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One Century of Logarithmic Forms

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1 Introduction

At the turn of any century it is very natural on the one hand for us to look back and see what were great achievements in mathematics and on the other to look forward and speculate about which directions mathematics might take. One hundred years ago Hilbert was in a similar situation and he raised on that occasion a famous list of 23 problems that he believed would be very significant for the future development of the subject. Hilbert's article on future problems in mathematics published in the *Comptes Rendus du Deuxième Congrès International des Mathématiciens* stimulated tremendous results and an enormous blossoming of the mathematical sciences overall. A significant part of Hilbert's discussion was devoted to number theory and Diophantine geometry and we have seen some wonderful achievements in these fields since then. In this survey, we shall recall how transcendence and arithmetical geometry have grown into beautiful and far-reaching theories which now enhance many different aspects of mathematics. Very surprisingly three of Hilbert's problems, which at first seemed very distant from each other, have now come together and have provided the catalyst for a vast interplay between the subjects in question. We shall concentrate on one of them, namely the seventh, and describe the principal developments in transcendence theory which it has initiated. This will lead us to the theory of linear forms in logarithms and to the generalization of the latter in the context of commutative group varieties. The theory has evolved to be the most crucial instrument towards a solution of the tenth problem of Hilbert on the effective solution of diophantine equations as well as many other well-known questions. The intimate relationships in this field become especially evident through a simple conjecture, the *abc*-conjecture, which seems to hold the key to much of the future direction of number theory. We shall discuss this at the end of this article.

2 Hilbert's seventh problem

Hilbert remarked in connection with the seventh problem that he believed that the proof of the transcendence of α^β for algebraic $\alpha \neq 0, 1$ and algebraic irrational β would be extremely difficult and that certainly the solution of this and analogous problems would lead to valuable new methods. Surprisingly, the problem was eventually solved independently, by different methods, by Gelfond and Schneider in 1934. Gelfond and Kuzmin had solved some particular cases of the conjecture a few years earlier and the solutions of Gelfond and Schneider used similar methods together with techniques introduced by Siegel in his well-known investigations on Bessel functions. The Gelfond–Schneider theorem shows that for any non-zero algebraic numbers α_1 and α_2 with $\log \alpha_1$ and $\log \alpha_2$ linearly independent over the rationals we have

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0.$$

In 1935 Gelfond considered the problem of establishing a lower bound for the absolute value of the linear form $L = \beta_1 T_1 + \beta_2 T_2$ evaluated at $(\log \alpha_1, \log \alpha_2)$ and succeeded in proving that its value Λ is bounded below by

$$\log |\Lambda| \gg -h(L)^\kappa$$

where $h(L)$ denotes the logarithmic height of the linear form and $\kappa > 5$. It was realized by Gelfond around 1940 that an extension of the theorem to linear forms in more than two variables would enable one to solve some of the most challenging problems in number theory and in the theory of diophantine equations. We mention here the *Liouville problem* of establishing effective lower bounds for the approximation of an algebraic number by rationals sharper than the bound obtained by Liouville himself. Other examples were the *Thue equation* and effective bounds for the size of solutions in *Siegel's great theorem on integral points* on algebraic curves. To great surprise one of the oldest and most exciting problems in number theory, Euler's famous *numeri idonii* problem, has also turned out to be intimately related to the theory of logarithmic forms.

The Liouville problem was mentioned by Davenport to Baker as a research topic in the early 60s. Baker's early papers made the first breakthrough in this area. The approach was through hypergeometric functions and Padé approximation theory and was related to some work of Thue and Siegel. After several significant results in diverse branches of transcendence theory, Baker was led to the famous class number problem of Gauss. A careful study of the work of Heilbronn, Gelfond, Linnik and others in this field convinced him that the most promising approach to this and many other fundamental questions in number theory was through linear forms in logarithms. Despite the fact that no significant progress had been made in this subject for many years, Baker succeeded in 1966 in establishing a definitive result; namely if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the rationals, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over the field of all algebraic numbers. The result and the method of proof was one of the most significant advances in number theory made in 20th century. Baker's theorem includes both the Hermite–Lindemann and the Gelfond–Schneider theorems as special cases. Baker's original paper also contained a quantitative result on lines similar to Gelfond's two-variable estimate mentioned earlier; this was sufficient to deal with the class number problem, the Thue problem, the Liouville problem, the elliptic curve problem and clearly had great potential for future research. It soon became clear that further progress on many critical problems would depend on sharp estimates for logarithmic forms, and between 1966 and 1975 Baker wrote a series of important papers on this subject. It was here that many of the instruments now familiar to specialists in the field were introduced, among them Kummer theory, the so-called Kummer descent and delta functions. In this context one should mention that both Stark and Feldman made substantial contributions.

3 Elliptic theory

Having published his solution to Hilbert's seventh problem, Schneider started to study elliptic and later also abelian functions. He had been motivated by a paper of Siegel on periods of elliptic functions. There Siegel had proved that for an elliptic curve with algebraic invariants g_2, g_3 not all periods can be algebraic. In particular he obtained the transcendence of non-zero periods when the elliptic curve has complex multiplication. In a fundamental paper Schneider established the transcendence of elliptic integrals of the first and second kind taken between algebraic points. As a special instance one obtains the transcendence of the value of the linear form $L = \alpha T_1 + \beta T_2$ at $(\omega, \eta(\omega))$ where ω is a non-zero period and $\eta(\omega)$ the corresponding quasi-period. Plainly the result gives Siegel's theorem without the additional hypothesis on complex multiplication. Schneider also applied the result to the modular j-function and found that it takes algebraic values at algebraic arguments if and only if the argument is imaginary quadratic. As was realized only recently, this opened a close connection with Hilbert's twelfth problem which emerged out of the famous *Jugendtraum* of Kronecker. For some forty years there was no obvious progress; then in 1970 Baker realized that the method which he had developed for dealing with linear forms in logarithms could be adapted to give the transcendence of non-zero values of linear forms in two elliptic periods and their associated quasi-periods. With this fundamental contribution he opened a new field of research. Masser and Coates succeeded a few years later in including the period $2\pi i$ and determining the dimension of the vector space generated by the numbers $1, 2\pi i, \omega_1, \omega_2, \eta(\omega_1), \eta(\omega_2)$. The main difficulty in going further and extending the results to an arbitrary number of periods lay in the use of determinants in Baker's method. It became clear that to achieve a breakthrough such aspects had to be modified significantly.

The situation was very similar in the case of abelian varieties which was first studied by Schneider in 1939 and which was subsequently investigated by Masser, Coates and Lang between 1975 and 1980. Again difficulties relating to the use of determinants presented a severe obstacle for progress. It was realized about that time that transcendence theory has much to do with algebraic groups and with the exponential map of a Lie group in particular. Lang was the first to consider a reformulation of the Gelfond–Schneider theorem and other classical results in the language of group varieties. With advice from Serre this was taken up by Waldschmidt and, amongst other things, he interpreted Schneider's result on elliptic integrals in the new language. This prepared the ground for the first successful attack, by Laurent, on Schneider's third problem concerning elliptic integrals of the third kind. However the difficulties relating to determinants referred to earlier blocked the passage to a complete solution. Likewise, for abelian integrals, Schneider had raised the question of extending his elliptic theorems to abelian varieties. As Arnold pointed out in his monograph on Newton, Hooke and Huyghens, the question is closely related to an unsolved problem of Leibniz emerging from celestical mechanics.

4 Group varieties

The situation changed completely when it was realized almost simultaneously by Brownawell, Chudnovsky, Masser, Nesterenko and Wüstholz that one way to deal with the difficulties referred to in the previous section was to use commutative algebra. Masser and Wüstholz started to apply the theory to commutative group varieties and the breakthrough was obtained by Wüstholz in 1981 when he succeeded in establishing the correct multiplicity estimates for group varieties. As a consequence he was able to formulate and prove the Analytic Subgroup Theorem. It says that if G is a commutative and connected algebraic group defined over $\overline{\mathbb{Q}}$ then an analytic subgroup defined over $\overline{\mathbb{Q}}$ contains a non-trivial algebraic point if and only if it contains a non-trivial algebraic subgroup over $\overline{\mathbb{Q}}$. The theorem generalizes that of Baker in a natural way and hence includes, as special cases, the classical theorems of Hermite, Lindemann and Gelfond–Schneider. It also implies directly the results of Schneider, Baker, Masser, Coates, Lang and Laurent mentioned above on elliptic and abelian functions and integrals. It further includes complete solutions to problems raised by Baker relating to elliptic logarithmics, to a problem mentioned by Waldschmidt on analytic homomorphisms as well as the outstanding third and fourth problems listed by Schneider at the end of his well-known book. The Analytic Subgroup Theorem was therefore one of the most significant results in modern transcendence theory; all the above consequences can be seen as questions concerning generalised logarithms defined in terms of suitable commutative algebraic groups. There are many other applications of the Theorem, in particular the work of Wolfart and Wüstholz on a question of Lang about the complex uniformisation of algebraic curves. There are also nice areas of application to Siegel modular functions, as studied by Cohen, Shiga and Wolfart, and to hypergeometric theory, as investigated by Wolfart, Beukers, Cohen and Wüstholz.

The multiplicitiy estimates on group varieties referred to earlier which were crucial to the proof of the Analytic Subgroup Theorem also lead to an improvement in the basic linear form estimate of Baker. This was noted independently by Wüstholz and by Philippon & Waldschmidt. Subsequently, in 1993, Baker & Wüstholz used the multiplicitiy estimates to establish a very sharp bound for logarithmic forms; it remains the best to date and it has served as an indispensible reference for practical applications to diophantine equations.

5 The quantitative theory

In the previous section we discussed the most natural version of the qualitative theory of logarithmic forms in the context of algebraic groups. Many of the most important applications, however, involve a quantitative form of the theory and this we shall discuss now. We begin with a report on the latest results concerning linear forms in ordinary logarithms. As mentioned earlier their derivation depends critically on the theory of multiplicity estimates on group varieties. Let now $\alpha_1, \ldots, \alpha_n$ be algebraic numbers, not 0 or 1, and let $\log \alpha_1, \ldots, \log \alpha_n$ be fixed determinations of the logarithms. Let K be the field generated by $\alpha_1, \ldots, \alpha_n$ over the rationals and let d be the degree of K. For each α in K and any given determination of $\log \alpha$ we define the modified height $h'(\alpha)$ by

$$h'(\alpha) = \frac{1}{d} \max(h(\alpha), |\log \alpha|, 1),$$

where $h(\alpha)$ is the logarithm of the standard Weil height of α. We consider the linear form

$$L = b_1 z_1 + \cdots + b_n z_n,$$

where $b_1, \ldots, b_n$ are integers, not all 0, and put

$$h'(L) = \frac{1}{d} \max(h(L), 1),$$

where $h(L)$ is the logarithmic Weil height of L, that is $d \log \max(|b_j|/b)$ with b given by the highest common factor of $b_1, \ldots, b_n$. In their 1993 paper Baker & Wüstholz showed that if $\Lambda = L(\log \alpha_1 \ldots, \log \alpha_n) \neq 0$ then

$$\log |\Lambda| > -C(n, d) h'(\alpha_1) \ldots h'(\alpha_n) h'(L),$$

where

$$C(n, d) = 18(n + 1)!\, n^{n+1} (32d)^{n+2} \log(2nd).$$

As we already indicated, the result gives the best lower bound for Λ known to date, and it is essential for computational diophantine theory. In this context we mention the important work of Győry and others who have applied the theory to large classes of diophantine equations; these include the so-called norm form, index form and discriminant form equations in particular. Recently a striking application was given by Bilu, Hanrot & Voutier (1999) in the realm of primitive divisors of Lucas and Lehmer numbers. Here the precision of the Baker–Wüstholz bound was critical in making the computations feasible.

We now discuss briefly the extent to which the classical quantitative theory of logarithmic forms has been carried over to deal with the general situation of commutative group varieties. The latest and most precise work in this field is due to Hirata-Kohno (1991) and the precision of her results is close to that established in the classical case. To indicate the form of Hirata-Kohno's main result, let K be a number field and G be a commutative group variety of dimension n defined over K. The basic theory refers to a non-vanishing linear form $L(z_1, \ldots, z_n) = \beta_1 z_1 + \cdots + \beta_n z_n$ as above with coefficients in K. The linear form is evaluated at a point $z_1 = u_1, \ldots, z_n = u_n$, where $u = (u_1, \ldots, u_n)$ is an element in the Lie algebra of G such that $\alpha = \exp_G(u)$ belongs to $G(K)$. Then Hirata-Kohno's work shows that there exists a positive constant C independent of u and L which can be determined effectively and which has the property that if $\Lambda = L(u) \neq 0$ then

$$\log |\Lambda| > -C \big(h'(\alpha)\big)^n \big(h'(L) + h'(\alpha)\big) \big(\max(1, \log(h'(L) h'(\alpha))\big)^{n+1}.$$

The result can be applied, in particular, to yield an alternative approach to Siegel's theorem on integral points on curves. It can also be used to give an effective bound for the height $h(P)$ of an integral point P depending on the height of a set of generators for the Mordell–Weil group; consequently one sees that in the cases when a suitable set of generators can be determined explicitly, all the integral points P on the curve can be fully computed. The method was successfully applied in the elliptic case by Stroeker & Tzanakis (1994) to give the complete set of solutions in integers x, y of the equation

$$y^2 = (x + 337)(x^2 + 337^2),$$

and by Gebel, Pethö & Zimmer (1994) to deal with the instance

$$y^2 = x^3 - 1642032x + 628747920.$$

Here an essential ingredient is some work of David (1992) which furnishes, in the elliptic case, an explicit estimate for the constant occurring in Hirata-Kohno's general result.

It emerged unexpectedly from the early work of Baker on logarithmic forms that if there is a rational linear dependence relation satisfied by logarithms of algebraic numbers then there exists such a relation with coefficients bounded in terms of the heights of the numbers. Motivated by Faltings' famous work on the Mordell conjecture, Masser & Wüstholz realized that Baker's observation, appropriately generalized to abelian varieties, could be applied to yield an effective Isogeny Theorem. The result significantly improves upon an essential aspect of Faltings' 1983 paper and it has initiated a substantial body of new theory on arithmetical properties of abelian varieties. For instance, Masser & Wüstholz obtained in this way an effective version of the well-known Tate conjecture, which was crucial to Faltings' work; they established discriminant estimates for endomorphism algebras; and they derived a solution to a problem of Serre on representations of Galois groups. This is currently a very active area of research.

6 The *abc*-conjecture

When Richard Mason became a graduate student in Cambridge in the early 80s, Baker suggested to him as a research topic the problem of generalizing the theory of logarithmic forms to function fields. This led Mason to an assertion about equations in polynomials of the form

$$a + b = c$$

which we now recognize as being the analogue of the *abc*-conjecture in the function field setting. The conjecture itself relates to relatively prime integers

a, b, c satisfying the above equation and it asserts that, for any $\epsilon > 0$,

$$\max(|a|, |b|, |c|) \ll N^{1+\epsilon},$$

where N, the conductor or radical of abc, denotes the product of all distinct prime factors of abc, and the constant implied by $\ll$ depends only on ϵ. The origin of the assertion lies in a conjecture of Szpiro on discriminants and conductors of elliptic curves; this was adapted by Oesterlé to give a conjecture as above but in a weaker form and it was Masser who formulated the precise statement as we recognize it today. The conjecture enables one in principle to resolve in integers x, y, z, l, m, n and given r, s, t, the exponential diophantine equation

$$rx^l + sy^m + tz^n = 0,$$

where l, m, n are positive and subject to $(1/l) + (1/m) + (1/n) < 1$; this includes the celebrated cases of Fermat and Catalan. Other consequences of the conjecture are the famous theorems of Roth and Faltings, and, if one assumes a generalised version for number fields, then it resolves, in principle, the problem of the non-existence of the Siegel zero for Dirichlet L-functions. The only non-trivial estimate for $\max(|a|, |b|, |c|)$ to date is due to Stewart & Yu Kunrui (1991). Their work is based on the Baker–Wüstholz archimedean bound for logarithmic forms quoted in Section 5 and on a natural non-archimedean analogue established by Yu Kunrui (1998).

In a very interesting recent paper, Baker (1998) described an intimate connnection between the abc-conjecture and the theory of logarithmic forms. He began by suggesting two refinements to the abc-conjecture, first

$$\max(|a|, |b|, |c|) \ll N(\log N)^{\omega}/\omega!,$$

and, secondly, for some absolute constant κ,

$$\max(|a|, |b|, |c|) \ll \epsilon^{-\kappa\omega(ab)} N^{1+\epsilon},$$

where the constants implied by $\ll$ are absolute, and where $\omega(n)$ signifies the number of distinct prime factors of the integer n and $\omega = \omega(abc)$. Baker went on to relate the second refinement with an estimate for the logarithmic form

$$\Lambda = u_1 \log v_1 + \cdots + u_n \log v_n,$$

in positive integers $v_1, \ldots, v_n$ and integers $u_1, \ldots, u_n$, not all 0. Indeed he showed that the second refinement is equivalent to the lower bound

$$\Xi \gg (N(v))^{-1}(\epsilon^{\kappa\omega(v)} a^{-\epsilon})^{1/(1+\epsilon)},$$

for the expression

$$\Xi = \min(1, |\Lambda|) \prod \min(1, p|\Lambda|_p),$$

where the product is taken over all primes p; here $v = v_1 \cdots v_n$ and $N(v)$ denotes the radical of v.

A slightly weaker version of the inequality is given by

$$\log \Xi \gg -\log u \log v,$$

with $u = \max |u_j|$, and the latter can be compared with the Baker–Wüstholz theorem which gives

$$\log |\Lambda| \gg -\log u \log v_1 \cdots \log v_n$$

with an implied constant depending only on n. Thus we see that a result in the direction of the *abc*-conjecture sufficient for all the major applications would follow if one could replace $|\Lambda|$ by Ξ and also the product $\log v_1 \times \cdots \times \log v_n$ by the sum $\log v_1 + \cdots + \log v_n$. Bearing in mind what has been achieved in connection with non-archimedean valuations, this would seem to present the most feasible line of attack for the future.

Bibliography

Baker, A. (1975), *Transcendental Number Theory*, Cambridge University Press, 1st ed., 1975; (3rd ed., Math. Library Series, 1990).

Baker, A. (1998), Logarithmic forms and the *abc*-conjecture. In *Number Theory: Diophantine, Computational and Algebraic Aspects*, de Gruyter, 37–44.

Baker, A. & A. Schinzel (1971), On the least integers represented by the genera of binary quadratic forms, *Acta Arith.*, **18**, 137–144.

Baker, A. & G. Wüstholz (1993), Logarithmic forms and group varieties, *J. Reine Angew. Math.*, **442**, 19–62.

Baker, A. & G. Wüstholz (1999), Number theory, transcendence and diophantine geometry in the next millennium. In *Mathematics: Frontiers and Perspectives*, American Mathematical Society, 1–12.

Bilu, Y., G. Hanrot & P.M. Voutier (1999), Existence of primitive divisors of Lucas and Lehmer numbers (with an appendix by M. Mignotte). Preprint.

Beukers, F. & J. Wolfart (1988), Algebraic values of hypergeometric functions. In *New Advances in Transcendence Theory*, A. Baker (ed.), Cambridge University Press, 68–81.

Cohen, Paula B. (1996), Humbert surfaces and transcendence properties of automorphic functions, *Rocky Mountain J. Math.*, **26**, 987–1001.

David, S. (1992), Minorations de formes linéaires de logarithmes elliptiques, *Publ. Math. Univ. Pierre et Marie Curie, Problèmes Diophantiens*, **106**, (1991/92), exposé no. 3.

Dvornicich, R. & U. Zannier (1994), Fields containing values of algebraic functions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **21**, 421–443.

Faltings, G. (1983), Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.*, **73**, 349–366.

Gebel, J., A. Pethö & H.G. Zimmer (1994), Computing integer points on elliptic curves, *Acta Arith.*, **68**, 171–192.

Hirata-Kohno, N. (1991), Formes linéaires de logarithmes de points algébriques sur les groupes algébriques, *Invent. Math.*, **104**, 401–433.

Masser, D.W. & G. Wüstholz (1993), Isogeny estimates for abelian varieties and finiteness theorems, *Annals Math.*, **137**, 459–472.

Mignotte, M. & Y. Roy (1997), Minorations pour l'équation de Catalan, *C. R. Acad. Sci. Paris*, **324**, 377–380.

Ribenboim, P. (1994), *Catalan's Conjecture*, Academic Press.

Shiga, H. & J. Wolfart (1995), Criteria for complex multiplication and transcendence properties of automorphic functions, *J. Reine Angew. Math.*, **463**, 1–25.

Stewart, C.L. & Kunrui Yu (1991) On the *abc* conjecture, *Math. Ann.*, **291**, 225–230.

Stroeker, R.J. & N. Tzanakis (1994), Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms, *Acta Arith.*, **67**, 177–196.

Wolfart, J. (1988), Werte hypergeometrischer Funktionen, *Invent. Math.*, **92**, 187–216.

Wüstholz, G. (1989), Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen, *Annals Math.*, **129** (1989), 501–517.

Yu, Kunrui (1998), p-adic logarithmic forms and group varieties I, *J. Reine Angew. Math.*, **502**, 29–92.