Preface

This volume contains the notes of the lectures delivered at the CIME course *Geometric Analysis and PDEs* during the week of June 11–16 2007 in Cetraro (Cosenza). The school consisted in six courses held by M. Gursky (*PDEs in Conformal Geometry*), E. Lanconelli (*Heat kernels in sub-Riemannian settings*), A. Malchiodi (*Concentration of solutions for some singularly perturbed Neumann problems*), G. Tarantello (*On some elliptic problems in the study of selfdual Chern-Simons vortices*), X.J. Wang (*The k-Hessian Equation*) and P. Yang (*Minimal Surfaces in CR Geometry*).

Geometric PDEs are a field of research which is currently very active, as it makes it possible to treat classical problems in geometry and has had a dramatic impact on the comprehension of three- and four-dimensional manifolds in the last several years. On one hand the geometric structure of these PDEs might cause general difficulties due to the presence of some invariance (translations, dilations, choice of gauge, etc.), which results in a lack of compactness of the functional embeddings for the spaces of functions associated with the problems. On the other hand, a geometric intuition or result might contribute enormously to the search for natural quantities to keep track of, and to prove regularity or a priori estimates on solutions. This two-fold aspect of the study makes it both challenging and complex, and requires the use of several refined techniques to overcome the major difficulties encountered. The applications of this subject are many while for the CIME course we had to select only a few, trying however to cover some of the most relevant ones, with interest ranging from the pure side (analysis/geometry) to the more applied one (physics/biology). Here is a brief summary of the topics covered in the courses of this school.

M. Gursky treated a class of elliptic equations from conformal geometry: the general aim is to deform conformally (through a dilation which depends on the point) the metric of a given manifold so that the new one possesses special properties. Classical examples are the uniformization problem of two-dimensional surfaces and the Yamabe problem in dimension greater or equal to three, where one requires the Gauss or the scalar curvature to become constant. After recalling some basic facts on these problems, which can be reduced to semilinear elliptic PDEs, Gursky turned to their fully nonlinear counterparts. These concern the prescription of the symmetric forms in the eigenvalues of the *Schouten tensor* (a combination of the Ricci tensor and the scalar curvature), and turn out to be elliptic under suitable conditions on their domain of definition (admissible functions). The solvability of these equations has concrete applications in geometry, since for example they might guarantee pinching conditions on the Ricci tensor, together with its geometric/topological consequences. After recalling some regularity estimates by Guan and Wang, existence was shown using blow-up analysis techniques. Finally, the functional determinant of conformally invariant operators in dimension four was discussed: the latter turns out to have a universal decomposition into three terms which respectively involve the scalar curvature, the Q-curvature and the Weyl tensor. Some conditions on the coefficients of these three terms guarantee coercivity of the functional, and in these cases existence of extremal metrics was obtained using a minimization technique.

E. Lanconelli covered some topics on existence and sharp estimates on heat kernels of subelliptic operators. Typically, in a domain or a manifold Ω of dimension n, k vector fields $X_1, \ldots, X_k$ are given (with $2 \leq k < n$) which satisfy the *Hörmander condition*, namely their Lie brackets span all of the tangent spaces to Ω. One considers then linear operators L (or their parabolic counterpart) whose principal part is given by $\sum_{i=1}^{k} X_i^2$. During the lectures, existence and regularity (Hörmander) theory for such operators was recalled, and in particular the role of the *Carnot-Caratheodory distance*, measured through curves whose velocities belong to the linear span of the X_i's. This distance is not homogeneous (at small scales), and it is very useful to describe the degeneracy of the operators in the above form. One of the main motivations for this study is the problem of prescribing the *Levi curvature* of boundaries of domains in $\mathbb{C}^n$, which for graphs amounts to solving a fully nonlinear degenerate equation, whose linearization is of the form previously described. Gaussian bounds for heat kernels were then given, first for constant coefficient operators modeled on Carnot groups, and then for general operators using the method of the parametrix. Finally, applications to Harnack type inequalities were derived in terms of the heat kernel bounds.

The course by A. Malchiodi on singularly perturbed Neumann problems dealt with elliptic nonlinear equations where a small parameter (the singular perturbation) is present in front of the principal term (the Laplacian). The study is motivated by considering a class of reaction-diffusion systems (in particular the Gierer-Meinhardt model) and the (focusing) nonlinear Schrödinger equation. First a finite-dimensional reduction technique, which incorporates the variational structure of the problem, was presented: by means of this method, existence of solutions concentrating at points of the boundary of the domain was studied. Here the geometry of the boundary is significant, as concentration occurs at critical points of the mean curvature. After this, existence of solutions concentrating at the whole boundary was proved: the phenomenon here is rather different, since the latter family has a diverging Morse index when the singular perturbation parameter tends to zero. Initially, accurate approximate solutions were constructed (depending on the second fundamental form of the domain), and then the invertibility of the linearized equation was shown (primarily using Fourier analysis), which made it possible to prove existence using local inversion arguments.

G. Tarantello's course focused on self-dual vortices in Chern-Simons theory. The physical phenomenon of superconductivity is described by a system of coupled gauge-field equations whose components stand for the wave function and the electromagnetic potential. A relatively well understood model is the abelian-Higgs (corresponding in a non-relativistic context to the Ginzburg-Landau) variant, for which much has been accomplished even away from the self-dual regime. A more sophisticated alternative is the Chern-Simons model, which compared to the previous one has the advantage of predicting the fact that gauge vortices carry electrical charge in addition to magnetic flux, although its mathematical description is at the moment less complete. After describing the main features of these models, Tarantello presented the approach of Taubes to the selfdual regime for the abelian-Higgs case, which reduces the system to a semilinear elliptic problem with exponential nonlinearities. This method partially extends to the Chern-Simons case, where some natural requirements on solutions can be proved, like their asymptotic behavior at infinity, integrability properties and the decay of their derivatives. The structure of C-S vortices is however more rich (and, as we remarked, far from being completely characterized) in comparison to the abelian-Higgs ones: in addition to the *topological solutions*, which have a well defined winding number at infinity, there are also non-topological solutions, which display different asymptotic behavior.

X.J. Wang treated k-Hessian equations, a class of fully nonlinear equations related to the problem of prescribing the Gauss curvature of a hypersurface, and to the Monge-Ampére equation, which is of interest in complex geometry. First the class of *admissible functions* was defined, where the equations are elliptic, and then existence for Dirichlet boundary value problems was obtained by means of a priori estimates and a continuation argument. Next, interior gradient estimates were derived, which imply Harnack inequalities, plus Sobolev-type inequalities for admissible functions which vanish on the boundary of the domain: the embedding which follows from the latter inequality possesses compactness properties analogous to the classical ones, and makes it possible to derive L^∞ estimates for solutions of equations with sufficiently integrable right-hand sides. These estimates make it possible to treat equations with nonlinear (subcritical or critical) reaction terms, where min-max methods can be applied. After this, the notion of k-admissibility was extended to non smooth functions using the concept of hessian measure, and applied to the existence of weak solutions and to potential-theoretical results. Finally, parabolic equations and several examples were treated.

The course by P. Yang concerned minimal surfaces in three-dimensional CR manifolds, which possess a subriemannian structure modeled on the Heisenberg group $\mathbb{H}^1$. In this setting the volume form of M is naturally defined in terms of the contact form θ as $\theta \wedge d\theta$: the *p-area* and the *p-mean curvature* (p standing for *pseudo*) of a two-dimensional surface were defined looking at the first and second variation of the volume form. p-minimal (regular) graphs in $\mathbb{H}^1$ were then considered, showing that they are ruled surfaces. A study of the *singular set* (where the tangent plane to the surface coincides with the contact plane, the kernel of θ) was then performed, and it was shown that it consists either of isolated points or of smooth curves: applications were given to the classification of entire p-minimal graphs. Existence of weak solutions (minimizers) to boundary value problems (of Plateau type) was considered, showing the uniqueness, comparison principles and geometric properties of the solutions. The regularity issue was then discussed, which is a rather delicate one since the global C^2 regularity of minimizers might fail in general.

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Finally, we would like to express our warm gratitude to the CIME Foundation, to the CIME Director Prof. P. Zecca, to the CIME Board Secretary Prof. E. Mascolo and to the CIME staff for their invaluable help and support, and for making the environment in Cetraro so stimulating and enjoyable.

Alice S.Y. Chang
Antonio Ambrosetti
Andrea Malchiodi

PDEs in Conformal Geometry

Matthew J. Gursky

1 Introduction

In these lectures I will discuss two kinds of problems from conformal geometry, with the goal of showing an important connection between them in four dimensions.

The first problem is a fully nonlinear version of the Yamabe problem, known as the σ_k-Yamabe problem. This problem is, in general, not variational (or at least there is not a natural variational interpretation), and the underlying equation is second order but possibly not elliptic. Moreover, in contrast to the Yamabe problem, there is very little known (except for some examples and counterexamples) when the underlying manifold is negatively curved.

The second problem we will discuss involves the study of a fourth order semilinear equation, and arose in the context of a natural variational problem from spectral theory. Despite their differences–higher order semilinear versus second order fully nonlinear, variational versus non-variational–both equations are invariant under the action of the conformal group, and we have to address the phenomenon of "bubbling." Therefore, in the first few sections of the notes we will present the necessary background material, including a careful explanation of the idea of a "standard bubble".

After covering the introductory material, we give a description of the σ_k-Yamabe problem, culminating in a sketch of the solution in the four-dimensional case. Modulo some technical regularity estimates, the proof is reduced to a global geometric result (Theorem 5.7) that is easy to understand.

M.J. Gursky
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
e-mail: mgursky@nd.edu
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In the last section of the notes we discuss the functional determinant of a four-manifold, a variational problem which is based on a beautiful calculation of Branson-Ørsted. We end with a sketch of the existence of extremals for the determinant functional for manifolds of positive scalar curvature. Here, the missing technical ingredient is a sharp functional inequality due to Adams (Theorem 65), but the proof is again reduced to Theorem 5.7. Therefore, we see the underlying unity of the two problems in a very concrete way.

In closing, I wish to express my gratitude to the Fondazione C.I.M.E. for their invitation and their support. The success of the meeting *Geometric Analysis and PDEs* was a result of the considerable efforts of the local organizers (especially Andrea Malchiodi), the scientific contributions of the participants, and the hospitality of our hosts in Cetraro.

2 Some Background from Riemannian Geometry

In this section we review some of the basic notions from Riemannian geometry, including the basic differential operators (gradient, Hessian, etc.) and curvatures (scalar, Ricci, etc.) This is not so much an introduction to the subject–which would be impossible in so short a space–but rather a summary of definitions and formulas.

2.1 Some Differential Operators

1. The Hessian

Let (M^n, g) be an n-dimensional Riemannian manifold, and let ∇ denote the Riemannian connection.

Definition 2.1. The *Hessian* of $f : M^n \to \mathbf{R}$ is defined by

$$\nabla^2 f(X,Y) = \nabla_X df(Y). \tag{1}$$

It is easy to see the Hessian is symmetric, bilinear form on the tangent space of M^n at each point. In a local coordinate system $\{x^i\}$, the *Christoffel symbols* are defined by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Using (1), in local coordinates we have

$$
\begin{aligned}
(\nabla^2 f)_{ij} &= \nabla_i \nabla_j f \\
&= \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k}.
\end{aligned}
$$

2. The Laplacian and Gradient

Definition 2.2. The *Laplacian* is the trace of the Hessian: Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space at a point; then

$$
\Delta f = \sum_i \nabla^2 f(e_i, e_i). \tag{2}
$$

In local coordinates $\{x^i\}$,

$$
\Delta f = g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right),
$$

where $g^{ij} = (g^{-1})_{ij}$. Another useful formula is

$$
\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{g} \frac{\partial f}{\partial x^j} \right),
$$

where $g = det(g_{ij})$.

The *gradient vector field* of f, denoted ∇f, is the vector field dual to the 1-form df; i.e., for each vector field X,

$$
g(\nabla f, X) = df(X).
$$

In local coordinates $\{x^i\}$,

$$
\nabla_j f = \sum_i g^{ij} \frac{\partial f}{\partial x^i}.
$$

3. The Curvature Tensor

For vector fields X, Y, Z, the Riemannian curvature tensor of (M, g) is defined by

$$
R(X, Y)Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z,
$$

where $[\cdot, \cdot]$ is the Lie bracket. With respect to a local coordinate system $\{x^i\}$, the curvature tensor is given by

$$R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j}=\sum_i R^i_{jkl}\frac{\partial}{\partial x^i}.$$

Let $\Pi \subset T_pM^n$ be a tangent plane with orthonormal basis $\{E_1, E_2\}$. The *sectional curvature* of Π is the number

$$K(\Pi) = \langle R(E_1, E_2)E_1, E_2\rangle.$$

($K(\Pi)$ does not depend on the choice of ON-basis.)

Example 1. For $\mathbf{R}^n$ with the Euclidean metric, all sectional curvatures are zero.

Example 2. Let $\mathbf{S}^n = \{\mathbf{x} \in \mathbf{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$ with the metric it inherits as a subspace of $\mathbf{R}^{n+1}$. Then all sectional curvatures are $+1$.

Example 3. Let $\mathbf{H}^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| < 1\}$, endowed with the metric

$$g = 4\sum_i \frac{(dx^i)^2}{(1-\|\mathbf{x}\|^2)^2}.$$

Then all sectional curvatures are -1.

The preceding examples are referred to as *spaces of constant curvature*, or *space forms*. A theorem of Hopf says that any complete, simply connected manifold of constant curvature is isometric to one of these examples (perhaps after scaling). Thus, curvature determines the local geometry of a manifold.

Another way of thinking about curvature is that it measures the failure of derivatives to commute:

Lemma 2.3. *In local coordinates,*

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = \sum_m R^m_{kij}\nabla_m f.$$

So third derivatives do not commute unless $R = 0$, i.e., the manifold is *flat*.

4. Ricci and Scalar Curvatures

Definition 2.4. The *Ricci curvature tensor* is the bilinear form $Ric : T_pM \times T_pM \to \mathbf{R}$ defined by

$$Ric(X,Y) = \sum_i \langle R(X,e_i)Y, e_i\rangle,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM.

In local coordinates, the components of Ricci are given by

$$R_{ij} = \sum_m R^m_{ijm}.$$

For spaces of constant curvature, the Ricci tensor is just a constant multiple of the metric:

$$\begin{aligned} \mathbf{S}^n: &\quad Ric = (n-1)g, \\ \mathbf{R}^n: &\quad Ric = 0, \\ \mathbf{H}^n: &\quad Ric = -(n-1)g. \end{aligned}$$

The Ricci tensor is symmetric: $Ric(X,Y) = Ric(Y,X)$. Therefore, at each point $p \in M$ we can diagonalize Ric with respect to an orthonormal basis of T_pM:

$$Ric = \begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_n \end{pmatrix}$$

where $(\rho_1, \dots, \rho_n)$ are the eigenvalues of Ric. To say that (M^n, g) has positive (negative) Ricci curvature means that all the eigenvalues of Ric are positive (negative).

In two dimensions, the Ricci curvature is determined by the Gauss curvature K:

$$Ric = Kg.$$

Definition 2.5. The *scalar curvature* is the trace of the Ricci curvature:

$$R = \sum_i Ric(e_i, e_i),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis.

If $\{\rho_1, \dots, \rho_n\}$ are the eigenvalues of the Ricci curvature at a point $p \in M$, then the scalar curvature is given by

$$R = \rho_1 + \cdots + \rho_n.$$

For the spaces of constant curvature, the scalar curvature is a constant function:

$$\begin{aligned} \mathbf{S}^n: &\quad R = n(n-1), \\ \mathbf{R}^n: &\quad R = 0, \\ \mathbf{H}^n: &\quad R = -n(n-1). \end{aligned}$$

Furthermore, in two dimensions the scalar curvature is twice the Gauss curvature:

$$R = 2K.$$

3 Some Background from Elliptic Theory

In this section we summarize some important results from functional analysis and the theory of partial differential equations.

1. Sobolev Spaces

These are important for discussing some of the PDE topics in these lectures. Let (M, g) be a compact Riemannian manifold. For $1 \leq k < \infty$ and $1 \leq p \leq \infty$, introduce the norms

$$\|u\|_{k,p}^p = \sum_{0 \leq j \leq k} \int |\nabla^j u|^p \, dV,$$

where $\nabla^j u$ denotes the iterated j^{th}-covariant derivative.

Example. For $k = 1$, $p = 2$,

$$\|u\|_{1,2}^2 = \int u^2 \, dV + \int |du|^2 \, dV.$$

The Sobolev space $W^{k,p}(M)$ is the completion of $C^\infty(M)$ in the norm $\| \cdot \|_{k,p}$.

Theorem 3.1. (Sobolev Embedding Theorems; see [GT83])
(i) If

$$\frac{1}{r} = \frac{1}{m} - \frac{k}{n},$$

then $W^{k,m}(M)$ is continuously embedded in $L^r(M)$:

$$\|u\|_r \leq C\|u\|_{k,m}.$$

(ii) Suppose $0 < \alpha < 1$ and

$$\frac{1}{m} \leq \frac{k - \alpha}{n}.$$

Then $W^{k,m}$ is continuously embedded in C^α.

(*iii*) *(Rellich-Kondrakov) If*

$$\frac{1}{r} > \frac{1}{m} - \frac{k}{n},$$

then the embedding $W^{k,m} \hookrightarrow L^r$ is compact: i.e., a sequence which is bounded in $W^{k,m}$ has a subsequence which converges in L^r.

2. Linear Operators

Consider the linear differential operator L:

$$Lu = a^{ij}(x)\partial_i\partial_j u + b^k(x)\partial_k u + c(x)u,$$

where the coefficients a^{ij}, b^k, c are defined in a domain $\Omega \subset \mathbf{R}^n$.

Definition 3.2. The operator L is *elliptic* in Ω if $\{a^{ij}(x)\}$ is positive definite at each point $x \in \Omega$. If there is a constant $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $\xi \in \mathbf{R}^n$ and $x \in \Omega$, then L is *strictly elliptic* in Ω. If, in addition, there is another constant $\Lambda > 0$ such that

$$\Lambda|\xi|^2 \geq a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

then we say that L is *uniformly elliptic* in Ω.

We can formulate a similar definition for operators defined on a Riemannian manifold; e.g., by introducing local coordinates. Of course, the laplacian $L = \Delta$ is an example of a linear, uniformly elliptic operator.

3. Weak Solutions

We say that $u \in W^{1,2}(M)$ is a *weak* solution of the equation

$$\Delta u = f(x) \tag{3}$$

if for each $\varphi \in W^{1,2}$,

$$\int -\langle \nabla u, \nabla \varphi \rangle \; dV = \int f\varphi \; dV. \tag{4}$$

Of course, a smooth solution of (3) satisfies (4) by virtue of Green's Theorem. Weak solutions of elliptic equations like (3) in fact satisfy much better estimates, as we shall see.

4. Elliptic Regularity

Theorem 3.3. (See [GT83]) *Suppose $u \in W^{1,2}$ is a weak solution of*

$$\Delta u = f$$

on M.
(i) If $f \in L^m$, then

$$\|u\|_{2,m} \leq C\big(\|f\|_m + \|u\|_m\big). \tag{5}$$

(ii) (Schauder estimates) If $f \in C^{\ell,\alpha}$ then

$$\|u\|_{C^{\ell+2,\alpha}} \leq C\big(\|f\|_{C^{\ell,\alpha}} + \|u\|_{C^{\ell,\alpha}}\big). \tag{6}$$

How are such estimates used?

- To prove the regularity of weak solutions.

Weak solutions are often easier to find, for example, by variational methods.

- To prove *a priori* estimates of solutions, that is, estimates which are necessarily satisfied by any solution of a given equation.

Often *a priori* estimates can be combined with a topological argument to establish existence.

Example. To illustrate some of these results we consider an equation that will be an important model for much of the subsequent material.

Theorem 3.4. *Suppose $u \geq 0$ is a (weak) solution of*

$$\Delta u + c(x)u = K(x)u^p, \tag{7}$$

where c, K are smooth functions, and

$$1 \leq p < \frac{n+2}{n-2}.$$

If

$$\int u^{\frac{2n}{(n-2)}}\, dV \leq B, \tag{8}$$

then u satisfies

$$\sup_M u \leq C(p, B).$$

In fact, we can estimate u with respect to any Hölder norm, all in terms of p and B. The main point here is that the assumption $p < \frac{n+2}{n-2}$ is *crucial*.

Proof. Using the preceding elliptic regularity theorem, we know that u satisfies

$$\begin{aligned} \|u\|_{2,m} &\leq C\big(\|\Delta u\|_m + \|u\|_m\big) \\ &\leq C\big(\|u^p\|_m + \|u\|_m\big) \\ &\leq C\big(\|u\|_{mp}^p + \|u\|_m\big). \end{aligned} \tag{9}$$

Denote

$$m_0 = \frac{2n}{n-2},$$

and choose m so that

$$mp = m_0.$$

It follows from (9) that

$$\|u\|_{2,m} \leq C(p, B).$$

We now use the Sobolev embedding theorem, which says

$$\|u\|_r \leq C\|u\|_{2,m}$$

where

$$\frac{1}{r} = \frac{1}{m} - \frac{2}{n} = \frac{n-2m}{mn},$$

or,

$$r = \frac{mn}{n-2m} = \frac{(\frac{m_0}{p})n}{n - 2(\frac{m_0}{p})}.$$

So, we've passed from one Lebesgue-space estimate to another. Have things improved?

The answer is yes, as long as

$$\frac{(\frac{m_0}{p})n}{n - 2(\frac{m_0}{p})} > m_0.$$

Solving this inequality, we see that it will hold provided p satisfies

$$p < \frac{n+2}{n-2}.$$

In this case, we iterate this process an arbitrary number of times, and conclude that

$$\|u\|_{2,m} \leq C(m,p,B) \quad \forall m >> 1.$$

Once m is large enough, though, we can once more appeal to the Sobolev embedding theorem, part (ii), and conclude that u is Hölder continuous–in particular, u is bounded as claimed.

Remarks.

1. For higher order regularity we turn to the Schauder estimates, since we actually proved that u is Hölder continuous. Iterating the Schauder estimates, we can prove the Hölder continuity of derivatives of all orders.

2. As we mentioned above, and will soon see by explicit example, the preceding result is false if $p = (n+2)/(n-2)$. However, it can be "localized": that is, if

$$\int_{B(x_0,r)} u^{\frac{2n}{(n-2)}} \, dV \leq \epsilon_0$$

for some $\epsilon_0 > 0$ small enough, then

$$\sup_{B(x_0,r/2)} u \leq C(r).$$

3. A Corollary of Theorem 3.4 is that weak solutions of (7) are regular, for all $1 \leq p \leq (n+2)/(n-2)$.

4 Background from Conformal Geometry

In this, the final section of the introductory material, we present some basic ideas from conformal geometry.

Definition 4.1. Let (M^n, g) be a Riemannian manifold. A metric h is *pointwise conformal* to g (or just *conformal*) if there is a function f such that

$$h = e^f g.$$

The function e^f is referred to as the *conformal factor*. We used the exponential function to emphasize the fact that we need to multiply by a positive function (since h must be positive definite). However, in some cases it will be more convenient to write the conformal factor differently.

We can introduce an equivalence relation on the set of metrics: $h \sim g$ iff h is pointwise conformal to g. The equivalence class of a metric g is called its *conformal class*, and will be denoted by $[g]$. Note that

$$[g] = \{e^f g \mid f \in C^\infty(M)\}.$$

Definition 4.2. Let (M, g) and (N, h) be two Riemannian manifolds. A diffeomorphism $\varphi : M \to N$ is called *conformal* if

$$\varphi^* h = e^f g.$$

We say that (M, g) and (N, h) are *conformally equivalent.* Note h and g are pointwise conformal if and only if the identity map is conformal.

Example 1. Let $\delta_\lambda(x) = \lambda^{-1} x$ be the dilation map on $\mathbf{R}^n$, where $\lambda > 0$. Then δ_λ is easily seen to be conformal; in fact,

$$\delta_\lambda^* ds^2 = \lambda^{-2} ds^2,$$

where ds^2 is the Euclidean metric.

Example 2. Let $P = (0, \ldots, 0, 1)$ be the north pole of $\mathbf{S}^n \subset \mathbf{R}^{n+1}$. Let $\sigma : \mathbf{S}^n \setminus \{P\} \to \mathbf{R}^{n+1}$ denote stereographic projection, defined by

$$\sigma(\zeta^1, \ldots, \zeta^n, \xi) = \big(\frac{\zeta^1}{1-\xi}, \ldots, \frac{\zeta^n}{1-\xi}\big).$$

Then $\sigma : (\mathbf{S}^n \setminus \{P\}, g_0) \to (\mathbf{R}^{n+1}, ds^2)$ is conformal, where g_0 is the standard metric on $\mathbf{S}^n$.

Since the composition of conformal maps is again conformal, we can use σ to construct conformal maps of $\mathbf{S}^n$ to itself: for $\lambda > 0$, let

$$\varphi_\lambda = \sigma^{-1} \circ \delta_\lambda \circ \sigma : \mathbf{S}^n \to \mathbf{S}^n.$$

Then

$$\varphi_\lambda^* g_0 = \Psi_\lambda^2 g_0,$$

where

$$\Psi_\lambda(\zeta, \xi) = \frac{2\lambda}{(1+\xi) + \lambda^2(1-\xi)}.$$

Note

$$\begin{aligned} (\zeta, \xi) = (\mathbf{0}, \mathbf{1}) &\Rightarrow \boldsymbol{\Psi}_\lambda \to \infty \quad \text{as } \lambda \to \infty, \\ (\zeta, \xi) \neq (\mathbf{0}, \mathbf{1}) &\Rightarrow \boldsymbol{\Psi}_\lambda \to \mathbf{0} \quad \text{as } \lambda \to \infty. \end{aligned}$$

The set of conformal maps of a given Riemannian manifold is a Lie group; the construction above shows that the conformal group of the sphere is *non-compact*. This fact distinguishes the sphere:

Theorem 4.3. (Lelong-Ferrand) *A compact Riemmanian manifold with non-compact conformal group is conformally equivalent to the sphere with its standard metric.*

This fact is the source of many of the analytic difficulties we will encounter in the PDEs we are about to describe.

1. Curvature and Conformal Changes of Metric

Let $h = e^{-2u}g$ be conformal metrics, and let $Ric(h), R(h)$ denote the Ricci and scalar curvatures of h, and $Ric(g), R(g)$ denote the Ricci and scalar curvatures of g. Then

$$\begin{aligned} Ric(h) &= Ric(g) + (n-2)\nabla^2 u + \Delta u g \\ &\quad + (n-2) du \otimes du - (n-2)|\nabla u|^2 g, \\ R(h) &= e^{2u}\big\{R(g) + 2(n-1)\Delta u \\ &\quad - (n-1)(n-2)|\nabla u|^2\big\}, \end{aligned}$$

where $\nabla^2 u$ and Δu denote the Hessian and laplacian of u with respect to g.

2. The Uniformization Theorem and Yamabe Problem.

Let (M^2, g) be a closed (no boundary), compact, two-dimensional Riemannian manifold. Let K denotes its Gauss curvature.

Theorem 4.4. (The Uniformization Theorem) *There is a conformal metric $h = e^{-2u}g$ with constant Gauss curvature.*

See ([Ber03], p. 254) for some historical background on the result. Let $K_h = const.$ denote the Gauss curvature of the metric h; then the sign of K_h is determined by the Gauss-Bonnet formula:

$$\begin{aligned} 2\pi\chi(M^2) &= \int K_h \, dA_h \\ &= K_h \cdot Area(h). \end{aligned}$$

Note the geometric/topological significance of the Uniformization Theorem: Since h has constant curvature, by the Hopf theorem the universal cover $\tilde{M}$ is isometric to either $\mathbf{S}^2, \mathbf{R}^2$, or $\mathbf{H}^2$, each case being determined by the sign of the Euler characteristic.

Now let (M^n, g) be a closed, compact, Riemannian manifold of dimension $n \geq 3$. In higher dimensions there are obstructions to being even *locally* conformal to a constant curvature metric. This leads to

Question: How do we generalize the Uniformization Theorem to higher dimensions?

A major theme of these lectures is the various ways one might answer this question (there are yet others). The first attempt we will discuss is the *Yamabe Problem*: Find a conformal metric $h = e^{-2u} g$ whose *scalar curvature* is constant.

By the formulas above, solving the Yamabe problem is equivalent to solving the semilinear PDE

$$2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + R(g) = \mu e^{-2u}$$

for some constant μ. This formula can be simplified if we write $h = v^{4/(n-2)} g$, where $v > 0$. Then v should satisfy

$$-\frac{4(n-1)}{(n-2)}\Delta v + R(g)v = \lambda v^{\frac{n+2}{n-2}}. \tag{10}$$

Notice the exponent! This equation is of the form

$$\Delta v + c(x)v = K(x)v^p,$$

where $p = (n+2)/(n-2)$. This is the critical case of the equation we considered in Theorem 3.4.

3. The Case of the Sphere

Recall the conformal maps of the sphere described above, $\varphi_\lambda : \mathbf{S}^n \to \mathbf{S}^n$. Then $h = \varphi_\lambda^* g_0 = \Psi_\lambda^2 g_0$ has the same scalar curvature as the standard metric. Therefore, writing

$$h = v_\lambda^{4/(n-2)} g_0,$$

where

$$v_\lambda = \Psi_\lambda^{\frac{(n-2)}{2}},$$

we have a family $\{v_\lambda\}$ of solutions to

$$-\frac{4(n-1)}{(n-2)}\Delta v_\lambda + n(n-1)v_\lambda = n(n-1)v_\lambda^{\frac{n+2}{n-2}}.$$

As we observed above, if P is the North pole, then

$$v_\lambda(P) \to \infty \quad \text{as } \lambda \to \infty,$$

whereas if $x \neq P$, then

$$v_\lambda(x) \to 0 \quad \text{as } \lambda \to \infty.$$

To summarize, there is good news and bad. The good news is that there are many solutions of the Yamabe equation. The bad news is that it will be impossible to prove *a priori* estimates for solutions of (10). Of course, the non-compactness of the set of solutions arises precisely because of the influence of the conformal group. Thus, on manifolds other than the sphere, one would expect that the set of solutions is compact. Put another way, ideally we would like to show that non-compactness implies the underlying manifold is $(\mathbf{S}^n, g_0)$.

2. The Yamabe Problem: A variational Approach.

There is an approach to solving the Yamabe problem by the methods of the calculus of variations. Define the functional $\mathcal{Y} : W^{1,2} \to \mathbf{R}$ by

$$\mathcal{Y}[v] = \frac{\int \left(\frac{4(n-1)}{(n-2)} |\nabla v|^2 + R(g) v^2 \right) \, dV}{\left(\int v^{\frac{2n}{(n-2)}} \, dV \right)^{(n-2)/n}}. \tag{11}$$

Using the formulas above, one can check that

$$\mathcal{Y}[v] = Vol(h)^{-(n-2)/n} \int R(h) \, dV(h),$$

where $h = v^{4/(n-2)} g$. The quantity on the right-hand side is called the *total scalar curvature* of h.

Lemma 4.5. *A function $v \in W^{1,2}$ is a critical point of $\mathcal{Y}$ iff v is a weak solution of the Yamabe equation.*

By critical point, we mean that

$$\frac{d}{dt} \mathcal{Y}(v + t\varphi) \big|_{t=0} = 0$$

for all $\varphi \in W^{1,2}$.

Recall that weak solutions of (10) are regular. Also, by the Sobolev embedding theorem the number

$$Y(M^n, [g]) = \inf_{v \in W^{1,2}} \mathcal{Y}(v) \tag{12}$$

is $> -\infty$. This number is called the *Yamabe invariant* of the conformal class of g.

Some historical notes: H. Yamabe claimed to have proved the existence of a minimizer of $\mathcal{Y}$, for all manifolds (M^n, g). However, N. Trudinger found a serious gap in his proof, which he was able to fix provided $Y(M^n, [g])$ was sufficiently small (for example, if $Y(M^n, [g]) \leq 0$). Subsequently, T. Aubin proved that for all n-dimensional manifolds

$$Y(M^n, [g]) \leq Y(\mathbf{S}^n, [g_0]), \tag{13}$$

and that whenever this inequality was strict, a minimizing sequence converges (weakly) to a (smooth) solution of the Yamabe equation. Aubin also showed that a strict inequality holds in (13) if (M^n, g) was of dimension $n \geq 6$ and not locally conformal to a flat metric.

Finally, the remaining cases were solved by Schoen: that is, he showed that when M^n has dimension $3, 4$, or 5, or if M is locally conformal to a flat metric, then the inequality (13) is strict unless (M^n, g) is conformally equivalent to (S^n, g_0). An excellent survey of the Yamabe problem can be found in [LP87].

5 A Fully Nonlinear Yamabe Problem

In this section we begin our discussion of the σ_k-*Yamabe problem*, a more recent attempt to generalize the Uniformization Theorem to higher dimensions. To do so, we need to introduce another notion of curvature:

Definition 5.1. The Schouten tensor of (M, g) is

$$A = \frac{1}{(n-2)} \left(Ric - \frac{1}{2(n-1)} R \cdot g \right). \tag{14}$$

Example. For spaces of constant curvature ± 1 (e.g. the sphere or hyperbolic space), the Schouten tensor is

$$A = \operatorname{diag} \left\{ \pm\frac{1}{2}, \ldots, \pm\frac{1}{2} \right\}.$$

From the perspective of conformal geometry, the Schouten is actually more natural than the Ricci tensor (but this takes some time to explain). Here's one indication: Suppose $\widehat{g} = e^{-2u} g$. Then the Schouten tensor of $\widehat{g}$ is given by

$$\widehat{A} = A + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g. \tag{15}$$

A complicated formula; but just think of it as saying

$$\widehat{A} = \nabla^2 u + \cdots$$

where $\cdots$ indicates lower order terms. Contrast this with the more complicated formulas for the Ricci tensor, which also involves the Laplace operator.

The equations we will consider involve symmetric functions of the eigenvalues of A. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A; suppose we choose a local basis which diagonalizes A:

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

Then define

$$\sigma_k(A) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{16}$$

i.e., σ_k is the k^{th} elementary symmetric polynomial in n variables. Note that

$$\sigma_1(A) = \mathrm{trace}(A) = \frac{R}{2(n-1)},$$

just a multiple of the scalar curvature. In general, the quantity $\sigma_k(A)$ is called the k^{th}*-scalar curvature*, or σ_k*-curvature*, of the manifold.

Now, we can rephrase the Yamabe problem in the following way: Given (M^n, g) find a conformal metric $\widehat{g} = e^{-2u} g$ with constant σ_1-curvature. This naturally leads to the σ_k*-Yamabe problem*: Given (M^n, g), find a conformal metric $\widehat{g} = e^{-2u} g$ such that the σ_k-curvature is constant. By the formula above, this is equivalent to solving the PDE

$$\sigma_k\big(A + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g\big) = \mu e^{-2ku} \tag{17}$$

for some constant μ. Note the exponential weight on the right comes from the fact that we are computing the eigenvalues of $\widehat{A}$ w.r.t. $\widehat{g}$.

These equations are closely related to the Hessian equations covered in Prof. Xu-Jia Wang's C.I.M.E. course. The differences will come from (1) The conformal invariance, and (2) The lower order (gradient) terms.

The σ_k-Yamabe problem was first formulated by J. Viaclovsky in his thesis [Via00]. Viaclovsky is also the author of a recent survey article on the subject, [Via06].

5.1 Ellipticity

Recall from Professor Wang's lectures that the *Hessian equations*

$$\sigma_k(\nabla^2 u) = f(x)$$

are elliptic provided u is *admissible*, or k-convex. That is, if

$$\sigma_j(\nabla^2 u) > 0, \quad 1 \leq j \leq k.$$

In particular, a necessary condition is that $f(x) > 0$. We will need to impose a similar ellipticity condition:

Definition 5.2. A metric g is *admissible* (or *k-admissible*) if the Schouten tensor satisfies

$$\sigma_j(A) > 0, \quad 1 \leq j \leq k$$

at each point of M^n.

What is the *geometric* meaning of admissibility? One can think of it as a kind of "positivity" condition on the Schouten tensor. When $k = n$, it means the Schouten tensor is positive definite; when $k = 1$, it means the trace (i.e., the scalar curvature) is positive. Here is a more precise result, due to Guan-Viaclovsky-Wang [GVW03]:

Theorem 5.3. *If (M^n, g) is k-admissible then*

$$Ric \geq \frac{2k-n}{2n(k-1)} R \cdot g.$$

In particular, if $k > n/2$ then admissibility means positive Ricci curvature. We can also define *negative admissibility*, which just means that $(-A)$ is k-convex.

As in the usual Yamabe problem, there is a non-compact family of solutions to the σ_k-Yamabe problem on S^n:

$$g_\lambda = \varphi_\lambda^* g_0 = \Psi_\lambda^2 g_0.$$

In particular, this gives an obstruction to proving *a priori* estimates (as it does for the Yamabe problem). Thus, we are faced with some of the same technical difficulties. However, there are some important technical differences between the σ_k- and classical Yamabe problems. For example, equation (17) does not have an easy variational description (though there are some important geometric cases where it does).

A more mysterious contrast arises when studying manifolds of negative curvature. If (M^n, g) has negative scalar curvature, the Yamabe problem is very easy to solve–indeed, the solution is unique. But for negative admissible metrics there are at this time no general existence results for the σ_k-Yamabe problem. In fact, Sheng-Trudinger-Wang showed by example that the local estimates of Guan-Wang are *false* for solutions in the negative cone (see [STW05]).

Finally, we remark that the condition of admissibility can be very restrictive: for example, the manifold $X^3 = S^2 \times S^1$ does not admit a k-admissible metric for $k = 2$ or 3. Of course, one can consider the Yamabe problem for *any* conformal class on X^3.

5.2 From Lower to Higher Order Estimates

Our goal is to explain the main issues involved in solving the σ_k-Yamabe problem, and sketch the proof of a particular case. As we shall see, the central problem is establishing *a priori* estimates. Owing to a fundamental result of Evans, Krylov ([Eva82], [Kry83]), plus the classical Schauder estimates, we only need to worry about estimating derivatives up to order $\underline{\text{two}}$. That is,

$$\begin{gathered} |u| + |\nabla u| + |\nabla^2 u| \leq C_2 \\ \Downarrow \\ |u| + |\nabla u| + \cdots + |\nabla^k u| \leq C(k, C_2). \end{gathered}$$

Of course, even C^2-estimates will fail without further assumptions, again because of the sphere. However, let's look closer: Let $\varphi_\lambda : S^n \to S^n$ be the 1-parameter family of conformal maps, and write

$$g_\lambda = \varphi_\lambda^* g_0 = e^{-2u_\lambda} g_0.$$

Note that as $\lambda \to \infty$, the conformal factor grows like

$$\max e^{-2u_\lambda} \sim \lambda^2,$$

while the gradient and Hessian of u grow like

$$\max |\nabla u_\lambda|^2 \sim \lambda^2, \quad \max |\nabla^2 u_\lambda| \sim \lambda^2.$$

In particular, for this family we have

$$|\nabla u|^2 + |\nabla^2 u| \approx \max e^{-2u_\lambda}.$$

So the optimal estimate one could hope for would be

$$\max \left(2^{nd} \text{ derivatives of } u \right) \leq C \max e^{-2u}. \tag{18}$$

It turns out that such an estimate always holds:

Theorem 5.4. (See Guan-Wang, [GW03]) *Assume $u \in C^4$ is an admissible solution of the σ_k-Yamabe equation on $B(1)$. Then*

$$\max_{B(1/2)} \left[|\nabla u|^2 + |\nabla^2 u|\right] \leq C(1 + \max_{B(1)} e^{-2u}).$$

In view of this result, and the Evans and Krylov results, we see that

$$\min_M u \geq C \Rightarrow \|u\|_{C^{k,\alpha}(M)} \leq C(k).$$

Therefore, if we can somehow rule out "bubbling", we obtain estimates of all orders. Once estimates are known, there are various topological methods to prove the existence of solutions. This shows the geometric nature of the problem: i.e., we need to detect the global geometry of the manifold in order to get estimates, hence existence.

5.3 An Existence Result: Four Dimensions

To finish our discussion of the σ_k-Yamabe problem, we want to sketch its solution in four dimensions. This case is special because, in 4-d, the integral

$$\int \sigma_2(A)\ dV$$

is conformally invariant. That is, if $\widehat{g} = e^{-2u} g$, then

$$\int \sigma_2(\widehat{A})\ d\widehat{V} = \int \sigma_2(A)\ dV.$$

You can check this by hand using the formulas above along with the fact that

$$d\widehat{V} = e^{-4u} dV.$$

Eventually, you will find that

$$\sigma_2(\widehat{A})\ d\widehat{V} = \sigma_2(A)\ dV + (\text{divergence terms}).$$

We will provide some details for the case $k = 2$; this was first treated by Chang-Gursky-Yang [CGY02b], and later by Gursky-Viaclovsky [GV04]. For $k = 3$ or 4, the scheme of the proof is essentially the same. However, the proof presented here is a simplified version of the original one, since we will use the local estimates of Guan-Wang (which appeared several years after [CGY02b]). As we emphasized above, existence eventually boils down to estimates: this is what we will prove.

To begin, let us write the equation in the case $k = 2$:

$$\sigma_2^{1/2}\big(A + \nabla^2 u + du \otimes du - \frac{1}{2}|du|^2 g\big) = f(x)e^{-2u}. \tag{19}$$

where $f \in C^\infty$. Using the definition of σ_2, this actually reads:

$$\begin{aligned} -|\nabla^2 u|^2 + (\Delta u)^2 + c_1 \nabla_i \nabla_j u \nabla_i u \nabla_j u \\ + c_2 \Delta u |\nabla u|^2 + c_3 |\nabla u|^4 + \cdots = f^2(x) e^{-4u}. \end{aligned}$$

We will prove:

Theorem 5.5. *Suppose (M^4, g) is (i) admissible, and (ii) not conformally equivalent to the round sphere. If $u \in C^4$ is a solution of (19), then there is a constant $C = C(g, f)$ such that*

$$\min u \geq -C.$$

Consequently,

$$\|u\|_{C^k} \leq C(k).$$

Proof. Suppose to the contrary there is a sequence of solutions $\{u_i\}$ of (19) with $\min u_i \to -\infty$. Let's imagine that there is a point P with

$$\min_M u_i = u_i(P)$$

and by introducing local coordinates we can identify P with the origin in $\mathbb{R}^4$ and think of u_i as being defined in a neighborhood Ω of 0. (In reality, the location of the minimum point will vary, but this doesn't affect the argument in a significant way).

It is time to use conformal invariance. Recall the dilations on Euclidean space are conformal. Define

$$w_i(x) = u(\epsilon_i x) + \log \frac{1}{\epsilon_i},$$

where $\epsilon_i > 0$ is chosen so that

$$w_i(0) = 0.$$

The w_i's are defined on $\frac{1}{\epsilon_i}\Omega$, and satisfy

$$\begin{aligned} \sigma_2^{1/2}\big(\epsilon_i^2 A + \nabla^2 w_i + dw_i \otimes dw_i - \frac{1}{2}|dw_i|^2 g\big) \\ = f(\epsilon_i x) e^{-2w_i}. \end{aligned}$$

After applying the local estimates of Guan-Wang, we can take a subsequence $\{w_i\}$ which converges in $C^{k,\alpha}_{loc}$ to a solution of

$$\sigma_2^{1/2}\big(\nabla^2 w + dw \otimes dw - \frac{1}{2}|dw|^2 g\big) = \mu e^{-2w}. \tag{20}$$

with $\mu > 0$.

We now appeal to the following uniqueness result

Lemma 5.6. (See Chang-Gursky-Yang, [CGY02a]) *Up to scaling, the unique solution of (20) is realized by*

$$e^{2w} ds^2 = (\sigma^{-1})^* g_0, \tag{21}$$

where σ is the stereographic projection map, ds^2 the Euclidean metric, and g_0 is the round metric on the sphere.

It is easy to check that each solution given by (21) satisfies

$$\int_{\mathbb{R}^4} \sigma_2(\widetilde{A})\ d\widetilde{V} = 4\pi^2.$$

where $\widetilde{g} = e^{2w} ds^2$. (Remember that $A = \mathrm{diag}\{1/2, \ldots, 1/2\}$, and $\mathrm{Vol}(S^4) = 8\pi^2/3$). Also, since our solution w comes from blowing up a little piece of the original manifold, for each $\widehat{g}_i = e^{-2u_i} g$ we must have

$$\int_{M^4} \sigma_2(\widehat{A}_i)\ d\widehat{V}_i \geq 4\pi^2.$$

The proof now follows from the following global geometric result:

Theorem 5.7. (See Gursky, [Gur99]) *If (M^4, g) has positive scalar curvature, then*

$$\int_{M^4} \sigma_2(A)\ dV \leq 4\pi^2,$$

and equality holds if and only if (M^4, g) is conformally equivalent to the sphere.

It follows that each (M^4, g_i) is conformally equivalent to the round sphere, a contradiction. Therefore, assuming the manifold (M^4, g) is not conformally the sphere, any sequence of solutions remains bounded, as claimed.

Important Remark. The following remark is for the benefit of experts: The proof of the preceding theorem does <u>not</u> use the Positive Mass Theorem! (Or, to be precise, it uses an extremely weak form). Therefore, we are not solving the σ_k-Yamabe problem by somehow reducing it to the classical Yamabe problem.

6 The Functional Determinant

In the final section we will introduce a higher order elliptic problem which has its origins in spectral theory. Although this problem is semilinear and not fully nonlinear, the structure of the Euler equation is related to the σ_2-Yamabe equation in 4-d. Moreover, for 4-manifolds of positive scalar curvature, the same result (Theorem 5.7) plays a crucial role in the existence theory.

Suppose (M^n, g) is a closed Riemannian manifold, and let Δ denote the Laplace-Beltrami operator associated to g. We can label the eigenvalues of $(-\Delta)$ (counting multiplicities) as

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \tag{22}$$

The *spectral zeta function* of (M^2, g) is defined by

$$\zeta(s) = \sum_{j \geq 1} \lambda_j^{-s}. \tag{23}$$

By Weyl's asymptotic law,

$$\lambda_j \sim j^{2/n}.$$

Consequently, (23) defines an analytic function for $\mathrm{Re}(s) > n/2$. In fact, one can meromorphically continue ζ in such a way that ζ becomes regular at $s = 0$ (see [RS71]). Note that formally–that is, if we take the definition in (23) literally–then

$$\begin{aligned} \zeta'(0) &= -\sum_{j \geq 1} \log \lambda_j \\ &= -\log \Big\{ \prod_{j \geq 1} \lambda_j \Big\} \\ &= -\log \det(-\Delta_g). \end{aligned} \tag{24}$$

In view of this *ansatz*, it is natural to define the regularized determinant of $(-\Delta_g)$ as

$$\det(-\Delta_g) = e^{-\zeta'(0)}. \tag{25}$$

6.1 The Case of Surfaces

Clearly, the determinant is not a local quantity. Therefore, it is rather remarkable that Polaykov ([Pol81a], [Pol81b]) was able to compute a closed formula for the *ratio* of the determinants of the laplacians of two conformally related surfaces:

Theorem 6.1. *Let $(\Sigma, g), (\Sigma, \hat{g} = e^{2w} g)$ be conformal surfaces. Then*

$$\log \frac{\det(-\Delta_{\hat{g}})}{\det(-\Delta_g)} = -\frac{1}{12\pi} \int_\Sigma \left[|\nabla w|^2 + 2Kw \right] \, dA, \tag{26}$$

where K is the Gauss curvature and dA the surface measure associated to (Σ^2, g).

Remarks.

1. The formula (26) naturally defines a (relative) action on the space of conformal metrics. That is, once we fix a metric g, we have the functional

$$\hat{g} \in [g] \mapsto \log \frac{\det(-\Delta_{\hat{g}})}{\det(-\Delta_g)}.$$

However, since the determinant is not scale-invariant, we should consider the *normalized functional determinant*

$$S[w] = \int_\Sigma \left[|\nabla w|^2 + 2Kw \right] \, dA - \big(\int_\Sigma K \, dA \big) \log \big(\fint_\Sigma e^{2w} \, dA \big), \tag{27}$$

so that

$$S[w] = -12\pi \log \frac{\det(-\Delta_{\hat{g}})}{\det(-\Delta_g)} + 2\pi \chi(\Sigma) \log \mathrm{Area}(\hat{g}),$$

while

$$S[w + c] = S[w].$$

2. A first variation shows that w is a critical point of S if and only if w satisfies

$$\Delta w + c e^{2w} = K, \tag{28}$$

where c is a constant. Now, if $\hat{g} = e^{2w} g$, then the Gauss curvature $\hat{K}$ of $\hat{g}$ is related to the Gauss curvature of g via

$$\Delta w + \hat{K} e^{2w} = K, \tag{29}$$

this is called the *Gauss curvature equation*. Comparing (28) and (29), we see that w is a critical point of S if and only if $(\Sigma, \hat{g})$ has constant Gauss curvature. In particular, a metric extremizes the functional determinant if and only if it uniformizes; i.e., it is a conformal metric of constant Gauss curvature.

3. In a series of papers ([Osg88b], [Osg88a]) Osgood-Phillips-Sarnak gave a proof of the Uniformization Theorem by showing that each conformal class on a surface admits a metric that extermizes the determinant. Like the Yamabe problem and its fully nonlinear version discussed earlier in the article, the main difficulty is the invariance of the determinant under the action of the conformal group. And like the analysis of the Yamabe problem, the solution involves the study of sharp functional inequalities. A very nice overview of the study of the functional determinant and related material can be found in [Cha].

6.2 Four Dimensions

The key property of the Laplacian that Polyakov exploited in his calculation was its *conformal covariance*:

$$\Delta_{e^{2w}g} = e^{-2w} \Delta_g. \tag{30}$$

More generally, we say that the differential operator $A = A_g : C^\infty(M^n) \to C^\infty(M^n)$ is *conformally covariant* of bi-degree (a, b) if

$$A_{e^{2w}g} \varphi = e^{-bw} A_g(e^{aw} \varphi). \tag{31}$$

In fact, this definition makes perfect sense for operators acting on smooth sections of bundles (spinors, forms, etc.) as well as on functions. Two examples of note are

Example 1. The conformal laplacian of (M^n, g), where $n \geq 3$, is

$$L = -\frac{4(n-1)}{(n-2)} \Delta + R, \tag{32}$$

where R is the scalar curvature. Then L is conformally covariant with

$$a = \frac{n-2}{2}, \; b = \frac{n+2}{2}.$$

Example 2. Let (M^4, g) be a four-dimensional Riemannian manifold. The *Paneitz operator* is

$$P = (\Delta)^2 + \mathrm{div}\{(\frac{2}{3}Rg - 2Ric) \circ d\}. \tag{33}$$

Then P is conformally covariant with

$$a = 4, \ b = 0.$$

An analogue of Polyakov's formula for conformally covariant operators defined on four-manifolds was computed by Branson-Ørsted in [Bra91]. To explain the Branson-Ørsted formula we need to introduce three functionals associated to a Riemannian 4-manifold (M^4, g). Each functional is defined on $W^{2,2}$, the Sobolev space of functions with derivatives up to order two in L^2.

The first functional is zeroth order in w:

$$I[w] = 4 \int w|W|^2 \ dV - \big(\int |W|^2 \ dV \big) \log ⨍ e^{4w} \ dV, \tag{34}$$

where W is the Weyl curvature tensor and dV the volume form of g. If $w \in W^{2,2}$, The Moser-Trudinger inequality ([GT83]) implies that

$$e^w \in L^p, \text{all } p \geq 1.$$

Therefore, $I : W^{2,2} \to \mathbb{R}$.

The second functional is analogous to the functional S defined in (27):

$$II[w] = \int wPw \ dV + 4 \int Qw \ dV - \big(\int Q \ dV \big) \log ⨍ e^{4w} \ dV, \tag{35}$$

where P is the Paneitz operator and Q is the Q-curvature:

$$Q = \frac{1}{12}(-\Delta R + R^2 - 3|Ric|^2). \tag{36}$$

Here we see the parallel between the Laplace-Beltrami operator/Gauss curvature of a surface and the Paneitz operator/Q-curvature of a 4-manifold.

The third functional is

$$III[w] = 12 \int (\Delta w + |\nabla w|^2)^2 \ dV - 4 \int (w \Delta R + R|\nabla w|^2) \ dV. \tag{37}$$

The geometric meaning of this functional is apparent if we rewrite it in terms of the scalar curvature $R_{\hat{g}}$ and volume form $d\hat{V}$ of the conformal metric $\hat{g} = e^{2w}g$:

$$III[w] = \frac{1}{3}\big[\int R_{\hat{g}}^2 \ d\hat{V} \big] - \int R^2 \ dV \big]. \tag{38}$$

Therefore, III is the L^2-version of the Yamambe functional in (11).

With these definitions, we can give the Branson-Ørsted formula: Suppose $A = A_g$ is a conformally covariant differential operator satisfying certain "naturality" conditions (see [Bra91] for details). Then there are numbers, $\gamma_i = \gamma_i(A), 1 \leq i \leq 3$, such that

$$F_A[w] = \log \frac{\det A_{e^{2w}g}}{\det A_g} = \gamma_1 I[w] + \gamma_2 II[w] + \gamma_3 III[w]. \tag{39}$$

We remark that the Branson-Ørsted formula is normalized; i.e., $F_A[w+c] = F_A[w]$.

Example 1. For the conformal laplacian, Branson-Ørsted calculated

$$\gamma_1(L) = 1,\ \gamma_2(L) = -4,\ \gamma_3(L) = -\frac{2}{3}. \tag{40}$$

Example 2. Later, in [Bra96], Branson calculated the coefficients for the Paneitz operator:

$$\gamma_1(L) = -\frac{1}{4},\ \gamma_2(L) = -14,\ \gamma_3(L) = \frac{8}{3}. \tag{41}$$

Neglecting lower order terms, the log det functional is of the form

$$\begin{aligned} \log \frac{\det A_{e^{2w}g}}{\det A_g} = \gamma_1 \int (\Delta w)^2\ dV + \gamma_3 \int [\Delta w + |\nabla w|^2]^2\ dV \\ + \kappa_A \log \fint e^{4w}\ dV + \cdots , \end{aligned} \tag{42}$$

where κ_A is given by

$$\kappa_A = -\gamma_1 \int |W|^2\ dV - \gamma_2 \int Q\ dV, \tag{43}$$

a conformal invariant. In particular, when γ_2 and γ_3 have the same sign (as they do for the conformal laplacian), the main issue from the variational point of view is the interaction of the highest order terms with the exponential term. However, when the signs of γ_2 and γ_3 differ, then the highest order terms are a non-convex combination of II and III, and the variational structure can be quite complicated.

6.3 The Euler Equation

As we observed above, critical points of the functional determinant on a surface corresponds to metrics of constant Gauss curvature. In four dimensions the geometric meaning of the Euler equation is less straightforward: Suppose $\hat{g} = e^{2w} g$ is a critical point of F_A; then the curvature of $\hat{g}$ satisfies

$$\gamma_1 |W_{\hat{g}}|^2 + \gamma_2 Q_{\hat{g}} + \gamma_3 \Delta_{\hat{g}} R_{\hat{g}} = -\kappa_A \cdot Vol(\hat{g})^{-1}. \tag{44}$$

The geometric significance of this condition is, at first glance, difficult to fathom. However, this equation in some sense includes all the significant curvature conditions studied in four-dimensional conformal geometry, as can be seen by considering special values of the γ_i's:

- Taking $\gamma_1 = \gamma_2 = 0$ and $\gamma_1 = 1$, equation (44) becomes

$$\Delta_{\hat{g}} R_{\hat{g}} = const. = 0, \tag{45}$$

which is equivalent to the Yamabe equation

$$R_{\hat{g}} = const.$$

- Taking $\gamma_1 = 0$ and $\gamma_2 = -12\gamma_3$, equation (44) becomes

$$\sigma_2(A_{\hat{g}}) = const., \tag{46}$$

that is, a critical point is a solution of the σ_2-Yamabe problem.

- Taking $\gamma_1 = \gamma_3 = 0$ and $\gamma_2 = 1$, then

$$Q_{\hat{g}} = const. \tag{47}$$

Thus, critical points are solutions of the *Q-curvature problem.*

Geometric properties of critical metrics were used in [Gur98] to prove various vanishing theorems, and as a regularization of the σ_2-Yamambe problem in [CGY02b].

Turning to analytic aspects of the Euler equation, it is clear from (42) that it is fourth order in w. A precise formula is

$$\mu e^{4w} = (\frac{1}{2}\gamma_2 + 6\gamma_3)\Delta^2 w + 6\gamma_3 \Delta |\nabla w|^2 - 12\gamma_3 \nabla^i \left[(\Delta w + |\nabla w|^2) \nabla_i w \right] \tag{48}$$

$$+ \gamma_2 R_{ij} \nabla_i \nabla_j w + (2\gamma_3 - \frac{1}{3}\gamma_2) R \Delta w + (2\gamma_3 + \frac{1}{6}\gamma_2) \langle \nabla R, \nabla w \rangle \tag{49}$$

$$+ (\gamma_1 |W|^2 + \gamma_2 Q - \gamma_3 \Delta R), \tag{50}$$

where R_{ij} are the components of the Ricci curvature and

$$\mu = -\frac{\kappa_A}{\int e^{4w}}. \tag{51}$$

To simplify this expression we can divide both sides of (49) by $6\gamma_3$, then rewrite the lower order terms to arrive at

$$\begin{aligned}(1+\alpha)\Delta^2 w = f(x)e^{4w} - \Delta|\nabla w|^2 + 2\nabla^i\big[(\Delta w + |\nabla w|^2)\nabla_i w\big] \\ + a^{ij}\nabla_i\nabla_j w + b^k\nabla_k w + c(x),\end{aligned} \tag{52}$$

where

$$\alpha = \frac{\gamma_2}{12\gamma_3}. \tag{53}$$

Although writing the equation in this form clearly reveals the divergence structure, for some purposes it is better to expand the terms on the right and write

$$\begin{aligned}(1+\alpha)\Delta^2 w = f(x)e^{4w} - 2|\nabla^2 w|^2 + 2(\Delta w)^2 + 2\langle\nabla|\nabla w|^2, \nabla w\rangle \\ + 2\Delta w|\nabla w|^2 + (\text{lower order terms}).\end{aligned} \tag{54}$$

In particular, we see that the right-hand side does not involve any third derivatives of the solution.

The regularity of extremal solutions of (49) was proved by Chang-Gursky-Yang in [CGY99]), and for general solutions by Uhlenbeck-Viaclovsky in [UV00]. Similar to the harmonic map equation in two dimensions, the main difficulty is that the right-hand side of (54) is only in L^1 when $w \in W^{2,2}$, ruling out the possibility of using a naive bootstrap argument to prove regularity.

6.4 Existence of Extremals

The most complete existence theory for extremals of the functional determinant was done by Chang-Yang in [CY95]:

Theorem 6.2. *Assume*

$$\gamma_2, \gamma_3 < 0, \tag{55}$$

and

$$\kappa_A < 8\pi^2(-\gamma_2). \tag{56}$$

Then $\sup_{W^{2,2}} F_A$ *is attained by some some* $w \in W^{2,2}$.

Remarks.

1. Recall that

$$\kappa_A = -\gamma_1 \int |W|^2 \, dV - \gamma_2 \int Q \, dV.$$

If $\gamma_1 > 0$, then

$$\kappa_A \leq -\gamma_2 \int Q \, dV.$$

Therefore, assuming $\gamma_2 < 0$, then (56) holds provided

$$\int Q \, dV < 8\pi^2. \tag{57}$$

By the definition of the Q-curvature,

$$Q = 2\sigma_2(A) - \frac{1}{12}\Delta R. \tag{58}$$

Therefore,

$$\int Q \, dV = 2 \int \sigma_2(A) \, dV.$$

In particular, for manifolds of positive scalar curvature, by Theorem 5.7 it follows that

$$\int Q \, dV \leq 8\pi^2, \tag{59}$$

with equality if and only if (M^4, g) is conformal to the round sphere. Thus, combining the existence result of Chang-Yang with the sharp inequality of Theorem 5.7, we conclude

Corollary 6.3. *If (M^4, g) has positive scalar curvature, then an extremal for F_L exists.*

2. It is easy to construct examples of 4-manifold–necessarily with negative scalar curvature–for which

$$\int Q \, dV >> 8\pi^2.$$

Thus, the existence theory for the functional determinant is quite incomplete. This shows another parallel with the σ_k-Yamabe problem (and contrast with the classical Yamabe problem): the case of negative curvature is much more difficult than the positive case.

3. Branson-Chang-Yang proved that on the sphere S^4, the functionals II and III are minimized by the round metric and its images under the conformal group [Bra]. In particular, the round metric is the unique extremal (up to conformal transformation) of F_L. Later, in [Gur97], Gursky showed that the round metric is the unique critical point.

6.5 Sketch of the Proof

In the following we give a sketch of the proof of Theorem 6.2. By Corollary 6.3, this will give the existence of extremals for F_A on any 4-manifold of positive scalar curvature.

To begin, we write the functional as

$$\begin{aligned} F_A[w] &= \gamma_1 I[w] + \gamma_2 II[w] + \gamma_3 III[w] \\ &= \gamma_1 \int (\Delta w)^2 + \gamma_2 \int (\Delta w + |\nabla w|^2)^2 + \kappa_A \log ⨍ e^{4(w-\bar{w}} + (l.o.t.). \end{aligned} \tag{60}$$

Next, divide by γ_2, and denote $\tilde{F} = (1/\gamma_2) F_A$:

$$\tilde{F}[w] = \int (\Delta w)^2 + \beta \int (\Delta w + |\nabla w|^2)^2 - \big(\frac{\kappa_A}{-\gamma_2}\big) \log ⨍ e^{4(w-\bar{w})} + (l.o.t.), \tag{61}$$

where

$$\beta = \gamma_3/\gamma_2 > 0. \tag{62}$$

Since $\gamma_2 < 0$, we are trying to prove the existence of *minimizers* of $\tilde{F}$.

Let us first consider the easy case, when $\kappa_A \leq 0$. Then

$$-\big(\frac{\kappa_A}{-\gamma_2}\big) \geq 0.$$

Also, by Jensen's inequality,

$$\log ⨍ e^{4(w-\bar{w})} \geq 0.$$

Therefore,

$$\tilde{F}[w] \geq \int (\Delta w)^2 + \beta \int (\Delta w + |\nabla w|^2)^2 + (l.o.t.) \tag{63}$$

Now suppose $\{w_k\}$ is a minimizing sequence for $\tilde{F}$; from (63) we conclude

$$C \geq \int (\Delta w_k)^2 + (l.o.t.),$$

which implies, for example by the Poincare inequality, that $\{w_k\}$ is bounded in $W^{2,2}$. It follows that a subsequence converges weakly to a minimizer $w \in W^{2,2}$.

For the more difficult case when $\kappa_A > 0$, first observe that by hypothesis, $\kappa_A < 8\pi^2(-\gamma_2)$. Therefore,

$$\frac{\kappa_A}{-\gamma_2} = 8\pi^2(1-\epsilon) \tag{64}$$

for some $\epsilon > 0$. The significance of the constant $8\pi^2$ is apparent from the following sharp Moser-Trudinger inequality due to Adams:

Proposition 6.4. (See [Ada]) *If (M^4, g) is a smooth, closed 4-manifold, then there is a constant $C_1 = C_1(g)$ such that*

$$\log \fint e^{4(w-\bar{w})} \leq \frac{1}{8\pi^2} \int (\Delta w)^2 + C_1. \tag{65}$$

Using Adams' inequality, we will show that the positive terms in $\tilde{F}$ dominate the logarithmic term. To see why, we argue in the following way: by the arithmetic-geometric mean,

$$2\beta xy \geq -\beta(1+\delta)x^2 - \beta(\frac{1}{1+\delta})y^2,$$

for any real numbers x, y, as long as $\beta, \delta > 0$. From this inequality it follows that

$$\int (\Delta w)^2 + \beta \int (\Delta w + |\nabla w|^2)^2 \geq \int (1-\delta\beta)(\Delta w)^2 + \beta(\frac{\delta}{1+\delta})|\nabla w|^4. \tag{66}$$

Therefore, by (64) and (66),

$$\tilde{F}[w] \geq \int (1-\delta\beta)(\Delta w)^2 + \beta(\frac{\delta}{1+\delta}) \int |\nabla w|^4 - 8\pi^2(1-\epsilon) \log \fint e^{4(w-\bar{w})} + (l.o.t.).$$

By Adams' inequality, the logarithmic term above can be estimated by

$$-8\pi^2(1-\epsilon) \log \fint e^{4(w-\bar{w})} \geq -(1-\epsilon) \int (\Delta w)^2 - C.$$

Substituting this above, we get

$$\tilde{F}[w] \geq \int (\epsilon - \delta\beta)(\Delta w)^2 + \beta(\frac{\delta}{1+\delta}) \int |\nabla w|^4 + (l.o.t.).$$

By choosing $\delta > 0$ small enough, we conclude

$$\tilde{F}[w] \geq \delta' \int \left[(\Delta w)^2 + |\nabla w|^4\right] - C.$$

Arguing as we did in the previous case, it follows that $\tilde{F}$ is bounded below, and a minimizing sequence converges (weakly) to a smooth extremal.

Remarks.

1. The lower order terms that we neglected in the proof can actually dominate the expression when $\gamma_3 = 0$, e.g., when studying the Q-curvature problem. In particular, there are known examples of manifolds for which the functional II in not bounded below.

2. When γ_2 and γ_3 have different signs–for example, when $A = P$, the Paneitz operator–the situation is even worse. In fact, F_P is never bounded from below. However, manifolds of constant negative curvature are always local extremals of F_P.

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