Preface

This is a book on holomorphic operator functions of a single variable and their applications, which is focussed on the relations between local and global theories. It is based on methods and technics of Complex analysis of scalar and matrix functions of several variables. The applications concern: interpolation, holomorphic families of subspaces and frames, spectral theory of polynomials with operator coefficients, holomorphic equivalence and diagonalization, and Plemelj-Muschelishvili factorization. The book also contains a theory of Wiener-Hopf integral equations with operator-valued kernels and a theory of infinite Töplitz matrices with operator entries.

We started to work on these topics long ago when one of us was a Ph.D. student of the other in Kishinev (now Cisinau) University. Then our main interests were in problems of factorization of operator-valued functions and singular integral operators. Working in this area, we realized from the beginning that different methods and tools from Complex analysis of several variables and their modifications are very useful in obtaining results on factorization for matrix and operator functions. We have in mind different methods and results concerning connections between local and global properties of holomorphic functions. The first period was very fruitful and during it we obtained the basic results presented in this book.

Then World Politics started to interfere in our joint work in the new area. For a long time the authors became separated. One emigrated to Israel, the other was a citizen of East Germany, and the authorities of the second country prevented further meetings and communications of the authors. During that time one of us became more and more involved in Complex analysis of several variables and finally started to work mainly in this area of mathematics. Our initial aims were for a while frozen. Later the political situation in the world changed and after the reunification of Germany the authors with pleasure continued the old projects.

During the time when our projects were frozen, the scientific situation changed considerably. There appeared in the literature new methods, results and applications. In order to cover the old and new material entirely in a modern form and terminology we decided to write this book. As always happens in such cases, during the writing new problems and gaps appear, and the material requires inclusion of additional material with new chapters containing new approaches, new results and plenty of unification and polishing. This work was done by the authors.

We hope the book will be of interest to a number of large groups of experts in pure and applied mathematics as well as for electrical engineers and physicists.

During the work on the book we obtained support of different kinds for our joint activities from the Tel-Aviv University and its School of Mathematical Sciences, the Family of Nathan and Lilly Silver Foundation, the Humboldt Foundation, the Deutsche Forschungsgemeinschaft and the Humboldt University in Berlin and its Institute of Mathematics. We would like to express our sincere gratitude to all these institutions for support and understanding. We would also like to thank the Faculty of Mathematics and Computer Sciences of the Kishinev University and the Institute of Mathematics and Computer Center of the Academy of Sciences of Moldova, where the work on this book was started.

Berlin, Tel-Aviv, November 2008 The authors

Introduction

The book. This book contains a theory and applications of operator-valued holomorphic functions of a single variable. (By *operators* we always mean bounded linear operators between complex Banach spaces.) The applications concern some important problems on factorization, interpolation, diagonalization and others. The book also contains a theory of Wiener-Hopf integral equations with operator-valued kernels and a theory of infinite Töplitz matrices with operator entries.

Our main attention is focussed on the connection between local and global properties of holomorphic operator functions. For this aim, methods from Complex analysis of several variables are used. The exposition of the material appears in style and terms of the latter field.

Multiplicative cocycles. Grauert's theory. The theory of multiplicative cocycles plays a central role in this book. It is a special case of the very deep and powerful theory of cocycles (fiber bundles) on Stein manifolds (any domain in $\mathbb{C}$ is a Stein manifold), which was developed in the 1950s by H. Grauert for cocycles with values in a (finite dimensional) complex Lie group. This theory then was generalized into different interesting directions. In 1968, L. Bungart obtained it for cocycles with values in a *Banach* Lie group, for example, the group of invertible operators in a Banach space.

One of the main statements of Grauert's theory is a principle which is now called the *Oka-Grauert principle*. Non-rigorously, this principle can be stated as follows: *If a holomorphic problem on a Stein manifold has no topological obstructions, then it has a holomorphic solution.* This important principle was first discovered in 1939 by K. Oka in the case of scalar functions.

For domains in the complex plane $\mathbb{C}$, Grauert's theory is much easier but still not simple. It is even not simple for the case of cocycles with values in the group of invertible complex $n \times n$-matrices when no topological obstructions appear.

For operators in infinite dimensional Banach spaces, we meet essential difficulties, which are due to the fact that the group of invertible operators in a Banach space need not be connected. This becomes a topological obstruction if the domain in $\mathbb{C}$ is not simply connected. So, for operator functions, the Oka-Grauert principle is meaningful also for domains in $\mathbb{C}$.

For the problem of Runge approximation, the Oka-Grauert principle claims the following: Runge approximation of a holomorphic invertible operator function by *holomorphic* invertible functions is possible if this is possible by *continuous* invertible functions. From this it follows that such a Runge approximation always holds when the domain is simply connected or the group of invertible operators is connected. The latter is the case for the group of invertible operators in a Hilbert space, and in particular, for the group of invertible complex $n \times n$-matrices.

For simply connected domains, the proof of the Runge approximation theorem for invertible operator functions is not difficult and can be obtained without the theory of cocycles. We show this at the end of Chapter 2. For general domains however, this proof is much more difficult (even in the case of matrix-valued functions) and will be given only in Chapter 5 in the framework of the theory of multiplicative cocycles.

A special type of multiplicative cocycles is given by two open sets D_1 and D_2 in $\mathbb{C}$ and an invertible holomorphic operator function on $D_1 \cap D_2$. For this type, the following is proved:

0.0.1 Theorem. *Let E be a Banach space, let $GL(E)$ be the group of invertible operators in E, let $D_1, D_2 \subseteq \mathbb{C}$ be two open sets, and let $A : D_1 \cap D_2 \to GL(E)$ be holomorphic. Assume that at least one of the following two conditions is satisfied:*

(i) *The union $D_1 \cup D_2$ is simply connected.*

(ii) *All values of A belong to the same connected component of $GL(E)$.*

Then there exist holomorphic operator functions $A_j : D_j \to GL(E)$, $j = 1, 2$, such that

$$A = A_1 A_2^{-1} \qquad \text{on } D_1 \cap D_2 \,. \tag{0.0.1}$$

If both topological conditions (i) and (ii) in Theorem 0.0.1 are violated, then the assertion of Theorem 0.0.1 is not true. A simple counterexample will be given in Section 5.6.2 for the case when $D_1 \cup D_2$ is an annulus.

The following operator version of the Weierstrass product theorem (on the existence of holomorphic functions with given zeros) is a straightforward consequence of Theorem 0.0.1.

0.0.2 Theorem. *Let E be a Banach space, let $GL(E)$ be the group of invertible operators in E, and let $GL_I(E)$ be the connected component in $GL(E)$ which contains the unit operator I. Let $D \subseteq \mathbb{C}$ be an open set and let Z be a discrete and closed subset of D. Suppose, for each $w \in Z$, a neighborhood $U_w \subseteq D$ of w with $U_w \cap Z = \{w\}$ and a holomorphic operator function $A_w : U_w \setminus \{w\} \to GL(E)$ are given. Further assume that at least one of the following two conditions is fulfilled:*

(i) *The set D is simply connected.*

(ii) *The values of each A_w, $w \in Z$, belong to $GL_I(E)$.*

Then there exist a holomorphic operator function $B : D\setminus Z \to GL(E)$ and a family of holomorphic operator functions $H_w : U_w \to GL(E)$ such that

$$H_w A_w = B \qquad \textit{on } U_w \setminus \{w\},, \quad w \in Z\,.$$

The classical Weierstrass product theorem we get for $E = \mathbb{C}$ and $H_w(z) = (z-w)^{\kappa_w}$, $\kappa_w \in \mathbb{N}^*$.

There are also a "right-sided" and a "two-sided" version of Theorem 0.0.2.

Contents. The book consists of an introduction and eleven chapters. Let us now describe in more detail the content of each chapter separately.

The first chapter contains the generalization to functions with values in Banach spaces of the traditional material from Complex analysis of one variable which is usually contained in the beginning of a basic course.

Chapter 2 starts with Pompeiju's integral formula for solutions of the inhomogeneous Cauchy-Riemann equation, the Runge approximation theorem, the Mittag-Leffler theorem, and the Weierstrass product theorem. Then, in Sections 2.6 and 2.7, we present the (less well known) "Anschmiegungsatz" of Mittag-Leffler and a strengthening of the Weierstrass product theorem. In the case of the Weierstrass product theorem and its generalization, in this chapter, we still restrict ourselves to scalar functions. It is one of the main goals of this book, to generalize these results to the case of operator functions, using Grauert's theory of cocycles.

Chapter 3 is dedicated to the splitting problem with respect to a contour for functions with values in a Banach space, as well as to the factorization problem for scalar functions with respect to a contour.

In Chapter 4 we generalize to finite meromorphic Fredholm operator functions the classical Rouché theorem from Complex analysis and the Smith factorization form. The proof is based on the local Smith form.

Chapter 5 is entirely dedicated to the theory of multiplicative cocycles, which were discussed in large before.

Chapter 6 contains a theory of families of subspaces of a Banach space E. First we introduce a complete metric on the set $G(E)$ of closed subspaces of E, the so-called *gap metric*. A **continuous family of subspaces of** E then will be defined as a continuous function with values in $G(E)$, and a **holomorphic** family of subspaces of E will be defined as a continuous family of subspaces which is locally the image of a holomorphic operator function. Vector functions with values in such a family are called **sections** of the family. Note that we do not require that the members of a holomorphic family be complemented in the ambient space. It may even happen they are not pairwise isomorphic. An example is given in Section 6.5.

First we prove the following results: any additive cocycle of holomorphic sections in a holomorphic family of subspaces splits; for any holomorphic operator function A whose image is a holomorphic family of subspaces, and any holomorphic section f of this family, there exists a global holomorphic vector function u that solves the equation $Au = f$; for any holomorphic family of subspaces there exists a global holomorphic operator function with this family as image. Proving this, the main difficulty is the solution of certain local problems (in this generality, published for the first time in this book). In terms of Complex analysis of several variables, the solution of these local problems means that any holomorphic family of subspaces is a so-called *Banach* coherent sheaf (a generalization of the notion of coherent sheaves). After solving this we proceed by standard methods that are well-known in Complex analysis of several variables.

Then we consider holomorphic families of subspaces, which we call **injective** and which have the additional property that, locally, the family can be represented as the image of a holomorphic operator function with zero kernel. We study the problem of a corresponding global representation. Here we need the theory of multiplicative cocycles from Chapter 5. It turns out that this is not always possible, but we have again an Oka-Grauert principle.

Then we study holomorphic families of complemented subspaces (which are injective), where we can prove more precise results than for arbitrary injective families. Again there is an Oka-Grauert principle.

At the end we consider the special case of families of subspaces which are finite dimensional or of finite codimension. Here there are no topological restrictions.

Chapters 7 and 8 are dedicated to factorization of operator functions with respect to a contour and the connection with Wiener-Hopf and Töplitz operators. This type of factorization was in fact considered for the first time in the pioneering works of Plemelj and of Muschelishvili. Because of that we call it **Plemelj-Muschelishvili factorization**. We start with the *local principle*, which quickly follows from the theory of multiplicative cocycles and which allows us to prove theorems on factorization for different classes of operator functions. The local principle reduces the problem to functions which are already holomorphic in a neighborhood of the contour.

For further applications we need a generalization of the theory of multiplicative cocycles. This is the topic of Chapter 9, where we introduce *cocycles with restrictions*. Let us offer an example (which is basic for all cocycles with restrictions). Suppose that in Theorem 0.0.1 an additional set $Z \subseteq D_1 \cup D_2$, discrete and closed in D, and positive integers m_w, $w \in Z$, are given. Assume that the function $A - I$ has a zero of order m_w at each $w \in D_1 \cap D_2 \cap Z$. Then the theory of cocycles with restrictions gives the additional information that the functions A_1 and A_2 in Theorem 0.0.1 can be chosen so that, for all $w \in D_j \cap Z$, $j = 1, 2$, the function $A_j - I$ has a zero of order m_w at w.

In Chapter 10, by means of the theory of cocycles with restrictions, we essentially improve the Weierstrass product Theorem 0.0.2: The functions H_w in this theorem now can be chosen so that, additionally, for each $w \in Z$, the function $H_w - I$ has a zero of an arbitrarily given order m_w at w. This has different consequences that are discussed in this short chapter.

Chapter 11 is dedicated to holomorphic equivalence and its applications to linearization and diagonalization. Let E be a Banach space, let $L(E)$ be the space of bounded linear operators in E, let $GL(E)$ be the group of invertible operators from $L(E)$, let $D \subseteq \mathbb{C}$ be an open set, and let Z be a discrete and closed subset of D. Then two holomorphic operator functions $A, B : D \setminus Z \to L(E)$ are called **(globally) holomorphically equivalent over** D if there exist holomorphic operator functions $S, T : D \to GL(E)$ such that $A = SBT$ on D.

In the first section, results are presented that explain the importance of the notion of holomorphic equivalence in spectral theory of linear operators and holomorphic operator functions. It contains the following two results: 1) For each relatively compact open subset Ω of D, each holomorphic operator function $A : D \to L(E)$, after an appropriate extension, becomes holomorphically equivalent to a function of the form $zI - T$, $z \in \Omega$, where T is a constant operator and I is the identical operator (Theorem 11.2.1). 2) Two operators $T, S \in L(E)$ with the spectra $\sigma(A)$ and $\sigma(B)$ are similar if and only if some extensions of the functions $zI - T$ and $zI - S$ are holomorphically equivalent over some neighborhood of $\sigma(A) \cup \sigma(B)$ (Corollary 11.2.3).

The remainder of this section is devoted to the relation between global and local holomorphic equivalence where two holomorphic operator functions are called **locally holomorphically equivalent** if, for each point, they are holomorphically equivalent over some neighborhood of this point. We prove that two meromorphic operator functions with meromorphic inverse are locally holomorphically equivalent if and only if they are globally holomorphically equivalent (Theorem 11.4.2), and we prove that any finite meromorphic Fredholm operator function is globally holomorphically equivalent to a diagonal function (Theorem 11.7.6). The local fact behind this is the Smith representation of matrices of germs of scalar holomorphic functions.

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