Theory of Motion from Hellenistic Time to the XX Century

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The idea of a *theory of motion* arises from the regularities observed in the motion of Stars and Planets on the Sky. It seemed therefore natural to conceive, since antiquity, regular motion as a uniform circular motion, eternally equal to itself, and to think that less regular motions were simply resolvable into combinations of uniform circular motions.

Even without using mathematical concepts it is possible to give a precise description of the very meaning of a "*motion composed by circular motions*".

Imagine a point at the extreme of a stick of length ℓ_1 which rotates uniformly with angular rotation speed ω_1 about a point O. At the extreme P of the stick is attached a second stick of length ℓ_2 and rotates at speed ω_2 about P, at its extreme a stick of length ℓ_3 rotates at speed ω_3 and so on. Then the motion of the endpoint of the last stick is a quite general motion "composed" by uniform circular motions.

The *angular velocities* $\omega_1, \omega_2, \ldots$ are called the *harmonics* and the motion is called *quasi periodic* with frequencies $\nu_1 = \omega_1/2\pi, \omega_2/2\pi, \ldots$. The circles on which the endpoints of the sticks rotate are called *epicycles*.

The entire body of ancient astronomy consists, as far as it has been legated to us, in imagining systems of epicycles and of angular velocities so arranged that the apparent (i.e. as seen by us on Earth) position of the celestial bodies is accurately represented. And accuracy reached by ancient astronomers is stunning even by today standards (as they could observe stars positions with an accuracy of the order of $1'$ of arc: which is an angle the rotation of the Sky covers in a time of the order of a second).

Planets are well described by up to 4 epicycles of suitable length rotating with a small number of basic angular velocities or by multiples of them.

The method to follow in order to represent motions is just to add, one after the other, as many epicycles as necessary. Not too many, in fact, for the eight planets (roughly 43).

Or *it should have been such*. The analysis is performed in the only fully extant treatise: the *Mathematical Syntaxis* (or *Almagest*) of Ptolemy. Copernicus was not the first to complain that Ptolemy had strayed off the path by using circular mo-

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tions rotating uniformly about an eccentric point, with the result that the motion seemed neither uniform nor representable by a combination of uniform motions. Worse the method for constructing the motions is not explained in general terms in the *Almagest*: and in the Renaissance observations had become so refined (and the equinox precession so important) that it was necessary to rebuild the Ephemerides of Hellenistic time. A quite hard task in absence of a general method to employ.

> Quoting Copernicus (*Commentariolus*): "*Nevertheless, what Ptolemy and several others legated to us about such questions, although mathematically acceptable, did not seem not to give rise to doubts and difficulties*" … "*So that such an explanation did not seem sufficiently complete nor sufficiently conform to a rational criterion*" … "*Having realized this, I often meditated whether, by chance, it would be possible to find a more rational system of circles with which it would be possible to explain every apparent diversity; circles, of course, moved on themselves with a uniform motion*".

Copernicus set out to correct this state and went back to the original proposition, dating at least as far back as Apollonius and traceable to Aristoteles and Plato, which gave the rigid prescription that motion should be a combination of uniform circular motions in the above sense. In this way he exposed a *general method* to build ephemerides: he did not achieve, however, a higher precision than Ptolemy nor was he able to represent the motion of the World with less circular motions (one can argue that he had a few more!). Nevertheless he did show a simple systematic method to interpolate the astronomical data, which was immediately adopted and opened the way to Galilei, Kepler, Newton and Laplace.

Before proceeding it is convenient to expose a few comments on the question *did Ptolemy really deviate from the path traced by the "fathers"*, followed since Apollonius and Hypparchus, replacing the beautiful circular motion with ugly new motions uniform around a point which is not the center of symmetry of the orbit? It is clear, or perhaps it should be clear, to whom read a few pages of Ptolemy that it is not likely that he had really deviated from the theory of circular motions: my feeling is that Ptolemy knew that the new motions that he was introducing were in fact also representable as above as accurately as wished: representing them as nonuniform was, I believe, a matter of convenience (just as referring the tables to the Earth rather than to the Sun). In other words the *Almagest* looks more like a commented *table* of ephemerides rather than as a book on Celestial Mechanics.

An analogy can be drawn from a reading of the modern "Bible" for the Ephemerides which is the *American Astronomical Almanac*: in spite of the great precision of the data and of the predictions it is very hard to see that behind the tables there is the theory of universal gravitation (this remains true even if one adds to the Almanac the equally ponderous *Explanatory supplement*).

Perhaps one could derive a theory on which were based the alleged ptolemaic *violations* of the principle of circular uniform motions and which Ptolemy did not describe in the *Almagest*.

Copernicus' idea of introducing epicycles upon epicycles, as many as needed for an accurate representation of the motion, is systematic and in modern language it coincides with the computation of the Fourier transform of the planetary coordinates with coefficients ordered by decreasing absolute value. Copernicus' work is strictly coherent with this principle, set in his early project quoted above. This freed astronomers and physicists from being bound to the "strange" constructions of Ptolemy: nonuniform and, more important, not based on any general systematic theory. And the new freedom arrived at a time in which astronomical observations were so much improving to pose serious challenges to a strict interpretation of the *Almagest*.

It opened the way to the universal principle of gravitation: universal in the sense that a single principle of astonishing simplicity allowed to construct faithful representations of the planetary motions even achieving unification of the latter with the earthly motion of freely falling bodies. But the new theory marked the triumph of the Hellenistic conception of motion (in spite of a somewhat widespread belief on the contrary). Laplace's *Mécanique Céleste* describes in detail the motion of Stars, Planets, Satellites (including the mysterious precession phenomena) in terms of motions which we call today *quasi periodic* and of which the motions of Hypparchus as well as of Ptolemy are a particular case.

But observations grew more and more precise and the mathematical analysis of Laplace needed improvements, i.e. more accurate computations. At the end of the XIX century Poincaré discovered that *one could not improve indefinitely the approximations* (although at the time the existent ones were still quite adequate and had recently led to major discoveries like the small star Ceres by Piazzi and Gauss (1801) and the new star Neptune by Le Verrier and Galle (1846)). The reason was quite simple: *there existed motions which could not be represented as combinations of circular motions*. The conceptual impact of this statement is simialr to that of the statement that the side and the diagonal of the square are not commensurate.

Therefore it is at the end of the XIX century that the Greek ideal of simplicity embodied in the uniform circular motion starts to be really challenged. Although one could still conceive, and many did, that the new motions would be quite exceptional and in a sense irrelevant for the classification of physical phenomena.

But chaotic motions, as we now call the new type of motions, were beginning to appear everywhere in Statistical Mechanics through he works of Boltzmann, Maxwell and others (although curiously sometimes they arose through the analysis of properties of circular motions, for instance in the case of Boltzmann). And also in mathematics through the work of Hopf, Birkhoff and others. But attention to them was diverted by the interest in the entirely new *Quantum Mechanics* as well as by the difficulty of even conceiving the chaotic motions, not to speak of subjecting them to a theory as well founded as the extremely refined theory of the ordered motions of Celestial Mechanics. It took more than half a century before they became really part of the cultural heritage of physicists.

In the 1950's Physicists were really confronted by the problem: Kolmogorov and Fermi-Pasta-Ulam discovered that regular motions might be frequent but at the same time they could coexist with the chaotic ones. In the 1960's the growing power of electronic computation and the further development of the ideas set down by Poincaré, Hopf and Birkhoff made easily visible and concrete the existence of chaotic motions. Together with the work of Lorenz and Ruelle-Takens this generated at the same time the hope that the motions could as well be rationally understood. The latter work gave a blow to the last surviving Ptolemaic theory: the theory of turbulence of Landau. Suddenly chaos became widely known, easily reproducible and the host of newly discovered "universal" phenomena (like the *period doubling* of Feigenbaum) and the subject of uncountably many experimental and theoretical works.

I want to conclude by giving an idea of what a chaotic motion is: imagine a mechanical system, like a double pendulum, or a rigid body rotating about a fixed point subject to an external force or a gas of 10^{23} atomic particles. We say that it is chaotic if one can define an observable quantity which, for simplicity, we take to assume only the value 0 or 1 and which has the following properties. Strting from a suitably chosen initial state and observing the evolution at fixed time intervals we see that it takes a sequence of values 0 or 1 which matches a *preselected* sequence of 0 and 1 obtained by the random tossing of a coin; and *viceversa* selecting randomly an initial state and observing the sequence of 0 and 1 that it produces one obtains a sequence that can be considered as obtained by random tossing of a coin. This is impossible if the observed motions are quasi periodic.

Chaotic motions are abundant in Nature: the axis of the Star Mars has an inclination that changes randomly if observed at time intervals of just a few million years (a short astronomical time) and changes by as much as 60°, sadly causing havoc in the season pace on that world. The motion of a ball on a billiard table with at least one pier is also chaotic if observed every few collisions between the ball and the obstacle. Chaos occurs very often in fluid motions: think of a fast flow from a pipe. The list can continue, becoming very long.

However one should be careful and avoid calling chaotic everything that looks irregular. After all at Hellenistic time it was believed that even the motion of the waves and of the air could be reduced to a combination of uniform circular motion. It is very instructive to write a computer program that simulates the motion of, say, 10 sticks attached by the extremes and rotating about them at pairwise incommensurate speeds: observing just the motion of the endpoint of the chain one could hardly be sure that the motion is not chaotic. There are therefore many tests of the chaoticity of a motion: and in the end it is a matter of precision and of time scales of observation the distinction between regular and chaotic, in spite of the fact that mathematically the distinction is very sharp. Again this reminds the phenomenon of incommensurability between the side and the diagonal of the square: although their ratio is mathematically irratinal in practice it can still be regarded as rational.

One can have a chaotic motion which looks for a very long time quasi periodic (for instance this is the case of the inclination of the axis of the star Mars) or motions that are very regular but which look chaotic over short time scales. If one thinks that the space time is discrete (of course we know that it *looks* continuum down to scales of the order at least of a billionth of the atomic scales so that discreteness will appear only on lower scales, if at all) than all confined motions will be *periodic* and therefore representable in terms of epicycles: however periodicity will manifest itself over time scales beyond imagination and we would not be interested in the phenomena that occur on such time scales. Not because of an Aesopian fox attitude but simply because they would be phenomena observable by no human being or by no successor of the human species living on any errant Star of the Universe.

References

For an extended version of this paper see:
G. Gallavotti: *Quasi periodic motions from Hypparcos to Kolmogorov*, Rendiconti Matematici dell'Accademia dei Lincei, **12**, 125–152, 2001. The paper can be found also on the 2001 page at http://ipparco.roma1.infn.it.

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