then,

$$\|T_F(f)\| \leq \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} \sum_{i=1}^n |\alpha_i|(t_i - t_{i-1}) = \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}\|f\|_1.$$

Hence, T_F has a unique extension $T_F \in \mathcal{L}(L^1(\mathbb{R}_+), X)$. Moreover,

$$\|T_F\| \leq \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}.$$

Conversely, if $T \in \mathcal{L}(L^1(\mathbb{R}_+), X)$, let $F(t) := T\chi_{[0,t)}$ for $t \geq 0$. Then for $t > s \geq 0$,

$$\|F(t) - F(s)\| = \|T\chi_{[s,t)}\| \leq \|T\|\ \|\chi_{[s,t)}\|_1 = \|T\|(t - s).$$

Thus, $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ and $\|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} \le \|T\|$. It follows from the definitions that $T = T_F$ and if $T = T_G$ then $F = G$. This shows that $F \mapsto T_F$ is an isometric isomorphism.

Finally, let $g \in L^1(\mathbb{R}_+)$ be a continuous function and let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$. Take $t > 0$, and let π be a partition of $[0, t]$ with partitioning points $0 = t_0 < t_1 < \ldots < t_n = t$ and intermediate points $s_i \in [t_{i-1}, t_i]$. Let

$$f_\pi := \sum_{i=1}^{n} g(s_i)\chi_{[t_{i-1},t_i)}.$$

Thus, $S(g, F, \pi) = T_F(f_\pi)$. As $|\pi| \to 0$, $\|f_\pi - g\chi_{[0,t)}\|_1 \to 0$, so

$$\int_0^t g(s)\, dF(s) = T_F(g\chi_{[0,t)}).$$

As $t \to \infty$, $\|g\chi_{[0,t)} - g\|_1 \to 0$, so

$$\int_0^\infty g(s)\, dF(s) = T_F(g). \qquad \square$$

We conclude this section by discussing convergence of functions and their Laplace-Stieltjes transforms. In fact, the Laplace-Stieltjes transform allows us to give a purely operator-theoretic proof of the following approximation theorem. Note, however, that the essential implication (i) $\Rightarrow$ (iv) can also be obtained with the help of Theorem 1.7.5 (which may easily be strengthened by merely considering convergence on a sequence of equidistant points).

Theorem 2.1.2. *Let $M > 0$, $F_n \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ with $\|F_n\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} \le M$ for all $n \in \mathbb{N}$, and $r_n = \mathcal{L}_S(F_n)$. The following are equivalent:*

(i) *There exist $a, b > 0$ such that $\lim_{n\to\infty} r_n(a + kb)$ exists for all $k \in \mathbb{N}_0$.*

(ii) *There exists $r \in C^\infty((0,\infty), X)$ such that $r_n \to r$ uniformly on compact subsets of $(0,\infty)$.*

(iii) $\lim_{n\to\infty} F_n(t)$ *exists for all $t \ge 0$.*

(iv) *There exists $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ such that $F_n \to F$ uniformly on compact subsets of $\mathbb{R}_+$.*

Moreover, if r and F are as in (ii) *and* (iv)*, then $r = \mathcal{L}_S(F)$.*

Proof. By the Riesz-Stieltjes Representation Theorem 2.1.1, there exist $T_n \in \mathcal{L}(L^1(\mathbb{R}_+), X)$ such that $\|T_n\| = \|F_n\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} \le M$, $T_n e_{-\lambda} = r_n(\lambda)$, and $T_n\chi_{[0,t]} = F_n(t)$ ($n \in \mathbb{N}$, $t \ge 0$, $\lambda > 0$). Each of the statements imply that the uniformly bounded family of operators T_n converges on a total subset of $L^1(\mathbb{R}_+)$

(see also Lemma 1.7.1). By equicontinuity (see Proposition B.15), for any uniformly bounded sequence of operators, the topology of simple convergence on a total subset equals the topology of simple convergence and the topology of uniform convergence on compact subsets. Thus there exists $T \in \mathcal{L}(L^1(\mathbb{R}_+), X)$ such that $T_n g \to Tg$ as $n \to \infty$ for all $g \in L^1(\mathbb{R}_+)$ (simple convergence). For all $b > 0$ the sets $K_b := \{\chi_{[0,t]} : \ 0 \le t \le b\}$ and $E_b := \{e_{-\lambda} : \frac{1}{b} \le \lambda \le b\}$ are compact in $L^1(\mathbb{R}_+)$ (continuous images of compact sets are compact). Hence, $T_n \to T$ uniformly on K_b and E_b (uniform convergence on compact subsets). Now the statements follow from the Riesz-Stieltjes representation. □

2.2 A Real Representation Theorem

In this section the range of the Laplace-Stieltjes transform $\mathcal{L}_S : F \mapsto \widehat{dF}$ acting on $\mathrm{Lip}_0(\mathbb{R}_+, X)$ will be characterized. Since $\lambda \mapsto \widehat{dF}(\lambda) = \lambda \widehat{F}(\lambda)$ is holomorphic and, by Proposition 1.7.2, functions like $\lambda \mapsto (\sin \lambda)x$ $(x \in X)$ cannot be in the range of $\mathcal{L}_S$, the range must be a proper subset of $C^\infty((0,\infty), X)$. The following observations will lead to a complete description of the range.

Let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ and $T_F := \Phi_S(F)$, where Φ_S is the Riesz-Stieltjes operator of Section 2.1. Define

$$r(\lambda) := \widehat{dF}(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t) \quad (\lambda > 0).$$

Then, by Theorem 1.10.6, $r \in C^\infty((0,\infty), X)$ and

$$r^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t} (-t)^n \, dF(t) = T_F k_{n,\lambda},$$

where $k_{n,\lambda}(t) := e^{-\lambda t}(-t)^n$ $(t \ge 0,\ \lambda > 0,\ n \in \mathbb{N}_0)$. Since $\|k_{n,\lambda}\|_1 = \int_0^\infty e^{-\lambda t} t^n \, dt = n!/\lambda^{n+1}$ and $\|T_F\| = \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+, X)}$, it follows that

$$\|r^{(n)}(\lambda)\| \le \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+, X)} n!/\lambda^{n+1}$$

for all $n \in \mathbb{N}_0$ and $\lambda > 0$. Thus, r is a C^∞-function whose Taylor coefficients satisfy

$$\|r\|_W := \sup_{\lambda > 0, k \in \mathbb{N}_0} \frac{\lambda^{k+1}}{k!} \|r^{(k)}(\lambda)\| \le \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+, X)}.$$

This shows that the Laplace-Stieltjes transform $\mathcal{L}_S : F \to \widehat{dF}$ maps $\mathrm{Lip}_0(\mathbb{R}_+, X)$ into the space

$$C_W^\infty((0,\infty), X) := \{r \in C^\infty((0,\infty), X) : \|r\|_W < \infty\}.$$

In 1936, Widder showed that the Laplace transform maps $L^\infty(\mathbb{R}_+, \mathbb{R})$ onto $C_W^\infty((0,\infty), \mathbb{R})$. The following result is the vector-valued version of Widder's classical theorem.

Theorem 2.2.1 (Real Representation Theorem). *The Laplace-Stieltjes transform* $\mathcal{L}_S$ *is an isometric isomorphism between* $\mathrm{Lip}_0(\mathbb{R}_+, X)$ *and* $C_W^\infty((0,\infty), X)$.

Proof. We have already shown that $\mathcal{L}_S$ maps $\mathrm{Lip}_0(\mathbb{R}_+, X)$ into $C_W^\infty((0,\infty), X)$ and that $\|\mathcal{L}_S(F)\|_W \le \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}$. If $\mathcal{L}_S(F) = \widehat{dF} = 0$ for some $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, then $T_F e_{-\lambda} = \int_0^\infty e^{-\lambda t}\, dF(t) = \widehat{dF}(\lambda) = 0$ for all $\lambda > 0$. Since the exponential functions $e_{-\lambda}$ ($\lambda > 0$) are total in $L^1(\mathbb{R}_+)$ (Lemma 1.7.1), it follows that $T_F = 0$. In particular, $T_F \chi_{[0,t]} = F(t) = 0$ for all $t \ge 0$. Thus, $\mathcal{L}_S$ is one-to-one.

The hard part of the proof is to show that $\mathcal{L}_S$ is onto. Let $r \in C_W^\infty((0,\infty), X)$. Define $T_k \in \mathcal{L}(L^1(\mathbb{R}_+), X)$ by

$$T_k f \;:=\; \int_0^\infty f(t)(-1)^k \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} r^{(k)}\left(\frac{k}{t}\right) dt \qquad (k \in \mathbb{N}_0).$$

The operators T_k are uniformly bounded by $\|r\|_W$ since $\|T_k f\| \le \|r\|_W \|f\|_1$ for all $f \in L^1(\mathbb{R}_+)$. We will show below that $T_k e_{-\lambda} \to r(\lambda)$ as $k \to \infty$ for all $\lambda > 0$. Since the exponential functions $e_{-\lambda}$ ($\lambda > 0$) are total in $L^1(\mathbb{R}_+)$ it then follows from Proposition B.15 that there exists $T \in \mathcal{L}(L^1(\mathbb{R}_+), X)$ with $\|T\| \le \|r\|_W$ such that $T_k f \to Tf$ for all $f \in L^1(\mathbb{R}_+)$. In particular,

$$r(\lambda) \;=\; \lim_{k\to\infty} T_k e_{-\lambda} \;=\; T e_{-\lambda}.$$

The Riesz-Stieltjes Representation Theorem 2.1.1 then yields the existence of some $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ with $\|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} = \|T\| \le \|r\|_W$ such that $Tg = \int_0^\infty g(t)\, dF(t)$ for all continuous functions $g \in L^1(\mathbb{R}_+)$. Hence, for all $\lambda > 0$,

$$r(\lambda) \;=\; Te_{-\lambda} \;=\; \int_0^\infty e^{-\lambda t}\, dF(t) \;=\; \widehat{dF}(\lambda).$$

Thus, $\mathcal{L}_S$ is onto and $\|\mathcal{L}_S(F)\|_W = \|\widehat{dF}\|_W = \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}$ for $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$.

It remains to be shown that $T_k e_{-\lambda} \to r(\lambda)$ as $k \to \infty$ for all $\lambda > 0$. Observe that

$$\begin{aligned}
T_k e_{-\lambda} \;&=\; \int_0^\infty e^{-\lambda t}(-1)^k \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} r^{(k)}\left(\frac{k}{t}\right) dt \\
&=\; (-1)^k \frac{1}{(k-1)!}\int_0^\infty \left(e^{-\lambda k/u} u^{k-1}\right) r^{(k)}(u)\, du \\
&=\; (-1)^k \frac{1}{(k-1)!}\Bigg[\sum_{j=0}^{k-1}(-1)^j \frac{d^j}{du^j}\left(e^{-\lambda k/u}u^{k-1}\right) r^{(k-j-1)}(u)\Bigg|_{u=0}^{\infty} \\
&\qquad\qquad + (-1)^k \int_0^\infty \frac{d^k}{du^k}\left(e^{-\lambda k/u}u^{k-1}\right) r(u)\, du\Bigg].
\end{aligned}$$

To discuss the derivatives of $u \mapsto e^{-\lambda k/u}u^{k-1}$, define $G(x,u) \;:=\; e^{-x/u}\left(\frac{u}{x}\right)^{k-1}$. Then $G(sx, su) = G(x,u)$ for all $s > 0$. Differentiating both sides of the last equality with respect to s and then setting $s = 1$ yields $x\frac{\partial G}{\partial x}(x,u)+u\frac{\partial G}{\partial u}(x,u) = 0$ or $\frac{1}{x}\frac{\partial G}{\partial u}(x,u) = -\frac{1}{u}\frac{\partial G}{\partial x}(x,u)$. This implies that

$$\frac{\partial}{\partial u}\left(e^{-x/u}\frac{u^{k-1}}{x^k}\right) = -\frac{\partial}{\partial x}\left(e^{-x/u}\frac{u^{k-2}}{x^{k-1}}\right).$$

By induction on j, it follows that

$$\frac{\partial^j}{\partial u^j}\left(e^{-x/u}\frac{u^{k-1}}{x^k}\right) = (-1)^j\frac{\partial^j}{\partial x^j}\left(e^{-x/u}\frac{u^{k-j-1}}{x^{k-j}}\right) \qquad (0 \le j \le k),$$

or

$$\frac{\partial^j}{\partial u^j}\left(e^{-x/u}u^{k-1}\right) = (-1)^j x^k u^{k-j-1}\frac{\partial^j}{\partial x^j}\left(\frac{e^{-x/u}}{x^{k-j}}\right). \tag{2.4}$$

Hence,

$$\begin{aligned} h(u) &:= \sum_{j=0}^{k-1}(-1)^j\frac{\partial^j}{\partial u^j}\left(e^{-x/u}u^{k-1}\right)r^{(k-j-1)}(u) \\ &= \sum_{j=0}^{k-1}x^k\frac{\partial^j}{\partial x^j}\left(\frac{e^{-x/u}}{x^{k-j}}\right)u^{k-j-1}r^{(k-j-1)}(u). \end{aligned}$$

Since

$$\|u^{k-j-1}r^{(k-j-1)}(u)\| \le \frac{\|r\|_W(k-j-1)!}{u},$$

one obtains that

$$\|h(u)\| \le \sum_{j=0}^{k-1}\frac{\|r\|_W(k-j-1)!}{u}x^k\left|\frac{\partial^j}{\partial x^j}\left(\frac{e^{-x/u}}{x^{k-j}}\right)\right|.$$

It follows that $\lim_{u\to\infty} h(u) = 0 = \lim_{u\to 0} h(u)$. Therefore, letting $x = \lambda k$,

$$T_k e_{-\lambda} = \frac{1}{(k-1)!}\int_0^\infty \frac{d^k}{du^k}\left(e^{-\lambda k/u}u^{k-1}\right)r(u)\,du.$$

Since by (2.4),

$$\frac{\partial^k}{\partial u^k}\left(e^{-x/u}u^{k-1}\right) = (-1)^k\frac{x^k}{u}\frac{\partial^k}{\partial x^k}\left(e^{-x/u}\right) = \frac{x^k}{u^{k+1}}e^{-x/u},$$

it follows that

$$\begin{aligned} T_k e_{-\lambda} &= \frac{\lambda^k k^k}{(k-1)!}\int_0^\infty e^{-\lambda k/u}\frac{1}{u^{k+1}}r(u)\,du \\ &= \frac{\lambda^k k^{k+1}}{k!}\int_0^\infty e^{-\lambda k t}t^{k-1}r\left(\frac{1}{t}\right)dt. \end{aligned}$$

Define $f(t) := \frac{1}{t} r(\frac{1}{t})$ and $s := \frac{1}{\lambda}$. Then

$$\begin{aligned} T_k e_{-\lambda} &= \frac{s}{k!}\left(\frac{k}{s}\right)^{k+1} \int_0^\infty e^{-kt/s} t^k f(t)\,dt \\ &= s(-1)^k \frac{1}{k!}\left(\frac{k}{s}\right)^{k+1} \hat{f}^{(k)}\left(\frac{k}{s}\right). \end{aligned}$$

Finally, one concludes from the Post-Widder Inversion Theorem 1.7.7 that

$$\lim_{k\to\infty} T_k e_{-\lambda} = s f(s) = r\left(\frac{1}{s}\right) = r(\lambda)$$

for all $\lambda > 0$. □

For later use in Section 2.5, we observe that in the Widder conditions it is not necessary to consider all values of k.

Proposition 2.2.2. *Let $r \in C^\infty((0,\infty), X)$, and suppose that $\lim_{\lambda\to\infty} r(\lambda) = 0$ and there exist $M > 0$ and infinitely many integers m such that $\sup_{\lambda>0} \|\frac{\lambda^{m+1}}{m!} r^{(m)}(\lambda)\| \le M$. Then $r \in C_W^\infty((0,\infty), X)$ and $\|r\|_W \le M$.*

Proof. It suffices to show that if $\|r^{(m)}(\lambda)\| \le Mm!/\lambda^{m+1}$, for all $\lambda > 0$, then $\|r^{(k)}(\lambda)\| \le Mk!/\lambda^{k+1}$ for all $\lambda > 0$ and $0 \le k < m$. Let

$$\tilde{r}(\lambda) := \frac{(-1)^m}{(m-1)!} \int_\lambda^\infty (\lambda-\mu)^{m-1} r^{(m)}(\mu)\,d\mu.$$

Note that the integral is absolutely convergent, $\tilde{r}^{(m)}(\lambda) = r^{(m)}(\lambda)$, and the substitution $t = \lambda/\mu$ gives

$$\|\tilde{r}(\lambda)\| \le Mm \int_\lambda^\infty \frac{(\mu-\lambda)^{m-1}}{\mu^{m+1}}\,d\mu = \frac{Mm}{\lambda}\int_0^1 (1-t)^{m-1}\,dt = \frac{M}{\lambda}.$$

Hence $r - \tilde{r}$ is a polynomial and $\lim_{\lambda\to\infty}(r-\tilde{r})(\lambda) = 0$, so $r = \tilde{r}$. It follows that

$$\begin{aligned} \|r^{(k)}(\lambda)\| &= \left\| \frac{(-1)^m}{(m-k-1)!} \int_\lambda^\infty (\lambda-\mu)^{m-k-1} r^{(m)}(\mu)\,d\mu \right\| \\ &\le \frac{Mm!}{(m-k-1)!} \int_\lambda^\infty \frac{(\mu-\lambda)^{m-k-1}}{\mu^{m+1}}\,d\mu \\ &= \frac{Mk!}{\lambda^{k+1}} \end{aligned}$$

for $\lambda > 0$ and $0 \le k < m$. □

Now it will be shown that the Laplace transform is an isometric isomorphism between $L^\infty(\mathbb{R}_+, X)$ and $C_W^\infty((0,\infty), X)$ if and only if the Banach space X has the

Radon-Nikodym property. Recall from Section 1.2 that X has the *Radon-Nikodym property* if every $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ is differentiable a.e., or equivalently if every absolutely continuous function $F : \mathbb{R}_+ \to X$ is differentiable a.e. As shown in Theorem 1.2.6 and Corollary 1.2.7, every separable dual space (for example, l^1) and every reflexive Banach space have the Radon-Nikodym property. However, $L^1(\mathbb{R}_+)$ and c_0 do not have the property (Propositions 1.2.9 and 1.2.10).

Theorem 2.2.3. *Let X be a Banach space. The following are equivalent:*

(i) *X has the Radon-Nikodym property.*

(ii) *The Laplace transform $\mathcal{L} : f \mapsto \hat{f}$ is an isometric isomorphism between $L^\infty(\mathbb{R}_+, X)$ and $C_W^\infty((0,\infty), X)$.*

(iii) *The Riesz operator $\Phi : f \mapsto R_f$, $R_f g := \int_0^\infty g(t) f(t)\, dt$ is an isometric isomorphism between $L^\infty(\mathbb{R}_+, X)$ and $\mathcal{L}(L^1(\mathbb{R}_+), X)$.*

Proof. Define the normalized antiderivative $I : L^\infty(\mathbb{R}_+, X) \to \mathrm{Lip}_0(\mathbb{R}_+, X)$ by $I(f) := F$, $F(t) := \int_0^t f(s)\, ds$ $(t \geq 0)$. Then I is one-to-one and $\|I(f)\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} \leq \|f\|_\infty$ for all $f \in L^\infty(\mathbb{R}_+, X)$. If I is onto, then X has the Radon-Nikodym property (see Proposition 1.2.2). Conversely, if X has the Radon-Nikodym property and $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ then $f(t) := F'(t)$ exists for almost all $t \geq 0$. Since $f(t) = \lim_{h\to 0} \frac{F(t+h)-F(t)}{h}$ a.e., one concludes that $\|f\|_\infty \leq \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}$. In particular, $f \in L^\infty(\mathbb{R}_+, X)$ and by Proposition 1.2.3, $F = I(f)$. Thus X has the Radon-Nikodym property if and only if I is an isometric isomorphism.

The Riesz-Stieltjes operator $\Phi_S : F \mapsto T_F$, where $T_F g = \int_0^\infty g(t)\, dF(t)$ for all continuous $g \in L^1(\mathbb{R}_+)$, is an isometric isomorphism between $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and $\mathcal{L}(L^1(\mathbb{R}_+), X)$, and the Laplace-Stieltjes transform

$$\mathcal{L}_S : F \mapsto \widehat{dF}, \quad \widehat{dF}(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t),$$

is an isometric isomorphism between $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and $C_W^\infty((0,\infty), X)$. When $F = I(f)$, $T_F g = \int_0^\infty g(t) f(t)\, dt$ for all $g \in L^1(\mathbb{R}_+)$, by Proposition 1.9.11 and continuity in L^1-norm. Now the statements follow from the fact that $\Phi = \Phi_S \circ I$ and $\mathcal{L} = \mathcal{L}_S \circ I$ on $L^\infty(\mathbb{R}_+, X)$. □

Example 2.2.4. a) Consider $X = L^1(\mathbb{R}_+)$. Let $F(t) := \chi_{[0,t]}$ $(t \geq 0)$ and $r(\lambda) := e_{-\lambda}$ $(\mathrm{Re}\,\lambda > 0)$, where $e_{-\lambda}(t) = e^{-\lambda t}$. Then $F \in \mathrm{Lip}_0(\mathbb{R}_+, L^1(\mathbb{R}_+))$ and

$$r(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t) = \widehat{dF}(\lambda).$$

Since F is nowhere differentiable (see Proposition 1.2.10), there does not exist $f \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}_+))$ such that

$$r(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt.$$

b) Consider $C_0(\mathbb{R}_+)$ as a subspace of $L^\infty(\mathbb{R}_+)$. Define $F : \mathbb{R}_+ \to C_0(\mathbb{R}_+)$ by $F(t)(s) := (t-s)\chi_{[0,t]}(s)$, and $f : \mathbb{R}_+ \to L^\infty(\mathbb{R}_+)$ by $f(t) := \chi_{[0,t]}$. Then $F \in \mathrm{Lip}_0(\mathbb{R}_+, C_0(\mathbb{R}_+))$ and $F(t) = \int_0^t f(s)\,ds$ as a Riemann integral in $L^\infty(\mathbb{R}_+)$, but F is nowhere differentiable and f is not measurable (see Examples 1.2.8 and 1.9.7). Moreover,

$$\frac{1}{\lambda}e_{-\lambda} = \int_0^\infty e^{-\lambda t}\,dF(t) = \int_0^\infty e^{-\lambda t} f(t)\,dt$$

as (improper) Riemann-Stieltjes and Riemann integrals, but $\lambda \mapsto \frac{1}{\lambda}e_{-\lambda}$ is not the Laplace transform of any function in $L^1(\mathbb{R}_+, L^\infty(\mathbb{R}_+))$. □

2.3 Real and Complex Inversion

We have shown in Section 2.2 that the Laplace-Stieltjes transform $\mathcal{L}_S$ is an isometric isomorphism between $\mathrm{Lip}_0(\mathbb{R}_+, X)$ and $C_W^\infty((0,\infty), X)$. In this section we will derive several representations of the inverse Laplace-Stieltjes transform $\mathcal{L}_S^{-1}$.

Theorem 2.3.1 (Post-Widder Inversion). *Let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, $r = \mathcal{L}_S(F)$, and $t > 0$. Then*

$$F(t) = \lim_{k\to\infty} (-1)^k \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \frac{d^k}{d\lambda^k}\left(\frac{r(\lambda)}{\lambda}\right)\Bigg|_{\lambda=k/t}.$$

Proof. Since $\omega(F) \le 0$ and $F(0) = 0$, it follows from (1.22) that

$$\frac{r(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda t} F(t)\,dt$$

for all $\lambda > 0$, where the integral is an absolutely convergent Bochner integral. Now the statement follows from Theorem 1.7.7. □

Applying Leibniz's rule $(f\cdot r)^{(k)} = \sum_{j=0}^k \begin{pmatrix} k \\ j \end{pmatrix} f^{(k-j)} r^{(j)}$ to $f(\lambda) := \frac{1}{\lambda}$ and r one can rewrite the Post-Widder inversion of the Laplace-Stieltjes transform as

$$F(t) = \lim_{k\to\infty} \sum_{j=0}^{k} (-1)^j \frac{1}{j!}\left(\frac{k}{t}\right)^j r^{(j)}\left(\frac{k}{t}\right) \quad (t > 0). \tag{2.5}$$

Compared to the Post-Widder inversion, it is remarkable that in the following Phragmén-Doetsch inversion formula only the values $r(k)$ for large $k \in \mathbb{N}$ are needed and that the convergence is uniform for all $t \ge 0$.

Theorem 2.3.2 (Phragmén-Doetsch Inversion). *Let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ and $r = \mathcal{L}_S(F)$. Then*

$$\left\| F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} r(kj) \right\| \le \frac{c}{k}\|r\|_W$$

for all $t \ge 0$ and $k \in \mathbb{N}$, where $c \approx 1.0159...$, and $\|r\|_W = \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}$.

Proof. By the Riesz-Stieltjes Representation Theorem 2.1.1 and the Real Representation Theorem 2.2.1, there exists $T \in \mathcal{L}(L^1(\mathbb{R}_+), X)$ such that $r(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t) = Te_{-\lambda}$ $(\lambda > 0)$, $T\chi_{[0,t]} = F(t)$ $(t \geq 0)$ and $\|T\| = \|r\|_W = \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}$. Thus,

$$\left\| F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} r(kj) \right\| \leq \|T\| \left\| \chi_{[0,t]} - \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j!} e^{tkj} e_{-kj} \right\|_1 .$$

Define $p_{k,t}(s) := 1 - e^{-e^{k(t-s)}} = \sum_{j=1}^\infty (-1)^{j+1} \frac{1}{j!} e^{tkj} e_{-kj}(s)$. Then,

$$\begin{aligned} \|\chi_{[0,t]} - p_{k,t}\|_1 &= \int_0^t |p_{k,t}(s) - 1|\, ds + \int_t^\infty |p_{k,t}(s)|\, ds \\ &= \int_0^t e^{-e^{k(t-s)}}\, ds + \int_t^\infty \left(1 - e^{-e^{k(t-s)}}\right) ds \\ &= \frac{1}{k} \int_1^{e^{kt}} \frac{e^{-u}}{u}\, du + \frac{1}{k} \int_0^1 \frac{1-e^{-u}}{u}\, du \\ &\leq \frac{1}{k} \left(\int_1^\infty \frac{e^{-u}}{u}\, du + \int_0^1 \frac{1-e^{-u}}{u}\, du \right) \end{aligned}$$

for all $t \geq 0$ and $k \in \mathbb{N}$. Now the claim follows from the fact that $\int_1^\infty \frac{1}{u} e^{-u}\, du + \int_0^1 \frac{1-e^{-u}}{u}\, du = -2\,\mathrm{Ei}(-1) + \gamma \approx 1.0159...$, where $\mathrm{Ei}(z)$ is the exponential integral and γ is Euler's constant (see [Leb72, Section 3.1]). □

The following corollary shows that the Phragmén-Doetsch inversion is invariant under exponentially decaying perturbations for small values of t.

Corollary 2.3.3. *Let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, $r = \mathcal{L}_S(F)$, and $q(\lambda) = r(\lambda) + a(\lambda)$ $(\lambda > 0)$, where $a : (0, \infty) \to X$ is a function such that $\limsup_{n\to\infty} \frac{1}{n} \log \|a(n)\| \leq -T$ for some $T > 0$. Then*

$$F(t) = \lim_{k\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} q(kj)$$

for all $0 \leq t < T$.

Proof. Let $0 < T_0 < T$ and choose k_0 such that $\|a(k)\| \leq e^{-T_0 k}$ for all $k \geq k_0$. Then,

$$\begin{aligned} &\left\| F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} q(kj) \right\| \\ &\leq \left\| F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} r(kj) \right\| + \left\| \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} a(kj) \right\| \\ &\leq \frac{2}{k} \|r\|_W + \sum_{j=1}^{\infty} \frac{1}{j!} e^{tkj} e^{-T_0 kj} \leq \frac{2}{k} \|r\|_W + e^{e^{-(T_0 - t)k}} - 1. \end{aligned}$$

□

The Post-Widder inversion and the Phragmén-Doetsch inversion are called real inversions of the Laplace-Stieltjes transform since they use only properties of $r(\lambda)$ for large real λ. For the following complex inversion formula we use the fact that if $r(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t)$ $(\lambda > 0)$ for some $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, then r admits a holomorphic extension for $\mathrm{Re}\,\lambda > 0$ which we denote by the same symbol (see Theorem 1.10.6). We shall give here a proof based on the Riesz-Stieltjes representation, but we shall give another, rather simple, proof in Section 4.2.

Theorem 2.3.4 (Complex Inversion). *Let $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ and $r = \mathcal{L}_S(F)$. Then*

$$F(t) = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{c-ik}^{c+ik} e^{\lambda t} \frac{r(\lambda)}{\lambda}\, d\lambda,$$

where the limit is uniform for $t \in [0,a]$ for any $a > 0$, and $c > 0$ is arbitrary.

Proof. By the Riesz-Stieltjes Representation Theorem 2.1.1, there exists $T \in \mathcal{L}(L^1(\mathbb{R}_+), X)$ such that $r(\lambda) = Te_{-\lambda}$ $(\mathrm{Re}\,\lambda > 0)$ and $F(t) = T\chi_{[0,t]}$ $(t \geq 0)$. Thus,

$$\left\| F(t) - \frac{1}{2\pi i} \int_{c-ik}^{c+ik} e^{\lambda t} \frac{r(\lambda)}{\lambda}\, d\lambda \right\| \leq \|T\| \left\| \chi_{[0,t]} - \frac{1}{2\pi i} \int_{c-ik}^{c+ik} e^{\lambda t} \frac{e_{-\lambda}}{\lambda}\, d\lambda \right\|_1 .$$

Now the statement follows from the next lemma. □

Lemma 2.3.5. *Let $t \geq 0$ and $a, c > 0$. Then the functions*

$$h_{k,t} := \frac{1}{2\pi i} \int_{c-ik}^{c+ik} e^{\lambda t} \frac{e_{-\lambda}}{\lambda}\, d\lambda$$

converge towards $\chi_{[0,t]}$ in $L^1(\mathbb{R}_+)$ as $n \to \infty$, uniformly for $t \in [0,a]$.

Proof. Let $\|h_{k,t} - \chi_{[0,t]}\|_1 = A_k + B_k$, where $A_k := \int_0^t |h_{k,t}(s) - 1|\, ds$ and $B_k := \int_t^\infty |h_{k,t}(s)|\, ds$. We show first that $\lim_{k\to\infty} A_k = 0$. The residue of the function $\lambda \mapsto e^{\lambda(t-s)}/\lambda$ at the point 0 is 1. By Cauchy's theorem,

$$h_{k,t}(s) - 1 = \frac{1}{2\pi i} \left(\int_{\Gamma_+} - \int_{\Gamma_-} - \int_{\Gamma_0} \right) \frac{e^{\lambda(t-s)}}{\lambda}\, d\lambda,$$

where $\Gamma_\pm := \{\lambda : \lambda = u \pm ik;\ 0 \leq u \leq c\}$, $\Gamma_0 := \{\lambda : \lambda = ke^{iu};\ \pi/2 \leq u \leq 3\pi/2\}$. Along Γ_+, and similarly along Γ_-, it follows from $0 \leq s \leq t$ that

$$\left| \int_{\Gamma_+} \frac{e^{\lambda(t-s)}}{\lambda}\, d\lambda \right| = \left| \int_0^c \frac{e^{(u+ik)(t-s)}}{u+ik}\, du \right| \leq c\frac{e^{c(t-s)}}{k}.$$

Along Γ_0, for $0 \leq s < t$,

$$\left| \int_{\Gamma_0} \frac{e^{\lambda(t-s)}}{\lambda}\, d\lambda \right| = \left| \int_{\pi/2}^{3\pi/2} e^{k(t-s)e^{iu}}\, du \right| \leq \int_{\pi/2}^{3\pi/2} e^{k(t-s)\cos u}\, du.$$

Hence,

$$\begin{aligned} A_k &= \int_0^t |h_{k,t}(t-s)-1|\,ds \\ &\le \int_0^t \left(\frac{ce^{cs}}{\pi k} + \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} e^{ks\cos u}\,du \right) ds \\ &\to 0 \end{aligned}$$

as $k \to \infty$, uniformly for $t \in [0,a]$ for all $a > 0$, by the monotone convergence theorem, or by explicit estimation.

In order to estimate B_k, we define $\widetilde{\Gamma}_\pm := \{\lambda : \lambda = u \pm ik\ ;\ c \le u \le k\}$, $\widetilde{\Gamma}_0 := \{\lambda : \lambda = k\sqrt{2}e^{iu}\ ;\ -\pi/4 \le u \le \pi/4\}$. By Cauchy's theorem,

$$h_{k,t}(s) = \frac{1}{2\pi i}\left(-\int_{\widetilde{\Gamma}_+} + \int_{\widetilde{\Gamma}_-} + \int_{\widetilde{\Gamma}_0} \right) \frac{e^{\lambda(t-s)}}{\lambda}\,d\lambda.$$

Along $\widetilde{\Gamma}_+$, and similarly along $\widetilde{\Gamma}_-$, it follows from $s - t \ge 0$ that

$$\begin{aligned} \left| \int_{\widetilde{\Gamma}_+} \frac{e^{\lambda(t-s)}}{\lambda}\,d\lambda \right| &= \left| \int_c^k \frac{e^{(u+ik)(t-s)}}{u+ik}\,du \right| \le \frac{1}{k}\int_c^k e^{-u(s-t)}\,du \\ &= \frac{e^{-c(s-t)} - e^{-k(s-t)}}{k(s-t)}. \end{aligned}$$

Along $\widetilde{\Gamma}_0$,

$$\begin{aligned} \left| \int_{\widetilde{\Gamma}_0} \frac{e^{\lambda(t-s)}}{\lambda}\,d\lambda \right| &= \left| \int_{-\pi/4}^{\pi/4} e^{k\sqrt{2}(t-s)e^{iu}}\,du \right| \le \int_{-\pi/4}^{\pi/4} e^{k\sqrt{2}(t-s)\cos u}\,du \\ &= 2\int_0^{\pi/4} e^{k\sqrt{2}(t-s)\cos(u)}\,du \le \frac{\pi}{2}e^{k\sqrt{2}(t-s)\cos(\pi/4)} = \frac{\pi}{2}e^{-k(s-t)}. \end{aligned}$$

Hence, for all $t \ge 0$,

$$\begin{aligned} \int_t^\infty |h_{k,t}(s)|\,ds &\le \frac{1}{\pi}\int_t^\infty \frac{e^{-c(s-t)} - e^{-k(s-t)}}{k(s-t)}\,ds \;+\; \frac{1}{4}\int_t^\infty e^{-k(s-t)}\,ds \\ &= \frac{1}{\pi}\int_0^\infty z_k(s)\,ds + \frac{1}{4k}, \end{aligned}$$

where $z_k(s) := \frac{1}{ks}(e^{-cs} - e^{-ks}) \le e^{-cs}$ for $k \ge c$ by the mean value theorem applied to e^{-x} over $[cs, ks]$. By the dominated convergence theorem, or by explicit estimation, $B_k \to 0$ as $k \to \infty$, uniformly for $t \in [0,a]$ for all $a > 0$. $\square$

2.4 Transforms of Exponentially Bounded Functions

So far in this chapter, Laplace transforms have been considered for bounded or globally Lipschitz continuous functions. We shall now adapt the results of the previous sections to functions with exponential growth at infinity, by an elementary "shifting" procedure (see Proposition 1.6.1 a) and Proposition 1.10.3). More precisely, for $\omega \in \mathbb{R}$ we consider the Laplace-Stieltjes transform acting on

$$\mathrm{Lip}_\omega(\mathbb{R}_+, X) := \Big\{ G : \mathbb{R}_+ \to X : \ G(0) = 0, \\ \|G\|_{\mathrm{Lip}_\omega(\mathbb{R}_+,X)} := \sup_{t>s\geq 0} \frac{\|G(t)-G(s)\|}{\int_s^t e^{\omega r}\,dr} < \infty \Big\}$$

and the Laplace transform acting on

$$L^\infty_\omega(\mathbb{R}_+, X) := \Big\{ g \in L^1_{loc}(\mathbb{R}_+, X) : \ \|g\|_{\omega,\infty} := \operatorname*{ess\,sup}_{t\geq 0} \|e^{-\omega t} g(t)\| < \infty \Big\}.$$

It is easy to see that

$$\|G\|_{\mathrm{Lip}_\omega(\mathbb{R}_+,X)} = \begin{cases} \sup_{0\leq s<t} \dfrac{\|G(t)-G(s)\|}{(t-s)e^{\omega t}} & \text{if } \ \omega \geq 0, \\ \sup_{0\leq s<t} \dfrac{\|G(t)-G(s)\|}{(t-s)e^{\omega s}} & \text{if } \ \omega \leq 0. \end{cases}$$

It is clear that the multiplication operator $M_\omega : g \mapsto e^{-\omega \cdot} g(\cdot)$ is an isometric isomorphism between $L^\infty_\omega(\mathbb{R}_+, X)$ and $L^\infty(\mathbb{R}_+, X)$, and we now set up the corresponding isomorphism between $\mathrm{Lip}_\omega(\mathbb{R}_+, X)$ and $\mathrm{Lip}_0(\mathbb{R}_+, X)$.

For $G \in \mathrm{Lip}_\omega(\mathbb{R}_+, X)$ and $f \in \mathrm{BSV}_{loc}(\mathbb{R}_+)$, it follows from the definition of the Riemann-Stieltjes integral that

$$\left\| \int_a^b f(t)\,dG(t) \right\| \leq \|G\|_{\mathrm{Lip}_\omega(\mathbb{R}_+,X)} \int_a^b |f(t)| e^{\omega t}\,dt \quad (0 \leq a \leq b). \tag{2.6}$$

Let

$$(I_\omega G)(t) := \int_0^t e^{-\omega s}\,dG(s).$$

Then (2.6) implies that

$$I_\omega G \in \mathrm{Lip}_0(\mathbb{R}_+, X) \ \text{ and } \ \|I_\omega G\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)} \leq \|G\|_{\mathrm{Lip}_\omega(\mathbb{R}_+,X)}.$$

Similarly if $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ and

$$(J_\omega F)(t) := \int_0^t e^{\omega s}\,dF(s),$$

then $J_\omega F \in \mathrm{Lip}_\omega(\mathbb{R}_+, X)$ and $\|J_\omega F\|_{\mathrm{Lip}_\omega(\mathbb{R}_+,X)} \le \|F\|_{\mathrm{Lip}_0(\mathbb{R}_+,X)}$. Moreover, $J_\omega I_\omega G = G$ and $I_\omega J_\omega F = F$, by Proposition 1.9.10. Hence, I_ω is an isometric isomorphism of $\mathrm{Lip}_\omega(\mathbb{R}_+, X)$ onto $\mathrm{Lip}_0(\mathbb{R}_+, X)$.

Note that if $G \in L^\infty_\omega(\mathbb{R}_+, X)$ then $\omega(G) \le \omega$ and $\mathrm{abs}(dG) \le \omega$ by Theorem 1.10.5. Thus, the Laplace-Stieltjes transform

$$(\mathcal{L}_{S,\omega}G)(\lambda) := \widehat{dG}(\lambda) = \int_0^\infty e^{-\lambda t}\, dG(t)$$

exists for $\lambda > \omega$. By Proposition 1.10.3,

$$(\mathcal{L}_{S,\omega}G)(\lambda) = (\mathcal{L}_S I_\omega G)(\lambda - \omega). \tag{2.7}$$

Let

$$C^\infty_W((\omega,\infty), X) := \Big\{ r \in C^\infty((\omega,\infty), X) : \\ \|r\|_W := \sup_{\lambda>\omega, k\in\mathbb{N}_0} \frac{(\lambda-\omega)^{k+1}}{k!} \|r^{(k)}(\lambda)\| < \infty \Big\}.$$

This is a Banach space, and it is clear that the shift $S_\omega : r \mapsto r(\cdot - \omega)$ is an isometric isomorphism of $C^\infty_W((0,\infty), X)$ onto $C^\infty_W((\omega,\infty), X)$. The equation (2.7) may be written as $\mathcal{L}_{S,\omega} = S_\omega \circ \mathcal{L}_S \circ I_\omega$.

Now we can give the following reformulation of the Real Representation Theorem 2.2.1.

Theorem 2.4.1. *Let $\omega \in \mathbb{R}$. The Laplace-Stieltjes transform is an isometric isomorphism of $\mathrm{Lip}_\omega(\mathbb{R}_+, X)$ onto $C^\infty_W((\omega,\infty), X)$. In particular, for $M > 0$ and $r \in C^\infty_W((\omega,\infty), X)$, the following are equivalent:*

(i) $\|(\lambda-\omega)^{k+1}\frac{1}{k!}r^{(k)}(\lambda)\| \le M \quad (\lambda > \omega,\ k \in \mathbb{N}_0)$.

(ii) *There exists $G : \mathbb{R}_+ \to X$ satisfying $G(0) = 0$ and $\|G(t+h) - G(t)\| \le M\int_t^{t+h} e^{\omega r}\, dr \ \ (t, h \ge 0)$, such that $r(\lambda) = \int_0^\infty e^{-\lambda t}\, dG(t)$ for all $\lambda > \omega$.*

Proposition 1.6.1 a) gives

$$\mathcal{L}_\omega = S_\omega \circ \mathcal{L} \circ M_\omega$$

where $\mathcal{L}$ and $\mathcal{L}_\omega$ are the Laplace transforms on $L^\infty(\mathbb{R}_+, X)$ and $L^\infty_\omega(\mathbb{R}_+, X)$. Hence Theorem 2.2.3 can be reformulated as follows.

Theorem 2.4.2. *Let $M > 0$, $\omega \in \mathbb{R}$. If X has the Radon-Nikodym property then for any $r \in C^\infty_W((\omega,\infty), X)$ the following are equivalent:*

(i) $\|(\lambda-\omega)^{k+1}\frac{1}{k!}r^{(k)}(\lambda)\| \le M \quad (\lambda > \omega,\ k \in \mathbb{N}_0)$.

(ii) *There exists $g \in L^1_{loc}(\mathbb{R}_+, X)$ with $\|g(t)\| \le Me^{\omega t}$ for almost all $t \ge 0$ such that $r(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt$ for all $\lambda > \omega$.*

As in Theorem 2.1.1 one shows that there exists an isometric isomorphism $\Phi_{S,\omega}$ between the spaces $\mathrm{Lip}_\omega(\mathbb{R}_+, X)$ and $\mathcal{L}(L^1_\omega(\mathbb{R}_+), X)$, where

$$L^1_\omega(\mathbb{R}_+) := \left\{ h \in L^1_{loc}(\mathbb{R}_+) : \ \|h\|_{\omega,1} := \int_0^\infty e^{\omega t} |h(t)|\, dt < \infty \right\}.$$

The isomorphism $\Phi_{S,\omega}$ assigns to every function $G \in \mathrm{Lip}_\omega(\mathbb{R}_+, X)$ an operator $T \in \mathcal{L}(L^1_\omega(\mathbb{R}_+), X)$ with $\|T\| = \|G\|_{\mathrm{Lip}_\omega(\mathbb{R}_+,X)}$ such that

$$Th = \int_0^\infty h(t)\, dG(t)$$

for all continuous functions $h \in L^1_\omega(\mathbb{R}_+)$, $T\chi_{[0,t]} = G(t)$ for all $t \geq 0$, and $Te_{-\lambda} = \widehat{dG}(\lambda)$ if $\mathrm{Re}\,\lambda > \omega$.

The inversion theorems in Section 2.3 all remain valid, with almost no changes in the proofs (the version of Theorem 2.3.4 for $\mathrm{Lip}_\omega(\mathbb{R}_+, X)$ can be deduced directly from the case $\omega = 0$ by using the isomorphism I_ω). Thus, if $r = \widehat{dF}$ for some $F \in \mathrm{Lip}_\omega(\mathbb{R}_+, X)$, then

$$F(t) = \lim_{k\to\infty} (-1)^k \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} \frac{d^k}{d\lambda^k} \left(\frac{r(\lambda)}{\lambda}\right)\bigg|_{\lambda=k/t}. \tag{2.8}$$

If $c > \max(\omega, 0)$, then

$$F(t) = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{c-ik}^{c+ik} e^{\lambda t} \frac{r(\lambda)}{\lambda}\, d\lambda, \tag{2.9}$$

where the limit exists uniformly on compact subsets of $\mathbb{R}_+$. Finally,

$$F(t) = \lim_{k\to\infty} \sum_{j=1}^\infty (-1)^{j+1} \frac{1}{j!} e^{tkj} r(kj), \tag{2.10}$$

where the limit exists uniformly on $\mathbb{R}_+$.

The following is a consequence of the Phragmén-Doetsch inversion (2.10).

Proposition 2.4.3. *Let $\varepsilon > 0$ and $f \in L^1_{loc}(\mathbb{R}_+, X)$ with $\mathrm{abs}(f) < \infty$. The following are equivalent.*

(i) $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \|\hat{f}(\lambda)\| \leq -\varepsilon$.

(ii) $f = 0$ *a.e. on* $[0, \varepsilon]$.

Proof. Let $F(t) := \int_0^t f(s)\, ds$ and $G(t) := \int_0^t F(s)\, ds$. Since $\mathrm{abs}(f) < \infty$, $\omega(F) < \infty$ by Theorem 1.4.3 and hence $G \in \mathrm{Lip}_\omega(\mathbb{R}_+, X)$ for some $\omega \in \mathbb{R}$. By Corollary 1.6.5 and Proposition 1.10.1,

$$\hat{f}(\lambda) = \lambda \widehat{F}(\lambda) = \lambda \widehat{dG}(\lambda) = \lambda^2 \widehat{G}(\lambda)$$

for $\operatorname{Re}\lambda > \omega$. Define

$$r(\lambda) := \frac{1}{\lambda}\hat{f}(\lambda) = \widehat{F}(\lambda) = \widehat{dG}(\lambda)$$

for $\lambda > \omega$. If (i) holds, then $\limsup_{\lambda\to\infty} \frac{1}{\lambda}\log\|r(\lambda)\| \leq -\varepsilon$. Let $0 < \xi < \varepsilon$. Then there exist $M, \lambda_0 > 0$ such that $\|r(\lambda)\| \leq Me^{-\lambda\xi}$ for all $\lambda > \lambda_0$. Let $t \in [0,\xi)$. Then, for $\lambda_0 < k \in \mathbb{N}$,

$$\left\|\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tkj} r(kj)\right\| \leq M \sum_{j=1}^{\infty} \frac{1}{j!} e^{(t-\xi)kj} = M\left(e^{e^{(t-\xi)k}} - 1\right) \to 0$$

as $k \to \infty$. Since $r = \widehat{dG}$, it follows from (2.10) that $G = 0$ on $[0,\xi)$ for all $0 < \xi < \varepsilon$. Thus, $G = 0$ on $[0,\varepsilon]$ and hence $f = 0$ a.e. on $[0,\varepsilon]$, by Proposition 1.2.2. This proves that (i) $\Rightarrow$ (ii).

Suppose that (ii) holds. Then $F = 0$ on $[0,\varepsilon]$. Thus

$$r(\lambda) = \int_0^\infty e^{-\lambda t} F(t)\,dt = \int_\varepsilon^\infty e^{-\lambda t} F(t)\,dt = e^{-\lambda\varepsilon}\int_0^\infty e^{-\lambda t} F(t+\varepsilon)\,dt.$$

Since $t \mapsto F(t+\varepsilon)$ is exponentially bounded, it follows that $\|\int_0^\infty e^{-\lambda t} F(t+\varepsilon)\,dt\| \leq C$ for some $C > 0$ and therefore $\|r(\lambda)\| \leq Ce^{-\varepsilon\lambda}$ for all sufficiently large λ. This proves that (ii) $\Rightarrow$ (i). $\square$

If $f \in L^1_{loc}(\mathbb{R}_+, X)$ with $\operatorname{abs}(f) < \infty$, then it follows from Corollary 1.6.5 and the exponential boundedness of F that there exist $M, \lambda_0 > 0$ such that $\|\hat{f}(\lambda)\| \leq M$ for all $\lambda > \lambda_0$. Thus, $\limsup_{\lambda\to\infty} \frac{1}{\lambda}\log\|\hat{f}(\lambda)\| \leq 0$. This and the previous proposition yield the following corollary.

Corollary 2.4.4. *Let $f \in L^1_{loc}(\mathbb{R}_+, X)$ with $\operatorname{abs}(f) < \infty$. Then the following are equivalent:*

(i) $\limsup_{\lambda\to\infty} \frac{1}{\lambda}\log\|\hat{f}(\lambda)\| = 0$.

(ii) *For every $\varepsilon > 0$, the restriction of f to $[0,\varepsilon]$ does not vanish a.e.*

2.5 Complex Conditions

It was shown in the previous section that a holomorphic function $q : \{\operatorname{Re}\lambda > \omega\} \to X$ has a Laplace-Stieltjes or multiplied Laplace representation

$$q(\lambda) = \int_0^\infty e^{-\lambda t}\,dF(t) = \lambda\int_0^\infty e^{-\lambda t} F(t)\,dt$$

if there exists a constant $M > 0$ such that the Taylor coefficients $\frac{1}{k!}q^{(k)}(\lambda)$ are bounded by $M/(\lambda-\omega)^{k+1}$ for all $\lambda > \omega$ and $k \in \mathbb{N}_0$. Since only properties of the function q along the real half-line (ω,∞) are involved, Widder's growth conditions are also referred to as "real conditions". In many instances, these real conditions are too difficult to be checked because all derivatives of q have to be considered, whereas the growth of q in a complex half-plane $\operatorname{Re}\lambda > \omega$ can be estimated. In these cases one can apply the following representation theorem.

Theorem 2.5.1 (Complex Representation). *Let $\omega \geq 0$, let $q : \{\operatorname{Re}\lambda > \omega\} \to X$ be a holomorphic function with $\sup_{\operatorname{Re}\lambda>\omega} \|\lambda q(\lambda)\| < \infty$ and let $b > 0$. Then there exists $f \in C(\mathbb{R}_+, X)$ with $\sup_{t>0} \|e^{-\omega t} t^{-b} f(t)\| < \infty$ such that $q(\lambda) = \lambda^b \hat{f}(\lambda)$ for $\operatorname{Re}\lambda > \omega$.*

Proof. Let $\alpha > \omega$ and define

$$f(t) := \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\alpha-iR}^{\alpha+iR} e^{\lambda t} \frac{q(\lambda)}{\lambda^b}\, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\alpha+ir)t} \frac{q(\alpha+ir)}{(\alpha+ir)^b}\, dr.$$

Observe that the latter integral is absolutely convergent, by the assumption on q, so the limit exists uniformly for t in compact subsets of $\mathbb{R}_+$. Hence, f is continuous on $\mathbb{R}_+$. By applying Cauchy's theorem over rectangles with vertices $\alpha \pm iR$, $\beta \pm iR$, and using the assumption on q, it is easy to see that the definition of f is independent of $\alpha > \omega$.

For $\alpha > \omega$ and $R > 0$, let $\Gamma_{\alpha,R}$ be the path consisting of the vertical half-line $\{\alpha + ir : \ r < -R\}$, the semicircle $\{\alpha + Re^{i\theta} : \ \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$, and the half-line $\{\alpha + ir : \ r > R\}$. By Cauchy's theorem,

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{\Gamma_{\alpha,R}} e^{\lambda t} \frac{q(\lambda)}{\lambda^b}\, d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{-R} e^{(\alpha+ir)t} \frac{q(\alpha+ir)}{(\alpha+ir)^b}\, dr \\ &\quad + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{(\alpha+Re^{i\theta})t} \frac{q(\alpha+Re^{i\theta})}{(\alpha+Re^{i\theta})^b} Re^{i\theta}\, d\theta \\ &\quad + \frac{1}{2\pi} \int_{R}^{\infty} e^{(\alpha+ir)t} \frac{q(\alpha+ir)}{(\alpha+ir)^b}\, dr. \end{aligned}$$

Hence,

$$\begin{aligned} \|f(t)\| &\leq \frac{Me^{\alpha t}}{\pi} \int_R^\infty \frac{dr}{r^{b+1}} + \frac{M}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{e^{(\alpha+R\cos\theta)t}}{R^b}\, d\theta \\ &= \frac{Me^{\alpha t}}{\pi b R^b} + \frac{Me^{\alpha t}}{\pi R^b} \int_0^{\pi/2} e^{Rt\cos\theta}\, d\theta, \end{aligned}$$

where $M := \sup_{\operatorname{Re}\lambda>\omega} \|\lambda q(\lambda)\|$. Choosing $R = 1/t$, we obtain that $\|f(t)\| \leq Ct^b e^{\alpha t}$ for some C independent of $\alpha > \omega$. Hence, $\|f(t)\| \leq Ct^b e^{\omega t}$.

Given λ with $\operatorname{Re}\lambda > \omega$, choose $\omega < \alpha < \operatorname{Re}\lambda$. By the dominated convergence theorem and Fubini's theorem,

$$\begin{aligned}\int_0^\infty e^{-\lambda t} f(t)\,dt &= \lim_{R\to\infty}\int_0^\infty e^{-\lambda t}\frac{1}{2\pi i}\int_{\alpha-iR}^{\alpha+iR} e^{zt}\frac{q(z)}{z^b}\,dz\,dt \\ &= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\alpha-iR}^{\alpha+iR}\frac{q(z)}{(\lambda-z)z^b}\,dz.\end{aligned}$$

By Cauchy's residue theorem around the path consisting of the semicircle $\{\alpha + Re^{i\theta} : -\pi/2 \le \theta \le \pi/2\}$ and the line-segment $\{\alpha + ir : -R \le r \le R\}$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\alpha-iR}^{\alpha+iR}\frac{q(z)}{(\lambda-z)z^b}\,dz &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{q(\alpha+Re^{i\theta})Re^{i\theta}}{(\lambda-\alpha-Re^{i\theta})(\alpha+Re^{i\theta})^b}\,d\theta + \frac{q(\lambda)}{\lambda^b} \\ &\to \frac{q(\lambda)}{\lambda^b}\end{aligned}$$

as $R \to \infty$, using the assumption on q. □

We mention that Theorem 2.5.1 does not hold for $b = 0$. In fact, Desch and Prüss [DP93] construct a scalar-valued holomorphic function q on $\mathbb{C}_+$ satisfying

$$\sup_{\operatorname{Re}\lambda>0}\|q(\lambda)\|(1+|\lambda|) < \infty$$

such that q is not the Laplace transform of a function $f \in L^\infty_{loc}(0,\infty)$.

On the other hand, if $\lambda q(\lambda)$ and $\lambda^2 q'(\lambda)$ are bounded on the right half-plane, then q is the Laplace transform of a bounded continuous function, as we show in the following corollary.

Corollary 2.5.2 (Prüss). *Let* $q : \{\operatorname{Re}\lambda > 0\} \to X$ *be holomorphic. If there exists* $M > 0$ *such that* $\|\lambda q(\lambda)\| \le M$ *and* $\|\lambda^2 q'(\lambda)\| \le M$ *for* $\operatorname{Re}\lambda > 0$, *then there exists a bounded function* $f \in C((0,\infty),X)$ *such that* $q(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$ *for* $\operatorname{Re}\lambda > 0$. *In particular,* $q \in C_W^\infty((0,\infty),X)$.

Proof. It follows from Theorem 2.5.1 that there are functions $f_i \in C(\mathbb{R}_+,X)$ ($i = 0,1$) and $C > 0$ such that $\|f_i(t)\| \le Ct$ for $t > 0$,

$$q(\lambda) = \lambda\int_0^\infty e^{-\lambda t} f_0(t)\,dt, \quad\text{and}\quad \lambda q'(\lambda) = \lambda\int_0^\infty e^{-\lambda t} f_1(t)\,dt$$

for $\operatorname{Re}\lambda > 0$. By Theorem 1.5.1,

$$q'(\lambda) = \int_0^\infty e^{-\lambda t} f_0(t)\,dt - \lambda\int_0^\infty e^{-\lambda t} t f_0(t)\,dt = \int_0^\infty e^{-\lambda t} f_1(t)\,dt.$$

Integration by parts (or Corollary 1.6.5) yields

$$\lambda\int_0^\infty e^{-\lambda t}\left(\int_0^t f_0(s)\,ds - t f_0(t)\right)dt = \lambda\int_0^\infty e^{-\lambda t}\int_0^t f_1(s)\,ds\,dt.$$

Since the Laplace transform is one-to-one, it follows that $tf_0(t) = \int_0^t f_0(s)\,ds - \int_0^t f_1(s)\,ds$. Thus, $f_0 \in C^1((0,\infty),X)$ and $tf_0'(t) = -f_1(t)$. Therefore, $\|f_0'(t)\| \le C$ for all $t > 0$ and

$$q(\lambda) = \lambda \int_0^\infty e^{-\lambda t} f_0(t)\,dt = \int_0^\infty e^{-\lambda t} f_0'(t)\,dt \quad (\mathrm{Re}\,\lambda > 0). \qquad \square$$

Remark 2.5.3. If $f \in L^\infty((0,\infty),X)$, then $r = \hat{f}$ is holomorphic on the right half-plane and

$$\begin{aligned} \|\lambda r(\lambda)\| &\le \frac{|\lambda|}{\mathrm{Re}\,\lambda}\|f\|_\infty, \\ \|\lambda^2 r'(\lambda)\| &\le \left(\frac{|\lambda|}{\mathrm{Re}\,\lambda}\right)^2 \|f\|_\infty \qquad (\mathrm{Re}\,\lambda > 0). \end{aligned}$$

In particular, $\lambda r(\lambda)$ and $\lambda^2 r'(\lambda)$ are bounded on each sector $\Sigma_\alpha = \{re^{i\gamma} : \; r > 0, \; |\gamma| < \alpha\}$ where $\alpha \in (0,\pi/2)$. In Corollary 2.5.2 the estimate is required uniformly on the right half-plane, which is more. On the other hand, continuity is obtained as an additional result. $\square$

We close this section with a characterization of Laplace transforms of functions in $L^1_{loc}(\mathbb{R}_+,X)$ with $\|f(t)\| \le Mt^n$ for some $M, n \ge 0$ and almost all $t \ge 0$ (if X has the Radon-Nikodym property) or the Laplace-Stieltjes transforms of functions $H : \mathbb{R}_+ \to X$ with $H(0) = 0$ and $\|H(t) - H(s)\| \le M \int_s^t r^n\,dr$ for some $M > 0$ and all $0 \le s \le t$ (for general X).

Corollary 2.5.4. *Let $M > 0$, $n \in \mathbb{N}_0$, and $r \in C^\infty((0,\infty),X)$. The following are equivalent:*

(i) $\|\frac{\lambda^{k+n+1}}{(k+n)!} r^{(k)}(\lambda)\| \le M \quad (\lambda > 0, \; k \in \mathbb{N}_0)$.

(ii) *There exists $H : \mathbb{R}_+ \to X$ satisfying $H(0) = 0$ and $\|H(t) - H(s)\| \le M \int_s^t r^n\,dr \;\; (0 \le s \le t)$, such that $r(\lambda) = \int_0^\infty e^{-\lambda t}\,dH(t)$ for all $\lambda > 0$.*

Proof. By the Real Representation Theorem 2.2.1, the statement holds for $n = 0$. Therefore, let $n \ge 1$. To show that (i) implies (ii), define

$$m(\lambda) := (-1)^n \int_\lambda^\infty \frac{1}{(n-1)!}(u-\lambda)^{n-1} r(u)\,du$$

for $\lambda > 0$. Then, $m^{(k)}(\lambda) = r^{(k-n)}(\lambda)$ for all $k \ge n$ and $\lambda > 0$. Since $\|\frac{\lambda^{k+1}}{k!} m^{(k)}(\lambda)\| = \|\frac{\lambda^{k+1}}{k!} r^{(k-n)}(\lambda)\| \le M$ for all $\lambda > 0$ and $k \ge n$, it follows from Proposition 2.2.2 that $m \in C^\infty_W((0,\infty),X)$ and $\|m\|_W \le M$. By Theorem 2.2.1, there exists $G : \mathbb{R}_+ \to X$ with $G(0) = 0$ and $\|G(t) - G(s)\| \le M|t-s|$ for all $t, s \ge 0$ such that $m(\lambda) = \int_0^\infty e^{-\lambda t}\,dG(t)$ for all $\lambda > 0$. By Theorem 1.5.1 and Proposition 1.9.10,

$$r(\lambda) = m^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t}(-t)^n\,dG(t) = \int_0^\infty e^{-\lambda t}\,dH(t),$$

where $H(t) := \int_0^t (-s)^n \, dG(s)$. Now the statement (ii) follows from $\|H(t)-H(s)\| = \| \int_s^t (-r)^n \, dG(r)\| \leq M \int_s^t r^n \, dr$ for all $0 \leq s \leq t$.

To show that (ii) implies (i), let $x^* \in X^*$. The function $x^* \circ H$ is locally Lipschitz continuous, hence absolutely continuous and differentiable a.e. If $h(t) := \frac{d}{dt}\langle H(t), x^*\rangle$, then $|h(t)| \leq Mt^n\|x^*\|$ and $\langle r(\lambda), x^*\rangle = \int_0^\infty e^{-\lambda t} h(t)\, dt$, by Proposition 1.9.11. Hence,

$$\begin{aligned} \left| \left\langle \frac{\lambda^{k+n+1}}{(k+n)!} r^{(k)}(\lambda), x^* \right\rangle \right| &= \left| \frac{\lambda^{k+n+1}}{(k+n)!} \int_0^\infty e^{-\lambda t} (-t)^k h(t)\, dt \right| \\ &\leq M\|x^*\|. \end{aligned}$$

Now (i) follows from the Hahn-Banach theorem. □

2.6 Laplace Transforms of Holomorphic Functions

In this section those functions are characterized which are Laplace transforms of holomorphic, exponentially bounded functions defined on some open sector $\Sigma_\alpha := \{re^{i\gamma} : r > 0, -\alpha < \gamma < \alpha\}$ for some $0 < \alpha \leq \pi/2$. The closure of Σ_α is denoted by $\overline{\Sigma}_\alpha$. We shall use the same notation for $0 < \alpha < \pi$. Note that $\Sigma_{\frac{\pi}{2}} = \mathbb{C}_+ := \{\operatorname{Re}\lambda > 0\}$.

Theorem 2.6.1 (Analytic Representation). *Let* $0 < \alpha \leq \frac{\pi}{2}$, $\omega \in \mathbb{R}$ *and* $q : (\omega, \infty) \to X$. *The following are equivalent:*

(i) *There is a holomorphic function* $f : \Sigma_\alpha \to X$ *such that* $\sup_{z\in\Sigma_\beta} \|e^{-\omega z} f(z)\| < \infty$ *for all* $0 < \beta < \alpha$ *and* $q(\lambda) = \hat{f}(\lambda)$ *for all* $\lambda > \omega$.

(ii) *The function* q *has a holomorphic extension* $\tilde{q} : \omega + \Sigma_{\alpha+\frac{\pi}{2}} \to X$ *such that* $\sup_{\lambda\in\omega+\Sigma_{\gamma+\frac{\pi}{2}}} \|(\lambda-\omega)\tilde{q}(\lambda)\| < \infty$ *for all* $0 < \gamma < \alpha$.

Proof. Assume that (i) holds. Let $0 < \beta < \alpha$. Then there exists $M > 0$ such that $\|f(z)\| \leq M|e^{\omega z}|$ for all $z \in \overline{\Sigma}_\beta \setminus \{0\}$. Define paths $\Gamma_\pm$ by $\Gamma_\pm := \{se^{\pm i\beta} : 0 \leq s < \infty\}$. By Cauchy's theorem,

$$\begin{aligned} q(\lambda) &= \int_0^\infty e^{-\lambda t} f(t)\, dt = \int_{\Gamma_\pm} e^{-\lambda z} f(z) dz \\ &= e^{\pm i\beta} \int_0^\infty e^{-\lambda s e^{\pm i\beta}} f(se^{\pm i\beta})\, ds \end{aligned} \tag{2.11}$$

for all $\lambda > \omega$. Let $0 < \varepsilon < \frac{\pi}{2} - \beta$, and let $\lambda \in \mathbb{C}$ with $-\frac{\pi}{2} - \beta + \varepsilon < \arg(\lambda - \omega) < \frac{\pi}{2} - \beta - \varepsilon$. Then $-\frac{\pi}{2} + \varepsilon < \arg((\lambda-\omega)e^{i\beta}) < \frac{\pi}{2} - \varepsilon$, so $\operatorname{Re}((\lambda-\omega)e^{i\beta}) \geq |\lambda-\omega| \sin\varepsilon$. It follows that

$$\|e^{-\lambda s e^{i\beta}} f(se^{i\beta})\| \leq M e^{-s|\lambda-\omega|\sin\varepsilon}.$$

Consequently, the integral

$$q_+(\lambda) := e^{i\beta} \int_0^\infty e^{-\lambda s e^{i\beta}} f(se^{i\beta})\, ds$$

is absolutely convergent and defines a holomorphic function in the region $-\frac{\pi}{2} - \beta + \varepsilon < \arg(\lambda - \omega) < \frac{\pi}{2} - \beta - \varepsilon$, with $\|(\lambda-\omega)q_+(\lambda)\| \le M/\sin\varepsilon$. Similarly,

$$q_-(\lambda) := e^{-i\beta} \int_0^\infty e^{-\lambda s e^{-i\beta}} f(se^{-i\beta})\, ds$$

defines a holomorphic function in the region $-\frac{\pi}{2} + \beta + \varepsilon < \arg(\lambda-\omega) < \frac{\pi}{2} + \beta - \varepsilon$, with $\|(\lambda-\omega)q_-(\lambda)\| \le M/\sin\varepsilon$. By (2.11), both q_+ and q_- are extensions of q, and together they define a holomorphic extension $\tilde{q}$ to $\omega + \Sigma_{\frac{\pi}{2}+\beta-\varepsilon}$, satisfying $\|(\lambda-\omega)\tilde{q}(\lambda)\| \le M/\sin\varepsilon$ in the sector. Since $\beta < \alpha$ and $0 < \varepsilon < \frac{\pi}{2} - \beta$ are arbitrary, this proves (ii).

Assume that (ii) holds. Let $0 < \gamma < \alpha$ and $\delta > 0$. There exists $M > 0$ such that $\|(\lambda-\omega)\tilde{q}(\lambda)\| \le M$ for all $\lambda \in (\omega + \overline{\Sigma}_{\gamma+\frac{\pi}{2}}) \setminus \{\omega\}$. Consider an oriented path Γ (depending on γ and δ) consisting of

$$\Gamma_\pm := \left\{\omega + re^{\pm i(\gamma+\pi/2)} : \delta \le r\right\} \text{ and } \Gamma_0 := \left\{\omega + \delta e^{i\theta} : -\gamma - \tfrac{\pi}{2} \le \theta \le \gamma + \tfrac{\pi}{2}\right\}.$$

Let $0 < \varepsilon < \gamma$ and consider $z \in \Sigma_{\gamma-\varepsilon}$. For $\lambda = \omega + re^{\pm i(\gamma+\pi/2)} \in \Gamma_\pm$,

$$\begin{aligned} \operatorname{Re}(\lambda z) &= \omega \operatorname{Re} z + r|z|\cos(\arg z \pm (\gamma + \pi/2)) \\ &\le \omega \operatorname{Re} z - r|z|\sin\varepsilon. \end{aligned}$$

Hence,

$$\|e^{\lambda z}\tilde{q}(\lambda)\| \le e^{\omega \operatorname{Re} z} e^{-r|z|\sin\varepsilon} \frac{M}{r} \quad (\lambda \in \Gamma_\pm). \tag{2.12}$$

This shows that

$$f(z) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda z} \tilde{q}(\lambda)\, d\lambda \tag{2.13}$$

is absolutely convergent, uniformly for z in compact subsets of Σ_γ, and therefore defines a holomorphic function in Σ_γ. By Cauchy's theorem, this function is independent of $\delta > 0$, and also independent of $\gamma < \alpha$ so long as $\arg z < \gamma$ (here we use the assumption on $\tilde{q}$ to estimate the integral of $e^{\lambda z}\tilde{q}(\lambda)$ over arcs $\{\omega + Re^{i\theta} : \gamma_1 + \frac{\pi}{2} \le |\theta| \le \gamma_2 + \frac{\pi}{2}\}$). Hence (2.13) defines a holomorphic function $f \in \Sigma_\alpha$.

To estimate $f(z)$, we choose $\delta = |z|^{-1}$, and choose γ and ε such that $\gamma < \alpha$ and $|\arg z| < \gamma - \varepsilon$. On Γ_0, $\lambda = \omega + |z|^{-1}e^{i\theta}$ $(-\gamma - \pi/2 \le \theta \le \gamma + \pi/2)$, so

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{\Gamma_0} e^{\lambda z}\tilde{q}(\lambda)\, d\lambda \right\| &\le \frac{1}{2\pi} \int_{-\gamma-\pi/2}^{\gamma+\pi/2} e^{\omega \operatorname{Re} z} e^{\cos(\arg z + \theta)} M\, d\theta \\ &\le Me^{1+\omega \operatorname{Re} z}. \end{aligned} \tag{2.14}$$

On $\Gamma_\pm$, $\lambda = \omega + re^{\pm i(\gamma+\pi/2)}$, and the estimate (2.12) gives

$$\begin{aligned}\left\|\frac{1}{2\pi i}\int_{\Gamma_\pm} e^{\lambda z}\tilde{q}(\lambda)\,d\lambda\right\| &\leq \frac{1}{2\pi}\int_{|z|^{-1}}^{\infty} e^{\omega\operatorname{Re} z}e^{-r|z|\sin\varepsilon}\frac{M}{r}\,dr \\ &= \frac{Me^{\omega\operatorname{Re} z}}{2\pi}\int_1^\infty \frac{e^{-r\sin\varepsilon}}{r}\,dr \\ &\leq \frac{Me^{\omega\operatorname{Re} z}}{2\pi\sin\varepsilon}. \qquad (2.15)\end{aligned}$$

Now (2.14) and (2.15) establish that

$$\sup_{z\in\Sigma_{\gamma-\varepsilon}} \|e^{-\omega z}f(z)\| < \infty$$

for any $0 < \varepsilon < \gamma < \alpha$.

Next we will show that $\hat{f}(\lambda) = q(\lambda)$ whenever $\lambda > \omega$. Given such λ, choose $0 < \delta < \lambda - \omega$, and $0 < \gamma < \alpha$. Then λ is to the right of the path Γ, and Fubini's theorem and Cauchy's residue theorem imply that

$$\begin{aligned}\hat{f}(\lambda) &= \int_0^\infty e^{-\lambda t}\frac{1}{2\pi i}\int_\Gamma e^{\mu t}\tilde{q}(\mu)\,d\mu\,dt \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{q}(\mu)}{\lambda-\mu}\,d\mu \\ &= \tilde{q}(\lambda) + \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\widetilde{\Gamma}_R}\frac{\tilde{q}(\mu)}{\lambda-\mu}\,d\mu,\end{aligned}$$

where $\widetilde{\Gamma}_R := \{\omega + Re^{i\theta} :\ -\gamma - \pi/2 \leq \theta \leq \gamma + \pi/2\}$. Then

$$\begin{aligned}\left\|\int_{\widetilde{\Gamma}_R}\frac{\tilde{q}(\mu)}{\lambda-\mu}\,d\mu\right\| &\leq \int_{-\gamma-\pi/2}^{\gamma+\pi/2}\frac{M}{|\omega + Re^{i\theta} - \lambda|}\,d\theta \\ &\to 0\end{aligned}$$

as $R \to \infty$. This proves that $\hat{f}(\lambda) = q(\lambda)$. □

When f is as in Theorem 2.6.1 (i), it is an easy consequence of Cauchy's integral formula for derivatives that

$$\sup_{z\in\Sigma_\beta}\left\|z^k e^{-\omega z}f^{(k)}(z)\right\| < \infty$$

for all $0 < \beta < \alpha$.

Recall from Sections 1.4 and 1.5 that, for $f \in L^1_{loc}(\mathbb{R}_+, X)$,

$$\begin{aligned}\omega(f) &:= \inf\left\{\omega\in\mathbb{R} :\ \sup_{t\geq 0}\|e^{-\omega t}f(t)\| < \infty\right\}, \\ \operatorname{abs}(f) &:= \inf\left\{\operatorname{Re}\lambda :\ \hat{f}(\lambda)\ \text{ exists}\right\}, \\ \operatorname{hol}(\hat{f}) &:= \inf\left\{\omega\in\mathbb{R} :\ \hat{f}\ \text{ has a holomorphic extension for }\ \operatorname{Re}\lambda > \omega\right\}.\end{aligned}$$

Moreover, $\text{hol}(\hat{f}) \leq \text{abs}(f) \leq \omega(f)$. We will now show that equalities hold when f is holomorphic and exponentially bounded on a sector.

Theorem 2.6.2. *Let $0 < \alpha < \pi/2$, let $f : \Sigma_\alpha \to X$ be holomorphic, and suppose that there exists $\omega \in \mathbb{R}$ such that $\sup_{z\in\Sigma_\alpha} \|e^{-\omega z} f(z)\| < \infty$. Then $\text{hol}(\hat{f}) = \text{abs}(f) = \omega(f)$.*

Proof. By Theorem 2.6.1, there exists $\gamma > 0$ such that $\hat{f}$ has a holomorphic extension (also denoted by $\hat{f}$) to $\omega + \Sigma_{\gamma+\pi/2}$ and $C := \sup_{\lambda\in\Sigma_{\gamma+\pi/2}} \|(\lambda-\omega)\hat{f}(\lambda)\| < \infty$. By definition of $\text{hol}(\hat{f})$, $\hat{f}$ also has a holomorphic extension to $\text{hol}(\hat{f}) + \Sigma_{\pi/2} = \{\text{Re}\,\lambda > \text{hol}(\hat{f})\}$.

Let $\omega' > \text{hol}(\hat{f})$. There exists $\gamma' > 0$ such that

$$\omega' + \overline{\Sigma}_{\gamma'+\pi/2} \subseteq (\omega + \Sigma_{\gamma+\pi/2}) \cup (\text{hol}(\hat{f}) + \Sigma_{\pi/2}).$$

Hence, $\hat{f}$ is holomorphic on $\omega' + \Sigma_{\gamma'+\pi/2}$ and continuous on the closure. Let

$$U := \left\{\lambda \in (\omega' + \overline{\Sigma}_{\gamma'+\pi/2}) \cap (\omega + \Sigma_{\gamma+\pi/2}) : \ |\lambda - \omega'| < 2|\lambda - \omega|\right\}.$$

If $\lambda \in U$, then $\|(\lambda-\omega')\hat{f}(\lambda)\| \leq 2C$. Moreover, $(\omega' + \overline{\Sigma}_{\gamma'+\pi/2}) \setminus U$ is compact. Hence, $\sup_{\lambda\in\omega'+\Sigma_{\gamma'+\pi/2}} \|(\lambda-\omega')\hat{f}(\lambda)\| < \infty$. It follows from Theorem 2.6.1, and the fact that the Laplace transform is one-to-one, that $\sup_{z\in\Sigma_\beta} \|e^{-\omega' z} f(z)\| < \infty$ for some $\beta > 0$, and in particular, $\omega(f) \leq \omega'$. Since this holds whenever $\omega' > \text{hol}(\hat{f})$, it follows that $\omega(f) \leq \text{hol}(\hat{f})$, completing the proof. □

In the remainder of this section we will consider asymptotic behaviour of $f(t)$ as $t \to \infty$ and as $t \to 0$. In the case of holomorphic functions defined on a sector it can be described completely in terms of the Laplace transform. This is not the case in general, and in Chapter 4 a systematic treatment of this question will be given. However, here we can use contour arguments directly on the basis of the representation formula (2.13).

First we show that asymptotic behaviour along one ray and on the whole sector are equivalent. This is a consequence of Vitali's theorem (Theorem A.5).

Proposition 2.6.3. *Let $0 < \alpha \leq \pi$ and let $f : \Sigma_\alpha \to X$ be holomorphic such that*

$$\sup_{z\in\Sigma_\beta} \|f(z)\| < \infty$$

for all $0 < \beta < \alpha$. Let $x \in X$.

a) *If $\lim_{t\to\infty} f(t) = x$, then $\lim_{\substack{|z|\to\infty \\ z\in\Sigma_\beta}} f(z) = x$ for all $0 < \beta < \alpha$.*

b) *If $\lim_{t\downarrow 0} f(t) = x$, then $\lim_{\substack{|z|\to 0 \\ z\in\Sigma_\beta}} f(z) = x$ for all $0 < \beta < \alpha$.*

Proof. a) Let $f_k(z) = f(kz)$. It follows from Vitali's theorem that $\lim_{k\to\infty} f_k(z) = x$ uniformly on compact subsets of Σ_α. Let $0 < \beta < \alpha$. Let $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that $\|f_k(z) - x\| \le \varepsilon$ whenever $z \in \Sigma_\beta,\ 1 \le |z| \le 2,\ k \ge k_0$. Let $z \in \Sigma_\beta,\ |z| \ge k_0$. Choose $k \in \mathbb{N}$ such that $k \le |z| < k+1$. Then

$$\|f(z) - x\| = \|f_k(z/k) - x\| \le \varepsilon.$$

This proves a).

b) This follows by applying a) to the function $z \mapsto f(z^{-1})$. □

Now we consider the asymptotic behaviour of $f(t)$ as $t \to \infty$ and $t \downarrow 0$.

Theorem 2.6.4 (Tauberian Theorem). *Consider the situation of Theorem* 2.6.1*, and let* $x \in X$*.*

a) *One has* $\lim_{t\downarrow 0} f(t) = x$ *if and only if* $\lim_{\lambda\to\infty} \lambda q(\lambda) = x$*.*

b) *Assume that* $\omega = 0$*. Then* $\lim_{t\to\infty} f(t) = x$ *if and only if* $\lim_{\lambda\downarrow 0} \lambda q(\lambda) = x$*.*

Proof. We can assume that $\omega = 0$ for both cases a) and b) by replacing $f(z)$ by $e^{-\omega z} f(z)$ otherwise. Replacing $f(t)$ by $f(t) - x$, we can also assume that $x = 0$. For simplicity, we shall denote the function $\tilde{q}$ of Theorem 2.6.1 by q.

Assume that $\lim_{\lambda\to\infty} \lambda q(\lambda) = x$. Let $0 < \gamma < \alpha$. By Proposition 2.6.3, $\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Sigma_{\gamma+\pi/2}}} \lambda q(\lambda) = x$. Let $\varepsilon > 0$. There exists $\delta_0 > 0$ such that $\|\lambda q(\lambda)\| \le \varepsilon$ whenever $|\lambda| \ge \delta_0,\ \lambda \in \Sigma_{\gamma+\frac{\pi}{2}}$. Let $0 < t \le 1/\delta_0$. Now we choose the contour Γ as in the proof of Theorem 2.6.1, (ii) $\Rightarrow$ (i), with $\delta = 1/t$. Then

$$\begin{aligned}
\left\| \frac{1}{2\pi i} \int_{\Gamma_0} e^{\lambda t} q(\lambda)\, d\lambda \right\| &= \left\| \frac{1}{2\pi i} \int_{-\gamma-\pi/2}^{\gamma+\pi/2} e^{e^{i\theta}} q\left(\frac{e^{i\theta}}{t}\right) \frac{ie^{i\theta}}{t}\, d\theta \right\| \\
&\le \frac{\varepsilon}{2\pi} \int_{-\gamma-\pi/2}^{\gamma+\pi/2} e^{\cos\theta}\, d\theta \\
&\le \varepsilon\, e,
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \frac{1}{2\pi i} \int_{\Gamma_\pm} e^{\lambda t} q(\lambda)\, d\lambda \right\| \\
&= \left\| \frac{1}{2\pi i} \int_{1/t}^{\infty} e^{t\cdot re^{\pm i(\gamma+\pi/2)}} q(re^{\pm i(\gamma+\pi/2)}) re^{\pm i(\gamma+\pi/2)}\ \frac{dr}{r} \right\| \\
&= \left\| \frac{1}{2\pi i} \int_{1}^{\infty} e^{se^{\pm i(\gamma+\pi/2)}} q(\frac{s}{t} e^{\pm(\gamma+\pi/2)}) \frac{s}{t} e^{\pm i(\gamma+\pi/2)}\ \frac{ds}{s} \right\| \\
&\to 0
\end{aligned}$$

as $t \downarrow 0$ by the dominated convergence theorem. It follows from the representation (2.13) that $\limsup_{t\downarrow 0} \|f(t)\| \le \varepsilon\, e$.

The converse implication is easy and does not depend on holomorphy. Assume that $\lim_{t\downarrow 0}\|f(t)\|=0$. Let $\varepsilon>0$. There exists $\tau>0$ such that $\|f(t)\|\le\varepsilon$ for all $t\in[0,\tau]$. Then

$$\begin{aligned}\limsup_{\lambda\to\infty}\|\lambda q(\lambda)\| &\le \limsup_{\lambda\to\infty}\left\{\|\lambda\int_0^\tau e^{-\lambda t}f(t)\,dt\|+\|\lambda\int_\tau^\infty e^{-\lambda t}f(t)\,dt\|\right\}\\ &\le \varepsilon+\limsup_{\lambda\to\infty}\lambda\int_\tau^\infty e^{-\lambda t}Me^{\omega t}\,dt\\ &= \varepsilon+\limsup_{\lambda\to\infty}M\frac{\lambda}{\lambda-\omega}e^{-(\lambda-\omega)\tau}=\varepsilon,\end{aligned}$$

where $\omega>\omega(f)$ and M is suitable. This completes the proof of a).

The assertion b) is proved in the same way as a). □

2.7 Completely Monotonic Functions

Throughout this section, X will be an ordered Banach space with normal cone (see Appendix C). Let $f:\mathbb{R}_+\to X$ be increasing. Then f is of bounded semivariation on each interval $[0,\tau]$, by Proposition 1.9.1. Assume that $\omega(f)<\infty$. Then the Laplace-Stieltjes transform

$$\widehat{df}(\lambda)=\lim_{\tau\to\infty}\int_0^\tau e^{-\lambda t}\,df(t)=:\int_0^\infty e^{-\lambda t}\,df(t) \tag{2.16}$$

converges on the half-plane $\{\mathrm{Re}\,\lambda>\mathrm{abs}(df)\}$, and defines a holomorphic function $\widehat{df}$ on $\{\mathrm{Re}\,\lambda>\mathrm{abs}(df)\}$. Recall from Theorem 1.10.5 that $\mathrm{abs}(df)<\infty$ if and only if $\omega(f)<\infty$.

Theorem 2.7.1. *Let $f:\mathbb{R}_+\to X$ be an increasing function. Assume that $-\infty<\mathrm{abs}(df)<\infty$. Then $\mathrm{abs}(df)$ is a singularity of $\widehat{df}$.*

Proof. Replacing $f(t)$ by $\int_0^t e^{-\mathrm{abs}(df)s}\,df(s)$, we can assume that $\mathrm{abs}(df)=0$. Assume that $\widehat{df}$ has a holomorphic extension to a neighbourhood of 0. Then there exists $\delta>0$ such that

$$\widehat{df}(-\delta)=\sum_{n=0}^\infty(-1)^n(1+\delta)^n\frac{(\widehat{df})^{(n)}(1)}{n!}.$$

Let $x^*\in X_+^*$. Then

$$\langle\widehat{df}(-\delta),x^*\rangle=\sum_{n=0}^\infty\frac{(1+\delta)^n}{n!}\int_0^\infty e^{-t}t^n\,d\langle f(t),x^*\rangle.$$

Since all expressions are positive we may interchange the sum and the integral and obtain

$$\begin{aligned}\int_0^\infty e^{\delta t}\, d\langle f(t), x^*\rangle &= \int_0^\infty e^{-t} e^{(1+\delta)t}\, d\langle f(t), x^*\rangle \\ &= \sum_{n=0}^\infty \frac{(1+\delta)^n}{n!} \int_0^\infty e^{-t} t^n\, d\langle f(t), x^*\rangle \\ &= \langle \widehat{df}(-\delta), x^*\rangle < \infty.\end{aligned}$$

Since X_+^* spans X^* (see Proposition C.2), it follows that $\operatorname{abs}(x^* \circ f) \le -\delta$ for all $x^* \in X^*$. It follows from (1.25) that $\operatorname{abs}(df) \le -\delta$, which contradicts the assumption. □

Corollary 2.7.2. *Let $f \in L^1_{loc}(\mathbb{R}_+, X)$ such that $f(t) \ge 0$ a.e. Assume that $-\infty < \operatorname{abs}(f) < \infty$. Then $\operatorname{abs}(f)$ is a singularity of $\hat{f}$. Hence, $\operatorname{hol}(\hat{f}) = \operatorname{abs}(f)$.*

Proof. This is immediate from Proposition 1.10.1 and Theorem 2.7.1. □

Our aim is to characterize functions of the form $\widehat{df}$ where $f : \mathbb{R}_+ \to X$ is increasing. Then

$$(-1)^n \widehat{df}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t} t^n\, df(t) \ge 0$$

for all $n \in \mathbb{N}_0,\ \lambda > \omega$. Thus $\widehat{df}$ is completely monotonic in the sense of the following definition.

Definition 2.7.3. *A function $r : (\omega, \infty) \to X$ is* completely monotonic *if r is infinitely differentiable and*

$$(-1)^n r^{(n)}(\lambda) \ge 0 \quad \text{for all } \lambda > \omega,\ n \in \mathbb{N}_0. \tag{2.17}$$

In the following, we shall assume that $\omega = 0$ for simplicity (otherwise, we can replace $r(\lambda)$ by $r(\lambda + \omega)$ and $f(t)$ by $\int_0^t e^{-\omega s}\, df(s)$). Recall that by Theorem 1.10.5 $\operatorname{abs}(df) \le 0$ if and only if $\omega(f) \le 0$.

Definition 2.7.4. *We say that* Bernstein's theorem holds *in X if for every completely monotonic function $r : (0, \infty) \to X$ there exists an increasing function $f : \mathbb{R}_+ \to X$ such that $\omega(f) \le 0$ and $r(\lambda) = \widehat{df}(\lambda)$ for all $\lambda > 0$.*

Bernstein's theorem does hold in $X = \mathbb{R}$; this is just Bernstein's classical theorem from 1928 [Ber28]. Here we will prove it, as a special case of Theorem 2.7.7, with the help of the Real Representation Theorem 2.2.1.

Definition 2.7.5. *The space X has the* interpolation property *if, given two sequences $(x_n)_{n\in\mathbb{N}},\ (y_n)_{n\in\mathbb{N}}$ in X such that*

$$x_n \le x_{n+1} \le y_{n+1} \le y_n \quad (n \in \mathbb{N}) \tag{2.18}$$

there exists $z \in X$ such that

$$x_n \leq z \leq y_n \quad \textit{for all } n \in \mathbb{N}. \tag{2.19}$$

Examples 2.7.6. a) *Assume that $X = Y^*$ where Y is an ordered Banach space with normal cone. Then X has the interpolation property.*

Proof. Let $x_n^* \leq x_{n+1}^* \leq y_{n+1}^* \leq y_n^*$ ($n \in \mathbb{N}$). Replacing x_n^* by $x_n^* - x_1^*$ and y_n^* by $y_n^* - x_1^*$ we can assume that $x_n^* \geq 0$. Define $z^* \in X^*$ by $\langle x, z^* \rangle = \sup_{n\in\mathbb{N}} \langle x, x_n^* \rangle$. Then z^* is linear and positive, and hence continuous (see Appendix C).

b) If X is reflexive, then X has the interpolation property. This follows from a).

c) Each von Neumann algebra (i.e., a $*$-subalgebra of $\mathcal{L}(H)$ which is closed in the strong operator topology, where H is a Hilbert space) has the interpolation property. This follows from a) and [Ped89, Theorem 4.6.17].

d) Every σ-order complete Banach lattice (i.e., a Banach lattice in which each countable order-bounded set has a supremum) has the interpolation property.

e) If X has order continuous norm (i.e., each decreasing positive sequence converges) then X has the interpolation property.

f) The space $C[0,1]$ does not have the interpolation property.

See the Notes for further comments on the interpolation property. □

Now we can formulate the following characterization, which is the main result of this section.

Theorem 2.7.7. *Bernstein's theorem holds in X if and only if X has the interpolation property.*

The proof of Theorem 2.7.7 will be carried out in several steps. On the way we will prove a characterization of completely monotonic functions which is valid without restrictions on the space. First, we study convex functions.

Let $J \subset \mathbb{R}$ be an interval. A function $F : J \to X$ is called *convex* if

$$F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t)$$

for all $s, t \in J$, $0 < \lambda < 1$. Many order properties of convex functions carry over from the scalar case since for $x \in X$ we have

$$x \geq 0 \quad \text{if and only if} \quad \langle x, x^* \rangle \geq 0 \quad \text{for all } x^* \in X_+^*.$$

For example, a twice differentiable function F is convex if and only if $F'' \geq 0$.

Lemma 2.7.8. *Let $[a,b]$ be a closed interval in the interior of J and let $F : J \to X$ be convex. Then F is Lipschitz continuous on $[a,b]$. Moreover, if $F(J) \subset X_+$ and $F(a) = 0$, then F is increasing on $[a,b]$.*

Proof. Let $c < a,\ d > b$ such that $[c, d] \subset J$. Then for $a \leq t < s \leq b$,

$$\frac{F(a) - F(c)}{a - c} \leq \frac{F(s) - F(t)}{s - t} \leq \frac{F(d) - F(b)}{d - b}.$$

Since the cone is normal this implies that F is Lipschitz continuous on $[a, b]$. The second assertion is easy to see. □

We notice in particular that every convex function defined on an open interval is continuous.

Let $-\infty < a < b \leq \infty$ and let $f : [a, b) \to X_+$ be increasing. Then f is Riemann integrable on $[a, t]$ whenever $a \leq t < b$ (see Corollary 1.9.6). Let

$$F(t) := \int_a^t f(s)\, ds \quad (a \leq t < b). \tag{2.20}$$

Then $F : [a, b) \to X_+$ is convex.

If X has the interpolation property, then the following converse result holds.

Proposition 2.7.9. *Assume that X has the interpolation property. Let $F : [a, b) \to X_+$ be convex such that $F(a) = 0$, where $-\infty < a < b \leq \infty$. Then there exists an increasing function $f : [a, b) \to X_+$ such that* (2.20) *holds.*

Proof. The following two properties follow from convexity:

a) Let $a \leq s < b$. Then the difference quotient

$$\frac{1}{h}\left(F(s + h) - F(s)\right)$$

is positive and increasing for $h \in (0, b - s)$.

b) Let $a \leq s < s + h \leq t < t + k < b$. Then

$$\frac{1}{h}\left(F(s + h) - F(s)\right) \leq \frac{1}{k}\left(F(t + k) - F(t)\right). \tag{2.21}$$

Put $f(a) = 0$. It follows from the interpolation property, a) and b) that for each $t \in (a, b)$ there exists $f(t) \in X$ such that

$$\frac{1}{h}\left(F(s + h) - F(s)\right) \leq f(t) \leq \frac{1}{k}\left(F(t + k) - F(t)\right) \tag{2.22}$$

whenever $a \leq s < s + h \leq t < t + k < b$. It follows from (2.21) and (2.22) that $f : [a, b) \to X_+$ is increasing.

Let $G(t) := \int_a^t f(s)\, ds$. We show that $F = G$. Let $a < t < b$. Let $a \leq t_0 < t_1 < \ldots < t_n = t$ be a partition of $[a, t]$. Setting $h_i := t_i - t_{i-1}$, we obtain from

(2.22) that

$$\begin{aligned}\sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) &\le \sum_{i=1}^{n} \frac{1}{h_i}\left(F(t_{i-1}+h_i) - F(t_{i-1})\right) h_i \\ &= \sum_{i=1}^{n} \left(F(t_i) - F(t_{i-1})\right) \\ &= F(t) - F(a) = F(t).\end{aligned}$$

It follows from the definition of the Riemann integral that $G(t) \le F(t)$. Also by (2.22),

$$\begin{aligned}\sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) &\ge \sum_{i=1}^{n} \frac{F(t_i) - F(t_{i-1})}{t_i - t_{i-1}}(t_i - t_{i-1}) \\ &= F(t).\end{aligned}$$

Hence $G(t) \ge F(t)$. □

Next, we prove a converse version of Proposition 2.7.9.

Proposition 2.7.10. *Assume that for every convex function $F : \mathbb{R}_+ \to X_+$ with $F(0) = 0$ and $\omega(F) = 0$ there exists an increasing function $f : \mathbb{R}_+ \to X_+$ such that $F(t) = \int_0^t f(s)\, ds$ $(t \ge 0)$. Then X has the interpolation property.*

Proof. Let $x_n \le x_{n+1} \le y_{n+1} \le y_n$ $(n \in \mathbb{N})$. We can assume that $x_1 \ge 0$ (replacing x_n by $x_n - x_1$ and y_n by $y_n - x_1$ otherwise). Define $f : \mathbb{R}_+ \to X$ by

$$f(t) := \begin{cases} x_n & \text{if } t \in [\frac{n-1}{n}, \frac{n}{n+1});\ n \ge 1, \\ y_n & \text{if } t \in [\frac{n+1}{n}, \frac{n}{n-1});\ n \ge 2, \\ y_1 & \text{if } t \in [2, \infty), \\ 0 & \text{if } t = 1. \end{cases}$$

Then $f \in L^1_{loc}(\mathbb{R}_+, X)$. Let $F(t) := \int_0^t f(s)\, ds$. Then $F : \mathbb{R}_+ \to X_+$ is convex and $F(0) = 0$. By assumption, there exists an increasing function $g : \mathbb{R}_+ \to X$ such that $F(t) = \int_0^t g(s)\, ds$ $(t \ge 0)$. Then

$$\frac{F(t-h) - F(t)}{-h} \le g(t) \le \frac{F(t+h) - F(t)}{h}$$

for all $t > 0$ and $h > 0$ small enough. It follows that $g(t) = F'(t)$ whenever F is differentiable at t. Consequently, $g(t) = x_n$ if $t \in (\frac{n-1}{n}, \frac{n}{n+1})$ and $g(t) = y_n$ if $t \in (\frac{n+1}{n}, \frac{n}{n-1})$. Hence, $x_n \le g(1) \le y_n$. Thus, $z := g(1)$ interpolates between the two sequences. □

For completeness, we also give the usual representation of convex functions as a corollary of Proposition 2.7.9.

Corollary 2.7.11. *Assume that X has the interpolation property. Let $F : (a,b) \to X$ be convex, and let $c \in (a,b)$. Then there exist $x \in X$ and an increasing function $f : (a,b) \to X$ such that*

$$F(t) = F(c) + (t-c)x + \int_c^t f(s)\,ds$$

for all $t \in (a,b)$.

Proof. We may assume that $c = 0$. It follows from convexity that

$$\frac{1}{t}\left(F(0) - F(-t)\right) \le \frac{1}{s}\left(F(s) - F(0)\right)$$

whenever $0 < s < b,\ 0 < t < -a$. Moreover, the left-hand difference quotient is decreasing in t, and the right-hand one is increasing in s. By the interpolation property, there exists $x \in X$ such that

$$\frac{1}{t}\left(F(0) - F(-t)\right) \le x \le \frac{1}{s}\left(F(s) - F(0)\right)$$

for all $0 < t < -a,\ 0 < s < b$. In particular, the function

$$G(t) := F(t) - F(0) - tx \quad (t \in (a,b))$$

is positive, convex and satisfies $G(0) = 0$.

By Proposition 2.7.9, there exist increasing functions $f_1 : [0,b) \to X_+$ and $f_2 : [0,-a) \to X_+$ such that

$$\begin{aligned} G(t) &= \int_0^t f_1(s)\,ds \quad \text{for } t \in [0,b) \quad \text{and} \\ G(-t) &= \int_0^t f_2(s)\,ds \quad \text{for } t \in [0,-a). \end{aligned}$$

We can assume that $f_1(0) = f_2(0) = 0$. Let $f(t) := f_1(t)$ for $t \in [0,b)$ and $f(t) := -f_2(-t)$ for $t \in (a,0)$. Then f is increasing and $G(t) = \int_0^t f(s)\,ds$ for all $t \in (a,b)$. □

Now we will study completely monotonic functions. We need the following formulas (2.23) and (2.24) (the latter is merely needed for $n = 1$ and $n = 2$). In the remainder of this section we shall sometimes use loose notation such as $\frac{r(\lambda)}{\lambda}$ to denote the function $\lambda \mapsto \frac{r(\lambda)}{\lambda}$, and $\left(\frac{r(\lambda)}{\lambda}\right)'$ and $\left(\frac{r(\lambda)}{\lambda}\right)^{(n)}$ to denote its derivatives of orders 1 and n.

Lemma 2.7.12. *Let $r \in C^\infty((0,\infty), X)$. Then*

$$\frac{(-1)^n}{n!}\lambda^{n+1}\left(\frac{r(\lambda)}{\lambda}\right)^{(n)} = \sum_{m=0}^{n} \frac{(-1)^m}{m!}\lambda^m r^{(m)}(\lambda) \tag{2.23}$$

and

$$\left(\lambda^{k+n}\left(\frac{r(\lambda)}{\lambda^n}\right)^{(k)}\right)^{(n)} = \lambda^k r^{(k+n)}(\lambda) \tag{2.24}$$

for all $\lambda > 0$, $k, n \in \mathbb{N}_0$. *In particular, if* r *is completely monotonic, then* $\lambda \mapsto r(\lambda)/\lambda$ *is also completely monotonic.*

Proof. The first formula (2.23) is immediate from Leibniz's rule. It follows that if r is completely monotonic, then $\lambda \mapsto r(\lambda)/\lambda$ is also completely monotonic.

We show by induction over n that (2.24) holds for all $k \in \mathbb{N}_0$. It is obvious for $n = 0$. Moreover,

$$\begin{aligned}\lambda^k r^{(k+1)}(\lambda) = \lambda^k\left(\lambda\frac{r(\lambda)}{\lambda}\right)^{(k+1)} &= \lambda^k\left\{\lambda\left(\frac{r(\lambda)}{\lambda}\right)^{(k+1)} + (k+1)\left(\frac{r(\lambda)}{\lambda}\right)^{(k)}\right\}\\ &= \left(\lambda^{k+1}\left(\frac{r(\lambda)}{\lambda}\right)^{(k)}\right)'\end{aligned}$$

for $\lambda > 0$. This shows that (2.24) holds for $n = 1$.

Now assume that (2.24) holds for a fixed $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then, applying (2.24) to r' yields

$$\lambda^k r^{(k+n+1)}(\lambda) = \left(\lambda^{k+n}\left(\frac{r'(\lambda)}{\lambda^n}\right)^{(k)}\right)^{(n)} \tag{2.25}$$

for $\lambda > 0$. Observe that

$$\begin{aligned}&\left(\lambda^{k+n+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right)'\\ &= \left(\lambda^n \cdot \lambda^{k+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right)'\\ &= n\lambda^{n-1}\left(\lambda^{k+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right) + \lambda^n\left(\lambda^{k+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right)'\\ &= n\lambda^{n-1}\left(\lambda^{k+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right) + \lambda^n\lambda^k(r(\lambda)/\lambda^n)^{(k+1)},\end{aligned}$$

by applying (2.24) for $n = 1$ to the function $r(\lambda)/\lambda^n$ instead of r. Hence,

$$\begin{aligned}&\left(\lambda^{k+n+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right)'\\ &= n\lambda^{n+k}\left(r(\lambda)/\lambda^{n+1}\right)^{(k)} + \lambda^{n+k}\left(r'(\lambda)/\lambda^n - nr(\lambda)/\lambda^{n+1}\right)^{(k)}\\ &= \lambda^{n+k}\left(r'(\lambda)/\lambda^n\right)^{(k)}\end{aligned}$$

for $\lambda > 0$. It follows from (2.25) that

$$\left(\lambda^{k+n+1}(r(\lambda)/\lambda^{n+1})^{(k)}\right)^{(n+1)} = \left(\lambda^{n+k}(r'(\lambda)/\lambda^n)^{(k)}\right)^{(n)} = \lambda^k r(\lambda)^{(k+n+1)}.$$

Thus, (2.24) holds when n is replaced by $n + 1$. □

Proposition 2.7.13. *Let* $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ *and let*

$$r(\lambda) = \lambda \widehat{dF}(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\, dF(t) \quad (\lambda > 0).$$

Then r *is completely monotonic if and only if* F *is convex and* $F(t) \geq 0$ $(t \geq 0)$.

Proof. Assume that r is completely monotonic. Note that $\frac{r(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda t}\, dF(t)$. Thus, by the Post-Widder formula (Theorem 2.3.1), for $t > 0$ we have $F(t) = \lim_{k\to\infty} F_k(t)$, where

$$F_k(t) := G_k(k/t), \quad G_k(\lambda) := \frac{(-1)^k}{k!} \lambda^{k+1} \left(r(\lambda)/\lambda^2\right)^{(k)}.$$

By Lemma 2.7.12, $\lambda \mapsto r(\lambda)/\lambda^2$ is completely monotonic, and it follows that $F_k(t) \geq 0$. We show that F_k is convex; i.e., that $F_k''(t) = -\left(kt^{-2}G_k'(k/t)\right)' \geq 0$.

Let $H(\lambda) := -\lambda^2 k G_k'(k\lambda)$. Then $F_k''(t) = \frac{d}{dt}H(1/t) = -t^{-2}H'(1/t)$. Thus it suffices to show that $H'(\lambda) \leq 0$ or equivalently $2\lambda k G_k'(k\lambda) + \lambda^2 k^2 G_k''(k\lambda) \geq 0$ for $\lambda > 0$. Letting $\mu := k\lambda$ we have to show that

$$(\mu G_k(\mu))'' = 2G_k'(\mu) + \mu G_k''(\mu) \geq 0 \quad (\mu > 0).$$

This is true since (2.24) for $n = 2$ gives

$$(\mu G_k(\mu))'' = \frac{(-1)^k}{k!} \left(\mu^{k+2}(r(\mu)/\mu^2)^{(k)}\right)'' = \frac{(-1)^k}{k!} \mu^k r^{(k+2)}(\mu) \geq 0 \quad (\mu > 0).$$

This proves one implication.

Conversely, suppose that F is convex and $F(t) \geq 0$ for all $t \geq 0$. Let $x^* \in X_+^*$. Then $x^* \circ F$ is convex, positive and Lipschitz continuous. There is an increasing, bounded function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(t) = \frac{d}{dt}\langle F(t), x^*\rangle$ a.e., and $\langle F(t), x^*\rangle = \int_0^t g(s)\, ds$ for all $t \geq 0$ (see Proposition 2.7.9). We may assume that $g(0) = 0$. By Proposition 1.10.1 and (1.22),

$$\langle r(\lambda), x^*\rangle = \lambda \langle \widehat{dF}(\lambda), x^*\rangle = \lambda \hat{g}(\lambda) = \widehat{dg}(\lambda) \quad (\lambda > 0).$$

Hence, $x^* \circ r$ is completely monotonic for all $x^* \in X_+^*$ and therefore r is completely monotonic. ☐

Next we prove a representation theorem for completely monotonic functions defined on $\mathbb{R}_+$ (and not merely $(0, \infty)$).

Proposition 2.7.14. *Let* $r \in C^\infty(\mathbb{R}_+, X)$ *such that* $(-1)^n r^{(n)}(\lambda) \geq 0$ $(\lambda \geq 0)$. *Then there exists a convex function* $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$ *such that* $F(t) \geq 0$ $(t \geq 0)$ *and*

$$r(\lambda) = \lambda \widehat{dF}(\lambda) \quad (\lambda > 0). \tag{2.26}$$

Proof. It follows from (2.23) that for $k \in \mathbb{N}$ and $\lambda > 0$,

$$p_k(\lambda) := \frac{(-1)^k}{k!}\lambda^{k+1}\left(\frac{r(\lambda)}{\lambda}\right)^{(k)} = \sum_{m=0}^{k}\frac{(-1)^m}{m!}\lambda^m r^{(m)}(\lambda) \geq 0.$$

Moreover, $\lim_{\lambda\downarrow 0} p_k(\lambda) = r(0)$. It follows from (2.24) for $n = 1$ that

$$p_k'(\lambda) = \frac{(-1)^k}{k!}\lambda^k r^{(k+1)}(\lambda) \leq 0 \quad (\lambda > 0).$$

Thus $0 \leq p_k(\lambda) \leq r(0)$ for all $\lambda > 0$. Since the cone is normal, this implies that the function $\frac{r(\lambda)}{\lambda}$ is in $C_W^\infty((0,\infty),X)$. By Theorem 2.2.1, there exists $F \in \mathrm{Lip}_0(\mathbb{R}_+,X)$ such that $\frac{r(\lambda)}{\lambda} = \widehat{dF}(\lambda)$ $(\lambda > 0)$. It follows from Proposition 2.7.13 that F is positive and convex. ☐

Theorem 2.7.15. *A function $r : (0,\infty) \to X$ is completely monotonic if and only if there exists a convex function $F : \mathbb{R}_+ \to X_+$ satisfying $F(0) = 0$ and $\omega(F) \leq 0$ such that*

$$r(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\, dF(t) \quad (\lambda > 0). \tag{2.27}$$

In that case, F is uniquely determined by r.

Proof. a) Assume that r is of the form (2.27). Let $x^* \in X_+^*$. Then there exists an increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$ and

$$\langle F(t), x^*\rangle = \int_0^t f(s)\, ds \quad (t \geq 0).$$

Thus

$$\langle r(\lambda), x^*\rangle = \int_0^\infty e^{-\lambda t}\, df(t) \quad (\lambda > 0).$$

Hence, $\langle r(\cdot), x^*\rangle$ is completely monotonic and

$$\langle (-1)^n r^{(n)}(\lambda), x^*\rangle = (-1)^n \left(\frac{d}{d\lambda}\right)^n \langle r(\lambda), x^*\rangle \geq 0.$$

Since $x^* \in X_+^*$ is arbitrary, it follows that r is completely monotonic.

b) Conversely, let r be completely monotonic. By Proposition 2.7.14, there exists a convex function $G \in \mathrm{Lip}_0(\mathbb{R}_+,X)$ such that $G(t) \geq 0$ $(t \geq 0)$ and $r(\lambda+1) = \lambda\int_0^\infty e^{-\lambda t}\, dG(t)$ $(\lambda > 0)$. Let

$$F(t) := \int_0^t (1-(t-s))e^s\, dG(s).$$

Then F is positive and convex. In fact, let $x^* \in X_+^*$. Then there exists an increasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\langle G(t), x^*\rangle = \int_0^t g(s)\,ds$ and $g(0) = 0$. By Proposition 1.9.10, Fubini's Theorem and (1.20),

$$\begin{aligned}\langle F(t), x^*\rangle &= \int_0^t e^s g(s)\,ds - \int_0^t (t-s)e^s g(s)\,ds \\ &= \int_0^t \left(e^s g(s) - \int_0^s e^r g(r)\,dr \right) ds \\ &= \int_0^t \int_0^s e^r\,dg(r)\,ds \quad (t \geq 0).\end{aligned}$$

Thus $x^* \circ F$ is positive and convex for all $x^* \in X_+^*$, so F is positive and convex. By Proposition 1.10.1 and (1.22),

$$\langle r(\lambda+1), x^*\rangle = \lambda \hat{g}(\lambda) = \int_0^\infty e^{-\lambda t}\,dg(t)$$

for $\lambda > 0$. By Proposition 1.10.3, for $\lambda > 1$,

$$\langle r(\lambda), x^*\rangle = \int_0^\infty e^{-\lambda t} e^t\,dg(t) = \int_0^\infty e^{-\lambda t}\,df(t),$$

where

$$f(t) := \int_0^t e^s\,dg(s) = e^t g(t) - \int_0^t e^s g(s)\,ds,$$

by (1.20). Since $\langle F(t), x^*\rangle = \int_0^t f(s)\,ds$, it follows that $r(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\,dF(t)$ for $\lambda > 1$. By Theorem 2.7.1, $\mathrm{abs}(dF)$ is a singularity of $\widehat{dF}$. Moreover, applying Proposition 2.7.14 to $r(\cdot + \delta)$ shows that r has a holomorphic extension to $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \delta\}$ for all $\delta > 0$, and hence to $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. It follows from the uniqueness of holomorphic extensions that $\mathrm{abs}(dF) \leq 0$ and $\widehat{dF}(\lambda) = r(\lambda)$ for $\lambda > 0$. By Theorem 1.10.5, $\omega(F) \leq 0$ (actually, $\omega(F) = 0$ unless $r \equiv 0$). Finally, uniqueness of F follows from the Post-Widder formula (Theorem 2.3.1). □

Theorem 2.7.16. *Assume that X has the interpolation property. Let $r : (0,\infty) \to X$ be completely monotonic. Then there exists an increasing function $f : \mathbb{R}_+ \to X_+$ such that $f(0) = 0$, $\omega(f) \leq 0$ and*

$$r(\lambda) = \int_0^\infty e^{-\lambda t}\,df(t) \quad (\lambda > 0).$$

Proof. By Theorem 2.7.15, there exists a convex function $F : \mathbb{R}_+ \to X_+$ satisfying $F(0) = 0$ and $\omega(F) \leq 0$ such that $r(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\,dF(t)$ for all $\lambda > 0$. By Proposition 2.7.9, there exists an increasing function $f : \mathbb{R}_+ \to X_+$ such that

$F(t) = \int_0^t f(s)\,ds$ $(t \geq 0)$. We can assume that $f(0) = 0$. Let $\omega > 0$. There exists $M \geq 0$ such that $\|F(t)\| \leq Me^{\omega t}$. Since f is increasing we have

$$\frac{t}{2} f(t/2) \leq \int_{t/2}^{t} f(s)\,ds \leq F(t).$$

It follows that $\omega(f) \leq 0$ (actually, $\omega(f) = 0$ unless $r \equiv 0$). By Proposition 1.10.2,

$$\int_0^\infty e^{-\lambda t}\,df(t) = \lambda \int_0^\infty e^{-\lambda t}\,dF(t) = r(\lambda) \quad (\lambda > 0). \qquad \square$$

Now we can prove Theorem 2.7.7.

Proof of Theorem 2.7.7. One direction is given by Theorem 2.7.16. In order to prove the other, assume that Bernstein's theorem holds in X. We show that X has the interpolation property. Let $F : \mathbb{R}_+ \to X_+$ be convex such that $F(0) = 0$ and $\omega(F) = 0$. By Proposition 2.7.10, it suffices to show that $F(t) = \int_0^t f(s)\,ds$ $(t \geq 0)$ for some increasing function $f : \mathbb{R}_+ \to X$. By Proposition 2.7.13, $r(\lambda) := \lambda \int_0^\infty e^{-\lambda t}\,dF(t)$ defines a completely monotonic function on $(0, \infty)$. By assumption, there exists an increasing function $f : \mathbb{R}_+ \to X$ such that

$$r(\lambda) = \int_0^\infty e^{-\lambda t}\,df(t).$$

We may assume that $f(0) = 0$. Let $H(t) := \int_0^t f(s)\,ds$. Using Proposition 1.10.2 and (1.22), $\lambda^2 \widehat{H}(\lambda) = \widehat{df}(\lambda) = r(\lambda) = \lambda^2 \widehat{F}(\lambda)$ for all $\lambda > 0$. It follows from the uniqueness theorem that $H(t) = F(t)$ for all $t \geq 0$. $\square$

If $r : (0, \infty) \to X$ is completely monotonic, there may be many increasing functions $f : \mathbb{R}_+ \to X_+$ such that $r = \widehat{df}$. However, if X has order continuous norm, then we may pick out a normalized version of f.

Let $f : \mathbb{R}_+ \to X$ be increasing and assume that X has order continuous norm. For $t \geq 0$ we define $f(t+) = \lim_{s \downarrow t} f(s)$, and for $t > 0$ we let $f(t-) = \lim_{s \uparrow t} f(s)$. We say that f *has a jump at* $t > 0$ if $f(t+) \neq f(t-)$.

Lemma 2.7.17. *Assume that X has order continuous norm and that $f : \mathbb{R}_+ \to X$ is increasing. Then the number of jumps of f is countable.*

Proof. Let $\tau > 0$ and $J := \{t \in (0, \tau) : \ f(t+) \neq f(t-)\}$. Let $\varepsilon > 0$ and $J_\varepsilon := \{t \in J : \ \|f(t+) - f(t-)\| \geq \varepsilon\}$. We claim that J_ε is finite. Otherwise there exist $t_n \in J_\varepsilon$ $(n \in \mathbb{N})$, $t_n \neq t_m$ for $n \neq m$. Let $x_n = f(t_n+) - f(t_n-)$. Then $\sum_{n=1}^m x_n \leq f(\tau) - f(0)$ for all $m \in \mathbb{N}$. Since X has order continuous norm, the sum $\sum_{n=1}^\infty x_n$ converges. Hence, $\|x_n\| \to 0$ as $n \to \infty$. This is a contradiction. Since $J = \bigcup_{n \in \mathbb{N}} J_{1/n}$, it follows that J is countable. $\square$

We continue to assume that X has order continuous norm. Let $f : \mathbb{R}_+ \to X$ be increasing. We define the *normalization* $f^* : \mathbb{R}_+ \to X$ of f by

$$f^*(t) = \begin{cases} f(0+) & \text{if } t = 0, \\ \frac{1}{2}\left(f(t+) + f(t-)\right) & \text{if } t > 0. \end{cases}$$

The function f is called *normalized* if $f = f^*$.

It follows from the definition of the Riemann-Stieltjes integral that

$$\int_0^t g(s)\, df(s) = \int_0^t g(s)\, df^*(s)$$

for every $t > 0$ and every continuous function $g : [0, t] \to \mathbb{C}$. In fact, one may take a sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ with intermediate points which avoid the jumps of f). Then $S(g, f, \pi_n) = S(g, f^*, \pi_n)$ for all $n \in \mathbb{N}$, and so

$$\int_0^t g(s)\, df(s) = \lim_{n\to\infty} S(g, f, \pi_n) = \lim_{n\to\infty} S(g, f^*, \pi_n) = \int_0^t g(s)\, df^*(s).$$

In conclusion, we obtain the following result.

Theorem 2.7.18 (Bernstein's theorem). *Assume that X has order continuous norm. Let $r : (0, \infty) \to X$ be completely monotonic. Then there exists a unique normalized increasing function $f : \mathbb{R}_+ \to X$ such that $f(0) = 0$, $\omega(f) \le 0$ and*

$$r(\lambda) = \int_0^\infty e^{-\lambda t}\, df(t) \quad (\lambda > 0).$$

Proof. Since X has the interpolation property, existence follows from Theorem 2.7.16. For uniqueness, suppose that $r(\lambda) = \int_0^\infty e^{-\lambda t}\, df(t)$ $(\lambda > 0)$. By Proposition 1.10.2, $r(\lambda) = \lambda \int_0^\infty e^{-\lambda t}\, dF(t)$ $(\lambda > 0)$ where $F(t) := \int_0^t f(s)\, ds$. It follows from Theorem 2.7.15 that F is uniquely determined by r. Since

$$F'(t+) := \lim_{h\downarrow 0} \frac{1}{h}\left(F(t+h) - F(t)\right) = f(t+)$$

if $t \ge 0$, and

$$F'(t-) := \lim_{h\downarrow 0} \frac{1}{h}\left(F(t) - F(t-h)\right) = f(t-)$$

if $t > 0$, the normalized function f is also unique. □

2.8 Notes

Section 2.1

Representation of operators from a space of the form $L^1(\Omega, \mu)$ into a Banach space

X by vector measures is a classical subject (see [DU77, Section III.1]). In view of the applications to Cauchy problems, Stieltjes integrals seem more appropriate than vector measures in our context. In the context of Laplace transform theory, the Riesz-Stieltjes Representation Theorem 2.1.1 appeared in a paper of Hennig and Neubrander [HN93] (see also [Neu94] and [BN94]). For a discussion of the representation of bounded linear operators in $\mathcal{L}(L^p(\mathbb{R}_+), X)$ as functions of bounded p'-variation $(1/p + 1/p' = 1, p' > 1)$, see the work of Weis [Wei93] and Vieten [Vie95].

Section 2.2
For real-valued functions, Theorem 2.2.1 was proved by Widder in 1936 [Wid36] (see also [Wid41]). In trying to extend scalar-valued Laplace transform theory to vector-valued functions, Hille [Hil48] remarks on several occasions that Widder's theorem can be lifted to infinite dimensions if the space is reflexive, but not in general (see [Hil48, p.213] or [Miy56]). In fact, it was shown by Zaidman [Zai60] (see also [Are87b] or Theorem 2.2.3) that Widder's theorem extends to a Banach space X if and only if X has the Radon-Nikodym property (for example, if X is reflexive). In 1965, Berens and Butzer [BB65] gave necessary and sufficient complex conditions for the Laplace-Stieltjes representability of functions in reflexive and uniformly convex Banach spaces. However, these results were of limited applicability. In general, important classes of Banach spaces that appear in studying evolution equations do not possess the Radon-Nikodym property. As a consequence, in the 1960s and 1970s Laplace transform methods were applied mainly to special vector-valued functions, like resolvents and semigroups, which have nice additional algebraic properties. In the theory of C_0-semigroups the link between the generator A and the semigroup T is given via the Laplace transform

$$(\lambda - A)^{-1}x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad (x \in X).$$

The crucial algebraic property which made it possible to extend classical Laplace transform results to this abstract setting is the algebraic semigroup law $T(t+s) = T(t)T(s)$, $(t, s \geq 0)$. Hille and Phillips comment in the foreword to [HP57] that *".... in keeping with the spirit of the times the algebraic tools now play a major role...."* and that *".... the Laplace-Stieltjes transform methods..... have not been replaced but rather supplemented by the new tools."* The major disadvantage of the "algebraic approach" to linear evolution equations becomes obvious if one compares the mathematical theories associated with them (for example, semigroup theories, cosine families, the theory of integro-differential equations, etc.). It is striking how similar the results and techniques are. Still, without a Laplace transform theory for functions with values in arbitrary Banach spaces, every type of linear evolution equation required its own theory because the algebraic properties of the operator families changed from one case to another. In the late 1970s, in search of a general analytic principle behind all these theories, the study of Laplace transforms of functions with values in arbitrary Banach spaces was revitalized by Sova (see [Sov77] up to [Sov82]). An important result for Laplace transforms in Banach spaces is Theorem 2.6.1, proved by Sova in 1979 [Sov79b], [Sov79c]. This analytic representation theorem is behind every generation result for analytic solution families of linear evolution equations.

The Real Representation Theorem 2.2.1 shows that the statement of Widder's Theorem extends to arbitrary Banach spaces if the Laplace transform is replaced by the Laplace-Stieltjes transform. It is due to [Are87b] where it was deduced from the scalar result by Widder [Wid41] by duality arguments. The proof of Theorem 2.2.1 given here is a modification of Widder's original proof given in [Wid41]; see [HN93]. Further extensions of these results are given in [Bob97a], [Bob97b], [Kis00], [Bob01] and [Cho02].

The characterization of the range of the Laplace-Stieltjes transform acting on $\mathrm{Lip}_0(\mathbb{R}_+, X)$ given in Theorem 2.2.1 is based on the Post-Widder inversion formula in Theorem 1.7.7. Corresponding to other inversion formulas, equivalent descriptions can be formulated. Employing the complex inversion formula (see [Sov80b], [BN94]), or the Phragmén-Doetsch inversion (see [PC98]), one can prove that the following growth and regularity conditions are equivalent.

Theorem 2.8.1. *Let $r : (0, \infty) \to X$ be continuous. The following are equivalent:*

(i) $r \in C^\infty((0, \infty), X)$ *and*

$$\sup_{\substack{\lambda>0\\ k\in\mathbb{N}_0}} \left\| \frac{\lambda^{k+1}}{k!} r^{(k)}(\lambda) \right\| < \infty.$$

(ii) $\lim_{\lambda\to\infty} r(\lambda) = 0$ *and r has an extension to a holomorphic function* $r : \{\mathrm{Re}\,\lambda > 0\} \to X$ *such that, for all* $\gamma > 0$, $\sup_{\mathrm{Re}\,\lambda>\gamma} \|r(\lambda)\| < \infty$ *and*

$$\sup_{\substack{s>0\\ k\in\mathbb{N}_0}} \left\| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{r(\gamma + it)}{(1 - ist)^{k+2}}\, dt \right\| < \infty.$$

(iii) $\sup_{\lambda>0} \|\lambda r(\lambda)\| < \infty$ *and*

$$\sup_{\substack{\lambda>0\\ k\in\mathbb{N}}} \left\| \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} e^{jk} \lambda r(j\lambda) \right\| < \infty.$$

For a discussion of the L^p-conditions

$$\int_0^\infty \left\| \left(\frac{k}{t}\right)^{k+1} \frac{1}{k!} r^{(k)}\left(\frac{k}{t}\right) \right\|^p dt \le M \quad \text{for all } k \ge 0,$$

and their connection to the representability of r as the Laplace transform of a function of bounded p-variation ($p > 1$), see [Wid41, Chapter VII], [Lev69], [Sov81a], [Wei93], and [Vie95]. It is shown in [KMV03] that a function $r \in C^\infty((0, \infty), X)$ is the finite Laplace-Stieltjes transform $r(\lambda) = \int_0^\tau e^{-\lambda t}\, dF(t)$ of a Lipschitz continuous function $F : [0, \tau] \to X$ with $\|F(t) - F(s)\| \le M|t - s|$ for all $0 \le t, s \le \tau$ if and only if

$$\sup_{k\in\mathbb{N}_0} \sup_{\lambda>k/\tau} \left\| \frac{\lambda^{k+1}}{k!} r^{(k)}(\lambda) \right\| \le M$$

and

$$\sup_{k\in\mathbb{N}} \sup_{\lambda\in(0,k/\tau)} \left\| \tau^{-k} e^{\lambda\tau} r^{(k)}(\lambda) \right\| < \infty.$$

Section 2.3

Theorem 2.3.2 goes back to Phragmén's proof of the Uniqueness Theorem 1.7.3 (see [Phr04]), and to Doetsch [Doe37] who recognized the usefulness of the formula as an inversion procedure (see also [Doe50, Volume I, Section 8.1]). The Phragmén-Doetsch inversion formula shows that a Laplace transformable function f is determined by the values of $\hat{f}(\lambda_n)$, where $\lambda_n = n \geq n_0$. An extension of the Phragmén-Doetsch inversion to arbitrary Müntz sequences $(\lambda_n) \subset \mathbb{R}_+$ (i.e., $\lambda_{n+1} - \lambda_n \geq 1$ and $\sum_{n=1}^{\infty} \lambda_n^{-1} = \infty$), has been obtained by Bäumer [Bäu03] (see also [BLN99]). There does not seem to be any inversion formula that holds for arbitrary uniqueness sequences (see Theorem 1.11.1). Corollary 2.3.3 is taken from [BN96] and is one of the key ingredients in the theory of asymptotic Laplace transforms (see [LN99], [LN01]). Whereas the complex inversion formula in Theorem 2.3.4 (the proof given here is from [HN93]) is in general affected by exponentially decaying perturbations of the Laplace transform, the following modification, due to Lyubich [Lyu66], gives a complex inversion formula which holds locally even if the transform undergoes such perturbations.

Theorem 2.8.2. *Let* $\tau > 0$, $\omega > 0$, $F \in \mathrm{Lip}_0(\mathbb{R}_+, X)$, *and* $q(\lambda) = \int_0^\infty e^{-\lambda t}\, dF(t) + a(\lambda)$ $(\lambda > 0)$, *where* $a \in L^1_{loc}(\mathbb{R}_+, X)$ *is a function with* $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \|a(\lambda)\| \leq -\tau$. *Then*

$$H(\mu) := \frac{1}{2\pi i} \int_\omega^\infty e^{\mu t} \frac{q(t)}{t}\, dt$$

is well defined for $\mathrm{Re}\,\mu < 0$, *has a holomorphic continuation to the sliced half-plane* $\{\mu : \mathrm{Re}\,\mu < \tau\} \setminus [0, \tau)$, *and*

$$F(t) = \lim_{\varepsilon\to 0} (H(t + i\varepsilon) - H(t - i\varepsilon)) \qquad \textit{for all } t \in [0, \tau).$$

Haase [Haa08] has given a different approach to Theorem 2.3.4 and Lemma 2.3.5.

Section 2.4

With the exception of Proposition 2.4.3 which is due to Doetsch (see [Doe50, Volume I, Section 14.3]) and Corollary 2.4.4, the results are straightforward reformulations of the main theorems of the sections 2.1–2.3. Using a Phragmén-Doetsch type inversion formula along sequences $(\lambda_n) \subset \mathbb{R}_+$ with $\lambda_{n+1} - \lambda_n \geq 1$ and $\sum_{n=1}^{\infty} \lambda_n^{-1} = \infty$ (Müntz sequences), one can strengthen the statement of Proposition 2.4.3 as follows (see [Bäu03]).

Theorem 2.8.3. *Let* $0 \leq \tau$ *and let* $f \in L^1_{loc}(\mathbb{R}_+, X)$ *with* $\mathrm{abs}(f) < \infty$. *Then the following are equivalent:*

(i) $f(t) = 0$ *almost everywhere on* $[0, \tau]$ *and* $\tau \in \mathrm{supp}(f)$.

(ii) *Every Müntz sequence* (β_n) *satisfies* $\limsup_{n\to\infty} \frac{1}{\beta_n} \log \|\hat{f}(\beta_n)\| = -\tau$.

(iii) *For every Müntz sequence* (β_n) *there exists a Müntz subsequence* (β_{n_k}) *such that*

$$\lim_{k\to\infty} \frac{1}{\beta_{n_k}} \log \|\hat{f}(\beta_{n_k})\| = -\tau.$$

(iv) *There exists a Müntz sequence* (β_n) *with* $\limsup_{n\to\infty} \frac{1}{\beta_n} \log \|\hat{f}(\beta_n)\| = -\tau$.

(v) $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \|\hat{f}(\lambda)\| = -\tau$.

As a consequence of these equivalences one obtains the following short proof of Titchmarsh's theorem (see [Bäu03], [BLN99] or [MB87, Section VI.7]).

Corollary 2.8.4 (Titchmarsh's Theorem). *Let* $k \in L^1[0, \tau]$ *with* $0 \in \mathrm{supp}(k)$ *and* $f \in L^1([0, \tau], X)$. *If* $k \star f = 0$ *on* $[0, \tau]$, *then* $f = 0$.

Proof. We extend k and f by zero to $\mathbb{R}_+$. Then, by Proposition 2.4.3 and Corollary 2.4.4, $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log|\hat{k}(\lambda)| = 0$ and $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log\|\widehat{k\star f}(\lambda)\| \le -T$. By taking subsequences, it follows from the theorem above that there exists a Müntz sequence (β_n) such that $\lim_{n\to\infty} \frac{1}{\beta_n} \log|\hat{k}(\beta_n)| = 0$ and

$$\begin{aligned} -\tau \ge \lim_{n\to\infty} \frac{1}{\beta_n} \log\|\widehat{k\star f}(\beta_n)\| &= \lim_{n\to\infty} \frac{1}{\beta_n} \log|\hat{k}(\beta_n)| + \lim_{n\to\infty} \frac{1}{\beta_n} \log\|\hat{f}(\beta_n)\| \\ &= \lim_{n\to\infty} \frac{1}{\beta_n} \log\|\hat{f}(\beta_n)\|. \end{aligned}$$

Thus, $f = 0$ on $[0,\tau]$. ☐

A function $k \in L^1_{loc}(\mathbb{R}_+)$ with $\operatorname{abs}(k) < \infty$ is a *regularizing function* if

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log|\hat{k}(\lambda)| = 0,$$

or, equivalently, if $0 \in \operatorname{supp}(k)$ (by Corollary 2.4.4). By the Titchmarsh-Foiaş theorem (see [BLN99]), the condition $0 \in \operatorname{supp}(k)$ is necessary and sufficient for the convolution operator $\mathcal{K} : f \to k * f$, $(k * f)(t) := \int_0^t k(t-s)f(s)\,ds$ to be an injective operator on $C(\mathbb{R}_+, X)$ with dense range in the Fréchet space $C_*(\mathbb{R}_+, X)$ of all continuous functions $g : \mathbb{R}_+ \to X$ such that $g(0) = 0$, equipped with the seminorms $\|g\|_n := \sup_{t\in[0,n]} \|g(t)\|$. Moreover, $\|f\|_{\mathcal{K},n} := \sup_{t\in[0,n]} \|\mathcal{K}f(t)\|$ defines a family of seminorms on $C(\mathbb{R}_+, X)$ and $\mathcal{K}$ extends to an isomorphism between the Fr echet completion $C^{[k]}(\mathbb{R}_+, X)$ of $C(\mathbb{R}_+, X)$ with respect to that family of seminorms and the Fréchet space $C_*(\mathbb{R}_+, X)$. Typical examples of regularizing functions are

$$k(t) = \frac{t^{b-1}}{\Gamma(b)} \quad \text{with} \quad \hat{k}(\lambda) = \frac{1}{\lambda^b}\ (b > 0), \quad \text{or}$$

$$k_\delta(t) = \frac{1}{2\pi i} \int_{\omega+i\mathbb{R}} e^{t\lambda-\lambda^\delta}\,d\lambda \quad \text{with} \quad \hat{k}_\delta(\lambda) = e^{-\lambda^\delta}\ (0 < \delta < 1).$$

Note that $k_{1/2}(t) = \frac{1}{2\sqrt{\pi}} t^{-3/2} e^{-1/4t}$ (see Lemma 1.6.7).

If k is a regularizing function, then the elements of the Fréchet space $C^{[k]}(\mathbb{R}_+, X)$ are called *k-generalized functions*. A k-generalized function u is said to be Laplace transformable if the continuous function $f := k * u \in C_*(\mathbb{R}_+, X)$ is Laplace transformable and the Laplace transform of u is defined as

$$\hat{u}(\lambda) := \frac{\hat{f}(\lambda)}{\hat{k}(\lambda)}.$$

Let $\mathcal{H} = \{\lambda : \operatorname{Re}\lambda > \omega\}$ and $m : \mathcal{H} \to \mathbb{C}$ be holomorphic. A meromorphic function $q : \mathcal{H} \to X$ is said to have an m-multiplied Laplace representation if there exists $f \in C_*(\mathbb{R}_+, X)$ with $\operatorname{abs}(f) \le \omega$ such that $mq = \hat{f}$ on $\mathcal{H}$. If $m = \hat{k}$ for some regularizing function k, then the meromorphic function q has a Laplace representation $q = \hat{u}$ for $u = \mathcal{K}^{-1} f \in C^{[k]}(\mathbb{R}_+, X)$ (see [Bäu97], [BLN99], and [LN99]).

Section 2.5

Theorem 2.5.1 is a standard result of Laplace transform theory. Corollary 2.5.2 is due to

Prüss [Prü93], the proof given here is from [BN94]. Corollary 2.5.4 is a special case of results in [DVW02] (see also [DHW97]).

Theorem 2.5.1 can be interpreted in terms of k-generalized functions and Laplace transforms (see the Notes of Section 2.4; we use the same notation here). Let $q : \mathcal{H} \to X$ be holomorphic with $\sup_{\lambda\in\mathcal{H}} \|\lambda q(\lambda)\| < \infty$. As shown in Theorem 2.5.1, for all $b > 0$ there exists $f \in C_*(\mathbb{R}_+, X)$ such that $q(\lambda) = \lambda^b \hat{f}(\lambda)$ on $\mathcal{H}$. Thus, $q(\lambda) = \hat{u}(\lambda) = \frac{\hat{f}(\lambda)}{\hat{k}(\lambda)}$, where $k(t) = \frac{1}{\Gamma(b)} t^{b-1}$ and $u = \mathcal{K}^{-1} f \in C^{[k]}(\mathbb{R}_+, X)$ coincides with the b-th (distributional) derivative of f. More generally, if q is a meromorphic function on some half-plane $\mathcal{H}$ with values in X for which $\lambda \mapsto \lambda \hat{k}_0(\lambda) q(\lambda)$ is holomorphic on $\mathcal{H}$ and

$$\sup_{\lambda\in\mathcal{H}} \|\lambda \hat{k}_0(\lambda) q(\lambda)\| < \infty$$

for some regularizing function k_0, then it follows from Theorem 2.5.1 that there exists $f \in C_*(\mathbb{R}_+, X)$ such that $\frac{1}{\lambda}\hat{k}_0(\lambda) q(\lambda) = \hat{k}(\lambda) q(\lambda) = \hat{f}(\lambda)$ or $q(\lambda) = \hat{u}(\lambda)$, where $k := 1 * k_0$ and $u \in C^{[k]}(\mathbb{R}_+, X)$ is a generalized function such that $f = k * u$. Notice that if k_i are regularizing functions and $k_1 * k_2 = k_3$, then $C^{[k_1]}(\mathbb{R}_+, X)$ is continuously embedded in $C^{[k_3]}(\mathbb{R}_+, X)$. Thus, a faster growing q will have a less regular u such that $q = \hat{u}$.

Section 2.6

Theorem 2.6.1 is due to Sova [Sov79b] and Theorem 2.6.2 is taken from [Neu89b].

Section 2.7.

In 1893, Stieltjes proved in a letter to Hermite that a bounded continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ is positive if and only if $\hat{f}^{(n)}(\lambda) \geq 0$ for all $n \in \mathbb{N}_0$ and all λ sufficiently large (see [BB05]). Bernstein proved his theorem in 1928 [Ber28].

The characterization of those ordered Banach spaces in which Bernstein's theorem (Theorem 2.7.7) holds is due to Arendt [Are94a].

The interpolation property is of particular interest for spaces of the form $C(K)$, where K is a compact space. Then it can be described in terms of K: the space $C(K)$ has the interpolation property if and only if K is an F-space (i.e., if $A, B \subset K$ are open and disjoint F_σ-sets, then $\overline{A} \cap \overline{B} = \emptyset$). Note that $C(K)$ is σ-order complete if and only if K is quasi-stonean (i.e., if $A \subset K$ is an open F_σ-set, then $\bar{A}$ is open). For example, $K := \beta\mathbb{N} \setminus \mathbb{N}$ is a F-space which is not quasi-stonean (where $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$). Whereas every quasi-stonean space K is totally disconnected (i.e. the connected component of each point x is $\{x\}$), there exist connected compact F-spaces. One reason why these spaces have been studied is that $C(K)$ has the Grothendieck property (see Section 4.3) if K is an F-space. We refer to the article by Seever [See68] for this and further information.

The interpolation property is also equivalent to two other vector-valued versions of classical theorems; namely, Riesz's representation theorem for positive functionals on $C[0,1]$ and Hausdorff's theorem on the moment problem. More precisely, the following is proved in [Are94a].

Theorem 2.8.5. *Let X be an ordered Banach space with normal cone. The following are equivalent:*

(i) *X has the interpolation property.*

(ii) *For every positive $T \in \mathcal{L}(C[0,1], X)$ there exists an increasing function $f : [0,1] \to X$ such that $Tg = \int_0^1 g(t)\, df(t)$ for all $g \in C[0,1]$.*

(iii) *For each completely monotonic sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *there exists an increasing function* $f : [0,1] \to X$ *such that* $x_n = \int_0^1 t^n \, df(t)$ $(n \in \mathbb{N})$.

Here, a sequence $x = (x_n)_{n\in\mathbb{N}}$ is called *completely monotonic* if $(-\Delta)^k x \geq 0$ for all $k \in \mathbb{N}$ where $\Delta : X^{\mathbb{N}} \to X^{\mathbb{N}}$ is given by $\Delta x = (x_{n+1} - x_n)_{n\in\mathbb{N}}$.

Bernstein's theorem in ordered Banach spaces with order continuous norm (Theorem 2.7.18) is proved in [Are87a] with the help of the classical scalar theorem. A first vector-valued version of Bernstein's theorem is due to Bochner [Boc42]. But Bochner considered convergence in order, whereas for our purposes norm convergence of Riemann-Stieltjes sums and improper integrals is essential in order to make the results applicable to operator theory. Here we deduce Bernstein's theorem from the Real Representation Theorem 2.2.1.

One can obtain Widder's theorem (the scalar case of Theorem 2.2.1) as an easy corollary of Bernstein's classical result (see [Wid71, Section 6.8]). However this argument is restricted to the scalar case. On the other hand, it is possible to deduce the vector-valued version of Theorem 2.2.1 from the scalar case by a duality argument (see [Are87b] and the Notes of Section 2.2).

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Vector-valued Laplace Transforms and Cauchy Problems
Second Edition
Arendt, W.; Batty, C.J.K.; Hieber, M.; Neubrander, F.
2011, XII, 540 p., Hardcover
ISBN: 978-3-0348-0086-0
A product of Birkhäuser Basel