Chapter 2. Abstract homotopy theory of n-fold spectra

In this chapter, it is shown that the category $\mathbf{Spt}^n$ of n-fold spectra carries closed model structures for both the strict (ie. levelwise) and stable weak equivalences. These results appear in the first section, and are proved by analogy with the corresponding results of Bousfield and Friedlander for the category of spectra [8].

We see, in the second section, that the diagonal functors $d : \mathbf{Spt}^n \to \mathbf{Spt}$ of Section 1.3 and 1.4 induce equivalences

$$\mathrm{Ho}(\mathbf{Spt}^n) \simeq \mathrm{Ho}(\mathbf{Spt})$$

of the homotopy categories associated to the respective stable closed model structures. In other words, the category of n-fold spectra is just another model for the ordinary stable category, albeit one which is quite useful. Applications will begin to appear in Chapter 3.

A quick review of the homotopy theories for simplicial presheaves and presheaves of spectra is presented in the third section of this chapter, and then the results of the first two sections are extended to the categories of presheaves of spectra and presheaves of n-fold spectra on an arbitrary Grothendieck site in the fourth section. The stable closed model structure for the category of presheaves of n-fold spectra is established by a method which parallels the proof given in [24] for the corresponding result about presheaves of spectra.

The equivalence of homotopy categories displayed above has a trivial analogy in an equivalence of the stable homotopy categories arising from presheaves of spectra and presheaves of n-fold spectra. Keep in mind, however, that the category of presheaves of spectra on a fixed site is not necessarily a model for the ordinary stable category. The results of the third section of this chapter imply that every Grothendieck topology gives rise to its own particular stable homotopy category, and so the collection of such theories (and models for each of them) is vast. These theories are not unrelated; in a coarse sense, the interplay between them is the subject of this book.

This chapter closes with a very short final section which "shows" that the stable homotopy categories of presheaves of spectra carry a symmetric monoidal structure with multiplication given by smash product, and with the ordinary sphere spectrum as unit. The method of proof is to defer completely to Adams [2], the point being that his constructions are natural in maps of spectra, as they are defined here.

2.1. Closed model category structures for n-fold spectra

Recall, from Chapter 1, that a bispectrum may be internally defined as a list of pointed simplicial sets $X^{i,j}$, $i, j \geq 0$, together with pointed simplicial set maps

$$S^1 \wedge X^{i,j} \xrightarrow{\sigma_1} X^{i+1,j} \qquad \text{and} \qquad S^1 \wedge X^{i,j} \xrightarrow{\sigma_2} X^{i,j+1}$$

such that the following diagram commutes for all i and j:

$$
\begin{array}{ccc}
S^1 \wedge X^{i+1,j} & \xrightarrow{\quad\sigma_2\quad} & X^{i+1,j+1} \\
{\scriptstyle S^1 \wedge \sigma_1}\big\uparrow & & \big\uparrow{\scriptstyle \sigma_1} \\
S^1 \wedge S^1 \wedge X^{i,j} \xrightarrow{\tau \wedge X^{i,j}} S^1 \wedge S^1 \wedge X^{i,j} \xrightarrow{S^1 \wedge \sigma_2} & & S^1 \wedge X^{i,j+1}
\end{array}
\qquad (2.1)
$$

Here, $\tau \wedge X^{i,j}$ is the map which flips the S^1 factors and is the identity on $X^{i,j}$. It's not difficult to see (by drawing the appropriate large diagram) that the commutativity of diagram (2.1) is equivalent to the commutativity of

$$
\begin{array}{ccc}
X^{i,j} & \xrightarrow{\sigma_2} & \Omega X^{i,j+1} \\
& & \searrow{\scriptstyle \Omega\sigma_1} \\
{\scriptstyle \sigma_1}\big\downarrow & & \Omega^2 X^{i+1,j+1} \\
& & \cong\big\downarrow{\scriptstyle \tau} \\
\Omega X^{i+1,j} & \xrightarrow{\Omega\sigma_2} & \Omega^2 X^{i+1,j+1}
\end{array}
\qquad (2.2)
$$

The map τ is the canonical isomorphism which interchanges loop factors.

More generally, an n-fold spectrum consists of pointed simplicial sets

$$X^{i_1,i_2,\ldots,i_n}, \qquad i_j \geq 0,$$

together with pointed simplicial set maps

$$\sigma_j : S^1 \wedge X^{i_1,\ldots,i_j,\ldots,i_n} \to X^{i_1,\ldots,i_j+1,\ldots,i_n},$$

$1 \leq j \leq n$ (called *bonding maps*), such that for each $1 \leq k < j \leq n$, and list of indices i_r, $r \neq j, k$, the array of simplicial sets

$$X^{i_1,\ldots,\overset{k}{*},\ldots,\overset{j}{*},\ldots,i_n},$$

together with the bonding maps σ_j and σ_k forms a bispectrum in the sense that diagrams analogous to (2.1) or (2.2) must commute. Write $\mathbf{Spt}^n$ for the category of n-fold spectra.

Suppose that X is a bispectrum. Then it's elementary to check that there is a bispectrum X_{Kan} with

$$X_{Kan}^{i,j} = S|X^{i,j}|,$$

and with bonding maps $\sigma_1 : S^1 \wedge X_{Kan}^{i,j} \to X_{Kan}^{i+1,j}$ and $\sigma_2 : S^1 \wedge X_{Kan}^{i,j} \to X_{Kan}^{i,j+1}$ defined, respectively, to be the composites

$$S^1 \wedge S|X^{i,j}| \xrightarrow{\eta \wedge X^{i,j}} S|S^1| \wedge |X^{i,j}| \xrightarrow{can} S|S^1 \wedge X^{i,j}| \xrightarrow{S|\sigma_1|} S|X^{i+1,j}|,$$

and

$$S^1 \wedge S|X^{i,j}| \xrightarrow{\eta \wedge X^{i,j}} S|S^1| \wedge |X^{i,j}| \xrightarrow{can} S|S^1 \wedge X^{i,j}| \xrightarrow{S|\sigma_2|} S|X^{i,j+1}|.$$

Furthermore, the natural maps $\eta : X^{i,j} \to S|X^{i,j}|$ define a natural map of bispectra

$$\eta : X \to X_{Kan}.$$

This construction is easily promoted to n-fold spectra for arbitrary n, giving

LEMMA 2.3. *Suppose that X is an n-fold spectrum. Then there is an n-fold spectrum X_{Kan} with*

$$X_{Kan}^{i_1,\ldots,i_n} = S|X^{i_1,\ldots,i_n}|.$$

Furthermore, the canonical weak equivalences

$$\eta : X^{i_1,\ldots,i_n} \to S|X^{i_1,\ldots,i_n}|$$

induce a natural map of n-fold spectra

$$\eta : X \to X_{Kan}.$$

Each $X_{Kan}^{i_1,\ldots,i_n}$ is a pointed Kan complex.

In the world of n-fold spectra, and given a pointed simplicial set K, the functors $X \mapsto X \wedge K$ and $X \mapsto \mathbf{hom}_*(K, X)$ have their obvious meanings. In other words, $X \wedge K$ is characterized by

$$(X \wedge K)^{i_1,\ldots,i_n} = X^{i_1,\ldots,i_n} \wedge K,$$

and by having bonding maps $\sigma_i \wedge K$. On the other hand,

$$\mathbf{hom}_*(K, X)^{i_1,\ldots,i_n} = \mathbf{hom}_*(K, X^{i_1,\ldots,i_n})$$

defines the spaces of $\mathbf{hom}_*(K, X)$, and this object has bonding maps

$$\mathbf{hom}_*(K, X^{i_1,\ldots,i_n}) \xrightarrow{\mathbf{hom}_*(K,\sigma_i)} \mathbf{hom}_*(K, \Omega X^{i_1,\ldots,i_n}) \xrightarrow{\tau}{\cong} \Omega\mathbf{hom}_*(K, X^{i_1,\ldots,i_n}),$$

where τ is the canonical isomorphism that switches function complex arguments.

An $(n+1)$-fold spectrum X (externally defined) consists of n-fold spectra X^i, $i \geq 0$, together with bonding maps of n-fold spectra

$$X^i \wedge S^1 \to X^{i+1}.$$

The usual adjointness relations apply, so that these bonding maps could be characterized as maps of n-fold spectra of the form

$$X^i \to \mathbf{hom}_*(S^1, X^{i+1}).$$

Say that a map $f : X \to Y$ of n-fold spectra is a *strict weak equivalence* (respectively *strict fibration*) if each of the maps

$$f : X^{i_1,\dots,i_n} \to Y^{i_1,\dots,i_n}$$

is a weak equivalence (respectively fibration) of simplicial sets. A *cofibration* is a map of n-fold spectra which has the left lifting property with respect to all maps which are strict fibrations and strict weak equivalences (aka. trivial strict fibrations).

PROPOSITION 2.4. *The category $\mathbf{Spt}^n$ of n-fold spectra, with the classes of cofibrations, strict fibrations and strict weak equivalences, satisfies the axioms for a closed model category.*

PROOF: We shall assume that $n \geq 1$, and that the category of $(n-1)$-fold spectra is a closed model category for these definitions. We shall also identify the category $\mathbf{Spt}^n$ with the category of spectrum objects in the category of $(n-1)$-fold spectra.

The category $\mathbf{Spt}^n$ is complete and cocomplete, so that the axiom **CM1** holds. The verification of the axioms **CM2** (weak equivalence) and **CM3** (retract) is clear.

The proof of the factorization axiom **CM5** is achieved by first showing that a map $i : A \to B$ of n-fold spectra has the left lifting property with respect to all (respectively trivial) strict fibrations if the following conditions are satisfied:

(1) $i^0 : A^0 \to B^0$ is a trivial cofibration (respectively cofibration) of $(n-1)$-fold spectra, and

(2) the map

$$(B^k \wedge S^1) \cup_{(A^k \wedge S^1)} A^{k+1} \to B^{k+1}$$

is a trivial cofibration (respectively cofibration) of $(n-1)$-fold spectra.

To show that any map $f : X \to Y$ has a factorization $f = qi$, where q is a trivial strict fibration and i is a cofibration, find a factorization

in degree 0, where i^0 is a cofibration and q^0 is a trivial strict fibration of $(n-1)$-fold spectra. Then the canonical map

$$g^1 : (Z^0 \wedge S^1) \cup_{(X^0 \wedge S^1)} X^1 \to Y^1$$

has a factorization

where α^1 is a cofibration and q^1 is a trivial strict fibration of $(n-1)$-fold spectra. The map q^1 is now given, and i^1 is defined to be the composite

$$X^1 \xrightarrow{(i^0 \wedge S^1)_*} (Z^0 \wedge S^1) \cup_{(X^0 \wedge S^1)} X^1 \xrightarrow{\alpha^1} Z^1.$$

This works, since cofibrations are closed under smashing with pointed simplicial sets, and closed under pushout, so that $(i^0 \wedge S^1)_*$ is a cofibration. Proceed inductively to obtain the factorization $f = qi$.

The other part of the factorization axiom is obtained by showing that f has a factorization $f = pj$, where p is a strict fibration and j is a map which is a strict weak equivalence and has the left lifting property with respect to all strict fibrations: one uses trivial cofibrations of $(n-1)$-fold spectra in place of the cofibrations in the argument just given.

A standard argument now shows that any map j of n-fold spectra which is a cofibration and a strict weak equivalence must be a retract of a map which is a strict weak equivalence and has the left lifting property with respect to all strict fibrations. Such a map j therefore has the left lifting property with respect to all fibrations, and axiom **CM4** follows. ∎

REMARK 2.5. The proof of the factorization axioms in the previous result implies that any cofibration of n-fold spectra is a retract of a map which is composed of maps which are cofibrations of $(n-1)$-fold spectra. It follows by induction on n that any cofibration $j : A \to B$ of n-fold spectra is a pointwise cofibration in the sense that all of the maps

$$j : A^{i_1,\dots,i_n} \to B^{i_1,\dots,i_n}$$

are cofibrations of simplicial sets. ∎

Suppose that X and Y are n-fold spectra. The *function complex*

$$\mathbf{hom}_*(X, Y)$$

is the pointed simplicial set whose m-simplices are the pointed maps of n-fold spectra of the form $X \ltimes \Delta^m \to Y$. It is a direct consequence of the corresponding result for pointed simplicial sets that there is an exponential law

$$\mathbf{hom}_*(X \wedge K, Y) \cong \mathbf{hom}_*(K, \mathbf{hom}_*(X, Y)).$$

PROPOSITION 2.6. *With the definitions of strict weak equivalence, cofibration, strict fibration and the definition of function complex given above, the category* $\mathbf{Spt}^n$ *of n-fold spectra is a proper closed simplicial model category.*

PROOF: Suppose that $p : X \to Y$ is a strict fibration of n-fold spectra, the map $i : U \to V$ is a cofibration of n-fold spectra, and that $j : K \to L$ is a trivial cofibration of pointed simplicial sets. Then the existence of the dotted arrow making the diagram commute is an equivalent condition for any of the following three (adjoint) diagrams:

$$
\begin{array}{ccc}
K & \xrightarrow{\quad\alpha\quad} & \mathbf{hom}_*(V, X) \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle (i^*, p_*)} \\
L & \xrightarrow[(\beta, \gamma)]{} & \mathbf{hom}_*(U, X) \times_{\mathbf{hom}_*(U,Y)} \mathbf{hom}_*(V, Y)
\end{array}
\tag{2.7}
$$

$$
\begin{array}{ccc}
(U \wedge L) \cup_{(U \wedge K)} (V \wedge K) & \xrightarrow{(\beta, \alpha)} & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
V \wedge L & \xrightarrow[\gamma]{} & Y
\end{array}
\tag{2.8}
$$

$$
\begin{array}{ccc}
U & \xrightarrow{\quad\beta\quad} & \mathbf{hom}_*(L, X) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle (p_*, j^*)} \\
V & \xrightarrow[(\gamma, \alpha)]{} & \mathbf{hom}_*(L, Y) \times_{\mathbf{hom}_*(K,X)} \mathbf{hom}_*(K, Y)
\end{array}
\tag{2.9}
$$

I have confused various maps notationwise with their adjoints in these diagrams. Note that the map

$$(p_*, j^*) : \mathbf{hom}_*(L, X) \to \mathbf{hom}_*(L, Y) \times_{\mathbf{hom}_*(K,X)} \mathbf{hom}_*(K, Y)$$

is a strict fibration, which is trivial if j or p is trivial. It follows that the map

$$(i^*, p_*) : \mathbf{hom}_*(V, X) \to \mathbf{hom}_*(U, X) \times_{\mathbf{hom}_*(U,Y)} \mathbf{hom}_*(V, Y)$$

is a fibration of pointed simplicial sets, which is trivial if i is a trivial cofibration or p is a trivial strict fibration.

Strict weak equivalences are closed under cobase change by cofibrations, by the corresponding property for simplicial sets and Remark 2.5. Similarly, strict weak equivalences are closed under base change by fibrations. ∎

Suppose that Z is an n-fold spectrum, and that $\{r_1, \ldots, r_n\}$ is a list of n integers. An n-fold spectrum $Z[r_1, \ldots, r_n]$ is defined by setting

$$Z[r_1, \ldots, r_n]^{i_1, \ldots, i_n} = Z^{i_1 + r_1, \ldots, i_n + r_n}.$$

This construction is functorial in n-fold spectra.

Suppose now that X is a strictly fibrant bispectrum. A bispectrum $\Omega_1 X$ is defined by

$$\Omega_1 X^{i,j} = \Omega X^{i,j},$$

and with bonding maps

$$\Omega \sigma_1 : \Omega X^{i,j} \to \Omega^2 X^{i+1,j},$$

for σ_1, and the composite

$$\Omega X^{i,j} \xrightarrow{\Omega \sigma_2} \Omega^2 X^{i,j+1} \xrightarrow[\cong]{\tau} \Omega^2 X^{i,j+1}$$

for σ_2. In other words,

$$\Omega_1 X^{*,j} = \Omega X^{*,j}$$

is a fake loop spectrum for each j, while

$$\Omega_1 X^{i,*} = \mathbf{hom}_*(S^1, X^{i,*})$$

is an honest loop object for each i. Similarly, there is a bispectrum $\Omega_2 X$ such that

$$\begin{cases} \Omega_2 X^{*,j} = \mathbf{hom}_*(S^1, X^{*,j}) & \text{for each } j, \text{ and} \\ \Omega_2 X^{i,*} = \Omega X^{i,*} & \text{for each } i. \end{cases}$$

Note that $\Omega_1 X$ and $\Omega_2 X$ are strictly fibrant.

The bonding maps σ_1 and σ_2 respectively determine maps of bispectra

$$\sigma_1 : X \to \Omega_1 X[1, 0] \qquad \text{and} \qquad \sigma_2 : X \to \Omega_2 X[0, 1].$$

Flipping loop factors determines a natural isomorphism of bispectra

$$\Omega_1 \Omega_2 X \xrightarrow[\cong]{\tau} \Omega_2 \Omega_1 X,$$

and there is a commutative diagram

$$
\begin{array}{c}
\Omega_2 X[0,1] \\
\sigma_1 \nearrow \qquad \searrow \Omega_2\sigma_1 \\
\Omega_1\Omega_2 X[1,1] \xrightarrow[\cong]{\ \tau\ } \Omega_2\Omega_1 X[1,1] \qquad (2.10)\\
\Omega_1\sigma_2 \searrow \qquad \nearrow \sigma_2 \\
\Omega_1 X[1,0]
\end{array}
$$

Similarly, if X is a strictly fibrant n-fold spectrum, where $n \geq 2$, and $1 \leq j \leq n$, there is an associated strictly fibrant n-fold spectrum $\Omega_j X$ with

$$
\begin{cases}
\Omega_j X^{i_1,\ldots,\overset{k}{*},\ldots,i_n} = \mathbf{hom}_*(S^1, X^{i_1,\ldots,\overset{k}{*},\ldots,i_n}) & \text{if } k \neq j, \text{ and}\\
\Omega_j X^{i_1,\ldots,\overset{k}{*},\ldots,i_n} = \Omega X^{i_1,\ldots,\overset{k}{*},\ldots,i_n} & \text{if } k = j.
\end{cases}
$$

Then there are natural maps

$$
\sigma_j : X \to \Omega_j X[0,\ldots,\overset{j}{1},\ldots 0]
$$

which are determined by the various bonding maps of X. If $i < j$, switching loop factors gives a canonical isomorphism

$$
\Omega_i\Omega_j X[0,\ldots,\overset{i}{1},\ldots,\overset{j}{1},\ldots,0] \xrightarrow[\cong]{\ \tau\ } \Omega_j\Omega_i X[0,\ldots,\overset{i}{1},\ldots,\overset{j}{1},\ldots,0]
$$

and a commutative diagram

$$
\begin{array}{c}
\Omega_j X[0,\ldots,\overset{j}{1},\ldots,0] \\
\sigma_i \nearrow \qquad \searrow \Omega_j\sigma_i \\
\Omega_i\Omega_j X[0,\ldots,\overset{i}{1},\ldots,\overset{j}{1},\ldots,0] \xrightarrow[\cong]{\ \tau\ } \Omega_j\Omega_i X[0,\ldots,\overset{i}{1},\ldots,\overset{j}{1},\ldots,0] \qquad (2.11)\\
\Omega_i\sigma_j \searrow \qquad \nearrow \sigma_j \\
\Omega_i X[0,\ldots,\overset{i}{1},\ldots,0]
\end{array}
$$

Suppose that Z is a strictly fibrant bispectrum. One defines natural isomorphisms

$$\tau_n : \Omega_2\Omega_1^n Z \to \Omega_1^n\Omega_2 Z$$

as follows. Let τ_1 be the isomorphism $\tau : \Omega_2\Omega_1 Z \to \Omega_2\Omega_1 Z$ that we constructed above. Suppose that τ_n defines an isomorphism

$$\tau_n : \Omega_2\Omega_1^n\Omega_1 Z \to \Omega_1^n\Omega_2\Omega_1 Z;$$

then τ_{n+1} is defined to be the composite of this map with the map

$$\Omega_1^n\tau : \Omega_1^n\Omega_2\Omega_1 Z \to \Omega_1^n\Omega_1\Omega_2 Z.$$

Let $Q_1 Z$ be the filtered colimit of the system

$$Z \xrightarrow{\sigma_1} \Omega_1 Z[1,0] \xrightarrow{\Omega_1\sigma_1} \Omega_1^2 Z[2,0] \to \dots$$

and let $\mu_1 : Z \to Q_1 Z$ denote the associated canonical map. Let $Q_2 Z$ denote the filtered colimit of the system

$$Z \xrightarrow{\sigma_2} \Omega_2 Z[0,1] \xrightarrow{\Omega_2\sigma_2} \Omega_2^2 Z[0,2] \to \dots$$

with canonical map $\mu_2 : Z \to Q_2 Z$.

The maps

$$\Omega_2\Omega_1 Z \xrightarrow{\tau} \Omega_1\Omega_2 Z$$

induce canonical isomorphisms

$$\Omega_2^m\Omega_1^n Z[n,m] \xrightarrow{\tau_{n,m}} \Omega_1^n\Omega_2^m Z[n,m],$$

which fit into commutative diagrams

$$
\begin{array}{ccc}
\Omega_2^m\Omega_1^{n-1} Z[n-1,m] & \xrightarrow{\tau_{n-1,m}} & \Omega_1^{n-1}\Omega_2^m Z[n-1,m] \\
\Big\downarrow{\Omega_2^m\sigma_1} & & \Big\downarrow{\sigma_1} \\
\Omega_2^m\Omega_1^n Z[n,m] & \xrightarrow{\tau_{n,m}} & \Omega_1^n\Omega_2^m Z[n,m] \\
\Big\uparrow{\sigma_2} & & \Big\uparrow{\Omega_1^n\sigma_2} \\
\Omega_2^{m-1}\Omega_1^n Z[n,m-1] & \xrightarrow{\tau_{n,m-1}} & \Omega_1^n\Omega_2^{m-1} Z[n,m-1].
\end{array}
$$

Taking filtered colimits in n and m therefore gives a commutative diagram

$$
\begin{array}{ccc}
 & Q_2Z & \\
Q_2\mu_1 \swarrow & & \searrow \mu_1 \\
Q_2Q_1Z \xrightarrow[\cong]{\ \tau\ } & & Q_1Q_2Z. \\
\mu_2 \searrow & & \nearrow Q_1\mu_2 \\
 & Q_1Z &
\end{array}
\qquad (2.12)
$$

Analogous definitions and techniques apply to n-fold spectra for higher n. If X is a strictly fibrant n-fold spectrum and $1 \le j \le n$, then the n-fold spectrum Q_jX is defined to be the filtered colimit of the system

$$
X \xrightarrow{\ \sigma_j\ } \Omega_j X[0,\dots,\overset{j}{1},\dots,0] \xrightarrow{\ \Omega_j\sigma_j\ } \Omega_j^2 X[0,\dots,\overset{j}{1},\dots,0] \to \dots
$$

with canonical map $\mu_j : X \to Q_jX$. The same methods as were used in the bispectrum case may then be used to show:

LEMMA 2.13. *Suppose that* $1 \le i < j \le n$. *Then there is a natural isomorphism of bispectra*

$$
\tau : Q_jQ_iX \xrightarrow{\ \cong\ } Q_iQ_jX,
$$

and a commutative diagram

$$
\begin{array}{ccc}
 & Q_jX & \\
Q_j\mu_i \swarrow & & \searrow \mu_i \\
Q_jQ_iX \xrightarrow[\cong]{\ \tau\ } & & Q_iQ_jX \\
\mu_j \searrow & & \nearrow Q_i\mu_j \\
 & Q_iX &
\end{array}
$$

The *stabilization* QX of X is the strictly fibrant n-fold spectrum which is defined by

$$
QX = Q_nQ_{n-1}\dots Q_1X.
$$

The map $\mu : X \to QX$ is defined to be the composite

$$
X \xrightarrow{\ \mu_1\ } Q_1X \xrightarrow{\ \mu_2\ } \dots \xrightarrow{\ \mu_{n-1}\ } Q_{n-1}\dots Q_1X \xrightarrow{\ \mu_n\ } Q_n\dots Q_1X.
$$

Lemma 2.13 implies that we can permute the Q_i's in the definition of QX and get a naturally isomorphic n-fold spectrum.

The first step in the analysis of QX will be to characterize the homotopy groups of the spaces $QX^{i_1,\dots,i_n}$.

The bonding maps

$$\sigma_r : X^{i_1,\dots,i_n} \to \Omega X^{i_1,\dots,i_r+1,\dots,i_n}$$

induce maps in homotopy groups of the form

$$\sigma_r : \pi_k X^{i_1,\dots,i_n} \to \pi_{k+1} X^{i_1,\dots,i_r+1,\dots,i_n}.$$

Furthermore, if $r < s$, there is a commutative diagram

$$
\begin{array}{ccc}
\pi_k X^{i_1,\dots,i_n} & \xrightarrow{\ \ \sigma_r\ \ } & \pi_{k+1} X^{i_1,\dots,i_r+1,\dots,i_n} \\
\downarrow{\sigma_s} & & \downarrow{(-1)\sigma_s} \\
\pi_{k+1} X^{i_1,\dots,i_s+1,\dots,i_n} & \xrightarrow{\ \ \sigma_r\ \ } & \pi_{k+2} X^{i_1,\dots,i_r+1,\dots,i_s+1,\dots i_n}
\end{array}
$$

It follows that there is an abelian group-valued functor

$$\pi_k \mathbf{X}^{i_1,\dots,i_n} : \mathbb{N}^{\times n} \to \mathbf{Ab}$$

which is defined by associating to the n-tuple $(s_1,\dots,s_n)$ the group

$$\pi_{k+(s_1+\cdots+s_n)} X^{i_1+s_1,i_2+s_2,\dots,i_n+s_n}.$$

The definition of this functor is completed by the assignment of the homomorphism

$$
\begin{array}{c}
\pi_{k+(s_1+\cdots+s_n)} X^{i_1+s_1,\dots,i_j+s_j,\dots,i_n+s_n} \\
\downarrow{(-1)^{(i_1+s_1)+\cdots+(i_{j-1}+s_{j-1})}\sigma_j} \\
\pi_{k+(s_1+\cdots+s_n)+1} X^{i_1+s_1,\dots,i_j+s_j+1,\dots,i_n+s_n}
\end{array}
$$

to the relation

$$(s_1,\dots,s_j,\dots,s_n) \to (s_1,\dots,s_j+1,\dots,s_n)$$

of $\mathbb{N}^{\times n}$.

Lemma 2.14. *There is a natural isomorphism*

$$\pi_k Q X^{i_1,\ldots,i_n} \cong \varinjlim_{\mathbb{N}\times n} \pi_k \mathbf{X}^{i_1,\ldots,i_n}.$$

Proof: The colimit

$$\varinjlim_{\mathbb{N}\times n} \pi_k \mathbf{X}^{i_1,\ldots,i_n}$$

may be identified up to natural isomorphism with the iterated colimit

$$\varinjlim_{1} \varinjlim_{2} \ldots \varinjlim_{n} \pi_k \mathbf{X}^{i_1,\ldots,i_n},$$

where $\varinjlim_j$ means "take the colimit in the j direction".

The natural map $\mu_j : X \to Q_j X$ induces an isomorphism

$$\varinjlim_{j} \pi_k \mathbf{X}^{i_1,\ldots,i_n} \xrightarrow[\cong]{\mu_j} \varinjlim_{j} \pi_k \mathbf{Q_j X}^{i_1,\ldots,i_n},$$

by the usual cofinality argument. But then any composite $X \to Q_n \ldots Q_1 X$ of instances of the μ_i maps induces an isomorphism

$$\varinjlim_{n} \ldots \varinjlim_{1} \pi_k \mathbf{X}^{i_1,\ldots,i_n} \cong \varinjlim_{n} \ldots \varinjlim_{1} \pi_k \mathbf{Q_n} \ldots \mathbf{Q_1 X}^{i_1,\ldots,i_n}. \qquad (2.15)$$

Finally, all arrows are isomorphisms in the diagram

$$\pi_k \mathbf{Q X}^{i_1,\ldots,i_n}$$

of abelian groups, so that there is a canonical isomorphism

$$\varinjlim_{\mathbb{N}\times n} \pi_k \mathbf{Q X}^{i_1,\ldots,i_n} \cong \pi_k Q X^{i_1,\ldots,i_n}. \qquad \blacksquare$$

Recall from Chapter 1 that the *diagonal spectrum dX* of an n-fold spectrum X is the spectrum having

$$dX^{nj+k} = X^{j+1,\ldots,\overset{k}{j+1},j,\ldots,j},$$

and with bonding map

$$\sigma : S^1 \wedge dX^{nj+k} \to dX^{nj+k+1}$$

given by σ_{k+1} for $0 \le k \le n-1$. This gives a diagonalization functor $d :$ $\mathbf{Spt}^n \to \mathbf{Spt}$, which will be shown in the next section to induce an equivalence of homotopy categories. We also know from the first chapter that there are many other candidates for d, but this one suffices.

Recall [8] that a map of ordinary spectra is a *stable equivalence* (or, according to the customary abuse, a *stable homotopy equivalence*) if it induces an isomorphism of all stable homotopy groups. A spectrum Z is *stably fibrant* if and only if it is strictly fibrant in the sense that that all of its constituent spaces Z^i are Kan complexes, and all of the bonding maps

$$\sigma : Z^i \to \mathbf{hom}_*(S^1, Z^{i+1})$$

are weak equivalences of simplicial sets.

COROLLARY 2.16. *Suppose that X is a strictly fibrant n-fold spectrum. Then the diagonal spectrum dQX of QX is a stably fibrant model of the diagonal spectrum dX of X in the sense that dQX is stably fibrant, and the map*

$$dX \to dQX$$

is a stable homotopy equivalence.

PROOF: The spaces $QX^{i_1,\dots,i_n}$ are Kan complexes by construction, and each of the bonding maps '

$$\sigma_j : QX^{i_1,\dots,i_j,\dots,i_n} \to \Omega QX^{i_1,\dots,i_j+1,\dots,i_n}$$

is a weak equivalence, so dQX is stably fibrant. The stable homotopy groups of dX may be identified up to natural isomorphism (ie. by cancelling signs) with groups

$$\varinjlim_{j} \pi_{i+kj} \mathbf{X}^{j,\dots,j}.$$

(see the proof of Lemma 1.28). Now use the previous result. ∎

A map $f : X \to Y$ of n-fold spectra is a *pointwise cofibration* if each of the maps

$$f : X^{i_1,\dots,i_n} \to Y^{i_1,\dots,i_n}$$

is a cofibration of simplicial sets. To fix ideas, remember that a cofibration of n-fold spectra is a somewhat more complex object than a pointwise cofibration. Pointwise cofibrations are nevertheless "right" for the construction of long exact sequences in stable homotopy groups.

It's worth recalling that, if $i : A \to X$ is a pointwise cofibration of spectra, then there is an exact sequence in stable homotopy groups

$$\pi_n^s A \xrightarrow{i} \pi_n^s X \xrightarrow{p} \pi_n^s X/A,$$

where $p : X \to X/A$ is the cofibre. The reason is the one that is learned in school: if $\alpha : K \to X^n$ is a map of pointed topological spaces such that $p\alpha$ is pointed homotopy trivial, then the cofibration sequence can be used to show that there is a map $\beta : S^1 \wedge K \to S^1 \wedge A^n$ such that $(S^1 \wedge i)\beta = S^1 \wedge \alpha$ as maps $S^1 \wedge K \to S^1 \wedge X^n$. Note that we have not required $i : A \to X$ to be a cofibration of spectra. This, together with the definition of the diagonal functor for n-fold spectra, leads immediately to

LEMMA 2.17. *Suppose that $i : A \to X$ is a pointwise cofibration of n-fold spectra and let $p : X \to X/A$ be the cofibre. Then there is an exact sequence in stable homotopy groups of the form*

$$\pi_k^s dX \xrightarrow{p} \pi_k^s d(X/A) \to \pi_{k-1}^s dA \xrightarrow{i} \pi_{k-1}^s dX \xrightarrow{p} \pi_{k-1}^s d(X/A).$$

If $p : X \to Y$ is a strict fibration of n-fold spectra with fibre F, then the diagonal

$$p : dX \to dY$$

is a strict fibration with fibre dF. Then one has:

Lemma 2.18. *Suppose that $p : X \to Y$ is a strict fibration of n-fold spectra with fibre F. Let $i : F \to X$ be the inclusion of the fibre. Then there is a long exact sequence in stable homotopy groups*

$$\pi_k^s dF \xrightarrow{i} \pi_k^s dX \xrightarrow{p} \pi_k^s dY \to \pi_{k-1}^s dF \xrightarrow{i} \pi_{k-1}^s dX.$$

Suppose that X is an n-fold spectrum, and let K be a pointed simplicial set. Let

$$\tau : \mathbf{hom}_*(K, \Omega X^{i_1,\ldots,i_n}) \xrightarrow{\cong} \Omega\mathbf{hom}_*(K, X^{i_1,\ldots,i_n})$$

denote the canonical isomorphism of function complexes, applied to the pointed simplicial sets $X^{i_1,\ldots,i_n}$. Then there is a commutative diagram

$$
\begin{array}{ccc}
\mathbf{hom}_*(K, \Omega X^{i_1,\ldots,i_j+1,\ldots,i_n}) & \xrightarrow{\tau} & \Omega\mathbf{hom}_*(K, X^{i_1,\ldots,i_j+1,\ldots,i_n}) \\
{\scriptstyle (\Omega\sigma_j)_*}\downarrow & & \downarrow{\scriptstyle \Omega(\sigma_j)_*} \\
\mathbf{hom}_*(K, \Omega^2 X^{i_1,\ldots,i_j+2,\ldots,i_n}) & \xrightarrow{\tau} & \Omega\mathbf{hom}_*(K, \Omega X^{i_1,\ldots,i_j+2,\ldots,i_n}) \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \Omega\tau} \\
\Omega\mathbf{hom}_*(K, \Omega X^{i_1,\ldots,i_j+2,\ldots,i_n}) & \xrightarrow{\Omega\tau} & \Omega^2\mathbf{hom}_*(K, X^{i_1,\ldots,i_j+2,\ldots,i_n})
\end{array}
$$

where, for example,

$$(\sigma_j)_* : \mathbf{hom}_*(K, X^{i_1,\ldots,i_j+1,\ldots,i_n}) \to \mathbf{hom}_*(K, \Omega X^{i_1,\ldots,i_j+2,\ldots,i_n})$$

is the map induced by the bonding map σ_j. It follows that the isomorphisms τ define a natural isomorphism of n-fold spectra

$$\tau : \mathbf{hom}_*(K, \Omega_j X[0,\ldots,\overset{j}{1},\ldots,0]) \xrightarrow{\cong} \Omega_j\mathbf{hom}_*(K, X)[0,\ldots,\overset{j}{1},\ldots,0].$$

Furthermore, the natural map

$$\sigma_j : \mathbf{hom}_*(K, X) \to \Omega_j\mathbf{hom}_*(K, X)[0,\ldots,\overset{j}{1},\ldots,0]$$

and the map τ appear in a commutative diagram

$$
\begin{array}{ccc}
\mathbf{hom}_*(K, X) & \xrightarrow{(\sigma_j)_*} & \mathbf{hom}_*(K, \Omega_j X[0,\ldots,\overset{j}{1},\ldots,0]) \\
& {\scriptstyle \sigma_j}\searrow & \downarrow{\scriptstyle \tau}{\scriptstyle \cong} \\
& & \Omega_j\mathbf{hom}_*(K, X)[0,\ldots,\overset{j}{1},\ldots,0].
\end{array}
$$

LEMMA 2.19. *Suppose that X is an n-fold spectrum, and let K be a pointed simplicial set. Then the isomorphisms*

$$\tau : \mathbf{hom}_*(K, \Omega_j X[0, \ldots, \overset{j}{1}, \ldots, 0]) \to \Omega_j \mathbf{hom}_*(K, X)[0, \ldots, \overset{j}{1}, \ldots, 0]$$

induce a natural isomorphism

$$\tau : \mathbf{hom}_*(K, Q_j X) \xrightarrow{\cong} Q_j \mathbf{hom}_*(K, X),$$

and there is a commutative diagram

$$
\begin{array}{ccc}
\mathbf{hom}_*(K, X) & \xrightarrow{\mathbf{hom}_*(K, \mu_j)} & \mathbf{hom}_*(K, Q_j X) \\
 & {\scriptstyle \mu_j} \searrow & \Big\downarrow {\scriptstyle \cong} \tau \\
 & & Q_j \mathbf{hom}_*(K, X),
\end{array}
$$

where $\mu_j : X \to Q_j X$ is the canonical map.

PROOF: The proof is a generalization of a step in the proof of Lemma 2.13. ∎

The following result is well known. Its proof is presented here as a prototype for later results (specifically Lemma 2.46).

LEMMA 2.20. *Suppose that $f : X \to Y$ is a stable equivalence of spectra, and that K is a pointed simplicial set. Then the induced map*

$$f \wedge K : X \wedge K \to Y \wedge K$$

is a stable equivalence.

PROOF: The first step is to show that, if Z is stably fibrant, then so is the spectrum $\mathbf{hom}_*(K, Z)$. But $\mathbf{hom}_*(K, Z)$ is strictly fibrant, and the canonical map

$$\mu : Z \to QZ$$

is a strict weak equivalence. Furthermore, by Lemma 2.19, there is a commutative diagram

$$
\begin{array}{ccc}
\mathbf{hom}_*(K, Z) & \xrightarrow{\mathbf{hom}_*(K, \mu)} & \mathbf{hom}_*(K, QZ) \\
 & {\scriptstyle \mu} \searrow & \Big\downarrow {\scriptstyle \cong} \tau \\
 & & Q\mathbf{hom}_*(K, Z),
\end{array}
$$

in which the map $\mathbf{hom}_*(K, \mu)$ is a strict weak equivalence; it follows that μ is a strict weak equivalence.

The functor $X \mapsto X \wedge K$ preserves strict weak equivalences. Choose a commutative diagram

$$\begin{array}{ccc} X_c & \xrightarrow{\pi_X} & X \\ {\scriptstyle f_c}\downarrow & & \downarrow{\scriptstyle f} \\ Y_c & \xrightarrow{\pi_Y} & Y \end{array}$$

in which the maps π_X and π_Y are strict fibrations and strict weak equivalences, and f_c is a cofibration which is necessarily stably trivial. Note that, since $\pi_X \wedge K$ and $\pi_Y \wedge K$ are strict weak equivalences, $f \wedge K$ is a stable equivalence if and only if $f_c \wedge K$ is a stable equivalence.

Observe that the spectra $X_c \wedge K$ and $Y_c \wedge K$ are cofibrant, and recall that Bousfield-Friedlander criterion (this is Lemma 4.5 of [8], but it is also easily proved) that a map $\theta : A \to B$ of cofibrant spectra is a stable equivalence if and only if the induced map

$$\theta^* : \mathbf{hom}_*(B, Z) \to \mathbf{hom}_*(A, Z)$$

of pointed simplicial sets is a weak equivalence for all stably fibrant spectra Z.

Now let Z be a stably fibrant spectrum, and form the diagram

$$\begin{array}{ccc} \mathbf{hom}_*(Y_c \wedge K, Z) & \xrightarrow{\cong} & \mathbf{hom}_*(Y_c, \mathbf{hom}_*(K, Z)) \\ {\scriptstyle (f_c \wedge K)^*}\downarrow & & \downarrow{\scriptstyle f_c^*} \\ \mathbf{hom}_*(X_c \wedge K, Z) & \xrightarrow[\cong]{} & \mathbf{hom}_*(X_c, \mathbf{hom}_*(K, Z)). \end{array}$$

The spectrum $\mathbf{hom}_*(K, Z)$ is stably fibrant from what we have seen above, and f_c is a stable equivalence, so that f_c^* is a weak equivalence, for all Z. But then $(f_c \wedge K)^*$ is a weak equivalence for all stably fibrant Z, and so $f_c \wedge K$ is a stable equivalence. ∎

COROLLARY 2.21.

(1) *The functor* $X \mapsto X \wedge K$ *preserves stably trivial cofibrations of spectra.*

(2) *The functor* $X \mapsto \mathbf{hom}_*(K, X)$ *preserves stable fibrations of spectra.*

(3) *The functor* $X \mapsto \mathbf{hom}_*(S^1, X)$ *preserves stable fibrations of spectra.*

REMARK 2.22. Lemma 2.20 is the non-trivial part of the assertion that the stable structure on the category of spectra satisfies Quillen's axiom **SM7** for a closed simplicial model category. It can also be proved by using the spectral sequence techniques of Section 4.2 below. ∎

A map $f : X \to Y$ of n-fold spectra is said to be a *stable equivalence* if the associated map of diagonals

$$f : dX \to dY$$

is a stable equivalence of spectra. In view of the proof of Lemma 2.14, f is a stable equivalence of n-fold spectra if and only if

$$f : QX_{Kan} \to QY_{Kan}$$

is a strict weak equivalence of n-fold spectra. We also have the following, which is another consequence of Lemma 2.20:

COROLLARY 2.23. *The functor $X \mapsto X \wedge K$ preserves stable equivalences of n-fold spectra.*

Following [8], one says that a map $f : X \to Y$ of n-fold spectra is a *stable fibration* if it has the right lifting property with respect to all cofibrations which are stable equivalences. A cofibration which is also a stable equivalence is said to be a *stably trivial cofibration*. Similarly, *stably trivial fibrations* are maps which are both stable fibrations and stable weak equivalences.

LEMMA 2.24. *The functor $X \mapsto X \wedge K$ preserves cofibrations of n-fold spectra.*

PROOF: The proof is by induction on n, and proceeds by identifying n-fold spectra with spectrum objects in the category of $(n-1)$-fold spectra.

Recall that every cofibration $i : U \to V$ is a retract of a map $j : U \to W$ such that

(1) the map $j : U^0 \to W^0$ is a cofibration of $(n-1)$-fold spectra,
(2) each canonical map

$$(S^1 \wedge W^k) \cup_{(S^1 \wedge U^k)} U^{k+1} \to W^{k+1}$$

is a cofibration of $(n-1)$-fold spectra.

Any map j satisfying these two properties is a cofibration of n-fold spectra, and the collection of all such maps is closed under smashing with K, by the inductive assumption. But then $i \wedge K$ is a retract of $j \wedge K$, and $j \wedge K$ is a cofibration, so that $i \wedge K$ is a cofibration. ∎

COROLLARY 2.25. *The functor $X \mapsto X \wedge K$ preserves stably trivial cofibrations of n-fold spectra.*

THEOREM 2.26. *With these definitions, the category $\mathbf{Spt}^n$ of n-fold spectra, with the classes of cofibrations, stable equivalences and stable fibrations, satisfies the axioms for a proper closed simplicial model category.*

PROOF: One verifies the Bousfield-Friedlander axioms **A.4**, **A.5** and **A.6** (see [8] again) to get the closed model structure.

Every strict weak equivalence is a stable equivalence by Lemma 2.14, whence **A.4**. Note that $\mu_X = \mu : X \to QX$ is a strict weak equivalence for strictly fibrant n-fold spectra X if and only if each of the maps

$$\mu_j : X \to \Omega_j X[0, \ldots, \overset{j}{1}, \ldots, 0]$$

is a strict weak equivalence, and this is certainly true for QX. Thus, μ_{QX} is a strict weak equivalence. On the other hand, the fact that $Q\mu_X$ is a strict weak equivalence is a consequence of the existence of the natural isomorphism (2.15), so that **A.5** holds. Finally, **A.6** is a consequence of Lemmas 2.17 and 2.18.

Suppose that $j : K \to L$ is a cofibration of pointed simplicial sets, $i : U \to V$ is a cofibration of $\mathbf{Spt}^n$, and that $p : X \to Y$ is a stable fibration of n-fold spectra. Then p is a strict fibration, so that the map

$$(i^*, p_*) : \mathbf{hom}_*(V, X) \to \mathbf{hom}_*(U, X) \times_{\mathbf{hom}_*(U,Y)} \mathbf{hom}_*(V, Y)$$

is a fibration of simplicial sets, by Proposition 2.6. Furthermore, the canonical map

$$\theta : (U \wedge L) \cup_{(U \wedge K)} (V \wedge K) \to V \wedge L$$

is a cofibration: see the diagrams (2.7) – (2.9). It follows that (i^*, p_*) is a stably trivial fibration if p is stably trivial (diagram (2.8)). On the other hand, Corollary 2.25 implies that θ is a stably trivial cofibration if i is stably trivial. Thus, Quillen's axiom **SM7** holds.

The properness assertion is axiom **A.6**. ∎

REMARK 2.27. A result of Bousfield and Friedlander (Theorem A.7 of [8]) implies that a map $f : X \to Y$ of n-fold spectra is a stable fibration if and only if f is a strict fibration and the diagram

$$
\begin{array}{ccc}
X & \overset{\gamma}{\longrightarrow} & QX_{Kan} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Qf_{Kan}} \\
Y & \underset{\gamma}{\longrightarrow} & QY_{Kan}
\end{array}
$$

is a homotopy fibre square in the strict category, where γ denotes the composite of canonical maps

$$X \xrightarrow{\eta} X_{Kan} \xrightarrow{\mu} QX_{Kan}.$$

In particular, it's rather useful to know that an n-fold spectrum Z is stably fibrant if and only if Z is strictly fibrant, and the map $\mu : Z \to QZ$ is a strict weak equivalence.   ∎

2.2. Equivalence of homotopy categories

An n-fold spectrum X which is *truncated at level* $(m_1, \ldots, m_n)$ consists of pointed simplicial sets $X^{i_1, \ldots, i_n}$ with $0 \leq i_j \leq m_j$, together with bonding maps

$$\sigma_j : S^1 \wedge X^{i_1 \ldots, i_j, \ldots, i_n} \to X^{i_1, \ldots, i_j+1, \ldots, i_n}$$

for $i_j < m_j$, such that the usual defining relations for an n-fold spectrum hold, where they make sense. Write

$$\mathbf{Spt}^n(m_1, \ldots, m_n)$$

for the category of n-fold spectra truncated at level $(m_1, \ldots, m_n)$. It is convenient to let some of the m_i's be infinite, and so

$$\mathbf{Spt}^n = \mathbf{Spt}^n(\infty, \ldots, \infty).$$

If $m_j \leq k_j$ for $1 \leq j \leq n$, there is a restriction functor

$$R : \mathbf{Spt}^n(k_1, \ldots, k_n) \to \mathbf{Spt}^n(m_1, \ldots, m_n).$$

This functor has a left adjoint

$$L : \mathbf{Spt}^n(m_1, \ldots, m_n) \to \mathbf{Spt}^n(k_1, \ldots, k_n)$$

such that the composite RL is the identity functor: one suspends in each variable inductively. Finally, each n-fold spectrum Z is naturally a colimit of n-fold spectra of the form LRZ, where the restrictions are defined with respect to some (in fact any) cofinal sequence consisting of n-tuple $(m_1, \ldots, m_n) \in \mathbb{N}^{\times n}$ — this is just a fancy way of saying that a map of an n-fold spectra is completely determined by its truncations.

LEMMA 2.28. *Suppose given an n-fold spectrum Z truncated at level*

$$(m_1, \ldots, m_n),$$

together with a map
$$f : S^1 \wedge Z^{m_1, \ldots, m_n} \to Y.$$

Then there is an n-fold spectrum $\tilde{Z}$ truncated at level $(m_1 + 1, m_2, \ldots, m_n)$ such that
$$\begin{cases} \tilde{Z}^{i_1, \ldots, i_n} = Z^{i_1, \ldots, i_n} & \text{if } i_j \leq m_j \\ \tilde{Z}^{m_1+1, m_2, \ldots m_n} = Y \end{cases}$$

and such that the bonding map

$$\sigma_1 : S^1 \wedge \tilde{Z}^{m_1,\ldots,m_n} \to \tilde{Z}^{m_1+1,m_2,\ldots,m_n}$$

coincides with f. Furthermore, the maps $\tilde{Z} \to X$ of n-fold spectra truncated at level $(m_1 + 1, m_2, \ldots, m_n)$ can be identified with the set of all pairs (g, θ), where $g : Z \to RX$ is a map of n-fold spectra truncated at level $(m_1, \ldots, m_n)$ and $\theta : Y \to X^{m_1+1,\ldots,m_n}$ is a map of pointed simplicial sets, such that the following diagram commutes:

$$
\begin{array}{ccc}
S^1 \wedge Z^{m_1,\ldots,m_n} & \xrightarrow{\ f\ } & Y \\
{\scriptstyle S^1 \wedge g}\big\downarrow & & \big\downarrow{\scriptstyle \theta} \\
S^1 \wedge X^{m_1,\ldots,m_n} & \xrightarrow[\ \sigma_1\ ]{} & X^{m_1+1,\ldots,m_n}.
\end{array}
$$

PROOF: The proof will be by induction on n; the case $n = 1$ is a triviality.

Assume that the object $\tilde{Z}^{*,\ldots,*,m_n}$ exists for the truncated $(n-1)$-fold spectrum $Z^{*,\ldots,*,m_n}$ and the map f. The map

$$Z^{*,\ldots,*,m_n-1} \wedge S^1 \xrightarrow{\ \sigma_n\ } Z^{*,\ldots,*,m_n}$$

of $(n-1)$-fold spectra truncated at level $(m_1, \ldots, m_{n-1})$ uniquely determines a map

$$L(Z^{*,\ldots,*,m_n-1}) \wedge S^1 \cong L(Z^{*,\ldots,*,m_n-1} \wedge S^1) \xrightarrow{\ \sigma_n\ } \tilde{Z}^{*,\ldots,*,m_n}$$

of $(n-1)$-fold spectra truncated at $(m_1 + 1, m_2, \ldots, m_{n-1})$. Here, L is the left adjoint of the restriction functor

$$R : \mathbf{Spt}^{n-1}(m_1 + 1, m_2, \ldots, m_{n-1}) \to \mathbf{Spt}^{n-1}(m_1, m_2, \ldots, m_{n-1}).$$

Similarly, the maps
$$Z^{*,\ldots,*,i} \wedge S^1 \xrightarrow{\ \sigma_n\ } Z^{*,\ldots,*,i+1}$$

for $i < m_n - 1$ induce morphisms

$$L(Z^{*,\ldots,*,i}) \wedge S^1 \cong L(Z^{*,\ldots,*,i} \wedge S^1) \xrightarrow{\ \sigma_n\ } L(Z^{*,\ldots,*,i+1}).$$

Thus, the collection of objects

$$L(Z^{*,\ldots,*,0}), \ldots, L(Z^{*,\ldots,*,m_n-1}), \tilde{Z}^{*,\ldots,*,m_n}$$

externally defines an n-fold spectrum truncated at $(m_1 + 1, m_2, \ldots, m_n)$ which has the desired properties. ∎

Corollary 2.29. *The diagonal functor*

$$d : \mathbf{Spt}^n \to \mathbf{Spt}$$

has a left adjoint

$$F : \mathbf{Spt} \to \mathbf{Spt}^n.$$

Furthermore, the canonical map

$$\eta : X \to d(FX)$$

is an isomorphism for all spectra X.

Corollary 2.30. *The diagonal functor d and its left adjoint F induce an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathbf{Spt}^n) \underset{F}{\overset{d}{\rightleftarrows}} \mathrm{Ho}(\mathbf{Spt}).$$

Proof: The functor F preserves stable equivalences, since the map

$$\eta : X \to d(FX)$$

is an isomorphism hence stable equivalence, by Corollary 2.29. But then the triangle identity

implies that the natural map of n-fold spectra

$$\epsilon_Z : Fd(Z) \to Z$$

is a stable equivalence.   ∎

2.3. Simplicial presheaves and presheaves of spectra
Suppose that $\mathcal{C}$ is an arbitrary Grothendieck site, and let

$$\mathbf{S}Pre(\mathcal{C}) \quad (\text{respectively } \mathbf{S}_* Pre(\mathcal{C}))$$

denote the category of simplicial presheaves (respectively pointed simplicial presheaves) on $\mathcal{C}$.

Recall [23] that a *cofibration* $f : X \to Y$ of (pointed) simplicial presheaves is a pointwise monomorphism in the sense that each map $f(U) : X(U) \to Y(U)$ of sections is a monomorphism of simplicial sets for all $U \in \mathcal{C}$.

A map $g : Z \to W$ in $\mathbf{S}Pre(\mathcal{C})$ is said to be a *local weak equivalence* if, for each object $U \in \mathcal{C}$ and each vertex x of $Z(U)$ the induced maps

$$\pi_n(|Z|_U|, x) \xrightarrow{f_*} \pi_n(|W|_U|, fx), \quad n \geq 0, \tag{2.31}$$

of presheaves of homotopy groups on the local site $C \downarrow U$ induce isomorphisms of associated sheaves. Note that, in the presence of enough points on $\mathcal{C}$, g is a weak equivalence if and only if each induced morphism of stalks $g_x : Z_x \to W_x$ is a weak equivalence of simplicial sets; for this reason, I often refer to local weak equivalences of simplicial presheaves in the sense just defined as *stalkwise weak equivalences*. A map g is said to be a *pointwise weak equivalence*, on the other hand, if each map of sections $g(U) : Z(U) \to W(U)$ is a weak equivalence of simplicial sets. One can express the defining condition (2.31) completely in terms of simplicial homotopy groups in the case where Z and W are presheaves of Kan complexes.

A *global fibration* is a map which has the right lifting property with respect to all maps which are cofibrations and local weak equivalences.

There is a natural choice of *function complex* $\mathbf{hom}(X, Y)$ for any simplicial presheaves X and Y. Explicitly, $\mathbf{hom}(X, Y)$ is the simplicial set whose n-simplices consist of all morphisms of simplicial presheaves

$$X \times \Delta^n \to Y.$$

In the case where X and Y are globally pointed, we are entitled to construct the *pointed function complex* $\mathbf{hom}_*(X, Y)$, which is the pointed simplicial set having n-simplices

$$X \ltimes \Delta^n \to Y.$$

For our purposes, the central theorem which relates these ideas is the following (see [23], [24]):

THEOREM 2.32. *With the definitions given above, the category $\mathbf{S}Pre(\mathcal{C})$ is a proper closed simplicial model category.*

A *presheaf of spectra* X on the site $\mathcal{C}$ consists of pointed simplicial presheaves X^n, together with pointed maps $\sigma : S^1 \wedge X^n \to X^{n+1}$, $n \geq 0$. The object X can alternatively be thought of as a spectrum object in the category of pointed simplicial presheaves. The presheaves of spectra on $\mathcal{C}$ arrange themselves into a category, which I denote by $\mathbf{Spt}Pre(\mathcal{C})$.

For each presheaf of spectra X, and as in the previous section, the assignment $X(U) \mapsto X_{Kan}(U)$ determines a functor $X \mapsto X_{Kan}$ from $\mathbf{Spt}Pre(\mathcal{C})$ to itself, and there is a natural map $\eta : X \to X_{Kan}$ which is given by the

natural weak equivalences $X^n(U) \to X^n_{Kan}(U)$ of simplicial sets. It's helpful to recall in particular that $X^n_{Kan}(U) = S|X^n(U)|$. Recall further that the stabilization construction $X(U) \mapsto QX(U)$ is natural, and determines a natural map $\mu : X \to QX$ of presheaves of spectra.

The *sheaf of stable homotopy groups* $\tilde{\pi}^s_j X$ for a presheaf of spectra X can be constructed by taking the sheaf associated to the presheaf

$$U \mapsto \pi^s_j |X(U)|,$$

where $\pi^s_j |X(U)|$ is the jth stable homotopy group of the realized (topological) spectrum $|X(U)|$. Note also that the map $\eta : X \to X_{Kan}$ induces natural isomorphisms of presheaves

$$\pi^s_j X \cong \pi^s_j X_{Kan},$$

and the map $\mu : X_{Kan} \to QX_{Kan}$ induces isomorphisms of the form

$$\pi^s_j X_{Kan} \cong \pi^s_j QX_{Kan} \cong \pi_{j+k} QX^k_{Kan}, \tag{2.33}$$

where the last is a presheaf of ordinary simplicial homotopy groups. It follows that there are natural isomorphisms of sheaves of groups

$$\tilde{\pi}^s_j X \cong \tilde{\pi}_{j+k} QX^k_{Kan},$$

for all integers j and k that make sense. A map $f : X \to Y$ of presheaves of spectra on the site $\mathcal{C}$ is said to be a *local stable equivalence* if the induced maps

$$f_* : \tilde{\pi}^s_j X \to \tilde{\pi}^s_j Y$$

of sheaves of stable homotopy groups are isomorphisms for all integers j. This condition can be rephrased in terms of any of the invariants appearing in (2.33).

Cofibrations in the category of presheaves of spectra can be defined in two ways. Say that a map $i : A \to B$ of $\mathbf{Spt}Pre(\mathcal{C})$ is a *cofibration* if all maps in sections $i : A(U) \to B(U)$, $U \in \mathcal{C}$, are cofibrations in the ordinary spectrum category $\mathbf{Spt}$. Equivalently, i is a cofibration if and only if

(1) the map $i^0 : A^0 \to B^0$ is a cofibration of simplicial presheaves, and
(2) all induced maps

$$S^1 \wedge B^n \cup_{S^1 \wedge A_n} A^{n+1} \to B^{n+1}$$

are cofibrations of simplicial presheaves on $\mathcal{C}$.

It is important to note that the definition of cofibration of presheaves of spectra is independent of topology on the site $\mathcal{C}$, as is the definition of cofibration of simplicial presheaves.

It has become standard to say that a map $p : Z \to W$ of presheaves of spectra is a *global fibration* if it has the right lifting property with respect to all morphisms which are simultaneously cofibrations and local stable equivalences. The governing principle behind these ideas is the following result [24]:

Theorem 2.34. *The category* $\mathbf{Spt}Pre(\mathcal{C})$ *of presheaves of spectra on a Grothendieck site* $\mathcal{C}$, *together with the classes of cofibrations, local stable equivalences and global fibrations, satisfies the axioms for a proper closed simplicial model category.*

The simplicial model structure comes from the smash products $X \wedge K$ of presheaves of spectra X with pointed simplicial sets K and the function complexes $\mathbf{hom}_*(X, Y)$, where the latter have n-simplices given by all maps of the form $X \ltimes \Delta^n \to Y$. The properness assertion follows from long exact sequence arguments, which will be repeated explicitly in a wider context below.

Remark 2.35. The proof of Theorem 2.34, once again, follows the script of Bousfield and Friedlander [8], and in particular depends on proving a series of assertions about the composite map

$$X \xrightarrow{\eta} X_{Kan} \xrightarrow{\mu} QX_{Kan},$$

which I shall denote by $\gamma : X \to QX_{Kan}$. The argument also implies that a map $f : X \to Y$ of presheaves of spectra is a global fibration if and only if all maps $f : X^n \to Y^n$ are global fibrations of simplicial presheaves, and all diagrams

$$
\begin{array}{ccc}
X^n & \xrightarrow{\gamma} & QX^n_{Kan} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Qf_{Kan}} \\
Y^n & \xrightarrow{\gamma} & QY^n_{Kan}
\end{array}
$$

are homotopy fibre squares in the simplicial presheaf category. In particular, a presheaf of spectra Z is globally fibrant if and only if each simplicial presheaf Z^n is globally fibrant, and each map $\nu : Z^n \to QZ^n$ is a local weak equivalence of simplicial presheaves. Observe that I am not claiming that the stable object QZ itself is globally fibrant. ∎

A *globally fibrant model* for a presheaf of spectra X is a local stable equivalence $\theta : X \to GX$, where GX is globally fibrant. Globally fibrant models always exist, by the factorization axioms, and any two globally fibrant models for X are non-canonically locally stably equivalent. There is an analogous definition of globally fibrant models for simplicial presheaves. In many cases of computational interest, globally fibrant models arise from Godement resolutions (see [23, p.76]). It's often useful to think of a globally fibrant model as a non-abelian version of an injective resolution.

Remark 2.36. The definitions of local weak equivalence of simplicial presheaves and local stable equivalence of presheaves of spectra both depend very much on the topology on the underlying site $\mathcal{C}$. The reader has already been

warned that local (stable) weak equivalences and stalkwise (stable) weak equivalences will mean the same thing in cases where the category of sheaves on the site $\mathcal{C}$ has enough points.

There is also the case where the category $\mathcal{C}$ effectively has no topology at all. The *chaotic topology* on the category $\mathcal{C}$ is the site whose covering sieves $R \subset \hom(\ ,U)$ contain the indentity 1_U, for all objects U of $\mathcal{C}$. There is no difference between sheaves and presheaves on $\mathcal{C}$ for this topology, and so in this case a map $f : X \to Y$ is a local weak equivalence of simplicial presheaves if and only if all induced maps in sections $f : X(U) \to Y(U)$, $U \in \mathcal{C}$, are weak equivalences of simplicial sets. Similarly, a map $g : Z \to W$ of presheaves of spectra on the site $\mathcal{C}$ with the chaotic topology is a local stable equivalence if and only if all maps $g : Z(U) \to W(U)$ are stable equivalences of spectra. I shall say that the local (stable) weak equivalences for the chaotic topology are *pointwise* (stable) weak equivalences. The associated closed model structures for $\mathcal{C}$-diagrams of simplicial sets and $\mathcal{C}$-diagrams of spectra are quite useful, and repeated applications will appear below. ∎

All of the results given both here and in the following section have analogues for (small) diagrams of presheaves of spectra or n-fold spectra. This arises from the fact that the category of J-diagrams within a given Grothendieck topos has the structure of a Grothendieck topos. An explicit construction of the topos of such J-diagrams is given as follows.

Suppose that $\mathcal{C}$ is a Grothendieck site and that J is a category, and let the topology (ie. collection of covering sieves) for the site $\mathcal{C}$ be denoted by $\mathcal{T}$.

The product category $\mathcal{C} \times J$ has a Grothendieck topology, whose covering sieves are those subfunctors of the representable functors $\hom(\ ,(x,i))$ which contain a collection of morphisms $R \times \{1_i\}$, where $R \subset \hom_{\mathcal{C}}(\ ,x)$ is a member of $\mathcal{T}$.

Suppose that $in_i : \mathcal{C} \to \mathcal{C} \times J$ is the functor defined by $x \mapsto (x,i)$, and write $F(i)$ for the composite $F \circ in_J$, where F is any presheaf on $\mathcal{C} \times J$. The covering sieves $R \times \hom_J(\ ,i)$, $R \in \mathcal{T}$, are cofinal among all covering sieves for a fixed object $(x,i) \in \mathcal{C} \times J$, and it follows that F is a sheaf on $\mathcal{C} \times J$ if and only if $F(i)$ is a sheaf on $\mathcal{C}$ for each $i \in J$. Thus, if one identifies F with a contravariant functor $F : J \to PreShv(\mathcal{C})$, then F represents a sheaf on $\mathcal{C} \times J$ if and only if each $F(i)$ is a sheaf on $\mathcal{C}$, so that the sheaf category $Shv(\mathcal{C} \times J)$ can be identified up to isomorphism with the category $Shv(\mathcal{C})^{J^{op}}$ of contravariant functors on J which take values in $Shv(\mathcal{C})$. In particular, the associated sheaf for a functor $F : J \to PreShv(\mathcal{C})$ can be constructed by applying the associated sheaf functor for presheaves on $\mathcal{C}$ to get a natural transformation $\eta : F \to \tilde{F}$ for contravariant functors on J, as can be seen directly from the universal property.

Recall that a map $f : X \to Y$ of simplicial presheaves on a site $\mathcal{C}$ is a (local) weak equivalence if and only if

(1) the presheaf map $f_* : \pi_0 X \to \pi_0 Y$ induces an isomorphism of associated sheaves, and

(2) for each $U \in \mathcal{C}$ and choice of base point $x \in X_0(U)$ the presheaf map

$$f_* : \pi_i(X|_U, x) \to \pi_i(Y|_U, f(x))$$

induces an isomorphism of associated sheaves on the associated comma category site $\mathcal{C} \downarrow U$.

The topology on the comma category site

$$(\mathcal{C} \times J) \downarrow (U, i) \cong (\mathcal{C} \downarrow U) \times (J \downarrow i)$$

is induced from the topology on $\mathcal{C} \downarrow U$ in the same way that the topology on $\mathcal{C} \times J$ is induced from the topology on $\mathcal{C}$. It follows that a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C} \times J$ is a local weak equivalence if and only if each of the associated maps $f : X(i) \to Y(i)$ is a weak equivalence of simplicial presheaves on the site $\mathcal{C}$.

These observations, together with the closed model structure for simplicial presheaves on an arbitrary Grothendieck site [23] now imply

PROPOSITION 2.37. *Suppose that $\mathcal{C}$ is an arbitrary Grothendieck site and that J is some index category. There is a closed simplicial model structure on the category $\mathbf{S}Pre(\mathcal{C})^J$ of J-diagrams in the category of simplicial presheaves on $\mathcal{C}$, such that*

(1) *a map $f : X \to Y$ is a weak equivalence if and only if each level map $f : X(j) \to Y(j)$, $j \in J$, is a local weak equivalence of simplicial presheaves on the site $\mathcal{C}$, and*

(2) *a map $g : U \to V$ is a cofibration if and only if each level map $g : U(j) \to V(j)$, $j \in J$ is a cofibration (ie. pointwise inclusion) of simplicial presheaves on the site $\mathcal{C}$.*

The fibrations in the category of J-diagrams in $\mathbf{S}Pre(\mathcal{C})$ are defined by a right lifting property, and are called global fibrations, as usual. One of the more important features of such objects for us in what follows is

LEMMA 2.38. *Suppose that $f : X \to Y$ is a global fibration in the category of J-diagrams of simplicial presheaves on the site $\mathcal{C}$. Then each map $f : X(j) \to Y(j)$, $j \in J$, is a global fibration of simplicial presheaves on $\mathcal{C}$.*

PROOF: The functor $\mathbf{S}Pre(\mathcal{C})^J \to \mathbf{S}Pre(\mathcal{C})$ which is defined by $X \mapsto X(j)$ has a left adjoint L_j which is defined, for $Z \in \mathbf{S}Pre(\mathcal{C})$, by

$$L_j Z(i) = \bigsqcup_{j \to i} Z.$$

The functor $L_j Z$ takes trivial cofibrations to trivial cofibrations. ∎

The following Corollary is a sample application of these ideas. This result has analogues in all of the categories of presheaves considered in this section.

Corollary 2.39. *Suppose that Z is a J-diagram of simplicial presheaves on the site $\mathcal{C}$. Then there is a map $\phi : Z \to GZ$ of J-diagrams of simplicial presheaves on $\mathcal{C}$ such that each map $\phi : Z(j) \to GZ(j)$, $j \in J$ is a local weak equivalence and such that each simplicial presheaf $GZ(j)$ is globally fibrant.*

Proof: Let the map $\phi : Z \to GZ$ be a globally fibrant model within the category of simplicial presheaves on the site $\mathcal{C} \times J$. The result follows from Lemma 2.38. ∎

Corollary 2.39 solves a homotopy coherence problem, in that it's not clear a priori that a collection of globally fibrant models $Z(j) \to GZ(j)$, randomly chosen for each simplicial presheaf $Z(j)$ fits together to give a J-diagram GZ, except up to homotopy.

2.4. Presheaves of n-fold spectra

A *presheaf of n-fold spectra* on the Grothendieck site $\mathcal{C}$ is a (contravariant) functor $X : \mathcal{C}^{op} \to \mathbf{Spt}^n$ which takes values in the category of n-fold spectra. In other words X consists of pointed simplicial presheaves

$$X^{i_1,\dots,i_n}, \quad i_j \geq 0,$$

together with pointed maps

$$\sigma_j : S^1 \wedge X^{i_1,\dots,i_j,\dots,i_n} \to X^{i_1,\dots,i_j+1,\dots,i_n}$$

of simplicial presheaves, which fit into commutative diagrams

$$
\begin{array}{ccc}
S^1 \wedge X^{i_1,\dots,i_j,\dots,i_k+1,\dots,i_n} & \xrightarrow{\ \sigma_j\ } & X^{i_1,\dots,i_j+1,\dots,i_k+1,\dots,i_n} \\
\big\uparrow{\scriptstyle S^1 \wedge \sigma_k} & & \big\uparrow{\scriptstyle \sigma_k} \\
S^1 \wedge S^1 \wedge X^{i_1,\dots,i_j,\dots,i_k,\dots i_n} & & \\
\big\uparrow{\scriptstyle \tau \wedge 1}{\scriptstyle \cong} & & \\
S^1 \wedge S^1 \wedge X^{i_1,\dots,i_j,\dots,i_k,\dots i_n} & \xrightarrow{\ S^1 \wedge \sigma_j\ } & S^1 \wedge X^{i_1,\dots,i_j+1,\dots,i_k,\dots,i_n}
\end{array}
$$

for all relevant $j < k$, where $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is the map which switches factors. Write $\mathbf{Spt}^n Pre(\mathcal{C})$ for the category of presheaves of n-fold spectra on the site $\mathcal{C}$. The notation is meant to suggest that we are dealing with n-fold spectrum objects in the category of simplicial presheaves on the site $\mathcal{C}$.

The axiomatic homotopy theory for presheaves of spectra was summarized in the previous section. A corresponding theory for presheaves of n-fold spectra is given here; the method is to promote the results established for n-fold spectra in the first two sections of this chapter to the presheaf level.

A map $f : X \to Y$ of presheaves of n-fold spectra is said to be a *strict weak equivalence* if each of the maps

$$f : X^{i_1,\dots,i_n} \to Y^{i_1,\dots,i_n} \tag{2.40}$$

is a local weak equivalence of simplicial presheaves on the site $\mathcal{C}$. The map f is said to be a *strict fibration* if each of the maps (2.40) is a global fibration of simplicial presheaves. The *cofibrations* of $\mathbf{Spt}^n Pre(\mathcal{C})$ are those maps which have the left lifting property with respect to all maps which are strict fibrations and strict weak equivalences.

Note, first of all, that Lemma 2.3 implies

LEMMA 2.41. *There is a functor $X \mapsto X_{Kan}$ from the category $\mathbf{Spt}^n Pre(\mathcal{C})$ to itself, such that each pointed simplicial set of sections*

$$X_{Kan}^{i_1,\dots,i_n}(U)$$

is a Kan complex. The natural weak equivalences of simplicial sets

$$\eta : X^{i_1,\dots,i_n}(U) \to X_{Kan}^{i_1,\dots,i_n}(U)$$

induce a natural strict weak equivalence of presheaves of n-fold spectra

$$\eta : X \to X_{Kan}.$$

For each pointed simplicial set K and each presheaf of n-fold spectra X, there are presheaves of n-fold spectra $X \wedge K$ and $\mathbf{hom}_*(K, X)$, which are given, respectively, at each multi-index by

$$X^{i_1,\dots,i_n} \wedge K \qquad \text{and} \qquad \mathbf{hom}_*(K, X^{i_1,\dots,i_n}).$$

Furthermore, there is an exponential law

$$\hom(X \wedge K, Y) \cong \hom(X, \mathbf{hom}_*(K, Y))$$

for morphisms of presheaves of n-fold spectra. In particular, a presheaf of $(n+1)$-fold spectra Z can be regarded (in $n + 1$ different ways) as a spectrum object in the category of presheaves of n-fold spectra, meaning that Z consists of presheaves of n-fold spectra Z^j, $j \geq 0$, together with bonding maps $\sigma : Z^j \wedge S^1 \to Z^{j+1}$ in $\mathbf{Spt}^n Pre(\mathcal{C})$.

Suppose that X and Y are presheaves of n-fold spectra. The *function complex* $\mathbf{hom}_*(X, Y)$ is the pointed simplicial set whose m-simplices are the maps $X \ltimes \Delta^m \to Y$ in $\mathbf{Spt}^n Pre(\mathcal{C})$, where

$$X \ltimes \Delta^n = X \wedge (\Delta^n \sqcup \{*\}).$$

There are exponential laws

$$\mathbf{hom}_*(X \wedge K, Y) \cong \mathbf{hom}_*(K, \mathbf{hom}_*(X, Y)) \cong \mathbf{hom}_*(X, \mathbf{hom}_*(K, Y)),$$

which are valid for any X and Y in $\mathbf{Spt}^n Pre(\mathcal{C})$ and any pointed simplicial set K.

PROPOSITION 2.42. *The category* $\mathbf{Spt}^n Pre(\mathcal{C})$, *together with the classes of cofibrations, strict fibrations, and strict weak equivalences, satisfies the axioms for a proper closed simplicial model category.*

PROOF: Subject to invoking Theorem 2.32 in place of the analogous result for simplicial sets, the proof of this result is the same as that for the corresponding results for n-fold spectra, namely Proposition 2.4 and Proposition 2.6. ∎

Say that a map $g : Z \to W$ of presheaves of n-fold spectra is a *local stable equivalence* if the induced map

$$d(g) : d(Z) \to d(W)$$

is a local stable equivalence of presheaves of spectra. Lemma 2.14 may be used to formulate this condition in terms of sheaves of homotopy groups. Every strict weak equivalence is a local stable equivalence.

Note in particular that if the natural map

$$\gamma : X \to QX_{Kan}$$

is defined to be the composite

$$X \xrightarrow{\eta} X_{Kan} \xrightarrow{\mu} QX_{Kan},$$

then the proof of Theorem 2.26 implies that the maps

$$Q\gamma : QX_{Kan} \to Q(QX_{Kan})_{Kan}$$

and

$$\gamma : QX_{Kan} \to Q(QX_{Kan})_{Kan}$$

are pointwise strict weak equivalences. In summary, we have

LEMMA 2.43. *The following hold for maps of presheaves of n-fold spectra on a Grothendieck site $\mathcal{C}$:*

(1) *Every strict weak equivalence is a local stable equivalence.*
(2) *The map* $\gamma : QX_{Kan} \to Q(QX_{Kan})_{Kan}$ *is a strict weak equivalence.*
(3) *The map* $Q\gamma : QX_{Kan} \to Q(QX_{Kan})_{Kan}$ *is a strict weak equivalence.*

It follows from Corollary 2.16 that g is a local stable equivalence if and only if the induced map

$$Qg_{Kan} : QZ_{Kan} \to QW_{Kan}$$

is a strict weak equivalence.

A map $f : A \to X$ of presheaves of n-fold spectra is said to be a *pointwise cofibration* if each of the maps

$$f : A^{i_1,\ldots,i_n} \to X^{i_1,\ldots,i_n}$$

is a cofibration of simplicial presheaves. One can show, by induction on n, that every cofibration of presheaves of n-fold spectra is a pointwise cofibration, just as was done for n-fold spectra. Thus, Lemma 2.17 implies that any cofibration of n-fold spectra $i : A \to X$ gives rise to an exact sequence of presheaves of abelian groups

$$\pi_k^s dX \xrightarrow{p} \pi_k^s d(X/A) \to \pi_{k-1}^s d(A) \xrightarrow{i} \pi_{k-1}^s dX \xrightarrow{p} \pi_{k-1}^s d(X/A),$$

and hence to an exact sequence of sheaves of abelian groups

$$\tilde{\pi}_k^s dX \xrightarrow{p} \tilde{\pi}_k^s d(X/A) \to \tilde{\pi}_{k-1}^s d(A) \xrightarrow{i} \tilde{\pi}_{k-1}^s dX \xrightarrow{p} \tilde{\pi}_{k-1}^s d(X/A),$$

where $p : X \to X/A$ is the canonical map to the cofibre. This gives

LEMMA 2.44. *Local stable equivalences in* $\mathbf{Spt}^n Pre(\mathcal{C})$ *are closed under cobase change by cofibrations. In other words, given a pushout diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\ g\ } & Z \\
{\scriptstyle i}\big\downarrow & & \big\downarrow \\
X & \xrightarrow[\ g_*\ ]{} & X \cup_A Z,
\end{array}
$$

where i is a cofibration, if g is a local stable equivalence, then so is g_*.

Global fibrations of simplicial presheaves are also pointwise fibrations. This fact has been alluded to in print before, usually with an injunction to see the proof of Lemma 3.4 in [23]: the essential point is that, if U is an object of the site $\mathcal{C}$, then the U-sections functor $Y \mapsto Y(U)$ has a left adjoint $X \mapsto X_U$, where

$$X_U(V) = \bigsqcup_{\phi : V \to U} X$$

defines the simplicial presheaf X_U associated to the simplicial set X (see p.68 of [23]). But then $X \mapsto X_U$ takes trivial cofibrations of simplicial sets to pointwise trivial cofibrations of simplicial presheaves, so the usual lifting property argument carries the day.

It follows that every strict fibration $p : X \to Y$ of presheaves of n-fold spectra consists (in part) of maps of n-fold spectra $p(U) : X(U) \to Y(U)$

which are strict fibrations, in all sections. It follows from Lemma 2.18 that, if $i : F \to X$ is the inclusion of the fibre of p, then there is an exact sequence of presheaves of stable homotopy groups

$$\pi_k^s dF \xrightarrow{i} \pi_k^s dX \xrightarrow{p} \pi_k^s dY \to \pi_{k-1}^s dF \xrightarrow{i} \pi_{k-1}^s dX,$$

and hence an exact sequence of sheaves of abelian groups

$$\tilde{\pi}_k^s dF \xrightarrow{i} \tilde{\pi}_k^s dX \xrightarrow{p} \tilde{\pi}_k^s dY \to \tilde{\pi}_{k-1}^s dF \xrightarrow{i} \tilde{\pi}_{k-1}^s dX.$$

This implies the following result:

LEMMA 2.45. *Local stable equivalences in $\mathbf{Spt}^n Pre(\mathcal{C})$ are closed under base change by strict fibrations.*

There are non-abelian analogues of the standard global ext constructions in the categories of pointed simplicial presheaves and presheaves of spectra. In particular, suppose that K and V are pointed simplicial presheaves. Then there is a pointed simplicial presheaf

$$\mathbf{Hom}_*(K, V),$$

which is defined for each $U \in \mathcal{C}$ by the equation

$$\mathbf{Hom}_*(K, V)(U) = \mathbf{hom}_*(K|_U, V|_U).$$

The simplicial presheaf $K|_U$, for example, is the restriction of K to the site $\mathcal{C} \downarrow U$ along the forgetful functor $Q : \mathcal{C} \downarrow U \to \mathcal{C}$. Furthermore, it's not hard to show that there are natural isomorphisms (ie. exponential laws)

$$\mathrm{hom}_*(X, \mathbf{Hom}_*(K, V)) \cong \mathrm{hom}_*(X \wedge K, V)$$

relating morphisms of pointed simplicial presheaves, and their function space counterparts

$$\mathbf{hom}_*(X, \mathbf{Hom}_*(K, V)) \cong \mathbf{hom}_*(X \wedge K, V).$$

This construction extends to give a presheaf of spectra $\mathbf{Hom}_*(K, Z)$ for any pointed simplicial presheaf K and presheaf of spectra Z. If X happens to be a presheaf of spectra, there is a corresponding exponential law

$$\mathbf{hom}_*(X, \mathbf{Hom}_*(K, Z)) \cong \mathbf{hom}_*(X \wedge K, Z).$$

LEMMA 2.46. *Suppose that K is a pointed simplicial presheaf on $\mathcal{C}$, and that $f : X \to Y$ is a local stable equivalence of presheaves of spectra on $\mathcal{C}$. Then the induced map*

$$f \wedge K : X \wedge K \to Y \wedge K$$

is a local stable equivalence of presheaves of spectra.

PROOF: It's a little bit fussy in cases where $Shv(\mathcal{C})$ does not have enough points, but one can show that the functor $U \mapsto U \wedge K$ preserves local weak equivalences of pointed simplicial presheaves. It follows that the functor

$$Z \mapsto \mathbf{Hom}_*(K, Z)$$

preserves global fibrations of pointed simplicial presheaves, and that the functor $X \mapsto X \wedge K$ preserves strict weak equivalences of presheaves of spectra.

Now suppose that the presheaf of spectra Z is globally fibrant. Then Z is strictly fibrant, and so $\mathbf{Hom}_*(K, Z)$ is also strictly fibrant. The canonical map $\mu : Z \to QZ$ is a strict weak equivalence of strictly fibrant presheaves of spectra, so that the map

$$\mathbf{Hom}_*(K, \mu) : \mathbf{Hom}_*(K, Z) \to \mathbf{Hom}_*(K, QZ)$$

is a strict weak equivalence. In effect, since Z^n and QZ^n are both globally fibrant, the standard construction that replaces a map between fibrant objects by a fibration, carried out within the closed model structure on the category of simplicial presheaves, guarantees that each simplicial presheaf map $\mu : Z^n \to QZ^n$ has a factorization

$$
\begin{array}{ccc}
Z^n & \xrightarrow{\ \ j\ \ } & W \\
& {\scriptstyle \mu} \searrow & \downarrow {\scriptstyle \pi} \\
& & QZ^n,
\end{array}
$$

where π is a global fibration and a local weak equivalence, and j is right inverse to a map $p : W \to Z^n$ which is a global fibration and a local weak equivalence. The maps π and p induce maps $\pi_* : \mathbf{Hom}_*(K, W) \to \mathbf{Hom}_*(K, QZ^n)$ and $p_* : \mathbf{Hom}_*(K, W) \to \mathbf{Hom}_*(K, Z^n)$ which have the right lifting property with respect to all cofibrations, and are therefore local weak equivalences. There is a commutative diagram

$$
\begin{array}{ccc}
\mathbf{Hom}_*(K, Z) & \xrightarrow{\ \mathbf{Hom}_*(K, \mu)\ } & \mathbf{Hom}_*(K, QZ) \\
& {\scriptstyle \mu} \searrow & \downarrow {\scriptstyle \cong}\, {\scriptstyle \tau} \\
& & Q\mathbf{Hom}_*(K, Z),
\end{array}
$$

so that the map

$$\mu : \mathbf{Hom}_*(K, Z) \to Q\mathbf{Hom}_*(K, Z)$$

is a strict weak equivalence. Thus, the presheaf of spectra $\mathbf{Hom}_*(K, Z)$ is globally fibrant under the assumption that Z is globally fibrant.

The rest of the argument follows the outline of proof given for Lemma 2.20. It is enough to assume that f is a cofibration and a local stable equivalence between cofibrant spectra, and show that the induced map

$$f^* : \mathbf{hom}_*(Y \wedge K, Z) \to \mathbf{hom}_*(X \wedge K, Z)$$

is a weak equivalence of simplicial sets for all globally fibrant presheaf of spectra Z. But f^* is canonically isomorphic to the map

$$\mathbf{hom}_*(Y, \mathbf{Hom}_*(K, Z)) \to \mathbf{hom}_*(X, \mathbf{Hom}_*(K, Z))$$

which is induced by f. The presheaf of spectra $\mathbf{Hom}_*(K, Z)$ is stably fibrant, so that the Bousfield-Friedlander criterion applies. ∎

Say that a map $f : X \to Y$ of presheaves of spectra which is both a local stable equivalence and a cofibration is a *locally trivial cofibration*, and that f is a *trivial global fibration* if it is both a local stable equivalence and a global fibration. Corresponding notions are defined for morphisms of presheaves of n-fold spectra. For any pointed simplicial presheaf K, the functor $X \mapsto X \wedge K$ preserves cofibrations of presheaves of spectra. This observation and Lemma 2.46 together imply

COROLLARY 2.47.

   (1) *The functor $X \mapsto X \wedge K$ preserves locally trivial cofibrations of presheaves of spectra.*

   (2) *The functor $X \mapsto \mathbf{Hom}_*(K, X)$ preserves global fibrations and trivial global fibrations of presheaves of spectra.*

COROLLARY 2.48. *The functor $X \mapsto X \wedge K$ preserves local stable equivalences of presheaves of n-fold spectra.*

LEMMA 2.49. *The functor $X \mapsto X \wedge K$ preserves cofibrations of n-fold spectra.*

PROOF: Use the corresponding fact for pointed simplicial presheaves (which appears in the course of proving Lemma 2.46), to give an inductive argument analogous to the proof of Lemma 2.24. ∎

COROLLARY 2.50. *The functor $X \mapsto X \wedge K$ preserves locally trivial cofibrations of presheaves of n-fold spectra.*

THEOREM 2.51. *The category $\mathbf{Spt}^n Pre(\mathcal{C})$, and the classes of cofibrations, local stable equivalences and global fibrations, together satisfy the axioms for a proper closed simplicial model category.*

Proof: The reader would have noticed that an analogy with the proof of Theorem 2.26 has already been set up. One verifies the closed model axioms by showing that the Bousfield-Friedlander axioms are verified, as they are done in Lemmas 2.43, 2.44 and 2.45. The simplicial model structure is a consequence of Theorem 2.32 and Corollary 2.50; the proof is the same as that for the corresponding result about n-fold spectra. ∎

The claims appearing in Remark 2.35 apply verbatim in the context of presheaves of n-fold spectra. Also, the constructions of the second section of this chapter apply without change to the presheaf case, giving:

Proposition 2.52. *The diagonal functor d and its left adjoint F induce an equivalence of homotopy categories*

$$\text{Ho}(\mathbf{Spt}^n Pre(\mathcal{C})) \xrightleftharpoons[F]{d} \text{Ho}(\mathbf{Spt} Pre(\mathcal{C})).$$

2.5. Smash products of presheaves of spectra

It's rather interesting at this point to compare the theory that has been described here with Adams' presentation of smash products of spectra [2, pp.158-190].

We can show that all of the diagonal functors $X \mapsto X_\psi$ from n-fold spectra to spectra which are associated to admissible paths $\psi : \mathbb{N} \to \mathbb{N}^{\times n}$ are naturally stable homotopy equivalent (Theorem 1.32). We also know that there is a closed model structure on the category of n-fold spectra such that any one of these diagonal functors induces an equivalence of the ordinary stable category with the homotopy category of n-fold spectra, and this for all n (Theorem 2.26, Corollary 2.30). These results go far beyond Adams' constructions, but a way has not yet been found to use them to demonstrate that smash product of spectra and the sphere spectrum together give the stable homotopy category the structure of a symmetric monoidal category. The central problem is that the natural equivalences of Theorem 1.32 are not known to be homotopy coherent in any sense, so there is no effective way of composing them, particularly in cases where one wants to analyze symmetries.

I shall not reproduce Adams' work here. In the case of bispectra, his methods can be used to show that each bispectrum X has an associated double telescope construction, which is a spectrum that comes equipped with canonical equivalences to each of the telescopes arising from admissible paths $\psi : \mathbb{N} \to \mathbb{N}^2$. The construction of the double telescope depends, fundamentally, on first extending a particular S^2-bundle over the boundary $\partial(I \times I)$ of the box $I \times I$ to the full box (for which one uses the calculation $\pi_1 BSO(2) = 0$), and then glueing the result to a space made up of telescopes of bonding maps. The construction is therefore natural in bispectra X, and induces a natural canonical stable equivalence $X_\psi \simeq X_\gamma$ for any pair of admissible paths $\psi, \gamma : \mathbb{N} \to \mathbb{N}^{\times 2}$.

Adams uses the same general trick of glueing an extended sphere bundle to some space made up of telescopes arising from certain n-fold spectra (for low values of n) to prove all of the technical results leading to the assertion that the stable category has a symmetric monoidal category. These constructions are therefore natural in the types of n-fold spectra that arise. In particular, we can infer the following:

THEOREM 2.53. *The smash product construction* $(X, Y) \mapsto d(X \wedge Y)$ *and the (constant) sphere spectrum* S *together give the stable homotopy category* $\mathrm{Ho}(\mathbf{Spt}Pre(\mathcal{C}))$ *of presheaves of spectra on a Grothendieck site* $\mathcal{C}$ *the structure of a symmetric monoidal category.*

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