Preface

Bond markets differ in one fundamental aspect from standard stock markets. While the latter are built up by a finite number of traded assets, the underlying basis of a bond market is the entire term structure of interest rates: an infinite-dimensional variable which is not directly observable. On the empirical side this necessitates curve-fitting methods for the daily estimation of the term structure. Pricing models on the other hand, are usually built upon stochastic factors representing the term structure in a finite-dimensional state space, making them computationally tractable.

This research monograph brings together curve-fitting methods and factor models for the term structure of interest rates within the Heath–Jarrow–Morton (henceforth HJM) framework [35], which basically unifies all continuous interest rate models. We provide appropriate consistency conditions and explore some important examples.

The HJM framework can be seen as a generic description of the arbitrage-free movements of the forward curve (the term structure of forward rates), driven by a Brownian motion. The no-arbitrage requirement leads to a restriction on the drift of any single forward rate process (HJM drift condition). The (infinite-dimensional) state variable in an HJM model is the entire forward curve. As such, any initial forward curve can be taken as model input. But, by the HJM drift condition, there is a limitation on the shapes of the curves produced by the model. For instance, a flat initial forward curve cannot move only by parallel shifts, unless it remains constant over time. This fact has been known by the experts for a long time. Such consistency considerations for HJM models and finite-dimensional manifolds of curves (such as the set of flat curves) form an essential part of this book.

On the empirical side there exist commonly used methods for fitting the current forward curve. Any curve-fitting method can be represented as a parametrized family of smooth curves $\mathcal{G} = \{G(\cdot, z) \mid z \in \mathcal{Z}\}$ with some finite-dimensional parameter set $\mathcal{Z}$. A lot of cross-sectional data, that is, daily estimations of z, is available, and one may ask for a suitable stochastic model for z that provides accurate prices for interest rate based derivatives. However, this requires the absence of arbitrage. In this book, we characterize all consistent $\mathcal{Z}$-valued state space Itô processes Z which, by definition, provide an arbitrage-free model when representing the parameter z. That is, the factor model $G(T-t, Z_t)$ has to satisfy the HJM drift condition. It turns out that selected common curve-fitting methods do not go well with the HJM framework.

We then consider the preceding consistency problem from a geometric point of view, as proposed by Björk and Christensen [8]. First, the HJM methodology is completely recaptured within a functional analytic framework, which also incorporates an infinite-dimensional driving Brownian motion. We then rigorously perform the change of parametrization due to Musiela [46] and arrive at a stochastic equation in a Hilbert space H, describing the arbitrage-free evolution of the forward curve. The family $\mathcal{G}$ can now be seen as a subset of H and the above consistency considerations yield a stochastic invariance problem for the previously derived stochastic equation. Under the assumption that $\mathcal{G}$ is a regular submanifold of H, we derive sufficient and necessary conditions for its invariance. Expressed in local coordinates they turn out to equal the HJM drift condition. In conclusion, we provide a general tool for exploiting the interplay between curve-fitting methods and HJM factor models.

Classical models, such as the Vasicek [56] and Cox–Ingersoll–Ross (henceforth CIR) [18] short rate models, and the popular Brace–Gątarek–Musiela (henceforth BGM) [13] LIBOR rate (short for London Interbank Offered Rate) model, are shown to fit well into our framework. By their very definition, affine HJM models are consistent with finite-dimensional linear submanifolds of H. A straight application of our results yields a complete characterization of all affine HJM models, as obtained by Duffie and Kan [23].

Many phenomena, say in physics or economics, are described by stochastic equations in finite or infinite dimension. Some parts of this book (Chapters 2 and 6) are of general form. The results can (and hopefully will) also be applied to fields other than interest rate theory.

As for the mathematical prerequisites, the reader is expected to be familiar with elementary stochastic calculus and probability theory. This book provides a short introduction both to the basic terminology in the theory of interest rates and to stochastic analysis and equations in infinite dimension.

This monograph is based on my Ph.D. thesis [27], which I wrote under the supervision of F. Delbaen at the Swiss Federal Institute of Technology (ETH) Zurich. It is a great pleasure to thank him here for his guidance throughout my Ph.D. studies. I am deeply grateful to T. Björk for his vital support, which made this publication possible. Further thanks go to B. J. Christensen, G. Da Prato, R. Cont, D. Heath, W. Schachermayer, D. Duffie and in particular to J. Zabczyk for their interest, support and hospitality. I thank Manuela for her love and encouragement over the last years. Financial support from Credit Suisse is gratefully acknowledged.

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