I. General Properties of Representations

In this chapter we give an account of the basic definitions and simplest theorems of the theory of linear representations. Some of these theorems are valid for both finite- and infinite-dimensional representations. This textbook, however, is devoted to the former case representations, and the reader is practically at no loss if he assumes that all representations considered here are *finite-dimensional* (except, of course, for those examples which are manifestly infinite-dimensional).

1. Invariant Subspaces

1.1. The study of the structure of linear representations begins with that of invariant subspaces.

Definition. Let $T: G \to \mathrm{GL}(V)$ be a linear representation of the group G in a vector space V. A subspace $U \subset V$ is said to be INVARIANT UNDER REPRESENTATION T (or G-INVARIANT, if it is clear which representation of G one has in mind) if

(1) $\quad T(g)u \in U \qquad \text{for all } g \in G \text{ and } u \in U.$

For example, let L be the representation of the additive group $\mathbf{R}$ in the space of all polynomials, given by the rule

$$(L(t)f)(x) = f(x - t).$$

Then the subspace of all polynomials of degree $\leq n$ is invariant under L for every n.

It is obvious that *sums and intersections of invariant subspaces are invariant.*

Suppose that the space V is finite-dimensional and that $(e) = (e_1, \ldots, e_n)$ is some basis in V such that $U = \langle e_1, \ldots, e_k \rangle$. Then the invariance of U under

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the given representation T of G means that in the basis (e) each operator $T(g)$, $g \in G$, is given by a matrix of the form

$$(2) \qquad T_{(e)}(g) = \left(\begin{array}{c|c} A(g) & C(g) \\ \hline \underbrace{0}_{k} & B(g) \end{array}\right)\Big\} \, k \quad .$$

1.2. With every invariant subspace U we can associate two linear representations of G, acting on the spaces U and V/U respectively. The first of these, called a SUBREPRESENTATION of T and denoted by T_U, is obtained by restricting the operators $T(g)$ to U:

$$(3) \qquad T_U(g) = T(g)\big|_U \qquad \text{for all } g \in G.$$

The second, called a QUOTIENT or FACTOR REPRESENTATION of T and denoted by $T_{V/U}$, is defined as follows:

$$(4) \qquad T_{V/U}(x+U) = T(g)x + U \qquad \text{for all } g \in G, \quad x \in V.$$

(Recall that the elements of the quotient space V/U are the cosets $x + U$ with $x \in V$.)

Definition (4) requires some further explanations. First of all, we have to verify that the right-hand side does not depend upon the choice of the representative x in a given coset. Replacing x by $x' = x + u$ with $u \in U$, we get

$$T(g)x' + U = T(g)x + T(g)u + U = T(g)x + U.$$

Here we have used in an essential manner the invariance of U, which guarantees that $T(g)u \in U$. Next, we have to check that $T_{V/U}(g)$ is a linear operator. By the addition rule for cosets,

$$\begin{aligned} T_{V/U}(g)((x+U)+(y+U)) &= T_{V/U}(g)(x+y+U) \\ &= T(g)(x+y) + U \\ &= T(g)x + T(g)y + U \\ &= (T(g)x + U) + (T(g)y + U) \\ &= T_{V/U}(g)(x+U) + T_{V/U}(g)(y+U). \end{aligned}$$

The homogeneity of $T_{V/U}$ is verified in a similar manner. Finally, we have to show that the map $g \mapsto T_{V/U}(g)$ is a homomorphism, i.e.,

$$T_{V/U}(g_1 g_2) = T_{V/U}(g_1) T_{V/U}(g_2) \qquad \text{for all } g_1, g_2 \in G.$$

But this is a straightforward consequence of the definition of $T_{V/U}$ and the equality $T(g_1 g_2) = T(g_1)T(g_2)$.

If the space V is finite-dimensional, T_U and $T_{V/U}$ can be conveniently described in terms of matrices. To this end, we pick a basis $(e_1, \ldots, e_n)$ of V

such that $U = \langle e_1, \dots, e_k \rangle$. Then the operators $T(g)$, $g \in G$, are described by the matrices (2). Here $A(g)$ and $B(g)$ are the matrices of the operators $T_U(g)$ and $T_{V/U}(g)$ in the bases $(e_1, \dots, e_k)$ of U and $(e_{k+1} + U, \dots, e_n + U)$ of V/U respectively.

To prove the assertion concerning the matrix $B(g)$, let $b_{ij}(g)$ (respectively $c_{ij}(g)$) denote the entry of $B(g)$ (respectively $C(g)$) lying on the i-th row and j-th column of the matrix $T_{(e)}(g)$. Then for every $j > k$

$$\begin{aligned} T_{V/U}(g)(e_j + U) &= T(g)e_j + U \\ &= \sum_{i=1}^{k} c_{ij}(g)e_i + \sum_{i=k+1}^{n} b_{ij}(g)e_i + U \\ &= \sum_{i=k+1}^{n} b_{ij}(g)e_i + U = \sum_{i=k+1}^{n} b_{ij}(g)(e_i + U), \end{aligned}$$

as it should be.

1.3. Definition. A linear representation $T: G \to \mathrm{GL}(V)$ is said to be IRREDUCIBLE if there are no nontrivial (i.e., different from 0 and V) subspaces $U \subset V$ invariant under T.

Examples.

1. Every one-dimensional representation is irreducible.

2. The identity representation of $\mathrm{GL}(V)$ is irreducible, since every nonnull vector in V can be taken into any other such vector by an invertible linear transformation.

3. The representation of $\mathbf{R}$ by rotations in the plane (see Example 1, 0.7) is also irreducible.

4. The representation of $\mathbf{R}$ by translations in the space of polynomials (see Example 2, 0.7) is not irreducible.

5. Let V be an n-dimensional vector space over the field K, and let $e_1, \dots, e_n$ be a basis in V. The representation M of the group S_n in V specified by the rule

$$M(\sigma)e_i = e_{\sigma(i)} \qquad (i = 1, \dots, n)$$

(see 0.2 and 0.4) is called a MONOMIAL REPRESENTATION OF S_n. It is not irreducible: for example, it leaves invariant the $(n-1)$-dimensional subspace

$$V_0 = \left\{ \sum x_i e_i \;\middle|\; \sum x_i = 0 \right\},$$

and also the one-dimensional subspace

$$V_1 = \left\langle \sum e_i \right\rangle.$$

If the characteristic of the field K is equal to zero, then $V_1 \not\subset V_0$, and hence

$$V = V_0 \oplus V_1.$$

We claim that in this case the representation $M_0 = M_{V_0}$ is irreducible. In fact, suppose $U \subset V_0$ is an invariant subspace. Let $x = \sum x_i e_i$ be a nonnull vector in U. Since $x \notin V_1$, at least two of the numbers x_i are distinct. Suppose, for the sake of definiteness, that $x_1 \neq x_2$. Then

$$M((12))x - x = (x_2 - x_1)(e_1 - e_2) \in U,$$

whence $e_1 - e_2 \in U$. Applying to $e_1 - e_2$ various operators $M(\sigma)$, we can obtain all vectors of the form $e_i - e_j$, and the latter span the subspace V_0. Thus $U = V_0$, as we needed to show.

1.4. Definition. The linear representation $T: G \to \mathrm{GL}(V)$ is said to be COMPLETELY REDUCIBLE if every invariant subspace $U \subset V$ has an invariant complement W. (Recall that W is called a COMPLEMENT of U if $V = U \oplus W$.)

Every irreducible representation is completely reducible (though from the point of view of the Russian [or English] language this may sound strange!) In fact, for an irreducible representation there are only two invariant subspaces: the entire representation space and the null subspace, which complement one another. Hence every invariant subspace has an invariant complement.

Notice that if U and W are complementary subspaces, then the restriction σ of the canonical map $V \to V/U$ to W is an isomorphism of the space W onto V/U (each coset of U in V contains exactly one element from W). If, in addition, U and W are invariant under the representation $T: G \to \mathrm{GL}(V)$, then σ commutes with the action of G:

$$\sigma T_W(g)x = T(g)x + U = T_{V/U}(g)\sigma x.$$

This implies that *the representations T_W and $T_{V/U}$ are isomorphic.*

Let us examine in more detail the finite-dimensional case. Let $T: G \to \mathrm{GL}(V)$ be a finite-dimensional linear representation of the group G. Let $U, W \subset V$ be complementary invariant subspaces. Pick bases $(e_1, \dots, e_k)$ in U and $(e_{k+1}, \dots, e_n)$ in W. Together they yield a basis $(e) = (e_1, \dots, e_n)$ in V. Relative to (e), the operators $T(g)$, for $g \in G$, are given by matrices of the form

$$(5) \qquad \left(\begin{array}{c|c} A(g) & 0 \\ \hline \underbrace{0}_{k} & B(g) \end{array}\right) \}\, k,$$

where $A(g)$ is the matrix of $T_U(g)$ in the basis $(e_1, \dots, e_k)$, and $B(g)$ is the matrix of $T_W(g)$ in the basis $(e_{k+1}, \dots, e_n)$, as well as the matrix of $T_{V/U}(g)$ (see 1.2 above).

Example. Consider the two-dimensional representations F and S of $\mathbf{R}$, given in the basis $(e) = (e_1, e_2)$ by the matrices

$$F_{(e)}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S_{(e)}(t) = \begin{pmatrix} \mathrm{e}^t & \mathrm{e}^t - 1 \\ 0 & 1 \end{pmatrix}.$$

In both cases $U = \langle e_1 \rangle$ is an invariant subspace. Does it admit an invariant complement?

In the first case, F_U and $F_{V/U}$ are trivial representations. Assuming that an invariant complement to U exists, F would be specified, in a suitable basis, by the (t-independent) matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., would be a trivial representation, which is not the case. Thus U has no invariant complement. In particular, F is not completely reducible.

For representation S, one can check that $S(t)(e_1 - e_2) = e_1 - e_2$ for all $t \in \mathbf{R}$. Consequently, $\langle e_1 - e_2 \rangle$ is an invariant subspace. Relative to the basis $(e_1, e_1 - e_2)$, S is given by the diagonal matrix

$$\begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & 1 \end{pmatrix}.$$

It is readily verified that $\langle e_1 \rangle$ and $\langle e_1 - e_2 \rangle$ are the only nontrivial subspaces invariant under S. This shows that S is completely reducible.

1.5. Theorem 1. *Every subrepresentation of a completely reducible representation is completely reducible.*

Proof. Let $T : G \to \mathrm{GL}(V)$ be a completely reducible representation, $U \subset V$ an invariant subspace, and U_1 an arbitrary invariant subspace contained in U. Since T is completely reducible, U_1 has an invariant complement W in V. Consider the subspace $W \cap U$. It is invariant and, as is readily verified, $U = U_1 \oplus (W \cap U)$, i.e., $W \cap U$ is a complement of U_1 in U. This proves the complete reducibility of the representation T_U. $\square$

Theorem 2. *The representation space of any completely reducible finite-dimensional representation admits a decomposition into a direct sum of minimal invariant subspaces.*

(We call an invariant subspace MINIMAL if it is minimal among the nonzero invariant subspaces.)

PROOF. We proceed by induction on the dimension of the representation space. Let $T : G \to \mathrm{GL}(V)$ be a completely reducible representation. If T is irreducible, the theorem is plainly true (and the sum reduces to one term). In the opposite case there exist nontrivial invariant subspaces. Let U be an arbitrary minimal invariant subspace, and let W be an invariant complement of U. By Theorem 1, the representation T_W is completely reducible. Applying the inductive hypothesis to T_W, we can assume that W decomposes into a direct sum of minimal invariant subspaces. Adding U to this decomposition, we obtain a decomposition of V into a direct sum of minimal invariant subspaces. □

Theorem 3. *Let $T : G \to \mathrm{GL}(V)$ be a linear representation. Let*

(6) $$V = V_1 + V_2 + \ldots + V_m$$

be a decomposition of the space V into a (not necessarily direct) sum of minimal invariant subspaces. Then T is completely reducible. Moreover, for every invariant subspace U there exist indices $i_1, \ldots, i_p$ such that

(7) $$V = U \oplus V_{i_1} \oplus \ldots \oplus V_{i_p}.$$

PROOF. It suffices to prove the second assertion of the theorem, since it is a stronger version of the first. Let U be an invariant subspace, and let $\{i_1, \ldots, i_p\}$ be a (possibly empty) maximal set of indices such that the subspaces $U, V_{i_1}, \ldots, V_{i_p}$ are linearly independent. We claim that (7) holds in this case. It suffices to show that

(8) $$V_i \subset U \oplus V_{i_1} \oplus \ldots \oplus V_{i_p}$$

for every $i \notin \{i_1, \ldots, i_p\}$. Since $V_i \cap (U \oplus V_{i_1} \oplus \ldots \oplus V_{i_p})$ is an invariant subspace contained in V_i, and since V_i is a minimal invariant subspace, either (8) holds or

(9) $$V_i \cap (U \oplus V_{i_1} \oplus \ldots \oplus V_{i_p}) = 0.$$

However, alternative (9) is impossible, because it would imply the linear independence of the subspaces $U, V_{i_1}, \ldots, V_{i_p}$, and V_i, thereby contradicting the choice of the set $\{i_1, \ldots, i_p\}$. This completes the proof of the theorem. □

Remarks.

1) Applying Theorem 3 to the subspace $U = 0$ we conclude that V itself is the direct sum of a number of subspaces V_i.

2) An invariant subspace is not necessarily the direct sum of a number of subspaces V_i. For instance, let T be the trivial representation in V. Then every subspace of V is invariant, and the minimal invariant subspaces are precisely the one-dimensional ones. Let $(e_1, \dots, e_n)$ be an arbitrary basis in V. Then $V = \langle e_1 \rangle \oplus \dots \oplus \langle e_n \rangle$ is a decomposition of V into a direct sum of minimal invariant subspaces. However, if $n > 1$, not every subspace is the linear span of a subset of basis vectors.

1.6. Some Examples. We consider three linear representations of the group $\mathrm{GL}(V)$, where V is an n-dimensional vector space over K.

1. *Representation by left multiplication in the algebra* $\mathrm{L}(V)$ *of all linear operators in* V:

$$\Lambda(\alpha)\xi = \alpha\xi \qquad (\alpha \in \mathrm{GL}(V), \quad \xi \in \mathrm{L}(V)).$$

Linear operators can be replaced by their matrices in some fixed basis of V. Then the definition of the representation Λ is accordingly modified to

$$\Lambda(A)X = AX \qquad (A \in \mathrm{GL}_n(K), \quad X \in \mathrm{L}_n(K)). \tag{10}$$

Let $\mathrm{L}^{(i)}$ denote the subspace of all matrices for which every column except the i-th contains only zeros. Obviously,

$$\mathrm{L}_n(K) = \mathrm{L}^{(1)} \oplus \mathrm{L}^{(2)} \oplus \dots \oplus \mathrm{L}^{(n)}. \tag{11}$$

On multiplying the matrix X on the left by the matrix A, every column of X is multiplied by A. This implies that the subspaces $\mathrm{L}^{(i)}$ are invariant under Λ. Moreover, $\Lambda_{\mathrm{L}^{(i)}}$ is isomorphic to the identity representation Id of $\mathrm{GL}_n(K)$ in the space of columns, and hence it is irreducible. According to Theorem 3, the representation Λ is completely reducible.

To the decomposition (11) there corresponds a decomposition of the space $\mathrm{L}(V)$ into a direct sum of minimal invariant subspaces. Such a decomposition is not unique: changing the basis in V, generally speaking, also changes the decomposition.

2. *The* ADJOINT REPRESENTATION *in the algebra* $\mathrm{L}(V)$:

$$\mathrm{Ad}(\alpha)\xi = \alpha\xi\alpha^{-1} \qquad (\alpha \in \mathrm{GL}(V), \quad \xi \in \mathrm{L}(V)).$$

This is indeed a representation:

$$\mathrm{Ad}(\alpha\beta)\xi = \alpha\beta\xi(\alpha\beta)^{-1} = \alpha\beta\xi\beta^{-1}\alpha^{-1} = \mathrm{Ad}(\alpha)\mathrm{Ad}(\beta)\xi.$$

In terms of matrices Ad is defined as follows:

$$\mathrm{Ad}(A)X = AXA^{-1} \qquad (A \in \mathrm{GL}_n(K), \quad X \in \mathrm{L}_n(K)). \tag{12}$$

One can show that there are only two nontrivial Ad-invariant subspaces: the one-dimensional subspace $\langle E\rangle$ and the (n^2-1)-dimensional subspace $\mathrm{L}_n^0(K)$ of the matrices with trace zero. If the characteristic of K is equal to zero, then $E \notin \mathrm{L}_n^0(K)$, and so

$$\mathrm{L}_n(K) = \langle E\rangle \oplus \mathrm{L}_n^0(K).$$

In this case the adjoint representation is completely reducible.

3. *The representation in the space* $\mathrm{B}(V)$ *of bilinear functions (forms) on* V:

$$(\Phi(\alpha)f)(x,y) = f(\alpha^{-1}x, \alpha^{-1}y) \qquad (\alpha \in \mathrm{GL}(V), \quad f \in \mathrm{B}(V)).$$

This is a natural definition if one is guided by the general principle that every one-to-one mapping σ of an arbitrary set X onto itself acts on functions of one or several X-valued arguments if one applies σ^{-1} simultaneously to all arguments (see 0.9). It is readily verified that $\Phi(\alpha)f$ is again a bilinear function.

The subspaces $\mathrm{B}^+(V)$ and $\mathrm{B}^-(V)$ of symmetric and skew-symmetric bilinear functions are invariant under Φ. If the characteristic of K is different from two, then

$$\mathrm{B}(V) = \mathrm{B}^+(V) \oplus \mathrm{B}^-(V), \tag{13}$$

and one can show that $\mathrm{B}^+(V)$ and $\mathrm{B}^-(V)$ are minimal. In this case Φ is completely reducible.

In terms of matrices, Φ is defined as follows:

$$\Phi(A)X = (A^{-1})'XA^{-1} \qquad (A \in \mathrm{GL}_n(K), \quad X \in \mathrm{L}_n(K)), \tag{14}$$

where $'$ stands for transposition of matrices.

To (13) there corresponds the decomposition

$$\mathrm{L}_n(K) = \mathrm{L}_n^+(K) \oplus \mathrm{L}_n^-(K),$$

where $\mathrm{L}_n^+(K)$ and $\mathrm{L}_n^-(K)$ denote the spaces of symmetric and skew-symmetric matrices respectively.

In the following we shall, as a rule, write $\alpha_* f$ instead of $\Phi(\alpha)f$, in agreement with the general notation adopted in 0.9.

Questions and Exercises

1. Prove that if the subspace U of the space of the representation T of G is invariant, then $T(g)U = U$ for all $g \in G$.

2. Find all subspaces of the space of polynomials that are invariant under the representation L of $\mathbf{R}$ given by the formula $(L(t)f)(x) = f(x-t)$.

3. Let F denote the representation of $\mathbf{C}$ in a complex n-dimensional space given by the formula $F(t) = \mathrm{e}^{t\alpha}$, where α is a linear operator whose characteristic polynomial has no multiple roots. Find all subspaces invariant under F.

4. Without resorting to computations, prove that the matrix $B(g)$ in formula (2) does not change on passing to a new basis $(f) = (f_1, \dots, f_n)$ of the space V if $f_i - e_i \in U$ for all $i > k$.

5. Let W_1 and W_2 be two invariant complements of the invariant subspace U in the space of the representation T. Prove that $T_{W_1} \simeq T_{W_2}$.

6. Prove that every quotient representation of a completely reducible representation is completely reducible.

7. Is the representation

a) of Exercise 2,

b) of Exercise 3

completely reducible?

8. Prove that the representation $t \mapsto \mathrm{e}^{t\alpha}$ of $\mathbf{C}$ is completely reducible if and only if the operator α is diagonalizable (i.e., it admits a basis of eigenvectors).

9. Prove that the identity representation of the orthogonal group O_n is irreducible for any n.

10. Prove that any monomial representation of the group S_n over a field of characteristic zero is completely reducible.

11. Prove that for $n \geq 4$ the restriction of the representation M_0 (see Example 5 of 1.3) to the subgroup A_n is irreducible.

12. Let $T\colon G \to \mathrm{GL}(V)$ be a completely reducible finite-dimensional linear representation. Show that for every invariant subspace $U \subset V$ there is a decomposition $V = V_1 \oplus \ldots \oplus V_m$ of V into a direct sum of minimal invariant subspaces and an $s \leq m$ such that $U = V_1 \oplus \ldots \oplus V_s$.

13. Prove that the subspaces invariant under the representation Λ of $\mathrm{GL}_n(K)$ defined by formula (10) are precisely the left ideals in the ring of matrices of order n.

14.* Prove that the adjoint representation of the group $\mathrm{GL}(V)$ possesses only the two nontrivial invariant subspaces indicated in 1.6.

15.* Prove that $\mathrm{B}^+(V)$ and $\mathrm{B}^-(V)$ are minimal $\mathrm{GL}(V)$-invariant subspaces of the space $\mathrm{B}(V)$ of bilinear functions.

2. Complete Reducibility of Representations of Compact Groups

In this section we are concerned only with finite-dimensional representations.

2.1. One of the basic problems of representation theory is that of describing all representations of a given group (over a given field). In the preceding section we have seen that the description of completely reducible representations reduces to that of the irreducible representations (see also Section 3). Here we will show that all real or complex representations of finite groups are completely reducible. This result will subsequently be generalized to compact topological groups; this class includes, for example, the orthogonal group O_n.

The idea of the proof of complete reducibility is to equip the representation space with an inner product invariant under the action of the group. Then, given an arbitrary invariant subspace, one finds an invariant complement for it by taking its orthogonal complement.

2.2. We proceed to implement the program formulated above.

Definition. A real linear representation $T: G \to \mathrm{GL}(V)$ is called ORTHOGONAL if on the space V there is a positive definite symmetric bilinear function f INVARIANT under T.

The invariance of f means that

$$f(T(g)x, T(g)y) = f(x, y) \tag{1}$$

for all $g \in G$ and all $x, y \in V$ or, equivalently, that

$$T(g)_* f = f \tag{2}$$

for all $g \in G$, where the asterisk indicates the natural action of an invertible linear operator on bilinear functions (see Example 3, 1.6). Taking f as an inner product, we turn V into a *Euclidean space* in which the operators $T(g)$, for $g \in G$, are orthogonal.

Similarly, the complex linear representation $T\colon G \to \mathrm{GL}(V)$ is called UNITARY if on the space V there is a positive definite Hermitian sesquilinear function f invariant under T. Taking f as an inner product, we turn V into a *Hermitian space* in which the operators $T(g)$, for $g \in G$, are unitary.

Proposition. *Every orthogonal or unitary representation is completely reducible.*

PROOF. Let $T\colon G \to \mathrm{GL}(V)$ be an orthogonal representation of the group G. Let $U \subset V$ be an arbitrary invariant subspace. Denote by U^o the orthogonal complement of U with respect to an invariant inner product on V. It is known that

$$V = U \oplus U^o.$$

For each $g \in G$ the operator $T(g)$ is orthogonal and preserves U. By a well-known property, it preserves U^o as well. Hence, U^o is an invariant subspace complementing U.

The proof for a unitary representation is identical. □

2.3. Theorem 1. *Every real (complex) linear representation of a finite group is orthogonal (respectively, unitary).*

PROOF. Let $T\colon G \to \mathrm{GL}(V)$ be a real linear representation of a finite group G. Pick an arbitrary positive definite symmetric bilinear function f_0 on V, and construct a new symmetric bilinear function f by the rule

$$f = \sum_{h \in G} T(h)_* f_0. \tag{3}$$

Since

$$f(x,x) = \sum_{h \in G} f_0(T(h)^{-1}x, T(h)^{-1}x) > 0$$

for every nonnull vector $x \in V$, f is positive definite. We claim that f is invariant under T. In fact, for every $g \in G$,

$$T(g)_* f = \sum_{h \in G} T(g)_* T(h)_* f_0 = \sum_{h \in G} T(gh)_* f_0.$$

Since the equation $gx = h$ has a unique solution in G for every fixed h, the last sum above differs from the one in (3) only in the order of its terms. Hence $T(g)_* f = f$, as claimed.

The proof for a complex representation is identical. □

Corollary. *Every real or complex linear representation of a finite group is completely reducible.*

In point of fact, every linear representation of a finite group over a field whose characteristic does not divide the order of the group is completely reducible. (For a proof see, for example, [4].)

2.4. A TOPOLOGICAL GROUP is, by definition, a group endowed with a topology such that the group operations

$$x \mapsto x^{-1} \qquad \text{and} \qquad (x, y) \mapsto xy$$

are continuous maps.

Examples of topological groups.

1. Any group with the discrete topology.

2. $\mathrm{GL}(V)$, where V is an n-dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$. The topology is defined as on any (open) subset of the vector space $\mathrm{L}(V)$. That is, the continuous functions in this topology are exactly the continuous functions of the matrix elements (relative to some fixed basis, the choice of which, however, does not affect the topology). Since the matrix elements of the operators α^{-1} and $\alpha\beta$ are continuous functions of the matrix elements of α and β, $\mathrm{GL}(V)$ is indeed a topological group.

3. Any subgroup of a topological group endowed with the induced topology. In particular, every group of linear transformations of a real or complex vector space is a topological group.

A topological group is said to be COMPACT if it is compact as a topological space.

Examples of compact topological groups.

1. Any finite group endowed with the discrete topology.

2. The orthogonal group O_n.

3. The unitary group U_n.

4. Any closed subgroup of a compact topological group.

To prove the compactness of the groups O_n and U_n, we remind the reader of the following general fact: a subset of a real or complex vector space is compact if and only if it is closed and bounded. O_n is singled out in the space $\mathrm{L}_n(\mathbf{R})$ of real matrices by the algebraic equations

$$\sum_k a_{ik}a_{jk} = \delta_{ij}$$

and consequently is closed in $\mathrm{L}_n(\mathbf{R})$. The same equations yield the bounds $|a_{ij}| \leq 1$, which prove the boundedness of O_n in $\mathrm{L}_n(\mathbf{R})$. The compactness of the unitary group is established in an analogous manner.

A real or complex linear representation $T\colon G \to \mathrm{GL}(V)$ of the topological group G is said to be CONTINUOUS if it is a continuous map of the underlying topological spaces. This means that the matrix elements of the operator $T(g)$ depend continuously on g.

Examples.

1. Any real or complex linear representation of a discrete topological group.

2. If V is a real or complex vector space, then all representations of $\mathrm{GL}(V)$ considered in 1.6 are continuous.

For example, let us prove the continuity of the representation Φ (Example 3, 1.6). To this end we use its matrix expression (formula (14), 1.6). Let $a_{ij}, \tilde{a}_{ij}, x_{ij}$, and y_{ij} denote the elements of the matrices A, A^{-1}, X, and $\Phi(A)X$, respectively. Then

$$y_{ij} = \sum_{k,\ell} \tilde{a}_{ki}x_{k\ell}\tilde{a}_{\ell j}.$$

We see that $\Phi(A)$ is the linear transformation with coefficients (matrix elements) $\tilde{a}_{ki}\tilde{a}_{\ell j}$, which obviously depend continuously on the elements of the matrix A. This means precisely that Φ is a continuous representation.

One usually considers only *continuous* linear representations of topological groups. For this reason from now on we shall omit, as a rule, the adjective "continuous."

2.5. From the point of view of the theory of (continuous) linear representations, compact topological groups are similar to discrete ones. In particular, we have

Theorem 2. *Every real (complex) linear representation of a compact topological group is orthogonal (respectively, unitary).*

Recalling the proposition of 2.2, we derive the following

Corollary. *Every real or complex linear representation of a compact topological group is completely reducible.*

To prove Theorem 2 one can proceed as in the proof of Theorem 1, but now integration replaces summation over a finite group. It is known (see [7], for example) that on every compact topological group G one can define an INVARIANT INTEGRATION, meaning that to each continuous function f on G one can assign a number, denoted by $\int_G f(x)\,dx$, such that the mapping $f \mapsto \int_G f(x)\,dx$ possesses the following properties:

1) $\int_G (a_1 f_1(x) + a_2 f_2(x))\,dx = a_1 \int_G f_1(x)\,dx + a_2 \int_G f_2(x)\,dx$ (*linearity*);

2) if f is nonnegative everywhere and does not vanish identically, then we have $\int_G f(x)\,dx > 0$ (*positivity*);

3) $\int_G f(gx)\,dx = \int_G f(xg)\,dx = \int_G f(x)\,dx$ for every $g \in G$ (*invariance*).

Such an integration is *unique* up to a constant factor. Usually this factor is chosen so that

4) $\int_G 1\,dx = 1$.

In what follows we shall assume that this last condition is satisfied.

Examples.

1. The invariant integration on a finite group G is defined by the formula

$$\int_G f(x)\,dx = \frac{1}{|G|} \sum_{x \in G} f(x).$$

2. The invariant integration on the group $\mathbf{T} \simeq \mathrm{U}_1$ is defined by the formula

$$\int_G f(x)\,dx = \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{i\phi})\,d\phi.$$

3. In Chapter III we will show that the group SU_2 can be identified with the three-dimensional sphere in such a manner that the left and right translations are isometries of the sphere. Under this identification, the invariant integration on SU_2 can be defined as integration over the sphere with respect to the usual measure, multiplied by a factor of $(2\pi^2)^{-1}$.

4. Let N be a closed normal subgroup of the compact group G. The invariant integration on the quotient group G/N can be defined in terms of the invariant integration on G itself as

$$\int_{G/N} f(y)\,dy = \int_G f(xN)\,dx.$$

In this way one can, for example, define the invariant integration on the group SO_3, which, as we will see in Chapter III, is isomorphic to the quotient group $\mathrm{SU}_2/\{E, -E\}$.

The PROOF OF THEOREM 2 for a real linear representation $T\colon G \to \mathrm{GL}(V)$ of a compact group G proceeds as follows. Let f_0 be an arbitrary positive definite symmetric bilinear function on V. One defines a new symmetric bilinear function f by the rule

$$f(x,y) = \int_G f_0(T(g)x, T(g)y)\,dg \qquad (x, y \in V).$$

Next, using the properties of invariant integration one shows that f is positive definite and invariant. In the case of a complex representation one proceeds in the same manner, but one replaces symmetric bilinear functions by Hermitian sesquilinear functions. □

2.6. We now give *an alternate proof of Theorem 2* which does not resort to integration on the group.

We remark that on multiplying the sum in (3) by the factor $|G|^{-1}$ the function f becomes the *center of mass* of the finite set $M = \{T(h)_* f_0 \mid h \in G\}$ in the vector space $\mathrm{B}^+(V)$ of symmetric bilinear functions on V. For each $g \in G$ the transformation $T(g)_*$ maps the set M into itself (permuting its points in some way), and consequently preserves its center of mass.

Our proof of Theorem 2 will also rest on the idea of using the center of mass, but we must first replace the elementary definition, appropriate for the finite case, by the notion of center of mass of a compact set of positive measure.

Let V be a real vector space. Let $K \subset V$ be a compact set of positive measure. By definition, the CENTER OF MASS of K is the point (vector)

$$c(K) = \mu(K)^{-1} \int_K x\,\mu(dx). \tag{4}$$

Here x is a vector variable and μ denotes the usual measure on V; μ is defined to within a constant factor, but, as formula (4) shows, this freedom in the choice of μ does not affect the result $c(K)$.

The integral in (4) can be defined either coordinate-wise, or directly, as a limit of integral sums. The first definition proves its existence, while the second establishes its independence of the choice of a coordinate system (basis) in V.

We now show that

(5) $\qquad c(\alpha K) = \alpha c(K) \qquad$ for all $\alpha \in \mathrm{GL}(V)$.

In fact,

$$\begin{aligned} c(\alpha K) &= \mu(\alpha K)^{-1} \int_{\alpha K} x\mu(dx) \\ &= (\det \alpha \cdot \mu(K))^{-1} \int_K \alpha x \cdot \det \alpha \cdot \mu(dx) \\ &= \mu(K)^{-1} \alpha \int_K x\mu(dx) = \alpha c(K). \end{aligned}$$

(Moving α in front of the integral sign is permitted thanks to the continuity and linearity of the transformation α.)

Another important property is that *the center of mass of a compact set K lies in the convex hull of K.*

Recall that the CONVEX HULL of an arbitrary set $K \subset V$ is defined as

$$\operatorname{conv} K = \{ \sum_{i=1}^{m} c_i x_i \mid x_i \in K,\ c_i \geq 0,\ \sum_{i=1}^{m} c_i = 1,\ m \text{ arbitrary} \}.$$

It is the smallest convex set containing K. One can show that *the convex hull of a compact set is closed* (see Appendix 3).

It follows from the definition of the integral that the center of mass $c(K)$ of the compact set K is a limit of vectors of the form

$$\mu(K)^{-1} \sum_{i=1}^{m} \mu(K_i) x_i,$$

where $x_i \in K_i \subset K$ and $\sum \mu(K_i) = \mu(K)$. Each such vector lies in $\operatorname{conv} K$, and since $\operatorname{conv} K$ is closed, $c(K) \in \operatorname{conv} K$, too.

2.7. PROOF OF THEOREM 2. Let $T\colon G \to \mathrm{GL}(V)$ be a real linear representation of the compact topological group G.

In the space $\mathrm{B}^+(V)$ of symmetric bilinear functions on V, consider the subset P of all positive definite functions. Obviously, P is closed under addition (the sum of two positive definite functions is again positive definite) and multiplication by positive numbers. This implies that P is convex. Moreover,

P is open, since in terms of matrices it is given by the condition that all principal minors be positive. Finally, it is plain that

(6) $\alpha_* P \subset P$

for all $\alpha \in \mathrm{GL}(V)$.

Let $K_0 \subset P$ be an arbitrary compact set of positive measure. Put

$$K = \bigcup_{h \in G} T(h)_* K_0$$

(cf. formula (3)). We claim that the set K enjoys the following properties:

(K1) $K \subset P$;

(K2) $T(g)_* K = K$ for all $g \in G$;

(K3) K is compact.

Property (K1) is a consequence of (6). (K2) follows from the equality

$$T(g)_* T(h)_* K_0 = T(gh)_* K_0.$$

To prove (K3), consider an arbitrary sequence $T(h_n)_* f_n$ ($h_n \in G, f_n \in K_0$) of elements of K. Since G and K_0 are compact, we can, passing to a subsequence if necessary, ensure that $h_n \to h \in G$ and $f_n \to f \in K_0$. Then $T(h_n)_* f_n \to T(h)_* f \in K$ (here we used the continuity of the representation T).

Now consider the center of mass $f = c(K)$ of K in the space $\mathrm{B}^+(V)$. Since $c(K) \in \operatorname{conv} K$ (see 2.6), (K1) and the convexity of P guarantee that $f \in P$, i.e., f is a positive definite symmetric bilinear function. Properties (K2) and (5) imply that $T(g)_* f = f$ for all $g \in G$, i.e., f is G-invariant. Thus, T is an orthogonal representation, as asserted.

The complex version of the theorem is proved in an analogous manner, with the difference that instead of $\mathrm{B}^+(V)$ one works with the space $\mathrm{H}^+(V)$ of positive definite Hermitian sesquilinear functions. Notice that $\mathrm{H}^+(V)$ is a real (and not complex) vector space, and the notion of center of mass introduced in 2.6 can be used in the indicated proof with no modifications.

Questions and Exercises

1. Let T be an orthogonal or unitary representation of the group G. Prove that all complex eigenvalues of the operators $T(g)$, $g \in G$, have modulus one.

2. Give an example of a nonunitary complex representation of $\mathbf{Z}$.

3. Let T be a real representation of $\mathbf{Z}_3$ in which the generator of $\mathbf{Z}_3$ goes into the linear operator with the matrix $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Find a positive definite symmetric bilinear function invariant under T.

4. Which of the following topological groups are compact: $\mathbf{Z}$, $\mathbf{Z}_m$, $\mathbf{T}$, $\mathrm{SL}_n(\mathbf{R})$?

5. Let $T\colon G \to \mathrm{GL}(V)$ be a continuous real or complex representation of the topological group G, and let $U \subset V$ be an invariant subspace. Show that the representations T_U and $T_{V/U}$ are continuous.

6. Let V be a real or complex vector space. Show that the adjoint representation of the group GL(V) (Example 2, 1.6) is continuous.

7.* Let K_1 and K_2 be compact sets of positive measure in the real vector space V. Prove that

a) if the symmetric difference of K_1 and K_2 has measure zero, then $c(K_1) = c(K_2)$;

b) if $\mu(K_1 \cap K_2) = 0$, then $c(K_1 \cup K_2)$ lies on the segment connecting $c(K_1)$ and $c(K_2)$.

8.* Give an example of a compact subset of positive measure in the set P of positive definite symmetric bilinear functions.

3. Basic Operations on Representations

Various methods of obtaining new representations from one or more other representations play an important role in representation theory. We have already encountered certain constructions of this sort, namely, composing a representation and a homomorphism (0.10), and passing to a subrepresentation or quotient representation (1.2). In this section we consider a number of other constructions that will be needed later.

3.1. The Contragredient or Dual Representation. Given any linear representation $T\colon G \to \mathrm{GL}(V)$, we define in a canonical manner the CONTRAGREDIENT or DUAL representation $T'\colon G \to \mathrm{GL}(V')$ in the dual space V' of V. (Recall that the elements of V' are the linear functions on V.)

Definition. $(T'(g)f)(x) = f(T(g)^{-1}x) \quad (g \in G,\ f \in V',\ x \in V)$.

T' is a subrepresentation of the representation T_* of G in the space of all K-valued functions on V (see 0.9).

For a finite-dimensional representation T, the contragredient representation can be described in terms of matrices as follows. Let $(e) = (e_1, \dots, e_n)$ and $(\varepsilon) = (\varepsilon_1, \dots, \varepsilon_n)$ be a basis in V and the dual basis in V' respectively, i.e., $\varepsilon_i(e_j) = \delta_{ij}$. Let

$$T_{(e)}(g) = [a_{ij}] \qquad \text{and} \qquad T'_{(\varepsilon)}(g) = [b_{ij}].$$

According to the definition,

$$(T'(g)\varepsilon_i)(T(g)e_j) = \varepsilon_i(e_j) = \delta_{ij}.$$

Since

$$T'(g)\varepsilon_i = \sum_k b_{ki}\varepsilon_k, \qquad T(g)e_j = \sum_\ell a_{\ell j}e_\ell,$$

we have that

$$(T'(g)\varepsilon_i)(T(g)e_j) = \sum_k b_{ki}a_{kj} = \delta_{ij}.$$

This means that $(T'_{(\varepsilon)}(g))'T_{(e)}(g) = E$ or, equivalently,

$$T'_{(\varepsilon)}(g) = ((T_{(e)}(g))')^{-1}. \tag{1}$$

In particular, if T is an orthogonal representation and the basis (e) is orthonormal (with respect to an invariant inner product), then $T_{(e)}(g)$ is an orthogonal matrix, and so $((T_{(e)}(g))')^{-1} = T_{(e)}(g)$. In this case $T'_{(\varepsilon)}(g) = T_{(e)}(g)$, and so $T' \simeq T$.

If T is a unitary representation and the basis (e) is orthonormal, then

$$T'_{(\varepsilon)}(g) = \overline{T_{(e)}(g)}, \tag{2}$$

where the bar denotes complex conjugation.

It also follows from formula (1) that $T'' \simeq T$ for every representation T.

Theorem 1. *Let T be an irreducible finite-dimensional representation. Then T' is irreducible.*

Proof. Let $U \subset V'$ be a T'-invariant subspace. Consider its annihilator

$$U^0 = \{\, x \in V \mid f(x) = 0 \quad \text{for all } f \in U \,\} \subset V.$$

It is a T-invariant subspace. In fact, for any $g \in G$, $x \in U^0$, and $f \in U$ we have

$$f(T(g)x) = (T'(g)^{-1}f)(x) = 0,$$

because $T'(g)^{-1}f \in U$. It is known from the theory of systems of linear equations that $\dim U^0 = \dim V - \dim U$. Since T is irreducible, U^0 is equal to 0 or V, and correspondingly U is equal to V' or 0. This means that V' contains no nontrivial T'-invariant subspaces, as we needed to show. ∎

3.2 Sums of Representations. Let

$$T_1\colon G \to \mathrm{GL}(V_1) \qquad \text{and} \qquad T_2\colon G \to \mathrm{GL}(V_2)$$

be two linear representations of the group G.

Definition. The SUM OF T_1 AND T_2 is the representation $T_1 + T_2$ of G in the space $V_1 \oplus V_2$ defined by the rule

$$(T_1 + T_2)(g)(x_1 + x_2) = T_1(g)x_1 + T_2(g)x_2$$
$$(g \in G,\ x_1 \in V_1,\ x_2 \in V_2).$$

The sum of an arbitrary finite number of representations is defined in a similar manner. A sum of representations is independent, up to an isomorphism, of the order of its summands.

This definition makes it clear that the spaces V_1 and V_2, canonically imbedded in $V_1 \oplus V_2$, are invariant under $T_1 + T_2$. Conversely, if the space V of a representation T of G can be written as the direct sum of two T-invariant subspaces V_1 and V_2, then T coincides with the sum of the representations T_{V_1} and T_{V_2}. In fact,

$$T(g)(x_1 + x_2) = T(g)x_1 + T(g)x_2 = T_{V_1}(g)x_1 + T_{V_2}(g)x_2$$

for all $x_1 \in V_1$, $x_2 \in V_2$, which is precisely the definition of the sum of the representations T_{V_1} and T_{V_2}. Analogous assertions are of course true for a sum of finitely many representations.

In terms of matrices, the sum T of the representations $T_i\colon G \to \mathrm{GL}(V_i)$, for $i = 1, 2, \ldots, m$, is described as follows. Let (e) be a basis in $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$ that is a union of bases $(e)_i$ in V_i. Then in block form

$$T(g)_{(e)} = \begin{pmatrix} T(g)_{(e)_1} & & & 0 \\ & T(g)_{(e)_2} & & \\ & & \ddots & \\ 0 & & & T(g)_{(e)_m} \end{pmatrix}.$$

The notion of a sum of representations is suitable for formulating properties of completely reducible representations.

Theorem 2. *Every completely reducible finite-dimensional linear representation is isomorphic to a sum of irreducible representations. Conversely, every sum of irreducible representations is completely reducible.*

This is simply a reformulation of Theorem 2 and of the first assertion of Theorem 3 of 1.5. □

Theorem 3. *Suppose the representation $T: G \to \mathrm{GL}(V)$ is isomorphic to a sum of irreducible representations $T_i: G \to \mathrm{GL}(V_i)$, $i = 1, \dots, m$. Then every subrepresentation of T as well as every quotient representation of T is isomorphic to a sum of some of the representations T_i.*

Proof. It suffices to prove the assertion for quotient representations, since every subrepresentation T_U of T is isomorphic to the quotient representation $T_{V/W}$, where W is an invariant complement of the subspace U.

Let U be an invariant subspace. By Theorem 3 of 1.5, it admits a complement of the form $V_{i_1} \oplus \dots \oplus V_{i_p}$, and then $T_{V/U} \simeq T_{V_{i_1} \oplus \dots \oplus V_{i_p}} \simeq T_{i_1} \oplus \dots \oplus T_{i_p}$. □

Corollary. *Let $T: G \to \mathrm{GL}(V)$ be a linear representation. Let $V_1, \dots, V_m$ be minimal invariant subspaces such that the representations $T_i = T_{V_i}$ are pairwise nonisomorphic. Then $V_1, \dots, V_m$ are linearly independent.*

Proof. Suppose this is not the case. Then there is a $k < m$ such that the subspaces $V_1, \dots, V_k$ are linearly independent, whereas $V_{k+1} \cap \sum_{i=1}^{k} V_i = \Delta \neq 0$. Since $\Delta \subset V_{k+1}$ and V_{k+1} is a minimal invariant subspace, $\Delta = V_{k+1}$, i.e., $V_{k+1} \subset \sum_{i=1}^{k} V_i$. But then, by Theorem 3, T_{k+1} is isomorphic to one of the representations $T_1, \dots, T_k$, which contradicts the hypothesis. □

We show next that the decomposition of a completely reducible representation into a sum of irreducible components is, in a certain sense, unique.

Theorem 4. *Let T be a linear representation. If*

$$T \simeq T_1 + \dots + T_m \simeq S_1 + \dots + S_p,$$

where T_i and S_j are irreducible representations, then $m = p$ and, for a suitable labeling, $T_i \simeq S_i$.

(Compare this result with the theorem asserting the uniqueness of the decomposition of a positive integer into prime factors.)

Proof. By hypothesis, the representation space V of T admits two decompositions into a direct sum of minimal invariant subspaces,

$$V = V_1 \oplus \dots \oplus V_m = U_1 \oplus \dots \oplus U_p,$$

such that $T_{V_i} \simeq T_i$ and $T_{U_j} \simeq S_j$.

The proof proceeds by induction on m. Applying Theorem 3 of 1.5 to the invariant subspace $U = U_1$, we deduce that $V = U \oplus V_{i_1} \oplus \dots \oplus V_{i_k}$ for certain $i_1, \dots, i_k$. Then

$$S_1 \simeq T_U \simeq T_{V/(V_{i_1} \oplus \dots \oplus V_{i_k})} \simeq T_{j_1} + \dots + T_{j_\ell},$$

where $\{j_1, \ldots, j_\ell\} = \{1, \ldots, m\} \setminus \{i_1, \ldots, i_k\}$. Since the representation S_1 is irreducible, $\ell = 1$. Now let us relabel the representations T_i so that $j_1 = 1$. Then $S_1 \simeq T_1$ and

$$V = U \oplus V_2 \oplus \ldots \oplus V_m.$$

This says that

$$T_{V/U} \simeq T_2 + \ldots + T_m.$$

On the other hand, it is clear that

$$T_{V/U} \simeq S_2 + \ldots + S_p.$$

Applying the inductive hypothesis to $T_{V/U}$, we conclude that $m = p$ and, after a suitable relabeling, $T_i \simeq S_i$ for all $i \geq 2$. Since $T_1 \simeq S_1$, the assertion of the theorem is also true for T. $\square$

We remark that Theorem 4 does not imply the uniqueness of the decomposition of the representation space into a direct sum of minimal invariant subspaces. Such uniqueness does not hold, as can be seen even in the case of a trivial representation (see the end of 1.5).

3.3. Products of Representations. Let $T: G \to \mathrm{GL}(V)$ and $S: G \to \mathrm{GL}(U)$ be two linear representations of the group G.

Definition. The PRODUCT OF T AND S is the representation TS of G in the space $V \otimes U$ defined by the rule

$$TS(g) = T(g) \otimes S(g).$$

(For the definitions of the tensor product for vector spaces and linear operators, see Appendix 2.)

Sometimes TS is referred to as the tensor product of the representations T and S. However, we reserve this term for another notion, defined below in 3.4.

Let us give the matrix interpretation of the product of finite-dimensional representations. To this end we pick bases

$$(e) = (e_1, \ldots, e_n) \subset V \qquad \text{and} \qquad (f) = (f_1, \ldots, f_m) \subset U$$

of the spaces V and U respectively. Each element $x \in V \otimes U$ can be uniquely expressed as

$$x = \sum x_{ij}(e_i \otimes f_j).$$

How is the matrix $X = [x_{ij}]$ transformed under the action of the operator $TS(g)$ on x?

Let $T_{(e)}(g) = [a_{ij}]$ and $S_{(f)}(g) = [b_{ij}]$. Then

$$T(g)e_i = \sum_k a_{ki}e_k, \qquad S(g)f_j = \sum_\ell b_{\ell j}f_\ell,$$

and

$$\begin{aligned} TS(g)x &= \sum_{i,j} x_{ij}(T(g)e_i \otimes S(g)f_j) \\ &= \sum_{i,j,k,\ell} x_{ij}a_{ki}b_{\ell j}(e_k \otimes f_\ell) \\ &= \sum_{i,j}(\sum_{k,\ell} a_{ik}x_{k\ell}b_{j\ell})(e_i \otimes f_j). \end{aligned}$$

Hence, X transforms according to the rule

$$X \mapsto T_{(e)}(g)XS_{(f)}(g)'.$$

We thus obtain the following matrix interpretation of the product of two representations:

$$(3) \qquad TS(g)X = T(g)XS(g)' \qquad (X \in \mathrm{L}_{n,m}(K)).$$

(Here T and S are regarded as matrix representations, and the representation space of TS is interpreted as the space of $(n \times m)$-matrices.)

Examples.

1. Let T be an n-dimensional linear representation of a group G in the space V, and $I = I_m$ the m-dimensional trivial representation of G in the space U. Let us examine the representation TI.

In terms of matrices, TI is given by the formula

$$TI(g)X = T(g)X \qquad (X \in \mathrm{L}_{n,m}(K)).$$

When $U = V'$, this coincides with the composition of the representation Λ of $\mathrm{GL}(V)$ considered in Example 1 of 1.6 and the representation $T\colon G \to \mathrm{GL}(V)$. In the general case one can show, proceeding exactly as in 1.6, that

$$TI_m \simeq mT \quad (= \underbrace{T + \ldots + T}_{m \text{ times}}).$$

The corresponding decomposition of $V \otimes U$ into a direct sum of invariant subspaces is readily described in invariant terms as well. It has the form

$$V \otimes U = (V \otimes f_1) \oplus \ldots \oplus (V \otimes f_m),$$

where $(f_1, \ldots, f_m)$ is a basis of U. The map

$$x \mapsto x \otimes f_i$$

is an isomorphism of the representations T and $(TI)_{V \otimes f_i}$.

2. The representation TT' is, according to (1) and (3), described in terms of matrices as

$$(5) \qquad TT'(g)X = T(g)XT(g)^{-1} \qquad (X \in \mathrm{L}_n(K)).$$

This shows that $TT' = \mathrm{Ad} \circ T$, where Ad is the adjoint representation of $\mathrm{GL}(V)$ (Example 2, 1.6).

3. The representation $(T')^2 = T'T'$ is given in terms of matrices by the formula

$$(6) \qquad (T')^2(g)X = T(g)'^{-1}XT(g)^{-1} \qquad (X \in \mathrm{L}_n(K)).$$

Consequently, $(T')^2 = \Phi \circ T$, where Φ is the natural representation of $\mathrm{GL}(V)$ in the space $\mathrm{B}(V) = V' \otimes V'$ (see Example 3, 1.6).

4. In the case where one of the representations T, S is *one-dimensional*, the product TS has a particularly simple meaning. Suppose, for example, that $T: G \to \mathrm{GL}(V)$ is an arbitrary representation of the group G, and $S: G \to \mathrm{GL}(U)$ is a one-dimensional representation, i.e., a homomorphism of G into K^*. Pick a nonnull vector $u_0 \in U$. The map

$$\sigma: x \mapsto x \otimes u_0$$

is an isomorphism of V onto $V \otimes U$. For every $g \in G$ we have

$$TS(g)\sigma x = T(g)x \otimes S(g)u_0 = S(g)T(g)x \otimes u_0 = \sigma S(g)T(g)x,$$

where $S(g)T(g)$ is understood as the product of the operator $T(g)$ by the scalar $S(g)$. Thus TS is isomorphic to the representation

$$g \mapsto S(g)T(g) \in GL(V).$$

The product of an arbitrary finite number of representations is defined in a natural manner. In particular, if T is a representation of G in a vector space V, then $T^kT'^\ell$ is a representation of G in the space of tensors of type (k, ℓ) over V. Such representations are often encountered in mathematical and physical applications of representation theory.

Examples 2 and 3 (see also the corresponding examples in 1.6) show that a product of irreducible representations is not necessarily irreducible. Decomposing such a product into a sum of irreducible representations is one of the most important problems of representation theory.

3.4. Tensor Products of Representations of Two Groups. Let $T\colon G \to \mathrm{GL}(V)$ and $S\colon H \to \mathrm{GL}(U)$ be two representations of the groups G and H.

Definition. The TENSOR PRODUCT OF T AND S is the representation $T \otimes S$ of the group $G \times H$ in the space $V \otimes U$, defined by the rule

$$(T \otimes S)(g, h) = T(g) \otimes S(h)$$

(here we should really write $(T \otimes S)((g, h))$!).

In the matrix interpretation (cf. 3.3), the tensor product of two finite-dimensional representations takes the form

$$(T \otimes S)(g, h)X = T(g)XS(h)'. \tag{7}$$

Here, in contrast to formula (3), the factors on the left and right of the matrix X are independent (even if $G = H$).

Let i_1 denote the canonical imbedding of the group G in $G \times H$, i.e., $i_1(g) = (g, e)$. Then $(T \otimes S)\circ i_1$ is a representation of G. It is clear from the definition that $(T\otimes S)\circ i_1 = TI$, where I is the trivial representation of G in U. Therefore (see Example 1 of 3.3),

$$(T \otimes S)\circ i_1 \simeq (\dim U)T.$$

Similarly, if i_2 denotes the canonical imbedding of H in $G \times H$, then

$$(T \otimes S)\circ i_2 = IS,$$

where I is now the trivial representation of H in V. Therefore,

$$(T \otimes S)\circ i_2 \simeq (\dim V)S.$$

A very important example. Let T be a representation of the group G in the n-dimensional space V. Consider the representation $T \otimes T'$ of $G \times G$ in $V \otimes V'$. In terms of matrices it is described as

$$(T \otimes T')(g_1, g_2)X = T(g_1)XT(g_2)^{-1} \qquad (X \in \mathrm{L}_n(K)) \tag{8}$$

(cf. Example 2, 3.3).

If one uses the canonical identification of the spaces $V \otimes V'$ and $\mathrm{L}(V)$ (see Appendix 2), then $T \otimes T'$ can be described in invariant form as

$$(T \otimes T')(g_1, g_2)\xi = T(g_1)\xi T(g_2)^{-1} \qquad (\xi \in \mathrm{L}(V)). \tag{9}$$

This follows from (8). In fact, the matrix assigned to a linear operator when that operator is viewed as an element of the tensor product $V \otimes V'$ coincides with its usual matrix (see Appendix 2), and to the product of matrices there corresponds the product of linear operators.

3.5. Extension of the Ground Field. Let K' be an extension of the field K.

The group $\mathrm{GL}_n(K)$ is then a subgroup of $\mathrm{GL}_n(K')$. Consequently, every n-dimensional matrix representation T of an arbitrary group G over K can also be regarded as an n-dimensional representation of G *over* K', and in this capacity we denote it by $E^K_{K'}T$. In exact terms, $E^K_{K'}T$ is the composition of the canonical imbedding of $\mathrm{GL}_n(K)$ in $\mathrm{GL}_n(K')$ with the representation T.

A similar operation can be defined for linear representations.

First of all, every vector space V over K can be included in a vector space $E^K_{K'}V$ over K' in such a way that a basis (e) of V is simultaneously a basis (over K') of $E^K_{K'}V$. Accordingly, every linear transformation α of V extends to a linear transformation $E^K_{K'}\alpha$ of $E^K_{K'}V$ that, in the basis (e), has the same matrix as α:

$$(E^K_{K'}\alpha)_{(e)} = \alpha_{(e)}.$$

This yields an imbedding $\mathrm{L}(V) \subset \mathrm{L}(E^K_{K'}V)$, which in turn induces a group imbedding

$$\mathrm{GL}(V) \subset \mathrm{GL}(E^K_{K'}V).$$

Now let T be a linear representation of G in V. Setting

$$(E^K_{K'}T)(g) = E^K_{K'}T(g),$$

we obtain a linear representation of G in $E^K_{K'}V$.

3.6. Let us examine in more detail the most important case for applications: $K = \mathbf{R}$, $K' = \mathbf{C}$. The operation $E^{\mathbf{R}}_{\mathbf{C}}$ is called COMPLEXIFICATION (of vector spaces, linear operators, and representations). For simplicity, we shall denote it by the index $\mathbf{C}$, writing $T_{\mathbf{C}}$, for example, instead of $E^{\mathbf{R}}_{\mathbf{C}}T$.

One reason why complexification is often useful is that in a complex vector space, in contrast to a real one, every linear operator has an eigenvector.

Example. By complexifying the representation of $\mathbf{R}$ through rotations of the Euclidean plane (Example 1, 0.7), we are allowed to write it in the form

$$t \mapsto \begin{pmatrix} \mathrm{e}^{it} & 0 \\ 0 & \mathrm{e}^{-it} \end{pmatrix}.$$

To do this we must pass from an orthonormal basis (e_1, e_2) of the Euclidean plane to the new basis $(e_1 - ie_2, e_1 + ie_2)$.

Theorem 5. *Two finite-dimensional real linear representations are isomorphic if and only if their complexifications are isomorphic.*

Proof. Let T_1 and T_2 be n-dimensional real representations of the group G. In matrix form, the fact that T_1 and T_2 are isomorphic means that there exists a real matrix C satisfying the following two conditions:

(C1) $CT_1(g) = T_2(g)C \qquad$ for all $g \in G$;

(C2) $\det C \neq 0$.

Similarly, the fact that the representations $(T_1)_{\mathbf{C}}$ and $(T_2)_{\mathbf{C}}$ are isomorphic means that there exists a complex matrix C satisfying the same two conditions. This clearly proves the implication $(T_1 \simeq T_2) \Rightarrow ((T_1)_{\mathbf{C}} \simeq (T_2)_{\mathbf{C}})$.

To prove the converse implication, we remark that condition (C1) is in fact a homogeneous system of linear equations with real coefficients for the entries of the matrix C. Its general solution has the form $t_1C_1 + t_2C_2 + \ldots + t_mC_m$, where $C_1, \ldots, C_m$ are linearly independent real matrices. The determinant $\det(t_1C_1 + t_2C_2 + \ldots + t_mC_m)$ is a polynomial d in $t_1, \ldots, t_m$ with real coefficients.

Suppose now that $(T_1)_{\mathbf{C}} \simeq (T_2)_{\mathbf{C}}$. Then there exist complex numbers $\tau_1, \ldots, \tau_m$ such that $d(\tau_1, \ldots, \tau_m) \neq 0$, and so d is not the zero polynomial. But in this case there also exist real numbers $\tau'_1, \ldots, \tau'_m$ such that $d(\tau'_1, \ldots, \tau'_m) \neq 0$. Therefore, $T_1 \simeq T_2$, as needed. □

3.7. What can be said about the connection between the invariant subspaces of the representation $T: G \to \mathrm{GL}(V)$ and those of its complexification $T_{\mathbf{C}}: G \to \mathrm{GL}(V_{\mathbf{C}})$? Obviously, *the complexification $U_{\mathbf{C}}$ of any T-invariant subspace $U \subset V$ is a $T_{\mathbf{C}}$-invariant subspace.* However, $V_{\mathbf{C}}$ may contain invariant subspaces which do not arise in this manner. For instance, in the example of 3.6, the representation T is irreducible, whereas $T_{\mathbf{C}}$ possesses one-dimensional invariant subspaces.

To answer the question posed above, we introduce the operation of *complex conjugation* in the space $V_{\mathbf{C}}$. Each vector $z \in V_{\mathbf{C}}$ can be uniquely written as $z = x + iy$, with $x, y \in V$. Put $\bar{z} = x - iy$. In a basis consisting of real vectors (i.e., vectors in V), the coordinates of $\bar{z}$ are the complex conjugates of the coordinates of z.

Complex conjugation is an anti-linear transformation, that is, $\overline{z+u} = \bar{z} + \bar{u}$ and $\overline{cz} = \bar{c}\bar{z}$ for $c \in \mathbf{C}$. It follows that it transforms every subspace of $V_{\mathbf{C}}$ into a subspace of the same dimension.

Lemma. *The subspace $W \subset V_{\mathbf{C}}$ is the complexification of some subspace $U \subset V$ if and only if $\overline{W} = W$.*

Proof. It is plain that if $W = U_{\mathbf{C}}$, then $\overline{W} = W$. Conversely, suppose that $\overline{W} = W$. Then the subspace W contains, together with each vector $z = x + iy \quad (x, y \in V)$, the vector $\bar{z} = x - iy$, and hence also the linear combinations $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2}(z - \bar{z})$. Consequently, $W = U_{\mathbf{C}}$, where $U = W \cap V$. □

Since the operators $T_{\mathbf{C}}(g)$, for $g \in G$, take real vectors into real vectors, they commute with complex conjugation:

$$T_{\mathbf{C}}(g)\bar{z} = \overline{T_{\mathbf{C}}(g)z} \qquad (z \in V_{\mathbf{C}}). \tag{10}$$

Therefore, $\overline{W}$ is an invariant subspace whenever W is invariant. Consider the subspaces $W + \overline{W}$ and $W \cap \overline{W}$. They are also G-invariant. Moreover, they coincide with their complex conjugates:

$$\overline{W + \overline{W}} = \overline{W} + W = W + \overline{W}, \qquad \overline{W \cap \overline{W}} = \overline{W} \cap W = W \cap \overline{W}.$$

By the preceding lemma, $W + \overline{W}$ and $W \cap \overline{W}$ are complexifications of G-invariant subspaces of V.

Theorem 6. *Let $T: G \to \mathrm{GL}(V)$ be an irreducible real linear representation. Then $T_{\mathbf{C}}$ is either irreducible or the sum of two irreducible representations. In the second case $V_{\mathbf{C}}$ decomposes into the direct sum of two complex-conjugate minimal invariant subspaces.*

Proof. Let W be a minimal invariant subspace of $V_{\mathbf{C}}$. Then $W + \overline{W}$ is the complexification of an invariant subspace of V, which must coincide with V in view of the irreducibility of the representation T. Hence $W + \overline{W} = V_{\mathbf{C}}$.

Now consider the invariant subspace $W \cap \overline{W} \subset W$. It must either coincide with W or be the null subspace. In the first case, $W = \overline{W} = V_{\mathbf{C}}$ and the representation $T_{\mathbf{C}}$ is irreducible. In the second case, $V_{\mathbf{C}} = W \oplus \overline{W}$, and $T_{\mathbf{C}}$ decomposes into the sum of two irreducible representations. □

Examples.

1. In the example considered in 3.6, the second alternative of Theorem 6 holds true.

2. In the Euclidean plane, consider an equilateral triangle $A_1A_2A_3$ centered at the origin. For each permutation $\sigma \in S_3$ we let $T(\sigma)$ denote the orthogonal transformation that takes the vertex A_i into $A_{\sigma(i)}$ $(i = 1, 2, 3)$. $T(\sigma)$ is either

the identity transformation, or the rotation by $2\pi/3$ in one of the two possible directions, or the reflection in one of the altitudes of the triangle $A_1A_2A_3$. We thus get a faithful two-dimensional real representation T of the group S_3. It is obviously irreducible.

Using Theorem 6 it is readily established that the complexification $T_{\mathbf{C}}$ is also irreducible. Otherwise it would decompose into the sum of two one-dimensional representations, and in a suitable basis all the operators $T_{\mathbf{C}}(\sigma)$, $\sigma \in S_3$, would be given by diagonal matrices. The latter is impossible, however, since diagonal matrices commute with one another, whereas the group S_3 is not commutative.

Assertions similar to Theorems 5 and 6 permit us to reduce all questions concerning real representations to questions concerning complex representations. Since complex representations are simpler to describe than real ones, they constitute the main object of representation theory.

3.8. Lifting and Factoring. In 0.10 we considered the composition of a linear transformation and a homomorphism. A particular case of that construction is the composition $S \circ p$ of a linear representation

$$S: G/N \to \mathrm{GL}(V)$$

of a quotient group G/N and the canonical homomorphism

$$p: G \to G/N.$$

We call it the LIFT OF THE REPRESENTATION S. The representation $S \circ p$ has the property that its kernel contains the subgroup N:

$$(S \circ p)(h) = \varepsilon \qquad \text{for all } h \in N.$$

Conversely, every linear representation T of G whose kernel contains N takes all elements of a given coset of N in G into the same operator and so can be "factored" through p, i.e., $T = S \circ p$, where S is a linear representation of the quotient group G/N. We call the transition from T to S FACTORING THE REPRESENTATION T WITH RESPECT TO THE SUBGROUP N.

We thus establish a *one-to-one correspondence between the linear representations of the quotient group G/N and those linear representations of G whose kernel contains N.*

Examples.

1. $G = \mathrm{GL}_n(K)$, $N = \mathrm{SL}_n(K)$. Since N is the kernel of the epimorphism

$$\det: \mathrm{GL}_n(K) \to K^*,$$

$G/N \simeq K^*$. Every linear representation S of K^* can be lifted to yield a linear representation T of $\mathrm{GL}_n(K)$. For example, if $S\colon t \mapsto t^m$ (a one-dimensional representation), then $T\colon A \mapsto (\det A)^m$.

2. $G = S_4$, $N = \{\varepsilon, (12)(34), (13)(24), (14)(23)\}$. ($N$ is called Klein's four-group.) Each coset of N in G contains exactly one permutation that keeps 1 fixed; hence, $G/N \simeq S_3$. This observation can be used to build a linear representation of S_4 from any given linear representation of S_3.

In particular, from the two-dimensional irreducible representation of S_3 constructed in Example 2 of 3.7 one obtains a two-dimensional irreducible representation of S_4.

3. All linear representations of the group $\mathbf{Z}_m = \mathbf{Z}/m\mathbf{Z}$ are obtained by factoring linear representations T of $\mathbf{Z}$ with the property

$$T(m) = T(1)^m = \varepsilon.$$

3.9. The considerations of 3.8 apply to the description of *one-dimensional linear representations.*

Let T be a one-dimensional representation of G. Then

$$T(ghg^{-1}h^{-1}) = T(g)T(h)T(g)^{-1}T(h)^{-1} = 1$$

for all $g, h \in G$. Consequently, $\operatorname{Ker} T$ contains the subgroup of G generated by all commutators $(g, h) = ghg^{-1}h^{-1}$. The latter is called the COMMUTATOR SUBGROUP OF G and is denoted (G, G). It is a normal subgroup of G, since the set of all commutators is invariant under any (in particular, any inner) automorphism a of G:

$$a((g, h)) = (a(g), a(h)).$$

(Recall that a normal subgroup is by definition a subgroup invariant under all inner automorphisms.)

Therefore, *every one-dimensional representation of the group G is the lift of a one-dimensional representation of the quotient group $G/(G,G)=A(G)$.* We remark that $A(G)$ is *abelian*. In fact, let p denote the canonical homomorphism of G onto $A(G)$. Then for any $g, h \in G$,

$$(p(g), p(h)) = p((g, h)) = 1,$$

whence $p(g)p(h) = p(h)p(g)$.

Example. Let us find all one-dimensional representations of the symmetric group S_n. To this end we compute its commutator subgroup. Since the commutator of any two permutations is an even permutation, $(S_n, S_n) \subset A_n$. We show that $(S_n, S_n) = A_n$.

It is a straightforward matter to check that the commutator of the transpositions (ik) and (jk) (with distinct i, j, k) is the triple cycle (ijk). Let $H \subset S_n$ denote the subgroup generated by all triple cycles. Using a permutation of the form (ijk) one can take 1 to any prescribed element of the set $\{1, 2, \ldots, n\}$; then, using a permutation of the form $(2jk)$, one can take 2 to any prescribed element of $\{1, 2, \ldots, n\}$ while keeping 1 in its place, and so on, up to and including $n-2$. This shows that for every $\sigma \in A_n$ there exists an $\eta \in H$ such that $\eta(i) = \sigma(i)$ for $i = 1, 2, \ldots, n-2$. Since η and σ have the same parity, one also has that $\eta(n-1) = \sigma(n-1)$ and $\eta(n) = \sigma(n)$. Hence, $H = A_n$, and since $(S_n, S_n) \supset H$, we conclude that $(S_n, S_n) = A_n$.

The quotient group $S_n/A_n \simeq \mathbf{Z}_2$ has two one-dimensional representations: one trivial, and the other taking the generator to -1. To the first there corresponds the trivial one-dimensional representation I of S_n, while to the second there corresponds the representation Π which takes all even permutations to 1 and all odd permutations to -1.

Questions and Exercises

1. Describe the dual of a trivial representation.

2. Prove that if the representation T' is irreducible, then so is T.

3. Prove that $(R+S)' \simeq R' + S'$ for any two representations R and S of a group G.

4. Prove that if the representation T is completely reducible, then so is T'.

5. Prove that the identity representation of $\mathrm{SL}_2(K)$ is isomorphic to its dual.

6. Let $T\colon G \to \mathrm{GL}(V)$ be a completely reducible representation, and let $U \subset V$ be an invariant subspace. Show that $T \simeq T_U + T_{V/U}$.

7. Let T_1, T_2, and S be completely reducible finite-dimensional linear representations. Show that if $T_1 + S \simeq T_2 + S$, then $T_1 \simeq T_2$.

8. Prove that $(T_1 + T_2)S \simeq T_1 S + T_2 S$ for any representations T_1, T_2, and S of G.

9. Prove that $TS \simeq ST$ for any representations T and S of G.

10. Describe the square of a representation in terms of matrices.

11.* Let V and U be complex vector spaces, and let $\alpha \in \mathrm{L}(V)$, $\beta \in \mathrm{L}(U)$. The product of the representations $t \mapsto \mathrm{e}^{t\alpha}$ and $t \mapsto \mathrm{e}^{t\beta}$ of $\mathbf{C}$ is necessarily of the form $t \mapsto \mathrm{e}^{t\gamma}$, where $\gamma \in \mathrm{L}(V \otimes U)$. Find the operator γ.

12. Let T and S be an irreducible representation and a one-dimensional representation, respectively, of the group G. Show that TS is irreducible.

13. Prove formula (9) without resorting to the matrix interpretation.

14. Interpret the representation $T \otimes T$ in terms of matrices, and compare it with T^2.

15. Prove that the complexification of any odd-dimensional irreducible real representation is irreducible.

16. Find all finite-dimensional representations of O_n whose kernels contain SO_n.

17. Find all one-dimensional representations of the group A_4.

18.* Prove that the commutator subgroup of $\mathrm{GL}_n(\mathbf{R})$ is equal to $\mathrm{SL}_n(\mathbf{R})$.

4. Properties of Irreducible Complex Representations

In this section we consider only finite-dimensional representations, except for 4.1.

4.1. Morphisms. The notion of an isomorphism of linear representations was defined in the Introduction. In group theory it is well known that, in addition to isomorphisms, homomorphisms also play an important role. Similarly, arbitrary linear maps, and not only isomorphisms, are important in linear algebra. In the theory of linear representations one considers an analogous generalization of the notion of isomorphism.

Definition. Let $T_1 \colon G \to \mathrm{GL}(V_1)$ and $T_2 \colon G \to \mathrm{GL}(V_2)$ be linear representations of the group G. A MORPHISM OF T_1 INTO T_2 is an arbitrary linear map $\sigma \colon V_1 \to V_2$ satisfying the condition

(1) $$\sigma T_1(g) = T_2(g)\sigma \qquad \text{for all } g \in G.$$

[Translator's note: Such a σ is also referred to as an INTERTWINING OPERATOR, and one says that σ INTERTWINES T_1 and T_2.]

Example. Let $V = U \oplus W$ be a decomposition of the representation space of T into a direct sum of invariant subspaces. Then the projection onto U parallel to W is a morphism of T into the representation T_U.

It follows from (1) that *the kernel* $\operatorname{Ker}\sigma$ *of the morphism* σ *is a subspace invariant under* T_1, *while its image* $\operatorname{Im}\sigma = \sigma(V_1)$ *is a subspace invariant under* T_2.

If the representations T_1 and T_2 are irreducible, only two cases are possible:

1) $\operatorname{Ker}\sigma = 0, \quad \operatorname{Im}\sigma = V_2$

or

2) $\operatorname{Ker}\sigma = V_1, \quad \operatorname{Im}\sigma = 0.$

In the first case σ is an isomorphism, while in the second it is the null map. We have thus proved

Theorem 1. *Every morphism of irreducible representations is either an isomorphism or the null map.* □

In spite of its extreme simplicity, Theorem 1 has important applications. In particular, with its help one can establish the following result, which will be used in what follows.

Theorem 2. *Suppose that the space* V *of the representation* T *splits into a direct sum of minimal invariant subspaces* $V_1, \ldots, V_m$ *such that the representations* $T_i = T_{V_i}$ *are pairwise nonisomorphic. Then every invariant subspace* $U \subset V$ *is the sum of a certain number of subspaces* V_i. (Cf. Remark 2 in 1.5.)

Proof. Representation T_U is completely reducible, being a subrepresentation of the completely reducible representation T. Consequently, U is a sum of minimal invariant subspaces. It suffices to consider the case where U is itself minimal. Suppose this is the case. Let p_i denote the projection of U onto V_i. It is a morphism of the irreducible representation T_U into T_i. It follows from the assumptions of the theorem that T_U can be isomorphic only to one of the representations T_i, say, to T_1. Then p_1 is an isomorphism, whereas $p_2 = \ldots = p_m = 0$. This means that $U = V_1$, which completes the proof of the theorem. □

4.2. The Schur Lemma. A morphism of a linear representation T into itself is called an endomorphism of T. In other words, an endomorphism of the representation T of the group G is a linear operator which commutes with all the operators $T(g)$, $g \in G$. An example is the identity operator ε.

Theorem 3 (Schur's Lemma). *Every endomorphism of an irreducible complex representation T is scalar, i.e., it has the form $c\varepsilon$, with $c \in \mathbf{C}$.*

Proof. Let σ be an endomorphism of T:

(2) $$\sigma T(g) = T(g)\sigma \qquad \text{for all } g \in G.$$

Let c be any of the eigenvalues of the operator σ. Then it follows from (2) that

$$(\sigma - c\varepsilon)T(g) = T(g)(\sigma - c\varepsilon) \qquad \text{for all } g \in G,$$

and so $\sigma - c\varepsilon$, too, is an endomorphism of T. Since $\det(\sigma - c\varepsilon) = 0$, Theorem 1 yields $\sigma - c\varepsilon = 0$, i.e., $\sigma = c\varepsilon$. □

Corollary. *Let T_1 and T_2 be isomorphic irreducible complex linear representations of the group G. Let σ be a fixed isomorphism of T_1 onto T_2. Then every morphism of T_1 into T_2 has the form $c\sigma$, where $c \in \mathbf{C}$.*

Proof. Let τ be a morphism of T_1 into T_2. Then $\sigma^{-1}\tau$ is an endomorphism of T_1. By Schur's Lemma, $\sigma^{-1}\tau = c\varepsilon$, with $c \in \mathbf{C}$, and so $\tau = c\sigma$. □

Schur's Lemma permits us to describe the invariant subspaces of a completely reducible complex representation in the situation opposite to the one considered in Theorem 2, namely when all irreducible components are mutually isomorphic.

Theorem 4. *Let T be an irreducible complex representation of the group G in the space V, and let I be the trivial representation of G in the space U. Then every minimal subspace $W \subset V \otimes U$ invariant under the representation TI has the form $V \otimes u_0$, where $u_0 \in U$.*

Proof. TI is isomorphic to the sum of a certain number of copies of the representation T (see Example 1, 3.3). By Theorem 3 of 3.2, $(TI)_W \simeq T$. Let σ be an arbitrary isomorphism of the representations $(TI)_W$ and T.

Pick a basis $(f_1, \ldots, f_m)$ of U. Every element of $V \otimes U$ can be uniquely written as $x_1 \otimes f_1 + \ldots + x_m \otimes f_m$, where $x_i \in V$. In particular, for every $w \in W$,

$$w = \sigma_1(w) \otimes f_1 + \ldots + \sigma_m(w) \otimes f_m.$$

It is clear that the vectors $\sigma_i(w)$ depend linearly on w, and that $\sigma_i((TI)(g)w) = T(g)\sigma_i(w)$ for all $g \in G$. Hence, σ_i is a morphism of the representation $(TI)_W$ into T. By the Corollary to Theorem 3, $\sigma_i = c_i\sigma$, with $c_i \in \mathbf{C}$. Consequently,

$$w = \sigma(w) \otimes (c_1 f_1 + \ldots + c_m f_m)$$

for all $w \in V$, and so indeed $W = V \otimes u_0$, where $u_0 = c_1 f_1 + \ldots + c_m f_m$. □

4.3. Irreducible Representations of Abelian Groups. One of the basic facts established in linear algebra is that every linear operator in a complex vector space possesses a one-dimensional invariant subspace. This implies that every irreducible complex linear representation of a cyclic group is one-dimensional. The next result generalizes this assertion.

Theorem 5. *Every irreducible complex linear representation of an abelian group is one-dimensional.*

PROOF. Let G be an abelian group. Let T be an irreducible complex representation of G. For $g_0, g \in G$ we have

$$T(g_0)T(g) = T(g_0 g) = T(g g_0) = T(g)T(g_0).$$

This means that $T(g_0)$ is an endomorphism of T. By Schur's Lemma, $T(g_0)$ is a scalar operator. Since this holds true for every $g_0 \in G$, it follows that any subspace is invariant under T. This forces T to be one-dimensional. $\square$

Corollary. *Every complex linear representation of an abelian group possesses a one-dimensional invariant subspace.*

PROOF. In fact, every minimal invariant subspace is, by Theorem 5, one-dimensional. $\square$

4.4. Tensor Products of Irreducible Representations.

Theorem 6. *The tensor product of two irreducible complex representations* $T\colon G \to \mathrm{GL}(V)$ *and* $S\colon H \to \mathrm{GL}(U)$ *of the groups* G *and* H *is an irreducible representation of the group* $G \times H$.
(For the definition of the tensor product of two representations, see 3.4.)

PROOF. The tensor product $T \otimes S$ is a representation of $G \times H$ in the space $V \otimes U$. Let $W \subset V \otimes U$ be a nonnull invariant subspace. We claim that $W = V \otimes U$.

It is obvious that W is invariant under the representation $TI = (T \otimes S) \circ i_1$ of G (see 3.4). By Theorem 4, W contains a subspace of the form $V \otimes u_0$, where $u_0 \in U$, $u_0 \neq 0$.

Now for each $x \in V$ consider the subspace

$$U(x) = \{\, u \in U \mid x \otimes u \in W \,\} \subset U.$$

It is H-invariant. In fact, if $x \otimes u \in W$, then also

$$x \otimes S(h)u = (T \otimes S)(e,h)(x \otimes u) \in W.$$

Moreover, $U(x) \ni u_0$. It follows from the irreducibility of the representation S that $U(x) = U$. This means that $x \otimes u \in W$ for all $x \in V$ and $u \in U$, and so $W = V \otimes U$, as claimed. $\square$

One can also prove the following converse of Theorem 4: *every irreducible complex linear representation of $G \times H$ is isomorpic to the tensor product of two irreducible representations of G and H.*

4.5 Spaces of Matrix Elements. Let $T: G \to \mathrm{GL}(V)$ be a complex linear representation. Let $(e) = (e_1, \ldots, e_n)$ be a basis of V. We put

$$T_{(e)}(g) = [T_{ij}(g)].$$

Definition. The functions $T_{ij} \in \mathbf{C}[G]$ are called the MATRIX ELEMENTS (or MATRIX COORDINATE FUNCTIONS) OF THE REPRESENTATION T relative to the basis (e).

Any linear combination of matrix elements

$$f = \sum_{i,j} c_{ij} T_{ij} \in \mathbf{C}[G] \qquad (c_{ij} \in \mathbf{C})$$

can be expressed invariantly (without using coordinates) as

(3) $$f(g) = \operatorname{tr} \xi T(g)$$

upon denoting by ξ the linear operator given in the basis (e) by the matrix $[c_{ji}]$. It follows from this invariant expression that *the linear span of the matrix elements does not depend on the choice of a basis.*

Definition. The SPACE OF MATRIX ELEMENTS OF THE REPRESENTATION T, denoted by $\mathrm{M}(T)$, is the linear span of the matrix elements of T (relative to some basis).

We emphasize that $\mathrm{M}(T)$ is a subspace of the space $\mathbf{C}[G]$ of all $\mathbf{C}$-valued functions on the group G.

We mention *two simple properties.*

1) *If* $T_1 \simeq T_2$, *then* $\mathrm{M}(T_1) = \mathrm{M}(T_2)$. In fact, in compatible bases the representations T_1 and T_2 are given by identical matrices.

2) *If* $T = T_1 + \ldots + T_m$, *then*

$$\mathrm{M}(T) = \mathrm{M}(T_1) + \ldots + \mathrm{M}(T_m).$$

In fact, in a suitable basis the operators $T(g)$, for $g \in G$, are given by block-diagonal matrices, the diagonal blocks of which are the matrices of the operators $T_1(g), \dots, T_m(g)$ (see 3.2).

The reason for the interest attached to *the spaces of matrix elements* of various linear representations of the group G is that *they are invariant under left and right translations.*

Specifically, let f be the function given by formula (3). Then

$$\begin{aligned} f(g_2^{-1} g g_1) &= \operatorname{tr} \xi\, T(g_2^{-1} g g_1) = \operatorname{tr} \xi\, T(g_2)^{-1} T(g) T(g_1) \\ &= \operatorname{tr}(T(g_1) \xi T(g_2)^{-1}) T(g) = \operatorname{tr} \eta\, T(g), \end{aligned} \tag{4}$$

where $\eta = T(g_1)\xi T(g_2)^{-1} = (T \otimes T')(g_1, g_2)\xi$ (see the Example in 3.4).

The result obtained can be interpreted as follows. Consider the map

$$\mu \colon \mathrm{L}(V) \to \mathbf{C}[G]$$

which takes each operator $\xi \in \mathrm{L}(V)$ into the function $f \in \mathbf{C}[G]$ given by (3), i.e.,

$$\mu(\xi)(g) = \operatorname{tr} \xi\, T(g) \qquad (\xi \in \mathrm{L}(V)). \tag{5}$$

Next, consider the linear representation Reg of the group $G \times G$ in $\mathbf{C}[G]$ defined by the rule

$$(\mathrm{Reg}(g_1, g_2) f)(g) = f(g_2^{-1} g g_1). \tag{6}$$

Reg combines the left and right regular representations of G.

Definition. The linear representation Reg of $G \times G$ in $\mathbf{C}[G]$ given by formula (6) is called the (TWO-SIDED) REGULAR REPRESENTATION.

Formula (4) says that μ *is a morphism of the representation* $T \otimes T'$ *into* Reg. The image of μ is precisely the space $\mathrm{M}(T)$ of matrix elements of T.

4.6. If T is an irreducible complex representation, then, by Theorem 4, $T \otimes T'$ is also irreducible, and so $\operatorname{Ker} \mu = 0$. We have thus proved

Theorem 7. *Let* $T \colon G \to \mathrm{GL}(V)$ *be an irreducible complex linear representation. Then the map* μ *defined by formula (5) is an isomorphism of the representations* $T \otimes T'$ *and* $\mathrm{Reg}_{\mathrm{M}(T)}$. □

Corollary 1. $\dim \mathrm{M}(T) = n^2$, *where* $n = \dim V$. □

Let I be the trivial representation G in a space V. Setting $g_1 = e$ or $g_2 = e$ in (4) we obtain

Corollary 2. *The map μ establishes an isomorphism of the representations IT' and $L_{\mathrm{M}(T)}$, as well as of the representations TI' and $R_{\mathrm{M}(T)}$.* □

Corollary 3. $L_{\mathrm{M}(T)} \simeq nT'$ *and* $R_{\mathrm{M}(T)} \simeq nT$. (See Example 1, 3.3.) □

Corollary 4. *Let T_1 and T_2 be nonisomorphic irreducible complex representations of the group G. Then the representations* $\mathrm{Reg}_{\mathrm{M}(T_1)}$ *and* $\mathrm{Reg}_{\mathrm{M}(T_2)}$ *of $G \times G$ are not isomorphic.*

PROOF. Suppose $\mathrm{Reg}_{\mathrm{M}(T_1)}$ and $\mathrm{Reg}_{\mathrm{M}(T_2)}$ are isomorphic. Then so are their restrictions to the subgroup $G \times \{e\}$, i.e., the representations $R_{\mathrm{M}(T_1)}$ and $R_{\mathrm{M}(T_2)}$ of G. However, by Corollary 3,

$$R_{\mathrm{M}(T_1)} \simeq n_1 T_1 \quad \text{and} \quad R_{\mathrm{M}(T_2)} \simeq n_2 T_2$$

(with $n_1 = \dim T_1$ and $n_2 = \dim T_2$), and hence $R_{\mathrm{M}(T_1)} \not\simeq R_{\mathrm{M}(T_2)}$, a contradiction. □

In view of Corollary 3 of 3.2, Corollary 4 implies

Corollary 5. *Let $T_1, \dots, T_q$ be pairwise nonisomorphic irreducible complex representations of the group G. Then the subspaces* $\mathrm{M}(T_1), \dots, \mathrm{M}(T_q)$ *of* $\mathbf{C}[G]$ *are linearly independent.* □

Next we find the *explicit form of the decomposition of the space* $\mathrm{M}(T)$ *into a direct sum of minimal left-invariant subspaces.*

Let $(e) = (e_1, \dots, e_n)$ be a basis of V, and $(\varepsilon) = (\varepsilon_1, \dots, \varepsilon_n)$ be the dual basis of V'. Relative to (e), the linear operator $e_j \otimes \varepsilon_i$ is given by the matrix E_{ji} whose only nonzero entry, equal to one, is in the (j, i) site. Consequently,

$$\mu(e_j \otimes \varepsilon_i) = T_{ij}. \tag{7}$$

Proceeding from the decomposition

$$V \otimes V' = \sum (e_j \otimes V')$$

of $V \otimes V'$ into a direct sum of minimal IT'-invariant subspaces (see Example 1 of 3.3) we obtain, using the isomorphism μ, the sought-for decomposition of the space $\mathrm{M}(T)$:

$$\mathrm{M}(T) = \sum \mu(e_j \otimes V').$$

A basis of the j-th component of this decomposition is provided by the entries of the j-th column of the matrix $[T_{ij}]$.

The decomposition of $\mathrm{M}(T)$ into a direct sum of minimal right-invariant subspaces is obtained in a similar manner. A basis of the i-th component of this second decomposition is provided by the entries of the i-th row of the matrix $[T_{ij}]$.

Example. Let G be a cyclic group of order m with generator a. Consider its one-dimensional representations

$$T_k(a^x) = \omega^{kx} \qquad (k = 0, 1, \ldots, m-1),$$

where $\omega = \mathrm{e}^{\frac{2\pi i}{m}}$. They are obviously pairwise nonisomorphic. Each space $\mathrm{M}(T_k)$ is one-dimensional: it is spanned by the function T_k. By Corollary 5, the functions $T_0, T_1, \ldots, T_{m-1}$ are linearly independent. This can also be verified directly: the matrix constructed from the values of these functions has the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(m-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \cdots & \omega^{(m-1)^2} \end{pmatrix},$$

and its determinant (a Vandermonde determinant) is different from zero.

4.7. Uniqueness of the Invariant Inner Product. As we saw in Section 2, introducing an invariant inner product in the representation space can be a very useful step. There arises naturally the problem of describing all such inner products. (By an inner product in a complex vector space we shall mean an arbitrary positive definite Hermitian sesquilinear function.)

To study this problem we need the following

Lemma. *Let f and f_0 be two inner products in the complex vector space V. Then there exists a linear operator σ such that*

$$f(x, y) = f_0(\sigma x, y) \tag{8}$$

for all $x, y \in V$.

Proof. Both sides of equation (8) are linear in x and anti-linear in y. Hence it suffices to check that (8) holds for vectors forming a basis. Let $(e) = (e_1, \ldots, e_n)$ be a basis of V orthonormal with respect to the inner product f_0.

Let σ denote the linear operator given in this basis by the matrix $[f(e_j, e_i)]$. Then

$$f_0(\sigma e_i, e_j) = f_0(\sum_k f(e_i, e_k)e_k, e_j) = f(e_i, e_j),$$

and so (8) holds with the indicated σ. □

Theorem 8. *Let $T: G \to \mathrm{GL}(V)$ be an irreducible unitary representation. Then the T-invariant inner product in V is unique up to a constant factor.*

PROOF. Let f_0 and f be two invariant inner products in V. Let σ be a linear operator satisfying condition (8). We prove that σ is an endomorphism of the representation T.

For arbitrary $g \in G$ and $x, y \in V$ we have

$$\begin{aligned} f_0(T(g)^{-1}\sigma T(g)x, y) &= f_0(\sigma T(g)x, T(g)y) \\ &= f(T(g)x, T(g)y) \\ &= f(x, y) = f_0(\sigma x, y); \end{aligned} \tag{9}$$

here we used the invariance of f and f_0 under $T(g)$. Therefore, $T(g)^{-1}\sigma T(g) = \sigma$, and so $\sigma T(g) = T(g)\sigma$.

Now, by Schur's Lemma, $\sigma = c\varepsilon$ for some $c \in \mathbf{C}$. But then $f = cf_0$, as we needed to show. □

Theorem 9. *Let $T: G \to \mathrm{GL}(V)$ be a unitary representation. Let $U, W \subset V$ be minimal invariant subspaces such that $T_U \not\simeq T_W$. Then U and W are orthogonal with respect to any invariant inner product in V.*

PROOF. Fix an invariant inner-product in V and denote the corresponding orthogonal projection of the subspace W onto U by p. It is easy to see that p is a morphism of the representation T_W into T_U. By Theorem 1, $p = 0$, which means precisely that W is orthogonal to U. □

Questions and Exercises

1. Prove that the image of an invariant subspace under a morphism of representations is an invariant subspace.

2.* Let G be a doubly-transitive group of permutations, i.e., a subgroup of the symmetric group S_n with the following property: for any i, j, k, ℓ such that $i \neq j$ and $k \neq \ell$ there exists a permutation $\sigma \in G$ such that $\sigma(i) = k$ and $\sigma(j) = \ell$. Let M be a monomial representation of S_n (see Example 5 of 1.3). Prove that every endomorphism of the representation $M|_G$ has the form $a\varepsilon + b\eta$, where $a, b \in \mathbf{C}$ and $\eta(e_i) = e_1 + \ldots + e_n$ for all i.

3. Let $T_1, \ldots, T_q$ be pairwise nonisomorphic complex linear representations of the group G. Prove that the set of all morphisms of the representation $\sum k_i T_i$ into the representation $\sum \ell_i T_i$ is a vector space of dimension $\sum k_i \ell_i$.

4.* Using Problems 2 and 3, prove that if G is a doubly-transitive group of permutations, then the representation $M_0|_G$ (see Example 5 of 1.3) is irreducible.

5. Find all automorphisms of the representation of $\mathbf{R}$ by rotations in the Euclidean plane.

6. Let T be an arbitrary complex representation of the abelian group G. Show that in the representation space of T there is a basis relative to which all operators $T(g)$, for $g \in G$, are given by triangular matrices.

7. Prove that every irreducible real representation of an abelian group is one- or two-dimensional.

8.* Let G be a finite group and R the right regular representation of G (see 0.9). Give a direct proof of the fact that the dimension of the space of all morphisms of R into an irreducible representation T is equal to $\dim T$. Applying Problem 3, deduce from this that $R \simeq \sum_{i=1}^{q} (\dim T_i) T_i$, where $T_1, \ldots, T_q$ is a complete list of irreducible complex representations of G.

9. Under the assumptions of Theorem 4, prove that every G-invariant subspace of $V \otimes U$ has the form $V \otimes U_0$, where U_0 is a subspace of U.

10. Prove that the matrix elements of an irreducible complex representation are linearly independent. Is this assertion true for real representations?

11. Let $T\colon G \to \mathrm{GL}(V)$ be an irreducible complex representation. Prove that the linear span of the set $\{\, T(g) \mid g \in G \,\} \subset \mathrm{L}(V)$ equals $\mathrm{L}(V)$ (Burnside's Theorem).

12. Prove that every irreducible representation of the group G over an arbitrary field is isomorphic to a subrepresentation of the right regular representation of G.

13. The same for the left regular representation.

14.* Prove Corollaries 4 and 5 of Theorem 7 for linear representations over an arbitrary field.

15.* Let $T: G \to \mathrm{GL}(V)$ be an irreducible orthogonal representation. Prove that the invariant inner product in V is unique up to a (positive) constant factor.