Preface

Although concepts that we now consider as part of topology had been expressed and used by mathematicians in the nineteenth century (in particular, by Riemann, Klein, and Poincaré), algebraic topology as a part of rigorous mathematics (i.e., with precise definitions and correct proofs) only began in 1900. At first, algebraic topology grew very slowly and did not attract many mathematicians; until 1920 its applications to other parts of mathematics were very scanty (and often shaky). This situation gradually changed with the introduction of more powerful algebraic tools, and Poincaré's vision of the fundamental role topology should play in all mathematical theories began to materialize. Since 1940, the growth of algebraic and differential topology and of its applications has been exponential and shows no sign of slackening.

I have tried in this book to describe the main events in that expansion prior to 1960. The choice of that terminal date does not correspond to any particular occurrence nor to an inflection in the development of the theory. However, on one hand, I wanted to limit the size of this book, which is already a large one; and, on the other hand, it is difficult to have a bird's eye view of an evolution that is still going on around us at an unabated pace. Twenty years from now it will be much easier to describe what happened between 1960 and 1980, and it will probably fill a book as large as this one.

There is one part of the history of algebraic and differential topology that I have not covered at all, namely, that which is called "low-dimensional topology." It was soon realized that some general tools could not give satisfactory results in spaces of dimension 4 at most, and, conversely, methods that were successful for those spaces did not extend to higher dimensions. I feel that a description of the discovery of the properties of these spaces deserves a book by itself, which I hope somebody will write soon.

The literature on algebraic and differential topology is very large, and to analyze each paper would have been unbearably boring. I have tried to focus the history on the emergence of ideas and methods opening new fields of research, and I have gone into some details on the work of the pioneers, even when their methods were later superseded by simpler and more powerful ones. As Hadamard once said, in mathematics simple ideas usually come last.

I assume that the reader is familiar with the elementary part of algebra and "general topology." Whenever I have had to mention striking applications of algebraic topology to other parts of mathematics, I have summarized the notions necessary to understand these applications.

CHAPTER II

The Build-Up of "Classical" Homology

§ 1. The Successors of Poincaré

It took about 30 years to construct a theory of homology applicable to curvilinear "polyhedra," embodying all the ideas of Poincaré and entirely rigorous. In a period in which the number of professional mathematicians was definitely on the increase, it is surprising that this new field of research at first attracted so few people. This is true even if one takes into account topological questions such as the theory of dimension or the theory of fixed points (see Part 2), which until 1920 were not directly linked to homology but attracted much more attention, owing to the spectacular use of simplicial methods by L.E.J. Brouwer (1881–1966). For many years Brouwer himself was completely isolated in Holland; in France, after Poincaré's death and until 1928 only Hadamard and Lebesgue were interested in these questions, but they did not use simplicial methods; Italian mathematicians do not seem to have been attracted at all to topology, nor the English until 1926. The progress in the build-up of homology is entirely due to (1) a handful of mathematicians in Germany, Austria–Hungary, and Denmark: P. Heegaard, M. Dehn (1878–1952), H. Tietze (1880–1964), E. Steinitz (1871–1928), and after 1920 H. Kneser (1898–1973), H. Künneth (1892–1974), W. Mayer (1887–1948), L. Vietoris (1891–), and H. Hopf (1894–1971); and (2) the three members of what may be called the "Princeton school": O. Veblen (1880–1960), J.W. Alexander (1888–1971), and S. Lefschetz (1884–1972).

The first treatise on this "classical" algebraic topology was Veblen's *Analysis Situs*, published in 1922 (but a preliminary version was given as "Colloquium lectures" in 1916); it was followed by the much more complete book *Topology* by Lefschetz (1930), the very popular *Lehrbuch der Topologie* of H. Seifert and W. Threllfall (1934), and the book by P. Alexandroff and H. Hopf entitled *Topologie* I (1935).*

* This was the first example of a projected treatise in several volumes, which stops with the first one; other conspicuous examples are the well-known books by Eilenberg–Steenrod [189] and Godement [208].

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§2. The Evolution of Basic Concepts and Problems

The emphasis Poincaré put on C^1-manifolds (or even analytic ones) was immediately abandoned by his successors. For them the closures of the *cells* of a triangulation are merely deduced by *homeomorphisms* from closures of bounded convex euclidean (rectilinear) polyhedra, so that all notions relative to triangulations are invariant under homeomorphisms. Furthermore, they generalized the notion of (curvilinear) "polyhedron" defined by Poincaré, and until 1925 they only considered the homology of what they called *complexes*. Unfortunately that word is given different meanings by the mathematicians who use it; for the sake of clarity we shall use a terminology that distinguishes these meanings, even if it does not coincide with the one used in the papers we describe. In §§2–5 of this chapter, a *triangulation* will only be defined for a *compact* space X: as with Poincaré, it will mean a *finite partition* T of X in cells of various dimensions, such that the frontier of a cell of T in X is the union of cells of T of strictly lower dimension. Each cell is given an *orientation*; but Poincaré's additional requirement that, for the maximal dimension p of the cells of T, each $(p-1)$-cell should be contained in the frontier of exactly two p-cells of T (see §4) is dropped.* The pair (X, T) (or, by abuse of language, X itself) will be called a *cell complex*; after §5 of this chapter, we shall say *finite cell complex*, since more general "cell complexes" will also be defined. The barycentric subdivision of Poincaré (chap. I, §3) naturally led to the introduction of *simplicial cell complexes*, where the cells of the triangulation T are (curvilinear) simplices and *each face* of a simplex of T *belongs to* T (and is not merely a *union* of simplices of T). This condition still leaves open the possibility that the intersection of the frontiers of two simplices of T of dimension k contains *more than one* simplex of T of dimension $k-1$.† To get the simplicial complexes obtained by barycentric subdivision that possibility must be excluded; it is easy to see‡ that this is equivalent to the condition that there exists a homeomorphism of X on a compact subset X′ of some $\mathbf{R}^N$ of

* In their first paper [21], Alexander and Veblen impose the condition that in a cell complex where p is the maximal dimension of the cells, every cell of dimension $q < p$ is contained in the frontier of at least one cell of dimension $q + 1$; this was later dropped.

† As an example, consider the usual description of the two-dimensional torus $\mathbf{T}^2$ as obtained by identification of opposite sides of a rectangle, and decompose the rectangle into two triangles by the diagonal.

‡ This is proved in [421], p. 46. If X_n is the union of all simplices of T of dimension $\leqslant n$ (later called the *n-skeleton* of X), the homeomorphism $X \to X'$ is defined by induction on the X_n. The set X_0 consists of the vertices of the simplices of T; each vertex is mapped onto a unit vector of the natural basis of $\mathbf{R}^N$, where N is the number of vertices. The extension of a homeomorphism of X_n onto X'_n to a homeomorphism of X_{n+1} onto X'_{n+1} is then reduced to the case in which X_n and X'_n are the frontiers of two simplices of dimension $n + 1$, in which case the extension is immediate by means of barycentric coordinates.

large dimension such that T is sent by that homeomorphism onto a triangulation T′ of X′ consisting of *rectilinear* simplices such that the faces of each simplex of T′ belong to T′. This is the definition of a *simplicial complex* (X, T) chosen by Lefschetz [304] and which we shall adopt (after § 5 of this chapter we will say *finite simplicial complex*); the simplicial complexes such as (X′, T′) will be called *euclidean simplicial complexes*, and in most questions we may only consider euclidean simplicial complexes; this has the advantage of avoiding all difficulties linked to the intersections of manifolds. Of course, barycentric subdivisions of euclidean simplicial complexes are also taken rectilinear.*

This enlarged concept of triangulation of course changes nothing in Poincaré's definition of the Betti numbers and torsion coefficients of the triangulation, nor in the algorithm for their computation. In fact, that algorithm is so obviously of an algebraic nature and uses so little of topology that, in the very first paper on topology published after Poincaré's *Compléments*, the *Enzyklopädie* article of Dehn and Heegaard [138], there is already an attempt to define "homology" in a purely algebraic context, where the "cells" are elements of finite sets without any topological properties, with an *ad hoc* system of axioms. This axiom system was slightly improved by Steinitz in 1908 [456], but he did not go beyond a notion of "orientation" within this context, and it was only Weyl in 1923 [484] who consistently pursued this idea and built up an algebraic "homology" theory; his axioms, however, like those of Steinitz, were so narrowly tailored to mimic the topological situation that they did not seem applicable to very different topological problems or to algebraic ones.

Weyl had already considered, in addition to Poincaré's incidence matrices, the **Z**-*modules* C_j having as bases the sets of oriented j-cells. In 1925 H. Hopf, at the beginning of his career, spent a year at Göttingen; E. Noether, who then was engaged in the process of liberating linear algebra from matrices and determinants, observed that the boundaries of j-chains defined a *homomorphism* of **Z**-modules

$$\mathbf{b}_j\colon C_j \to C_{j-1} \tag{1}$$

such that

$$\mathbf{b}_{j-1} \circ \mathbf{b}_j = 0, \tag{2}$$

and that the consideration of Betti numbers and torsion coefficients amounted to that of the **Z**-modules

$$H_j = \operatorname{Ker} \mathbf{b}_j / \operatorname{Im} \mathbf{b}_{j+1}; \tag{3}$$

Hopf accordingly used these *homology modules* when writing his 1928 paper

* The orientation of a simplex may be defined by choosing an *order* among its vertices; two orderings give the same orientation if they are deduced from one another by an even permutation.

on the Lefschetz trace formula (Part 2, chap. III, § 2). Independently, in 1926, Vietoris also needed to get rid of matrices in order to define homology for more general spaces than simplicial complexes (see below, chap. IV, § 2), and he used the definition (3) of homology groups for a simplicial complex, without relating it to general notions of linear algebra [475].

This seemingly innocuous modification was to have important consequences, both for the ulterior development of algebraic topology and later for algebra itself (see chap. IV), since it was clear that the definition of homology modules could at once be extended to *arbitrary* (finite or infinite) sequences $C_\bullet = (C_j)_{j \geqslant 0}$ of modules over any ring, and to module homomorphisms $\mathbf{b}_j$ ($j \geqslant 1$) satisfying (2) (when $\mathbf{b}_0$ is taken by convention to be the unique homomorphism $C_0 \to \{0\}$). We shall say that such a system $(C_j, \mathbf{b}_j)$ is a *chain complex*; Mayer, in 1929 [336], was apparently the first to consider such systems, with the additional restriction that the C_j are *free* modules with *finite* bases; he calls them "complexes."*

In particular, he considered the following situation, suggested to him by Vietoris: each C_j has a basis, union of two subsets B_j^1, B_j^2 such that, if C_j^1, C_j^2, and C_j^3 are the $\mathbf{Z}$-modules having as bases B_j^1, B_j^2, and $B_j^1 \cap B_j^2$, respectively, the sequences (C_j^1), (C_j^2), and (C_j^3) are again differential graded modules for the restrictions of the homomorphisms $\mathbf{b}_j$. Mayer looked for a relation between the homology modules H_j^1, H_j^2, H_j^3, and H_j of (C_j^1), (C_j^2), (C_j^3), and (C_j), respectively; he proved that $H_j = E_j \oplus G_{j-1}$, where $G_j \subset H_j^3$ consists of the classes of cycles that are boundaries both in C_j^1 and C_j^2, and E_j consists of the classes of the sums of a cycle of C_j^1 and a cycle of C_j^2. In 1930 Vietoris [476] completed Mayer's result and showed that

$$E_j \simeq (H_j^1 \oplus H_j^2)/(H_j^3/G_j). \tag{4}$$

These results, later incorporated into what became known as the Mayer–Vietoris exact sequence (chap. IV, § 6,B), were to have many applications in algebraic topology.†

The first example of a chain complex different from the classical modules of "chains" of a triangulation was linked to a more abstract conception of those chains, which appeared simultaneously around 1926 in papers by Alexander [14], Alexandroff [22], and M.H.A. Newman [356] and was characterized by van der Waerden [477] as "pure combinatorial topology."

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* We shall also use the notation $C_\bullet$ for the *direct sum* $\bigoplus_{j \geqslant 0} C_j$ (in modern terminology, this is a *differential graded module*) when no confusion can arise. For rings of coefficients which are principal ideal rings, it is equivalent to saying that each C_j is free or that their direct sum is free; we will also say in that case that the *chain complex* $C_\bullet = (C_j)$ is *free*. More special "abstract" free chain complexes, mimicking the simplicial complexes, were introduced by Tucker, and used by the American school around 1940, in particular for the definition of cohomology.

† Mayer himself gave an application of his results to the usual torus $\mathbf{T}^2$ considered as union of two cylinders, their intersection being also the union of two disjoint cylinders ([336], p. 41).

The underlying idea is that rectilinear euclidean simplices and their orientation are entirely determined by the sequence of their *vertices*. Disregarding anything else, we shall therefore define a *combinatorial complex* as a set V equipped with a (finite or infinite) set $\mathfrak{S}$ of *finite subsets*, the *combinatorial simplices*, submitted only to the restriction that if $S \in \mathfrak{S}$ and $S' \subset S$, then also $S' \in \mathfrak{S}$; the dimension, faces, and orientation of these combinatorial simplices are defined in an obvious way. The module C_j of *alternating j-chains* of the combinatorial complex is then the set of finite linear combinations with integral coefficients

$$\sum_i x^i(a_i^0, a_i^1, \ldots, a_i^j), \tag{5}$$

where $x^i \in \mathbf{Z}$, and $(a_i^0, a_i^1, \ldots, a_i^j)$ is the sequence (in an arbitrary order) of the *distinct* vertices of a j-dimensional simplex, with the identification

$$(a_i^{\pi(0)}, a_i^{\pi(1)}, \ldots, a_i^{\pi(j)}) = \operatorname{sgn}(\pi)(a_i^0, a_i^1, \ldots, a_i^j) \tag{6}$$

for any permutation π of $\{0, 1, \ldots, j\}$. The boundary operator (1) is then defined by*

$$\mathbf{b}_j \cdot (a_i^0, a_i^1, \ldots, a_i^j) = \sum_{k=0}^{j} (-1)^k (a_i^0, \ldots, \widehat{a_i^k}, \ldots, a_i^j) \tag{7}$$

and makes (C_j) into a *chain complex*, the homology of which is, by definition, the homology of the combinatorial complex $(X, \mathfrak{S})$. Another equivalent definition of C_j consists in choosing a total order on X, and considering only in (5) the sequences such that $a_i^0 < a_i^1 < \cdots < a_i^j$ for this order; this shows that C_j is a *free* $\mathbf{Z}$-module.

To each euclidean simplicial complex (X, T) is thus associated a *finite* combinatorial complex $(V, \mathfrak{S})$, where V is the finite set of *all* vertices of *all* simplices of T, and $\mathfrak{S}$ is the subset of $\mathfrak{P}(V)$ consisting of the sets of vertices of all simplices of T. It is clear that there is an isomorphism of the chain complex of (X, T) onto the chain complex of $(V, \mathfrak{S})$, commuting with the boundary operators, and therefore giving a natural isomorphism of the homology of (X, T) onto the homology of $(V, \mathfrak{S})$. Conversely, it is easily shown ([308], p. 97) that for each *finite* combinatorial complex, there exist euclidean simplicial complexes to which it is associated; they are called the *realizations* of the combinatorial complex, and it can be proved that any two realizations of the same combinatorial complex are homeomorphic.

It is possible to define for a *combinatorial complex* $K = (V, \mathfrak{S})$ a notion that reduces to the classical "barycentric subdivision" for simplicial complexes: the *first derived complex* K′ of K is a combinatorial complex, where the set of vertices is the set $\mathfrak{S}$ of *combinatorial simplices* of K: a combinatorial p-simplex of K′ is a set $S_1 \subset S_2 \subset \cdots \subset S_{p+1}$ of $p + 1$ distinct simplices of K, *totally*

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* Eilenberg and Mac Lane introduced the convention that a "hat" above a letter means that this letter should be omitted in the sequence in which it is inserted.

ordered by inclusion, the dimensions of which form an arbitrary strictly increasing sequence ([308], p. 164).

Another chain complex emerged with the consideration of "singular simplices," which we will introduce in §3. In a combinatorial complex K, and with the same notations as above, the module C'_j of j-chains of this chain complex consists again of the linear combinations (5), but in which this time a_i^0, a_i^1, ..., a_i^j are vertices of a combinatorial simplex $S \in \mathfrak{S}$ but are *not necessarily distinct*; such sequences $(a_i^0, \ldots, a_i^j)$ with repetitions are called *degenerate simplices* of K. The identification (6) is not applied to degenerate simplices: if the boundary of a degenerate j-simplex is again defined by (7), the right-hand side is a combination of degenerate $(j-1)$-simplices.

It is clear that there is a natural injection $h: C_j \to C'_j$, and a retraction $r: C'_j \to C_j$ obtained by replacing the coefficients of the degenerate simplices by 0; both mappings commute with the boundary operators, and therefore yield homomorphisms $H_j \to H'_j$ and $H'_j \to H_j$ for the homology modules, but it is not immediately obvious that these homomorphisms are *bijective*. This was taken for granted by both Alexander [9] and Lefschetz [304] and the proof was only provided in 1938 by Tucker [471], who showed that if a chain (5) consisting only of *degenerate* simplices is a *cycle*, it is also a *boundary*. The use of the chain complex (C'_j) by these authors was never very explicit; with the work of Eilenberg on singular homology (chap. IV, §2) it gave way to a much less hybrid type, namely, the chain complex (C''_j), where the j-chains are simply the linear combinations of *all* sequences $(a_i^0, a_i^1, \ldots, a_i^j)$ consisting of vertices of the same simplex (distinct or not), but *no* identification is made; the boundary operator is still given by (7). There is a natural surjection $C''_j \to C_j$, the kernel of which is generated by the degenerate simplices and the differences

$$(a_i^{\pi(0)}, a_i^{\pi(1)}, \ldots, a_i^{\pi(j)}) - \operatorname{sgn}(\pi)(a_i^{(0)}, a_i^{(1)}, \ldots, a_i^{(j)}).$$

The elements of C''_j are the *ordered j-chains* of the combinatorial complex; the proof that the homology of (C''_j) is naturally isomorphic to that of (C_j) was initially made by using a homotopy operator, and is an easy consequence of the method of acyclic models (chap. IV, §5,G).

Another novelty in homology was introduced by Tietze [466]* and taken up by Alexander and Veblen [21], the *homology modulo* 2, where the coefficients of the cells in a chain are integers mod 2. This dispenses altogether with any consideration of orientation of the cells, and the "incidence matrices" now have coefficients in the field $\mathbf{F}_2$ of two elements, hence are equivalent to matrices (ρ_{ij}) with $\rho_{ij} = 0$ if $i \neq j$ and $\rho_{ii} = 0$ or 1 (or, equivalently, the homology modules are now vector spaces over $\mathbf{F}_2$). This does not give new topological invariants, since the dimension of that vector space for dimension p is

* This seems to be the first paper that questions the validity of Poincaré's arguments, and points to pathologies in the theory of differential manifolds ([466], pp. 32, 36, and 41)

the sum of the p-dimensional Betti number and of the number of torsion coefficients in dimensions p and $p-1$ which are not divisible by 2; it is now possible to generalize the duality theorem for *nonorientable* n-dimensional compact connected triangulated manifolds, that expresses the isomorphism of the p-dimensional and the $(n-p)$-dimensional homology vector spaces over $\mathbf{F}_2$. Later Alexander considered more generally "homology modulo m" for any integer m [14], and Lefschetz realized that the "homology with division" of Poincaré was simply "rational homology" with coefficients in the field $\mathbf{Q}$ ([302], p. 234), but this still did not yield any invariant not expressible by the known ones.

These attempts testify to the persistence of the search for a system of numerical or algebraic invariants of a topological space that would entirely characterize it up to homeomorphism, on the model of what Jordan and von Dyck had succeeded in doing (with insufficient proofs) for surfaces ([373], p. 139); we saw in chapter I that the introduction by Poincaré of homology and of the fundamental group was certainly motivated in part by this search. But even for dimension 3, where the Poincaré conjecture remained undecided, it was soon realized that the fundamental group was not sufficient to determine an orientable manifold up to homeomorphism. This followed from the study of a remarkable family of three-dimensional, compact, connected orientable manifolds, first defined by Tietze in 1908, and now called the *lens spaces* ([466], § 20). For an odd prime p and an integer q such that $0 \leqslant q \leqslant p-1$, the lens space $\mathrm{L}(p, q)$ is defined by Tietze as the quotient space $\mathbf{D}_3/\mathrm{R}$, where $\mathbf{D}_3$ is the ball $|x| \leqslant 1$ in $\mathbf{R}^3$, and R is the equivalence relation whose classes consist of the one-element sets $\{x\}$ for $|x| < 1$, and of the orbits of the cyclic group $\mathbf{Z}/p\mathbf{Z}$, acting on the sphere $\mathbf{S}_2$: $|x| = 1$ by the action

$$(k,(\varphi,\theta))\mapsto\left(\varphi+\frac{2kq\pi}{p},(-1)^k\theta\right) \tag{8}$$

φ and θ being the usual longitude and latitude. Later another equivalent definition of $\mathrm{L}(p,q)$ was formulated as the space of orbits of the group $\mathbf{Z}/p\mathbf{Z}$ acting on the sphere $\mathbf{S}_3$: this sphere is considered to be the manifold $|z_1|^2 + |z_2|^2 = 1$ in the space $\mathbf{C}^2$, and the action is

$$(k,(z_1,z_2))\mapsto(\omega^k z_1,\omega^{kq}z_2) \tag{9}$$

with $\omega = e^{2\pi i/p}$. The fundamental group of $\mathrm{L}(p,q)$ is $\mathbf{Z}/p\mathbf{Z}$, and the homology modules are $\mathrm{H}_1 = \mathbf{Z}/p\mathbf{Z}$, $\mathrm{H}_2 = 0$, so that the value of q is irrelevant; nevertheless, Tietze suspected (but could not prove) that, for instance, L(5, 1) and L(5, 2) are not homeomorphic. This was proved in 1919 by Alexander [10], using a construction of L(5, 1) and L(5, 2) different from that of Tietze, whose paper was not mentioned;* another proof was provided by de Rham in 1931 (see Part 2, chap. VI, § 3,A), using the notion of linking coefficient (Part 2, chap. I, § 3).

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* See [421], p. 216.

More urgent than this ultimate and more and more elusive goal* was the immediate necessity to prove conclusively that the homology modules defined by two different triangulations of the same compact, connected space X are isomorphic (the *invariance problem*), which would show that these homology modules only depend on the homeomorphism class of the triangulable space. A "natural" method would have been to show that, for two triangulations T, T′ of X, there existed two suitable subdivisions of T and of T′ that could be deduced from one another by a homeomorphism of X; this was given the name *Hauptvermutung* in algebraic topology by H. Kneser [274], but for a long time it could only be proved for complexes of dimension 2, and remained undecided for higher dimensions. It was finally shown much later [349] that the "Hauptvermutung" is true for dimension 3, but counterexamples exist for dimension $\geqslant 5$. The invariance property must therefore be proved by independent means.

During that period the concepts of deformation, homotopy, and isotopy finally acquired a precise meaning. The words *homotopy* and *isotopy* were coined by Dehn and Heegaard in their *Enzyklopädie* article with a purely combinatorial definition adapted to their "abstract" conception of homology ([138], pp. 205–207), and they were not retained by later workers, with the exception of Steinitz.† Brouwer seems to have been the first to give our present definition of homotopy ([89], p. 462): two continuous mappings $f\colon X \to Y$, $g\colon X \to Y$ are homotopic if there exists a continuous mapping $F\colon X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ in X.

The final touches to the homology theory of cell complexes were brought about by the theory of intersections (§ 4), the introduction of product spaces (§ 5), and, finally, the concept of relative homology (§ 6). Around 1930 algebraic topology was ready for further extensions and new concepts.

§ 3. The Invariance Problem

There are two proofs of the independence of homology from the triangulation of a simplicial complex. Both are essentially due to Alexander ([9] and [14]);‡ they both use the new ideas of *simplicial mapping* and *simplicial approximation*, and the first one is also based on a new concept, the *singular chains*.

Simplicial mappings are a natural extension to *n* dimensions of the classical

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* It has finally been proved by A.A. Markov [332] that there *cannot exist* any algorithm (in the sense of the theory of recursive functions) that would allow one to determine if two euclidean simplicial complexes X, Y of dimension $\geqslant 4$ are homeomorphic or not. He considers the fundamental groups $\pi_1(X)$, $\pi_1(Y)$; these groups may be *any* group finitely generated and finitely presented, and the algorithm would enable one to decide if two such groups are isomorphic or not. But it is known that no such algorithm exists.

† Steinitz only uses the Dehn–Heegaard notion of "homotopy" to introduce an abstract notion of "orientation"; Dehn and Heegaard themselves do not seem to have used it at all for questions of homology.

‡ For a third, indirect, proof by Alexander, see § 6.

notion of *piecewise linear* function of a real variable. Let X be a euclidean simplicial complex; a *simplicial mapping* f of X into a euclidean simplicial complex Y is a continuous mapping such that for *all p*, and any p-simplex S of X, $f(S)$ is *contained* in a q-simplex of Y for a number $q \leqslant p$ and the restriction of f to S is *affine*. This restriction is therefore entirely determined by the values of f at the vertices of S, which must be vertices of a q-simplex of Y for a number $q \leqslant p$, *not necessarily distinct*, but otherwise entirely arbitrary.

For *combinatorial complexes* X, Y (§ 2) a *simplicial mapping* will be a map $f\colon X \to Y$ such that for *all p* and any p-simplex S of X (which are finite subsets of X here), $f(S)$ is contained in a q-simplex of Y for some $q \leqslant p$. Thus there is a one-to-one correspondence between the simplicial mappings of a euclidean simplicial complex X into a euclidean simplicial complex Y, and the simplicial mappings of the combinatorial complex associated to X into the combinatorial complex associated to Y. By linearity, if $(C'_j(X))$ and $(C'_j(Y))$ are the chain complexes of ordinary *and degenerate* simplices in X and Y, one deduces for each j, from a simplicial mapping $f\colon X \to Y$, a homomorphism $\tilde{f}_j\colon C'_j(X) \to C'_j(Y)$, by the formula

$$\tilde{f}_j((a_0, a_1, \ldots, a_j)) = (f(a_0), f(a_1), \ldots, f(a_j)) \tag{10}$$

for each (ordinary or degenerate) j-simplex $(a_0, a_1, \ldots, a_j)$. If $g\colon Y \to Z$ is a second simplicial mapping and $h = g \circ f\colon X \to Z$, h is also a simplicial mapping and $\tilde{h} = \tilde{g} \circ \tilde{f}$. Furthermore, it can easily be shown that for the boundary operators

$$\mathbf{b}_j \circ \tilde{f}_j = \tilde{f}_{j-1} \circ \mathbf{b}_j; \tag{11}$$

hence $\tilde{f}_{\bullet} = (\tilde{f}_j)$ is a homomorphism of *chain complexes*, which yields a homomorphism $f_*\colon (H'_j(X)) \to (H'_j(Y))$ of graded homology modules. If it is composed with the natural homomorphisms $(H_j(X)) \to (H'_j(X))$ and $(H'_j(Y)) \to (H_j(Y))$ (§ 2), this also gives a homomorphism $(H_j(X)) \to (H_j(Y))$, which topologists identified with f_* even before it had been proved that the natural maps $H_j \to H'_j$ are isomorphisms. The homomorphism $f_*\colon (H_j(X)) \to (H_j(Y))$ can also be defined directly, if the definition (10) is modified by taking $\tilde{f}_j((a_0, a_1, \ldots, a_j)) = 0$ when the simplex $(f(a_0), f(a_1), \ldots, f(a_j))$ is degenerate. With the same notations as above, $(g \circ f)_* = g_* \circ f_*$.

The idea of simplicial approximation is due to Brouwer ([89], p. 459). He considered two euclidean simplicial complexes X, Y (satisfying some additional conditions that we disregard) and a continuous map $f\colon X \to Y$ such that for *any* simplex σ of X $f(\sigma)$ is contained in a simplex of Y. Then for any $\varepsilon > 0$ there is a triangulation T′ of X obtained by repeated barycentric subdivisions of the given one T, and a map $g\colon X \to Y$ that coincides with f at the vertices of the new triangulation, is such that $|f(x) - g(x)| \leqslant \varepsilon$ for all $x \in X$ and is an *affine* map in every simplex of T′. We shall return in Part 2, chaps. I, II, and III to describe the way he used this result with great virtuosity to prove his famous theorems without linking them to homology.

In his first proof Alexander realized that he could extend Brouwer's method to *all* euclidean simplicial complexes X, Y and to an *arbitrary* continuous map $f: X \to Y$.* The stars of the triangulation of Y form an open covering of Y; replacing the triangulation of X by one obtained by repeated barycentric subdivisions, it can be assumed that for the new triangulation the image of any *star* of X is contained in one of the *stars* of Y. Thus, if $(a_k)_{1 \leqslant k \leqslant N}$ are the vertices of the new triangulation of X, and $(b_j)_{1 \leqslant j \leqslant N'}$ are the vertices of the triangulation of Y, let $b_{\varphi(k)}$ be one of the vertices of Y whose star contains the image of the star of a_k. If $j + 1$ vertices a_k are the distinct vertices of a j-simplex of X, the $b_{\varphi(k)}$ are (not necessarily distinct) vertices of a q-simplex of Y with $q \leqslant j$. If a simplicial mapping g is defined by $g(a_k) = b_{\varphi(k)}$ for all k, g is a simplicial approximation of f, with $|f(x) - g(x)| \leqslant 3\delta$, where δ is the maximum diameter of the simplices of Y. Furthermore, for any $x \in X$, $f(x)$ and $g(x)$ are the extremities of a segment contained in Y, and therefore f and g are *homotopic*.

The notion of *singular chain* also arose from the need to consider continuous maps $f: X \to Y$ between euclidean simplicial complexes, both having arbitrary dimensions. It was first mentioned by Dehn–Heegaard in their *Enzyklopädie* article [138]; they of course realized that phenomena such as the Peano curve implied that the image $f(E)$ of a cell E may exhibit the weirdest pathology, so they included in their conception not only the image $f(E)$, but also the cell E itself in rather vague terms;† they do not seem to have made any use of it to prove anything.

In [9] Alexander had the idea‡ that the singular simplices might be used to define new kinds of chains by linear combination, and be provided with boundary operators with which one could define new homology modules that *ipso facto* would be independent of any triangulation; the invariance problem would then be solved if he could define isomorphisms of these modules on the homology modules of an arbitrary triangulation. At least this is what we may guess from the context of his paper, for his definition of singular cells is simply translated from Dehn–Heegaard. He never said when two images of different p-cells by two continuous mappings should be identified, nor what the boundary of a singular cell should be. This vagueness was only partly improved in the successive versions of Alexander's proof given (this time for cell complexes) by Veblen ([474], p. 102), van der Waerden [477], and Lefschetz ([304], chap. II); it was only in a short note published in 1933 that Lefschetz, "to clear up misconceptions," defined a singular cell on a space X [305]: he considers pairs (e_p, f), where e_p is a p-dimensional oriented convex polyhedron in some $\mathbf{R}^N$, and $f: \bar{e}_p \to X$ is a continuous mapping; singular p-cells are classes of such

* Although Alexander did not mention any paper on algebraic topology with the exceptions of Poincaré's and his own joint paper with Veblen [21], it is quite certain that he knew Brouwer's work, for it is quoted in a 1913 paper by Veblen.

† "Wir nennen C'_n, aufgefasst als das Abbild eines bestimmten C_n, *einen n-dimensionalen Komplex mit Singularitäten*, ..., und geben ihm die Bezeichnung $C'_n(C_n)$" (p. 164).

‡ Alexander only considered homology mod 2 on manifolds.

pairs for the equivalence relation $(e_p, f) \equiv (e'_p, f')$, where $f' = f \circ u$ and u is an *affine bijection* $u: e'_p \to e_p$.* If $\mathbf{b}_p$ denotes the boundary map for euclidean polyhedra, then the equivalence

$$(\mathbf{b}_p e_p, f | \mathbf{b}_p e_p) \equiv (\mathbf{b}_p e'_p, f' | \mathbf{b}_p e'_p)$$

holds when the equivalence relation is extended to "singular chains," linear combinations of singular cells;† this clearly defines a boundary operator for these new "chains," and from these data one deduces by the standard method, applicable to all chain complexes, homology modules that this time obviously only depend on the space X up to homeomorphism. To make things precise, we shall attach the qualification "topological" or the index "top" to the notions entering in the homology of singular chains. For any continuous mapping g: X → Y the image by $\tilde{g}$ of a singular cell on X is defined by

$$\tilde{g}(e_p, f) = (e_p, g \circ f) \tag{12}$$

and therefore this can be extended by linearity to a homomorphism $\tilde{g}_p$: $C_p^{top}(X) \to C_p^{top}(Y)$ of singular chains, permuting with boundary operators and yielding a homomorphism g_*: $(H_j^{top}(X)) \to (H_j^{top}(Y))$ of graded homology modules with the relation $(g_1 \circ g_2)_* = g_{1*} \circ g_{2*}$ for two continuous mappings.

Granted this clarification, Alexander's method may be stated as follows: For a triangulation T of a euclidean simplicial complex X, there is a homomorphism

$$H_j \to H_j^{top} \tag{13}$$

from the homology defined by chains of T to the homology of singular chains, defined in a natural way: each p-simplex E_p of the triangulation T is identified to the singular p-simplex (E_p, Id.), and its boundary with the (singular) boundary of that singular simplex. What has to be shown is that (13) is *bijective*, or equivalently that: (A) every topological p-cycle w_p is topologically homologous to a p-cycle of T; (B) every p-cycle z_p of T that is a topological boundary is also a boundary of T.

Some preliminary results are needed. First is the fact that the homology of T is naturally identified with the homology of any triangulation T′ deduced from T by barycentric subdivision. We have seen in chap. I that Poincaré had already given a substantially correct proof of that result, and others were proposed by Tietze ([466], p. 42), Alexander himself ([9], p. 153), Veblen ([474], p. 90), and Lefschetz ([304], p. 68). This invariance by subdivision is immediately extended to the homology of singular chains,‡ and has as a consequence the fact that in the proof of A (resp. B) the singular chain w_p [resp.

* This allows one to take all the e_p equal to the same simplex, which will be done later (chap. IV, § 2).

† In addition, Lefschetz imposed the relation $(-e_p, f) = -(e_p, f)$, where $-e_p$ is the simplex e_p with opposite orientation.

‡ A subdivision of a singular cell (e_p, f) consists of the singular simplices $(e_p^i, f | e_p^i)$, where (e_p^i) is the family of p-simplices of a subdivision of e_p.

the singular $(p + 1)$-chain of which z_p is the topological boundary] may consist of singular simplices whose images in X are *arbitrarily small.*

The second result that emerges from Veblen's invariance proof ([474], p. 102) and more clearly from Lefschetz's, is, at last, a correct statement and proof of the invariance of homology under homotopy (the irrelevance of "deformation," so long taken for granted, as we have seen in chap. I). Writing $S_p(\mathbf{R}^N)$ the $\mathbf{Z}$-module of the simplicial p-chains in $\mathbf{R}^N$, linear combinations with integral coefficients of the oriented euclidean simplices in $\mathbf{R}^N$, start with an elementary simplicial subdivision* of a product $\Delta_p \times I$ in $\mathbf{R}^{N+1}$ (a "prism"), where Δ_p is a p-dimensional euclidean simplex in $\mathbf{R}^N$ and $I =]0, 1[$. With suitable orientations, we obtain the relation between p-chains in $\mathbf{R}^{N+1}$

$$\mathbf{b}_{p+1}(\Delta_p \times I) = \Delta_p \times \{1\} - \Delta_p \times \{0\} + (\mathbf{b}_p(\Delta_p) \times I) \tag{14}$$

and by linearity this gives in the $\mathbf{Z}$-module $S_p(\mathbf{R}^{N+1})$ the relation

$$\mathbf{b}_{p+1}(P_p(z_p)) = z_p \times \{1\} - z_p \times \{0\} + P_{p-1}(\mathbf{b}_p z_p), \tag{15}$$

where, for each integer q, $z_q \mapsto P_q(z_q)$ is the linear map of $S_q(\mathbf{R}^N)$ into $S_{q+1}(\mathbf{R}^{N+1})$ that coincides with the map $\Delta_q \mapsto \Delta_q \times I$ on each q-simplex Δ_q. From (15), by applying to both sides the homomorphism deduced from a continuous map $F: X \times I \to X$ as shown above, where X is a euclidean simplicial complex, this immediately gives the first example of a *homotopy formula* for singular p-chains z_p in X (cf. chap. IV, § 5,F):

$$\tilde{f}(z_p) - \tilde{g}(z_p) = \mathbf{b}_{p+1}(\tilde{F}(z_p \times I)) - \tilde{F}((\mathbf{b}_p z_p \times I)), \tag{16}$$

where $f(x) = F(x, 1)$, $g(x) = F(x, 0)$, from which it follows at once that if z_p is a singular *p-cycle*, $\tilde{f}(z_p)$ and $\tilde{g}(z_p)$ are topologically homologous.

To prove A, after subdividing the singular simplices of w_p in order to be able to apply the Alexander construction of simplicial approximations described above, one shows that there exists a *homotopy* of w_p on another singular chain w'_p, whose singular simplices (e_p, g) are such that g is an *affine* map of e_p into a p-simplex of T sending vertices of e_p into vertices of that p-simplex. This would clinch the matter, except that the affine map g is not necessarily bijective.

This difficulty was ignored by Alexander and van der Waerden; Veblen's proof is very obscure and he does not seem to have distinguished, for cycles of T, between the concepts of "topologically homologous to 0" (i.e., being boundary of a singular chain) and "homologous to 0 in T" (i.e., being boundary of a chain of T). Lefschetz realized that w'_p is not identified with a p-chain of

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* If $A_0 A_1 \cdots A_p$ is the sequence of vertices of Δ_p, identified with $\Delta_p \times \{0\}$, and $B_0 B_1 \cdots B_p$ is the sequence of the vertices $(A_j, 1)$ of $\Delta_p \times \{1\}$, the subdivision consists of the $(p + 1)$-simplices

$$(-1)^k A_0 A_1 \cdots A_k B_k B_{k+1} \cdots B_p$$

for $0 \leqslant k \leqslant p$; this generalizes the decomposition of a prism into tetrahedra, which goes back to Euclid.

T but with a p-chain in the module C'_p of chains of ordinary *and degenerate* simplices of T; but we have seen that he identified the homology of (C'_j) and of (C_j), a result that was only proved in 1938.

The proof of proposition B is very similar. The singular $(p + 1)$-chain of which z_p is the topological boundary is subdivided in such a way that a homotopy of that $(p + 1)$-chain can be defined as above, with the added proviso that the vertices of z_p remain invariant under the homotopy; z_p is then identified to the boundary of a $(p + 1)$-chain w_{p+1} of C'_{p+1}, which may contain degenerate simplices; but as z_p does not contain degenerate simplices, it is also the boundary of the $(p + 1)$-chain obtained by deleting from w_{p+1} the degenerate simplices, and that is a chain of T.

Alexander's second proof [14] did not use singular simplices any more, and relies exclusively on simplicial approximation. It was enough to show that if two *euclidean* simplicial complexes X, X' are homeomorphic, and T (resp. T') is a triangulation of X (resp. X'), then the homology modules $H_\bullet(T)$ and $H_\bullet(T')$ are isomorphic. Let $f: X \to X'$ be a homeomorphism, with inverse $g = f^{-1}: X' \to X$. Let (T_i) [resp. (T'_j)] be the sequence of successive barycentric subdivisions of T (resp. T'); the maximum diameter of the simplices of T_i (resp. T'_j) tends to 0; hence, for each index i, there is an index j and a simplicial approximation g_{ij} of g, from T'_j to T_i; similarly, there is an index $k > i$ and a simplicial approximation f_{jk} of f, from T_k to T'_j. The composite $h_{ik} = g_{ij} \circ f_{jk}$ is then a simplicial map of T_k into T_i; suppose i and k large enough; then, owing to the relation $g \circ f = 1_X$, for every p, every p-simplex σ of T_i, and every p-simplex $\tau \subset \sigma$ of T_k [which is a p-simplex of the $(k - i)$-th barycentric subdivision of σ], h_{ik} sends every vertex of τ to a vertex of σ.

Let $\tilde{f}_{jk}: C'_p(T_k) \to C'_p(T'_j)$ and $\tilde{g}_{ij}: C'_p(T'_j) \to C'_p(T_i)$ be the homomorphisms of modules of ordinary *and degenerate* p-chains corresponding to the simplicial maps f_{jk} and g_{ij}, and $\tilde{h}_{ik} = \tilde{g}_{ij} \circ \tilde{f}_{jk}: C'_p(T_k) \to C'_p(T_i)$ their composite. On the other hand, let sd_{k-i} be the homomorphism of $C'_p(T_i)$ into $C'_p(T_k)$ that associates to every p-simplex of T_i the *sum* of the p-simplices of T_k contained in it, with the same orientation; then $\tilde{h}_{ik}(\mathrm{sd}_{k-i}(\sigma)) = \sigma + \theta_{ik}$, where θ_{ik} is a *degenerate* p-chain. This lemma is proved by induction on p, being obvious by definition for $p = 0$. The assumption on h_{ik} implies that for any p-simplex τ of T_k contained in σ, either $\tilde{h}_{ik}(\tau) = \pm\sigma$ or $\tilde{h}_{ik}(\tau)$ is a degenerate p-simplex; hence $\tilde{h}_{ik}(\mathrm{sd}_{k-i}(\sigma)) = c\,.\,\sigma + \theta_{ik}$, where c is a constant and θ_{ik} is a degenerate chain. But as $\tilde{h}_{ik} \circ \mathrm{sd}_{k-i}$ is a simplicial map,

$$\tilde{h}_{ik}(\mathrm{sd}_{k-i}(\mathbf{b}_p\sigma)) = c\,.\,\mathbf{b}_p\sigma + \mathbf{b}_p\theta_{ik}$$

and $\mathbf{b}_p\theta_{ik}$ is degenerate. On the other hand, the induction hypothesis implies

$$\tilde{h}_{ik}(\mathrm{sd}_{k-i}(\mathbf{b}_p\sigma)) = \mathbf{b}_p\sigma + \theta'_{ik},$$

where θ'_{ik} is degenerate; the comparison of the two formulas gives $c = 1$.*

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* This lemma is a special case of the Sperner lemma, proved two years later by the same method ([30], p. 376).

Assuming, as Alexander does, that the homology $H_\bullet(T_i)$ can be identified to the homology $H'_\bullet(T_i)$ of ordinary *and degenerate* chains, $(h_{ik})_* = (g_{ij})_* \circ (f_{jk})_*$ is the *identity* in $H_\bullet(T_i)$; similarly a "right inverse" is obtained for $(f_{jk})_*$, which proves that $(f_{jk})_*$ is an isomorphism; the theorem then results from the fact that $H_\bullet(T)$ (resp. $H_\bullet(T')$) is isomorphic to $H_\bullet(T_k)$ (resp. $H_\bullet(T'_j)$).

§4. Duality and Intersection Theory on Manifolds

A. The Notion of "Manifold"

After the invariance problem had been solved, two main items remained in the implementation of the program outlined by Poincaré: a rigorous proof of the duality theorem and a complete theory of intersections, barely begun by Poincaré (chap. I, §2). Obvious examples show that in neither question can one work with a general cell complex; some restrictions have to be introduced in order to make available the arguments Poincaré used for his "manifolds."

We have seen (chap. I, §2) that the concept of a C^r-manifold for $r \geqslant 1$ was clear to Poincaré. In what follows we will systematically use the name n-dimensional C^0-*manifold* to designate what is also called a locally euclidean space, namely, a Hausdorff space in which any point has a compact neighborhood homeomorphic to a closed ball in $\mathbf{R}^n$.* The triangulability of C^r-manifolds for $r \geqslant 1$ was only proved in 1930 (chap. III, §2); but (except for $n \leqslant 3$) the triangulability of C^0-manifolds remained undecided until about 1960, when counterexamples were found for dimensions $\geqslant 5$. In the meantime, in order to use simplicial methods, topologists had to settle for more tractable definitions of "manifolds."

In fact, several definitions were proposed ([308], pp. 342–343); the first one was described by Veblen ([474], pp. 91–95) and it is a definition that is based on a *given triangulation* T into "cells" [in the sense of Poincaré (chap. I, §2)] of the compact space X, but Veblen did not investigate its invariance under homeomorphism. The definition generalizes Poincaré's condition that for the maximal dimension n of the cells of T, each $(n-1)$-cell should be in the frontier of exactly two n-cells: for any k-cell C ($k \leqslant n-1$), let $Z^{n-k-1}(C)$ be the union of the j-cells ($j \leqslant n-k-1$) that are in the frontiers of the n-cells having C in their frontier but the closures of which do not meet the closure of C. Then (X, T) is a manifold (without boundary) in Veblen's sense if, for all $k \leqslant n-1$ and all k-cells C, $Z^{n-k-1}(C)$ is homeomorphic to the sphere S_{n-k-1}.

However, since (as Poincaré had shown in his fifth *Complément*) the homology of a sphere is not enough to characterize it up to homeomorphism, it was not possible to verify Veblen's condition by purely combinatorial means, and, in particular, it was not at all obvious that it would be satisfied by a triangu-

* Of course, to be sure that this definition is meaningful, one has to invoke Brouwer's theorem on the invariance of dimension (Part 2, chap. II, §1).

lated C^r-manifold, so that the proof of Poincaré's duality theorem for these manifolds, given by Veblen, could not be considered as conclusive. This observation was first made in print by Vietoris in 1928; he therefore proposed to consider only what he called *h-manifolds*, defined by induction on the dimension in the following way: such a manifold is a compact n-dimensional simplicial complex (X, T) for which the frontier of the star of each vertex of T is an $(n-1)$-dimensional h-manifold with the homology of $\mathbf{S}_{n-1}$. A similar (unpublished) observation was made by Alexander, who proposed to weaken Veblen's condition by requiring only that the $Z^{n-k-1}(C)$ be cell complexes with the same homology as $\mathbf{S}_{n-k-1}$. This definition was adopted by Lefschetz in 1929 and by most of the later writers under the name of *combinatorial manifolds*; it is easily shown that they are the same as Vietoris' h-manifolds.

B. Computation of Homology by Blocks

We have seen in § 3 that after Poincaré it was essentially known that the homology of a cell complex is naturally isomorphic to the homology of a simplicial subdivision of the complex. But if one starts with a *simplicial* complex (X, T) and regroups simplices into "blocks" [as in Poincaré's construction of "dual cells" (chap. I, § 3)], it is useful to know conditions that allow the computation of the homology of the complex to be performed by using *only* these "blocks" of simplices.

This question was analyzed by Seifert and Threlfall in their book [421]. They defined a *system of blocks* by giving, for each $p \geqslant 0$, a *basis* of the $\mathbf{Z}$-submodule K_p of the (free) $\mathbf{Z}$-module C_p of p-chains of T, satisfying for each $p \geqslant 0$ the two following conditions (where as usual Z_p and B_p are the submodules of cycles and boundaries in C_p):

1. $\mathbf{b}_p K_p = K_{p-1} \cap B_{p-1}$;
2. $Z_p = (K_p \cap Z_p) + \mathbf{b}_{p+1} K_{p+1}$.

This implies that $Z_p = B_p + (K_p \cap Z_p)$, hence, for the homology groups

$$H_p = Z_p/B_p \simeq (K_p \cap Z_p)/(K_p \cap B_p) = (K_p \cap Z_p)/\mathbf{b}_{p+1} K_{p+1}. \qquad (17)$$

In other words, (K_p) is a *chain complex* for the same boundary operator as (C_p), and the homology of (K_p) is isomorphic to the homology of (X, T). This is useful not only for proving Poincaré duality (see below), but also for practical computation of homology modules for explicitly given complexes.

C. Poincaré Duality for Combinatorial Manifolds

The simplest proof of Poincaré duality for an oriented combinatorial manifold X with a *simplicial* triangulation T is the one described by Pontrjagin ([374], p. 186). He considered the barycentric subdivision T′ of T, and "regrouped," as did Poincaré, the simplices of T′ into "dual cells," forming the dual triangulation T* of T. Any such "dual cell" E of dimension k has a frontier F such that the pair $(\bar{E}, F)$ has the same relative homology (§ 6) as the pair $(\mathbf{D}_k, \mathbf{S}_{k-1})$ consisting of the unit ball $\mathbf{D}_k$ in $\mathbf{R}^k$ and its frontier $\mathbf{S}_{k-1}$. This follows from the definition of combinatorial manifolds, and implies that the "dual cells" of T* form a *system of blocks* with which one may compute the homology of X: only condition 2 of B needs a proof, which can be done most simply by descending induction on the dimension k ([421], p. 235). It is still necessary to check the relation between the incidence matrices of T and of T*, but this follows easily from the definitions.

The proofs of Vietoris and Lefschetz ([304], pp. 135–140) are similar: start from a combinatorial manifold X, whose triangulation T into "cells" is related to a simplicial triangulation T′ by the fact that each "cell" is the *star* of a vertex of T′ (the *center* of the star), and T′ is the barycentric subdivision of T. Then assume that the frontier of a k-star is a union of stars of dimension $\leqslant k - 1$, that each k-star, for $k \leqslant n - 1$, is in the frontier of a $(k + 1)$-star, and finally that the homology of the frontier of a k-star is isomorphic to the homology of S_{k-1}. This implies, as in the particular case of a simplicial complex, that the "dual" cells obtained by the Poincaré construction have the same properties, and the Poincaré duality follows as before.

In their 1934 book Seifert and Threlfall showed that it is possible to replace in this proof the definition of combinatorial manifold by a definition independent of the triangulation: it is enough to suppose that the compact space X is *triangulable* and that it is an n-dimensional *generalized manifold* in the sense defined in 1933 by Lefschetz [306] and Čech [122] (Part 2, chap. IV, § 3); here this simply means that for any $x \in X$, the relative homology (§ 6) $H_q(X, X - \{x\}H; \mathbf{Z})$ is 0 for $q \neq n$ and isomorphic to $\mathbf{Z}$ for $q = n$ ([421], pp. 236–241).*

The duality theorems proved by Lefschetz and Čech in these papers of 1933 applied to generalized manifolds that were not necessarily triangulable, and therefore had to be proved by other methods (see Part 2, chap. IV, § 3).

D. Intersection Theory for Combinatorial Manifolds

When Poincaré's construction of "dual cells" is possible, it is easy to extend in a "cell complex" X his definition of the "Kronecker index" $N(V_1, V_2)$ (chap. I, § 2) to a "Kronecker index" $N(a_p, b^*_{n-p})$, where a_p is a p-cell of the complex and b^*_{n-p} its dual cell, both oriented: one transcribes the definition of Poincaré using the oriented vector spaces that are the directions of one of the simplices constituting a_p (resp. b^*_{n-p}) with vertex at the intersection of a_p and b^*_{n-p} ([369], p. 242). This was done in 1923 by Veblen and Weyl [484]. Assuming that the homology of X could be computed by using both the given "cell complex" and its dual, they defined in that way a bilinear form on the product $H_p \times H_{n-p}$ of the homology modules, and this form determines a duality between H_p and H_{n-p}. Actually there was very little to add to Poincaré's arguments to reach that conclusion, and it is a bit surprising that he did not do it himself, even taking into account the clumsy character of the linear algebra he had at his disposal.

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* These conditions are satisfied by C^r manifolds for $r \geqslant 1$.

The papers by Alexander and Lefschetz on intersections, which date from about the same time as those of Veblen and Weyl, are much more ambitious. Both authors started their mathematical careers in algebraic geometry, which Alexander abandoned almost immediately in favor of topology. Lefschetz, on the contrary, kept a continued and vigorous interest in the topic for more than 30 years, and we shall see later (Part 2, chap. VII, § 1,B) how, by expanding the ideas of Picard and Poincaré, he "planted the harpoon of algebraic topology into the body of the whale of algebraic geometry," to use his own words ([296], p. 13). But it should be emphasized here that if in the hands of Lefschetz algebraic geometry was transformed by this injection of algebraic topology, the latter, as we shall see presently and later, received from him impulses inspired by algebraic geometry just as valuable as the ones it gave in return (see [435]).

In the type of algebraic geometry begun around 1870 by Clebsch, Brill, and M. Noether and followed by Halphen, Picard, Humbert, Zeuthen and the Italian school, algebraic subvarieties of a complex projective space and algebraic families of such varieties were a fundamental tool. Under ill-defined conditions, for two subvarieties V, W of a third variety X, the combinations $V + W$, $V - W$, kV (for an integer k) were considered as subvarieties (or "virtual" subvarieties) when V and W have the same dimension, as well as the "product" $V.W$ when $\dim V + \dim W \geqslant \dim X$; in the best cases $V + W$ would be the set-theoretic union and $V.W$ the set-theoretic intersection, but the complexity of the general definitions ruled out any possibility of dealing with varieties (or classes of "equivalent" varieties in some sense) as elements of a group or a ring. We may wonder if Poincaré was not inspired by these would be algebraic operations when he introduced his "chains" of varieties in algebraic topology. At any rate this analogy was central in Lefschetz's early work, and Poincaré's algebraic manipulations probably appealed to him more than the complicated geometric constructions of the Italians; following ideas of Picard he combined algebraic and topological arguments in an original way and obtained remarkable new results (see Part 2, chap. VII, § 1,B); but since he was as reckless as Poincaré (and the Italians) in his use of "intuition," none of these results could be supported at that time by a convincing proof, which he and others could only supply 10 years later.

When Alexander and Lefschetz shifted their investigations to cell complexes and combinatorial manifolds, they naturally were led to generalize the concept of intersection to arbitrary cycles. But Alexander did not make any effort to clarify nor even to define that concept, which apparently he considered "intuitive" enough; his short notes on the subject [13] were only bent on showing by examples that the formulas giving intersections of cycles on two manifolds X, Y could be essentially different* even if the homology modules of X and Y are isomorphic.

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* In today's terminology, the intersection *rings* (or cohomology *rings*) of X and Y are not isomorphic. Other examples were given by de Rham ([388], p. 104).

Lefschetz took the matter much more seriously. Although at first he did not express it in this form, what he needed was, for an n-dimensional compact, connected, and oriented combinatorial manifold X, and for any two integers p, q such that $0 \leqslant p, q \leqslant n$, a bilinear mapping

$$(z_p, z_q) \mapsto z_p . z_q$$

of $\mathrm{H}_p \times \mathrm{H}_q$ into H_{p+q-n} (replaced by 0 if $p + q < n$), such that

$$z_q . z_p = (-1)^{(n-p)(n-q)} z_p . z_q \tag{18}$$

and

$$(z_p . z_q) . z_r = z_p . (z_q . z_r) \tag{19}$$

for any three integers p, q, r in $[0, n]$. Of course, when z_p and z_q are homology classes of cycles that are submanifolds and whose intersection is a $(p + q - n)$-dimensional submanifold, the homology class of that submanifold should be $z_p . z_q$ up to sign; furthermore, when $q = n - p$, $z_p . z_{n-p}$ is a scalar multiple λz_0 of the homology class z_0 of any point of X, and the scalar λ should be the "Kronecker index" defined by Poincaré, which Lefschetz wrote $(z_p . z_{n-p})$.

We shall see later (chapter IV, §4) that once cohomology was introduced, it was easy to define the products $z_p . z_q$ for manifolds using the "cup-product" of Whitney; here therefore, we shall only give a sketchy description of the direct methods which Lefschetz initially used in [300] and [301] to define the products $z_p . z_q$.

His idea was to consider *singular cycles* C_p, C_q having, respectively, z_p, z_q as homology classes, and deduce from them a $(p + q - n)$-singular cycle having $z_p . z_q$ as homology class. As could be expected, all he could actually do was to define a whole family of singular cycles, all homologous to each other, by a fairly complicated approximation process, of which he published two variants.

Both variants start with the definition of the oriented intersection "product" P . Q of two oriented convex polyhedra of respective dimensions p, q contained in a third one R of dimension n with $p + q \geqslant n$; P, Q, R are open in the respective linear affine varieties $\mathrm{V_P}$, $\mathrm{V_Q}$, and $\mathrm{V_R}$ they generate. If $\mathrm{P} \cap \mathrm{Q} = \varnothing$, take $\mathrm{P} . \mathrm{Q} = 0$; otherwise, P . Q is only defined when $\mathrm{V_P} \cap \mathrm{V_Q}$ has dimension $s = p + q - n$; $\mathrm{P} \cap \mathrm{Q}$ is then a convex polyhedron open in $\mathrm{V_P} \cap \mathrm{V_Q}$, and there is a way of assigning to $\mathrm{V_P} \cap \mathrm{V_Q}$ an orientation canonically dependent on those of $\mathrm{V_P}$, $\mathrm{V_Q}$ and $\mathrm{V_R}$;* P . Q is then the convex polyhedron $\mathrm{P} \cap \mathrm{Q}$ with that orientation, and

$$\mathrm{Q} . \mathrm{P} = (-1)^{(n-p)(n-q)} \mathrm{P} . \mathrm{Q}. \tag{20}$$

When P and Q satisfy all these conditions, they are said to be "in general position."

* Orienting an n-dimensional vector space means choosing a decomposable n-vector spanning that space. Let u, v, w be decomposable multivectors orienting the directions of $\mathrm{V_P}$, $\mathrm{V_Q}$, $\mathrm{V_R}$; then w defines a "regressive" product $u \vee v$, and that s-vector orients the direction of $\mathrm{V_P} \cap \mathrm{V_Q}$.

Now let X be an n-dimensional euclidean simplicial complex, which is a compact, connected, oriented *combinatorial manifold* with triangulation T. If C_0 is a p-chain and C'_0 is a q-chain of T, the "intersection product" $C_0 . C'_0$ can be defined by linearity provided that when a p-simplex of C_0 and a q-simplex of C'_0 have a nonempty intersection, they are contained in the closure of the same n-simplex of T and are in general position; $C_0 . C'_0$ is then a $(p + q - n)$-chain.

Now suppose C is a singular p-chain and C′ is a singular q-chain on X. Their *geometric intersection* is by definition the intersection of their images in X; assume that $p + q \geqslant n$, and that the geometric intersections of C with $\mathbf{b}_q C'$ and of $\mathbf{b}_p C$ with C′ are empty. The general idea is, after suitable subdivisions of T, C, and C′, to apply to C and C′ a refined (and somewhat complicated) version of the Alexander approximation process (§ 3), in order to obtain a p-chain C_0 of T and a q-chain C'_0 of T which satisfy the above condition and are such that $\mathbf{b}_p C_0 \cap C'_0 = C_0 \cap \mathbf{b}_q C'_0 = \emptyset$. In his first version [300] Lefschetz had to introduce an additional condition on chains C_0 and C'_0 in "general position" in order to ensure that the relation

$$\mathbf{b}_s(C_0 . C'_0) = \mathbf{b}_p C_0 . C'_0 + (-1)^{n-q} C_0 . \mathbf{b}_q C'_0 \tag{21}$$

holds; when the above condition on the boundaries of C and C′ is added the approximation process yields an *s-cycle* $C_0 . C'_0$. He could then easily show that the homology class of that cycle did not depend on the approximation used as long as that approximation deformed the singular chains by an amount smaller than a fixed quantity depending only on T, C, and C′.

In the second variant [301] he had the idea of using the "intersection product" P . Q of oriented convex polyhedra *only* when P is a p-cell of a (not necessarily simplicial) triangulation T of X, and Q is a q-cell of the *dual* triangulation T*. There is then no need to suppose "general position" for P and Q: automatically, either $P \cap Q = \emptyset$ or $P \cap Q$ is a $(p + q - n)$-dimensional convex polyhedron, and the application of the approximation process is greatly simplified.

However, in both variants, it is still necessary to prove that the homology class $z_p . z_q$ obtained is also independent of the triangulation chosen on X, and in both cases this necessitates a long and complicated argument.

Today the properties of the intersection products $z_p . z_q$ are expressed by saying that they define, by linearity, a structure of (associative and anti-commutative) *ring*, on the direct sum

$$H_{\bullet} = \bigoplus_{0 \leqslant p \leqslant n} H_p \tag{22}$$

of the homology modules, and that this ring is an invariant of the complex X under homeomorphism. It is a curious reflection on the clumsiness of algebra before van der Waerden that this formulation, which seems so obvious to us, was only given by Hopf in 1930 [242] (perhaps again under the influence of E. Noether). Alexander and Lefschetz, in the case of homology over the rationals, picked up bases (z_p^i) in each H_p, wrote out the expressions of the intersection products

$$z_p^i \cdot z_q^j = \sum_k \alpha_k^{ij} z_{p+q-n}^k \tag{23}$$

and then limited themselves to saying that the systems (α_k^{ij}) of rational numbers are "tensors" invariant under homeomorphisms!

§ 5. Homology of Products of Cell Complexes

Even the cartesian product of two arbitrary sets (excepting of course subsets of the $\mathbf{R}^n$) was not a notion in common use at the end of the nineteenth century (although it had been explicitly defined by Cantor). Only among algebraic geometers did it occasionally occur, for instance when Cayley considered the product of two algebraic curves or C. Segre the product of two complex projective spaces of arbitrary dimension; then these products were immediately given a structure of algebraic variety. The first mathematician who introduced the concept of a topological space, product of two given topological spaces, was apparently Steinitz in 1908 [456], but the investigation of the relations between the topology of the two factor spaces and the topology of their product was only begun independently by Künneth ([289], [290]) and Lefschetz [301] in 1923.

Both limited themselves to euclidean (rectilinear) compact connected cell complexes; actually, once the invariance problem had been solved (§ 3), computation of the homology of X × Y for two such cell complexes X, Y was an exercise in elementary linear algebra; for simplicity we shall describe it in the algebraic language of today. From the given triangulations T(X), T(Y) of X and Y into convex polyhedra (not necessarily simplices) a similar triangulation T(X × Y) is derived by taking all products A × B for A ∈ T(X) and B ∈ T(Y), and the **Z**-module $S_p(X \times Y)$ of p-chains of T(X × Y) is just the direct sum

$$S_p(X \times Y) = \bigoplus_{0 \leqslant k \leqslant p} (S_k(X) \otimes S_{p-k}(Y)). \tag{24}$$

Now the "reduction" of Poincaré's incidence matrices amounts to a decomposition

$$S_p(X) = Z_p(X) \oplus F_p(X) \tag{25}$$

into a direct sum of two submodules such that the boundary map $\mathbf{b}_p$ is 0 in $Z_p(X)$ and is an injection $F_p(X) \to Z_{p-1}(X)$ in $F_p(X)$; by the theory of invariant factors there are bases (e_{p-1}^j) of $Z_{p-1}(X)$ and (f_p^i) of $F_p(X)$ such that for those bases the matrix of $\mathbf{b}_p$ considered as an injection of $F_p(X)$ into $Z_{p-1}(X)$ is the matrix (ρ_{ij}) defined in chap. I, § 3, after removal of the zero columns. Now for the boundary map in T(X × Y), if A is a k-cell of T(X) and B is a $(p-k)$-cell of T(Y),

$$\mathbf{b}_p(A \times B) = \mathbf{b}_k A \times B + (-1)^k A \times \mathbf{b}_{p-k} B. \tag{26}$$

As $S_p(X \times Y)$ splits into a direct sum of the **Z**-modules

$$F_k(X) \otimes F_{p-k}(Y), \qquad F_k(X) \otimes Z_{p-k}(Y), \qquad Z_k(X) \otimes F_{p-k}(Y),$$
$$Z_k(X) \otimes Z_{p-k}(Y)$$

for $0 \leqslant k \leqslant p$, the matrix of $\mathbf{b}_p\colon S_p(X \times Y) \to S_{p-1}(X \times Y)$ splits accordingly into *blocks*, all of which are trivially written down immediately, with the exception of the matrix of

$$\mathbf{b}_p\colon F_k(X) \otimes F_{p-k}(Y) \to (Z_{k-1}(X) \otimes F_{p-k}(Y)) \oplus (F_k(X) \otimes Z_{p-k-1}(Y)); \quad (27)$$

but the "reduction" of that matrix is at once brought down to the "reduction" of 2×2 matrices. One thus obtains a regular algorithm for computing the homology modules of $X \times Y$ when one knows those of X and Y; in particular, one has for the Betti numbers the "Künneth formula"

$$b_p(X \times Y) = \sum_{0 \leqslant k \leqslant p} b_k(X) b_{p-k}(Y) \quad (28)$$

from which one deduces at once for the Euler–Poincaré characteristics

$$\chi(X \times Y) = \chi(X)\chi(Y). \quad (29)$$

It is just as easy to compute the intersection ring of $X \times Y$ when X and Y are oriented combinatorial manifolds; from formula (26) it follows that the cartesian product of two cycles is a cycle; if we denote by $z_p \times z'_q$ the homology class of the cartesian product of a cycle of class z_p in X and of a cycle of class z'_p in Y, then

$$(z_k \times z'_{p-k}).(u_h \times u'_{q-h}) = (z_k . u_h) \times (z'_{p-k} . u'_{q-h}). \quad (30)$$

§6. Alexander Duality and Relative Homology

Until 1920 homology had only been defined for finite cell complexes (connected or not). In a remarkable paper [11] published in 1922 (the first draft of which goes back to 1916) Alexander broke new ground by considering the homology of *open* subsets of an $\mathbf{R}^n$; at the same time he showed how the Brouwer theorems, proved by him without reference to homology (Part 2, chaps. I and II), could be inserted into the theory of homology and extended in that way.

He considered a subspace of a sphere $\mathbf{S}_n$ ($n \geqslant 2$), which is a *compact* (connected or not) *curvilinear cell complex* X of dimension $m < n$ (for instance, a closed Jordan curve, for $n = 2$ and $m = 1$). He first had to define the homology of the open set $\mathbf{S}_n - X$. Alexander did not do this formally, but considered a simplicial subdivision T *of* $\mathbf{S}_n$ and the sequence of triangulations T_j obtained by successive barycentric subdivisions of T; p-chains *of* $\mathbf{S}_n - X$ are then p-chains of *any* T_j that are linear combinations of p-simplices *contained in* $\mathbf{S}_n - X$. In order to add a p-chain C of T_j and a p-chain C′ of T_k for $k > j$ (both combinations of p-simplices contained in $\mathbf{S}_n - X$), he replaced each simplex in C by the *sum* of the simplices of T_k into which it is decomposed. (Actually Alexander worked in homology mod. 2, in which he did not have to bother with signs.) Boundary operators and homology are then defined as usual, but *a priori* the homology modules might not be finitely generated.

Another equivalent method is to define the homology of an *arbitrary* open subset U of $\mathbf{R}^n$ by an extension of the concept of *triangulation* for such a set: this time it means a *locally finite* partition T of U into cells of various dimensions, with the condition that the frontier of any cell of T is a *finite* union of cells of T of strictly lower dimension;* more generally, from now on we shall call a space equipped with such a triangulation a *cell complex*. The p-chains are then defined as linear combinations of a *finite* number of cells of T, and boundaries and homology are defined as for finite cell complexes [30]. Alexander's remarkable result was that if $X \subset \mathbf{S}_n$ is a *finite curvilinear cell complex*, all Betti numbers (mod 2) of $\mathbf{S}_n - X$ are *finite* and satisfy *Alexander duality*

$$\dim H_p(X; \mathbf{F}_2) = \dim H_{n-p-1}(\mathbf{S}_n - X; \mathbf{F}_2) \quad \text{for } 1 \leqslant p \leqslant n-2, \tag{31}$$

$$\dim H_0(\mathbf{S}_n - X; \mathbf{F}_2) = \dim H_{n-1}(X; \mathbf{F}_2) + 1,$$

$$\dim H_{n-1}(\mathbf{S}_n - X; \mathbf{F}_2) = \dim H_0(X; \mathbf{F}_2) - 1 \tag{32}$$

[with $H_p(X; \mathbf{F}_2) = 0$ by convention when p is larger than the dimensions of the simplices of X].

The very ingenious and rather intricate proof relies on splitting X into a union of two (curvilinear) cell complexes Y, Z, and , from the knowledge of the cases in which X is replaced by Y, Z or $Y \cap Z$, to deduce the result for X: a typical "Mayer–Vietoris" procedure (although, as we have seen in §2, the papers of Mayer and Vietoris were only published 7 years later). This is applied in three steps, each one using the results of the preceding one.

The first step concerns the case in which X is homeomorphic to a closed cube of any dimension $m \leqslant n$; then it is shown that $H_k(\mathbf{S}_n - X) = 0$, except for $k = 0$ [a generalization of Brouwer's "no separation" theorem (Part 2, chap. II, §4)]. The procedure consists in an induction on m, splitting X into two half cubes and using contradiction, by an infinite iteration of the splitting into cubes with diameters tending to 0. The second step is devoted to the case in which X is homeomorphic to a sphere $\mathbf{S}_m$; induction on m, splitting X into two closed hemispheres with an intersection homeomorphic to $\mathbf{S}_{m-1}$, to each

* To prove the existence of such a triangulation for an *arbitrary* open subset U of $\mathbf{R}^n$, one may consider the closed n-dimensional cubes of $\mathbf{R}^n$ having as vertices the points of $2^{-k}\mathbf{Z}^n$, and 2^{-k} as lengths of their edges; let C_k be the set of those cubes contained in U, and C'_k the subset of C_k consisting of cubes having no common interior point with the cubes of C_{k-1}; the triangulation T is obtained by decomposing each cube belonging to the union of the C'_k for all integers $k \geqslant 1$ into disjoint open cubes of all dimensions $\leqslant n$. This method was already used by Runge in $\mathbf{R}^2$ ([30], p. 143); it was considered as well known by Brouwer ([89], p. 316).

of which the first case can be applied. This already contains as a particular case the Jordan–Brouwer theorem (Part 2, chap. II, § 3) for $m = n - 1$.

The general case (for which X may be taken as a curvilinear *simplicial* complex) is treated in the third step, by a double induction on $m = \dim X$ and on the number of simplices of maximum dimension m in X. Write $b'_p(X) = \dim H_p(X; \mathbf{F}_2)$ for simplicity. First split X into the union of the closure Y of one m-simplex and the closed complement Z of that simplex and prove that

$$b'_p(X) = b'_p(Z) \quad \text{and} \quad b'_{n-p-1}(S_n - X) = b'_{n-p-1}(S_n - Z) \quad \text{for } p \leqslant m - 2. \tag{33}$$

By induction, assume that

$$b'_m(Z) - b'_{m-1}(Z) = b'_{n-m-1}(S_n - Z) - b'_{n-m}(S_n - Z) \tag{34}$$

and show that

$$b'_m(X) - b'_{m-1}(X) = b'_{n-m-1}(X) - b'_{n-m}(X) \tag{35}$$

[add 1 on the right-hand sides of (34) and (35) when $m = n - 1$]. Finally, after having split off *all simplices of dimension m*, one gets a cell complex Z_0 of dimension $\leqslant m - 1$; the induction hypothesis shows that $0 = b'_m(Z_0) = b'_{n-m-1}(S_n - Z_0)$ and $b'_{m-1}(Z_0) = b'_{n-m}(S_n - Z_0)$. The final step consists in proving that $b'_m(X) = b'_{n-m-1}(S_n - X)$ (with 1 added on the left-hand side if $m = n - 1$) by looking at the $(m - 1)$-simplices of Z_0 and at the m-simplices of X of which these $(m - 1)$-simplices are faces; finally $b'_{m-1}(X) = b'_{n-m}(S_n - X)$ by (35).

At the end of the paper Alexander observed that relations (31) and (32) yield a third proof of the independence of homology from the triangulation used to compute it, since the triangulations of X and of S_n are independent of each other.

Alexander's paper was, on one hand, the starting point of investigations by several mathematicians (Vietoris, Alexandroff, Lefschetz, Pontrjagin, Čech) aiming at generalizing homology modules to spaces other than compact complexes or open subsets of $\mathbf{R}^n$; we shall describe these developments in chap. IV, § 2.

On the other hand, it led Lefschetz to introduce the new and important concept of *relative homology*. In his first publication on that subject [302] he introduced this notion for homology with rational coefficients and for very general spaces, thus linking with the generalizations we just mentioned. In his book [304] Lefschetz separated the homology of finite cell complexes from its generalizations and allowed coefficients in $\mathbf{Z}$, $\mathbf{Z}/m\mathbf{Z}$, or $\mathbf{Q}$. If K is a finite euclidean simplicial complex, L is a union of simplices of K, the boundary $\mathbf{b}_L C$ of a chain C of K *relative to* L (or *mod* L) is the sum of the simplices of K in the expression of $\mathbf{b}C$ that *are not contained in* L, so that a p-chain C of K is a *cycle mod* L (resp. is *homologous to* 0 *mod* L) if its (usual) boundary is a combination of simplices of L [resp. if there is a $(p + 1)$-chain of K whose (usual) boundary is the sum of C and of simplices of L]; hence, the definition of the *homology modules of* K *mod* L, which Lefschetz wrote $H_p(K, L)$. More generally, he also gave the corresponding definitions when K is an "open complex," an open subset of a compact euclidean simplicial complex K′ that is a union of simplices of K′.

With these definitions Lefschetz first considered the case in which K is an n-dimensional euclidean simplicial complex, and L is a subcomplex of K (which is automatically closed in K since the faces of a simplex of L have to be in L). He then proved that the homology of K mod L only depends on the topology of K − L (first appearance of what later will be called "excision"); for that purpose he adapted the homotopy process used by Alexander in his first proof of invariance (§3) in such a way that a singular chain having its frontier in L is deformed into another chain whose frontier remains in L ([304], p. 86). Lefschetz also investigated the relations between the homology of K, the homology of L, and the homology of K mod L ([304], pp. 149–150), which later took their final form in the exact sequence of relative homology (chap. IV, §6,B).

Supposing that K − L is also an orientable combinatorial manifold, Lefschetz first generalized to the homology of K mod L the Poincaré duality. He denoted by K* the union of the duals of the simplices of K that are not simplices of L; then he showed that K* is a compact simplicial complex, and that its incidence matrix of $(n - p)$-chains and $(n - p - 1)$-chains is, up to sign, the transpose of the incidence matrix of the $(p + 1)$-chains and p-chains of K − L.

From this he was able to deduce, by an entirely different method, Alexander's duality theorem, at least in the special case in which L is a *subcomplex* of S_n [for a suitable (curvilinear) triangulation of S_n] by showing that in this case the relative homology module $H_p(S_n, L)$ is isomorphic to the "absolute" homology module $H_{p-1}(L)$ for $1 < p < n$, by an argument that again is essentially part of the exact sequence of relative homology ([304], pp. 143–144).

Finally, Lefschetz took up the more general situation in which $L_1 \subset L$ are two subcomplexes of K, and by a more refined argument, he can show that if $L_2 = L - L_1$, then for the Betti numbers

$$b_p(K - L_1, L_2) = b_{n-p}(K - L_2, L_1) \tag{36}$$

and a corresponding relation holds for the torsion coefficients ([304], pp. 141–142).