The Gravitational Million-Body Problem

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Astrophysics Introduction

When Newton studied dynamics and calculus he was motivated, at least in part, by a desire to understand the movements of the planets. The mathematical model which he reached may be described by the equations

$$\ddot{\mathbf{r}}_i = -G \sum_{j=1, j\neq i}^{j=N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \tag{1.1}$$

where $\mathbf{r}_j$ is the position vector of the jth body at time t, m_j is its mass, G is a constant, and a dot denotes differentiation with respect to t. Now, over 300 years later, our motivation has a very different emphasis. The study of the solar system has lost none of its importance or fascination, but it now competes with the study of star clusters, galaxies, and even the structure of the universe as a whole. What is remarkable is that Newton's model is as central to this extended field of study as it was in his own time.

Stellar dynamics

It may seem surprising that the single, simple set of equations (1.1) forms a good first approximation for modelling many astrophysical systems, such as the solar system, star clusters, whole galaxies as well as clusters of galaxies (Fig. 1.1). The reason is that gravity, being an attractive long-range force, dominates everything else in the Universe. The only other long-range force, electromagnetism, is generally not important on very large scales, since positive and negative charges tend to screen

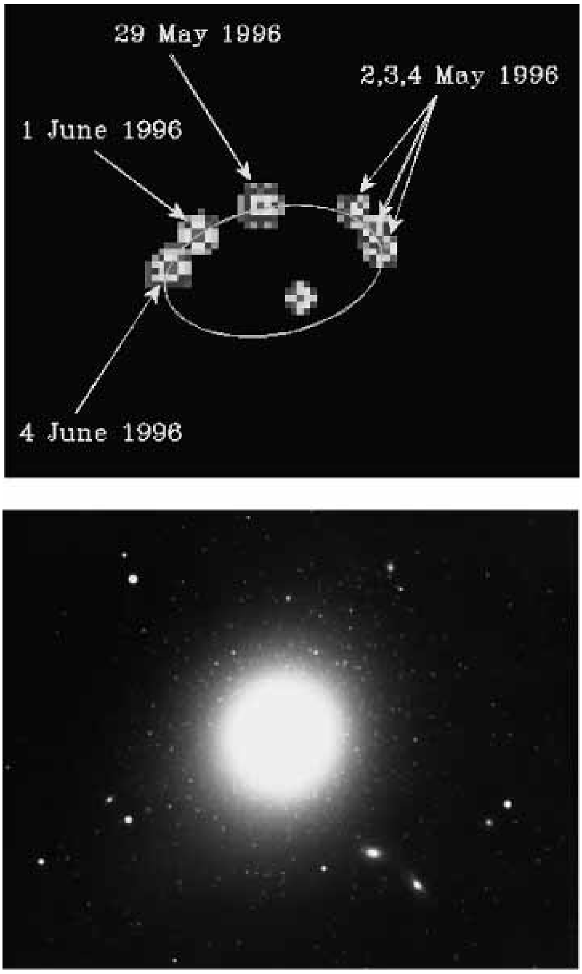

Fig. 1.1. Two extreme astrophysical many-body problems. *Top* A binary star, ζ^1 Ursae Majoris, imaged interferometrically (*Credit: J. Benson et al., NPOI Group, USNO, NRL*). *Bottom* The galaxy M87. This is not a straightforward *N*-body problem, as much of the mass is not stellar. Many of the spots of light in the fringes of the galaxy are individual globular clusters (*Credit: NOAO/AURA/NSF*).

each other. Short-range forces, such as gas pressure, are usually only important on small scales, such as in the interiors of stars. On large scales, comparable to the size of a galaxy, pressure is rarely important. On intermediate scales we have giant gas clouds in which both pressure and magnetic fields can play a role.

This dominance of gravity on cosmic scales is a fortunate feature of our Universe. It implies that it is relatively simple to perform detailed computer simulations of many astronomical systems. Mastery of the much more complicated physics of other specialisations in astronomy, such as plasma astrophysics, radiative transfer, or nuclear astrophysics, is often not immediately necessary. Nor, in applications to star clusters, is it usually necessary to use the more refined theory of general relativity. The velocities are too low, except in a few situations involving degenerate stars (neutron stars). Even the motions of stars round the black hole at the centre of our

Galaxy (Ghez *et al.* 2000), though it takes place on a human time scale, can be understood with classical dynamics.

Stellar dynamics can be defined as studying the consequences of Eq. (1.1) in astrophysical contexts. Traditionally, these equations were discovered by studying the motions of the Moon and planets, and for the next few centuries they were applied mainly to planetary dynamics. Before the advent of electronic computers, most effort went into developing analytical approximations to the nearly regular motion of the planets. This field, known as celestial mechanics, had an important influence on developments both in physics and mathematics. An example is the study of chaos, which first arose as an annoying complexity barring attempts to make long-time predictions in celestial mechanics.

The solar system, however, is a very regular system. All planets move in orbits close to the ecliptic, and all revolve in the same direction. The orbits are well-separated, and consequently no close encounters take place. When we look around us on larger scales, that of star clusters and galaxies, then no such regularity applies. In our Galaxy, most stars move in the galactic disk, revolving in the same sense as the Sun, but close encounters are not excluded. In many globular star clusters, there is not even a preferred direction of rotation, and all stars move as they please in any direction. Does this mean that analytic approximations are not very useful for such systems?

The answer is: 'it depends on what you would like to know'. In general, the less regular a situation is, the larger the need for a computer simulation. For example, during the collision of two galaxies each star in each of the galaxies is so strongly perturbed that it becomes very difficult to predict the overall outcome with pen and paper. This is indeed an area of research which had to wait for computers to even get started, in the early seventies (Toomre & Toomre 1972; but see the reference to Holmberg in Chapter 3). On the other hand, when we want to understand the conditions in a relatively isolated galaxy, such as our own Milky Way, then there is some scope for pen-and-paper work. For example, a rich variety of analytic as well as semi-numerical models has been constructed for a range of galaxies. However, even in this case we often have to switch to numerical simulations if we want to obtain more precise results.

Two flavours of stellar dynamics

The field of stellar dynamics can be divided into two subfields, traditionally called collisional and collisionless stellar dynamics. The word 'collision' is a bit misleading here, since it is used to describe a close encounter between two or more stars, not a physical collision. After all, physical collisions are excluded in principle as soon as we have made the approximation of point particles, as we will do throughout most of the present book, with the exception of the last few chapters, where we will take up the question of real stellar traffic accidents.

Collisional stellar dynamics is concerned about the long-term effects of close (as well as not-so-close) stellar encounters. The evolution of a star cluster is governed by the slow diffusion of ‘heat’ through the system from the inside towards the edge. The heat transport occurs through the frequent interactions of pairs of stars, in a way similar to the heat conduction in the air in a room, which is caused by collisions between pairs of gas molecules. The main difference here is that individual stars in a star cluster have mean free paths that are much longer than the size of the system. In other words, little heat exchange takes place during a system-crossing time scale.

Collisionless stellar dynamics is the subfield of stellar dynamics in which the heat flow due to pairwise interactions of stars is neglected. For small systems this approximation is appropriate when we consider the evolution of the star system on a time scale which does not exceed the crossing time by a very large amount. For a system with very many particles, such as a galaxy, the collisionless approximation is generally valid even on time scales comparable to the age of the Universe (which is comparable to the age of most galaxies).

Life in a collisional stellar system

So far, we have talked about the astrophysical many-body problem in a rather generalised way. We have dealt with stars or planets moving in the gravitational field of their mutually attractive forces, and have ignored their internal structure, but we have not focused on any particular kind of system to which these idealisations apply. Fortunately, there are systems in nature which approach the idealisations of collisional stellar dynamics to a remarkable degree. These are the globular clusters, among the oldest components of our galactic system, and going their own way in wide orbits around the Galaxy (see Webbink 1988). They are nearly spherical because they don’t rotate much by astronomical standards (see Merritt *et al.* 1997, Davoust & Prugniel 1990). They are largely isolated from perturbing influences of the galactic disk, and therefore form ideal laboratories for stellar dynamics. Typically they have about a million stars each (Table 1.1), with a range of individual stellar masses (see Paresce & De Marchi 2000).

Other galaxies also have globular clusters (Fig. 1.1, bottom; Ashman & Zepf 1998). In some galaxies they are not the oldest but some of the *youngest* stellar components (see Lançon & Boily 2000). For instance, the cluster NGC 1866, which lies in the Large Magellanic Cloud, has a mass of about $10^5 M_\odot$ but is only about 10^8yrs old (Fischer *et al.* 1992, Testa *et al.* 1999). Even in our own galaxy, the most massive young clusters are found in the apparently hostile environment of the Galactic Centre (Figer *et al.* 1999), and they are as massive as a modest old globular cluster (though they will certainly not last as long, see Kim *et al.* 2000, Portegies Zwart *et al.* 2001a).

Table 1.1. *Basic facts about the globular clusters of the Galaxy*

Number known	147
Median distance from Galactic Centre	9.3kpc
Median absolute V magnitude	−7.27
Median concentration	1.50
Median core relaxation time	3.39×10^8yr
Median relaxation time at the half-mass radius	1.17×10^9yr
Median core radius	1.32pc
Median half-mass radius	3.08pc
Median tidal radius	34.5pc
Median mass	$8.1 \times 10^4 M_\odot$
Median line-of-sight velocity dispersion	5.50km/s

The data are based on tables in Harris (1996; for updated online versions see physun.physics.mcmaster.ca/Globular.html), Mandushev *et al.* (1991) and Pryor & Meylan (1993), and different data are based on somewhat different samples. The terms *concentration*, *relaxation time*, *core* and *tidal radius* are defined elsewhere in the book. Units are discussed in Box 1.1.

The realisation that the formation of globular clusters is still going on in the Universe around us has taken hold only relatively recently, and has helped to rejuvenate the study of these stellar systems. For the most part, though, we concentrate on the old and better observed systems in our own Galaxy.

Box 1.1. **Units**

Consider a stellar system whose total mass and energy are M and E, respectively. Then we may choose M for the unit of mass, and in simple cases a common and useful unit of length is the *virial radius* $R_v = -\frac{GM^2}{4E}$. (For example, for a binary star this is roughly the diameter of the relative orbit.) Then, if we want units in which the constant of gravitation is unity, the unit of time is required to be $U_t = R_v^{3/2}/\sqrt{GM}$. Another common unit of time is the *crossing time*, which is defined to be $2\sqrt{2}U_t$.

Now in the evaluation of these expressions R_v and M will often be in units favoured by astrophysicists (e.g. parsecs and solar masses, respectively), while G is most familiar in one of the standard physical systems, such as SI. The evaluation of U_t then requires conversion to a common system of units. At this point, because of the enormous distances and time scales which prevail in astronomy, the formulae bristle with enormous exponents, until the final result is re-expressed in astrophysically sensible units, and the numbers become reasonable once again. Indeed, it is one of the minor pleasures and surprises of a theorist's life, and a clue that he is on the right lines, when sensible numbers emerge at the end of a long and tiresome calculation studded with huge powers of ten. Sir Harold Jeffreys once remarked that 'incorrect geophysical hypotheses usually fail by extremely large margins' (Jeffreys 1929). The same is true in astrophysics.

To avoid this unpleasantness, it is much simpler to calculate entirely within the astrophysical system, *provided that you know the value of G in these units*. The following remarks will therefore save much time. Consider units such that masses are measured in $M_\odot$, speeds are measured in km/s, and distances are measured in parsecs (pc). Then $G = 1/232$ approximately. Also, the corresponding unit of time is 10^6 yr approximately.

In integrating the N-body equations numerically, still other units are used. Part of the reason for this is to avoid carrying around the values of physical constants, such as G, that would be needed in any physical system, partly to avoid the huge and unreadable numbers that would result from an inappropriate choice of physical units, but mainly so as not to obscure the intrinsic scalability of many simple N-body calculations.

In simulations it is natural to choose units such that $G = 1$, and, following Hénon (1971), the units of length and mass are conventionally chosen so that the total mass of the system is $M = 1$ and the total energy is $E = -1/4$. These choices might seem bizarre, but it then follows that, if the system is in virial equilibrium (Chapter 9), the potential energy is $W = -1/2$, and the virial radius is then $R_v = 1$. These units are referred to as N-body units, and are in widespread, but not quite universal, use. They become rather ambiguous when external fields or many binary stars are included in the computations.

In order to get a feel for the physical presence of a globular cluster, imagine that we live in the very core of a dense globular cluster (Fig. 1.2). The density of stars there can easily be as high as $10^6 M_\odot/\text{pc}^3$. This is a factor 10^6 higher than in our own neighbourhood, in the part of the galactic disk where the Sun happens to reside. Therefore, we can get an impression of the night sky by bringing each star that we normally see at night closer to us by a factor 10^2. Each star would thus become brighter by a factor 10^4, which corresponds to a difference of ten magnitudes (in the astronomical system where a factor ten corresponds to $\Delta m = 2.5$).

The brightest stars would thus appear at magnitudes at or above $m \sim -10$, comparable to that of the full Moon. They would be too bright to look at directly, because their size would be so much smaller than that of the Moon (they would still look point-like). It would be easy to read books by the light of the night sky. This is what the huge stellar density in these systems means from the 'human' point of view. Later on we shall consider what effect it has on the stars.

A recipe for a star cluster

Looking at Fig. 1.2, we cannot immediately tell how such an assembly of stars has got there, or what its fate will be. One might even guess, as Eddington (1926a) once did, that the stars cannot 'escape the fate of ultimately condensing into one confused mass'. Nowadays, though many details are unclear, we would argue that the clusters

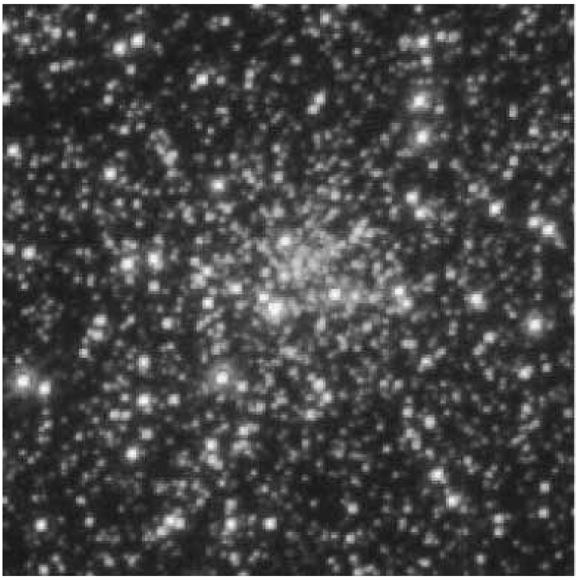

Fig. 1.2. The central region of the globular cluster M15. Image by courtesy of P. Guhathakurta, UC Santa Cruz.

have survived for almost as long as the universe itself, and that they are only now slowly dying off as their stars escape. Here we paint a few details on this picture, showing what we think are the stages through which a cluster would have to pass in order to arrive at something resembling those we see around us.

To make a star cluster, in practice, one should start out with a large gas cloud, which under the right conditions undergoes internal collapse on several length and time scales, to produce a large number of stars. This is a process which is being modelled with increasing sophistication (see, for example, the beautiful movie at www.ukaff.ac.uk/movies.shtml, or Klessen & Burkert 2001). Nevertheless, the physical processes related to star formation are very complex, and at present only partly understood. Therefore, let us perform a thought experiment in which we will cheat a bit. Let us take a bucket full of ready-made single stars, and sprinkle them into a limited region of space, in order to watch just the stellar dynamics between the stars at play.

The simplest way to place the stars in space is to start from rest. As soon as the stars are allowed to start moving, the whole system will begin to collapse under the mutual gravitational attraction of all the stars. If the stars are sprinkled in at random, the contraction will not proceed very far, since the individual star–star interactions will cause deflections from a purely radial infall. The stars will pass at some distance from the centre, each one at a somewhat different time, and then move out again. As a consequence, the whole system will breath a few times: globally shrinking and expanding. However, phase coherence will be lost very soon, and after a few breathing motions, the stars will be pretty well mixed. Some stars will be lost, spilled into the surrounding space, never to return, but most stars will remain bound to the system.

After those first few crossing times, not much will happen for quite a while. During a typical crossing time, a typical star will move through the system on a rosette-shaped orbit, while almost conserving its energy and angular momentum. On a longer time scale, though, the cumulative effect of many distant encounters, and occasional closer encounters, will affect the orbit. If we wait long enough, the memory of the original energy and angular momentum of the star will be lost. This time scale is called the relaxation time scale.

Even if the system was set off in a non-spherical distribution, such as that of a pancake, say, the original order will be erased after a few relaxation time scales, and the system will become spherical. There will also be a steady trickle of stars that escape from the system. In addition, energy will be transported from the centre to the outer regions, and as a result the inner regions tend to contract. Before long, the centre will gain in density, by many orders of magnitude, a phenomenon called core collapse.

After core collapse, double stars will be formed, which provide a source of energy for the system, empowering the ongoing evaporation of the system, until the whole star cluster is dissolved. While the star cluster slowly evaporates, the system will undergo core oscillations, with vast swings in central density, if the number of stars is large enough, well over 10^4. These so-called gravothermal oscillations are the result of the fact that the inner regions, having much shorter relaxation times than the rest of the system, grow impatient in the post-collapse phase, and start collapsing all over again. Interestingly, these oscillations exhibit mathematical chaos, one more example of the gravitational many-body problem providing a stage for some of the most modern forms of applied mathematics.

This general picture, of a star cluster being formed, coming to rest, undergoing a slow form of core 'collapse', followed by evaporation, will be spun out in much greater detail throughout the rest of the book. But before we take a closer look at the specific million-body problem, let us see how much the picture can change if we treat the stars as stars and not as point masses.

Dynamics of dense stellar systems

Most stars do not interact much with their environment, after they leave the cradle of the interstellar cloud they are born in. Some stars are born single, and stay that way throughout their life, although they may have acquired a planetary system during the late stages of their formation. However, most stars are members of a double star or an even more complex multiple system (triples, quadruples, etc.). In such a system, when two stars are sufficiently close, all kind of interesting interactions may take place, including the transfer of mass from one star to another, and even the spiral-in and eventual merging of two or more stars.

Although binary-star evolution is much more complex than single-star evolution, it can generally still be studied in isolation from its wider environment. After all, the typical separation between stars in the solar neighbourhood (itself typical for our Galaxy) is some hundred million times larger than the diameters of individual stars. The exception to this rule occurs in unusually dense stellar systems. Examples are star clusters, both in the disk of the galaxy and outside (the globular clusters), and the nuclei of galaxies.

Especially in the cores of globular clusters and in the nuclei of galaxies, the densities of stars can easily exceed that of the solar neighbourhood by a factor of a million or more (Fig. 1.2). At any given time, such a system is still dilute enough to make physical collisions unlikely, even during many crossing times, thus allowing point mass dynamics to provide useful first-order approximations. However, when viewed over a time scale of billions of years, such collisions become unavoidable in many of these systems.

In recent years much progress has been made in the study of physical collisions, both theoretically in computer simulations, and observationally by looking for 'star wrecks' as tell-tale signs of violent encounters. For example, observations with the Hubble Space Telescope have shown us the presence of so-called blue stragglers, right down in the centre of the most crowded star clusters. Blue stragglers are unusual types of stars that are at least compatible with being the products of stellar collision. Earlier, millisecond pulsars and X-ray binaries already have hinted to us more indirectly about the sagas of their formation and subsequent interactions.

In order to interpret this wealth of observational information, vigorous attempts are being made by several groups to make theoretical models for the evolution of dense stellar systems. The first step is to determine the long-time behaviour of a large system of point masses (a million or so), a classical problem that is still far from solved, and that has given rise to fascinating new insights, even over the last ten years, including the surprising discovery of the presence of mathematical chaos in the late stages of its evolution. It is this step that is the main theme for the present book.

The second step is to integrate our understanding of the dynamics of a system of point masses with the extra complexity introduced by the non-point-behaviour, in the form of stellar evolution, physical collisions between stars, mass loss, etc. This integration gives rise to an extremely complex picture of the ecology of star clusters, as we will briefly discuss in the last part of this book.

Why study the dynamics of globular clusters?

The dynamical study of globular clusters has had a peculiar history, which stretches over the entire twentieth century and beyond. Even up to the 1960s, however, many astrophysicists regarded it as a quiet backwater, even though it had attracted the attention of the best theorists of each generation: Jeans, Eddington, Chandrasekhar, Spitzer, Hénon, Lynden-Bell, Ostriker, ... Perhaps they were initially attracted to it because of what Lyman Spitzer called its 'appealing but deceptive simplicity'.

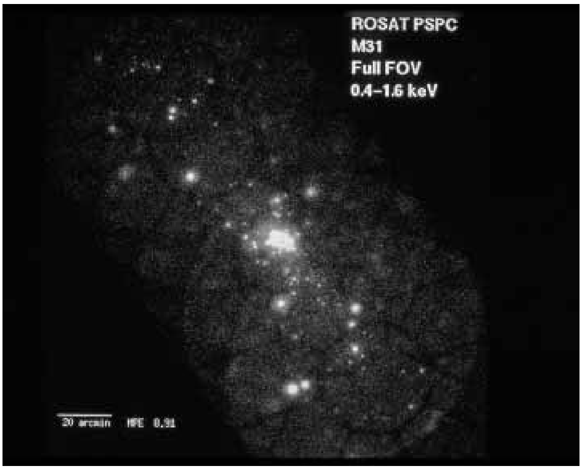

Fig. 1.3. A recent X-ray image of the Andromeda galaxy, M31. More than 5% of the sources lie in globular clusters belonging to Andromeda. Original colour image by courtesy of the ROSAT Mission (www.xray.mpe.mpg.de/) and the Max-Planck-Institut für extraterrestrische Physik (www.mpe.mpg.de/).

This dynamical study of the clusters went in parallel with the astrophysical study of the stars inside them, as these were seen to be excellent test beds for checking the theory of stellar evolution, the chemical history of the Galaxy, the age of the Universe, and so on. In fact there was hardly any major problem of astronomy – from star formation to cosmology – in which globular star clusters did not have something important to say, and this remains true to the present day.

Until the early 1970s these parallel tracks in the study of globular clusters – the dynamical and the astrophysical – proceeded entirely independently. The two communities had nothing in common and did not even need to talk to each other. By the 1990s all this had changed. What happened?

More than anything else, it was the discovery of variable X-ray sources in globular clusters in the 1970s which brought down the barriers between the two communities (Fig. 1.3). Though their origin is still not quite clear, it was soon realised that dynamical processes were important. Dynamicists also realised that here was a new process which could have a profound influence on their understanding of dynamical evolution. Suddenly the dynamics of globular clusters had rejoined the mainstream of astrophysics, and in the 1970s there was not much that seemed more glamorous than X-ray astronomy.

The integration of the dynamical and stellar studies of globular clusters has progressed steadily in the intervening decades, and there is even a new breed of 'cross-over' astrophysicists who are at home in both camps. This grand unification of cluster studies has been strengthened by a stream of exciting discoveries: binary stars (through the rapid explosion of data on radial velocities, and other techniques), millisecond pulsars, white dwarfs, blue stragglers in dense clusters, core collapse, and so on. They are ideal targets for searching for extrasolar planets, or should be (Gilliland *et al.* 2000).

The sheer visual beauty of globular clusters, and the richness of the problems they present, must not lead us to lose sight of their place in the bigger picture. In particular, they allow us to study the behaviour of stars in dense environments, and will prove to be an important stepping stone for the understanding of the even denser stellar systems in galactic nuclei – surely one of the major problems of astrophysics for the twenty-first century. Nor are we here restricting attention to *active* galactic nuclei. For instance, the nucleus of the quiescent nearby galaxy M33 can really be regarded as a gigantic globular cluster sitting at the centre of its parent galaxy, and may well have exhibited the kind of core collapse studied in the context of globular clusters (Hernquist *et al.* 1991).

Sidney van den Bergh (1980) once expressed this philosophy very nicely: ‘One of the dangers of living on a beautiful isle, such as Vancouver Island, is that it is very easy to become insular. By the same token it is all too easy for us, working in various exciting areas of cluster studies, to forget about the broader impact that such work might have on other areas of astronomy. The justification of our work on clusters is not just that it is fun but also that such investigations often help to illuminate other branches of science.’

Problems

(1) Compute the virial radius (Box 1.1) of a binary consisting of two stars of mass m_1 and m_2 in a circular relative orbit of radius a.

(2) Compute the speed of a star in a circular orbit at the median tidal radius (Table 1.1) around a cluster of median mass. Do this in astrophysical, physical and N-body units.

(3) Using the fact that the magnitude, m, and luminosity, L, of a source at a distance d are related by

$$m = -2.5 \log_{10} \frac{L}{L_\odot} + 5 \log_{10} \frac{d}{10\text{pc}} + M_\odot,$$

where $L_\odot$ and $M_\odot$ are the luminosity and absolute magnitude of the Sun, find the mean surface brightness (within the tidal radius) of the median globular cluster (using data from Table 1.1). Express your result in magnitudes per square arc second.