Chapter 1

INTRODUCTION

This book is an example of fruitful interaction between (non-classical) propositional logics and (classical) model theory which was made possible due to categorical logic. Its main aim consists in investigating the existence of model-completions for equational theories arising from propositional logics (such as the theory of Heyting algebras and various kinds of theories related to propositional modal logic). The existence of model-completions turns out to be related to proof-theoretic facts concerning interpretability of second order propositional logic into ordinary propositional logic through the so-called 'Pitts' quantifiers' or 'bisimulation quantifiers'. On the other hand, the book develops a large number of topics concerning the categorical structure of finitely presented algebras, with related applications to propositional logics, both standard (like Beth's theorems) and new (like effectiveness of internal equivalence relations, projectivity and definability of dual connectives such as difference). A special emphasis is put on sheaf representation, showing that much of the nice categorical structure of finitely presented algebras is in fact only a restriction of natural structure in sheaves. Applications to the theory of classifying toposes are also covered, yielding new examples.

The book has to be considered mainly as a research book, reporting recent and often completely new results in the field; we believe it can also be fruitfully used as a complementary book for graduate courses in categorical and algebraic logic, universal algebra, model theory, and non-classical logics.

1. Motivating example

The origin of this work goes back to a surprising Theorem of A.M. Pitts, cf. [Pi2], stating that the second order intuitionistic propositional calculus IpC^2 can be interpreted into ordinary intuitionistic propositional calculus IpC. More precisely, THEOREM 1.1 (A.M. PITTS) *For each propositional variable x and for each formula t of IpC, there exist formulas $\exists^x t$ and $\forall^x t$ of IpC (effectively computable from t) containing only variables not equal to x which occur in t, and such that for any formula u not involving x, we have*

$$\vdash_{IpC} \exists^x t \to u \qquad \textit{iff} \qquad \vdash_{IpC} t \to u$$

and

$$\vdash_{IpC} u \to \forall^x t \qquad \textit{iff} \qquad \vdash_{IpC} u \to t.$$

Although the above result looks like a purely proof-theoretical fact, it can be interpreted model-theoretically in a quite interesting way as a statement about the theory of Heyting algebras. We summarize the main point below. Using the identification of intuitionistic formulas with the terms in the first order theory of Heyting algebras we can characterize semantically the 'Pitts' quantifiers' $\exists^x$ and $\forall^x$, as follows. For a formula $t(\vec{y}, x)$ of IpC, and a tuple of elements $\vec{a}$ from a Heyting algebra H, we have that

$$H \models (\exists^x t)(\vec{a}) = 1 \quad \text{iff} \quad H[\mathbf{x}]/t(\vec{a}, \mathbf{x}) \text{ is an extension of } H$$

where $H[\mathbf{x}]/t(\vec{a}, \mathbf{x})$ is the Heyting algebra of polynomials $H[\mathbf{x}]$ divided by the congruence generated by the condition $t(\vec{a}, \mathbf{x}) = 1$. Moreover

$$H \models (\forall^x t)(\vec{a}) = 1 \quad \text{iff} \quad H[\mathbf{x}] \models t(\vec{a}, \mathbf{x}) = 1.$$

The proof of these characterizations easily follows from Pitts' Theorem using any presentation for H.

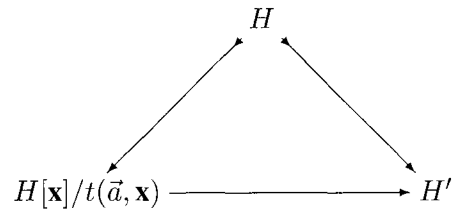
This explanation of Pitts' quantifiers in terms of Heyting algebras can be used in order to show that the first order theory of Heyting algebras admits a model completion. In fact, it turns out that the system of equations and inequations with parameters $\vec{a}$ from H

$$t(\vec{a}, x) = 1 \;\&\; u_1(\vec{a}, x) \neq 1 \;\&\; \ldots \;\&\; u_m(\vec{a}, x) \neq 1 \tag{1.1}$$

is solvable in an extension of H iff the quantifier-free formula

$$(\exists^x t)(\vec{a}) = 1 \;\&\; (\forall^x (t \to u_1))(\vec{a}) \neq 1 \;\&\; \ldots$$
$$\ldots \;\&\; (\forall^x (t \to u_m))(\vec{a}) \neq 1 \tag{1.2}$$

is true in H. If the formula (1.2) is true, we can take $H[\mathbf{x}]/t(\vec{a}, \mathbf{x})$ as an extension of H in which the system (1.1) has a solution. Conversely, if the system (1.1) is solvable in an extension H', then we have a factorization



showing that $H[\mathbf{x}]/t(\vec{a},\mathbf{x})$ is an extension of H in which the system (1.1) has the solution $x = \mathbf{x}$. This, together with the above characterization of Pitts' quantifiers, shows that formula (1.2) is true in H. Thus the class of existentially closed Heyting algebras is an elementary class and, as the above quantifier-elimination procedure is effective, it can easily be shown that the related first order theory is decidable. In Section 4.7, we shall provide examples of this decision procedure, together with a list of some basic properties of existentially closed Heyting algebras.

In this way Pitts' Theorem implies that the first order theory of Heyting algebras admits a model completion. The interesting point is that the converse is also true, in a quite general setting. In order to explain what we mean by this, we need a category-theoretic formulation of Pitts' Theorem. In this equivalent formulation, Theorem 1.1 just says that the opposite of the category of finitely presented Heyting algebras is a *Heyting category*.

The notion of Heyting category ([MR1], [MR2] or logos in [Pi1]) is a quite standard notion in categorical logic: Heyting categories are just 'Lindenbaum categories' for many-sorted intuitionistic first-order theories. A Heyting category is a category with finite limits in which finite joins, images and dual images among subobjects exist and are pullback-stable. Such a structure is needed in order to interpret first-order intuitionistic logic: terms are interpreted as arrows, formulas as subobjects and images and dual images along projections correspond to quantifiers. With each first-order many sorted intuitionistic theory, a Heyting category, built up in a completely syntactic way, can be associated: objects are formulas, arrows are equivalence classes (with respect to provable equivalence) of formulas which are provably functional in the restricted domains given by the source and the target of the arrow they define. Conversely, with each Heyting category, a first-order many sorted intuitionistic theory can be associated: we have one sort for each object, one term for each arrow, no relation symbols, and, as axioms, all the formulas which are 'internally true' in the given Heyting category. The two inverse passages are bijective, modulo the standard notion of equivalence between categories and modulo some natural notion of equivalence between theories.

Thus, using this category-theoretic formulation of Pitts' Theorem, we can say that the fact that HA_{fp}^{op}, i.e. the opposite to the category of finitely presented Heyting algebras, has enough categorical structure to classify internally a first-order intuitionistic theory implies (and actually it is equivalent to, see below) the fact that the first-order theory of Heyting algebras admits a model completion.

This is a rather interesting kind of connection: it says that the existence of a *classical* theory (the model completion) is equivalent to the existence of a suitable *intuitionistic* theory. Notice that the connection is not completely trivial, in the sense that it can be shown that the first-order intuitionistic theory classified by HA^{op}_{fp} is a theory speaking about Heyting algebras, but it differs considerably from the model completion of the theory of Heyting algebras. The two theories are indeed almost contradictory, for instance the statement

$$\forall x \forall y \ (x \vee y = 1 \Rightarrow (x = 1 \text{ or } y = 1))$$

is false in any existentially closed (non degenerate) Heyting algebra, but it is true in the theory classified by the opposite to the category of finitely presented Heyting algebras.

2. An overview of the book

We describe here the main *strategy* of the book. In Chapter 3 there is the proof of a theorem which generalizes the above observations for Heyting algebras. We take into consideration an arbitrary equational theory T satisfying a certain assumption (see next Section) which is rather strong in general, but which is often satisfied in varieties of algebras arising from logic. Under this assumption, we prove (Theorem 3.11) that

T admits a model completion iff $\mathbf{T}$ *is an r-Heyting category,*

where $\mathbf{T}$ is the opposite of the category $Alg(T)_{fp}$ of finitely presented T-algebras. In other words T admits a model completion iff the category $\mathbf{T}$ derived from T has some nice categorical structure.

The notion of r-Heyting category is obtained from the notion of Heyting category by replacing 'subobject' by 'regular subobject' everywhere in the definition. This modification is due to the fact that we prefer not to assume that monos are all regular in $\mathbf{T}$ (i.e. that epis are quotients in $Alg(T)_{fp}$), an assumption which holds for Heyting algebras as a consequence of the Beth property (BP), cf. Theorem 2.14, but which may fail in other cases.

In the following three Chapters, we apply Theorem 3.11 to two kinds of varieties of algebras: Heyting algebras and modal algebras. In both cases we adopt a similar strategy. Theorem 3.11 says that, under suitable assumptions, the existence of a model completion for T is equivalent to the existence of a certain categorical structure in $\mathbf{T}$. Usually it is not easy to decide directly whether $\mathbf{T}$ is an r-Heyting category. But, as this is a purely categorical property, we can study it in any category equivalent to $\mathbf{T}$. The strategy we adopt for an equational theory T can be summarized in the following four steps:

1 *Embedding.* Find an r-Heyting category $\mathcal{E}$ and an embedding

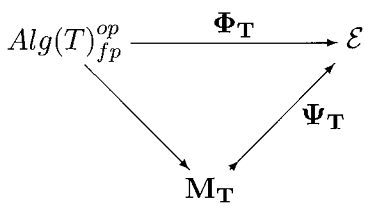
$$\mathbf{\Phi_T} : Alg(T)^{op}_{fp} \longrightarrow \mathcal{E}$$

which is conservative, preserves finite limits and all the other r-Heyting category structure that exists in $Alg(T)^{op}_{fp}$.

Conservativity ensures that the operations that can be performed in $Alg(T)^{op}_{fp}$ and are preserved by $\mathbf{\Phi_T}$ satisfy automatically any exactness properties that these operations satisfy in $\mathcal{E}$. In particular the operations of left $(\exists_f)$ and right $(\forall_f)$ adjoint to the pullback functors f^* (operating on regular subobjects, see Section 2.3) in $Alg(T)^{op}_{fp}$, if they exist, they automatically satisfy the Beck-Chevalley condition.

In the applications the category $\mathcal{E}$ is (equivalent to) the category of sheaves on the opposite of the category of finite T-algebras with the canonical topology.

2 *Duality.* Identify the image of $\mathbf{\Phi_T}$ in $\mathcal{E}$, i.e. describe in a convenient way a subcategory $\mathbf{M_T}$ of $\mathcal{E}$ so that we have a factorization of $\mathbf{\Phi_T}$



with the first component being an equivalence of categories and $\mathbf{\Psi_T}$ being an inclusion.

In the applications this point is slightly reversed. It is usually more natural to define a 'duality' functor in the opposite direction, i.e. $\mathbf{M_T} \longrightarrow Alg(T)^{op}_{fp}$.

3 *Combinatorial condition for existence of adjoints.* Now the existence of the adjoints is reduced to the verification whether the existing adjoints in $\mathcal{E}$ when applied to objects coming from $Alg(T)^{op}_{fp}$ give objects coming from $Alg(T)^{op}_{fp}$, as well.

In applications, with the help of an appropriate description of $\mathbf{M_T}$, this can be reduced to an equivalent condition of a combinatorial nature, expressed in terms of Ehrenfeucht-Fraissé games on finite Kripke models.

4 *Verification of combinatorial conditions.* Last, but not least, the combinatorial conditions should be verified to establish whether the adjoints do exist, if they do $Alg(T)^{op}_{fp}$ is an r-Heyting category.

We believe that this method is general and can be applied in other similar contexts.

Chapter 2

PRELIMINARY NOTIONS

1. Basic algebraic structures

In this section we recall the main algebraic structures which will be investigated within the book. They are structures that provides an algebraic semantics for propositional logics. They are usually obtained by enriching posets by some algebraic operations. We are mainly interested in Heyting and modal algebras i.e. those algebras that provide counterparts of superintuitionistic and modal logics.

A partially ordered set (*poset*, for short) is a set P equipped with a reflexive, transitive and antisymmetric binary relation $\leq$. For such a poset, the *infimum* (resp. *supremum*) of a family $\{a_i\}_{i \in I}$ of elements of P is an element (it may or may not exists, but if it exists it is unique) $\bigwedge_i a_i \in P$ (resp. $\bigvee_i a_i \in P$) such that for all $b \in P$, we have

$$(\forall i \in I \; b \leq a_i) \qquad \text{iff} \qquad b \leq \bigwedge_i a_i$$

(or

$$(\forall i \in I \; a_i \leq b) \qquad \text{iff} \qquad \bigvee_i a_i \leq b)$$

respectively). In case the index I is empty, the above conditions say that the infimum of the empty set is the maximum element of P and the supremum of the empty set is just the minimum.

We recall some facts about *adjoints among posets*, although they can be deduced from the general results about categories given in the Appendix, it is worth having a direct knowledge of what happens in this special case. The right adjoint f_* (resp. left adjoint f^*) to an order-preserving map $f : P \to Q$ among posets, is an order-preserving map in the opposite direction, satisfying

$$f(a) \leq b \qquad \text{iff} \qquad a \leq f_*(b)$$

(or

$$b \leq f(a) \qquad \text{iff} \qquad f^*(b) \leq a$$

respectively) for all $a \in P, b \in Q$. Such a right (left) adjoint may not exists, but if it exists it is unique. It is easily seen that left adjoints preserve existing suprema and right adjoints existing infima: the latter, for instance, is shown by an easy chain of equivalences as follows

$$\frac{\frac{\frac{\frac{a \leq f_*(\bigwedge_i b_i)}{f(a) \leq \bigwedge_i b_i}}{\forall i\ f(a) \leq b_i}}{\forall i\ a \leq f_*(b_i)}}{a \leq \bigwedge_i f_*(b_i)}$$

yieldying $f_*(\bigwedge_i b_i) = \bigwedge_i f_*(b_i)$ as a is arbitrary. If P is complete (i.e. iff all suprema -or equivalently all infima- exist), then any order-preserving map $f : P \to Q$ has a right adjoint iff it preserves suprema and has a left adjoint iff it preserves infima. Such adjoints are easily seen to be given by the following formulas:

$$f_*(b) = \bigvee_{f(a) \leq b} a \qquad f^*(b) = \bigwedge_{b \leq f(a)} a$$

for all $b \in Q$.

A (meet) *semilattice* is a commutative idempotent monoid, i.e. a structure $(M, \wedge, \top)$ satisfying the equations

$$a \wedge b = b \wedge a, \quad a \wedge \top = a, \quad a \wedge a = a, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad (2.1)$$

for all $a, b, c \in M$. Putting

$$a \leq b \qquad \text{iff} \qquad a \wedge b = a$$

we can define a partial order in any semilattice; the operation $\wedge$ turns out to be the infimum (also called *meet*) of the pair $\{a, b\}$ and $\top$ turns out to be the maximum element. In fact, one can equivalently define a semilattice as a partially ordered set in which infima exist for all finite sets of elements (this includes the maximum element, seen as the infimum over the empty set).

Many important further operations can be characterized with respect to the partial order so introduced: in order to obtain the notion of a *lattice*[1] one simply has to require that also suprema (called *joins* as well) exist for all finite sets; equivalently, a lattice is a semilattice with another binary operation $\vee$

[1]Notice that we always require the presence of $\bot$ and $\top$ in a lattice (this is different from some common literature). Sometimes, we also use the notation $\mathbf{1}$ for $\top$ and $\mathbf{0}$ for $\bot$.

and another constant $\bot$ satisfying equations (2.1) (with $\wedge, \top$ replaced by $\vee, \bot$ respectively) and moreover the following absorption laws

$$a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a.$$

A lattice is said to be *distributive* iff it satisfies one of the two (equivalent) equations

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

In a given semilattice M it may happens that for $a, b \in M$ the supremum of the set $\{c \mid a \wedge c \leq b\}$ exists; such an element is called the *relative pseudocomplement* of a relative to b (or the *implication* of a and b, using logical terminology) and is written as $a \to b$. Otherwise said, $a \to b$, if it exists, it is the unique element satisfying the condition

$$a \wedge c \leq b \qquad \text{iff} \qquad c \leq a \to b$$

for all c. A *Brouwerian semilattice* is a semilattice in which all implications among pairs of elements exist and a *Heyting algebra* is a Brouwerian semilattice which is also a distributive lattice. Brouwerian semilattices (hence also Heyting algebras) may be equivalently introduced for instance through the equations

$$\begin{array}{rr} a \wedge (a \to b) = a \wedge b & b \wedge (a \to b) = b \\ a \to (b \wedge c) = (a \to b) \wedge (a \to c) & a \to a = \top. \end{array}$$

This shows that Heyting algebras form an equational class, i.e. a *variety*. An important example of a Heyting algebra is given by the open sets of a topological space; here the partial order is inclusion, (finite) meets and joins are intersections and unions, whereas implication of the open subsets a and b is the interior of $a' \cup b$ (where a' is the complement of a). The most important example for us is given by the downward closed subsets $\mathcal{D}(P)$ of a poset P ($a \subseteq P$ is downward closed iff $p \in a$ and $q \leq p$ imply $q \in a$): here the partial order, joins and meets are again inclusion, intersections and unions, respectively, whereas the implication of a and b is

$$a \to b \ = \ \{p \in P \mid \forall q \leq p \ (q \in a \Rightarrow q \in b)\}.$$

A finite distributive lattice is always a Heyting algebra, because a finite distributive lattice is complete and, thanks to distributivity, for any element a, the order preserving map $a \wedge (-)$ preserves suprema, so that it has a right adjoint $a \to (-)$. For the same reason, a finite Brouwerian semilattice is always a Heyting algebra: in fact joins exists and are distributive as $a \wedge (-)$ preserves them (being a left adjoint).

In a Heyting algebra H, *negation* is introduced through

$$\neg a = a \to \bot;$$

such operation satisfies many usual laws, but not all the classical ones (for instance, only three of the four De Morgan identities hold). A *Boolean algebra* is a Heyting algebra in which we have $\neg\neg a = a$ (or, equivalently, $a \vee \neg a = \top$) for all a.

In a distributive lattice lattice D certain elements play a special role, they are the join-irreducible ones. We say that a is *join-irreducible* iff for all $n \geq 0, b_1, \ldots, b_n \in D$, we have

$$\text{if } a \leq b_1 \vee \cdots \vee b_n \text{ then for some } 1 \leq i \leq n, \quad a \leq b_i$$

(notice that join-irreducible elements are non-zero, i.e. different from $\bot$, by taking $n = 0$ in the above definition). In distributive lattices of the kind $\mathcal{D}(P)$, where P is a finite poset, join-irreducible elements are those of the kind $\downarrow p$ for $p \in P$ (here $\downarrow p$ is $\{q \in P \mid q \leq p\}$). In a Boolean algebra B, join-irreducible elements are called *atoms* and turn out to be just the minimal non-zero elements. A Boolean algebra B may have no atoms (in this case we say that it is *atomless*), or, at the extreme opposite, it may happen that for any non-zero $b \in B$ there is an atom $a \leq b$ (in this case, we say that B is *atomic*).

Distributive lattices and Boolean algebras (also Brouwerian semilattices, but we shall not use this further result) are *locally finite varieties*, namely varieties in which finitely generated algebras are finite; this is easily seen, e.g. in the case of Boolean algebras, from the fact that if the set G generates the algebra B, then every element of B admits a representation of the kind $\bigwedge_i \bigvee_j x_{ij}$ where i, j range over finite sets of indices and where x_{ij} is either g or $\neg g$ for some $g \in G$. This is not true for Heyting algebras since the free Heyting algebra on one generator is infinite.

Modal algebras are just Boolean algebras endowed with a further unary 'necessity' operator $\Box$ satisfying the conditions:

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad \Box\top = \top;$$

the 'possibility' operator $\Diamond$ is introduced in any modal algebra through the definition $\Diamond a = \neg\Box\neg a$. We shall mainly deal with **K4**-*algebras*, i.e. modal algebras in which the operator $\Box$ satisfies the further axiom

$$\Box a \leq \Box\Box a.$$

We shall meet in the book many interesting varieties of $K4$-algebras; for the moment let us only mention **S4**-*algebras* (or interior algebras or topological Boolean algebras), which are characterized by the further axiom

$$\Box a \leq a.$$

The main examples of modal algebras we are interested in are obtained through frames. A *frame* is a pair (X, R), where X is a set endowed with a relation (the *accessibility* relation of the frame); a *transitive frame* is a frame in which R is assumed also to be transitive and a *preordered frame* is a transitive frame in which R is also reflexive. Given a frame (resp. a transitive frame, a preordered frame) (X, R), we can turn $\mathcal{P}(X)$ into a modal algebra (resp. into a $\mathbf{K4}$-algebra, into an $\mathbf{S4}$-algebra) by putting, for $a \subseteq X$

$$\Box_R a = \{p \in X \mid \forall q \in X(pRq \Rightarrow q \in a)\};$$

the corresponding definition of the possibility operator is

$$\Diamond_R a = \{p \in X \mid \exists q \in X(pRq \ \&\ q \in a)\}.$$

Let us mention how to describe quotients in Heyting and $K4$-algebras. The central notion to this respect is the notion of a *filter* F, which makes sense at the level of a semilattice R (although it becomes fully operative only when there are implications): this is a subset of R satisfying the following requirements

- $\top \in F$;
- if $a_1, a_2 \in F$, then $a_1 \wedge a_2 \in F$;
- if $a_1 \in F$ and $a_1 \leq a_2$, then $a_2 \in F$.

Given a subset $S \subseteq R$, there exists the minimum filter $[S]$ containing S, which is given by

$$[S] = \{b \in R \mid \exists n \geq 0, \exists a_1, \ldots, a_n \in S \text{ s.t. } a_1 \wedge \cdots \wedge a_n \leq b\}.$$

In particular, the minimum (or *principal*) filter containing an element a is just $[a] = \{b \mid a \leq b\}$. For the case of modal algebras, the relevant notion is the notion of *modal filter*, which is an ordinary filter satisfying the further condition

- if $a \in F$, then $\Box a \in F$.

A formula for the minimum modal filter $[S]_m$ containing a set S can be easily given; it simplifies considerably for the case of $K4$-algebras where we have

$$[S] = \{b \in R \mid \exists n \geq 0, \exists a_1, \ldots, a_n \in S \text{ s.t. } \Box^+ a_1 \wedge \cdots \wedge \Box^+ a_n \leq b\}$$

(here $\Box^+ a_i$ stands for $a_i \wedge \Box a_i$). Consequently, the principal modal filter corresponding to an element a is just $[a]_m = [\Box^+ a]$.

In Heyting algebras, the lattice of filters and the lattice of congruences are isomorphic; given a congruence $\simeq$, we can associate to it the filter $\{a \mid a \simeq \top\}$ and given a filter F we can associate to it the congruence $a \simeq b$ iff $a \leftrightarrow b \in F$

Chapter 3

MODEL COMPLETIONS

1. r-Heyting categories

In this section we introduce the notions of r-regular and r-Heyting categories and study some of their basic properties. Roughly speaking, these notions are obtained from the extensively studied notions of regular and Heyting category (see e.g. [MR1], [MR2]]) 'by replacing monos with regular monos and regular epis by epis'. In case all subobjects are regular, the two notions coincide (this is evident from Proposition 3.3 below), so, for instance, any topos is r-regular and also r-Heyting. In case not all monos are regular, the two concepts are quite distinct: posets and order-preserving maps, for instance, form an r-Heyting category which is not even regular. As we saw in Proposition 2.14, regularity of monos in the opposite of the category of finitely presented algebras follows from some appropriate version of Beth theorem, which is often true (e.g. it holds in all varieties of Heyting and of **K4**-algebras, see [Ma5] and Section 5.6 below). Up to some extent, the theory of r-regular and r-Heyting categories goes parallel to that of regular and Heyting categories: some of the properties established in this section, for instance, are obtained through adaptations of standard arguments.

The best way to introduce r-regular categories is probably through stable factorization systems. Given a category **C**, a pair of classes of arrows $\langle \mathcal{E}, \mathcal{M} \rangle$ is said to be a *stable factorization system* for **C** iff the following four conditions are satisfied (see [FK], but we follow the equivalent formulation of [CJKP]):

(i) both $\mathcal{E}$ and $\mathcal{M}$ contain identities and are closed under left and right composition with isomorphisms;

(ii) each map f in **C** can be written as $m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;

(iii) whenever we have a commutative square,

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[m]{} & D \end{array}$$

with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, there is a unique $w : B \longrightarrow C$ such that $w \circ e = u$ and $m \circ w = v$;

(iv) whenever we have a pullback square

$$\begin{array}{ccc} A & \xrightarrow{u'} & B \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle e} \\ C & \xrightarrow[u]{} & D \end{array}$$

the fact that $e \in \mathcal{E}$ implies that $e' \in \mathcal{E}$.

The decomposition in (ii) is said to be a *factorization* for f; this factorization is unique in the sense that if $f = m \circ e$ can be factored as well as $m' \circ e'$, for $m' \in \mathcal{M}$ and $e' \in \mathcal{E}$, then using (iii), it can be shown that there is an invertible map w such that $w \circ e = e'$ and $m' \circ w = m$. In a factorization system, both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition [CJKP]: let us recall how to show it for $\mathcal{M}$ (for $\mathcal{E}$ the proof is analogous). It is sufficient to have a characterization of arrows in $\mathcal{M}$, from which the desired property easily follows. So let us prove that for all $f : C \to D$, f belongs to $\mathcal{M}$ iff f is *orthogonal* to $\mathcal{E}$, i.e. iff the following condition is satisfied:

- 'for every $e \in \mathcal{E}$ and for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[f]{} & D \end{array}$$

there is a unique $w : B \longrightarrow C$ such that $w \circ e = u$ and $f \circ w = v$.'

One side is just (iii); for the other side, take a factorization $m \circ e$ of f and consider the square

$$\begin{array}{ccc} C & \xrightarrow{e} & D' \\ {\scriptstyle id_C}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[f]{} & D \end{array}$$

to get w such that $w \circ e = id_C$. As $e \circ w$ and $id_{D'}$, both fills the diagonal of the square

$$\begin{array}{ccc} C & \xrightarrow{e} & D' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle m} \\ D' & \xrightarrow[m]{} & D \end{array}$$

they are equal by (iii), so e is iso and $f \in \mathcal{M}$ by (i).

Having established that an arrow belongs to $\mathcal{M}$ iff it is orthogonal to $\mathcal{E}$, it is not difficult to see that if $C \xrightarrow{m_1} D \xrightarrow{m_2} E$ are both orthogonal to $\mathcal{E}$, so is $m_2 \circ m_1$.

We say that a category **C** is *r-regular* iff it has finite limits and moreover taking $\mathcal{E}$ =all epis and $\mathcal{M}$ =all regular monos, we get a stable factorization system for **C**. As conditions (i) and (iii) are trivially true in this case (by the definition of epi and regular mono), **C** is r-regular iff it has finite limits, each arrow has an epi/regular mono factorization and epis are stable under pullbacks.

PROPOSITION 3.1 *If* **C** *is r-regular, then the pullback functors operating on regular subobjects have left adjoints satisfying the Beck-Chevalley condition.*

Proof. The statement of the Proposition says that for every arrow $f : B \to A$ in **C**, for every regular subobject $S \hookrightarrow B$, there is a regular subobject $\exists_f(S) \hookrightarrow A$ satisfying the condition

$$\exists_f(S) \leq T \quad \text{iff} \quad S \leq f^*(T)$$

for every regular subobject $T \hookrightarrow A$. The Beck-Chevalley condition says that for every pullback square

$$\begin{array}{ccc} C & \xrightarrow{p_1} & B_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ B_2 & \xrightarrow[f_2]{} & A \end{array} \tag{3.1}$$

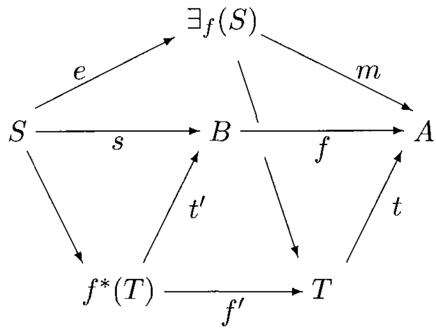
and for every regular subobject $S \hookrightarrow B_1$, the equation

$$f_2^*(\exists_{f_1}(S)) = \exists_{p_2}(p_1^*(S)) \tag{3.2}$$

holds in $Sub_r(B_2)$.

Let us fix a morphism $f : B \to A$ and a regular mono $s : S \hookrightarrow B$ in $\mathbf{C}$. We take as $\exists_f(S)$ the second component of the factorization of $f \circ s$. We can form a commuting diagram

(3.3)



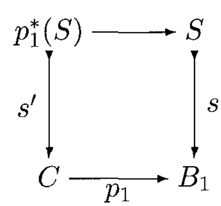
of all the named arrows; the arrows without names might not exist, but if they do, they are unique making the obvious shapes to commute, ($f^*(T)$ is a pullback of t along f). Now if $\exists_f(S) \leq T$, i.e. if $\exists_f(S) \to T$ exists in (3.3), then the two arrows

$$S \xrightarrow{s} B \xrightarrow{f} A \qquad S \xrightarrow{e} \exists_f(S) \longrightarrow T \xrightarrow{t} A$$

are equal, hence by the universal property of the pullback, $S \to f^*(T)$ exists in (3.3), showing that $S \leq f^*(T)$.

On the other hand, if $S \leq f^*(T)$ i.e. $S \to f^*(T)$ exists in (3.3), then the outer penthagon in (3.3) commutes, and $\exists_f(S) \to T$ exists, by the property (iii) of the definition of a stable factorization system, as e is an epi, and t is a regular mono.

It remains to show the Beck-Chevalley condition. Let us consider a pullback square (3.1) and let $S \overset{s}{\hookrightarrow} B_1$ be a regular mono. Take the further pullback



What we have to show is that the factorization of $p_2 \circ s'$ is just (up to an isomorphism) the factorization of $f_1 \circ s$ pulled back along f_2. But it is a general property of stable factorization systems that in a pullback square

$$\begin{array}{ccc} Z & \xrightarrow{q_1} & Y_1 \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ Y_2 & \xrightarrow[g_2]{} & X \end{array}$$

the factorization of q_2 is obtained by taking the factorization $m \circ e$ of g_1 and by successively pulling back m and e: this property is essentially due to condition (iv) ensuring that members of $\mathcal{E}$ are pullback-stable, whereas pullback-stability of members of $\mathcal{M}$ follows from conditions (i)-(iii) [CJKP] (in our case, anyway, stability of regular monos under pullbacks is a general fact). □

We shall reverse Proposition 3.1 in order to get an alternative definition of r-regular category. First, we prove a Lemma:

LEMMA 3.2 *Let* **C** *a category with finite limits, and* $f : B \longrightarrow A$ *a morphism in* **C**. *Then*

(i) *f is epi iff for every regular subobject $S \hookrightarrow A$, we have that $id_B \leq f^*(S)$ implies $id_A \leq S$;*

(ii) *if, moreover, the pullback functor $f^* : Sub_r(A) \longrightarrow Sub_r(B)$ has a left adjoint $\exists_f : Sub_r(B) \longrightarrow Sub_r(A)$, then f is epi iff $id_A \leq \exists_f(id_B)$.*

Proof. Ad (i). Let $f : B \longrightarrow A$ be an epi and let $S \hookrightarrow A$ be a regular mono such that $id_B \leq f^*(S)$; this means that we have a pullback square

$$\begin{array}{ccc} B & \longrightarrow & S \\ {\scriptstyle id_B}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & A \end{array}$$

so $S \hookrightarrow A$ is epi as a second component of an epic arrow. Being also a regular mono, it is an isomorphism. Vice versa, suppose that $id_B \leq f^*(S)$ implies $id_A \leq S$ for all $S \in Sub_r(A)$ and suppose that $g_1 \circ f = g_2 \circ f$ for some parallel arrows of domain A. Let $S \overset{s}{\hookrightarrow} A$ be the equalizer of g_1, g_2; by the universal property of equalizers, we have a unique map $f' : B \longrightarrow S$ such that $s \circ f' = f$. As s is mono, it turns out that the diagram

Chapter 4

HEYTING ALGEBRAS

We shall develop a duality for the category finitely presented Heyting algebras HA_{fp}. Using a combinatorial description $\mathbf{M_H}$ of HA_{fp}^{op} we shall prove that it is a Heyting category and hence, according to Theorem 3.8, the theory of Heyting algebras T_H admits a model completion T_H^*. Then we shall study some further properties of HA_{fp}^{op} and we shall derive some conclusions from these studies for intuitionistic propositional logic IpC. We introduce a sheaf semantics for second order logic and show how to use it to eliminate quantifiers in T_H^*.

1. Basic definitions

In this Chapter, we shall mainly deal with finite posets, to be indicated with the letters $P, Q, \ldots$; their elements will be usually written as $p, q, \ldots$ and their ordering (reflexive, transitive and antisymmetric) relations will be written simply as $\leq$, leaving the more explicit notation $\leq_P, \leq_Q, \ldots$ for contexts requiring such further specification. A poset P is called *rooted* iff it has a greatest element $\rho(P)$ (sometimes we indicate it simply as ρ instead of $\rho(P)$). If a basic finite poset L (the poset of 'labels') is fixed, we call an *L-evaluation* or simply an *evaluation* a pair $\langle P, u \rangle$, where P is a rooted finite poset and $u : P \longrightarrow L$ is an order-preserving map. This notion has a strict relation with finite Kripke models. In fact if $\langle L, \leq \rangle$ is $\langle \mathcal{P}(\vec{p}), \supseteq \rangle$ (where $\vec{p}$ is a finite list of propositional letters), then an L-evaluation $u : P \longrightarrow L$ is the same as a Kripke model for the propositional intuitionistic language built up from $\vec{p}$.[1]

We define for every $n \in \omega$ and for every pair of L-evaluations $u : P \longrightarrow L$ and $v : Q \longrightarrow L$, the notions of being *n-equivalent* (written $u \sim_n v$) and of

[1]According to our conventions, we have that (for $p, q \in P$) if $p \leq q$ then $u(p) \supseteq u(q)$, that is we use $\leq$ where standard literature uses $\geq$.

being n-*less than or equal to* (written $u \leq_n v$). These notions are motivated by the fact (implicit in what is proved in the next section) that in the case of Kripke models being n-equivalent means exactly to satisfy the same formulas up to implicational degree n, see Exercise 1. Similarly, $u \leq_n v$ means that u satisfies all formulas up to implicational degree n that v satisfies. We define also for two L-evaluations u, v the notions of being *infinitely equivalent* (written $u \sim_\infty v$) and *infinitely less than or equal to* (written $u \leq_\infty v$). All these notions are parallel to the analogous definitions introduced in [Fi1], [Fi2] for modal logic, see Chapter 5. We prefer to introduce them by means of Ehrenfeucht-Fraissé games.

Let $u : P \to L$ and $v : Q \to L$ be two L-evaluations. The *game* we are interested in has two players, player I and player II. Player I can choose either a point in P or a point in Q and player II must answer by choosing a point in the other poset and the only rule is that, if $\langle p \in P, q \in Q\rangle$ is the last move played, then in the successive move the two players can only choose points $\langle p', q'\rangle$ such that $p' \leq p$ and $q' \leq q$. If $\langle p_1, q_1\rangle, \ldots, \langle p_i, q_i\rangle, \ldots$ are the points chosen at the end of the game, after infinitely many moves, player II wins iff for every $i = 1, 2, \ldots$, we have that $u(p_i) = v(q_i)$. We say that

- $u \sim_\infty v$ iff *player II has a winning strategy*;
- $u \sim_n v$ (for $n > 0$) iff *player II has a winning strategy for the first* n-*moves*, i.e. he has a winning strategy provided we stipulate that the game terminates after n moves;
- $u \sim_0 v$ iff $u(\rho(P)) = v(\rho(Q))$;
- $u \leq_\infty v$ iff *player II has a winning strategy in the modified game*, where the word 'modified' refers to the fact that we have an additional rule forcing player I to play in the domain of u the first move;
- $u \leq_n v$ (for $n > 0$) iff *player II has a winning strategy for the first* n *moves in the modified game*;
- $u \leq_0 v$ iff $u(\rho(P)) \leq v(\rho(Q))$.

The relations $\sim_n$ and $\sim_\infty$ are clearly equivalence relations, whereas the relations $\leq_n$ and $\leq_\infty$ are only reflexive and transitive. Notice also that for every n, $u \sim_n v$ implies $u \sim_0 v$ because $\leq_L$ is antisymmetric. The straightforward proposition below (to be often used without explicit mention in the following) provides equivalent definitions. We fix a notation: if $u : P \to L$ is an L-evaluation and $p \in P$, u_p is u restricted in the domain to the downward closed subset $\downarrow p = \{p' : p' \leq p\}$.

PROPOSITION 4.1 *Given two L-evaluations* $u : P \to L, v : Q \to L$, *and* $n \in \omega$, *we have*

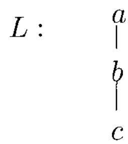
(i) $u \sim_{n+1} v$ *iff* $\forall p \in P\ \exists q \in Q\ (u_p \sim_n v_q)$ *and vice versa;*

(ii) $u \leq_{n+1} v$ *iff* $\forall p \in P\ \exists q \in Q\ (u_p \sim_n v_q)$*;*

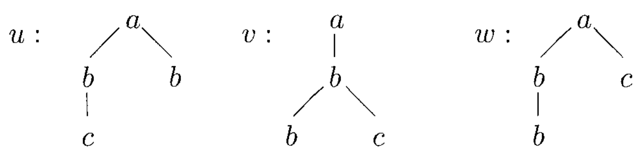
(iii) $u \leq_\infty v$ *iff* $(\forall p \in P\, \exists q \in Q$ *such that* $u_p \sim_\infty u_q)$ *iff* $(\exists q \in Q\ (u \sim_\infty v_q))$*;*

(iv) $u \sim_\infty v$ *iff* $(u \leq_\infty v$ *and* $v \leq_\infty u)$*.* □

Example Let L be a three element partial order with set of nodes $\{a, b, c\}$ given by the following Hasse diagram



Consider the following three L-evaluations on three different four element posets:



In the above picture, we put in place of points of the poset the values of evaluations at these points.

Then $u \sim_2 v$ but $u \not\sim_3 v$, and $u \sim_1 w$ but $v \not\sim_2 w$.

Notice that saying that '$u \sim_\infty v$' is equivalent to say that '$u \sim_n v$ holds for every n'. In fact, on one side it is evident that if there exists an infinite strategy, then there are also strategies for n games for every n. Vice versa, suppose that there are all such finite strategies and suppose that player I chooses a point $p \in P$. Our posets are *finite* so that, as for every $n \geq 0$ there exists $q_n \in Q$ such that $u_p \sim_n v_{q_n}$, there is also $q \in Q$ (independent of n) such that $u_p \sim_n v_q$ for every n. Player II answers this q and continuing this way, it is clear that we can define the desired winning infinite strategy.

With each L-evaluation $u : P \to L$ we associate for every $n \in \omega$ a set $Type_n(u)$ of $\sim_n$-equivalence classes by: $Type_n(u) = \{[u_p]_n : p \in P\}$ (where, of course, $[u_p]_n$ is the $\sim_n$-equivalence class of u_p). An important, although simple, fact is given by the following proposition:

PROPOSITION 4.2 *Fix a finite poset L and $n \in \omega$; then there are only finitely many equivalence classes of L-evaluations with respect to $\sim_n$.*

Proof. This is evident for $n = 0$. For $n > 0$, we argue by induction as follows. By Proposition 4.1(i), we have that $u \sim_n v$ iff $Type_{n-1}(u) = Type_{n-1}(v)$, hence there cannot be more non $\sim_n$-equivalent L-evaluations than sets of $\sim_{n-1}$ equivalence classes.[2] □

For infinite equivalence the result of the previous Proposition is not true, but on the other hand infinite equivalence can be characterized in terms of open maps (see Proposition 4.4 below). Recall from Chapter 2 that an order-preserving map $h : P \to Q$ is called *open* iff for all $p \in P, q \in Q$, if $q \leq h(p)$ then there exists $p' \leq p$ such that $h(p') = q$. Notice that if Q is rooted and h open, h is surjective iff the inverse image of the root of Q is non-empty. As shown in Chapter 2, open maps are exactly Birkhoff duals of Heyting algebras homomorphisms.

LEMMA 4.3 *Given an L-evaluation $u : P \to L$ and an open map $h : Q \to P$, we have that $(u \circ h) \leq_\infty u$. Moreover, if h is also surjective, then $(u \circ h) \sim_\infty u$.*

Proof. Clearly it is sufficient to prove the second part of the claim (for $(u \circ h) \sim_\infty u_{h(p)}$, hence $(u \circ h) \leq_\infty u$ follows). So suppose that h is surjective. We have the following infinite strategy for player II: if player I plays $q \in Q$, the answer is $h(q)$, if he plays in P the answer is suggested by the openness of h (or by the surjectivity of h in the first move), so that we reach only positions of the kind $\langle q, h(q) \rangle$. It is evident that in this way player II wins. □

PROPOSITION 4.4 *For two L-evaluations $u : P \to L$, $v : Q \to L$, we have that $u \sim_\infty v$ iff there is a commutative square*

$$\begin{array}{ccc} R & \xrightarrow{\;h\;} & P \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle u} \\ Q & \xrightarrow[\;v\;]{} & L \end{array}$$

such that R is a finite rooted poset and h, k are open surjective maps. Moreover $u \leq_\infty v$ iff there is a commutative square like the above one, with the only difference that now k is not required to be surjective.

Proof. Suppose that $u \sim_\infty v$. Take $R = \{\langle p, q \rangle : u_p \sim_\infty v_q\}$. Order is the restriction of the product order on $P \times Q$ and h, k are the two projections, restricted in their domains. R is clearly rooted, the square commutes and h, k are surjective and open. The latter can be shown as follows (e.g. for h).

[2]This is only an exponential strict upper bound, because not all sets of $\sim_{n-1}$-equivalence classes are legal, i.e. are of the form $Type_{n-1}(u)$ for some u (for instance, for $n = 1$ only subsets of L having a greatest element are legal and the situation becomes more involved for larger n, see [Ur] , [Gh1]).

Suppose that $\langle p, q\rangle \in R$ and that $p' \leq p$; we have that there exists $q' \leq q$ such that $u_{p'} \sim_\infty v_{q'}$. For such q' we have that $\langle p', q'\rangle \in R$, $\langle p', q'\rangle \leq \langle p, q\rangle$ and $h(p', q') = p'$.

Conversely, suppose that there is a commutative square with the required properties. Now $u \sim_\infty (u \circ h)$, by Lemma 4.3 and similarly $v \sim_\infty (v \circ k)$, hence $u \sim_\infty v$.

The characterization of $u \leq_\infty v$ follows from the above characterization of $\sim_\infty$ and Proposition 4.1 (iii). □

It is interesting to know that, given an L-evaluation u, for sufficiently large n (strictly depending on u) infinite equivalence to u is the same as n-equivalence to u. By $ht(P)$, we denote the height of the finite poset P, i.e. the length of a maximal chain in P. We have

PROPOSITION 4.5 *Let $u : P \to L$ be an L-evaluation. For $n(P) = 2\,ht(P) - 1$ we have that for every $v : Q \to L$, $u \sim_{n(P)} v$ iff $u \sim_\infty v$.*

Proof. The claim is clear for $ht(P) = 1$, because in this case P is a one-point poset and $u \sim_1 v$ means that v is constant. Suppose that $ht(P) > 1$ and that $u \sim_{n(P)} v$: we define an infinite strategy for player II. We recall that, as $u \sim_{n(P)} v$, for every $q \in Q$ there must exist a point $p \in P$ such that $u_p \sim_{n(P)-1} v_q$. Such p may or may not be the root of P. Player II behaves as follows: as long as player I plays a point $q \in Q$, such that $u \sim_{n(P)-1} v_q$, player II answers the root of u. After an initial (possibly empty) sequence of such moves we reach a position $\langle \rho(P), q\rangle$ with $u \sim_{n(P)-1} v_q$. If now player I tries with $p \in P$ (different from $\rho(P)$, otherwise answer is obviously q), then there exists $q' \leq q$ such that $u_p \sim_{n(P)-2} u_{q'}$: player II answers such q' and wins by induction hypothesis. If player I tries with $q' \in Q$ such that there exists $p \neq \rho(P)$ so that $u_p \sim_{n(P)-1} v_{q'}$, then player II answers such p and wins again by induction hypothesis. □

The next Lemma will complete the list of basic properties of the relations $\sim_n$ and $\leq_n$. Before stating the Lemma, we introduce a useful construction on L-evaluations. Suppose that $u : P \to L$, $v : Q \to L$ are L-evaluations such that $v \leq_0 u$. The *grafting* of v below the root of P is an L-evaluation $v \triangleleft u : P' \to L$ defined as follows. P' is $P + Q$ (disjoint union as sets) with the following order $\leq'$: $q \leq_{P'} p$ iff either ($p = \rho(P)$) or ($p, q \in P$ and $q \leq_P p$) or ($p, q \in Q$ and $q \leq_Q p$). Moreover, $v \triangleleft u$ acts as u on P and as v on Q (this is order-preserving because $v \leq_0 u$). It is immediately seen that:

LEMMA 4.6 *$(v \triangleleft u) \circ i_Q = v$ (where i_Q is the open inclusion of Q into P'), hence $v \leq_\infty (v \triangleleft u)$. Moreover, if for some $n \in \omega$, $v \leq_n u$, then $u \sim_n (v \triangleleft u)$*

□

Chapter 5

DUALITY FOR MODAL ALGEBRAS

In this Chapter we shall develop a duality for finitely presented modal algebras in a similar way we have developed a duality for finitely presented Heyting algebras in the previous Chapter. We point out below to some slight differences between these dualities.

The duality for modal algebras has a parameter **S**, being an equational theory containing the equational theory of **K**4-algebras with the finite model property and (AP) for finite algebras. In varieties of **K**4-algebras, the principal congruences do not correspond to elements of algebras, but rather to elements of form $a \wedge \Box a$. This is a source of some additional technical complications. In order to define properly the dual category $\mathbf{M_S}$ we need to use games to define two kinds of relations $\sim_n$, $\approx_n$ which reflects the difference between arbitrary elements of algebras and those that are of form $a \wedge \Box a$. The first relation serves to define morphisms and the second to define objects in the dual categories. As one could see from an exercise at the end of the previous Chapter the index n in the relation $\sim_n$ was reflecting the implicational degree of the intuitionistic formulas. This time the index is related to the nesting of the necessity connective $\Box$. The site on which the sheaves are defined in the dual category differ slightly, as well. The frames we consider are the duals of all finite **S**-algebras not only of those which are subdirectly irreducible.

In the exercises the reader will find some hints on how one can develop a similar duality for a theory **S** without (AP) for finite algebras.

1. Frames, evaluations and games

In this Section we define frames, evaluations and games on evaluations. Games give rise to some equivalence relations. We also study some basic properties of these notions that will be used later.

Recall from Section 2.1 that $\mathbf{K4}$ denote the first order equational theory in the language $\wedge$, $\vee$, $\rightarrow$, $\neg$, 0, 1, $\Box$ consisting of the following axioms:

(i) axioms of Boolean algebras;

(ii) $\Box 1 = 1$ and $\Box(p \wedge q) = \Box p \wedge \Box q$;

(iii) $\Box p \leq \Box\Box p$.

The connective $\Diamond$ is defined in the usual way as $\neg\Box\neg$. *Modal formulas* are terms in the above language.

If $\mathbf{S}$ is an equational theory containing $\mathbf{K4}$ then by $Alg(\mathbf{S})$ we denote the category of all $\mathbf{S}$-algebras (i.e. algebras satisfying the axioms of $\mathbf{S}$), and by $Alg(\mathbf{S})_{fin}$ and $Alg(\mathbf{S})_{fp}$ we denote the full subcategories of $Alg(\mathbf{S})$ of finite and finitely presented $\mathbf{S}$-algebras, respectively.

If X, Y are sets and $f : X \rightarrow Y$ is a function then Y^X denotes the set of functions from X to Y, and $f^\circ : 2^Y \rightarrow 2^X$ is the dual map of 'composing with f'. The Boolean operations on 2^X are denoted by $\wedge_X, \vee_X, \rightarrow_X, \neg_X, 0_X, 1_X$.

We will use the same letter for Boolean algebras and their universes. If B is a boolean algebra, and together with the operation $\Box : B \rightarrow B$ satisfies the axioms of $\mathbf{K4}$, then the corresponding $\mathbf{K4}$-algebra will be denoted by $(B, \Box)$. Recall that by a *frame* we mean a pair (X, R) where X is a finite set and R is a binary relation on X. The relation R is called the *accessibility relation* of the frame (X, R). If R is transitive, the frame is called *transitive* as well. From now on all frames are assumed to be transitive. The *frame algebra* of (X, R) is the $\mathbf{K4}$-algebra $(2^X, \wedge_X, \vee_X, \rightarrow_X, \neg_X, 0_X, 1_X, \Box_R)$ (to be denoted $(2^X, \Box_R)$), where for $v : X \rightarrow 2 \in 2^X$ and $x \in X$, $\Box_R(v)(x) = 1$ iff for all $y \in X,\ \ x\,R\,y$ implies $v(y) = 1$.

By a *morphism of frames* $f : (X, R) \rightarrow (Y, S)$ we mean a function $f : X \rightarrow Y$ which is an open frame map, i.e.

(i) $x\,R\,x'$ implies $f(x)\,S\,f(x')$, for $x, x' \in X$;

(ii) for any $x \in X$ and $y \in Y$, if $f(x)\,S\,y$ then there is $x' \in X$ such that $x\,R\,x'$ and $f(x') = y$.

It can be easily checked that $f : (X, R) \rightarrow (Y, S)$ is a frame morphism iff the dual map $f^\circ : (2^Y, \Box_S) \rightarrow (2^X, \Box_R)$ is a homomorphism of $\mathbf{S}$-algebras.

In this way we have defined a category $\mathbf{F}$ of finite frames and morphisms of frames. The category $\mathbf{F}$, as remarked in Section 2.1 is equivalent to the dual of the category of finite $\mathbf{K4}$-algebras. More precisely, we have a functor

$$Alg(\mathbf{K4})_{fin} \longrightarrow \mathbf{F}^{op} \tag{5.1}$$

$$(B, \Box) \longmapsto (at(B), R_\Box)$$

where $at(B)$ is the set of atoms of the boolean algebra B and $R_\Box$ is the binary relation on $at(B)$ defined by $x\, R_\Box\, x'$ iff $x \leq \Diamond x'$. The functor acts on morphisms in the obvious way. Moreover, we have a functor

$$\mathbf{F}^{op} \longrightarrow Alg(\mathbf{K4})_{fin} \tag{5.2}$$

$$(X, R) \longmapsto (2^X, \Box_R)$$

associating with a finite frame its frame algebra and with every frame morphism its dual map. They are essential inverses one to the other establishing the mentioned duality $Alg(\mathbf{K4})_{fin} \simeq \mathbf{F}^{op}$. Note that via this duality the empty frame $(\emptyset, \emptyset)$ correspond to the **K4**-algebra in which $0 = 1$.

This duality restricts to some subcategories. Let **S** be an equational theory containing **K4**, $\mathbf{F_S}$ be the full subcategory of the category **F** corresponding via the above duality to the subcategory $Alg(\mathbf{S})_{fin}$, i.e. we have $Alg(\mathbf{S})_{fin} \simeq \mathbf{F_S}^{op}$. By an **S**-*frame* we mean an object of $\mathbf{F_S}$. For some theories **S**, the objects of the category $\mathbf{F_S}$ can be described in a simple way in terms of their accessibility relations. We list below some of them (notice that any equational theory **S** containing **K4** can be axiomatized by equations of the form $t = 1$, where t is any term, hence it is possible to introduce **S** simply by specifying a set of terms, which could be viewed as a set of axioms of a logic):

Logic **S**	Axioms	Description of the accessibility relation
S4	**K4** + $\Box p \to p$	reflexive
S4.2	**S4** + $\Diamond\Box p \to \Box\Diamond p$	reflexive and locally confluent
S4.3	**S4** + $\Box(\Box p_1 \to p_2) \vee \Box(\Box p_2 \to p_1)$	reflexive and locally linear
Grz	**S4** + $(\Box(\Box(p \to \Box p) \to p) \to p)$	partial order
G	**K4** + $(\Box(\Box p \to p) \to \Box p)$	irreflexive
S5	**S4** + $\Diamond p \to \Box\Diamond p$	equivalence relation

A frame (X, R) is *locally confluent* iff for any $x, y, z \in X$, if (xRy and xRz) then there is $t \in X$ such that (yRt and zRt). A frame (X, R) is *locally linear* iff for any $x, y, z \in X$, if (xRy and xRz) then (yRz or zRy).

In all the above cases, the class of frame algebras of the finite frames indicated in the third column generates the variety $Alg(\mathbf{S})$, where **S** is axiomatized as shown in the second column. The fact that $Alg(\mathbf{S})_{fin}$ generates $Alg(\mathbf{S})$ means that **S** has *the finite model property*. The finite model property can be equivalently stated by asking that finitely generated free **S**-algebras embed into products of finite **S**-algebras. In view of Proposition 2.19, the same property, if true, extends in our case to finitely presented algebras. In order to establish the finite model property, specific techniques are needed: in Section 2.1 we saw only how to get it quickly for the basic **K4**-case; we won't even sketch in the sequel the proofs for other cases (including the cases mentioned in the above table), the reader is referred to [CZ] as an excellent textbook on such questions. In fact, there is no need for our purposes to enter in such (interesting but non trivial) field. For instance, we never need to know that **G**-algebras (also called *diagonalizable* algebras) can be axiomatized by the single equation $\Box(\Box p \to p) \to \Box p = 1$, we could simply define **G**-algebras as the algebras belonging to the variety generated by the class of finite irreflexive frame algebras. Only proof-theoretic (or at least decidability) questions would be sensitive to the existence of nice axiomatizations, but such questions are outside the scope of this book. On the other hand, we shall almost exclusively deal with systems **S** having the finite model property, so that all such systems are fully specified once the class of finite **S**-frame algebras is given.

By an *L-evaluation*, or simply *evaluation*, $v : (X, R) \to L$ we mean a function $v : X \to L$, where (X, R) is a frame and L is a finite set. Note that if $L = \mathcal{P}(p_1, \ldots, p_n)$ then an L-evaluation is nothing but a usual Kripke model which forces modal formulas in variables $p_1, \ldots, p_n$ (see the exercises for the description of the forcing relation).

We fix an arbitrary equational theory **S** containing **K4** for the rest of the section. Consequently, from now on by a frame we mean an **S**-frame. Let (X, R) be a frame, $Y \subseteq X$, $S = R \cap (Y \times Y)$. Then (Y, S) is a *generated subframe* of (X, R) iff for any $y \in Y$ and $x \in X$, if yRx then $x \in Y$. We have the following easy Lemma (for further information concerning classes of finite frames see the exercises):

LEMMA 5.1 (i) *If (Y,S) is a generated subframe of an* **S**-*frame (X,R) then (Y,S) is an* **S**-*frame as well;*

(ii) *If* (X_i, R_i) *are* **S**-*frames for i=1,...,n then the disjoint sum* $\coprod_{i=1}^{n}(X_i, R_i)$ *is an* **S**-*frame;*

(iii) *If* $f : (X, R) \to (Y, S)$ *is a surjective morphism of frames and* (X, R) *is an* **S**-*frame then* (Y, S) *is an* **S**-*frame as well.*

We shall define three relations on evaluations in terms of some games. They are variants of the Ehrenfeucht-Fraissé games adopted by K.Fine to the context of modal logic. In fact, we will describe only one game and the others will be obtained as slight modifications of its. Let $v : (X, R) \to L$ and $u : (Y, S) \to L$ be two L-evaluations, n be a natural number or ∞. The n-game on v and u is played by two players, Player I and Player II. In the first move Player I chooses one frame, either (X, R) or (Y, S) and a node in it. Player II answers by choosing a node in the other frame. After k moves the players already constructed sequences $< x_1, \ldots, x_k >$, $< y_1, \ldots, y_k >$ with $x_i\, Rx_{i+1}$, $y_i\, Sy_{i+1}$ for every $i < k$. Then in the $(k+1)$-st move Player I chooses one of the frames, say (X, R), and a node $x_{k+1} \in X$ such that $x_k\, R\, x_{k+1}$. Player II chooses a node in the other frame, say $y_{k+1} \in Y$, such that $y_k\, S\, y_{k+1}$. If Player II can't make a move[1] at his turn then he immediately loses. Otherwise, the game terminates after m moves if either Player I can't make a move or $m = n$ (if $n = \infty$ then $m = n$ means that both players plays infinitely many moves). Then Player II wins iff for all natural $k \leq m$, $v(x_k) = u(y_k)$.

We define for $n \geq 1$

- $v \approx_n u$ iff Player II has a winning strategy in n-game on v and u.
- $v \leq_n u$ iff Player II has a winning strategy in n-game on v and u modified so that Player I must play the first move in X.

For $n \geq 0$ and $x \in X, y \in Y$ we define

- $(v, x) \sim_n (u, y)$ iff Player II has a winning strategy in $n+1$-game on v and u modified so that the first moves of the players must be x and y.

Notice that the relations $\leq_n$ and $\approx_n$ are defined on L-evaluations whereas the relation $\sim_n$ is defined on L-evaluations with a distinguished nodes. The relations $\sim_n$ and $\approx_n$ are clearly equivalence relations, whereas the relation $\leq_n$ is only reflexive and transitive. The equivalence class of a pair (v, x) with respect to the equivalence relation $\sim_n$ will be denoted by $[(v, x)]_n$. The straightforward proposition below provides equivalent definitions of these relations.

PROPOSITION 5.2 *Given two L-evaluations $v : (X, R) \to L$, $u : (Y, S) \to L$, nodes $x \in X$, $y \in Y$, and $n \in \omega$. We have:*

(i) $(v, x) \sim_0 (u, y)$ *iff* $v(x) = u(y)$;

(ii) $(v, x) \sim_{n+1} (u, y)$ *iff* $v(x) = u(y)$, *and* $\forall x' \in X\ \exists y' \in Y$ (*if* $x\, R\, x'$ *then* $y\, S\, y'$ *and* $(v, x') \sim_n (u, y')$), *and vice versa;*

(iii) $v \leq_{n+1} u$ *iff* $\forall x \in X\ \exists y \in Y\ (v, x) \sim_n (u, y)$;

(iv) $v \approx_{n+1} u$ *iff* $v \leq_{n+1} u$ *and* $v \geq_{n+1} u$.

(v) $(v, x) \sim_\infty (u, y)$ *iff* $v(x) = u(y)$, *and* $\forall x' \in X\ \exists y' \in Y$ (*if* $x\, R\, x'$ *then* $y\, S\, y'$ *and* $(v, x') \sim_\infty (u, y')$), *and vice versa;*

[1]Since accessibility relations of our frames need not to be reflexive it may happen that there is no point accessible from a given point.