10. Pricing Derivatives on a Single Interest-Rate Curve

Increasingly, problems do not rule out practice, but support it. Instead of finding that practice is too difficult, that we have too many problems, we see that the problems themselves are the jewels, and we devote ourselves to be with them in a way we never dreamt of before.

Charlotte Joko Beck, "Nothing Special: Living Zen", 1995, HarperCollins.

In this chapter, we present a sample of financial products we believe to be representative of a large portion of the interest-rate market. We will use different models (mostly the LFM and the G2++ model) for different problems, and try to clarify the advantages of each model. All the discounted payoffs will be calculated at time $t = 0$.

Before starting, we remark upon the possible use of an approximated LIBOR market model (LFM) for pricing some of the products we will consider.

It is possible to freeze part of the drift of the LFM dynamics so as to obtain a geometric Brownian motion. This is what was done for example in Section 6.14 to derive approximated formulas for terminal correlations. In that section, we derived such a dynamics under the T_γ-forward-adjusted measure Q^γ:

$$dF_k(t) = \bar{\mu}_{\gamma,k}(t) F_k(t)\, dt + \sigma_k(t) F_k(t)\, dZ_k(t)\,, \tag{10.1}$$

where

$$\mu_{\gamma,k}(t) := -\sum_{j=k+1}^{\gamma} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(0)}{1+\tau_j F_j(0)}\,, \quad k < \gamma\,,$$
$$\mu_{\gamma,\gamma}(t) := 0\,, \quad k = \gamma\,,$$
$$\mu_{\gamma,k}(t) := \sum_{j=\gamma+1}^{k} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(0)}{1+\tau_j F_j(0)}\,, \quad k > \gamma\,,$$
$$\bar{\mu}_{\gamma,k}(t) := \sigma_k(t)\mu_{\gamma,k}(t).$$

This dynamics gives access, in some cases, to a number of techniques which have been developed for the basic Black and Scholes setup, for example, in equity and FX markets. Moreover, this "freezing-part-of-the-drift" technique can be combined with drift interpolation so as to allow for rates that are not in the fundamental (spanning) family corresponding to the particular LFM being implemented.

We detail this possible "interpolate and freeze (part of the) drift" approach in case of accrual swaps in Section 10.11.1, but the method is rather general and, when used in combination with other possible approximations, can be used for other products.

Finally, even if one keeps on using Monte Carlo evaluation, the frozen-drift approximation leads to a process (geometric Brownian motion) that is much easier to propagate in time and requires no small discretization step in the propagation, allowing instead for "one-shot" simulation also over long periods of time.

In the following, we assume we are given a set of dates $T_\alpha, \ldots, T_i \ldots, T_{\beta+1}$ with associated year fractions $\tau_\alpha, \ldots, \tau_i \ldots, \tau_{\beta+1}$.

10.1 In-Advance Swaps

An in-advance swap is an IRS that resets at dates $T_{\alpha+1}, \ldots, T_\beta$ and pays at the same dates, with unit notional amount and with fixed-leg rate K. More precisely, the discounted payoff of an in-advance swap (of "payer" type) can be expressed via

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i) - K).$$

The value of such a contract is, therefore,

$$\mathbf{IAS} = E\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i) - K)\right],$$

where we omit arguments in the "IAS" notation for brevity.

Before calculating the expectations, it is convenient to make some adjustments. We shall use the following identity (obtained easily via iterated conditioning, as seen in Proposition 2.8.1):

$$E[XD(0,T)] = E\left[\frac{XD(0,S)}{P(T,S)}\right] \quad \text{for all} \quad 0 < T < S, \tag{10.2}$$

where X is a T-measurable random variable.

To value the above contract, notice that

$$E\left\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i)-K)\right\}$$
$$= E\left\{\sum_{i=\alpha+1}^{\beta} D(0,T_i)\left[\frac{1}{P(T_i,T_{i+1})} - (1+\tau_{i+1}K)\right]\right\}$$
$$= E\left\{\sum_{i=\alpha+1}^{\beta} \left[\frac{D(0,T_{i+1})}{P(T_i,T_{i+1})^2} - D(0,T_i)(1+\tau_{i+1}K)\right]\right\}$$

$$= \sum_{i=\alpha+1}^{\beta} \left\{P(0,T_{i+1})E^{i+1}\left[\frac{1}{P(T_i,T_{i+1})^2}\right] - P(0,T_i)(1+\tau_{i+1}K)\right\}$$
$$= \sum_{i=\alpha+1}^{\beta} \left\{P(0,T_{i+1})E^{i+1}\left[(1+\tau_{i+1}F_{i+1}(T_i))^2\right] - P(0,T_i)(1+\tau_{i+1}K)\right\}.$$

Computing the expected value is an easy task, since we know that, under Q^{i+1}, F_{i+1} has the driftless (martingale) lognormal dynamics

$$dF_{i+1}(t) = \sigma_{i+1}(t)F_{i+1}(t)dZ_{i+1}(t) ,$$

so that, remembering the resulting lognormal distribution of $F_{i+1}^2(T_i)$, one has

$$E^{i+1}\left(F_{i+1}^2(T_i)\right) = F_{i+1}^2(0)\exp\left[\int_0^{T_i} \sigma_{i+1}^2(t)dt\right] = F_{i+1}^2(0)\exp(v_{i+1}^2)$$

where the v's have been defined in (6.18) and are deduced from cap prices. We obtain

$$\mathbf{IAS} = \sum_{i=\alpha+1}^{\beta} \{P(0,T_{i+1})\left[1+2\tau_{i+1}F_{i+1}(0) + \tau_{i+1}^2F_{i+1}^2(0)\exp(v_{i+1}^2)\right]$$
$$-(1+\tau_{i+1}K)P(0,T_i)\}. \qquad (10.3)$$

Contrary to the plain-vanilla case, this price depends on the volatility of forward rates through the caplet volatilities v. Notice however that correlations between different rates are not involved in this product, as one expects from the additive and "one-rate-per-time" nature of the payoff.

10.2 In-Advance Caps

An in-advance cap is composed by caplets resetting at dates $T_{\alpha+1},\ldots,T_\beta$ and paying at the same dates, with unit notional amount and strike rate K.

More precisely, the discounted payoff of an in-advance cap can be expressed via

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i)-K)^+.$$

The value of such a contract is, therefore,

$$\mathbf{IAC} = E\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_{i+1}(F_{i+1}(T_i)-K)^+\right].$$

The payoff is the same as in the case of in-advance swaps, except for the positive-part operator.

10.2.1 A First Analytical Formula (LFM)

We apply the same reasoning we used for in-advance swaps, obtaining:

$$\begin{aligned}
\mathbf{IAC} &= \sum_{i=\alpha+1}^{\beta} P(0,T_{i+1})E^{i+1}\left[(1+\tau_{i+1}F_{i+1}(T_i))(F_{i+1}(T_i)-K)^+\right] \\
&= \sum_{i=\alpha+1}^{\beta} P(0,T_{i+1})\left(E^{i+1}[(F_{i+1}(T_i)-K)^+]\right. \\
&\qquad\qquad \left.+\tau_{i+1}E^{i+1}\left[F_{i+1}(T_i)(F_{i+1}(T_i)-K)^+\right]\right) \\
&= \sum_{i=\alpha+1}^{\beta} P(0,T_{i+1})\left[\mathrm{Bl}(K,F_{i+1}(0),v_{i+1})+\tau_{i+1}g(K,F_{i+1}(0),v_{i+1})\right],
\end{aligned}$$

$$g(K,F,v) := F^2\exp[v^2]\,\Phi\!\left(\frac{3v}{2}-\frac{1}{v}\ln\frac{K}{F}\right) - FK\,\Phi\!\left(\frac{v}{2}-\frac{1}{v}\ln\frac{K}{F}\right),$$

where "Bl" and v have been defined in (1.26), (6.18) and above. In-advance caps do not depend on the correlation of different rates but just on the caplet volatilities v, as one expects again from the additive and "one-rate-per-time" nature of the payoff.

10.2.2 A Second Analytical Formula (G2++)

The above expectations can also be easily computed under the Gaussian G2++ model, by exploiting the lognormal distribution of bond prices. After lengthy but straightforward calculations we obtain:

$$
\mathbf{IAC} = \sum_{i=\alpha+1}^{\beta} P(0,T_i) \left[\frac{P(0,T_i)}{P(0,T_{i+1})} e^{\Sigma(T_i,T_{i+1})^2} \Phi\left(\frac{\ln \frac{P(0,T_i)}{\widetilde{K}_i P(0,T_{i+1})} + \frac{3}{2}\Sigma(T_i,T_{i+1})^2}{\Sigma(T_i,T_{i+1})} \right) \right.
$$

$$
\left. - \widetilde{K}_i \Phi\left(\frac{\ln \frac{P(0,T_i)}{\widetilde{K}_i P(0,T_{i+1})} + \frac{1}{2}\Sigma(T_i,T_{i+1})^2}{\Sigma(T_i,T_{i+1})} \right) \right],
$$

where $\widetilde{K}_i = 1 + K\tau_i$ and

$$
\Sigma(T,S)^2 = \frac{\sigma^2}{2a^3}\left[1-e^{-a(S-T)}\right]^2\left[1-e^{-2aT}\right] + \frac{\eta^2}{2b^3}\left[1-e^{-b(S-T)}\right]^2\left[1-e^{-2bT}\right]
$$

$$
+ 2\rho\frac{\sigma\eta}{ab(a+b)}\left[1-e^{-a(S-T)}\right]\left[1-e^{-b(S-T)}\right]\left[1-e^{-(a+b)T}\right].
$$

10.3 Autocaps

We adopt the same notation, terminology and conventions as in Section 6.4, and take $\alpha = 0$. An autocap is similar to a cap, but at most $\gamma \le \beta$ caplets can be exercised, and they *have* to be automatically exercised when in the money. Therefore, the discounted payoff can be written as

$$
\sum_{i=1}^{\beta} \tau_i \left[F(T_{i-1};T_{i-1},T_i) - K\right]^+ D(0,T_i)\, 1\{A_i\},
$$

$$
A_i = \{\text{at most } \gamma \text{ among } F_1(T_0),\ldots,F_i(T_{i-1}) \text{ are larger than } K\},
$$

where $1\{A\}$ denotes the indicator function for the set A.

The pricing of this contract can be obtained by considering the risk-neutral expectation E of its discounted payoff:

$$
E\left[\sum_{i=1}^{\beta} \tau_i (F_i(T_{i-1}) - K)^+ D(0,T_i)\, 1\{A_i\}\right]
$$

$$
= P(0,T_\beta) \sum_{i=1}^{\beta} \tau_i E^\beta \left[\frac{(F_i(T_{i-1}) - K)^+\, 1\{A_i\}}{P(T_i,T_\beta)}\right],
$$

where we have used (10.2) (equivalently, the remarks of Section 2.8).

Notice that the A_i term depends not only on the forward rate $F_i(T_{i-1})$, but also on $F_1(T_0), ..., F_{i-1}(T_{i-2})$. Therefore, a "path-dependent" feature is introduced in the contract. If we attempt to price this contract by a Monte Carlo method, in order to compute the discounted payoff we need to generate paths under Q^β for the vector (whose dimension decreases over time)

$$
F_{\beta(t)}(t), \ldots, F_\beta(t)\,,
$$

where we recall that $t \in (T_{\beta(t)-2}, \ T_{\beta(t)-1}]$. These paths can be deduced from discretizing the dynamics (6.14). In our setting, such dynamics reads ($k = \beta(t), \beta(t)+1, \dots, \beta$)

$$dF_k(t) = -\sigma_k(t) F_k(t) \sum_{j=k+1}^{\beta} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j(t)}{1+\tau_j F_j(t)} dt + \sigma_k(t) F_k(t) dZ_k^{\beta}(t). \quad (10.4)$$

Taking logs and using the Milstein scheme, analogously to what was done for swaptions in Section 6.10, yields the desired simulated paths:

$$\ln F_k^{\Delta t}(t+\Delta t) = \ln F_k^{\Delta t}(t) - \sigma_k(t) \sum_{j=k+1}^{\beta} \frac{\rho_{k,j}\tau_j\sigma_j(t)F_j^{\Delta t}(t)}{1+\tau_j F_j^{\Delta t}(t)} \Delta t - \frac{\sigma_k(t)^2}{2}\Delta t$$
$$+\sigma_k(t)(Z_k^{\beta}(t+\Delta t) - Z_k^{\beta}(t)). \quad (10.5)$$

Actually, here too, one can improve the scheme by resorting to more refined shocks, in the spirit of Remark 6.10.1.

10.4 Caps with Deferred Caplets

These are caps for which all caplets payments occur at the final time T_β. The discounted payoff is, therefore,

$$\sum_{i=1}^{\beta} \tau_i (F_i(T_{i-1}) - K)^+ D(0, T_\beta) \, .$$

The pricing of this "deferred" cap can be obtained by considering the risk-neutral expectation E of its discounted payoff:

$$E\left[\sum_{i=1}^{\beta} \tau_i (F_i(T_{i-1}) - K)^+ D(0, T_\beta)\right] = P(0, T_\beta) \sum_{i=1}^{\beta} \tau_i E^{\beta}\left[(F_i(T_{i-1}) - K)^+\right].$$

The expected value can be computed through a Monte Carlo method based on the discretized Q^β dynamics (10.5).

10.4.1 A First Analytical Formula (LFM)

The above formula requires to compute the expected values

$$E^{\beta}\left[(F_i(T_{i-1}) - K)^+\right].$$

This is a case where the LFM with partially frozen drift can be of help in deriving analytical approximations. Indeed, consider the approximate LFM dynamics (10.1) with $\gamma = \beta$. Then, the above expectation is easily computed as a Black and Scholes call-option price (see Appendix B):

$$\exp\left(\int_0^{T_{i-1}} \bar{\mu}_{\beta,i}(t)dt\right) F_i(0)\,\Phi\left(\frac{\ln\frac{F_i(0)}{K} + \int_0^{T_{i-1}}\left[\bar{\mu}_{\beta,i}(t) + \frac{\sigma_i^2(t)}{2}\right]dt}{\sqrt{\int_0^{T_{i-1}}\sigma_i^2(t)dt}}\right)$$
$$-K\Phi\left(\frac{\ln\frac{F_i(0)}{K} + \int_0^{T_{i-1}}\left[\bar{\mu}_{\beta,i}(t) - \frac{\sigma_i^2(t)}{2}\right]dt}{\sqrt{\int_0^{T_{i-1}}\sigma_i^2(t)dt}}\right).$$

Replacing the expectation with such expression in the above summation, we obtain an analytical formula for the price of the cap with deferred caplets.

10.4.2 A Second Analytical Formula (G2++)

The expectations

$$E^\beta\left[(F_i(T_{i-1}) - K)^+\right]$$

can also be easily computed under the Gaussian G2++ model, by again exploiting the lognormal distribution of bond prices. After lengthy but straightforward calculations we obtain:

$$\frac{1}{\tau_i}\left[\frac{P(0,T_{i-1})}{P(0,T_i)} e^{\psi(0,T_{i-1},T_i,T_\beta,1)}\Phi\left(\frac{\ln\frac{P(0,T_{i-1})}{\widetilde{K}P(0,T_i)} + \psi(0,T_{i-1},T_i,T_\beta,\frac{3}{2})}{\sqrt{\psi(T_{i-1},T_i,T_i,2)}}\right)\right.$$
$$\left. - \widetilde{K}\Phi\left(\frac{\ln\frac{P(0,T_{i-1})}{\widetilde{K}P(0,T_i)} + \psi(0,T_{i-1},T_i,T_\beta,\frac{1}{2})}{\sqrt{\psi(T_{i-1},T_i,T_i,2)}}\right)\right],$$

where $\widetilde{K} := 1 + \tau_i K$ and

$$\begin{aligned}
&\psi(T,S,\tau,\lambda)\\
&:= \frac{\sigma^2}{2a^3}\left[1 - e^{-a(S-T)}\right]\left[1 - e^{-2aT}\right]\left[e^{-a(\tau-T)} - 1 + \lambda - \lambda e^{-a(S-T)}\right]\\
&+ \frac{\eta^2}{2b^3}\left[1 - e^{-b(S-T)}\right]\left[1 - e^{-2bT}\right]\left[e^{-b(\tau-T)} - 1 + \lambda - \lambda e^{-b(S-T)}\right]\\
&+ \frac{\rho\sigma\eta}{ab(a+b)}\left[1 - e^{-a(S-T)}\right]\left[1 - e^{-(a+b)T}\right]\left[e^{-b(\tau-T)} - 1 + \lambda - \lambda e^{-b(S-T)}\right]\\
&+ \frac{\rho\sigma\eta}{ab(a+b)}\left[1 - e^{-b(S-T)}\right]\left[1 - e^{-(a+b)T}\right]\left[e^{-a(\tau-T)} - 1 + \lambda - \lambda e^{-a(S-T)}\right].
\end{aligned}$$

Notice that, using the previous notation, we can write $\psi(T_{i-1},T_i,T_i,2) = \Sigma(T_{i-1},T_i)^2$.

10.5 Ratchets (One-Way Floaters)

We give a short description of one-way floaters in the following. We assume a unit nominal amount.

- Institution A pays to B (a percentage γ of) a reference floating rate (plus a constant spread S) at dates $\mathcal{T} = \{T_1, \ldots, T_\beta\}$. Formally, at time T_i institution A pays to B

$$(\gamma F_i(T_{i-1}) + S)\tau_i \,.$$

- Institution B pays to A a coupon that is given by the reference rate plus a spread X at dates $\mathcal{T}$, floored and capped respectively by the previous coupon and by the previous coupon plus an increment Y. Formally, at time T_i with $i > 1$, institution B pays to A the coupon

$$c_i = \begin{cases} (F_i(T_{i-1}) + X)\tau_i & \text{if } \; c_{i-1} \le (F_i(T_{i-1}) + X)\tau_i \le c_{i-1} + Y, \\ c_{i-1} & \text{if } \; (F_i(T_{i-1}) + X)\tau_i < c_{i-1}, \\ c_{i-1} + Y & \text{if } \; (F_i(T_{i-1}) + X)\tau_i > c_{i-1} + Y, \end{cases}$$

  At the first payment time T_1, institution B pays to A the coupon

$$(F_1(T_0) + X)\tau_1.$$

The discounted payoff as seen from institution A is

$$\sum_{i=1}^{\beta} D(0, T_i) \left[c_i - (\gamma F_i(T_{i-1}) + S)\tau_i\right]$$

and the value to A of the contract is the risk-neutral expectation

$$E\left\{\sum_{i=1}^{\beta} D(0, T_i) \left[c_i - (\gamma F_i(T_{i-1}) + S)\tau_i\right]\right\}$$
$$= P(0, T_\beta) \sum_{i=1}^{\beta} E^\beta \left[\frac{c_i - (\gamma F_i(T_{i-1}) + S)\tau_i}{P(T_i, T_\beta)}\right].$$

Since the forward-rate dynamics under Q^β of

$$F_{\beta(t)}(t), \ldots, F_\beta(t)$$

is known as from (10.4), a Monte Carlo pricing can be carried out in the usual manner.

10.6 Constant-Maturity Swaps (CMS)

10.6.1 CMS with the LFM

A constant-maturity swap is a financial product structured as follows. We assume a unit nominal amount. Let us denote by $\mathcal{T} = \{T_0, \dots, T_n\}$ a set of payment dates at which coupons are to be paid. We assume, for simplicity, such dates to be one-year spaced.

- At time T_{i-1} (in some variants at time T_i), $i \geq 1$, institution A pays to B the c-year swap rate resetting at time T_{i-1}. Formally, at time T_{i-1} institution A pays to B

  $$S_{i-1,i-1+c}(T_{i-1})\, \tau_i \,,$$

  where, as usual,

  $$S_{i-1,i-1+c}(t) = \frac{P(t,T_{i-1}) - P(t,T_{i-1+c})}{\sum_{k=i}^{i-1+c} \tau_k P(t,T_k)} \,. \tag{10.6}$$

- Institution B pays to A a fixed rate K.

The net value of the contract to B at time 0 is

$$\begin{aligned} &E\left(\sum_{i=1}^{n} D(0,T_{i-1})(S_{i-1,i-1+c}(T_{i-1}) - K)\tau_i\right) \\ &= \sum_{i=1}^{n} \tau_i P(0,T_{i-1}) \left[E^{i-1}\left(S_{i-1,i-1+c}(T_{i-1})\right) - K\right] \\ &= \sum_{i=1}^{n} \tau_i \left(P(0,T_n)\, E^n\left(\frac{S_{i-1,i-1+c}(T_{i-1})}{P(T_{i-1},T_n)}\right) - KP(0,T_{i-1})\right). \end{aligned} \tag{10.7}$$

We need only compute either

$$E^{i-1}\left[S_{i-1,i-1+c}(T_{i-1})\right] \quad \text{or} \quad E^n[S_{i-1,i-1+c}(T_{i-1})/P(T_{i-1},T_n)]$$

for all i's. At first sight, one might think to discretize equation (6.38) for the dynamics of the forward swap rate and compute the required expectation through a Monte Carlo simulation. However, notice that forward rates appear in the drift m^α of such equation, so that we are forced to evolve forward rates anyway. As a consequence, we can use equation (10.6) jointly with Monte Carlo simulated forward-rate dynamics, and do away with the dynamics (6.38), thus directly recovering the swap rate $S_{i-1,i-1+c}(T_{i-1})$ from the T_{i-1} values of the (Monte Carlo generated) spanning forward rates

$$F_i(T_{i-1}), F_{i+1}(T_{i-1}), \dots, F_{i-1+c}(T_{i-1}).$$

Analogously to the autocaps case, such forward rates can be generated according to the usual discretized (Milstein) dynamics (10.5) based on Gaussian shocks and under the unique measure Q^n.

10.6.2 CMS with the G2++ Model

It is possible to price a CMS with the G2++ model (4.4). See the related Section 11.2.2 on quanto CMS's in the next chapter. We do not repeat things here, since the CMS pricing procedure can be easily deduced from the procedure for the more general quanto-CMS case.

10.7 The Convexity Adjustment and Applications to CMS

10.7.1 Natural and Unnatural Time Lags

As with so many things, it was simply a matter of time.
The Time Trapper, Zero Hour – End of an Era, LSH 61, 1994, DC Comics

To appropriately introduce the convexity-adjustment technique, we quickly recall the pricing formulas for swaps. We begin by a plain-vanilla swap with natural time lag.

Consider an IRS that resets at dates $T_\alpha, T_{\alpha+1}, \dots, T_{\beta-1}$ and pays at dates $T_{\alpha+1}, \dots, T_\beta$, with unit notional amount. The fact that the payment indexed by the LIBOR rate resetting at time T_i for the maturity T_{i+1} occurs precisely at time T_{i+1} is referred to as a "natural time lag". This renders the swap price independent of the volatility of rates. Indeed, let us consider only the variable swap leg. The discounted value of this leg can be expressed either via the swap rate or via forward rates. In effect, the discounted payoff is given by

$$D(0,T_\alpha)S_{\alpha,\beta}(T_\alpha)\sum_{i=\alpha+1}^{\beta}\tau_i P(T_\alpha,T_i),$$

which is equivalent to

$$\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i F_i(T_{i-1}).$$

The value of such a leg is easily computed in both cases as

$$\begin{aligned}E\left[\sum_{i=\alpha+1}^{\beta} D(0,T_i)\tau_i F_i(T_{i-1})\right] &= \sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i E^i\left[F_i(T_{i-1})\right]\\ = \sum_{i=\alpha+1}^{\beta} P(0,T_i)\tau_i F_i(0) &= \sum_{i=\alpha+1}^{\beta}\left[P(0,T_{i-1})-P(0,T_i)\right]\\ &= P(0,T_\alpha)-P(0,T_\beta).\end{aligned}$$

From the last formula notice that, as is well known, neither volatility nor correlation of rates affect this financial product.

Now, let us reconsider in-advance swaps. Consider the variable leg of an IRS that resets at dates $T_{\alpha+1}, \dots, T_\beta$ and pays *at the same dates*, with unit notional amount. We say this swap has an "unnatural time lag". This term is justified by seeing that the price of such a leg depends on volatility. Indeed, see formula (10.3) with $K = 0$.

Contrary to the plain-vanilla case, the in-advance-swap price depends on the volatility of forward rates through their average volatilities v, which are usually deduced inverting cap prices through Black's formula.

The "natural/unnatural" terminology reflects the above calculations. A *natural time lag* for the variable leg of a swap makes the value of such a leg *independent of the rates volatility*. On the contrary, an *unnatural time lag* makes the value of the variable leg volatility dependent.

As a corollary, we can derive the corresponding formulas for forward-rate agreements. Suppose we are now at time 0, and at time T_2 the contract pays the LIBOR rate resetting at time $T_1 < T_2$ and maturing at T_2. As usual, we denote this rate by $L(T_1, T_2) = F_2(T_1)$ and we denote by τ the year fraction between T_1 and T_2. The contract value is therefore, consistently with the general FRA notation previously established,

$$\begin{aligned}
-\mathbf{FRA}(0, T_1, T_2, 0) &= E[D(0, T_2)\tau F_2(T_1)] \\
&= P(0, T_2)\tau E^2[F_2(T_1)] = P(0, T_2)\tau F_2(0) \\
&= P(0, T_1) - P(0, T_2).
\end{aligned}$$

If we have an in-advance FRA, this time the contract pays at time T_1 the LIBOR rate resetting at the same time $T_1 < T_2$ and maturing at T_2. By reasoning in an analogous way to the case of in-advance swaps, we obtain

$$\begin{aligned}
\mathbf{IAFRA} &= P(0, T_2)\left[1 + 2\tau F_2(0) + \tau^2 F_2^2(0)\exp(v_2^2(T_1))\right] - P(0, T_1) \\
&= P(0, T_1)\left[1 + \frac{\tau\ F_2(0) + \tau^2 F_2^2(0)\exp(v_2^2(T_1))}{1 + \tau F_2(0)}\right] - P(0, T_1) \\
&= P(0, T_2)\tau F_2(0)\left(1 + \tau F_2(0)\exp(v_2^2(T_1))\right) \\
&\approx P(0, T_2)\tau F_2(0)\left(1 + \tau F_2(0) + \tau v_2^2(T_1) F_2(0)\right) \\
&= P(0, T_1)\tau F_2(0) + P(0, T_2)\tau^2 F_2^2(0) v_2^2(T_1)
\end{aligned} \tag{10.8}$$

10.7.2 The Convexity-Adjustment Technique

The time is out of joint. O cursed spite,
That ever I was born to set it right
Hamlet, I.5

The convexity-adjustment technique can be attempted any time there is an unnatural time lag. We consider its application to a single payment.

Assume a swap rate is involved, and that the payment $\tau_i S_{\alpha,\beta}(T_{i-k})$ is due at time T_i, $i-k \le i \le \alpha < \beta$.

Remark 10.7.1. **(CMS)** This is typical of constant-maturity swaps (CMS) where we have $i = \alpha$ and $k = 1$ or $k = 0$. This is the case where the convexity adjustment works well and is also supported by the output of more sophisticated models like, for instance, the G2++ model. If k is large, the correction can be quite wrong. Therefore, in such cases, the correction discussed here should be considered with care.

We are far from the "usual" IRS case, because the rate being exchanged at each payment instant is a *swap* rate rather than a *LIBOR* rate.

A first adjustment

The forward swap rate $S_{\alpha,\beta}$ is originally defined as related to an IRS that pays at times $\alpha+1,\ldots,\beta$: $S_{\alpha,\beta}(T)$ at time T, $T \le T_\alpha$, is the fixed rate such that the fixed leg of the above IRS has value equal to that of the floating leg. In case of reimbursement of the notional amount, such a value at time T is always $P(T,T_\alpha)$ (see Definition 1.5.2 and the subsequent comments), so that we can write ("FL" stands for "floating leg")

$$\mathrm{FL}_{\alpha,\beta}(T) = P(T,T_\alpha) = S_{\alpha,\beta}(T)\sum_{i=\alpha+1}^{\beta}\tau_i P(T,T_i) + P(T,T_\beta)\ .$$

Now rewrite the same expression with the discount factors coming from a flat yield curve fixed at a level y (annually compounded) at time T_α, ("FFL" stands for Flat Floating Leg):

$$\mathrm{FFL}_{\alpha,\beta}(T;y) = S_{\alpha,\beta}(T)\sum_{i=\alpha+1}^{\beta}\tau_i \frac{P(T,T_\alpha)}{(1+y)^{\tau_{\alpha,i}}} + \frac{P(T,T_\alpha)}{(1+y)^{\tau_{\alpha,\beta}}}\ ,$$

where $\tau_{\alpha,i}$ denotes the year fraction between T_α and T_i.

If one allows for the first-order expansion

$$\delta S_{\alpha,\beta}(T) = (1+S_{\alpha,\beta}(T))^{\delta} - 1\,,$$

and takes $T_i = i\delta$, $\tau_{\alpha,i} = (i-\alpha)\delta$ and $\tau_i = \delta$, it is easy to see that the above flat floating leg coincides with $P(T,T_\alpha)$ only for $y = S_{\alpha,\beta}(T)$,

$$\mathrm{FFL}_{\alpha,\beta}(T;S_{\alpha,\beta}(T)) = P(T,T_\alpha).$$

Therefore, the value of y around which the flat-curve approximation is to be considered is the forward swap rate $S_{\alpha,\beta}(T)$, in that it is the flat rate that agrees with the non-flat case as far as the price of the floating leg is concerned:

$$\mathrm{FFL}_{\alpha,\beta}(T; S_{\alpha,\beta}(T)) = \mathrm{FL}_{\alpha,\beta}(T).$$

Consider now the expectation

$$\begin{aligned} E_0^T &\left[P(0,T)\mathrm{FFL}_{\alpha,\beta}(T; S_{\alpha,\beta}(T)) - \mathrm{FFL}_{\alpha,\beta}(0; S_{\alpha,\beta}(0))\right] \\ &= E_0^T\left[P(0,T)\frac{P(T,T_\alpha)}{P(T,T)} - P(0,T_\alpha)\right] = 0\,. \end{aligned} \tag{10.9}$$

At this point, we proceed by defining the following quantity $\Phi_{\alpha,\beta}(y)$ through a slight approximation of the argument of the above expectation, where we assume $P(0,T)P(T,T_\alpha) \approx P(0,T_\alpha)$:

$$P(0,T)\mathrm{FFL}_{\alpha,\beta}(T,y) \approx S_{\alpha,\beta}(T)\sum_{i=\alpha+1}^{\beta}\tau_i\frac{P(0,T_\alpha)}{(1+y)^{\tau_{\alpha,i}}}+\frac{P(0,T_\alpha)}{(1+y)^{\tau_{\alpha,\beta}}} =: \Phi_{\alpha,\beta}(T,y).$$

We expand Φ through a second-order Taylor expansion around $y = S_{\alpha,\beta}(0)$, we evaluate the resulting expression at $y = S_{\alpha,\beta}(T)$, we then solve for $S_{\alpha,\beta}(T) - S_{\alpha,\beta}(0)$ and introduce a further approximation:

$$\begin{aligned} S_{\alpha,\beta}(T) - S_{\alpha,\beta}(0) &\approx \frac{\Phi_{\alpha,\beta}(T,S_{\alpha,\beta}(T)) - \Phi_{\alpha,\beta}(T,S_{\alpha,\beta}(0))}{\Phi'_{\alpha,\beta}(T,S_{\alpha,\beta}(0))} \\ &\quad - \frac{(S_{\alpha,\beta}(T)-S_{\alpha,\beta}(0))^2}{2}\frac{\Phi''_{\alpha,\beta}(T,S_{\alpha,\beta}(0))}{\Phi'_{\alpha,\beta}(T,S_{\alpha,\beta}(0))} \\ &\approx \frac{\Phi_{\alpha,\beta}(T,S_{\alpha,\beta}(T)) - \Phi_{\alpha,\beta}(0,S_{\alpha,\beta}(0))}{\Phi'_{\alpha,\beta}(0,S_{\alpha,\beta}(0))} \\ &\quad - \frac{(S_{\alpha,\beta}(T)-S_{\alpha,\beta}(0))^2}{2}\frac{\Phi''_{\alpha,\beta}(0,S_{\alpha,\beta}(0))}{\Phi'_{\alpha,\beta}(0,S_{\alpha,\beta}(0))}\,, \end{aligned} \tag{10.10}$$

where the superscript $'$ denotes partial derivative with respect to y.

Now take expectation on both sides under the measure Q^T. The first term on the right-hand side has expectation zero, due to equation (10.9).

We further assume that we can approximate the true Q^T-dynamics of $S_{\alpha,\beta}$ by its lognormal $Q^{\alpha,\beta}$-dynamics (6.36), an approximation that has been shown to work well in most situations for the LFM when T is close to T_α (see the tests on the distributions of the swap rate described at the end of Section 8.2 and the related results in Section 8.3). We obtain:

$$\begin{aligned} E_0^T\left[(S_{\alpha,\beta}(T)-S_{\alpha,\beta}(0))^2\right] &\approx E_0^{\alpha,\beta}\left[(S_{\alpha,\beta}(T)-S_{\alpha,\beta}(0))^2\right] \\ &= S_{\alpha,\beta}(0)^2(e^{v^2_{\alpha,\beta}(T)}-1) \approx S^2_{\alpha,\beta}(0)\,v^2_{\alpha,\beta}(T), \end{aligned}$$

where

$$v^2_{\alpha,\beta}(T) = \int_0^T (\sigma^{(\alpha,\beta)}(t))^2\,dt$$

is the average variance of the forward swap rate in the interval $[0, T]$ times the interval length. Now, we can evaluate (10.10) by taking expectation on both sides:

$$E_0^T\left[S_{\alpha,\beta}(T)\right] \approx S_{\alpha,\beta}(0) - \tfrac{1}{2}S_{\alpha,\beta}^2(0)v_{\alpha,\beta}^2(T)\frac{\Phi''_{\alpha,\beta}(0, S_{\alpha,\beta}(0))}{\Phi'_{\alpha,\beta}(0, S_{\alpha,\beta}(0))}\,. \quad (10.11)$$

A second adjustment

A second adjustment we can consider is based on neglecting the final reimbursement of the notional amount in the above IRS. We thus define Ψ as Φ without notional reimbursement,

$$\Psi_{\alpha,\beta}(y) := S_{\alpha,\beta}(T)\sum_{i=\alpha+1}^{\beta}\tau_i\frac{P(0,T_\alpha)}{(1+y)^{\tau_{\alpha,i}}}.$$

Assuming that also

$$E_0^T[\Psi_{\alpha,\beta}(S_{\alpha,\beta}(T)) - \Psi_{\alpha,\beta}(S_{\alpha,\beta}(0))] \approx 0,$$

as for Φ when taking expectations on both sides of (10.10), and using again a second-order expansion, it follows that

$$\boxed{E_0^T\left[S_{\alpha,\beta}(T)\right] \approx S_{\alpha,\beta}(0) - \tfrac{1}{2}S_{\alpha,\beta}^2(0)v_{\alpha,\beta}^2(T)\frac{\Psi''_{\alpha,\beta}(S_{\alpha,\beta}(0))}{\Psi'_{\alpha,\beta}(S_{\alpha,\beta}(0))}}\,, \quad (10.12)$$

where the ratio $\Psi''_{\alpha,\beta}(S_{\alpha,\beta}(0))/\Psi'_{\alpha,\beta}(S_{\alpha,\beta}(0))$ is independent of T.

This is the formula that is usually considered in the market for convexity adjustments (especially for CMS), see for example Hull (1997), in particular formula (16.13) and the related Example 16.8. The approximation works well when T is not too far away from T_α, as implied by the "Q^T vs $Q^{\alpha,\beta}$" dynamics approximation for the forward swap rate.

Let us now apply this formula to specific situations.

Floating leg with swap-rate-indexed payments

Suppose we need to compute the present value of our generic payment,

$$E[\tau_i D(0,T_i)S_{\alpha,\beta}(T_{i-k})]\,.$$

Move under the T_i-forward measure, to obtain

$$P(0,T_i)E^i[\tau_i S_{\alpha,\beta}(T_{i-k})]\,.$$

The first rougher approximation is to treat the measure Q^i as if it were the swap measure $Q^{\alpha,\beta}$, under which S can be modeled through the lognormal martingale

$$dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\, dW_t\,.$$

Under this approximation, we would then have

$$E[\tau_i D(0,T_i) S_{\alpha,\beta}(T_{i-k})] \approx \tau_i P(0,T_i) S_{\alpha,\beta}(0).$$

The convexity adjustment (10.12) leads to the following modification of this last formula:

$$\begin{aligned}
&E[\tau_i D(0,T_i) S_{\alpha,\beta}(T_{i-k})] \\
&\quad = \tau_i P(0,T_i) E^i[S_{\alpha,\beta}(T_{i-k})] \approx \tau_i P(0,T_i) E^{i-k}[S_{\alpha,\beta}(T_{i-k})] \\
&\quad \approx \tau_i P(0,T_i) \left[S_{\alpha,\beta}(0) - \tfrac{1}{2} S^2_{\alpha,\beta}(0) v^2_{\alpha,\beta}(T_{i-k}) \frac{\Psi''_{\alpha,\beta}(S_{\alpha,\beta}(0))}{\Psi'_{\alpha,\beta}(S_{\alpha,\beta}(0))} \right].
\end{aligned}$$

As anticipated in Remark 10.7.1, this approximation turns out to work well only for small values of k. Therefore, if k is large, the correction should be considered with due care.

We now check that, in case of an in-advance FRA, this formula is consistent with the value found earlier by exact evaluation. We take $\alpha = i = 1$, $k = 0$, $\beta = 2$ and $\tau_{1,2} = \tau$, so that $S_{\alpha,\beta}(t) = F(t; T_1, T_2)$ and

$$\Psi_{1,2}(y) = \frac{C}{(1+\tau y)},$$

with C a suitable constant, and where we have used simple compounding instead of annual compounding. Notice that

$$\frac{\Psi''_{1,2}(y)}{\Psi'_{1,2}(y)} = \frac{-2\tau}{(1+\tau y)},$$

so that the convexity-adjustment formula (10.12) yields

$$\begin{aligned}
&\tau P(0,T_1) \left[F_2(0) + \tau \frac{F_2^2(0) v_2^2(T_1)}{1 + \tau F_2(0)} \right] \\
&= \tau P(0,T_1) F_2(0) + \tau^2 P(0,T_2) F_2^2(0) v_2^2(T_1),
\end{aligned}$$

which is the same result found, at first order in v_2^2, by exact evaluation in (10.8).

10.7.3 Deducing a Simple Lognormal Dynamics from the Adjustment

We can easily adjust the approximate driftless dynamics

$$dS_{\alpha,\beta}(t) = \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\, dW_t\,,$$

for which

$$E_0^T\left[S_{\alpha,\beta}(T_{i-k})\right] = S_{\alpha,\beta}(0)\,,$$

to a new dynamics

$$dS_{\alpha,\beta}(t) = \mu^{\alpha,\beta} S_{\alpha,\beta}(t)\,dt + \sigma^{(\alpha,\beta)}(t) S_{\alpha,\beta}(t)\,dW_t\,, \tag{10.13}$$

for which

$$E_0^T\left[S_{\alpha,\beta}(T_{i-k})\right] = S_{\alpha,\beta}(0) - \tfrac{1}{2} S^2_{\alpha,\beta}(0) v^2_{\alpha,\beta}(T_{i-k}) \frac{\Psi''_{\alpha,\beta}(S_{\alpha,\beta}(0))}{\Psi'_{\alpha,\beta}(S_{\alpha,\beta}(0))}\,,$$

consistently with the convexity-adjustment evaluation. Since the dynamics (10.13) produces

$$E_0^T\left[S_{\alpha,\beta}(T_{i-k})\right] = S_{\alpha,\beta}(0)\exp(\mu^{\alpha,\beta} T_{i-k}) \approx S_{\alpha,\beta}(0)(1 + \mu^{\alpha,\beta} T_{i-k}),$$

at first order in $\mu^{\alpha,\beta} T_{i-k}$, it suffices to set

$$\mu^{\alpha,\beta} = -\tfrac{1}{2} S_{\alpha,\beta}(0) \frac{v^2_{\alpha,\beta}(T_{i-k})}{T_{i-k}} \frac{\Psi''_{\alpha,\beta}(S_{\alpha,\beta}(0))}{\Psi'_{\alpha,\beta}(S_{\alpha,\beta}(0))}.$$

Notice that in case the instantaneous forward-swap-rate volatility $\sigma^{\alpha,\beta}$ is assumed to be constant, we have

$$\frac{v^2_{\alpha,\beta}(T_{i-k})}{T_{i-k}} = (\sigma^{\alpha,\beta})^2.$$

This approximation is however rather rough and should not be used to evaluate nonlinear payoffs, unless a considerable amount of testing has been performed and acceptable errors are found.

10.7.4 Application to CMS

We have seen before that a constant-maturity swap has a floating leg that pays at times $T_{\alpha+1}, \ldots, T_\beta$ the swap rates

$$S_{\alpha,\alpha+c}(T_\alpha), S_{\alpha+1,\alpha+1+c}(T_{\alpha+1}), \ldots, S_{\beta-1,\beta-1+c}(T_{\beta-1})\ .$$

Therefore, at each payment instant $T_{\alpha+k+1}$, such leg pays a certain pre-specified c-year swap rate resetting at the previous instant $T_{\alpha+k}$. In some variants, instead, it pays at $T_{\alpha+k+1}$ a certain pre-specified swap rate resetting at the same instant. We will consider here the first version.

The value of the generic CMS payment is given by

$$\begin{aligned}
&E\left[D(0,T_{i+1})\tau_{i+1} S_{i,i+c}(T_i)\right] = \tau_{i+1} P(0,T_{i+1}) E^{i+1} S_{i,i+c}(T_i) \\
&\approx \tau_{i+1} P(0,T_{i+1}) \left[S_{i,i+c}(0) - \tfrac{1}{2} S^2_{i,i+c}(0) v^2_{i,i+c}(T_i) \frac{\Psi''_{i,i+c}(S_{i,i+c}(0))}{\Psi'_{i,i+c}(S_{i,i+c}(0))} \right],
\end{aligned}$$

see also Example 16.8 in Hull (1997). The CMS price is then obtained by adding terms for i ranging from α to $\beta - 1$. Recall that the adjustment used here has been derived under a number of approximations. As such, it can be improved. Indeed, the classical adjustment has been found to be not completely satisfactory by some traders, especially in some market situations involving volatility smiles. For a recent work on CMS adjustments see for example Pugachevsky (2001).

10.7.5 Forward Rate Resetting Unnaturally and Average-Rate Swaps

We consider now the following problem, which can have several applications. Consider two time instants s, u and a payment date T, $s < u < T$. Assume we have a contract that pays at time T the spot LIBOR rate resetting at time s for the maturity u:

$$L(s, u) = F(s; s, u).$$

In case $T = u$ we have a natural time lag. Indeed, the contract value at time 0 is the risk-neutral expectation of the discounted payoff

$$E_0[D(0,T)F(s; s, T)] = P(0,T)E_0^T[F(s; s, T)] = P(0,T)F(0; s, T)$$

and does not depend on volatility specifications.

If $T > u$, the above formula no longer holds. However, we can still evaluate the contract as follows.

Consider the no-arbitrage forward-rate dynamics for $F(t) = F(t; s, u)$ under the T-forward-adjusted measure Q^T:

$$dF(t) = -\sigma_{s,u}\sigma_{u,T}\tau(u,T)F(t)\frac{F(t; u, T)}{1 + \tau(u,T)F(t; u, T)}\,dt + \sigma_{s,u}F(t)\,dW_t^T,$$

where $\sigma_{s,u}$ is the instantaneous volatility of $F(t) = F(t; s, u)$ and $\sigma_{u,T}$ is the instantaneous volatility of $F(t; u, T)$, and both are assumed to be constant (otherwise they can be replaced with the square roots of the average variances of $F(t; s, u)$ and $F(t; u, T)$, respectively, over $[0, s]$). The quantity $\tau(a, b)$ denotes in general the time between dates a and b in years.

We assume unit correlation between $F(t; s, u)$ and $F(t; u, T)$, since usually T and u are close. If this is not the case, a ρ parameter can be included in the drift of the above process.

With the usual deterministic-percentage-drift approximation we can write

$$dF(t) = -\sigma_{s,u}\sigma_{u,T}\tau(u,T)F(t)\frac{F(0; u, T)}{1 + \tau(u,T)F(0; u, T)}\,dt + \sigma_{s,u}F(t)\,dW_t^T.$$

This new process has lognormal distribution under the T-forward measure and it can be easily seen that its expected value, conditional on the information available at time 0, under the T-forward measure, is

$$E_0^T[F(s;s,u)] = F(0;s,u)\exp\left(-\tau(u,T)\sigma_{s,u}\sigma_{u,T}\, s\,\frac{F(0;u,T)}{1+\tau(u,T)F(0;u,T)}\right)\ .$$

We are now able to price the discounted payoff

$$E_0[D(0,T)L(s,u)] = P(0,T)E_0^T[F(s;s,u)]$$
$$= P(0,T)F(0;s,u)\exp\left(-\tau(u,T)\sigma_{s,u}\sigma_{u,T}\, s\,\frac{F(0;u,T)}{1+\tau(u,T)F(0;u,T)}\right).$$

As an example, consider a contract that pays at a future time T the average value of the 3-month ($3m$) LIBOR rates in the days $t_1 < t_2 < \ldots < t_n$, $t_n < T$, with δ_i denoting the year fraction between t_i and $t_i + 3m$. This is a possible example of a leg of an average-rate swap.

If the notional is N, the contract price is

$$E_0\left[D(0,T)\frac{\sum_{i=1}^n \delta_i N L(t_i, t_i+3m)}{n}\right]$$
$$= \frac{P(0,T)}{n}\, N\sum_{i=1}^n \delta_i E_0^T\left[F(t_i;t_i,t_i+3m)\right]$$

and is given by

$$\frac{P(0,T)}{n}\, N\sum_{i=1}^n \delta_i F(0;t_i,t_i+3m)$$
$$\cdot\exp\left[-\sigma_{t_i,t_i+3m}\sigma_{t_i+3m,T}\; t_i\,\frac{\tau(t_i+3m,T)F(0;t_i+3m,T)}{1+\tau(t_i+3m,T)F(0;t_i+3m,T)}\right].$$

Notice that the correction to the "brute-force" formula

$$\frac{P(0,T)}{n}\, N\sum_{i=1}^n \delta_i F(0;t_i,t_i+3m)$$

is multiplicative for each term and is given by the exponentials. The correction effect is to (slightly) reduce the "brute-force" value, since the exponents are negative. The correction might be not negligible for large values of the volatilities.

The difficulty in applying the above formula lies in the fact that the forward rate $F(0;t_i+3m,T)$ can be rather atypical as for expiry or maturity dates. Therefore, apart from few exceptions, its volatility cannot be recovered exactly from market cap prices. However, a synthetic volatility deduced from volatilities of "smaller" forward rates nested in $F(0;t_i{+}3m,T)$ can be used for this purpose, or arguments similar to those of Section 6.16 can be employed.

At a first stage, the above formula can be used to have a feeling on the order of magnitude of the adjustment due to second-order effects, and to decide whether these should be taken into account or not.

10.8 Captions and Floortions

A caption is an option that gives its holder the right to enter at a future time T_γ a cap whose first caplet resets at date $T_\alpha \geq T_\gamma$ and whose subsequent caplets reset at times $T_{\alpha+1}, \ldots, T_{\beta-1}$ with T_β the last payment date. The strike rate for this cap will be denoted by K. The price the holder of the caption will pay for this future cap is fixed as the caption strike and will be denoted by X. We can therefore express the caption payoff as a call payoff on the underlying cap.

We assume a unit notional amount. The T_γ value of the underlying cap described above is given by the usual Black formula (see for instance Section 6.4.3), computed at time T_γ instead of time 0,

$$\sum_{i=\alpha+1}^{\beta} \tau_i P(T_\gamma, T_i)\,\mathrm{Bl}(K, F_i(T_\gamma), \sqrt{T_{i-1} - T_\gamma}\; V(T_\gamma, T_{i-1})),$$

where the average volatility $V(\cdot,\cdot)$ was defined in Section 6.5.

The caption discounted payoff, expressed as a call payoff, can be written as

$$D(0, T_\gamma) \left\{ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\gamma, T_i)\,\mathrm{Bl}(K, F_i(T_\gamma), \sqrt{T_{i-1} - T_\gamma}\; V(T_\gamma, T_{i-1})) - X \right\}^+ .$$

The caption value is given by the risk-neutral expectation of this payoff, which in turn is given by

$$P(0, T_\gamma) E^\gamma \left\{ \sum_{i=\alpha+1}^{\beta} \tau_i P(T_\gamma, T_i)\,\mathrm{Bl}(K, F_i(T_\gamma), \sqrt{T_{i-1} - T_\gamma}\, V(T_\gamma, T_{i-1})) - X \right\}^+ .$$

Once again, the expected value can be computed through a Monte Carlo method, given the simulated values of

$$F_{\gamma+1}(T_\gamma), F_{\gamma+2}(T_\gamma), \ldots, F_\beta(T_\gamma)$$

under Q^γ, obtained through the usual discretized Milstein dynamics (10.5).

10.9 Zero-Coupon Swaptions

In this section we introduce zero-coupon swaptions and explain an approximated analytical method to price them. A payer (receiver) zero-coupon swaption is a contract giving the right to enter a payer (receiver) zero-coupon IRS at a future time. A zero-coupon IRS is an IRS where a single fixed payment is due at the unique (final) payment date T_β for the fixed leg in exchange for a stream of usual floating payments $\tau_i L(T_{i-1}, T_i)$ at times T_i in $T_{\alpha+1}, T_{\alpha+2}, \ldots, T_\beta$ (usual floating leg). In formulas, the discounted payoff of a payer zero-coupon IRS is, at time $t \le T_\alpha$:

$$D(t, T_\alpha) \left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta) \tau_{\alpha,\beta} K \right],$$

where $\tau_{\alpha,\beta}$ is the year fraction between T_α and T_β. The analogous payoff for a receiver zero-coupon IRS is obviously given by the opposite quantity.

Taking risk-neutral expectation, we obtain easily the contract value as

$$P(t, T_\alpha) - P(t, T_\beta) - \tau_{\alpha,\beta} K P(t, T_\beta),$$

which is the typical value of a floating leg minus the value of a fixed leg with a single final payment.

The value of the strike rate K that renders the contract fair is obtained by equating to zero the above value and solving in K. One obtains $K = F(t; T_\alpha, T_\beta)$. Indeed, we could have reasoned as follows. The value of the swap is independent of the number of payments on the floating leg, since the floating leg always values at par, no matter the number of payments (see Section 1.5.2 and the related remarks). Therefore, we might as well have taken a floating leg paying only in T_β the amount $\tau_{\alpha,\beta} L(T_\alpha, T_\beta)$. This would have given us again a standard swaption, standard in the sense that the two legs of the underlying IRS have the same payment dates (collapsing to T_β) and the unique reset date T_α. In such a one-payment case, the swap rate collapses to a forward rate, so that we should not be surprised to find out that the forward swap rate in this particular case is simply a forward rate.

An option to enter a payer zero-coupon IRS is a payer zero-coupon swaption, and the related payoff is

$$D(t, T_\alpha) \left[\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i) \tau_i F_i(T_\alpha) - P(T_\alpha, T_\beta) \tau_{\alpha,\beta} K \right]^+,$$

or, equivalently, by expressing the F's in terms of discount factors,

$$D(t, T_\alpha) \left[1 - P(T_\alpha, T_\beta) - P(T_\alpha, T_\beta) \tau_{\alpha,\beta} K \right]^+,$$

which in turn can be written as

$$D(t, T_\alpha) \tau_{\alpha,\beta} P(T_\alpha, T_\beta) \left[F(T_\alpha; T_\alpha, T_\beta) - K \right]^+.$$

Notice that, from the point of view of the payoff structure, this is merely a caplet. As such, it can be priced easily through Black's formula for caplets. The problem, however, is that such a formula requires the integrated percentage volatility of the forward rate $F(\cdot; T_\alpha, T_\beta)$, which is a forward rate over a non-standard period. Indeed, $F(\cdot; T_\alpha, T_\beta)$ is not in our usual family of spanning forward rates, unless we are in the trivial case $\beta = \alpha+1$. Therefore, since the market provides us (through standard caps and swaptions) with volatility data for standard forward rates, we need a formula for deriving the integrated percentage volatility of the forward rate $F(\cdot; T_\alpha, T_\beta)$ from volatility data of the standard forward rates $F_{\alpha+1}, \ldots, F_\beta$. The reasoning is once again based on the "freezing the drift" procedure, leading to an approximately lognormal dynamics for our standard forward rates.

Denote for simplicity $F(t) := F(t; T_\alpha, T_\beta)$ and $\tau := \tau_{\alpha,\beta}$.

We begin by noticing that, through straightforward algebra, we have (write everything in terms of discount factors to check)

$$1 + \tau F(t) = \prod_{j=\alpha+1}^{\beta} (1 + \tau_j F_j(t)).$$

It follows that

$$\ln(1 + \tau F(t)) = \sum_{j=\alpha+1}^{\beta} \ln(1 + \tau_j F_j(t)),$$

so that

$$d\ln(1 + \tau F(t)) = \sum_{j=\alpha+1}^{\beta} d\ln(1 + \tau_j F_j(t)) = \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Now, since

$$dF(t) = \frac{1 + \tau F(t)}{\tau} d\ln(1 + \tau F(t)) + (\ldots)dt,$$

we obtain from the above expression

$$dF(t) = \frac{1 + \tau F(t)}{\tau} \sum_{j=\alpha+1}^{\beta} \frac{\tau_j dF_j(t)}{1 + \tau_j F_j(t)} + (\ldots)dt.$$

Take variance (conditional on the information up to time t) on both sides:

$$\mathrm{Var}\left(\frac{dF(t)}{F(t)}\right) = \left[\frac{1 + \tau F(t)}{\tau F(t)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} \sigma_i(t) \sigma_j(t) F_i(t) F_j(t)}{(1 + \tau_i F_i(t))(1 + \tau_j F_j(t))} dt.$$

Now freeze all $t's$ to zero except for the σ's, and integrate over $[0, T_\alpha]$:

$$(v^{zc}_{\alpha,\beta})^2 := \left[\frac{1 + \tau F(0)}{\tau F(0)}\right]^2 \sum_{i,j=\alpha+1}^{\beta} \frac{\tau_i \tau_j \rho_{i,j} F_i(0) F_j(0)}{(1 + \tau_i F_i(0))(1 + \tau_j F_j(0))} \int_0^{T_\alpha} \sigma_i(t) \sigma_j(t) dt.$$

To price the zero-coupon swaption it is then enough to put this quantity into the related Black's formula:

$$\mathbf{ZCPS} = \tau P(0, T_\beta)\mathrm{Bl}(K, F(0), v^{zc}_{\alpha,\beta}).$$

We can check the accuracy of this formula against the usual Monte Carlo pricing based on the exact dynamics of the forward rates. In the tests all swaptions are at-the-money. We have done this under the data of case (3.a) of the volatility tests of Section 8.2, and in other situations. Under the data of Section 8.2, we considered first the case $T_\alpha = 2y$, $T_\beta = 19y$. We obtained the implied volatility $v^{zcMC}_{\alpha,\beta}/\sqrt{T_\alpha}$ by inverting the Monte Carlo price through Black's formula:

$$\mathbf{MCZCPS} = \tau P(0, T_\beta)\mathrm{Bl}(F(0), F(0), v^{zcMC}_{\alpha,\beta}).$$

We found, in this case:

$$\frac{v^{zcMC}_{\alpha,\beta}}{\sqrt{T_\alpha}} = 0.1410, \quad \frac{v^{zc}_{\alpha,\beta}}{\sqrt{T_\alpha}} = 0.1455.$$

A two-side 98% window for the Monte Carlo volatility defined as in Section 8.2 is in this case [0.1404 0.1416]. Our algebraic approximation falls out of the 98% window, but of a small amount if compared with the distance from the volatility of the corresponding plain-vanilla European swaption. In fact, the standard at-the-money plain-vanilla swaption with the same initial reset date and final payment date, whose algebraic approximation has been found to be accurate in Section 8.2, has volatility

$$\frac{v^{\mathrm{LFM}}_{\alpha,\beta}}{\sqrt{T_\alpha}} = 0.0997.$$

We have also considered the case $T_\alpha = 10y$, $T_\beta = 19y$. We obtained

$$\frac{v^{zcMC}_{\alpha,\beta}}{\sqrt{T_\alpha}} = 0.1081, \quad \frac{v^{zc}_{\alpha,\beta}}{\sqrt{T_\alpha}} = 0.1114.$$

Now a two-side 98% window for the Monte Carlo volatility defined as in Section 8.2 is [0.1076 0.1086]. Again, our algebraic approximation falls out of the 98% window of a small amount when compared with the discrepancy with respect to the corresponding standard swaption, resulting in a volatility

$$\frac{v^{\mathrm{LFM}}_{\alpha,\beta}}{\sqrt{T_\alpha}} = 0.0897.$$

In the two examples above we notice that the at-the-money standard swaption has always a lower volatility (and hence price) than the corresponding at-the-money zero-coupon swaption. We may wonder whether this is a general feature. Indeed, we have the following.

Remark 10.9.1. **(Comparison between zero-coupon swaptions and corresponding standard swaptions).** A first remark is due for a comparison between the zero-coupon swaption volatility $v^{zc}_{\alpha,\beta}$ and the corresponding European-swaption approximation $v^{\mathrm{LFM}}_{\alpha,\beta}$. If we rewrite the latter as

$$(v^{\text{LFM}}_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \rho_{i,j}\lambda_i\lambda_j \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt, \quad \lambda_i = \frac{w_i(0)F_i(0)}{S_{\alpha,\beta}(0)},$$

it is easy to check that

$$(v^{\text{ZC}}_{\alpha,\beta})^2 = \sum_{i,j=\alpha+1}^{\beta} \rho_{i,j}\mu_i\mu_j \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt,$$

where

$$\mu_i = \frac{P(0,T_\alpha)}{P(0,T_i)}\lambda_i \geq \lambda_i,$$

the discrepancy increasing with the payment index i. It follows that, for positive correlations, the zero-coupon swaption volatility is always larger than the corresponding plain vanilla swaption volatility, the difference increasing with the tenor $T_\beta - T_\alpha$, for each given T_α.

A final remark concerns the possibility to price zero-coupon swaptions with other models.

Remark 10.9.2. **(Pricing zero-coupon swaptions with other models).** Zero-coupon swaptions can be priced analytically under all short-rate models admitting explicit formulas for European options on zero-coupon bonds and, accordingly, for caplets. For instance, under the CIR++ model (3.76) we can use formula (3.79), whereas under the G2++ model (4.4) we can resort to formula (4.29).

10.10 Eurodollar Futures

A Eurodollar-futures contract gives its owner the payoff

$$X\,(1 - L(S_1, S_2))$$

at the future time $S_1 < S_2$, where X is a notional amount, and the year fraction between S_1 and S_2 is denoted by τ. The fair price of this contract at time t is

$$\begin{aligned} V_t = E_t[X\,(1 - L(S_1,S_2))] &= X\,(1 - E_t[L(S_1,S_2)]) \qquad (10.14)\\ &= X\left(1 + \frac{1}{\tau} - \frac{1}{\tau}E_t\left[\frac{1}{P(S_1,S_2)}\right]\right), \end{aligned}$$

and takes into account continuous rebalancing (see for example Sandmann and Sondermann (1997) and their reference to the related work of Cox Ingersoll and Ross).

The problem is computing the expectation

$$E_t\left[\frac{1}{P(S_1,S_2)}\right] = E_t\left[\frac{P(S_1,S_1)}{P(S_1,S_2)}\right].$$

If we were under the S_2-forward-adjusted measure this would be simply

$$\frac{P(0,S_1)}{P(0,S_2)} = 1 + \tau F(0;S_1,S_2)\,,$$

and the price would reduce to

$$X(1 - F(0;S_1,S_2)).$$

Instead, we need the expectation under the risk-neutral measure.

Since we need to compute

$$E_t[L(S_1,S_2)] = E_t[F(S_1;S_1,S_2)],$$

the result will depend on the interest-rate model we are using.

10.10.1 The Shifted Two-Factor Vasicek G2++ Model

We can use the two-additive-factor Gaussian model described in Chapter 4. Consistently with the notation adopted there, recall that

$$\begin{aligned}
P(t,T) &= \frac{P^M(0,T)}{P^M(0,t)}\exp\{\mathcal{A}(t,T)\},\\
\mathcal{A}(t,T) &= \frac{1}{2}[V(t,T) - V(0,T) + V(0,t)] - \frac{1-e^{-a(T-t)}}{a}x(t)\\
&\quad - \frac{1-e^{-b(T-t)}}{b}y(t),
\end{aligned}$$

with V defined as in(4.10), so that

$$\begin{aligned}
E_t\left[\frac{1}{P(T_1,T_2)}\right] &= \frac{P^M(0,T_1)}{P^M(0,T_2)}E_t\left[\exp\left\{-\mathcal{A}(T_1,T_2)\right\}\right]\\
= \frac{P^M(0,T_1)}{P^M(0,T_2)}\exp\Bigg\{&-\frac{1}{2}[V(T_1,T_2) - V(0,T_2) + V(0,T_1)]\\
&+\frac{1-e^{-a(T_2-T_1)}}{a}\,x(t)e^{-a(T_1-t)}\\
&+\frac{1-e^{-b(T_2-T_1)}}{b}\,y(t)e^{-b(T_1-t)}
\end{aligned}$$

$$+\left(\frac{1-e^{-a(T_2-T_1)}}{a}\right)^2 \frac{\sigma^2}{4a}\left[1-e^{-2a(T_1-t)}\right]$$
$$+\left(\frac{1-e^{-b(T_2-T_1)}}{b}\right)^2 \frac{\eta^2}{4b}\left[1-e^{-2b(T_1-t)}\right]$$
$$+\frac{(1-e^{-a(T_2-T_1)})(1-e^{-b(T_2-T_1)})}{ab}$$
$$\left.\cdot\rho\frac{\sigma\eta}{a+b}\left[1-e^{-(a+b)(T_1-t)}\right]\right\}.$$

By substituting this algebraic formula in (10.14) one has the value of the Eurodollar-futures contract. Notice that if this is evaluated at time 0, since $x_0 = y_0 = t = 0$ the above formula simplifies a little. Typically $X = 100$ and $\tau = 0.25$.

We have then considered a set of parameters coming from a typical calibration of the G2++ model to swaptions volatilities and to the zero-coupon curve of the Euro market. The values of these parameters are: $a = 0.0234$; $b = 0.0015$; $\sigma = 0.0081429$; $\eta = 0.0020949$; $\rho = -0.2536$.

We have finally computed prices for increasing maturities T_1 (from three months to ten years), while keeping $T_2 = T_1 + 0.25$, and we considered the differences

$$\text{Spread}(T_1) := E_0[L(T_1, T_1 + 0.25)] - F(0; T_1, T_1 + 0.25)$$

as T_1 increases. Such differences, in basis points (hundredths of a percentage point), are shown in Figure 10.1 below.

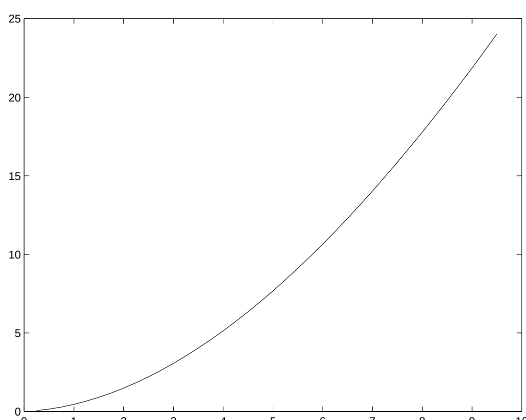

Fig. 10.1. Spread(T_1) in basis points plotted against T_1

The (upward concave) qualitative behaviour of the correction agrees with what is usually experienced in the market.

10.10.2 Eurodollar Futures with the LFM

Since we need to compute

$$E_t[L(S_1, S_2)] = E_t[F(S_1; S_1, S_2)],$$

we need the dynamics of the forward rate $F(\cdot; S_1, S_2)$ under the risk-neutral measure. This can be obtained starting from the martingale dynamics under the numeraire $P(\cdot, S_2)$ and moving to the bank-account numeraire via the "change of numeraire toolkit". This new dynamics involves the bond-price dynamics of $P(\cdot, S_2)$, which in turn can be expressed in terms of spanning forward rates. Therefore, in the relevant dynamics, correlation and volatilities of all spanning forward rates are involved. Subsequently, the forward-rates dynamics need be discretized and a Monte Carlo method can be applied to compute the relevant expectation under the risk-neutral measure.

In detail, assume we have a set of expiry/maturity dates $\{T_0, T_1, \dots, T_M\}$ for a family of spanning forward rates, with $T_{M-1} = S_1$ and $T_M = S_2$. As we explained in Section 6.3, the forward-rate dynamics under the risk-neutral measure is given by (6.16).

Here, we would need to model the instantaneous forward rate f in the initial interval $(t, T_{\beta(t)-1}]$ to close the equations, but if we discretize these equations (for the logarithm of F's) with a Milstein scheme *exactly at the time instants* $\{T_0, T_1, \dots, T_M\}$, we are in no need to model f. One sees easily that this is the same as discretizing the LFM dynamics (6.17) under the spot LIBOR measure whose numeraire is the discretely-rebalanced bank account. As usual, a Monte Carlo method, based on the jointly Gaussian distributions of the shocks for different components, can be applied to propagate all $F's$ up to time $T_{M-1} = S_1$ in order to evaluate the final expectation

$$E_0[F_M(T_{M-1})] = E_0[F(S_1; S_1, S_2)] \,.$$

Again, we can freeze part of the drift in the Spot-LIBOR-measure dynamics (6.17) thus obtaining

$$dF_k(t) = \sigma_k(t)\, F_k(t) \sum_{j=\beta(t)}^{k} \frac{\tau_j\, \rho_{j,k}\, \sigma_j(t)\, F_j(0)}{1 + \tau_j F_j(0)}\, dt + \sigma_k(t)\, F_k(t)\, dZ_k^d(t)\,,$$

which is a geometric Brownian motion. Under this dynamics, the above expected value is easily computed in terms of the now deterministic percentage drift.

10.11 LFM Pricing with "In-Between" Spot Rates

Let us assume the current time to be $t = 0$, and let us denote by $\mathcal{T} = \{T_0, \dots, T_n\}$ a set of payment dates, at which coupons of a certain financial instrument are to be paid. Such dates are assumed to be equally δ-spaced for simplicity. We also denote by $\mathcal{G} = \{g_1, \ldots, g_m\}$ the set of future dates at which a reference rate (typically the six-month LIBOR rate) is quoted in the market up to time T_n.

We denote by $L(t)$ the relevant reference rate at time t with maturity $t + \delta$. Forward-rate dynamics will be considered under the forward-adjusted measure Q^n corresponding to the final payment time T_n.

Consider a financial product whose payoff depends on the (for example daily) evolution of the reference rate L in each reset/payment interval.

In order to Monte Carlo price this product based on the forward-rate dynamics of the LFM, we need to recover at any time t the reference rate, which we assume to be the δ spot rate $L(t) = F(t; t, t + \delta)$, from the family of spanning forward rates at our disposal at times $T_{\beta(t)-2}$ and $T_{\beta(t)-1}$, i.e. at the dates in $\mathcal{T}$ that are closest to the current time t: $T_{\beta(t)-2} < t \leq T_{\beta(t)-1}$. In particular, we have both

$$L(T_{\beta(t)-2}) = F_{\beta(t)-1}(T_{\beta(t)-2})$$

and

$$L(T_{\beta(t)-1}) = F_{\beta(t)}(T_{\beta(t)-1}) \ .$$

How do we obtain $L(t)$ from $L(T_{\beta(t)-2})$ and $L(T_{\beta(t)-1})$? We have faced this problem earlier in Sections 6.17.1 and 6.17.2, proposing both a "drift interpolation" and a "bridging" technique.

We now present some particular products depending on "in-between" rates, and we will tacitly assume that "in-between" rates have been obtained through one of these methods.

10.11.1 Accrual Swaps

We give a short description of accrual swaps in the following. We assume a unit nominal amount.

- Institution A pays to B (a percentage γ of) the reference rate L (plus a spread S) at dates $\mathcal{T}$. Formally, at time T_i institution A pays to B
$$(\gamma L(T_{i-1}) + S)\tau_i \ ,$$
where τ_i is the year fraction between the payment dates T_{i-1} and T_i.
- Institution B pays to A, at time T_i, a percentage α of the reference rate plus a spread Q, times the relative number of days between T_{i-1} and T_i where the reference rate L was in the corridor $L_1 \leq L \leq L_2$. Formally, at time T_i, institution B pays to A the coupon
$$c(T_i) = (\alpha L(T_{i-1}) + Q)\, \tau_i \, \frac{\sum_{g \in \mathcal{G} \cap [T_{i-1}, T_i)} 1\{L_1 \leq L(g) \leq L_2\}}{\#\{\mathcal{G} \bigcap [T_{i-1}, T_i)\}} \ , \quad (10.15)$$
where, as usual, # denotes the number of elements of a set (cardinality).

When the simulated paths for L are available, we are able to evaluate the accrual swap. The discounted payoff as seen from institution A is

$$\sum_{i=1}^{n} D(0,T_i)\tau_i \Bigg[(\alpha L(T_{i-1}) + Q) \frac{\sum_{g\in\mathcal{G}\cap[T_{i-1},T_i)} 1\{L_1 \le L(g) \le L_2\}}{\#\{\mathcal{G}\bigcap[T_{i-1},T_i)\}} - (\gamma L(T_{i-1}) + S) \Bigg],$$

so that the value to A of the accrual swap is the risk-neutral expectation

$$E\Bigg\{ \sum_{i=1}^{n} D(0,T_i)\tau_i \Bigg[(\alpha\ L(T_{i-1}) + Q)\ \frac{\sum_{g\in\mathcal{G}\cap[T_{i-1},T_i)} 1\{L_1 \le L(g) \le L_2\}}{\#\{\mathcal{G}\bigcap[T_{i-1},T_i)\}} -(\gamma L(T_{i-1}) + S) \Bigg] \Bigg\}$$

$$= P(0,T_n) \sum_{i=1}^{n} \tau_i E^n \Bigg\{ \frac{1}{P(T_i,T_n)} \cdot \Bigg[(\alpha\ L(T_{i-1}) + Q) \frac{\sum_{g\in\mathcal{G}\cap[T_{i-1},T_i)} 1\{L_1 \le L(g) \le L_2\}}{\#\{\mathcal{G}\bigcap[T_{i-1},T_i)\}} - (\gamma L(T_{i-1}) + S) \Bigg] \Bigg\}$$

Both the forward-rate dynamics and the related approximated L dynamics under Q^n are known and a Monte Carlo pricing can be carried out.

Analytical Formula for Accrual Swaps

Alternatively, we may study an analytical formula based on a drift approximation in the LIBOR market model, similar to the one used in deriving approximated swaptions volatilities and terminal correlations in Chapter 6. We proceed as follows.

We concentrate on the non-trivial leg, paid by B to A. Let us focus on the single discounted payment occurring at time T_i. It will suffice to add up all contributions after each one has been priced. We have seen above the payment at time T_i to be given by (10.15). Instead of expressing every coupon under the terminal measure, let us write

$$E_0[D(0,T_i)c(T_i)] = P(0,T_i)E_0^i[c(T_i)].$$

Our task is then reduced to computing the expected value $E_0^i[c(T_i)]$, which, in turn, amounts to computing, by additive decomposition, expected values such as

$$E_0^i[L(T_{i-1})\,1\{L_1 \le L(u) \le L_2\}], \quad E_0^i[1\{L_1 \le L(u) \le L_2\}]$$

under the T_i-forward-adjusted measure Q^i and for $T_{i-1} \le u < T_i$. These may be rewritten in terms of forward rates as

$$E_0^i[F_i(T_{i-1})\, 1\{L_1 \le F(u; u, u+\delta) \le L_2\}], \quad Q^i\{L_1 \le F(u; u, u+\delta) \le L_2\},$$

where we have expressed the expected value of an indicator function directly as a probability.

Now, in order to handle such expressions we consider approximated forward-rate dynamics. Actually, no approximation is needed for F_i under Q^i, since its drift is zero and we have a nice geometric Brownian motion. Instead, we act on the dynamics of $F(t; u, u+\delta)$. Since $F(\cdot; u, u+\delta)$ is not in our fundamental family of forward rates, we use the drift-interpolation technique seen in Section 6.17.1. If we set $F_u(t) = F(t; u, u+\delta)$ for brevity, by applying formula (6.69), with partially frozen coefficients and a few rearrangements, we obtain (notice that $T_i \le u + \delta < T_{i+1}$)

$$\begin{aligned} dF_u(t) &= \mu(t) F_u(t)\, dt + \sigma(t; u, u+\delta) F_u(t)\, dZ^i(t), \\ \mu(t) &:= \frac{(u + \delta - T_i) F_{i+1}(0)}{1 + \tau_{i+1} F_{i+1}(0)}\, \sigma(t; u, u+\delta) \rho_{i,i+1} \sigma_{i+1}(t) \end{aligned} \qquad (10.16)$$

where $\sigma(t; u, u+\delta)$ is the instantaneous volatility of the related forward rate, and is usually obtained by some kind of interpolation from the "standard rates" volatilities σ_k's.

Our approximation has produced a fundamental effect. The process (10.16) is now a geometric Brownian motion, and we can apply a standard "Black and Scholes technology" to our pricing problem.

Let us recall the following Black and Scholes fundamental setup. Assume we are given two asset prices following correlated geometric Brownian motions under the relevant measure,

$$\begin{aligned} dS_t &= \mu_1(t) S_t\, dt + v_1(t) S_t\, dZ_1(t), \\ dA_t &= \mu_2(t) A_t\, dt + v_2(t) A_t\, dZ_2(t), \quad dZ_1\, dZ_2 = \rho\, dt \end{aligned}$$

(all coefficients being deterministic). We can easily calculate, through laborious but straightforward computations, for $T < u$,

$$\begin{aligned} E_0[S(T)\, 1\{L_1 \le A(u) \le L_2\}] &= S(0) \exp\left(\int_0^T \mu_1(s) ds \right) \\ \cdot \Bigg[\Phi\left(\frac{\ln(L_2/A(0)) - \int_0^u \left(\mu_2(s) - \frac{v_2(s)^2}{2}\right) ds - \rho \int_0^T v_1(s) v_2(s) ds}{\sqrt{\int_0^u v_2(s)^2 ds}} \right) \\ - \Phi\left(\frac{\ln(L_1/A(0)) - \int_0^u \left(\mu_2(s) - \frac{v_2(s)^2}{2}\right) ds - \rho \int_0^T v_1(s) v_2(s) ds}{\sqrt{\int_0^u v_2(s)^2 ds}} \right) \Bigg]. \end{aligned}$$

We can apply this formula to our case, by setting $T = T_{i-1}$, $S(t) = F_i(t)$, $A(t) = F_u(t)$, $\mu_1(t) = 0$, $\mu_2(t) = \mu(t)$, $v_1(t) = \sigma_{i+1}(t)$, $v_2(t) = \sigma(t; u, u+\delta)$, $\rho = \rho_{i,i+1}$. This provides us with the terms

$$E_0^i[F_i(T_{i-1})\, 1\{L_1 \leq F(u; u, u+\delta) \leq L_2\}],$$

where the impact of correlation is evident. On the other hand, we may compute

$$E_0[1\{L_1 \leq A(u) \leq L_2\}] = \left[\Phi\left(\frac{\ln(L_2/A(0)) - \int_0^u \left(\mu_2(s) - \frac{v_2(s)^2}{2}\right)ds}{\sqrt{\int_0^u v_2(s)^2 ds}}\right) - \Phi\left(\frac{\ln(L_1/A(0)) - \int_0^u \left(\mu_2(s) - \frac{v_2(s)^2}{2}\right)ds}{\sqrt{\int_0^u v_2(s)^2 ds}}\right)\right],$$

from which terms

$$Q^i\{L_1 \leq F(u; u, u+\delta) \leq L_2\}$$

are readily computed. Now, putting all the pieces together, we obtain the accrual-swap price. The "frozen drift" approximation guarantees us that for short maturities this formula should work well. However, we have seen that the drift "freezing approximation" above usually does not take us far away from the lognormal distribution even for large maturities, as we have observed in the density plots given in Chapter 8. Problems might only occur with pathological or very large volatilities.

Finally, we would like to point out that the "freezing part of the drift" method can usually be used to transform the distributionally unknown LFM dynamics into the geometric-Brownian-motion dynamics of the basic Black and Scholes lognormal setup. As a consequence, this method can be of help in all cases where forward rates play the role of underlying assets under the relevant measure and where the basic Black and Scholes setup leads to analytical formulas. Before adopting the thus derived approximated formulas, however, one should test them against a Monte Carlo pricing, carried out through the true LFM dynamics, in a sufficiently large number of market situations.

10.11.2 Trigger Swaps

A trigger swap is an interest-rate swap periodically paying a certain *reference* rate against a fixed payment. This swap "comes to life" or "terminates" when a certain *index* rate hits a prespecified level. It is somehow similar to barrier options in the FX or equity markets. Usually, the two rates coincide, but the index rate is observed at a higher frequency than the payment frequency. For example, the index rate and the reference rate can both coincide with the six-month LIBOR rate, which can be observed daily for the indexing and every six months for the payments.

There are four standard basic types of trigger swaps: Down and Out (DO), Up and Out (UO), Down and In (DI), Up and In (UI). Let the prespecified level be H.

- DO: The initial index rate is above H. The swap terminates its payments ("goes OUT") as soon as the index rate hits the level H (from above, i.e. going "DOWN").
- UO: The initial index rate is below H. The swap terminates its payments ("goes OUT") as soon as the index rate hits the level H (from below, i.e. going "UP").
- DI: The initial index rate is above H. The swap starts its payments ("goes IN") as soon as the index rate hits the level H (from above, i.e. going "DOWN").
- UI: The initial index rate is below H. The swap starts its payments ("goes IN") as soon as the index rate hits the level H (from below, i.e. going "UP").

The payoff from a DO trigger swap can be expressed formally as follows. As for accrual swaps, we assume the current time to be $t = 0$, and we denote by $\mathcal{T} = \{T_0, \ldots, T_n\}$ a set of payment dates, at which payments occur. Such dates are assumed to be equally δ-spaced. We also denote by $\mathcal{G} = \{g_1, \ldots, g_m\}$ the set of future dates at which the reference rate (typically the six-month LIBOR rate) is quoted in the market up to time T_n.

We assume the index rate and reference rate to coincide. We denote by $L(t)$ the reference rate at the generic time instant t with maturity $t + \delta$. Forward-rate dynamics will be considered under the forward-adjusted measure Q^n corresponding to the final payment time T_n.

We assume unit nominal amount. If the swap is still alive at time $t = T_{i-1}$, then at time T_i the following will occur:

- Institution A pays to B the fixed rate K at time T_i *if* at all previous instants in the interval $(T_{i-1}, T_i]$ the index rate L is above the triggering barrier H. Formally, if the swap is still alive at time T_{i-1}, at time T_i institution A pays to B
$$\begin{aligned} &K\tau_i \prod_{g \in \mathcal{G} \cap (T_{i-1}, T_i]} 1\{L(g) > H\} \\ &= K\tau_i \, 1\Big\{ \min\{L(g),\ g \in \mathcal{G} \cap (T_{i-1}, T_i]\} > H \Big\} , \end{aligned}$$
where τ_i is the year fraction between the payment dates T_{i-1} and T_i.
- Institution B pays to A (a percentage α of) the reference rate L at the last reset date T_{i-1} (plus a spread Q) *if* at all previous instants of the interval $(T_{i-1}, T_i]$ the index rate L is above the triggering barrier H. Formally, at time T_i institution B pays to A

$$(\alpha L(T_{i-1}) + Q)\, \tau_i\, 1\{\min\{L(g),\ g \in \mathcal{G} \cap (T_{i-1}, T_i]\} > H\}$$

The complete discounted payoff as seen from institution A can be expressed as

$$\sum_{i=1}^{n} D(0,T_i)\, (\alpha L(T_{i-1}) + Q - K)\ \tau_i\, 1\Big\{ \min\{L(g),\ g \in \mathcal{G} \cap (T_0, T_i]\} > H \Big\}$$

and the contract value to institution A is

$$E\left[\sum_{i=1}^{n} D(0,T_i)\, (\alpha L(T_{i-1}) + Q - K)\ \tau_i\, 1\Big\{ \min\{L(g),\ g \in \mathcal{G} \cap (T_0, T_i]\} > H \Big\} \right]$$
$$= P(0,T_n) \sum_{i=1}^{n} \tau_i\, E^n \left[\frac{(\alpha L(T_{i-1}) + Q - K)\ 1\Big\{ \min\{L(g),\ g \in \ldots\} > H \Big\}}{P(T_i, T_n)} \right].$$

Once again, it is enough to recover spot rates $L(T_i) = F_{i+1}(T_i)$ and discount factors $P(T_i, T_n)$ by generating for all i's spanning forward rates

$$F_{i+1}(T_i), F_{i+2}(T_i), \ldots, F_n(T_i)$$

under Q^n according to the usual discretized (Milstein) dynamics (analogously to the autocaps case (10.5)), and apply either the "drift interpolation" or the "bridging" technique of Sections 6.17.1 and 6.17.2 to recover in-between rates $L(g)$.

10.12 LFM Pricing with Early Exercise and Possible Path Dependence

Here we shortly present Longstaff and Schwartz's (2000) method for pricing early-exercise (and possibly path-dependent) products through Monte Carlo simulation in the LFM. Indeed, what we will present here can be intended as a solution of the following two different and yet related problems.

1. How can we use Monte Carlo for early-exercise (non path-dependent) products? This can be necessary when in presence of non-Markovian dynamics or of large dimensionality of the underlying process, as we shall see in a moment.
2. How can we price derivatives that show at the same time path dependence and early-exercise features, even in the favorable cases of low dimensionality and Markovian dynamics?

In Chapter 3 we observed that the pricing of early-exercise products can be carried out through binomial/trinomial trees, and that Monte Carlo is instead suited to treat path-dependent products. Here, before proposing a recent promising extension of the Monte Carlo method, we shortly recall what was already remarked there, in the beginning of Sections 3.11.2 and 3.11.3.

Trees can be used for early-exercise products when the fundamental underlying variable is low-dimensional (say one or two-dimensional), as happens typically with short-rate models. In such cases, the tree is the ideal instrument, given its "backward-in-time" nature. We know the value of the payoff in each final node, and move backward in time, by updating the value of continuation through discounting. At each node of the tree we can compare the backwardly propagated value of continuation with the payoff evaluated at that node, and decide whether exercise is to be considered or not at that point. After the exercise decision has been taken, the backward induction restarts and we continue to propagate backwards the updated value. When we reach the initial node of the tree, at time 0, we have (an approximation of) the price of our early-exercise product. Thus trees are ideally suited to "travel backward in time".

The other family of products that is usually considered is the family of "path-dependent" payoffs. Such products can be exercised only at a final date, but their final payoffs depend on the history of the underlying variable up to the final time, and not only on the value of the underlying variable at maturity. For path-dependent products, the Monte Carlo method is ideally suited, since it works through forward propagation in time of the underlying variable, by simulating its transition density between dates where the underlying-variable history matters to the final payoff. Monte Carlo is thus ideally suited to "travel forward in time".

In principle, trees have problems mainly in two situations. The first case concerns high dimensionality. If the underlying variable follows a high-dimensional process (in practice with dimension larger than two or three, as in case of the LFM, for example), the tree is practically impossible to consider, since the computational time grows roughly exponentially with the dimension. Moreover, there are also difficulties in handling correlations and other aspects, so that trees become extremely difficult to use.

The second case where trees have major problems is with path-dependent products. When we try and propagate backwards the contract value from the final nodes we are immediately in trouble, since to value the payoff at a given node (and at any final node in particular) we need to know the past history of the underlying variable. But this past history is not determined yet, since we move backward in time. This method, therefore, is not applicable in a standard way.

Actually, there are ad-hoc procedures to render trees able to price particular path-dependent products in the basic Black and Scholes setting, for example barrier and lookback options. However, in general there is no consolidated and realistic recipe on how using a tree for path-dependent payoffs, and moreover, when dealing with interest-rate derivatives models, we are usually outside the Black and Scholes framework.

As for the Monte Carlo method, it does better with respect to high dimensionality, in that computational time grows roughly *linearly* with the dimension, and it is also suited to parallel computing. However, there are problems with early exercise. Since we propagate trajectories forward in time, we have no means to know whether it is optimal to continue or to exercise at a certain time. Therefore, Monte Carlo cannot be used, in its original formulation, for the large range of products involving early exercise features.

However, Longstaff and Schwartz (2000) have proposed an approximated method to make Monte Carlo techniques work also in presence of early-exercise features. The resulting method is very promising, since it allows for the pricing of instruments with high-dimensional underlying variables, path dependence and early exercise at the same time.

Clearly, the method needs further testing beyond what shown in Longstaff and Schwartz (2000), especially on practical cases concerning interest-rate models. Still, test results in Longstaff and Schwartz (2000) look rather encouraging so as to justify a general exposition of the method and of its possible developments even before extensive testing has been carried out. This generality, together with the potential of the method of not exceeding the true value of the early-exercise contract, could result in a supremacy of Monte Carlo over trees and finite-difference methods in general, especially if numerically efficient Monte Carlo methods are brought into play.

Recently, research on improvements of the basic Monte Carlo setup, based on weighted paths and other techniques have received considerable attention in the literature, thus further strengthening the interest in the Longstaff and Schwartz (2000) method.

We now review this method for a generic product whose final payoff depends on a (possibly multi-dimensional) underlying variable X.

Assume we have a product that can be exercised at times $t_1, \ldots, t_N$, whose immediate-exercise value at each time t_k depends on part of the history of an underlying process $X(t)$ up to time t_k itself. Typically, the value can depend on $X(s_1), \ldots, X(s_{j_k})$, where the times $s_1 < s_2 < \ldots < s_{j_k} \leq t_k$ are the ones contributing to the immediate-exercise payoff at time t_k. In detail, we assume that, if exercised at time t_k, the product pays immediately the Cash flow from Exercise (CE) given by

$$\mathrm{CE}(t_k) := \mathrm{CE}(t_k; X(s_1), \ldots, X(s_{j_k})).$$

This value has to be compared with the backwardly Cumulated discounted cash flows from Continuation (CC) at the same time, namely the value of the contract at t_k when this has not been exercised before or at t_k itself,

$$\mathrm{CC}(t_k) := \mathrm{CC}(t_k; X(s_1), \ldots, X(s_{j_k})).$$

We assume we are computing prices under a generic numeraire asset $U(t)$. In their paper, Longstaff and Schwartz (2000) take the bank account as fundamental numeraire, and work under the risk-neutral measure.

The method can be summarized through the following scheme:

1. Choose a number of paths, np.
2. (Choice of the basis functions). For each time t_k, choose i_k basis functions

$$\phi_1(t_k, x_1, \ldots, x_{j_k}), \ldots, \phi_{i_k}(t_k, x_1, \ldots, x_{j_k})$$

   that will be used in approximating the continuation value as a function of the past and present values $X(s_1), X(s_2), \ldots, X(s_{j_k})$ of the underlying variable up to time t_k (see step 7 below).
3. (Simulating the underlying variables). Simulate np paths for both the underlying variable X and the numeraire U from time t_1 to time t_n. Make sure of including the reset times $s_1, \ldots, s_{j_n}$ among the dates at which X and U are simulated. Typically, this simulation is "exact" if the transition distributions of X and U are known, like, for example, in the case of geometric Brownian motion or linear-Gaussian processes as in Hull and White's models. Alternatively, a numerical discretization scheme for SDEs such as the Euler or Milstein schemes can be employed if this transition density is not known. In any case, denote by

$$X^j(t_k), U^j(t_k)$$

   the simulated values of X and U respectively under the j-th scenario at time t_k. More generally, the superscript on a stochastic quantity will denote the quantity itself under the scenario given by the superscript index.
4. (Computing the payoff at final time). Set

$$\mathrm{CC}^j(t_n) := \mathrm{CE}(t_n; X^j(s_1), \ldots, X^j(s_{j_n})).$$

   (The backwardly Cumulated discounted cash flow from Continuation at final time is simply the exercise value at that time).
5. (Positioning the initial step at final time). Set $k = n$. We position ourselves at the final exercise time. Now the iterative part of the scheme begins.
6. (Consider only scenarios where the immediate-exercise value of the contract is positive). Set

$$I_{k-1} := \{j \in \{1, 2, \ldots, np\} : \mathrm{CE}(t_{k-1}; X^j(s_1), \ldots, X^j(s_{j_{k-1}})) > 0\}.$$

   We thus focus only on scenarios where the exercise value is strictly positive at the current evaluation time t_{k-1};
7. (Regressing the discounted continuation value on the chosen basis functions). In this step, we aim at approximating the discounted continuation value at current time t_{k-1} as a linear combination of the basis functions

$$\phi_1(t_{k-1}, x_1, \ldots, x_{j_{k-1}}), \ldots, \phi_{i_{k-1}}(t_{k-1}, x_1, \ldots, x_{j_{k-1}})$$

through a regression, so as to estimate the combinators λ in

$$\frac{U^j(t_{k-1})}{U^j(t_k)}\mathrm{CC}^j(t_k) = \sum_{h=1}^{i_{k-1}} \lambda_h(t_{k-1})\ \phi_h(t_{k-1}, X^j(s_1), \ldots, X^j(s_{j_{k-1}})),$$

where $j \in I_{k-1}$.
On the left-hand side of the above equation, we have the continuation value an instant later discounted back at current time t_{k-1} through the chosen numeraire U. Notice that if the numeraire is the bank account $B(t) = \exp(rt)$, with deterministic constant r, as in Longstaff and Schwartz (2000), then the U's ratio reduces to $\exp(-r(t_k - t_{k-1}))$.
On the right-hand side of the same equation, we have a linear combination of the chosen basis functions, corresponding ideally to a truncated L^2 expansion. The step could be made exact with an infinite expansion ($i_{k-1} = \infty$), when the conditional expectation defining the actual continuation value above behaves nicely in an L^2 sense. See Longstaff and Schwartz (2000) for further details.

8. Store the exercise flag (EF) over scenarios at time t_{k-1}:

$$\mathrm{EF}(j, t_{k-1}) := 1\Big\{\mathrm{CE}(t_{k-1}; X^j(s_1), \ldots, X^j(s_{j_{k-1}})) > \sum_{h=1}^{i_{k-1}} \lambda_h(t_{k-1})\ \phi_h(t_{k-1}, X^j(s_1), \ldots, X^j(s_{j_{k-1}}))\Big\}.$$

This flag is set to one when exercise is the convenient choice, and to zero when continuation is in order. Again, $1\{\cdots\}$ denotes the indicator function of the set between curly brackets.
When $\mathrm{EF}(j, t_{k-1})$ is one, set all its subsequent values to zero, $\mathrm{EF}(j, t_h) := 0$ for all $h > k - 1$.

9. Set

$$\mathrm{CC}^j(t_{k-1}) := \frac{U^j(t_{k-1})}{U^j(t_k)}\mathrm{CC}^j(t_k) \quad \text{if} \quad \mathrm{EF}(j, t_{k-1}) = 0 \ \ \text{(continuation)},$$

and set

$$\mathrm{CC}^j(t_{k-1}) := \mathrm{CE}(t_{k-1}; X^j(s_1), \ldots, X^j(s_{j_{k-1}})) \ \text{ if } \ \mathrm{EF}(j, t_{k-1}) = 1 \ \ \text{(exercise)}$$

10. If $k = 0$ stop, otherwise replace k with $k - 1$ and restart from point 6.

10.13 LFM: Pricing Bermudan Swaptions

Bermudan swaptions are options to enter an IRS not only at its first reset date, but also at subsequent reset dates of the underlying IRS, at least in some of the simplest formulations.

Let again $\mathcal{T} = \{T_1, \dots, T_n\}$ be a set of reset and payment dates. Recall that we denote by $\mathbf{PS}(T_i, T_k, \{T_k, \dots, T_n\}, K)$ the price at time T_i of a (payer) swaption maturing at time T_k, which gives its holder the right to enter at time T_k an interest-rate swap with first reset date T_k and payment dates $T_{k+1}, \dots, T_n$ at the fixed strike rate K. We will abbreviate this price by $\mathbf{PS}_{k,n}(T_i)$. This price is known as a function of the present value for basis point $C_{k,n}(T_i)$ and of the forward swap rate $S_{k,n}(T_i)$ through Black's formula for swaptions.

Definition 10.13.1. (Bermudan Swaption). *A (payer) Bermudan swaption is a swaption characterized by three dates $T_k < T_h < T_n$, giving its holder the right to enter* at any time T_l in-between T_k and T_h *(included) into an interest-rate swap with first reset in T_l, last payment in T_n and fixed rate K. Thus, the swap start and length depend on the instant T_l when the option is exercised. We denote by $\mathbf{PBS}_{k,h,n}(T_i)$ the value of such a Bermudan swaption at time T_i, with $T_i \leq T_k$.*[1]

Pricing Bermudan swaptions with the LFM has to be handled through tailor-made methods, since the model is not ideally suited for the implementation of recombining lattices. A possible alternative to the tailor-made techniques is the general Longstaff Schwartz "Monte Carlo Regression" (LSMC) approach reviewed in Section (10.12), which is indeed quite general and usually results in good approximations. However, the method itself has to be tailored (choice of the basis functions, ...) when applied to Bermudan swaptions in the LFM.

10.13.1 Longstaff and Schwartz's Approach

As we just noticed, the LSMC method can be used to price Bermudan swaptions in the LFM. Longstaff and Schwartz (2000), however, tested the LSMC method (in the section "valuing swaptions in a string model" of their paper) by actually considering a version of the so called string model. In practice, when working in a finite set of expiries/maturities, string models are often equivalent to the LIBOR market model (LFM). For more details on string models, see for example Santa Clara and Sornette (2001) or Longstaff, Santa Clara and Schwartz (2001). In the specific application of string models we are considering here, Longstaff and Schwartz (2000) used directly bond-prices dynamics and bond-prices volatilities instead of forward-rates volatilities. For completeness, we here illustrate their procedure.

A Bermudan swaption is considered, where the underlying swap starts at the initial time with given reset and payment dates. The swaption's holder has the right to exercise the option at some fixed dates and enter the swap, whose life span decreases as time moves forward.

[1] There are other types of Bermudan swaptions, but for our purposes the type described here suffices.

The underlying swap, with a ten-year maturity, resets semi-annually, and exercise can occur at any reset date after one year, one year included and ten years excluded. There are therefore nineteen exercise dates. In propagating the zero-coupon-bond prices,

$$P(\cdot,\ 0.5y), P(\cdot, 1y), P(\cdot,\ 1.5y), \ldots, P(\cdot, 10y),$$

the LSMC method starts from the twenty-dimensional vector above, and the dimension decreases by one each six months. The bond-price dynamics has as percentage risk-neutral drift the risk-free rate, which is approximated with the corresponding six-month continuously-compounded rate

$$r(t) \approx -2 \ln P(t, t + 0.5y),$$

thus closing the set of equations in P for the discretized approximate dynamics once the volatility has been assigned. Indeed, the approximate dynamics reads now

$$dP(t, T_i) = -2 \ln P(t, t + 0.5y) P(t, T_i) dt + \sigma_{P_i}(t) P(t, T_i) dZ_i(t), \quad T_i = 0.5i,$$

$i = 1, \ldots, 20$. When these equations are discretized at times T_i, we obtain a closed set of equations, since the drift rate now involves a bond price in the family.

The Z_i's are correlated Brownian motions under the risk-neutral measure. In the simulation it is assumed that

$$dZ_i dZ_j = \exp(-k|i - j|) dt,$$

where k is a positive constant, and the Z vector is kept twenty-dimensional.

At the exercise time T_i, the basis functions of the algorithm are selected as:

$$1, P(\cdot, T_i), \ldots, P(\cdot, T_{20}), \frac{1 - P(T_i, T_{20})}{\sum_{j=i+1}^{20} 0.5 P(T_i, T_j)}, \left[\frac{1 - P(T_i, T_{20})}{\sum_{j=i+1}^{20} 0.5 P(T_i, T_j)}\right]^2,$$

$$\left[\frac{1 - P(T_i, T_{20})}{\sum_{j=i+1}^{20} 0.5 P(T_i, T_j)}\right]^3,$$

where the last three terms are simply the underlying swap rate $S_{i,20}(T_i)$ and its second and third powers.

At the first exercise time ($i = 3$), there are 22 basis functions, their number decreasing as time goes by. Longstaff and Schwartz state that adding further functions does not change the option value, so that one can infer the valuation to be correct, given that the approximated value never exceeds the real value. See also the related discussion in Longstaff and Schwartz (2000).

Notice that Longstaff and Schwartz have assumed a deterministic bond-price percentage volatility. This is not really consistent with the LFM distribution for lognormal forward rates. Therefore, as already mentioned above, the model analyzed in this section is not a LFM, from a theoretical point of view.

10.13.2 Carr and Yang's Approach

Carr and Yang (1997) use simulations to develop a Markov-chain approximation for the valuation of Bermudan swaptions in the LFM. Their method stems from the observation that, given the tenor structure

$$T_1, \ldots, T_n,$$

one can represent the whole yield curve along the structure by just knowing the evolution of a chosen numeraire. Take for example the numeraire $P(\cdot, T_n)$, associated with the terminal measure Q^n. At a time T_i in the tenor structure, the whole (Zero-bond) curve

$$P(T_i, T_{i+1}), P(T_i, T_{i+2}), \ldots, P(T_i, T_n)$$

can be obtained as follows. Recall that by definition of numeraire we have

$$\frac{P(T_i, T_j)}{P(T_i, T_n)} = E^n_{T_i}\left[\frac{P(T_j, T_j)}{P(T_j, T_n)}\right],$$

or

$$P(T_i, T_j) = P(T_i, T_n)\, E^n_{T_i}\left[\frac{1}{P(T_j, T_n)}\right], \tag{10.17}$$

so that we can compute each $P(T_i, T_j)$ by knowing the current value of the numeraire $P(\cdot, T_n)$ and its distribution under its own measure Q^n. The exercise decision, at any instant, can thus be reduced to knowledge of the distributional properties of the single process $P(\cdot, T_n)$.

Based on this observation, Carr and Young (1997) found a way to construct a Markov chain approximating the migration of $P(\cdot, T_n)$ in between areas of a selected partition of $[0, 1]$. Partitioning $[0, 1]$ in $I_1(t), I_2(t), \ldots, I_{l(t)}(t)$, so that $[0, 1]$ is given by the disjoint union of the sets I, the Markov chain is constructed as follows.

First, simulate spanning forward-rate dynamics $F_{i+1}(t)^j, \ldots, F_n(t)^j$ under several scenarios, each scenario denoted by a superscript j, up to a generic time $t = T_i$. Second, obtain the numeraire bond price $P(t, T_n)^j$ from these simulations under each scenario j. Third, define the transition matrix between "state" h at time $t = T_i$ and "state" k at time $t + \Delta = T_{i+1}$ as

$$p_{h,k}(t) := \frac{\#\{j:\ P(t, T_n)^j \in I_h(t) \ \text{ and } \ P(t + \Delta, T_n)^j \in I_k(t)\}}{\#\{j:\ P(t, T_n)^j \in I_h(t)\}}.$$

Then one defines $\bar{P}_h(t, T_n)$ as the average of the $P(t, T_n)^j$'s in $I_h(t)$,

$$\bar{P}_h(t, T_n) := \frac{\sum_{j:\ P(t,T_n)^j \in I_h(t)} P(t, T_n)^j}{\#\{j:\ P(t, T_n)^j \in I_h(t)\}}.$$

Consider the chain $X(t)$ with states $\{1, 2, \ldots, l(t)\}$ and probability $p_{h,k}(t)$ of going from $X(t) = h$ to $X(t + \Delta) = k$. Our $\bar{P}_h(t, T_n)$ can be considered as a discrete-space approximation of the numeraire $P(t, T_n)$ when $X(t) = h$.

The chain X summarizes the true dynamics of $P(t, T_n)$ into a Markov process that can be used for approximately simulating $P(t, T_n)$. We can therefore simulate the whole yield curve in the spirit of the relationship (10.17). Then backward induction becomes possible by using the Markov chain instead of the original paths for the numeraire.

We move backwards in time by means of the process $\bar{P}_{X(t)}(t, T_n)$ in place of the process $P(t, T_n)$, with $\bar{P}_{X(t)}(t, T_n)$ that assumes only a finite set of possible values at each instant. The transition probabilities allow us to roll back the relevant expectations and the Bermudan swaption can be easily priced through backward induction, see Carr and Yang (1997) for the details and for numerical tests.

A similar approach has been suggested in Clewlow and Strickland (1998) for a Gaussian multi-factor Heath-Jarrow-Morton model (and not the LFM), where again the early-exercise opportunity is evaluated in terms of a single variable. This variable is taken to be the fixed leg of the underlying interest-rate swap. Since the floating leg is always valued on par at reset dates, this choice amounts roughly to considering the value of the underlying interest-rate swap as fundamental single process at the reset dates.

The approximate specification of the early-exercise region as a function of the underlying variable is found by using a single-factor extended Vasicek (Hull and White) approximation of the multi-factor model. With the one-factor model one obtains the approximate early-exercise region via a recombining tree for the short rate, by determining the critical values of the underlying interest-rate swap at the early-exercise dates through backward induction on the tree. Choosing only one factor allows for a richer discretization in time and this yields an accurate exercise region.

Once the exercise decision has been estimated as a function of the underlying swap through the tree, one runs a Monte Carlo simulation for the original multi-factor model, where each early-exercise opportunity, when encountered, is evaluated as the (known) approximate function of the underlying swap.

This method seems to be robust. It provides one with a lower bound for the Bermudan swaption price, due to the sub-optimal exercise region, as in the LSMC method. A similar method for the LFM has been proposed by Andersen (1999), and we review it in the following.

10.13.3 Andersen's Approach

Andersen (1999) proposed a method similar to that of Clewlow and Strickland (1998). Again, the early-exercise region is extracted by a low-dimensional parameterization, consisting of a small number of key variables (these including the underlying interest-rate swap as in Clewlow and Strickland), but the approximated early-exercise region, as a function of these variables, is not determined through a *one-factor* model. Rather, an optimization on a separate simulation *for the whole multi-factor model* is considered in order to determine this function. The method can be summarized as follows. We adopt the notation introduced earlier in Definition 10.13.1.

We now provide a scheme summarizing a possible formulation of Andersen's method for approximately computing $\mathbf{PBS}_{k,h,n}(T_k)$.

1) Choose a function f approximating for each T_l the optimal exercise flag $\mathcal{I}(T_l)$, depending for example on the nested European swaptions and on a function $H = H(T_l)$ to be determined,

$$\mathcal{I}(T_l) \approx f(\mathbf{PS}_{l,n}(T_l), \mathbf{PS}_{l+1,n}(T_l), \ldots, \mathbf{PS}_{h,n}(T_l), H(T_l)).$$

The optimal exercise flag $\mathcal{I}(T_l)(\omega)$ at time T_l, under the path ω, is defined to be one when exercise is optimal at T_l along the trajectory ω and 0 when the continuation value at T_l is larger than the exercise value along ω. As usual, ω is omitted in the notation.

2) Simulate, through the LFM dynamics for the forward LIBOR rates, in a set of scenarios indexed by j, all the variables

$$\mathbf{PS}^j_{l,n}(T_l),\ \mathbf{PS}^j_{l+1,n}(T_l), \ldots,\ \mathbf{PS}^j_{h,n}(T_l),\ B^j_d(T_l)$$

entering in f's expression above, for all $l = k, k+1, \ldots, h$. The last quantity is the discrete-bank-account numeraire that is used for discounting, i.e.

$$B_d(T_l) = \prod_{m=1}^{l} [1 + \tau_m F_m(T_{m-1})],$$

which is determined by the simulated forward-rate dynamics of the LFM, with T_0 denoting the initial time. Notice that the first variable $\mathbf{PS}^j_{l,n}(T_l)$ involves the interest-rate swap whose swap rate is $S_{l,n}(T_l)$, which was the (unique) "early-exercise flag" variable in the Clewlow and Strickland method.

3) Compute by backward induction all values of $H(T_l)$ from $T_l = T_h$ to $T_l = T_k$ as follows:

- 3.a) The final $H(T_h)$ has to be known from the requirement

$$f(\mathbf{PS}_{h,n}(T_h), H(T_h)) = 1\{\mathbf{PS}_{h,n}(T_h) > 0\}.$$

This is to say that at the last possible exercise date we simply exercise if the underlying European swaption has strictly positive value, as should be. Set $m = h$.

- 3.b) Find $H(T_{m-1})$ as follows. For each simulated path j, solve the optimization problem

$$\begin{aligned} H^j(T_{m-1}) = \arg\sup_H \Big\{ & f(\mathbf{PS}^j_{m-1,n}(T_{m-1}), \mathbf{PS}^j_{m,n}(T_{m-1}), \ldots \\ & \ldots, \mathbf{PS}^j_{h,n}(T_{m-1}), H)\ \mathbf{PS}^j_{m-1,n}(T_{m-1}) \\ & + \frac{B^j_d(T_{m-1})}{B^j_d(T_m)} (1-f)\ \mathbf{PBS}^j_{m,h,n}(T_m) \Big\}, \end{aligned}$$

where we omit f's arguments in the second half of the expression for brevity, and where the expression between curly brackets basically reads as:
if (exercise(H)) then (current underlying European swaption)
else (present value of one-period ahead Bermudan swaption).
We thus look for the value of H in the exercise strategy that maximizes the option value in each scenario.
Notice also that we can write

$$\frac{B_d(T_{m-1})}{B_d(T_m)} = \frac{1}{1 + \tau_m F_m(T_{m-1})}.$$

Let $\mathbf{PBS}^j_{m-1,h,n}(T_{m-1})$ be the supremum corresponding to the above $H^j(T_{m-1})$.
Average over all scenarios j and find $H(T_{m-1})$ from the $H^j(T_{m-1})$'s.

- 3.c) If $m-1$ equals k then move to point 4), otherwise decrease m by one and restart from point 3.b).

4) Now that H is known at all times, compute the Bermudan-swaption price $\mathbf{PBS}_{k,h,n}(T_k)$ through a new simulation with a larger number of paths and with the approximated exercise function given by f.

Andersen (1999) proposed as possible examples of approximate early-exercise function f two possibilities. First, one can set

$$\mathcal{I}(T_l) = 1\{\mathbf{PS}_{l,n}(T_l) > H(T_l)\}.$$

With this choice we say that early exercise will depend on the longest nested European swaption exceeding a level H. A second possibility is setting

$$\mathcal{I}(T_l) = 1\big\{\mathbf{PS}_{l,n}(T_l) > H(T_l) \text{ and } \max_{p=l+1,\dots,h} \mathbf{PS}_{p,n}(T_l) \le \mathbf{PS}_{l,n}(T_l)\big\}.$$

This choice is more refined than the previous one and amounts to adding the requirement that all the other nested future European swaptions, when valued at T_l, have a lower value than the current longest one. This intuitively amounts to saying that, in the context of European swaptions evaluated now, the most convenient is the current longest one. Then, as before, the option is to be exercised if this longest swaption exceeds a level $H(T_l)$.

The second choice is more refined but also more computationally demanding. Indeed, with the first choice, f depends only on the present value per basis point C and on the underlying swap rate (both defining the relevant European swaption), so that backward induction concerns only these two variables and memory requirements are not a problem.

Andersen (1999) also made several considerations on the possible computational efficiency of the method and on low memory requirements. The first Monte Carlo simulation involved in steps 1)-3) usually requires a low number of paths, whereas the evaluation in step 4) requires usually a higher number of scenarios. For other considerations and numerical results, see Andersen (1999). We also mention that Pedersen (1999), among several other issues, considers a comparison of the Andersen method with the Longstaff and Schwartz Monte Carlo method summarized in Section 10.13.1.