2

The rotation group

In this Chapter we give a short account of the main properties of the three-dimensional rotation group $SO(3)$ and of its universal covering group $SU(2)$. The group $SO(3)$ is an important subgroup of the Lorentz group, which will be considered in the next Chapter, and we think it is useful to give a separate and preliminary presentation of its properties. After a general discussion of the general characteristics of $SO(3)$ and $SU(2)$, we shall consider the corresponding Lie algebra and the irreducible representations of these groups. All the group concepts used in the following can be found in the previous Chapter.

2.1 Basic properties

The three-dimensional rotations are defined as the linear transformations of the vector $\mathbf{x} = (x_1, x_2, x_3)$

$$x'_i = \sum_j R_{ij} x_j \, , \tag{2.1}$$

which leave the square of $\mathbf{x}$ invariant:

$$x'^2 = x^2 \, . \tag{2.2}$$

Explicitly, the above condition gives

$$\sum_i x'^2_i = \sum_{ijk} R_{ij} R_{ik} x_j x_k = \sum_j x_j^2 \, , \tag{2.3}$$

which implies

$$R_{ij} R_{ik} = \delta_{jk} \, . \tag{2.4}$$

In matrix notation Eqs. (2.1) and (2.4) can be written as

$$\mathbf{x}' = R\mathbf{x} \tag{2.5}$$

G. Costa and G. Fogli, *Symmetries and Group Theory in Particle Physics*,
Lecture Notes in Physics 823, DOI: 10.1007/978-3-642-15482-9_2,

and

$$\tilde{R}R = I \,, \tag{2.6}$$

where $\tilde{R}$ is the transpose of R. Eq. (2.6) defines the *orthogonal* group $O(3)$; the matrices R are called *orthogonal* and they satisfy the condition

$$\det R = \pm 1 \,. \tag{2.7}$$

The condition

$$\det R = +1 \tag{2.8}$$

defines the *special orthogonal group* or *rotation group* $SO(3)$[1]. The corresponding transformations do not include space inversions, and can be identified with pure rotations.

A real matrix R satisfying Eqs. (2.6), (2.8) is characterized by 3 independent parameters, i.e. the dimension of the group is 3. One can choose different sets of parameters: a common parametrization, which will be considered explicitly in Section 2.4, is in terms of the three Euler angles. Another useful parametrization consists in associating to each matrix R a point of a sphere of radius π in the Euclidean space $\mathcal{R}^3$ (Fig. 2.1). For each point P inside the sphere there is a corresponding unique rotation: the direction of the vector OP individuates the axis of rotation and the lenght of OP fixes the angle ϕ $(0 \leq \phi \leq \pi)$ of the rotation around the axis in counterclockwise sense. However, if $\phi = \pi$, the same rotation corresponds to the antipode P' on the surface of the sphere. We shall come back to this point later. Writing $OP = \phi\mathbf{n}$ where

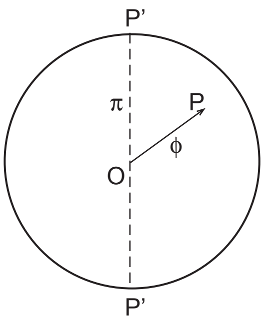

Fig. 2.1. *Parameter domain of the rotation group.*

$\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector, each matrix R can be written explicitly in terms of the parameters ϕ and n_1, n_2, n_3 (only two of the n_i are independent, since $\sum n_i^2 = 1$). With the definitions $c_\phi = \cos\phi$, $s_\phi = \sin\phi$, one can write explicitly:

[1] For a detailed analysis see E.P. Wigner, *Group Theory and its applications to the quantum mechanics of atomic spectra*, Academic Press (1959).

$$R = \begin{pmatrix} n_1^2(1-c_\phi)+c_\phi & n_1n_2(1-c_\phi)-n_3s_\phi & n_1n_3(1-c_\phi)+n_2s_\phi \\ n_1n_2(1-c_\phi)+n_3s_\phi & n_2^2(1-c_\phi)+c_\phi & n_2n_3(1-c_\phi)-n_1s_\phi \\ n_1n_3(1-c_\phi)-n_2s_\phi & n_2n_3(1-c_\phi)+n_1s_\phi & n_3^2(1-c_\phi)+c_\phi \end{pmatrix} . \tag{2.9}$$

From Eq. (2.9), one can prove that the product of two elements and the inverse element correspond to analytic functions of the parameters, i.e. the rotation group is a Lie group.

If one keeps only the orthogonality condition (2.6) and disregard (2.8), one gets the larger group $O(3)$, which contains elements with both signs, $\det R = \pm 1$. The groups consists of two disjoint sets, corresponding to $\det R = +1$ and $\det R = -1$. The first set coincides with the group $SO(3)$, which is an *invariant subgroup* of $O(3)$: in fact, if R belongs to $SO(3)$ and R' to $O(3)$, one gets

$$\det(R'RR'^{-1}) = +1 \, . \tag{2.10}$$

The group $O(3)$ is then neither simple nor semi-simple, while one can prove that $SO(3)$ is simple.

The elements with $\det R = -1$ correspond to *improper* rotations, i.e. rotations times *space inversion* I_s, where

$$I_s\mathbf{x} = -\mathbf{x} \qquad \text{i.e.} \qquad I_s = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} . \tag{2.11}$$

The element I_s and the identity I form a group $\mathcal{J}$ which is abelian and isomorphic to the permutation group $\mathcal{S}_2$. It is an invariant subgroup of $O(3)$. Each element of $O(3)$ can be written in a unique way as the product of a proper rotation times an element of $\mathcal{J}$, so that $O(3)$ is the *direct product*

$$O(3) = SO(3) \otimes \mathcal{J} \, . \tag{2.12}$$

It is important to remark that the group $SO(3)$ is *compact*; in fact its parameter domain is a sphere in the euclidean space $\mathcal{R}^3$, i.e. a compact domain. From Eq. (2.12) it follows that also the group $O(3)$ is compact, since both the disjoint sets are compact.

The rotation group $SO(3)$ is *connected*: in fact, any two points of the parameter domain can be connected by a continuous path. However, not all closed paths can be shrunk to a point. In Fig. 2.2 three closed paths are shown. Since the antipodes correspond to the same point, the path in case b) cannot be contracted to a point; instead, for case c), by moving P' on the surface, we can contract the path to a single point P. Case c) is then equivalent to case a) in which the path can be deformed to a point. We see that there are only two classes of closed paths which are distinct, so that we can say that the group $SO(3)$ is *doubly connected*. The group $O(3)$ is *not connected*, since it is the union of two disjoint sets.

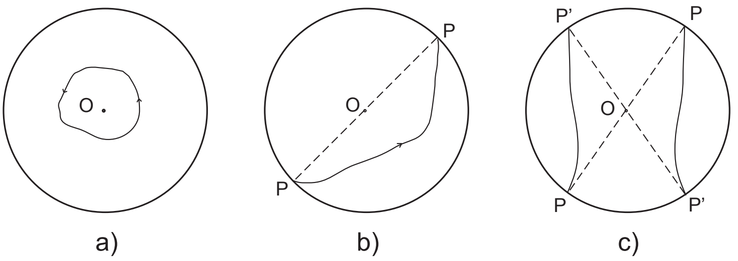

Fig. 2.2. *Different paths for $SO(3)$.*

Since the group $SO(3)$ is not simply connected, it is important to consider its universal covering group, which is the *special unitary group* $SU(2)$ of order $r = 3$. The elements of the group $SU(2)$ are the complex 2×2 matrices u satisfying

$$uu^\dagger = u^\dagger u = I\,, \tag{2.13}$$

$$\det u = 1\ , \tag{2.14}$$

where $u^\dagger$ is the adjoint (conjugate transpose) of u. They can be written, in general, as

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{2.15}$$

where a and b are complex parameters restricted by the condition

$$|a|^2 + |b|^2 = 1\,. \tag{2.16}$$

Each matrix u is then specified by 3 real parameters. Defining

$$a = a_0 + ia_1\ , \tag{2.17}$$

$$b = a_2 + ia_3\ , \tag{2.18}$$

Eq. (2.16) becomes

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1\ . \tag{2.19}$$

The correspondence between the matrices u and the matrices R of $SO(3)$ can be found replacing the orthogonal transformation (2.5) by

$$h' = uhu^\dagger\ , \tag{2.20}$$

where

$$h = \boldsymbol{\sigma} \cdot \mathbf{x} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}\ . \tag{2.21}$$

and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{2.22}$$

which satisfy the relation

$$\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij} . \tag{2.23}$$

Making use of eqs. (2.1), (2.20), one can express the elements of the matrix R in terms of those of the matrix u in the form:

$$R_{ij} = \tfrac{1}{2}\mathrm{Tr}(\sigma_i u \, \sigma_j u^\dagger) . \tag{2.24}$$

Since this relation remains unchanged replacing u by $-u$, we see that for each matrix R of $SO(3)$ there are two corresponding matrices u and $-u$ of $SU(2)$.

The group $SU(2)$ is *compact* and *simply connected*. In fact, if we take the real parameters a_0, a_1, a_2, a_3 to characterize the group elements, we see that the parameter space, defined by Eq. (2.19), is the surface of a sphere of unit radius in a 4-dimensional euclidean space. This domain is compact and then also the group $SU(2)$ is compact. Moreover, all the closed paths on the surface can be shrunk continuously to a point, so that the group $SU(2)$ is simply connected. Since $SU(2)$ is homomorphic to $SO(3)$ and it does not contain simply connected subgroups, according to the definition given in Subsection 1.2.1, $SU(2)$ is the *universal covering group* of $SO(3)$. The kernel of the homomorphism is the invariant subgroup $\mathcal{E}(I, -I)$, and then the factor group $SU(2)/\mathcal{E}$ is isomorphic to $SO(3)$.

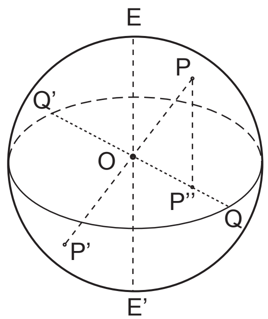

Fig. 2.3. *Parameter space for $SU(2)$ and $SO(3)$.*

The homomorphism between $SO(3)$ and $SU(2)$ can be described in a pictorial way, as shown in Fig. 2.3. The sphere has to be thought of as a 4-dimensional sphere, and the circle which divides it into two hemispheres as a three-dimensional sphere. The points E, E' correspond to the elements I and $-I$ of $SU(2)$. In general, two antipodes, such as P and P', correspond to a pair of elements u and $-u$. Since both u and $-u$ correspond to the same element of $SO(3)$, the parameter space of this group is defined only by the surface of one hemisphere; this can be projected into a three-dimensional sphere, and we see that two opposite points Q, Q' on the surface (on the circle in Fig. 2.3) correspond now to the same element of the group.

2.2 Infinitesimal transformations and Lie algebras of the rotation group

In order to build the Lie algebra of the rotation group, we consider the infinitesimal transformations of $SO(3)$ and $SU(2)$ in a neighborhood of the unit element. There is a *one-to-one correspondence* between the infinitesimal transformations of $SO(3)$ and $SU(2)$, so that the two groups are *locally isomorphic*. Therefore, the groups $SO(3)$ and $SU(2)$ have the same Lie algebra. We can build a basis of the Lie algebra in the following way. We can start from the three independent elements R_1, R_2, R_3 of $SO(3)$ corresponding to the rotations through an angle ϕ around the axis x_1, x_2, x_3 respectively. From Eq. (2.9) we get

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \; R_2 = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \; R_3 = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.25}$$

and, according to Eq. (1.30), we obtain the *generators* of the Lie algebra. For the sake of convenience, we use the definition:

$$J_k = i \frac{dR_k(\phi)}{d\phi}\bigg|_{\phi=0} \qquad (k = 1, 2, 3) \,, \tag{2.26}$$

so that the three generators are given by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.27}$$

We can take J_1, J_2, J_3 as the basis elements of the Lie algebra; making use of the relation between linear Lie algebras and connected Lie groups (see Subsection 1.2.2), the three rotations (2.25) can be written in the form:

$$R_1 = e^{-i\phi J_1} \,, \quad R_2 = e^{-i\phi J_2} \,, \quad R_3 = e^{-i\phi J_3} \,. \tag{2.28}$$

In general, a rotation through an angle ϕ about the direction $\mathbf{n}$ is represented by

$$R = e^{-i\phi\, \mathbf{J}\cdot\mathbf{n}} \,. \tag{2.29}$$

We have considered the specific case of a three-dimensional representation for the rotations R_i and the generators J_i; in general, one can consider J_i as hermitian operators and the R_i as unitary operators in a n-dimensional linear vector space. One can check that J_1, J_2, J_3 satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \,, \tag{2.30}$$

which show that the algebra has *rank* 1. The structure constants are given by the antisymmetric tensor ϵ_{ijk}, and Eq. (1.35) reduces to

$$g_{ij} = -\delta_{ij} \,. \tag{2.31}$$

Since the condition (1.42) is satisfied, the algebra is simple and the Casimir operator (1.36) becomes, with a change of sign,

$$C = J^2 = J_1^2 + J_2^2 + J_3^2 \,. \tag{2.32}$$

The above relations show that the generators J_k have the properties of the *angular momentum operators*[2].

2.3 Irreducible representations of $SO(3)$ and $SU(2)$

We saw that the group $SO(3)$ can be defined in terms of the orthogonal transformations given in Eq. (2.1) in a 3-dimensional Euclidean space. Similarly, the group $SU(2)$ can be defined in terms of the unitary transformations in a 2-dimensional complex linear space

$$\xi'^i = \sum_j u_{ij} \xi^j \,. \tag{2.33}$$

This equation defines the self-representation of the group. Starting from this representation, one can build, by reduction of *direct products*, the higher irreducible representations (IR's). A convenient procedure consists in building, in terms of the basic vectors, higher tensors, which are then decomposed into *irreducible tensors*. These are taken as the bases of irreducible representations; in fact, their transformation properties define completely the representations (for details see Appendix B).

However, starting from the basic vector $\mathbf{x} = (x_1, x_2, x_3)$, i.e. from the three-dimensional representation defined by Eq. (2.1), one does not get all the irreducible representations of $SO(3)$, but only the so-called *tensorial* IR's which correspond to integer values of the angular momentum j. Instead, all the IR's can be easily obtained considering the universal covering group $SU(2)$. The basis of the self-representation consists, in this case, of two-component vectors, usually called *spinors*[3], such as

[2] We recall that the eigenvalues of J^2 are given by $j(j+1)$; see e.g. W. Greiner, *Quantum Mechanics, An Introduction*, Springer-Verlag (1989).

[3] Strictly speaking, one should call the basis vectors ξ "spinors" with respect to $SO(3)$ and "vectors" with respect to $SU(2)$.

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} , \tag{2.34}$$

which transforms according to (2.33), or in compact notation

$$\xi' = u\xi . \tag{2.35}$$

We call ξ *controvariant* spinor of rank 1. In order to introduce a scalar product, it is useful to define the *covariant* spinor η of rank 1 and components η_i, which transforms according to

$$\eta' = \eta u^{-1} = \eta u^\dagger , \tag{2.36}$$

so that

$$\eta\xi = \eta'\xi' = \sum_i \eta_i \xi^i . \tag{2.37}$$

In terms of the components η_i:

$$\eta_i{}' = \sum_j u^\dagger_{ji}\eta_j = \sum_j u^*_{ij}\eta_j . \tag{2.38}$$

Taking the complex conjugate of (2.33)

$$\xi'^{*i} = \sum_j u^*_{ij}\xi^{*j} , \tag{2.39}$$

we see that the component ξ_i transform like ξ^{*i}, i.e.

$$\xi^{*i} \equiv \xi_i . \tag{2.40}$$

The representation u^* is called the *conjugate representation*; the two IR's u and u^* are equivalent. One can check, using the explicit expression (2.15) for u, that u and u^* are related by a similarity transformation

$$u^* = SuS^{-1} , \tag{2.41}$$

with

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.42}$$

We see also that the spinor

$$\overline{\xi} = S^{-1}\xi^* = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix} \tag{2.43}$$

transforms in the same way as ξ. Starting from ξ^i and ξ_i we can build all the higher irreducible tensors, whose transformation properties define all the IR's of SU(2). Besides the tensorial representations, one obtains also the *spinorial* representations, corresponding to half-integer values of the angular momentum j.

We consider here only a simple example. The four-component tensor

$$\zeta_i^j = \xi^j \xi_i \tag{2.44}$$

can be splitted into a scalar quantity

$$\mathrm{Tr}\{\zeta\} = \sum_i \xi^i \xi_i \tag{2.45}$$

and a traceless tensor

$$\hat{\zeta}_i^j = \xi^j \xi_i - \tfrac{1}{2}\delta_i^j \sum_k \xi^k \xi_k \ , \tag{2.46}$$

where δ_i^j is the Kronecker symbol.

The tensor $\hat{\zeta}_i^j$ is not further reducible. It is equivalent to the 3-vector $\mathbf{x}$; in fact, writing it as a 2×2 matrix $\hat{\zeta}$, it can be identified with the matrix h defined in (2.21)

$$\hat{\zeta} = \boldsymbol{\sigma} \cdot \mathbf{x} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} . \tag{2.47}$$

Its transformation properties, according to those of ξ, η, are given by

$$\hat{\zeta}' = u\hat{\zeta}u^\dagger \ . \tag{2.48}$$

Let us consider the specific case

$$u = \begin{pmatrix} e^{-\frac{1}{2}i\phi} & 0 \\ 0 & e^{\frac{1}{2}i\phi} \end{pmatrix} . \tag{2.49}$$

Using for $\hat{\zeta}$ the expression (2.47), we get from (2.49):

$$\begin{aligned} x_1' &= \cos\phi\, x_1 - \sin\phi\, x_2 \ , \\ x_2' &= \sin\phi\, x_1 + \cos\phi\, x_2 \ , \\ x_3' &= x_3 \ . \end{aligned} \tag{2.50}$$

The matrix (2.49) shows how a spinor is transformed under a rotation through an angle ϕ and it corresponds to the 3-dimensional rotation

$$R_3 = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.51}$$

given in eq. (2.25). In particular, taking $\phi = 2\pi$, we get $\xi' = -\xi$, i.e. *spinors change sign* under a rotation of 2π about a given axis. The angles ϕ and $\phi + 2\pi$ correspond to the same rotation, i.e. to the same element of $SO(3)$; on the other hand, the two angles correspond to two different elements of $SU(2)$, i.e.

to the 2×2 unitary matrices u and -u. Then to each matrix R of $SO(3)$ there correspond two elements of $SU(2)$; for this reason, often the matrices u and -u are said to constitute a "double-valued" representation[4] of $SO(3)$.

In general, we can distinguish two kinds of IR's

$$D(u) = +D(-u) \ , \tag{2.52}$$
$$D(u) = -D(-u) \, , \tag{2.53}$$

which are called *even* and *odd*, respectively. The direct product decomposition $u \otimes u \otimes ... \otimes u$, where the self-representation u appears n times, gives rises to even and odd IR's according to whether n is even or odd. The even IR's are the *tensorial* representations of $SO(3)$, the odd IR's are the *spinorial* representations of $SO(3)$.

The IR's are usually labelled by the eigenvalues of the squared angular momentum operator J^2, i.e. the Casimir operator given in Eq. (2.32), which are given by $j(j+1)$, with j integer or half integer. An IR of $SO(3)$ or $SU(2)$ is simply denoted by $D^{(j)}$; its dimension is equal to $2j+1$. Even and odd IR's correspond to *integer* and *half-integer* j, respectively.

The basis of the $D^{(j)}$ representation consists of $(2j+1)$ elements, which correspond to the eigenstates of J^2 and J_3; it is convenient to adopt the usual notation $\mid j, m>$ specified by[5]

$$\begin{aligned} J^2 \mid j,m> &= j(j+1) \mid j,m> \ , \\ J_3 \mid j,m> &= m \mid j,m> \ . \end{aligned} \tag{2.54}$$

Finally, we want to mention the IR's of $O(3)$. We have seen that the group $O(3)$ can be written as the direct product

$$O(3) = SO(3) \otimes \mathcal{J} \ , \tag{2.12}$$

where $\mathcal{J}$ consists of I and I_s. Since $I_s^2 = I$, the element I_s is represented by

$$D(I_s) = \pm I \ . \tag{2.55}$$

According to (2.12), we can classify the IR's of $O(3)$ in terms of those of $SO(3)$, namely the $D^{(j)}$'s. In the case of *integer* j, we can have two kinds of IR's of $O(3)$, according to the two possibilities

$$D^{(j)}(I_s R) = +D^{(j)}(R) \ , \tag{2.56}$$
$$D^{(j)}(I_s R) = -D^{(j)}(R) \ . \tag{2.57}$$

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[4] See e.g. J.F. Cornwell, *Group Theory in Physics*, Vol. 1 and 2, Academic Press (1984); M. Hamermesh, *Group Theory and its Applications to Physical Problems*, Addison-Wesley (1962).

[5] M.E. Rose, *Elementary Theory of Angular Momentum*, John Wiley and Sons (1957).

Only the second possibility corresponds to *faithful* representations, since the improper rotations I_sR are distinguished from the proper rotations R. We denote the two kinds of IR's (2.56), (2.57) by $D^{(j+)}$, $D^{(j-)}$, respectively. The bases of these IR's are called *tensors* and *pseudotensors*; in general, one calls tensors (scalar, vector, etc.) the basis of $D^{(0+)}$, $D^{(1-)}$, $D^{(2+)}$, ..., and pseudotensors (pseudoscalar, axial vector, etc.) the basis of $D^{(0-)}$, $D^{(1+)}$, $D^{(2-)}$, etc.

In the case of half-integer j, since the IR's of $SO(3)$ are double-valued, i.e. each element R is represented by $\pm D^{(j)}(R)$ also for the improper element, one gets

$$I_sR \rightarrow D^{(j)}(I_sR) = \pm D^{(j)}(R) \; . \tag{2.58}$$

Then, for each half-integer j, there is one double-valued IR of $O(3)$.

2.4 Matrix representations of the rotation operators

For the applications in many sectors of physics one needs the explicit expressions of the rotation matrices in an arbitrary representation. Following the notation established in the literature, it is useful to specify a rotation R in terms of the so-called Euler angles α, β, γ. For their definition, we consider a fixed coordinate system (x, y, z). Any rotation R can be regarded as the result of three successive rotations, as indicated in Fig. 2.4.

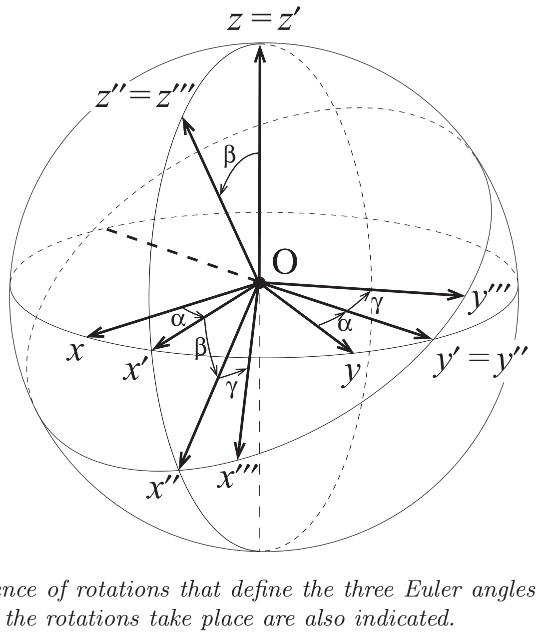

Fig. 2.4. *Sequence of rotations that define the three Euler angles α, β and γ. The planes in which the rotations take place are also indicated.*

1. Rotation $R_\alpha \equiv R_\alpha(z)$ through an angle α $(0 \leq \alpha < 2\pi)$ about the z-axis, which carries the coordinate axes (x, y, z) into $(x', y', z' = z)$;
2. Rotation $R'_\beta \equiv R_\beta(y')$ through an angle β $(0 \leq \beta \leq \pi)$ about the y'-axis, which carries the system (x', y', z') into $(x'', y'' = y', z'')$;
3. Rotation $R''_\gamma \equiv R_\gamma(z'')$ through an angle γ $(0 \leq \gamma < 2\pi)$ about the z''-axis, which carries the system (x'', y'', z'') into $(x''', y''', z''' = z'')$.

The three rotations can be written in the form

$$R_\alpha = e^{-i\alpha J_z} \, , \quad R'_\beta = e^{-i\beta J_{y'}} \, , \quad R''_\gamma = e^{-i\gamma J_{z''}} \, , \tag{2.59}$$

where J_z, $J_{y'}$ and $J_{z''}$ are the components of $\mathbf{J}$ along the z, y' and z'' axes. The complete rotation R is then given by:

$$R(\alpha, \beta.\gamma) = R''_\gamma R'_\beta R_\alpha = e^{-i\gamma J_{z''}} e^{-i\beta J_{y'}} e^{-i\alpha J_z} \; . \tag{2.60}$$

We leave as an exercise the proof that the three rotations can be carried out in the *same* coordinate system if the order of the three rotations in inverted, i.e. Eq. (2.60) can be replaced by

$$R(\alpha, \beta, \gamma) = R_\alpha R_\beta R_\gamma = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \; . \tag{2.61}$$

The procedure for determining the rotation matrices, i.e. the matrix representations of the rotation operator R, is straightforward, even if the relative formulae may appear to be rather involved. One starts from the basis $\mid j, m>$ given in Eq. (2.54) and considers the effect of a rotation R on it:

$$R \mid j, m> = \sum_{m'} D^{(j)}_{m'm}(\alpha, \beta, \gamma) \mid j, m'> \; . \tag{2.62}$$

An element of the rotation matrix $D^{(j)}$ is given by

$$D^{(j)}_{m'm}(\alpha, \beta, \gamma) = <j, m' \mid e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} \mid j, m> = e^{-i\alpha m'} d^{(j)}_{m'm}(\beta) e^{-i\gamma m} \, , \tag{2.63}$$

where one defines

$$d^{(j)}_{m'm}(\beta) = <j, m' \mid e^{-i\beta J_y} \mid j, m> \; . \tag{2.64}$$

There are different ways of expressing the functions $d^j_{m'm}$; we report here Wigner's expression[6]:

$$\begin{aligned} d^{(j)}_{m'm}(\beta) = \sum_s & \frac{(-)^s [(j+m)!(j-m)!(j+m')!(j-m')!]^{1/2}}{s!(j-s-m')!(j+m-s)!(m'+s-m)!} \times \\ & \times \left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2s} \left(-\sin\frac{\beta}{2}\right)^{m'-m+2s} \, , \end{aligned} \tag{2.65}$$

[6] See e.g. M.E. Rose, *Elementary Theory of Angular Momentum*, John Wiley and Sons (1957).

where the sum is over the values of the integer s for which the factorial arguments are equal or greater than zero.

It is interesting to note that, in the case of integral angular momentum, the d-functions are connected to the well-known spherical harmonics by the relation:

$$d^{\ell}_{m,0}(\theta) = \sqrt{\frac{4\pi}{2\ell+1}} Y^m_{\ell}(\theta,\phi) e^{-im\phi} , \tag{2.66}$$

where θ and ϕ are the angles of the spherical coordinates. In fact, $Y^m_{\ell}(\theta,\phi)$ represents the eigenfunction corresponding to the state $| \, j, m>$ of a particle with orbital angular momentum $j = \ell$.

In Appendix A we collect the explicit expressions of the spherical harmonics and the d-functions for the lowest momentum cases.

2.5 Addition of angular momenta and Clebsch-Gordan coefficients

An important application of the IR's of the rotation group is related to the addition of angular momenta and the construction of the relevant orthonormal bases[7].

We start from the IR's $D^{(j_1)}$ and $D^{(j_2)}$ and the direct product decomposition

$$D^{(j_1)} \otimes D^{(j_2)} = D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus .... \oplus D^{(|j_1-j_2|)} . \tag{2.67}$$

The IR's on the r.h.s. correspond to the different values of total angular momenta obtained by the quantum addition rule

$$\mathbf{J} = \mathbf{J_1} + \mathbf{J_2} , \tag{2.68}$$

with $|J_1 - J_2| \leq J \leq J_1 + J_2$. From the commutation properties of angular momentum operators, one finds that the eigenvectors

$$| \, j_1, j_2; m_1, m_2> \equiv | \, j_1, m_1> \otimes \, | \, j_2, m_2> \tag{2.69}$$

constitute an orthogonal basis for the direct product representation, while the eigenvectors

$$| \, j_1, j_2; j, m> \tag{2.70}$$

are the bases of the IR's on the r.h.s. of eq. (2.67). One can pass from one basis to the other by a unitary transformation, which can be written in the form

[7] For a detailed treatment of this subject see e.g. W. Greiner, *Quantum Mechanics, An Introduction*, Springer-Verlag (1989) and M. Hamermesh, *Group Theory and its Applications to Physical Problems*, Addison-Wesley (1962).

$$| j_1, j_2; j, m> = \sum_{m_1,m_2} C(j_1, j_2, j; m_1, m_2, m) \mid j_1, j_2; m_1, m_2> \ . \tag{2.71}$$

The elements of the transformation matrix are called *Clebsh-Gordan coefficients* (or simply *C-coefficients*), defined by

$$C(j_1, j_2, j; m_1, m_2, m) = < j_1, j_2; m_1, m_2 \mid j_1, j_2; j, m> \ , \tag{2.72}$$

where $m = m_1 + m_2$. With the standard phase convention the C-coefficients are real and they satisfy the orthogonality relation (replacing m_2 by $m - m_1$):

$$\sum_{m_1} C(j_1, j_2, j; m_1, m - m_1) C(j_1, j_2, j'; m_1, m - m_1) = \delta_{jj'} \ . \tag{2.73}$$

Moreover, the transformation (2.71) is *orthogonal*, and the inverse transformation can be easily obtained:

$$| j_1, j_2; m_1, m_2 > = \sum_{j,m} C(j_1, j_2, j; m_1, m - m_1) \mid j_1, j_2; j, m> \ . \tag{2.74}$$

We report the values of the Clebsh-Gordan coefficients for the lowest values of j_1 and j_2 in Appendix A, while for other cases and for a general formula we refer to specific texbooks[8].

In connection with the C-coefficients it is convenient to quote without proof the *Wigner-Eckart theorem* which deals with matrix elements of tensor operators. An irreducible tensor operator of rank J is defined as a set of $(2J+1)$ functions T_{JM} (where $M = -J, -J+1, ..., J-1, J$) which transform under the $(2J+1)$ dimensional representations of the rotation group in the following way:

$$R T_{JM} R^{-1} = \sum_{M'} D^J_{M'M}(\alpha, \beta, \gamma) T_{JM'} \ . \tag{2.75}$$

The Wigner-Eckart theorem states that the dependence of the matrix element $< j', m' \mid T_{JM} \mid j, m >$ on the quantum number M, M' is entirely contained in the C-coefficients:

$$< j', m' \mid T_{JM} \mid j, m > = C(j, J, j'; m, M, m') < j' \mid T_J \mid j > \ . \tag{2.76}$$

We note that the C-coefficient vanishes unless $m' = M + m$, so that one has $C(j, J, j'; m, M, m') = C(j, J, j'; m, m' - m)$. The matrix element on the r.h.s. of the above equation is called reduced matrix element.

[8] See e.g. D.R. Lichtenberg, *Unitary Symmetry and Elementary Particles*, Academic Press (1970); M.E. Rose, *Elementary Theory of Angular Momentum*, John Wiley and Sons (1957).

Problems

2.1. Give the derivation of Eq. (2.24) and write explicitly the matrix R in terms of the elements of the u matrix.

2.2. Consider the Schrödinger equation $H \,|\, \psi > = E \,|\, \psi >$ in which the Hamiltonian H is invariant under rotations. Show that the angular momentum J commutes with H and then it is conserved.

2.3. The πN scattering shows a strong resonance at the kinetic energy about 200 MeV; it occurs in the P-wave ($\ell = 1$) with total angular momentum $J = \frac{3}{2}$. Determine the angular distribution of the final state.

2.4. Prove the equivalence of the two expressions for a general rotation R given in Eqs. (2.60) and (2.61).

2.5. Consider the eigenstates $|\frac{1}{2}, \pm\frac{1}{2} >$ of a particle of spin $\frac{1}{2}$ and spin components $\pm\frac{1}{2}$ along the z-axis. Derive the corresponding eigenstates with spin components along the y-axis by a rotation about the x-axis.

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