# Chapter 2

# The Essential Theorem

In this chapter we give our new proof of the Rubio de Francia extrapolation theorem, Theorem 1.4, and discuss how our proof allows a number of powerful generalizations. For the convenience of the reader we restate it here.

**Theorem 1.4.** *Given an operator $T$, suppose that for some $p_0$, $1 \leq p_0 < \infty$, and every $w \in A_{p_0}$, there exists a constant $C$ depending on $[w]_{A_{p_0}}$ such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx.$$

*Then for every $p$, $1 < p < \infty$, and every $w \in A_p$ there exists a constant depending on $[w]_{A_p}$ such that*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p} w(x)\, dx.$$

Before giving our proof of Theorem 1.4 we want to describe briefly earlier proofs. The original proof of Rubio de Francia [193, 194, 195] is quite complex and depends on a connection between vector-valued estimates and weighted norm inequalities. A more direct proof that depends only on weighted norm inequalities was given by García-Cuerva [83] (see also [88]). However, this approach requires two complicated lemmas on the structure of $A_p$ weights, and the proof itself is divided into two cases, depending on whether $p > p_0$ or $p < p_0$. A more refined version of this proof appears in Grafakos [92] and in Dragičević, Grafakos, Pereyra and Petermichl [65]. (We will consider this proof again below.)

As we noted in Section 1.2 above, Rubio de Francia and García-Cuerva used the iteration algorithm of Rubio de Francia: given a positive, sublinear operator $T$ that is bounded on $L^p(w)$, define a new operator $\mathcal{R}$ by

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{T^k h}{2^k \|T\|_{L^p(w)}^k}.$$

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A simpler proof of Theorem 1.4 that avoided the iteration algorithm was given by Duoandikoetxea [68] when $p_0 > 1$. This proof has two steps: first prove that the desired inequality holds for $1 < p < p_0$ and $w \in A_1$, and then use this to prove the full result. The proof requires the very deep property of $A_p$ weights that if $w \in A_p$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$ (Theorem 1.3).

## 2.1 The new proof

Our proof of the Rubio de Francia extrapolation theorem is simpler and more direct than any previous proof, since it yields the desired inequality directly without cases or intermediate steps, and uses only the iteration algorithm and basic properties of $A_p$ weights.

*Proof of Theorem* 1.4. Fix $p$, $1 < p < \infty$, and $w \in A_p$. We first introduce two versions of the iteration algorithm. Since $w \in A_p$, $M$ is bounded on $L^p(w)$, so given $h \in L^p(w)$ we can define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^p(w)}},$$

where for $k \geq 1$, $M^k = M \circ \cdots \circ M$ is $k$ iterations of the maximal operator, and $M^0 h = |h|$. The operator $\mathcal{R}$ has the following properties:

- for all $x$, $|h(x)| \leq \mathcal{R}h(x)$;
- $\|\mathcal{R}h\|_{L^p(w)} \leq 2\|h\|_{L^p(w)}$;
- $\mathcal{R}h \in A_1$ with $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^p(w)}$.

The first two follow immediately from the definition; to see the third, note that since $M$ is sublinear we have that

$$M(\mathcal{R}h)(x) \leq \sum_{k=0}^{\infty} \frac{M^{k+1} h(x)}{2^k \|M\|^k_{L^p(w)}} \leq 2\|M\|_{L^p(w)} \mathcal{R}h(x).$$

Now define the operator $M'f = M(fw)/w$. Since $w^{1-p'} \in A_{p'}$, $M$ is bounded on $L^{p'}(w^{1-p'})$ and so $M'$ is bounded on $L^{p'}(w)$. Therefore, we can define another iteration algorithm:

$$\mathcal{R}'h(x) = \sum_{k=0}^{\infty} \frac{(M')^k h(x)}{2^k \|M'\|^k_{L^{p'}(w)}}.$$

(Again, $(M')^0 h = |h|$.) Arguing exactly as before we have that:

- for all $x$, $|h(x)| \leq \mathcal{R}'h(x)$;
- $\|\mathcal{R}'h\|_{L^{p'}(w)} \leq 2\|h\|_{L^{p'}(w)}$;

- $M'(\mathcal{R}'h)(x) \le 2\|M'\|_{L^{p'}(w)}\mathcal{R}'h(x)$, and so $\mathcal{R}'h\,w \in A_1$ with $[\mathcal{R}'h\,w]_{A_1} \le 2\|M'\|_{L^{p'}(w)}$.

Given the two iteration algorithms, the proof is now straightforward. Fix $f \in L^p(w)$. By duality there exists a non-negative function $h \in L^{p'}(w)$, $\|h\|_{L^{p'}(w)} = 1$, such that

$$\begin{aligned}\|Tf\|_{L^p(w)} &= \int_{\mathbb{R}^n} |Tf(x)|h(x)w(x)\,dx \\ &\le \int_{\mathbb{R}^n} |Tf(x)|\mathcal{R}f(x)^{-1/p_0'}\mathcal{R}f(x)^{1/p_0'}\mathcal{R}'h(x)w(x)\,dx,\end{aligned}$$

where we have used that $h \le \mathcal{R}'h$, and if $p_0 = 1$ we let $1/p_0' = 0$. Since $\mathcal{R}f$, $\mathcal{R}'h\,w \in A_1$, by the reverse factorization property, Proposition 1.2 $(c)$, $(\mathcal{R}f)^{1-p_0}\mathcal{R}'h\,w \in A_{p_0}$. Therefore, by Hölder's inequality with respect to the measure $\mathcal{R}'h\,w$ (if $p_0 > 1$), by our hypothesis, and since $|f| \le \mathcal{R}f$,

$$\begin{aligned}\|Tf\|_{L^p(w)} &\le \left(\int_{\mathbb{R}^n} |Tf(x)|^{p_0}\mathcal{R}f(x)^{1-p_0}\mathcal{R}'h(x)w(x)\,dx\right)^{1/p_0} \\ &\qquad\qquad \times \left(\int_{\mathbb{R}^n} \mathcal{R}f(x)\mathcal{R}'h(x)w(x)\,dx\right)^{1/p_0'} \\ &\le C\left(\int_{\mathbb{R}^n} |f(x)|^{p_0}\mathcal{R}f(x)^{1-p_0}\mathcal{R}'h(x)w(x)\,dx\right)^{1/p_0} \\ &\qquad\qquad \times \left(\int_{\mathbb{R}^n} \mathcal{R}f(x)\mathcal{R}'h(x)w(x)\,dx\right)^{1/p_0'} \\ &\le C\int_{\mathbb{R}^n} \mathcal{R}f(x)\mathcal{R}'h(x)w(x)\,dx.\end{aligned}$$

Again by Hölder's inequality and since $\mathcal{R}$ is bounded on $L^p(w)$ and $\mathcal{R}'$ is bounded on $L^{p'}(w)$,

$$\begin{aligned}\|Tf\|_{L^p(w)} &\le C\left(\int_{\mathbb{R}^n} \mathcal{R}f(x)^p w(x)\,dx\right)^{1/p}\left(\int_{\mathbb{R}^n} \mathcal{R}'h(x)^{p'}w(x)\,dx\right)^{1/p'} \\ &\le C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}\left(\int_{\mathbb{R}^n} h(x)^{p'}w(x)\,dx\right)^{1/p'} \\ &= C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}. \qquad \square\end{aligned}$$

Beyond its simplicity, an important feature of our proof of the Rubio de Francia extrapolation theorem is that it makes clear exactly what the essential ingredients are. They are three-fold: norm inequalities for the Hardy-Littlewood maximal operator, duality, and the reverse factorization property of $A_p$ weights. More precisely, we need the following:

(a) $M$ is sublinear, positive and bounded on $L^p(w)$ if $w \in A_p$;

(b) $M'$ is sublinear, positive and bounded on $L^{p'}(w)$ if $w \in A_p$;

(c) $L^{p'}(w)$ is the dual space of $L^p(w)$;

(d) Hölder's inequality;

(e) if $w_1, w_2 \in A_1$, then $w_1 w_2^{1-p} \in A_p$.

Properties (a) and (b) let us define the iteration algorithm, and Properties (c), (d), and (e) are all that we use in the second part of the proof.

This list of essential properties can be simplified further. Property (b) follows from (a) and another structural property of $A_p$ weights:

(f) $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$.

Furthermore, in the proof we do not use that $L^{p'}(w)$ is the dual space of $L^p(w)$; it suffices to assume that it is the associate space, thereby giving us the reverse of Hölder's inequality. We can avoid explicitly using duality if we simply define the function $h = |Tf|^{p-1}/\|Tf\|^{p-1}_{L^p(w)}$. With some minor modifications to the proof we can actually take $h = |Tf|^{p-1}$.

As we mentioned in Chapter 1, Property (e) is usually subsumed into the Jones factorization theorem, but we emphasize that we only need the reverse factorization property, and not the converse, which is the heart of this result and more difficult to prove.

Conspicuously missing from this list of properties is any mention of the operator $T$: we do not assume that $T$ is linear or even sublinear. In the original proofs of Rubio de Francia and García-Cuerva, $T$ was assumed to be sublinear; it was later noted that this hypothesis is superfluous provided that $T$ is well defined on the union of $L^p(w)$ for all $1 < p < \infty$ and $w \in A_p$.

## 2.2 Extensions of the extrapolation theorem

A very important feature of our proof is that we can adapt it to prove a number of non-trivial extensions of the Rubio de Francia extrapolation theorem. The following are the principal generalizations which we will consider in Chapters 3 and 4.

### Generalized maximal operators

The Hardy-Littlewood maximal operator is defined in terms of averages over cubes, as are the Muckenhoupt $A_p$ classes. However, the maximal operator can be generalized to averages over other families of sets: dyadic cubes, rectangles with sides parallel to the coordinate axes, etc. For each such maximal operator there is a corresponding $A_p$ class, and in many important examples the maximal operator satisfies one-weight norm inequalities with respect to this class. In each of these cases there is an extrapolation theorem: the original proofs of Rubio de Francia and García-Cuerva both go through, as each author noted.

Our approach makes this extension immediate: since the structural properties (e) and (f) automatically hold for these generalized $A_p$ classes, our proof extends at once if we assume that the maximal operator satisfies property (a). We will develop these ideas carefully in Chapter 3 and use this approach throughout Part I. (In Part II we will restrict ourselves to the Hardy-Littlewood maximal operator.)

## Elimination of the operator

Since we make no assumptions on the operator $T$, we can reinterpret Theorem 1.4 as follows: if an $L^{p_0}(w)$ inequality holds for pairs of the form $(|Tf|, |f|)$, then an $L^p(w)$ inequality also holds for such pairs. In fact, we can eliminate the operator $T$ entirely, and restate the extrapolation theorem for pairs of non-negative functions $(f, g)$: given a suitably chosen family of pairs of functions $(f, g)$, if for some $p_0$ and all $w \in A_{p_0}$, $\|f\|_{L^{p_0}(w)} \leq \|g\|_{L^{p_0}(w)}$, then for all $p$ and $w \in A_p$, $\|f\|_{L^p(w)} \leq \|g\|_{L^p(w)}$.

This perspective was first described in passing in [54], but it was not fully exploited until later in a series of papers by the authors and their collaborators (see [40, 44, 45, 57, 93]). The advantage of our approach is that a number of different results become special cases of a single extrapolation theorem. We consider three important examples. For clarity we state them in terms of operators, but below we will treat them in full generality in terms of pairs of functions.

### Weak type inequalities

Given an operator $T$, suppose that for some $p_0$ and all $w \in A_{p_0}$, $T : L^{p_0}(w) \to L^{p_0,\infty}(w)$. Let

$$E_\lambda = \{x \in \mathbb{R}^n : |Tf(x)| > \lambda\};$$

then we can rewrite the weak type $(p_0, p_0)$ inequality as

$$\|\lambda \chi_{E_\lambda}\|_{L^{p_0}(w)} \leq C\|f\|_{L^{p_0}(w)}.$$

Hence, if we apply extrapolation to the family of pairs $(\lambda\chi_{E_\lambda}, |f|)$, we get that for all $p$ and $w \in A_p$ that $\|\lambda\chi_{E_\lambda}\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$, or equivalently, that $T$ is of weak type $(p, p)$.

This idea first appeared in [93]. Extrapolation for weak type inequalities was proved by both Rubio de Francia and García-Cuerva [83, 195]. However, each gave a separate proof by adapting the proof in the case of strong type inequalities.

**Vector-valued inequalities**

Given an operator $T$, suppose that it is bounded on $L^{p_0}(w)$ for all $w \in A_{p_0}$. Then by extrapolation it is bounded on $L^q(w)$ whenever $w \in A_q$. If we let $f = \{f_i\}$, we can extend $T$ to a vector-valued operator by defining $Tf = \{Tf_i\}$. Then we immediately have that

$$\int_{\mathbb{R}^n} \|Tf(x)\|_{\ell^q}^q w(x)\,dx \leq C \int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^q w(x)\,dx.$$

We can, therefore, apply extrapolation again, this time to the pairs $\big(\|Tf\|_{\ell^q}, \|f\|_{\ell^q}\big)$, and conclude that for all $p$ and $w \in A_p$,

$$\int_{\mathbb{R}^n} \|Tf(x)\|_{\ell^q}^p w(x)\,dx \leq C \int_{\mathbb{R}^n} \|f(x)\|_{\ell^q}^p w(x)\,dx.$$

Extrapolation to vector-valued inequalities was proved by Rubio de Francia [195] as part of the original extrapolation theorem. He was led to this extension because his proof relied on the connection between vector-valued inequalities and weighted norm inequalities. In [88] it was noted in passing that extrapolation can be used to prove vector-valued inequalities, but no details were given. Later authors only considered the scalar case.

**Rescaling**

There are two versions of the extrapolation theorem that yield $L^p(w)$ inequalities for weights $w$ that are not in $A_p$. Rubio de Francia [195] observed that his proof of the extrapolation theorem could be modified to prove the following: given $r > 1$, suppose that for some $p_0 \geq r$ the operator $T$ is bounded on $L^{p_0}(w)$ whenever $w \in A_{p_0/r}$. Then for all $p > r$, $T$ is bounded on $L^p(w)$ whenever $w \in A_{p/r}$. (See also Duoandikoetxea [68].) As an application, Rubio de Francia used this result to proved weighted Littlewood-Paley inequalities. Other operators that satisfy such inequalities include the square function $g_\lambda^*$ (see [153, 219]) and singular integrals with rough kernels (see [67, 119, 196, 228]).

This version of the extrapolation theorem is an immediate consequence of the general result for pairs of functions. We can restate the $L^{p_0}$ inequality as an $L^{p_0/r}$ inequality: $\||Tf|^r\|_{L^{p_0/r}(w)} \leq C\||f|^r\|_{L^{p_0/r}(w)}$. Hence, we can apply extrapolation to the pairs $(|Tf|^r, |f|^r)$ to get the desired inequality for all $p$ and $w \in A_{p/r}$.

An extrapolation theorem for $A_\infty$ weights was introduced in [44]: given a pair of operators $S$ and $T$, suppose that for some $p_0$, $0 < p_0 < \infty$, and for all $w \in A_\infty$, $\|Tf\|_{L^{p_0}(w)} \leq C\|Sf\|_{L^{p_0}(w)}$. Then for all $p$, $0 < p < \infty$, $\|Tf\|_{L^p(w)} \leq C\|Sf\|_{L^p(w)}$ whenever $w \in A_\infty$. The original proof of this result did not use Rubio de Francia extrapolation; the proof was direct and had two steps, similar to the proof of Theorem 1.4 due to Duoandikoetxea [68]. However, $A_\infty$ extrapolation is an immediate corollary of the general extrapolation theorem for pairs of functions.

Since the $A_p$ classes are nested, $w \in A_\infty$ is equivalent to $w \in A_{p_0/r}$ for some $r$, $0 < r < p_0$. Therefore, this result follows by the same rescaling argument as before.

As we noted in Chapter 1, inequalities of this type were introduced by Coifman and Fefferman [25], who showed that if $T$ is a Calderón-Zygmund singular integral, then for all $p$ and $w \in A_\infty$, $\|Tf\|_{L^p(w)} \le C\|Mf\|_{L^p(w)}$. This and related estimates were originally proved using good-$\lambda$ inequalities, but in [44] we showed that they can also be proved using extrapolation. As there is no standard terminology, we refer to all such inequalities involving pairs of operators as Coifman-Fefferman inequalities.

In Chapter 3 we will prove the extrapolation theorem for pairs of functions, and we will also provide the details on the above applications. Throughout this monograph we will state and prove all the extrapolation theorems in this generality.

## Sharp constants

Initially, little attention was paid to the exact constant obtained via extrapolation: the primary concern was to establish weighted $L^p$ estimates. However, beginning with the work of Buckley [15], there has been increasing interest in the best constants in weighted norm inequalities (in terms of the $A_p$ constant of the weight). In particular, the results of Astala, Iwaniec and Saksman [6] on the Beltrami equation (discussed in Section 1.2) showed that sharp constants had important consequences. Sharp constants for singular integrals and other operators have been considered by a number of authors: see [48, 49, 65, 121, 127, 129, 130, 179, 181, 182, 184]. For every operator except the Hardy-Littlewood maximal operator, sharp constants were proved for a specific value of $p$ (usually but not universally $p = 2$) and then extrapolation was used to find the best constant for all other values of $p$.

If the constant in the initial $L^{p_0}$ inequality is $N_{p_0}([w]_{A_{p_0}})$, where $N_{p_0}$ is an increasing function with values in $[1,\infty)$, then it can be shown that the constants gotten for $L^p$ inequalities are

$$\begin{cases} 2^{1/p_0} N_{p_0}(C_{n,p,p_0}[w]_{A_p}) & p > p_0, \\ 2^{1/p_0'} N_{p_0}(C_{n,p,p_0}[w]_{A_p}^{\frac{p_0-1}{p-1}}) & p < p_0. \end{cases}$$

These bounds are sharp in the sense that for many operators (e.g. the Hilbert transform) the resulting constants are the best possible. These bounds were first obtained by Petermichl and Volberg [184] for $p > p_0 = 2$, and then for all $p$ and $p_0$ by Dragičević, *et al.* [65]. These proofs required a careful adaptation of the two case proof of García-Cuerva. A simpler proof was given by Grafakos [92].

The singular weakness of our proof is that it does not yield these sharp constants. A close examination of the proof shows that the constant is

$$N_{p_0}(C_{n,p_0,p}[w]_{A_p}^{1+\frac{p_0-1}{p-1}}).$$

This estimate depends on the best constant for $\|M\|_{L^p(w)}$ due to Buckley [15] (see also the recent proof by Lerner [127]) and details are left to the reader. This appears to be intrinsic to our proof: since we treat all values of $p$ simultaneously we must use both iteration algorithms, and this yields a larger constant. However, by treating the cases separately we can refine our approach to get the sharp constants, and we do so in the context of Muckenhoupt bases. Our proof uses some ideas from recent work of Duoandikoetxea [69].

## Off-diagonal extrapolation

We can extend our proof of Theorem 1.4 to prove extrapolation for "off-diagonal" inequalities. More precisely, given $p$, $q$, $1 < p < q < \infty$, we say that $w \in A_{p,q}$ if for every cube $Q$,

$$\left(\fint_Q w(x)^q \, dx\right)^{1/q} \left(\fint_Q w(x)^{-p'} \, dx\right)^{1/p'} \leq K < \infty.$$

Suppose an operator $T$ is such that for some $p_0$, $q_0$, and every $w \in A_{p_0,q_0}$, $T : L^{p_0}(w^{p_0}) \to L^{q_0}(w^{q_0})$. Then for all pairs $(p,q)$ such that $1/p - 1/q = 1/p_0 - 1/q_0$, and all $w \in A_{p,q}$, $T : L^p(w^p) \to L^q(w^q)$. In addition, though we do not explore it in detail, our approach yields many of the same extensions and generalizations described above (such as vector-valued inequalities) in the off-diagonal case.

Off-diagonal extrapolation was first proved by Harboure, Macías and Segovia [96] by adapting the proof of García-Cuerva. Our proof simplifies and extends theirs. It is applicable to the study of the so-called fractional operators, for instance the fractional integral operators, also known as the Riesz potentials.

## Extrapolation for arbitrary pairs of operators

In the proof of Theorem 1.4 we can replace $M$ (and so $M'$) not just with a more general maximal operator as we discussed above, but with an arbitrary pair of positive, sublinear operators, $T$, $T'$. If we do so, however, we need to replace $A_p$ weights with weight classes associated to these operators. This generalization was implicit in Coifman, Jones and Rubio de Francia [26], and was made explicit by Jawerth [108], who replaced the maximal operator with a positive sublinear operator $T$. (Also see Bloom [13].) Later, Hernández [102] and Ruiz and Torrea [198] extended the argument to two arbitrary operators. A version of this technique was used by Watson [229] to prove norm inequalities for a family of rough operators.

Using our approach we can easily deduce what we need to assume about the operators $T$ and $T'$, though the situation is complicated by the fact that property $(e)$, reverse factorization, does not necessarily hold in this context. We will give a precise statement and proof in Chapter 3. We consider the special case of extrapolation for the one-sided $A_p$ weights associated with the one-sided maximal operators. These weights were introduced by Sawyer [206], and extensively explored by Martín-Reyes, *et al.* [139, 140, 141, 142]. Extrapolation results were proved in [134, 141, 198].

## Limited range extrapolation

The conclusion of Theorem 1.4 yields that the operator $T$ is bounded on $L^p(w)$, $1 < p < \infty$. Therefore, extrapolation cannot be applied to operators $T$ that are only bounded, for instance, on $L^p$ if $1 < p_- < p < p_+ < \infty$. Operators of this type include the Riesz transforms and square functions associated with divergence form elliptic operators; see [7, 8] for precise definitions and results.

A restricted range extrapolation theorem can be gotten, however, by restricting the class of weights to $A_p \cap RH_s$ for some $s > 1$ depending on $p$. Results of this kind have been obtained by Johnson and Neugebauer [109] and by Duoandikoetxea *et al.* [70]. Here we use our techniques to prove a limited range extrapolation theorem that generalizes the extrapolation result in [9] and includes the above results as special cases.

We note in passing that a different kind of limited range extrapolation theorem was proved by Passarelli di Napoli [168].

## Extrapolation to Banach function spaces

Since our proof of Theorem 1.4 only uses a basic property of $L^p(w)$—the existence of an associate space—we can replace $L^p(w)$ by more general Banach function spaces. Given a Banach function space $\mathbb{X}$, with modest assumptions we have that $M$ is bounded on $\mathbb{X}$ and $M'$ is bounded on its associate space $\mathbb{X}'$. Thus we can show that if $T$ is bounded on $L^p(w)$ whenever $w \in A_p$, then $T$ is bounded on $\mathbb{X}$. If $\mathbb{X}$ is rearrangement invariant, then we can also get estimates for $T$ on the weighted spaces $\mathbb{X}(w)$.

Further, as we noted above, by a clever choice of the "dual function" $h$ we do not have to use duality explicitly. This point of view can be extended to let us extrapolate to so-called modular spaces (see [156]) and so obtain weighted modular inequalities,

$$\int_{\mathbb{R}^n} \Phi(|Tf(x)|)w(x)\,dx \le C \int_{\mathbb{R}^n} \Phi(|f(x)|)w(x)\,dx,$$

where $\Phi$ is a Young function, as a consequence of weighted $L^p$ inequalities. Such inequalities have been considered extensively by many authors: see [90, 114, 115] for details and further references. We will consider all of these results and their applications in Chapter 4.

This extension of extrapolation is new: the idea of extending extrapolation to modular inequalities and Banach function spaces first appeared in [57] where the authors and their collaborators used it to prove $A_\infty$ extrapolation theorems for rearrangement invariant Banach function spaces and modular spaces. In [40] extrapolation results were obtained for variable $L^p$ spaces, Banach function spaces that are not rearrangement invariant. Here we unite and extend both results.

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