CHAPTER 2

CONVEX HULLS

In this Chapter special results in the geometry of convex hulls are developed which are required for the analytic approximation theory in Chapter III. The main result of this chapter is the Integral Representation Theorem 2.12 which, simply stated, represents a continuous function $f : B \to \mathbf{R}^q$, with values in the convex hull of a connected open set $X \subset \mathbf{R}^q$, as a Riemann integral whose integrand is a continuous function with values in X: for all $b \in B$,

$$f(b) = \int_0^1 h(t, b)\, dt,$$

where $h\colon [0,1] \times B \to X$ is continuous. Over each point $b \in B$, the map h is defined to be a suitable reparametrization of a contractible loop in X which strictly surrounds the point $f(b)$. A parametrized family of contractible loops (parametrized by the space B) together with a specific parametrized family of contractions is called a C-structure with respect to f. A key observation, Lemma 2.2, important for subsequent applications, is that the space of C-structures is itself a contractible space. Employing the lemma, one is able to glue together local C-structures in a neighbourhood of each point $b \in B$ to obtain a global C-structure over B, with respect to which one constructs the map h in the above Riemann integral.

§1. Contractible Spaces of Surrounding Loops

Let $\rho\colon X \to \mathbf{R}^q$, $q \geq 1$, be continuous where X is path connected. Denote by $\mathrm{Conv}\, X$ the convex hull of the ρ-image of X in $\mathbf{R}^q$: the subset of $\mathbf{R}^q$ consisting of all convex linear combinations $\sum_{i=1}^N p_i \rho(x_i)$, $N \geq 1$, where $x_i \in X$ and $\sum_{i=1}^N p_i = 1$, $p_i \in [0,1]$, $i = 1, 2, \ldots, N$. (A convenient reference for the principal properties of convex hulls is Hörmander [23]). Let $\mathrm{IntConv}\, X$ denote the *interior* (possibly empty) of $\mathrm{Conv}\, X$ in $\mathbf{R}^q$. In case X is not path connected and $x \in X$, we employ the notation $\mathrm{Conv}(X, x)$, respectively $\mathrm{IntConv}(X, x)$, to denote the convex hull, respectively the interior of the convex hull, of the ρ-image of the path component in X to which x belongs. Note that in case $\rho\colon X \to \mathbf{R}^q$ is the inclusion of an open path connected subspace then the convex hull of X coincides with the interior of

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the convex hull of X. Indeed if $z \in \operatorname{Conv} X$ then there is an integer k, $0 \leq k \leq q$, and an affine k-simplex σ_k which contains z and whose vertices lie in X. Since X is open one easily constructs an affine q-simplex Δ_q whose vertices lie in X and which contains σ_k in its interior. More generally, if $\rho\colon X \to \mathbf{R}^q$ is a microfibration (V Lemma 5.7) then $\operatorname{Conv}(X, x) = \operatorname{IntConv}(X, x)$ for all $x \in X$.

Ample Sets. A continuous map $\rho\colon X \to \mathbf{R}^q$ is *ample* if for all $x \in X$, $\operatorname{Conv}(X, x) = \mathbf{R}^q$. That is, X is ample if the convex hull of the ρ-image of each path component of X is all of $\mathbf{R}^q$. Examples of open ample sets $X \subset \mathbf{R}^q$ which occur in the theory are $X = \mathbf{R}^q \setminus L$, where L is a smooth (or stratified) submanifold of codimension ≥ 2.

Nowhere Flat Sets. For the purposes of applying convex integration theory to the construction of solutions of non-linear systems of partial differential equations, the following affine notion is also required. A subset X in $\mathbf{R}^q$ is *nowhere flat* if for each affine $(q-1)$-dimensional hyperplane H of $\mathbf{R}^q$, the intersection $H \cap X$ is nowhere dense in X. For example, the unit sphere in $\mathbf{R}^q$ (in the Euclidean metric) is nowhere flat in $\mathbf{R}^q$, $q \geq 2$. Evidently, any affine subspace of $\mathbf{R}^q$ is flat i.e., not nowhere flat.

In case $X \subset \mathbf{R}^q$ is nowhere flat then $X \subset \bigcup_{x \in X} \overline{\operatorname{IntConv}(X, x)}$ in $\mathbf{R}^q$. Indeed, each $x \in X$ is a vertex of a non-degenerate affine q-simplex in $\mathbf{R}^q$, all of whose vertices are points in X.

It is useful for what follows to remark the obvious connection between Riemann integration and convex hulls. Let $f\colon [0,1] \to \mathbf{R}^q$ be continuous and let $X = \operatorname{im} f \subset \mathbf{R}^q$. Evidently a Riemann sum for the integral $\int_0^1 f(t)\,dt$ is a point of $\operatorname{Conv} X$. Furthermore, since X is compact, $\operatorname{Conv} X$ is compact and hence the integral $\int_0^1 f(t)\,dt \in \operatorname{Conv} X$. Geometrically, in case $[0,1]$ is parametrized by arc-length of the C^1-regular curve f, $\int_0^1 f(t)\,dt$ is the barycentre of X.

Surrounding Loops. Let $\rho\colon X \to \mathbf{R}^q$ be continuous; fix $x \in X$. Suppose $g\colon [0,1] \to X$ is a continuous loop at x: $g(0) = g(1) = x$. A point $z \in \operatorname{Conv}(X, x)$ is *surrounded*, respectively *strictly surrounded*, by the loop g if z lies in the convex hull, respectively the interior of the convex hull, of the composed path $\rho \circ g(t) \in \mathbf{R}^q$, $t \in [0,1]$. The main interest is in loops g in X that surround z and which are homotopically trivial i.e. there is a base point preserving homotopy of loops in X which connects g to the constant path C_x. Such loops are easily constructed as follows. Let $z \in \operatorname{Conv}(X, x)$, respectively $z \in \operatorname{IntConv}(X, x)$. There is a loop λ in X, based at x, such that $\operatorname{im} \rho \circ \lambda$ contains the vertices of a simplex that contains z, respectively contains the vertices of a q-simplex in $\mathbf{R}^q$ whose interior contains z. Then the product $g = \lambda * \lambda^{-1}$ is a contractible loop that surrounds z, respectively strictly surrounds z.

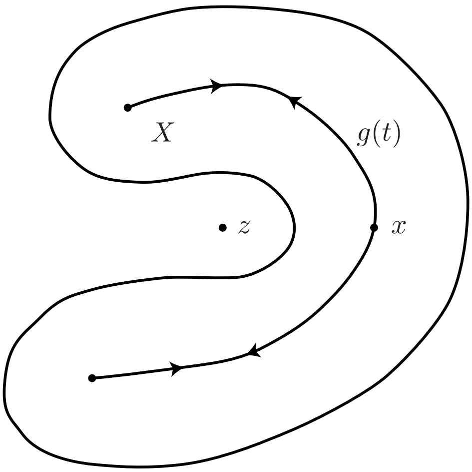

FIGURE 2.1

For each $z \in \mathrm{Conv}(X, x)$ let $X_x^z \subset C^0([0,1], X) \times C^0([0,1]^2, X)$ be the subspace, in the compact-open topology, of pairs of contractible loops at x which surround z, together with a contraction of the loop to C_x: pairs (g, G) where $g\colon [0,1] \to X$, $g(0) = g(1) = x$, g surrounds z and $G\colon [0,1]^2 \to X$ is a base point preserving homotopy which contracts the loop g to the constant path C_x. Thus, for all $(t,s) \in [0,1]^2$,

$$G(t,0) = x;\ G(t,1) = g(t);\ G(0,s) = G(1,s) = x. \tag{2.1}$$

Analogously, $\mathrm{int}\, X_x^z$ is the space of pairs (g, G) as above where g strictly surrounds z.

The space $\mathrm{int}\, X_x^z$ is an example of a space of C-structures (cf.§2) on an $\mathbf{R}^q$-bundle, in this case the bundle $\mathbf{R}^q \to *$ over a point, with respect to $z \in \mathrm{IntConv}(X, x)$, $x \in X$. Note that the loop g of the pair (g, G) is included for convenience; g is the map at time 1 of the homotopy G. The following elementary result is essential for the proof of the existence of C-structures in §2, Proposition 2.3.

Lemma 2.1. *For all $(x,z) \in X \times \mathrm{Conv}(X,x)$, respectively all $(x,z) \in X \times \mathrm{IntConv}(X,x)$, the space X_x^z, respectively $\mathrm{int}\, X_x^z$ is contractible.*

Proof. We prove the lemma for the space X_x^z; the proof for the space $\mathrm{int}\, X_x^z$ is similar and is omitted. We show that X_x^z deformation retracts in a canonical way to each point in X_x^z: there is a continuous map $\mathcal{D}\colon X_x^z \times X_x^z \times [0,1] \to X_x^z$ such that for all $u, v \in X_x^z$, the corresponding map $\mathcal{D}_u\colon X_x^z \times [0,1] \to X_x^z$,

$\mathcal{D}_u(v,t) = \mathcal{D}(u,v,t)$, satisfies the property:

$$\mathcal{D}_u(v,1) = v;\ \mathcal{D}_u(v,0) = u. \tag{2.2}$$

Thus for each $u \in X_x^z$ the map $\mathcal{D}_u$ is a deformation retract of X_x^z to u which depends continuously on $u \in X_x^z$.

Briefly, the map $\mathcal{D}$ is constructed canonically as follows. Let $u = (g,G), v = (h,H) \in X_x^z$ and let $*$ denote the product operator on the space of loops. Employing the contractions H, G one constructs a canonical homotopy of closed loops in X which connects h, g, and which, at each stage, consists of closed loops which surround z:

$$h \sim h * C_x \sim h * g \sim C_x * g \sim g. \tag{2.3}$$

Furthermore, again employing the contractions H, G, each of the surrounding loops in the above homotopy is canonically contractible to the constant path C_x. Specifically, with respect to the two middle homotopies in (2.3), the product of paths in the t variable, $H(t,u)*G(t,su)$, $0 \le u \le 1$, is a homotopy which connects C_x $(u = 0)$ to the loops which surround z, $h(t) * G(t,s)$, $0 \le s \le 1$, $(u = 1)$. Similarly $H(t,(1-s)u) * G(t,u)$, $0 \le u \le 1$, is a homotopy which connects C_x $(u = 0)$ to the loops which surround z, $H(t,(1-s)) * g(t)$, $0 \le s \le 1$, $(u = 1)$. With respect to the two end homotopies of (2.3), standard homotopies which connect $h, h * C_x$, respectively which connect $C_x, C_x * g$, are employed to construct a homotopy of contractions which connects $H_s, H_s * C_x$, respectively which connects $C_x * G_s$, G_s, $0 \le s \le 1$. Precise details are left to the reader.

□

§2. C-Structures for Relations in Affine Bundles

Let $p \colon E \to B$ be an affine $\mathbf{R}^q$-bundle over a second-countable paracompact base space B (for example, a manifold B). In the applications $p \colon E \to B$ occurs mainly as the restriction over a submanifold $B \subset X^\perp$ of a naturally defined affine $\mathbf{R}^q$-bundle $p_\perp^r \colon X^{(r)} \to X^\perp$ associated to a smooth fiber bundle $p \colon X \to V$ and to a codimension 1 tangent hyperplane field τ on the base space V (cf. Chapter VI). The affine structure means that the transition functions of the bundle take their values in the group of affine transformations of $\mathbf{R}^q$. $\Gamma(E)$ denotes the space of continuous sections of the bundle E, in the compact-open topology. Let $\rho \colon \mathcal{R} \to E$ be a continuous map referred to as a *relation* over E. For example $\mathcal{R} \subset E$ and ρ is the inclusion map. $\Gamma(\mathcal{R})$ denotes the space of continuous sections of the map $p \circ \rho \colon \mathcal{R} \to B$ in the compact-open topology. If $\alpha \in \Gamma(\mathcal{R})$ then $\rho \circ \alpha \in \Gamma(E)$. $\Gamma_K(E)$, $\Gamma_K(\mathcal{R})$ are the corresponding spaces of sections over the subspace $K \subset B$.

For each $b \in B$ let $E_b = p^{-1}(b)$, the $\mathbf{R}^q$-fiber over the base point b; $\mathcal{R}_b = \mathcal{R} \cap \rho^{-1}(E_b)$, the subspace of $\mathcal{R}$ lying over b (possibly empty). If $a \in \mathcal{R}_b$ then $\mathrm{Conv}(\mathcal{R}_b, a)$ denotes the convex hull (in the fiber E_b) of the ρ-image of the path component of $\mathcal{R}_b$ to which a belongs (cf. §1).

Ample Relations. A relation $\rho\colon \mathcal{R} \to E$ is *ample* if for all pairs $(b, a) \in B \times \mathcal{R}_b$, $\mathrm{Conv}(\mathcal{R}_b, a) = E_b$ (by convention this includes the case $\mathcal{R}_b$ is empty).

C-structures. Let $\rho\colon \mathcal{R} \to E$ be a relation over E. Suppose $f \in \Gamma(E)$, $\beta \in \Gamma(\mathcal{R})$ satisfy the property that for all $b \in B$,

$$f(b) \in \mathrm{IntConv}(\mathcal{R}_b, \beta(b)).$$

A *C-structure* (C = contractible) over a subset $K \subset B$, with respect to f, β, is a pair consisting of a contractible loop of sections in $\Gamma_K(E)$, based at β_K, which fiberwise strictly surrounds the section f_K together with a contraction of the loop to β_K: a pair (g, G) where $g\colon [0,1] \to \Gamma_K(\mathcal{R})$ is continuous, $g(0) = g(1) = \beta_K$ (the restriction of β to K) such that for all $b \in K$ the path $g_b\colon [0,1] \to \mathcal{R}_b$, $g_b(t) = g(t, b)$, strictly surrounds $f(b)$; $G\colon [0,1]^2 \to \Gamma_K(\mathcal{R})$ is a (fiberwise) base point preserving contraction of g to β_K: for all $t, s \in [0,1]$,

$$G(t,1) = g(t);\ G(t,0) = \beta_K;\ G(0,s) = G(1,s) = \beta_K. \tag{2.4}$$

The set of all C-structures (g, G) over K, with respect to f, β, is topologized as a subspace of $C^0([0,1], \Gamma_K(\mathcal{R})) \times C^0\left([0,1]^2, \Gamma_K(\mathcal{R})\right)$, in the compact-open topology. In this bundle theoretic context, if $\rho\colon X \to \mathbf{R}^q$ is continuous then the space $\mathrm{int}\, X_x^z$ (§1, Surrounding Paths) is precisely the space of C-structures with respect to $x \in X$, $z \in \mathrm{IntConv}(X, x)$, for the trivial bundle over a point $\mathbf{R}^q \to *$. The proof of Lemma 2.1 obviously admits a parametric version which is stated as the following lemma.

Lemma 2.2. *For each $K \subset B$, the space of C-structures over K with respect to f, β is contractible.*

Proof. The proof is analogous to the proof of Lemma 2.1, for which all surrounding maps and contracting homotopies carry an additional K-space of parameters. Indeed from Lemma 2.1, $\mathrm{int}\, X_x^z$ is canonically contractible, via the deformation retract $\mathcal{D}$, to each point of $\mathrm{int}\, X_x^z$. To prove the lemma one applies $\mathcal{D}$ fiberwise to the relation $\mathcal{R}$ over E: for each $b \in K$ the deformation retract $\mathcal{D}$ of $\mathrm{int}\, X_x^z$ is applied to the case $X = \mathcal{R}_b$ with respect to $z = f(b) \in E_b$, $x = \beta(b) \in \mathcal{R}_b$. In this way $\mathcal{D}$ induces a canonical deformation retract $\mathcal{D}_K$ of the space of C-structures over K with respect to f, β to each C-structure in the space. A useful, though schematic, description of the deformation $\mathcal{D}_K$ is described succinctly as follows (cf. (2.3)). Full details are left to the reader. If (h, H), (g, G), are C-structures over K with respect to f, β then,

$$h \sim h * C_\beta \sim h * g \sim C_\beta * g \sim g.$$

where $C_\beta \colon [0,1] \to \Gamma_K(\mathcal{R})$ is the constant section β. In particular, let (g_0, G_0), (g_1, G_1) be C-structures over K with respect to f, β. There is a homotopy of C-structures (g_t, G_t), $t \in [0,1]$, over K with respect to f, β which connects (g_0, G_0), (g_1, G_1). □

The principal result in this section is the following proposition which establishes the existence of C-structures over each $K \subset B$ in case $\mathcal{R} \subset E$ is open. In case $\mathcal{R} \subset E$ is open then for each $b \in B$ and $a \in \mathcal{R}_b$, $\mathrm{Conv}(\mathcal{R}_b, a) = \mathrm{IntConv}(\mathcal{R}_b, a)$. Since $f(b) \in \mathrm{IntConv}(\mathcal{R}_b, \beta(b))$, it follows that pointwise there is a C-structure over each point $b \in B$ with respect to $f(b) \in E_b$, $\beta(b) \in \mathcal{R}_b$ i.e. there is a contractible loop g in $\mathcal{R}_b$ based at $\beta(b)$ that strictly surrounds $f(b)$ in the fiber E_b. The following proposition constructs these strictly surrounding contractible loops continuously over the base space B.

Proposition 2.3. *Let $\mathcal{R} \subset E$ be open. Suppose $\beta \in \Gamma(\mathcal{R})$ (a fiberwise base point map) and $f \in \Gamma(E)$ satisfy the property that for all $b \in B$, $f(b) \in \mathrm{Conv}(\mathcal{R}_b, \beta(b))$. There is a C-structure (ψ, H) globally defined over B with respect to f, β. Explicitly, there is a continuous map $\psi \colon [0,1] \to \Gamma(\mathcal{R})$, $\psi(0) = \psi(1) = \beta$, such that for all $b \in B$, the path $\psi_b \colon [0,1] \to \mathcal{R}_b$, $\psi_b(t) = \psi(t, b)$, strictly surrounds $f(b)$. Furthermore, there is a base point preserving homotopy $H \colon [0,1]^2 \to \Gamma(\mathcal{R})$ which contracts ψ to the constant path of sections C_β: For all $t, s \in [0,1]^2$,*

$$H(t,1) = \psi(t);\ \ H(t,0) = \beta;\ \ H(0,s) = H(1,s) = \beta.$$

Proof. The local existence of C-structures in a neighbourhood of each point of B is proved in Lemma 2.4 below. The contractibility of the space of C-structures, Lemma 2.2, is employed in an essential way to patch together these local constructions of C-structures to obtain a globally defined C-structure over all of B. The details are as follows.

Throughout this text we employ the convenient notation, due to Gromov [18]: if $Z \subset Y$ then $\mathfrak{Op}\, Z$ denotes an open neighbourhood (i.e., an "opening") of Z in Y. $\mathfrak{Op}\, Z$ is employed in the sense of a germs: with no change of notation, $\mathfrak{Op}\, Z$ may be replaced by a smaller neighbourhood, if necessary, during the course of a proof.

Lemma 2.4. *For all $b \in B$, each C-structure over $\{b\}$ with respect to f, β extends to a C-structure over $\mathfrak{Op}\, b$ with respect to f, β.*

Proof. Fix $b \in B$. By hypothesis, $f(b) \in \mathrm{IntConv}(\mathcal{R}_b, \beta(b))$. Let (g, G) be a C-structure over b with respect to $f(b), \beta(b)$ (as noted above C-structures exist over each point $b \in B$). Employing the hypothesis that $\mathcal{R} \subset E$ is open, one extends as follows the C-structure (g, G) over the point b to a C-structure (h, H) with respect to f, β, over a suitably small neighbourhood $\mathfrak{Op}\, b$. Since we are working locally near b we assume that the bundle $p \colon E \to B$ is the product bundle: $p \colon E = B \times \mathbf{R}^q \to B$.

Let $L = \mathrm{im}\, g = \mathrm{im}\, G$, a compact set in $\mathcal{R}_b$. Since $\mathcal{R}$ is open there is a neighbourhood $\mathfrak{Op}\, b$ such that $\mathfrak{Op}\, b \times L \subset \mathcal{R}$. For each $y \in \mathfrak{Op}\, b$ let $(g(y), G(y))$ be the translate of the C-structure (g, G) over $\{b\}$ to the fiber $\mathcal{R}_y$. For $\mathfrak{Op}\, b$ sufficiently small, for all $y \in \mathfrak{Op}\, b$ the loop $g(y)$ strictly surrounds $f(y)$, and also the line segment $\ell(y)$ in E_y that joins $\beta(y)$ to $(y, \beta(b))$ satisfies $\ell(y) \subset \mathcal{R}_b$.

Let $h \colon [0,1] \to \Gamma_{\mathfrak{Op}\, b}(\mathcal{R})$, $h(0) = h(1) = \beta_{\mathfrak{Op}\, b}$, be the family of loops obtained by conjugating the translated loop $g(y)$ with the parametrized line segment $\ell(y)$: for all $y \in \mathfrak{Op}\, b$, $t \in [0,1]$,

$$h(t, y) \;=\; \ell(y)(t) * g(y)(t) * \ell(y)^{-1}(t), \tag{2.5}$$

Similarly, let $H_0 \colon [0,1]^2 \to \Gamma_{\mathfrak{Op}\, b}$ be the contraction of h obtained by conjugating $G(y)$ with $\ell(y)$: for all $y \in \mathfrak{Op}\, b$, $(t, s) \in [0,1]^2$,

$$H_0(t, s, y) \;=\; \ell(y)(t) * G(y)(t, s) * \ell(y)^{-1}(t), \tag{2.6}$$

followed by a contraction of the segment $\ell(y)$ to the base point $\beta(y)$. Evidently the pair (h, H) is a C-structure over $\mathfrak{Op}\, b$ with respect to f, β. $\square$

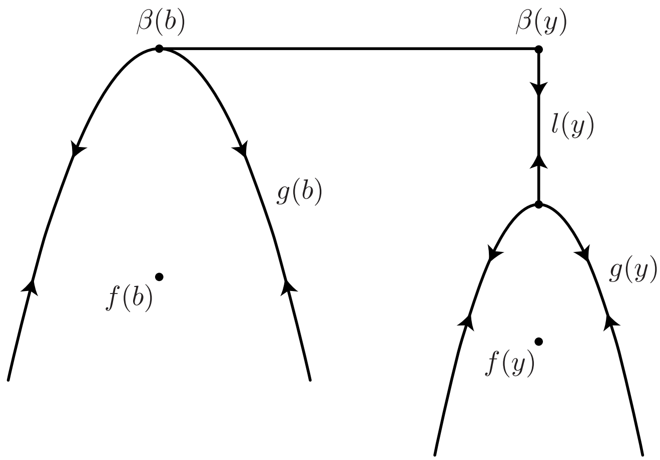

FIGURE 2.2

Employing Lemma 2.2, one patches together the local C-structures constructed above over $\mathfrak{Op}\, b$ for all $b \in B$. The inductive step for this process is as follows.

Lemma 2.5. *Let $K, L \subset B$ be closed and let (g_0, G_0), (g_1, G_1) be C-structures over $\mathfrak{Op}\, K$, $\mathfrak{Op}\, L$ respectively with respect to f, β. There is a C-structure (h, H) over $\mathfrak{Op}\,(K \cup L)$ with respect to f, β such that $(h, H) = (g_0, G_0)$ over (a smaller) $\mathfrak{Op}_1 K$ and $(h, H) = (g_1, G_1)$ over $\mathfrak{Op}\,(L \setminus \mathfrak{Op}\,(K \cap L))$.*

Proof. In case $K \cap L = \emptyset$ then one may assume $\mathfrak{Op}\, K \cap \mathfrak{Op}\, L = \emptyset$ in which case the lemma is trivial. Suppose now $M = K \cap L \neq \emptyset$. Applying Lemma 2.2, there is a homotopy of C-structures (g_s, G_s), $s \in [0, 1]$, over $\mathfrak{Op}\, M$ with respect to f, β which connects the C-structures (g_0, G_0) to (g_1, G_1) on $\mathfrak{Op}\, M$. Furthermore since $K \cap (L \setminus \mathfrak{Op}\, M) = \emptyset$, there are disjoint neighbourhoods $\mathfrak{Op}_1 K$, $\mathfrak{Op}_2(L \setminus \mathfrak{Op}\, M)$ in the paracompact space B.

Let $\lambda\colon B \to [0, 1]$ be continuous such that $\lambda = 0$ on $\mathfrak{Op}_3 K$ and $\lambda = 1$ on $B \setminus \mathfrak{Op}_1 K$, where $\overline{\mathfrak{Op}_3 K} \subset \mathfrak{Op}_1 K$. Let $H\colon [0, 1]^2 \to \Gamma(\mathcal{R})$ be the homotopy such that: (i) $H = G_0$ on $\mathfrak{Op}_3 K$; (ii) $H = G_1$ on $\mathfrak{Op}_2(L \setminus \mathfrak{Op}\, M)$; ($iii$) for all $(s, t, b) \in [0, 1]^2 \times \mathfrak{Op}\, M$, $H(s, t, b) = G_{\lambda(b)}(s, t, b)$. One verifies that H is well defined hence continuous. Setting $h(t, b) = H(t, 1, b)$ it follows that (h, H) is a C-structure over $\mathfrak{Op}\,(K \cup L)$. □

Returning to the proof of the proposition, according to Lemma 2.4 there is a C-structure over $\mathfrak{Op}\, b$ for each $b \in B$. Hence there are countable locally finite open covers $\{W_i\}$, $\{U_i\}$ of the base space B such that for all i, $\overline{W}_i \subset U_i$, and there is a C-structure over U_i. Inductively on n, let $K = \bigcup_{i=1}^{i=n} \overline{W}_i$, a closed set in B, and suppose (ψ_n, G_n) is a C-structure over $\mathfrak{Op}\, K$. Applying Lemma 2.5 to the closed set K and to $\overline{W}_{n+1}$, there is a C-structure (ψ_{n+1}, G_{n+1}) over $\mathfrak{Op}\,(K \cup \overline{W}_{n+1})$ such that $(\psi_{n+1}, G_{n+1}) = (\psi_n, G_n)$ over $\mathfrak{Op}\, K$. Since the covers $\{W_i\}$, $\{U_i\}$ are locally finite, it follows that the maps,

$$\psi = \lim_{n\to\infty} \psi_n\colon [0, 1] \to \Gamma(\mathcal{R}); \quad G = \lim_{n\to\infty} G_n\colon [0, 1]^2 \to \Gamma(\mathcal{R}),$$

are well-defined and continuous. Consequently, (ψ, G) is a C-structure over B with respect to $f \in \Gamma(E)$, $\beta \in \Gamma(\mathcal{R})$. □

Proposition 2.3 admits several refinements which are required for the full development of convex integration theory. These refinements are stated below as a series of complements to the above proposition. In all these complements, $\mathcal{R} \subset E$ is open.

Complement 2.6 (Relative Theorem). *Let N be a neighbourhood of $\beta(B)$ in E. Suppose there is a closed subspace K of B such that $f = \beta$ on K. There is a*

C-structure (h, H) over B with respect to f, β, such that over $\mathfrak{Op}\, K$, the image of H lies in N:

$$H\left([0,1]^2 \times \mathfrak{Op}\, K\right) \subset N.$$

Proof. The point here is that in case $b \in K$ the C-structure (h, H) that is constructed over $\mathfrak{Op}\, b$ in Lemma 2.4 may be chosen to satisfy the additional property that $\operatorname{im} H \subset N$. Passing to a locally finite subcover of the open cover $\{\mathfrak{Op}\, b\}_{b \in K}$ of K, the inductive proof procedure of Proposition 2.3 applies to construct a C-structure (h_1, H_1) over $\mathfrak{Op}\, K$ such that the image of H_1 lies in N. Applying Lemma 2.5 to the C-structure (h_1, H_1) over $\mathfrak{Op}\, K$ and to any C-structure over B one obtains a C-structure (h, H) over B which equals (h_1, H_1) over (a smaller) $\mathfrak{Op}\, K$. □

Remark 2.7. If one replaces "strictly surrounds" by "surrounds" in the definition of a C-structure then the relative theorem above can be improved to state that, in case $f = \beta$ on $\mathfrak{Op}\, K$, then one can choose H to be the constant homotopy equal to the base point section β over (a smaller) $\mathfrak{Op}\, K$. However the strictly surrounding property for C-structures is indispensable for the general theory, for example in the proof of the Integral Representation theorem below.

Suppose $p\colon E \to B$ is a smooth (i.e., C^∞) affine $\mathbf{R}^q$-bundle. Since $\mathcal{R}$ is open in E and since the "strictly surrounds" property of a C-structure is an open condition, the standard smooth approximation theorems apply to prove the following smooth refinement of Proposition 2.3.

Complement 2.8 (C^∞-Structures). *Suppose, in addition, $p\colon E \to B$ is a smooth affine $\mathbf{R}^q$-bundle. Let $f \in \Gamma(E)$, $\beta \in \Gamma(\mathcal{R})$ satisfy the hypothesis of Proposition 2.3, where, in addition, β is a smooth section. There is a C-structure (h, H) over B, with respect to f, β, such that H (and hence h) is a smooth map i.e., the evaluation map $H\colon [0,1]^2 \times B \to E$ is a smooth map.*

Complement 2.9 (Parameters). *Let P be a compact Hausdorff space (a parameter space). Let $f\colon P \to \Gamma(E)$, $\beta\colon P \to \Gamma(\mathcal{R})$ be continuous maps such that for all $(p, b) \in P \times B$, the following convex hull property obtains:*

$$f(p, b) \in \operatorname{IntConv}(\mathcal{R}_b, \beta(p, b)). \tag{2.7}$$

There is P-parameter family of C-structures (h, H) over B with respect to f, β. That is, there are continuous maps,

$$h\colon P \times [0,1] \to \Gamma(\mathcal{R}); \quad H\colon P \times [0,1]^2 \to \Gamma(\mathcal{R}),$$

such that, for each $p \in P$, (h_p, H_p) is a C-structure over B with respect to the sections $f(p), \beta(p)$. (One employs the notation, $h_p(t) = h(p, t)$; $H_p(t, s) = H(p, t, s)$.)

Proof. Let $\mathrm{id}\times p\colon P\times E\to P\times B$ be the pullback of the affine bundle $p\colon E\to B$ along the projection map onto the second factor, $\pi\colon P\times B\to B$. Then $P\times\mathcal{R}$, the pullback of $\mathcal{R}$, is open in $P\times E$, and the maps f,β induce obvious sections (same notation) in $\Gamma(P\times E)$, respectively $\Gamma(P\times\mathcal{R})$. Employing (2.7), the hypothesis of Proposition 2.3 is satisfied with respect to the maps f,β above, from which the Complement follows. □

Corollary 2.10 (Ample Relations). *Let $\mathcal{R}$ be open and ample in E, and suppose $\mathcal{R}$ admits a section $\beta\in\Gamma(\mathcal{R})$. For each section $f\in\Gamma(E)$ there is a C-structure (h,H) globally defined over B, with respect to f,β.*

The point here is that the convex hull hypothesis of Proposition 2.3 is automatically satisfied for any $f\in\Gamma(E)$. Indeed, since $\mathcal{R}$ is ample in E, it follows by definition that, for each $b\in B$, the convex hull of each path component of $\mathcal{R}_b$ is E_b.

§3. The Integral Representation Theorem

Let $X\subset\mathbf{R}^q$, $x\in X$, and let $z\in\mathrm{IntConv}(X,x)$. The space $\mathrm{int}\,X^z_x$ is precisely the space of C-structures with respect to $z\in\mathbf{R}^q$, $x\in X$, for the trivial bundle over a point, $\mathbf{R}^q\to *$. In what follows, C-structures in $\mathrm{int}\,X^z_x$ are employed to obtain a representation of the point z as a Riemann integral whose integrand is a function with values in X. The main result of this section, the Integral Representation Theorem 2.12, establishes this Riemann integral representation continuously over the base space, in the context of an affine $\mathbf{R}^q$-bundle.

Proposition 2.11. *Let $X\in\mathbf{R}^q$, $x\in X$, and let $z\in\mathrm{IntConv}(X,x)$. Each C-structure $(g,G)\in\mathrm{int}\,X^z_x$ can be reparametrized to a C-structure (h,H) such that $z=\int_0^1 h(t)\,dt$.*

Proof. Let $(g,G)\in\mathrm{int}\,X^z_x$ be a C-structure and let $0=s_0<s_1\cdots<s_{n+1}=1$ be a partition of the interval $[0,1]$ such that z is contained in the interior of the convex hull of the points $g(s_i)$, $1\le i\le n$. For each i, $1\le i\le n$, let $d\mu_i$ be a positive measure on $[0,1]$ such that $\int_0^1 d\mu_i=1$, and $d\mu_i\approx\delta(s-s_i)$. ($\delta$ is the Dirac delta function). For example, $d\mu_i$ is represented by a positive, continuous density function f_i on $[0,1]$ such that $\int_0^1 f_i(s)\,ds=1$, and $f_i(s)$ is concentrated near s_i, $1\le i\le n$.

Let $b_i=\int_0^1 g\,d\mu_i$, $1\le i\le n$. Let $\epsilon>0$. If each $d\mu_i$ is a sufficiently close approximation to $\delta(s-s_i)$ then, with respect to a norm $\|\ \|$ on $\mathbf{R}^q$,

$$\|b_i-g(s_i)\|<\epsilon,\quad 1\le i\le n,\tag{2.8}$$

Since z is in the interior of the convex hull of the points $g(s_i)$, $1\le i\le n$, it follows from (2.8) that for $\epsilon>0$ sufficiently small, $\mathfrak{Op}\,z$ is contained in the convex hull of the points b_i, $1 \le i \le n$. One may therefore assume,

$$z = \sum_{i=0}^{n} p_i\, b_i, \quad \text{where} \sum_{i=1}^{n} p_i = 1,\ p_i \in [0,1],\ 1 \le i \le n. \tag{2.9}$$

We remark here that one may choose $n = q+1$ and $g(s_i) \in \mathbf{R}^q$, $1 \le i \le q+1$, to be the vertices of a $q+1$-simplex, from which it follows that the above barycentric coordinates $(p_i)_{1 \le i \le q+1}$ are unique and strictly positive. This is important for proving the continuity of these coordinates in the parametric (bundle) version Theorem 2.12 below.

The continuous measure $d\mu = \sum_{i=1}^{n} p_i\, d\mu_i$ on the interval $[0,1]$ is positive, $\int_0^1 d\mu = 1$, and with respect to $d\mu$ one has the following integral representation:

$$\begin{aligned} \int_0^1 g\, d\mu \ &= \ \sum_{i=1}^{n} p_i \int_0^1 g\, d\mu_i \\ &= \sum_{i=1}^{n} p_i\, b_i = z. \end{aligned} \tag{2.10}$$

Employing a simple change of coordinates, one obtains the integral representation (2.10) with respect to Lebesgue measure on $[0,1]$. Explicitly, let $\lambda(t) = \int_0^t d\mu$. Evidently, $\lambda(0) = 0$, $\lambda(1) = 1$, and $d\lambda/dt > 0$ on $[0,1]$. Define $h = g \circ \lambda^{-1}$. Clearly $h(0) = h(1) = x$; h strictly surrounds z and, employing the change of coordinates $s = \lambda(t)$, it follows from (2.9) that,

$$\int_0^1 h(s)\, ds \ = \ \int_0^1 g \circ \lambda^{-1}(s)\, ds \ = \ \int_0^1 g\, d\mu \ = \ z. \tag{2.11}$$

Let $H\colon [0,1]^2 \to X$ be the reparametrization, $H(t,s) = G(\lambda^{-1}(t), s)$. Hence (h, H) is a C-structure in $\operatorname{int} X_x^z$ for which the integral representation (2.11) obtains. □

Theorem 2.12 (Integral Representation). *Let $p\colon E \to B$ be an affine $\mathbf{R}^q$-bundle over a second-countable paracompact space B. Let $\mathcal{R} \subset E$ be open and suppose $\beta \in \Gamma(\mathcal{R})$ and $f \in \Gamma(E)$ satisfy the property that, for all $b \in B$,*

$$f(b) \in \operatorname{Conv}(\mathcal{R}_b, \beta(b)).$$

Each C-structure (g, G) over B with respect to f, β can be reparametrized to a C-structure (h, H) such that for all $b \in B$ (recall $h\colon [0,1] \to \Gamma(\mathcal{R})$), $f(b) = \int_0^1 h(t,b)\, dt$.

Proof. Applying Proposition 2.3, there exists a C-structure (g, G) over B with respect to the sections $f \in \Gamma(E)$, $\beta \in \Gamma(\mathcal{R})$. The proof consists of suitably reparametrizing a C-structure (g, G) to obtain a C-structure (h, H) over B for which the above integral representation of f obtains. The map $g\colon [0,1] \to \Gamma(\mathcal{R})$ satisfies the property that $g(0) = g(1) = \beta$, and for each $b \in B$ the corresponding path $g_b\colon [0,1] \to \mathcal{R}_b$ strictly surrounds $f(b)$ in the fiber E_b. Consequently (since "strictly surrounding" is an open condition), for each $b \in B$ there is a neighbourhood U of b and a partition $0 < s_1 < s_2 \cdots < s_{q+1} < 1$ of the interval $[0,1]$, such that for all $y \in U$ the sequence of points $g_y(s_1), g_y(s_2), \ldots, g_y(s_{q+1})$, spans an affine q-simplex $\Delta(y)$ in the fiber E_y, and $f(y)$ is an interior point of $\Delta(y)$. In particular the barycentric coordinates of $f(y) \in \Delta(y)$ are strictly positive continuous functions on U (cf. the remark following (2.9) above).

Recall, Proposition 2.11, the positive, continuous measures $d\mu_i$ on $[0,1]$, $\int_0^1 d\mu_i = 1$, and such that $d\mu_i \approx \delta(s - s_i)$, $1 \le i \le q+1$. For each $y \in U$ let $b_i(y) = \int_0^1 g_y(s)\, d\mu_i$, $1 \le i \le q+1$. If $d\mu_i$ is a sufficiently close approximation to $\delta(s - s_i)$, $1 \le i \le q+1$, and if U is a sufficiently small neighbourhood of b, then, for all $y \in U$, the sequence of points $b_1(y), b_2(y), \ldots, b_{q+1}(y)$ also spans an affine q-simplex $\Delta'(y)$ in the fiber E_y, and $f(y)$ is an interior point of $\Delta'(y)$. In particular, the barycentric coordinates of $f(y)$ in the q-simplex $\Delta'(y)$ are strictly positive continuous functions of $y \in U$. Consequently, there is a neighbourhood $W \equiv W(b)$ of b, $\overline{W} \subset U$, and globally defined continuous functions, compactly supported in U, $p_i\colon B \to [0,1]$, $1 \le i \le q+1$, such that the sequence of functions (p_i) on W are the barycentric coordinates of the section f: for all $y \in W$,

$$f(y) = \sum_{i=1}^{q+1} p_i(y)\, b_i(y); \quad \sum_{i=1}^{q+1} p_i(y) = 1. \tag{2.12}$$

Let $\{W_j\}$ be a countable locally finite subcover of the above open cover $\{W(b)\}_{b \in B}$ of the base space B. Thus, for each index j, (2.12) applies to the section f over W_j. Explicitly, in the above notation (the index j corresponds to W_j) for all $y \in W_j$,

$$f(y) = \sum_{i=1}^{q+1} p_i^j(y)\, b_i^j(y); \quad \sum_{i=1}^{q+1} p_i^j(y) = 1.$$

Let $\{q_j\colon B \to [0,1]\}_{j \ge 1}$ be a partition of unity subordinate to the cover $\{W_j\}$. Also, for each index j, one employs the notation, $d\mu_i^j$, $1 \le i \le q+1$, to denote the measures above on $[0,1]$, with respect to the open set W_j. Thus $b_i^j(y) = \int_0^1 g(s, y)\, d\mu_i^j$, $1 \le i \le q+1$. Let $d\mu$ be the B-parameter family of measures on the interval $[0,1]$ defined as follows. For each $b \in B$,

$$d\mu(b) = \sum_{j=1}^{\infty} \sum_{i=1}^{q+1} q_j(b)\, p_i^j(b)\, d\mu_i^j. \tag{2.13}$$

Evidently, $d\mu(b)$ is a positive measure on $[0,1]$, continuous in the parameter $b \in B$, such that for all $b \in B$, $\int_0^1 d\mu(b) = 1$. Furthermore, employing (2.13), $f \in \Gamma(E)$ has the following integral representation with respect to the B-parameter of measures $d\mu(b)$. For each $y \in B$,

$$\begin{aligned}\int_0^1 g(s,y)\,d\mu(y) &= \sum_{j=1}^{\infty} q_j(y) \sum_{i=1}^{q+1} p_i^j(y) \int_0^1 g(s,y)\,d\mu_i^j \\ &= \sum_{j=1}^{\infty} q_j(y) \sum_{i=1}^{q+1} p_i^j(y)\, b_i^j(y) = \sum_{j=1}^{\infty} q_j(y)\, f(y) \,=\, f(y).\end{aligned} \tag{2.14}$$

We now change coordinates to obtain an integral representation for the section f with respect to Lebesgue measure. Let $\lambda\colon [0,1] \times B \to [0,1]$, be the continuous function, $\lambda(t,b) = \int_0^t d\mu(b)$. Thus for all $b \in B$, $\lambda(0,b) = 0$, $\lambda(1,b) = 1$, and the derivative $\partial\lambda/\partial t\colon [0,1] \times B \to [0,1]$ is a continuous, positive function. Consequently, for each $b \in B$, the inverse function $\lambda^{-1}(t,b)$ exists and $\lambda^{-1}\colon [0,1] \times B \to [0,1]$ is a continuous function.

Let $h\colon [0,1] \to \Gamma(\mathcal{R})$ be the map, $h(s,b) = g(\lambda^{-1}(s,b),b)$. Evidently, h is continuous, $h(0) = h(1) = \beta$, and employing the change of coordinates $s = \lambda(t,b)$, it follows from (2.14) that for each $b \in B$,

$$\begin{aligned}\int_0^1 h(s,b)\,ds &= \int_0^1 g(\lambda^{-1}(s,b),b)\,ds \\ &= \int_0^1 g(t,b)\,d\mu(b) \,=\, f(b).\end{aligned} \tag{2.15}$$

Let $H\colon [0,1]^2 \to \Gamma(\mathcal{R})$ be the corresponding reparametrization, $H(t,s,b) = G(\lambda^{-1}(t,b),s,b)$. Hence (h,H) is a C-structure over B which satisfies the integral representation (2.15) for the section $f \in \Gamma(\mathcal{R})$. □

The Integral Representation Theorem 2.12 admits a series of complements which are derived from the corresponding Complements 2.6 to 2.10. Again, $\mathcal{R} \subset E$ is open for all these complements.

Complement 2.13 (Relative Theorem). *Let N be a neighbourhood of $\beta(B)$ in E. Suppose there is a closed subspace K of B such that $f = \beta$ on K. There is a C-structure (h,H) over B such that,*

(i) *For each $b \in B$, $f(b) = \int_0^1 h(t,b)\,dt$.*

(ii) *Over $\mathfrak{Op}\, K \subset B$, the image of H lies in N:*

$$H([0,1]^2 \times \mathfrak{Op}\, K) \subset N.$$

Proof. Applying Complement 2.6, one obtains a C-structure (h_0, H_0) over B with respect to f, β, and which satisfies (ii) above. Applying Theorem 2.12 to this initial C-structure (h_0, H_0), one obtains a C-structure (h, H) over B which satisfies both of the properties (i), (ii). □

Complement 2.14 (C^∞-structures). *Suppose $p\colon E \to B$ is a smooth affine $\mathbf{R}^q$-bundle, and that $f \in \Gamma(E)$, $\beta \in \Gamma(\mathcal{R})$ are smooth sections. There is a C-structure (h, H) over B with respect to f, β such that H (and hence h) is a smooth map and such that for all $b \in B$, $f(b) = \int_0^1 h(t, b)\, dt$.*

Proof. Applying Complement 2.7, there is a smooth C-structure (h_0, H_0) over B with respect to f, β. The complement follows from Theorem 2.12 applied to this initial C-structure (h_0, H_0), subject to the following modifications to ensure a smooth reparametrization. One may assume that the measures $d\mu_i$ on the interval $[0, 1]$, $1 \le i \le q + 1$, are defined by positive smooth density functions. Hence the local barycentric coordinate functions p_i, of the smooth section f in (2.12), are smooth functions, $1 \le i \le q + 1$. Consequently, with respect to a smooth partition of unity $\{q_j\}_{j \ge 1}$, the measures $d\mu(b)$ defined by (2.13) are smooth, from which it follows that the change of coordinates map, $\lambda(t, b) = \int_0^1 d\mu(b)$, is smooth. One concludes that the reparametrized C-structure (h, H), $H(t, s, b) = G(\lambda^{-1}(t, b), s, b)$, satisfies the property that H (and hence h) is a smooth map. □

Complement 2.15 (Parameters). *Let P be a compact Hausdorff space (a parameter space). Let $f\colon P \to \Gamma(E)$, $\beta\colon P \to \Gamma(\mathcal{R})$ be continuous maps such that for all $(p, b) \in P \times B$, the following convex hull property obtains:*

$$f(p, b) \in \operatorname{Conv}(\mathcal{R}_b, \beta(p, b)).$$

There is a P-parameter family of C-structures (h, H) over B with respect to f, β such that for all $(p, b) \in P \times B$, $f(p, b) = \int_0^1 h(p, t, b)\, dt$.

Proof. Let $\mathrm{id} \times p\colon P \times E \to P \times B$ be the pullback of the affine bundle $p\colon E \to B$ along the projection map onto the second factor, $\pi\colon P \times B \to B$. Applying Complement 2.9, one obtains a C-structure (h_0, H_0) over $P \times B$, with respect to f, β. The complement follows from Theorem 2.12 applied to this initial C-structure (h_0, H_0) over $P \times B$. □

Corollary 2.16 (Ample Relations). *Let $\mathcal{R}$ be open and ample in E and suppose $\mathcal{R}$ admits a section $\beta \in \Gamma(\mathcal{R})$. For each section $f \in \Gamma(E)$ there is a C-structure (h, H) over B with respect to f, β such that, for all $b \in B$, $f(b) = \int_0^1 h(t, b)\, dt$.*

Proof. Applying Complement 2.10, it follows that for each $f \in \Gamma(E)$, there is a C-structure (h_0, H_0) over B with respect to f, β. The complement follows from Theorem 2.12 applied to this initial C-structure (h_0, H_0). □

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