Preface to the Second English Edition

Meanwhile the fourth edition of the German version appeared. In this second English edition we adapt the content closely to this German edition. We also followed several suggestions and tried to improve the language and typography. We thank Shari Scott for her support.

Heidelberg, August 2008

Eberhard Freitag
Rolf Busam

Preface to the English Edition

This book is a translation of the forthcoming fourth edition of our German book "Funktionentheorie I" (Springer 2005). The translation and the LaTeX files have been produced by Dan Fulea. He also made a lot of suggestions for improvement which influenced the English version of the book. It is a pleasure for us to express to him our thanks. We also want to thank our colleagues Diarmuid Crowley, Winfried Kohnen and Jörg Sixt for useful suggestions concerning the translation.

Over the years, a great number of students, friends, and colleagues have contributed many suggestions and have helped to detect errors and to clear the text.

The many new applications and exercises were completed in the last decade to also allow a partial parallel approach using computer algebra systems and graphic tools, which may have a fruitful, powerful impact especially in complex analysis.

Last but not least, we are indebted to Clemens Heine (Springer, Heidelberg), who revived our translation project initially started by Springer, New York, and brought it to its final stage.

Heidelberg, Easter 2005

Eberhard Freitag
Rolf Busam

II

Integral Calculus in the Complex Plane $\mathbb{C}$

In Section I.5 we already encountered the problem of finding a primitive function for a given analytic function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, i. e., an analytic function $F : D \to \mathbb{C}$ such that $F' = f$.

In general, one may ask: Which functions $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, have a primitive? Recall that in the real case any *continuous* function $f : [a, b] \to \mathbb{R}$, $a < b$, has a primitive, namely, for example the integral

$$F(x) := \int_a^x f(t)\, dt \ .$$

Whether one uses the notion of a RIEMANN integral or the integral for regulated functions is irrelevant in this connection.

In the complex case the situation however is different. We shall see that a function that has a primitive must itself already be analytic, and that is, as we already know, a much stronger condition than just continuity. To explore the similarities with and differences from real analysis we will attempt to construct a primitive using an integration process

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta \ , \quad z_0 \ \text{ fixed} \ .$$

For this we first have to introduce a suitable complex integral, the *complex line integral*. In contrast to the real case this not only depends on the starting and end points, but also on the choice of the curve connecting them. One obtains a primitive only when one can prove its independence of this choice.

The *Cauchy Integral Theorem* (A.L. CAUCHY, 1814, 1825) is the main result in this direction. However, as it can be extracted from a letter of C.F. GAUSS to F.W. BESSEL sent on December 18, 1811, GAUSS already knew the statement of CAUCHY's Integral Theorem (C.F. GAUSS, Werke 8, 90-92).

An extension of the CAUCHY Integral Theorem is provided by the *Cauchy Integral Formulas* (A.L. CAUCHY, 1831), which are themselves a special case of the *Residue Theorem*, which is a powerful tool for function theory. However, we shall only get to the Residue Theorem in the next chapter.

E. Freitag and R. Busam, *Complex Analysis*,
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II.1 Complex Line Integrals

A complex-valued function

$$f : [a,b] \longrightarrow \mathbb{C} \quad (a,b \in \mathbb{R}\ ,\ a<b)$$

on a real interval is called *integrable*, if $\operatorname{Re} f,\ \operatorname{Im} f : [a,b] \to \mathbb{R}$ are integrable functions in the sense of real analysis. (For instance, in the RIEMANN sense or in the sense of a regulated function. Which notion of integral is to be used is not important, it is only essential that all *continuous functions* are integrable.) Then one defines the integral

$$\int_a^b f(x)\,dx := \int_a^b \operatorname{Re} f(x)\,dx + \mathrm{i}\int_a^b \operatorname{Im} f(x)\,dx$$

and furthermore

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx\ , \qquad \int_a^a f(x)\,dx := 0\ .$$

The usual rules of calculation with RIEMANN integrals, or with integrals of regulated functions, then can be extended to complex-valued functions:

(1) The integral is $\mathbb{C}$-linear: For continuous functions $f, g : [a,b] \to \mathbb{C}$ the following holds:

$$\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\ ,$$

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx \quad (\lambda \in \mathbb{C})\ .$$

(2) If f is continuous and F is a primitive of f, i. e. $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(3)

$$\left| \int_a^b f(x)\,dx \right| \ \le\ \int_a^b |f(x)|\ dx\ \le\ (b-a)C\ , \qquad \text{if } |f(x)| \le C$$

for all $x \in [a,b]$. This inequality holds for step functions from the triangle inequality, the general case follows by approximation.

(4) *Substitution rule:* Let $M_1, M_2 \subset \mathbb{R}$ be intervals, $a, b \in M_1$ and

$\varphi : M_1 \longrightarrow M_2$ continuously differentiable and $f : M_2 \longrightarrow \mathbb{C}$ continuous.

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(y)\,dy = \int_a^b f\big(\varphi(x)\big)\varphi'(x)\,dx\ .$$

Proof. If F is a primitive of f, then $F \circ \varphi$ is a primitive of $(f \circ \varphi)\varphi'$. $\square$

(5) *Partial integration*

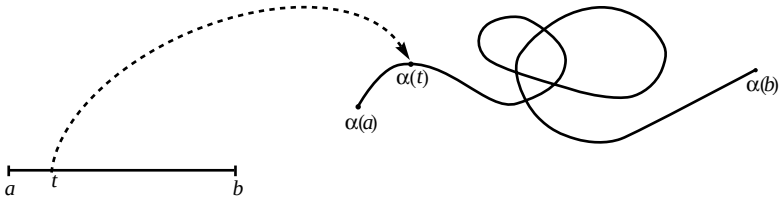
$$\int_a^b u(x)v'(x)\,dx \;=\; uv\Big|_a^b - \int_a^b u'(x)v(x)\,dx\ .$$

Here u and $v : [a,b] \to \mathbb{C}$ are continuously differentiable functions. The proof follows from the product formula $(uv)' = uv' + u'v$. □

Definition II.1.1 *A **curve** is a continuous map*

$$\alpha : [a,b] \longrightarrow \mathbb{C}\ , \quad a < b\ ,$$

*from a compact real interval into the complex plane. We call $\alpha(a)$ the **starting point**, and $\alpha(b)$ the **end point** of α.*



Examples.

(1) The straight line connecting $z, w \in \mathbb{C}$ is parametrized by

$$\alpha : [0,1] \longrightarrow \mathbb{C}\ , \quad \alpha(t) = z + t(w-z) \quad (\alpha(0) = z\ ,\ \alpha(1) = w)\ .$$

(2) The k-fold unit circle, $k \in \mathbb{Z}$, is

$$\varepsilon_k : [0,1] \longrightarrow \mathbb{C}\ , \quad \varepsilon_k(t) = \exp(2\pi \mathrm{i} k t)\ .$$

Definition II.1.2 *A curve is called **smooth**, if it is continuously differentiable.*

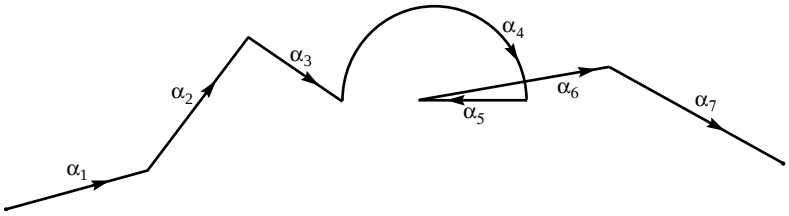
Definition II.1.3 *A curve is called **piecewise smooth**, if there is a partition*

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that the restrictions

$$\alpha_\nu := \alpha \mid [a_\nu, a_{\nu+1}]\ , \quad 0 \le \nu < n\ ,$$

are smooth.



Definition II.1.4 *Let*

$$\alpha : [a, b] \longrightarrow \mathbb{C}$$

be a smooth curve and

$$f : D \longrightarrow \mathbb{C} , \quad D \subset \mathbb{C} ,$$

a continuous function, whose domain of definition contains the image of the curve α, i.e. $D \supset \alpha\big([a, b]\big)$. Then one defines

$$\int_\alpha f := \int_\alpha f(\zeta)\, d\zeta := \int_a^b f\big(\alpha(t)\big)\alpha'(t)\, dt ,$$

and calls this complex number the ***line integral*** *or* ***contour integral*** *of f along α.*

If α is only piecewise smooth, there exists a partition

$$a = a_0 < \cdots < a_n = b ,$$

such that the restrictions

$$\alpha_\nu : [a_\nu, a_{\nu+1}] \longrightarrow \mathbb{C} , \quad 0 \leq \nu < n ,$$

are smooth. In this case we define

$$\int_\alpha f(\zeta)\, d\zeta := \sum_{\nu=0}^{n-1} \int_{\alpha_\nu} f(\zeta)\, d\zeta .$$

It is obvious that this definition does not depend on the choice of the partition. By the *arc length* of a smooth curve we mean

$$l(\alpha) := \int_a^b |\alpha'(t)|\, dt .$$

The length of a piecewise smooth curve is

$$l(\alpha) := \sum_{\nu=0}^{n-1} l(\alpha_\nu) .$$

Examples.

(1) The length of the straight line connecting z and w is

$$l(\alpha) = |z - w| \ .$$

(2) The arc length of a k-fold unit circle is

$$l(\varepsilon_k) = 2\pi\,|k| \ .$$

Now we shall list the fundamental properties of complex line integrals. The proofs all follow immediately from properties (1) – (5) of the integral $\int_a^b f(x)\,dx$.

Remark II.1.5 *The complex line integral has the following properties:*

1. $\int_\alpha f$ *is* $\mathbb{C}$*-linear in* f.
2. *The "standard estimate" states*

$$\left|\int_\alpha f(\zeta)\,d\zeta\right| \leq C \cdot l(\alpha), \ \textit{if}\ |f(\zeta)| \leq C \ \textit{for all}\ \zeta \in \text{Image}\,\alpha \ .$$

3. *The line integral generalizes the ordinary Riemann integral (or the integral of regulated functions). If*

$$\alpha : [a,b] \longrightarrow \mathbb{C} \ , \quad \alpha(t) = t \ ,$$

then $\alpha'(t) = 1$, *and for any continuous* $f : [a,b] \to \mathbb{C}$ *one has:*

$$\int_\alpha f(\zeta)\,d\zeta = \int_a^b f(t)\,dt \ .$$

4. *Parameter invariance of the line integral:*
Let $\alpha : [c,d] \to \mathbb{C}$ *be a piecewise smooth curve and*

$$f : D \longrightarrow \mathbb{C}, \quad \text{Image}\,\alpha \subset \mathrm{D} \subset \mathbb{C} \ ,$$

a continuous function, and

$$\varphi : [a,b] \longrightarrow [c,d] \quad (\ a < b \ , \ c < d \ )$$

a continuously differentiable function with $\varphi(a) = c$, $\varphi(b) = d$. *Then we have*

$$\int_\alpha f(\zeta)\,d\zeta = \int_{\alpha\circ\varphi} f(\zeta)\,d\zeta \ .$$

5. *Let*

$$f : D \longrightarrow \mathbb{C} \ , \ D \subset \mathbb{C} \ \textit{open} \ ,$$

be a continuous function, which has a primitive F *(i.e.* $F' = f$*). Then for any piecewise smooth curve* α *in* D

$$\int_\alpha f(\zeta)\,d\zeta = F\big(\alpha(b)\big) - F\big(\alpha(a)\big) \ .$$

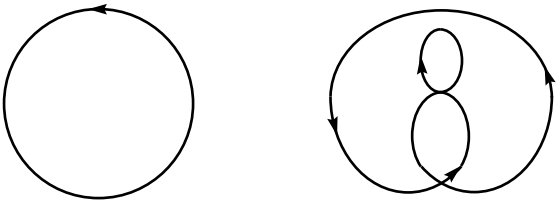
The last point in the remark implies:

Theorem II.1.6 *If a continuous function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, has a primitive then*

$$\int_\alpha f(\zeta)\, d\zeta = 0$$

for ***any*** *closed piecewise smooth curve α in D.*

(A curve $\alpha : [a, b] \to \mathbb{C}$ is called *closed,* if $\alpha(a) = \alpha(b)$.)



Remark II.1.7 *Let $r > 0$ and*

$$\alpha(t) = r \exp(\mathrm{i}t)\ , \quad 0 \le t \le 2\pi\ ,$$

(a circle with the "counterclockwise" orientation). Then for $n \in \mathbb{Z}$

$$\int_\alpha \zeta^n\, d\zeta = \begin{cases} 0 & \text{for } n \neq -1\ , \\ 2\pi\mathrm{i} & \text{for } n = -1\ . \end{cases}$$

Corollary II.1.7$_1$ In the domain $D = \mathbb{C}^\bullet$ the (continuous) function

$$f : D \longrightarrow \mathbb{C}\ , \quad z \longmapsto \frac{1}{z}\ ,$$

does not have a primitive.

Otherwise, because of II.1.6, the integral along any closed curve in $\mathbb{C}^\bullet$ would have to vanish. However,

$$\int_\alpha \frac{1}{\zeta}\, d\zeta \ = \ 2\pi\mathrm{i}$$

for the circle line (counterclockwise oriented)

$$\begin{aligned} \alpha : [0, 2\pi] &\longrightarrow \mathbb{C}^\bullet\ , \\ t &\longmapsto r \exp(\mathrm{i}t) \quad (r > 0)\ . \end{aligned}$$

Proof of II.1.7. In case of $n \neq -1$ the function $f(z) = z^n$ has the primitive $F(z) = \dfrac{z^{n+1}}{n+1}$. Therefore its integral along any closed curve vanishes. For $n = -1$, however, we have

$$\int_\alpha \zeta^{-1} d\zeta = \int_0^{2\pi} (re^{\mathrm{i}t})^{-1}\, r\mathrm{i}e^{\mathrm{i}t}\, dt = \mathrm{i}\int_0^{2\pi} dt = 2\pi\mathrm{i}\ .$$

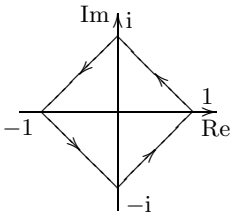
□

A different proof of the above formula uses the principal branch of the logarithm, which makes a "jump of $2\pi\mathrm{i}$" while crossing the negative real axis (see I.5.8).

Exercises for II.1

1. The figure on the right shows a closed curve α, Give an explicit parametrization for α and calculate

$$\frac{1}{2\pi\mathrm{i}} \int_\alpha \frac{1}{z}\, dz\ .$$



2. Let $\alpha : [0, \pi] \to \mathbb{C}$ be defined by

$$\alpha(t) := \exp(\mathrm{i}t)$$

and $\beta : [0, 2] \to \mathbb{C}$ by

$$\beta(t) = \begin{cases} 1 + t(-\mathrm{i} - 1) & \text{for } t \in [0, 1]\ , \\ 1 - t + \mathrm{i}(t - 2) & \text{for } t \in [1, 2]\ . \end{cases}$$

Sketch α and β, and calculate

$$\int_\alpha \frac{1}{z}\, dz \qquad \text{and} \qquad \int_\beta \frac{1}{z}\, dz\ .$$

3. Prove the transformation invariance of the line integral, II.1.5, (4).
4. Sketch the following curve α ("figure eight")

$$\alpha(t) := \begin{cases} 1 - \exp(\ \ \mathrm{i}t) & \text{for } t \in [0, 2\pi]\ , \\ -1 + \exp(-\mathrm{i}t) & \text{for } t \in [2\pi, 4\pi]\ . \end{cases}$$

5. Compute

$$\int_\alpha z \exp(z^2)\, dz\ ,$$

where

(a) α is the line between the point 0 and the point $1 + \mathrm{i}$,

(b) α is the piece of the parabola with equation $y = x^2$, which lies between the points 0 and $1 + \mathrm{i}$.

6. Compute

$$\int_\alpha \sin z\, dz\ ,$$

where α is the piece of the parabola with equation $y = x^2$, which lies between the points 0 and $-1 + \mathrm{i}$.

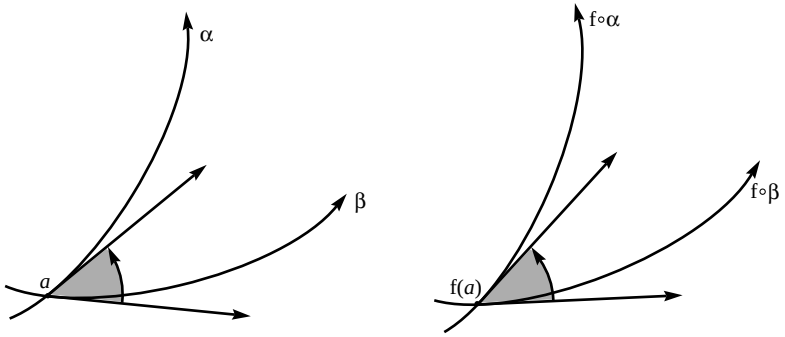
7. Let $[a,b]$ and $[c,d]$ ($a<b$ and $c<d$) be compact intervals in $\mathbb{R}$.
Show: There is an affine map
$$\begin{aligned}\varphi : [a,b] &\longrightarrow [c,d] \ , \\ t &\longmapsto \alpha t + \beta \ ,\end{aligned}$$
with $\varphi(a)=c$ and $\varphi(b)=d$.

8. Let $R>0$ be a positive number. We consider the curve
$$\beta(t) = R\,\exp(\mathrm{i}t) \ , \qquad 0 \le t \le \frac{\pi}{4} \ .$$
Show:
$$\left| \int_\beta \exp(\mathrm{i}z^2)\,dz \right| \ \le \ \frac{\pi(1-\exp(-R^2))}{4R} \ < \ \frac{\pi}{4R} \ .$$

9. Let $\alpha : [a,b] \to \mathbb{C}$ be continuously differentiable and assume that the function $f : \text{Image } \alpha \to \mathbb{C}$ is continuous.
Show: For any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property:
If $\{a_0,\dots,a_N\}$ and $\{c_1,\dots,c_N\}$ are finite subsets of $[a,b]$ with
$$a = a_0 \le c_1 \le a_1 \le c_2 \le a_2 \le \cdots \le a_{N-1} \le c_N \le a_N = b$$
and
$$a_\nu - a_{\nu-1} < \delta \text{ for } \nu = 1,\dots,N \ ,$$
then
$$\left| \int_\alpha f(z)\,dz - \sum_{\nu=1}^{N} f\big(\alpha(c_\nu)\big)\cdot\big(\alpha(a_\nu)-\alpha(a_{\nu-1})\big) \right| \ < \ \varepsilon \ .$$
(Approximation of the line integral by a RIEMANN sum.)

10. By splitting f into its real and imaginary parts, represent the complex line integral $\int_\alpha f(z)\,dz$ in terms of *real* integrals.
Result: If $f = u + \mathrm{i}v$, $\alpha(t) = x(t) + \mathrm{i}y(t)$, $t \in [a,b]$, then
$$\begin{aligned}\int_\alpha f(z)\,dz &= \int_\alpha (u\,dx - v\,dy) + \mathrm{i}\int_\alpha (v\,dx + u\,dy) \\ &= \int_a^b \Big[\ u\big(x(t),y(t)\big)\,x'(t) \ - \ v\big(x(t),y(t)\big)\,y'(t)\ \Big]\,dt \\ &\quad + \mathrm{i}\int_a^b \Big[\ v\big(x(t),y(t)\big)\,x'(t) \ + \ u\big(x(t),y(t)\big)\,y'(t)\ \Big]\,dt \ .\end{aligned}$$

11. A smooth curve is called *regular* if its derivative does not vanish anywhere. Assume that there are given an analytic function $f : D \to \mathbb{C}$, $D \subset \mathbb{C}$ open, and a point $a \in D$ with $f'(a) \neq 0$, and also two regular curves $\alpha, \beta : [-1,1] \to D$ with $\alpha(0) = \beta(0) = a$. One may then consider the oriented angle $\angle\big(\alpha'(0),\beta'(0)\big)$ (see I.1, Exercise 4). This is the angle between the two intersecting curves. Show that the two image curves $f\circ\alpha$ and $f\circ\beta$ intersect with the same angle at their intersection point $f(a) = f\big(\alpha(0)\big) = f\big(\beta(0)\big)$.



Thus an analytic function is "angle- and orientation-preserving" at any point at which its derivative does not vanish (see also Exercise 18 in I.5).

II.2 The Cauchy Integral Theorem

By an interval $[a, b]$ we will always mean a real interval. And we shall always understand, without mentioning it, that expressions like

$$a \leq b \ , \quad a < b \ , \quad [a, b]$$

imply that a and b are real.

Definition II.2.1 *A set $D \subset \mathbb{C}$ is called* ***arcwise connected****, if for any two points $z, w \in D$ there is a piecewise smooth curve joining z and w and lying entirely inside D, such that*

$$\alpha : [a, b] \longrightarrow D \ , \quad \alpha(a) = z \ , \quad \alpha(b) = w \ .$$

Remark II.2.2 *Every arcwise connected set $D \subset \mathbb{C}$ is connected, i.e. every locally constant function on D is constant.*

Proof. Let $f : D \to \mathbb{C}$ be locally constant. If f is not constant (this is an indirect proof), then there exist points $z, w \in D$ with $f(z) \neq f(w)$. Join z and w by a piecewise smooth curve within D

$$\alpha : [a, b] \longrightarrow D \ .$$

Since α is continuous

$$g(t) = f\big(\alpha(t)\big)$$

is locally constant. Therefore $g'(t) = 0$, and so $g = \text{const}$. But we have

$$g(a) = f(z) \neq f(w) = g(b) \ . \qquad \square$$

It should be mentioned that for *open* sets D the converse of II.2.2 also holds, although we will not make use of this.

Definition II.2.3 *By a* ***domain*** *we understand an* ***arcwise connected non-empty open set*** $D \subset \mathbb{C}$.

Remark. The connected subsets of $\mathbb{R}$ are known to be exactly the *intervals.* The concept of a domain is thus a generalization of the notion of an open interval. However, the domains in $\mathbb{C}$ can be much more complicated.

Let

$$\alpha : [a,b] \longrightarrow \mathbb{C} \text{ and } \beta : [b,c] \longrightarrow \mathbb{C} , \qquad a \leq b \leq c ,$$

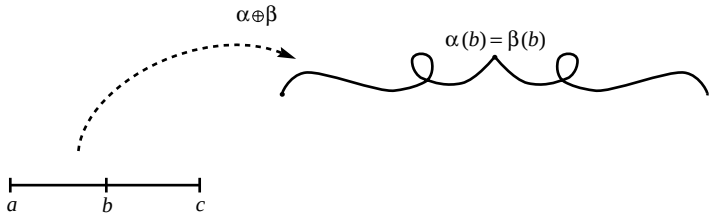
be two piecewise smooth curves with the property

$$\alpha(b) = \beta(b) .$$

Then the formula

$$\alpha \oplus \beta : [a,c] \longrightarrow \mathbb{C} ,$$
$$(\alpha \oplus \beta)(t) = \begin{cases} \alpha(t) & \text{for } a \leq t \leq b , \\ \beta(t) & \text{for } b \leq t \leq c , \end{cases}$$

also defines a piecewise smooth curve. The curve $\alpha \oplus \beta$ is called the *composition* of α and β.



If f is a continuous function, whose domain of definition contains the images of α and β, then

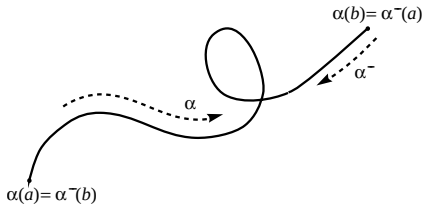
$$\int_{\alpha \oplus \beta} f(\zeta)\, d\zeta = \int_{\alpha} f(\zeta)\, d\zeta + \int_{\beta} f(\zeta)\, d\zeta .$$

For any curve

$$\alpha : [a,b] \longrightarrow \mathbb{C}$$

the *reciprocal* curve is defined by

$$\alpha^{-} : [a,b] \longrightarrow \mathbb{C} ,$$
$$t \mapsto \alpha(b + a - t) .$$



Obviously we have the *reversal rule*

$$\int_{\alpha^{-}} f(\zeta)\, d\zeta = -\int_{\alpha} f(\zeta)\, d\zeta$$

for all continuous functions f with Image α in the domain of definition of f.

Convention. We shall assume, unless the contrary is explicitly mentioned, that curves are *piecewise smooth*.

Theorem II.2.4 *For a continuous function*

$$f : D \longrightarrow \mathbb{C} \ , \quad D \subset \mathbb{C} \text{ a domain} \ ,$$

the following three statements are equivalent:

(a) *f has a primitive.*
(b) *The integral of f along any closed curve in D vanishes.*
(c) *The integral f along any curve in D depends only on the beginning and end points of the curve.*

Proof.
(a) $\Rightarrow$ (b): Theorem II.1.6.
(b) $\Rightarrow$ (c): Let

$$\alpha : [a, b] \longrightarrow D \text{ and } \beta : [c, d] \longrightarrow D$$

be two curves with the same starting and end points. We have to show

$$\int_\alpha f = \int_\beta f \ .$$

There is no loss of generality in assuming $b = c$, since by II.1.5, (4) one may replace β by the curve

$$t \longmapsto \beta(t + c - b) \ , \quad b \leq t \leq b + (d - c) \ .$$

Now, we can consider the closed curve $\alpha \oplus \beta^-$, and obtain

$$0 = \int_{\alpha \oplus \beta^-} f = \int_\alpha f - \int_\beta f \ .$$

(c) $\Rightarrow$ (a): We fix a point $z_* \in D$ and consider

$$F(z) = \int_{z_*}^{z} f(\zeta) \, d\zeta$$

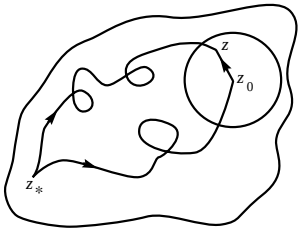
as the integral of f along some curve connecting z_* with z within D. The assumption ensures that the integral does not depend on the choice of the curve.

Claim. $F' = f$. For the proof, we consider an arbitrary, but for the moment fixed point $z_0 \in D$ and show $F'(z_0) = f(z_0)$. Since D is open, there is a full disk $U_\varrho(z_0)$ around z_0 in D. For $z \in U_\varrho(z_0)$, by definition, we have

$$F(z) = \int_{z_*}^{z} f(\zeta) \, d\zeta = \int_{z_*}^{z_0} f(\zeta) \, d\zeta + \int_{z_0}^{z} f(\zeta) \, d\zeta = F(z_0) + \int_{z_0}^{z} f(\zeta) \, d\zeta \ ,$$

where we can take the integral from z_0 to z along the line segment connecting them:

$$\sigma(z_0, z)(t) := z_0 + t(z - z_0) \ , \quad 0 \le t \le 1 \ .$$



Since $\int_{\sigma(z_0,z)} d\zeta = z - z_0$ we have

$$F(z) = F(z_0) + f(z_0)(z - z_0) + r(z) \quad \text{with}$$
$$r(z) = \int_{\sigma(z_0,z)} \big(f(\zeta) - f(z_0)\big)\, d\zeta \ .$$

By the continuity of f at z_0 there is for any $\varepsilon > 0$ a δ, $0 < \delta < \varrho$, such that for all $z \in D$ with $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| < \varepsilon \ .$$

Therefore the usual estimate for integrals implies

$$|r(z)| \le |z - z_0| \cdot \varepsilon \ .$$

But this means that F is complex differentiable at z_0 and $F'(z_0) = f(z_0)$. Since $z_0 \in D$ was arbitrary, F must be a primitive for f. □

The existence of a primitive is thus reduced to the question of the vanishing of line integrals along closed curves. In the next section we shall prove a vanishing theorem for differentiable functions and special closed curves, namely triangular paths.

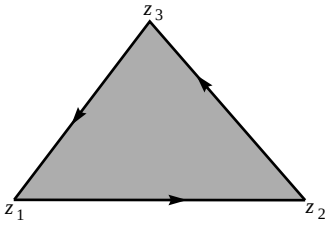
Let $z_1, z_2, z_3 \in \mathbb{C}$ be three points in the complex plane. The *triangle spanned* by z_1, z_2, z_3 is the point set

$$\Delta := \{\ z \in \mathbb{C}\ ; \quad z = t_1 z_1 + t_2 z_2 + t_3 z_3,\ 0 \le t_1, t_2, t_3,\ t_1 + t_2 + t_3 = 1\ \} .$$

Clearly this set is convex, i.e. with any pair of points in Δ the line segment connecting them also lies in Δ, and Δ is, in fact, the smallest convex set containing z_1, z_2 and z_3 (their convex hull).

By the *triangular path* $\langle z_1, z_2, z_3 \rangle$ we mean the closed curve

$$\langle z_1, z_2, z_3 \rangle \ = \ \alpha := \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \ , \qquad \text{with}$$
$$\alpha_1(t) = z_1 + (t - 0)\,(z_2 - z_1) \ , \quad 0 \le t \le 1 \ ,$$
$$\alpha_2(t) = z_2 + (t - 1)\,(z_3 - z_2) \ , \quad 1 \le t \le 2 \ ,$$
$$\alpha_3(t) = z_3 + (t - 2)\,(z_1 - z_3) \ , \quad 2 \le t \le 3 \ .$$



We obviously have

$$\text{Image } \alpha \subset \Delta \qquad (\text{or precisely Image } \alpha = \text{Boundary } \Delta).$$

The following theorem is the key for solving the problem of the existence of a primitive. It is sometimes called the *Fundamental Lemma* of complex analysis.

Theorem II.2.5 (Cauchy Integral Theorem for triangular paths, E. Goursat, 1883/84, 1899; A. Pringsheim, 1901) *Let*

$$f : D \longrightarrow \mathbb{C} \ , \ D \subset \mathbb{C} \text{ open} \ ,$$

be an analytic function (i.e. complex differentiable at any point $z \in D$). Let z_1, z_2, z_3 be three points in D, such that the triangle they span is also contained in D; then

$$\boxed{\int\limits_{\langle z_1, z_2, z_3 \rangle} f(\zeta)\, d\zeta = 0 \ .}$$

Proof. We shall inductively construct a sequence of triangular paths

$$\alpha^{(n)} = \langle z_1^{(n)}, z_2^{(n)}, z_3^{(n)} \rangle \ , \quad n = 0, 1, 2, 3, \ldots \qquad ,$$

in the following steps:

(a) $\alpha^{(0)} := \alpha = \langle z_1, z_2, z_3 \rangle$.

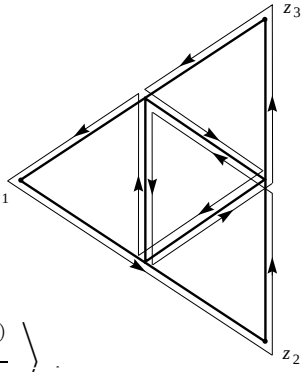
(b) $\alpha^{(n+1)}$ is one of the following four triangular paths

$$\alpha_1^{(n)} : \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2} \ , \ z_2^{(n)} \ , \ \frac{z_2^{(n)} + z_3^{(n)}}{2} \right\rangle \ ,$$

$$\alpha_2^{(n)} : \left\langle \frac{z_2^{(n)} + z_3^{(n)}}{2} \ , \ z_3^{(n)} \ , \ \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle \ ,$$

$$\alpha_3^{(n)} : \left\langle \frac{z_1^{(n)} + z_3^{(n)}}{2} \ , \ z_1^{(n)} \ , \ \frac{z_1^{(n)} + z_2^{(n)}}{2} \right\rangle \ ,$$

$$\alpha_4^{(n)} : \left\langle \frac{z_1^{(n)} + z_2^{(n)}}{2} \ , \ \frac{z_2^{(n)} + z_3^{(n)}}{2} \ , \ \frac{z_1^{(n)} + z_3^{(n)}}{2} \right\rangle \ .$$



Thus we choose

$$\alpha^{(n+1)} = \alpha_1^{(n)} \text{ or } \alpha_2^{(n)} \text{ or } \alpha_3^{(n)} \text{ or } \alpha_4^{(n)} \ .$$

So we are partitioning the triangle using lines parallel to the sides and passing through their midpoints. Obviously the triangles corresponding to the triangular paths $\alpha_\nu^{(n)}$ and $\alpha^{(n)}$ are entirely contained in $\Delta = \Delta^{(0)}$, and we have

$$\int_{\alpha^{(n)}} = \int_{\alpha_1^{(n)}} + \int_{\alpha_2^{(n)}} + \int_{\alpha_3^{(n)}} + \int_{\alpha_4^{(n)}} \ .$$

(c) We can and do choose $\alpha^{(n+1)}$ such that

$$\left| \int_{\alpha^{(n)}} f \right| \leq 4 \left| \int_{\alpha^{(n+1)}} f \right| .$$

From this follows

$$\left| \int_{\alpha} f(\zeta)\, d\zeta \right| \leq 4^n \left| \int_{\alpha^{(n)}} f(\zeta)\, d\zeta \right| .$$

The closed triangles $\Delta^{(n)}$ are nested

$$\Delta = \Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \cdots$$

($\Delta^{(n)}$ is the triangle corresponding to the triangular path $\alpha^{(n)}$). By CANTOR's theorem for nested "intervals" there is a point z_0, which is contained in all these triangles. We then use the fact that f is complex differentiable there:

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + r(z) \qquad \text{with} \qquad \lim_{z \to z_0} \frac{r(z)}{|z - z_0|} = 0 .$$

Since the affine part $z \mapsto f(z_0) + f'(z_0)(z - z_0)$ has a primitive, we have

$$\int_{\alpha^{(n)}} f(\zeta)\, d\zeta = \int_{\alpha^{(n)}} r(\zeta)\, d\zeta$$

and therefore

$$\left| \int_{\alpha} f(\zeta)\, d\zeta \right| \leq 4^n \left| \int_{\alpha^{(n)}} r(\zeta)\, d\zeta \right| .$$

We shall now prove that the right-hand side converges to 0 for $n \to \infty$. Let $\varepsilon > 0$. There exists $\delta > 0$ with

$$|r(z)| \leq \varepsilon\, |z - z_0| \text{ for all } z \in D \text{ with } |z - z_0| < \delta .$$

If n is large enough, $n \geq N$, then

$$\Delta^{(n)} \subset U_\delta(z_0) .$$

In addition,

$$|z - z_0| \leq l(\alpha^{(n)}) = \frac{1}{2^n} l(\alpha) \quad \text{for} \quad z \in \Delta^{(n)} .$$

We get

$$\left| \int_{\alpha} f(\zeta)\, d\zeta \right| \leq 4^n \cdot l(\alpha^{(n)}) \cdot \varepsilon l(\alpha^{(n)}) = l(\alpha)^2 \cdot \varepsilon$$

for any positive ε and thus

$$\int_{\alpha} f(\zeta)\, d\zeta = 0 . \qquad \square$$

For non-analytic functions this theorem is false. For example the integral of $f(z) = |z|^2$ along a triangle path is usually different from 0, as one verifies by direct computation.

Definition II.2.6 *A **star-shaped domain** is an open set $D \subset \mathbb{C}$ with the following property: There is a point $z_* \in D$ such that for each point $z \in D$ the whole line segment joining z_* and z is contained in D:*

$$\{\, z_* + t(z - z_*) \;;\; t \in [0,1] \,\} \subset D \;.$$

The point z_* is not uniquely determined, and is called a (possible) *star center.*

Remark. Since one can join any two points through the star center, a star domain is arcwise connected, and therefore a domain.

Examples.

(1) Each convex domain is star-shaped, in particular, *any open disk is star-shaped.* Each point of the convex domain can be chosen as the star center.

(2) The plane slit along the negative real axis is star-shaped. (As star centers we can take points $x \in \mathbb{R}$, $x > 0$, and only such points.)

(3) An open disk $U_r(a)$, from which we remove finitely many line segments which join a boundary point b with a point on the straight line between a and b.

(4) $D = \mathbb{C}^\bullet = \mathbb{C} \setminus \{0\}$ is not star-shaped since any $z_* \in \mathbb{C}^\bullet$ cannot be a star center for the point $z := -z_*$ "cannot be seen from" z_*.

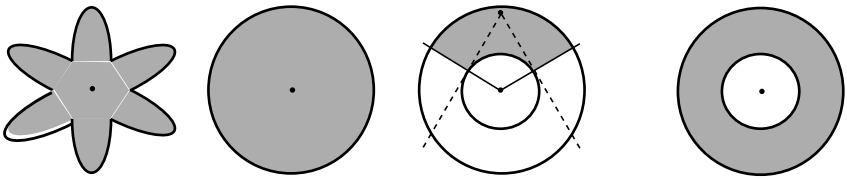
(5) The *annulus* $\mathcal{R} = \{\, z \in \mathbb{C} \;;\quad r < |z| < R \,\}$, $0 < r < R$, is not star-shaped.

(6) The ring sector

$$\left\{\, z = z_0 + \zeta \varrho e^{\mathrm{i}\varphi} \;;\;\; r < \varrho < R \,, 0 < \varphi < \beta \;\right\} \subset \mathcal{R} \;,\;\; \zeta, z_0 \in \mathbb{C} \;,\; |\zeta| = 1 \;,$$

is star-shaped, if $\beta < \pi$ and $\cos\frac{\beta}{2} > \frac{r}{R}$.

(7) In the following figure the three left domains are star-shaped, the right one is not.



Theorem II.2.7 (Cauchy Integral Theorem for star domains)

Version 1. *Let*

$$f : D \longrightarrow \mathbb{C}$$

be an analytic function on a star domain $D \subset \mathbb{C}$. Then the integral f along any closed curve in D vanishes.

Version 2. *Each analytic function f defined on a star domain D has a primitive in D.*

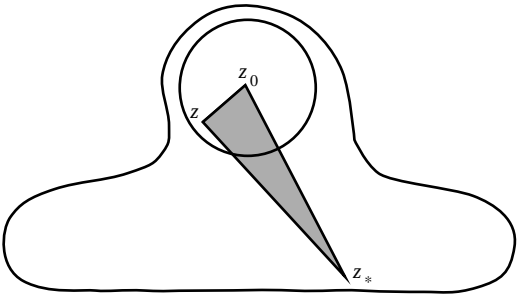
Corollary *In arbitrary domains $D \subset \mathbb{C}$ an analytic function has, at least locally, a primitive, i.e. for each point $a \in D$ there is an open neighborhood $U \subset D$ of a, such that $f \mid U$ has a primitive.*

Taking into account II.2.4, both versions of the theorem are clearly equivalent. We will proof the second version. So, let $z_* \in D$ be a star center and F be defined by

$$F(z) = \int_{z_*}^{z} f(\zeta)\, d\zeta \ ,$$

where the integral is taken *along the line segment connecting* z_* with z. If $z_0 \in D$ is an arbitrary point, then the line segment connecting z_0 and z does not have to lie in D. But there does exist a disk around z_0 which is entirely contained in D. It is easy to see then that:

If z is a point in this disk, then the entire triangle spanned by z_*, z_0 and z is contained in D.



Then Cauchy's integral theorem for triangular paths implies

$$\int_{z_*}^{z_0} + \int_{z_0}^{z} + \int_{z}^{z_*} = 0 \ .$$

(In each case integration is taken along the connecting line segments.) Now we can repeat word-for-word the proof of II.2.4, (c) $\Rightarrow$ (a).

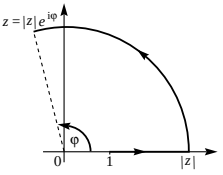
Proof of the Corollary. The proof is clear, since for each $a \in D$ there is an open disk $U_\varepsilon(a)$ with $U_\varepsilon(a) \subset D$, and disks are convex, and thus certainly star-shaped. □

Thus we have achieved a solution to our existence problem for star domains.

As an application of II.2.7 we get a new construction of the principal branch of the logarithm as a primitive of $1/z$ in the star domain $\mathbb{C}_-$, namely

$$L(z) := \int_1^z \frac{1}{\zeta}\, d\zeta \ .$$

One integrates along some curve connecting 1 with z in $\mathbb{C}_-$. Since the functions L and Log have the same derivatives, and coincide at a point ($z = 1$), we have $L(z) = \mathrm{Log}(z)$ for $z \in \mathbb{C}_-$. If one chooses as the curve the line segment from 1 to $|z|$ and then the arc from $|z|$ to $z = |z|e^{\mathrm{i}\varphi}$, we obtain the form we already know



$$L(z) = \int_1^{|z|} \frac{1}{t}\, dt + \mathrm{i} \int_0^{\varphi} dt = \log|z| + \mathrm{i}\,\mathrm{Arg}\, z \ .$$

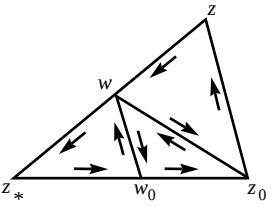
The following variant of II.2.7 is a useful tool:

Corollary II.2.7$_1$ Let $f : D \to \mathbb{C}$ be a continuous function in a star domain D with center z_*. If f is complex differentiable at every point $z \neq z_*$, then f has a primitive in D.

Proof. As one can see from the proof of II.2.7, it is enough to show that

$$\int_{z_*}^{z_0} + \int_{z_0}^{z} + \int_{z}^{z_*} = 0 \ ,$$

where we may assume that the triangle Δ spanned by z_*, z_0 and z is entirely contained within D.



Moreover, we can assume $z_* \neq z$ and $z_* \neq z_0$. Let w, resp. w_0, be an arbitrary point different from z_* on the line segment between z_* and z, resp. z_* and z_0. From the Cauchy integral theorem for triangular paths (II.2.5 above) the integrals along the paths $\langle w_0, z_0, w\rangle$ and $\langle z_0, z, w\rangle$ vanish. On the other hand, we have

$$\int_{\langle z_*, z_0, z\rangle} = \int_{\langle z_*, w_0, w\rangle} + \int_{\langle w_0, z_0, w\rangle} + \int_{\langle z_0, z, w\rangle} = \int_{\langle z_*, w_0, w\rangle} \ .$$

The assertion now follows by passing to the limit

$$w \to z_* \ , \quad w_0 \to z_* \ . \qquad \square$$

Definition II.2.8 *A domain $D \subset \mathbb{C}$ is called an* ***elementary domain****, if any analytic function defined on D has a primitive in D.*

Any star domain is thus an elementary domain. For example, $\mathbb{C}_-$, the plane cut along the negative real axis, is an elementary domain. In this connection it is of interest to note:

Theorem II.2.9 *Let $f : D \to \mathbb{C}$ be an analytic function on an elementary domain, let f' also be analytic*[1]*, and $f(z) \neq 0$ for all $z \in D$. Then there exists an analytic function $h : D \to \mathbb{C}$ with the property*

$$f(z) = \exp\big(h(z)\big) \ .$$

The function h is called an *analytic branch of the logarithm* of f.

Corollary II.2.9$_1$ *Under the assumptions in II.2.9, there exists for any $n \in \mathbb{N}$ an analytic function $H : D \to \mathbb{C}$ with $H^n = f$.*

Proof of the Corollary. Set $H(z) = \exp\big(\ \frac{1}{n} h(z)\ \big)$. □

Proof of Theorem II.2.9. Let F be a primitive of f'/f. Then one can check immediately that, with

$$G(z) = \left(\frac{\exp\big(F(z)\big)}{f(z)} \right) \ ,$$

one has $G'(z) = 0$ for all $z \in D$. Therefore

$$\exp\big(F(z)\big) = C\ f(z) \text{ for all } z \in D$$

with some nonzero constant C. Since $\exp : \mathbb{C} \to \mathbb{C}^\bullet$ is surjective one can write this in the form $C = \exp(c)$. The function

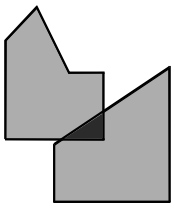
$$h(z) = F(z) - c$$

has the desired property. □

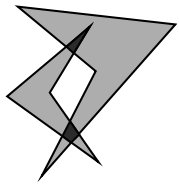
Since the function $f(z) = 1/z$ does not have a primitive in the punctured plane $\mathbb{C}^\bullet$, we see that $\mathbb{C}^\bullet$ is not an elementary domain; however it is not true that any elementary domain must be star-shaped, as the following construction shows:

Remark II.2.10 *Let D and D' be two elementary domains. If $D \cap D'$ is non-empty and connected, then $D \cup D'$ is also an elementary domain.*

Corollary. *Slitted annuli are elementary domains.*



⟵ elementary domain
non-elementary domain ⟶



Proof of II.2.10. Let $f : D \cup D' \to \mathbb{C}$ be analytic. By assumption there exist primitives

[1] Actually, this assumption is unnecessary by II.3.4.

$$F_1 : D \longrightarrow \mathbb{C}, \quad F_2 : D' \longrightarrow \mathbb{C} .$$

The difference $F_1 - F_2$ must be locally constant in $D \cap D'$, and therefore constant since $D \cap D'$ is connected. By addition of a constant if necessary, one may assume

$$F_1 \mid D \cap D' = F_2 \mid D \cap D' .$$

The functions F_1 and F_2 now glue to a single function

$$F : D \cup D' \longrightarrow \mathbb{C} . \qquad \square$$

The following is also immediately clear:

Remark II.2.11 *Let*

$$D_1 \subset D_2 \subset D_3 \subset \cdots$$

be an increasing sequence of elementary domains. Then their union

$$D = \bigcup_{n=1}^{\infty} D_n$$

is also an elementary domain.

It can be shown (in a non-trivial way) that with the two constructions above give all elementary domains starting from disks.

We shall later obtain a simple topological characterization of elementary domains (see Appendix C of Chapter IV):

Elementary domains are precisely the so-called ***simply connected domains****. (Intuitively these are the domains "without holes").*

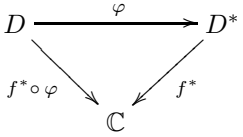
For practical purposes this characterization of elementary domains is not so important. For this reason we postpone the proof of this theorem. More elementary domains can be obtained by means of *conformal mappings* (cf. I.5.13).

Remark II.2.12 *Let $D \subset \mathbb{C}$ be an elementary domain and*

$$\varphi : D \longrightarrow D^*$$

a globally conformal mapping of D onto the domain D^*. We assume that its derivative is analytic. Then D^* is also an elementary domain.

Proof. We have to show: Any analytic function $f^* : D^* \to \mathbb{C}$ has a primitive F^*. This can naturally be reduced to checking the corresponding statement for D.



For if $f^* : D^* \to \mathbb{C}$ is analytic, then so is $f^* \circ \varphi : D \to \mathbb{C}$ analytic. But then

$$(f^* \circ \varphi)\varphi' : D \longrightarrow \mathbb{C}$$

is analytic, and so has a primitive F. (Here we have to assume that φ' is also analytic. This condition is, as we shall see in following sections, automatically satisfied.) In fact, $F^* := F \circ \varphi^{-1}$ is analytic (φ^{-1} is analytic too!) and $F^{*\prime} = f^*$. □

Exercises for II.2

1. Which of the following subsets of $\mathbb{C}$ are domains?
   (a) $\{z \in \mathbb{C}; \quad |z^2 - 3| < 1\}$,
   (b) $\{z \in \mathbb{C}; \quad |z^2 - 1| < 3\}$,
   (c) $\{z \in \mathbb{C}; \quad ||z|^2 - 2| < 1\}$,
   (d) $\{z \in \mathbb{C}; \quad |z^2 - 1| < 1\}$,
   (e) $\{z \in \mathbb{C}; \quad z + |z| \neq 0\}$,
   (f) $\{\, z \in \mathbb{C};\ 0 < x < 1,\ 0 < y < 1 \,\} - \bigcup_{n=2}^{\infty} \{\, x + \mathrm{i}y;\ x = 1/n,\ 0 < y \leq 1/2 \,\}$.

2. Let $z_0, \ldots, z_N \in \mathbb{C}$ ($N \in \mathbb{N}$). Define the line segments connecting z_ν with $z_{\nu+1}$ ($\nu = 0, 1, \ldots, N-1$) by

$$\alpha_\nu : [\nu, \nu+1] \longrightarrow \mathbb{C} \text{ with } \alpha_\nu(t) = z_\nu + (t - \nu)(z_{\nu+1} - z_\nu) \ .$$

   Then $\alpha := \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_{N-1}$ defines a curve $\alpha : [0, N] \to \mathbb{C}$. One calls α the *polygonal path,* which joins z_0 with z_N (along $z_1, z_2, \ldots, z_{N-1}$).

   Show: An open set $D \subset \mathbb{C}$ is connected (and thus a domain) if and only if any two points of D can be connected by a polygonal path α inside D (i.e. Image $\alpha \subset D$).

3. Let $a \in \mathbb{C}$, $\varepsilon > 0$. The punctured disk

$$\dot{U}_\varepsilon(a) := \{\, z \in \mathbb{C} \,; \quad 0 < |z - a| < \varepsilon \,\} \ ,$$

   is a domain.

   Deduce: If $D \subset \mathbb{C}$ is a domain and $z_1, \ldots, z_m$ are finitely many points, then the set $D' := D \setminus \{z_1, \ldots, z_m\}$ is also a domain.

4. Let $\emptyset \neq D \subset \mathbb{C}$ be open. The continuous function

$$f : D \longrightarrow \mathbb{C} \ , \quad z \longmapsto \overline{z} \ ,$$

   has no primitive in D.

5. For $\alpha : [0, 1] \to \mathbb{C}$ with $\alpha(t) = \exp(2\pi i t)$ compute

$$\int_\alpha 1/|z| \, dz \ , \qquad \int_\alpha 1/(|z|^2) \, dz \ , \qquad \text{and show} \qquad \left| \int_\alpha 1/(4 + 3z) \, dz \right| \leq 2\pi \ .$$

6. Let
$$D := \{\, z \in \mathbb{C};\quad 1 < |z| < 3 \,\}$$
and $\alpha : [0,1] \to D$ be defined by $\alpha(t) = 2\exp(2\pi i t)$. Calculate
$$\int_\alpha \frac{1}{z}\, dz\ .$$

7. For $a, b \in \mathbb{R}_+^\bullet$, let $\alpha, \beta : [0,1] \to \mathbb{C}$ be defined by
$$\alpha(t) := a\cos 2\pi t + \mathrm{i}a \sin 2\pi t\ ,$$
$$\beta(t) := a\cos 2\pi t + \mathrm{i}b \sin 2\pi t\ .$$
(a) *Show:*
$$\int_\alpha \frac{1}{z}\, dz = \int_\beta \frac{1}{z}\, dz\ .$$
(b) *Show using* (a)
$$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2 \sin^2 t}\, dt = \frac{2\pi}{ab}\ .$$

8. Let $D_1, D_2 \subset \mathbb{C}$ be star domains with the common star center z_*. Then $D_1 \cup D_2$ and $D_1 \cap D_2$ are also star domains with respect to z_*.

9. Which of the following domains are star-shaped?
(a) $\{\, z \in \mathbb{C};\quad |z| < 1 \text{ and } |z+1| > \sqrt{2} \,\}$,
(b) $\{\, z \in \mathbb{C};\quad |z| < 1 \text{ and } |z-2| > \sqrt{5} \,\}$,
(c) $\{\, z \in \mathbb{C};\quad |z| < 2 \text{ and } |z+\mathrm{i}| > 2 \,\}$.
In each case determine the set of all star centers.

10. Show that the "sickle-shaped domain"
$$D = \{\, z \in \mathbb{C}\ ;\quad |z| < 1\ ,\ |z - 1/2| > 1/2 \,\}$$
is an elementary domain.

11. Let $0 < r < R$ and f be the function
$$f : \overset{\bullet}{U}_R(0) \longrightarrow \mathbb{C}\ ,$$
$$z \longmapsto \frac{R+z}{(R-z)z}\ .$$
Show that $f(z) = \dfrac{1}{z} + \dfrac{2}{R-z}$, and, by integrating along the curve α,
$$\alpha : [0, 2\pi] \longrightarrow \mathbb{C}\ ,\quad \alpha(t) = r\exp(\mathrm{i}t)\ ,$$
that
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos t + r^2}\, dt = 1\ .$$
Show in a similar manner:
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R\cos t}{R^2 - 2Rr\cos t + r^2}\, dt = \frac{r}{R^2 - r^2}\ ,\ \text{if } 0 \le r < R\ .$$

12. **Lemma on polynomial growth**
Let P be a nonconstant polynomial of degree n:

$$P(z) = a_n z^n + \cdots + a_0 \ , \quad a_\nu \in \mathbb{C} \ , \ 0 \le \nu \le n \ , \ n \ge 1 \ , \ a_n \neq 0 \ .$$

Then, for all $z \in \mathbb{C}$ with the property

$$|z| \ge \varrho := \max\left\{1, \frac{2}{|a_n|} \sum_{\nu=0}^{n-1} |a_\nu|\right\} \ ,$$

we have

$$\boxed{\frac{1}{2}\,|a_n|\,|z|^n \le |P(z)| \le \frac{3}{2}\,|a_n|\,|z|^n \ .}$$

Corollary. Any root of the polynomial P lies in the open disk with radius ρ centered at the origin.

13. **A proof of the Fundamental Theorem of Algebra**
Let P be a nonconstant polynomial of degree n,

$$P(z) = a_n z^n + \cdots + a_0 \ , \quad a_\nu \in \mathbb{C}, \ 0 \le \nu \le n, \ n \ge 1, \ a_n \neq 0 \ .$$

We have $P(z) = z(a_n z^{n-1} + \cdots + a_1) + a_0 = zQ(z) + a_0$.
Assumption: $P(z) \neq 0$ for all $z \in \mathbb{C}$.
Then for $z \neq 0$ we have

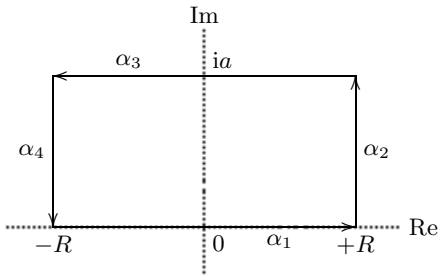
$$\frac{1}{z} = \frac{P(z)}{zP(z)} = \frac{zQ(z) + a_0}{zP(z)} = \frac{Q(z)}{P(z)} + \frac{a_0}{zP(z)} \ .$$

By integration along $\alpha(t) = R\exp(\mathrm{i}t)$, $0 \le t \le 2\pi$, $R > 0$, it follows that

$$2\pi\mathrm{i} = \int_\alpha \frac{a_0}{zP(z)}\, dz \ .$$

By using the lemma on growth of polynomials, derive a contradiction (consider the limit $R \to \infty$).

14. Let $a \in \mathbb{R}$, $a > 0$. Consider the "rectangular path" α sketched in the figure.



$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 .$$

Since

$$f(z) = e^{-z^2/2}$$

is analytic in $\mathbb{C}$, and $\mathbb{C}$ is star-shaped, it follows from the CAUCHY integral theorem for star domains that

$$0 = \int_\alpha f(z)\, dz = \int_{\alpha_1} f(z)\, dz + \int_{\alpha_2} f(z)\, dz + \int_{\alpha_3} f(z)\, dz + \int_{\alpha_4} f(z)\, dz \ .$$

Show:

$$\lim_{R\to\infty}\left|\int_{\alpha_2} f(z)\,dz\right| = \lim_{R\to\infty}\left|\int_{\alpha_4} f(z)\,dz\right| = 0$$

and deduce that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\mathrm{i}a)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \quad (=\sqrt{2\pi})\ .$$

$$I(a) := \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+\mathrm{i}a)^2}dx := \lim_{R\to\infty}\int_{-R}^{R} e^{-\frac{1}{2}(x+\mathrm{i}a)^2}dx$$

is therefore independent of a and has the value $\sqrt{2\pi}$.

Corollary. (The FOURIER transform of $x \mapsto e^{-x^2/2}$)

$$\boxed{\int_0^{\infty} e^{-x^2/2}\cos(ax)\,dx \;=\; \frac{1}{2}\sqrt{2\pi}\,e^{-a^2/2}\ .}$$

15. Let $D \subset \mathbb{C}$ be a domain with the property

$$z \in D \implies -z \in D$$

and $f : D \to \mathbb{C}$ a continuous and even function ($f(z) = f(-z)$). Moreover, for some $r > 0$ let the closed disk $\overline{U}_r(0)$ be contained in D. Then

$$\int_{\alpha_r} f = 0 \ \text{ for } \ \alpha_r(t) := r\exp(2\pi\mathrm{i}t)\ ,\ 0 \le t \le 1\ .$$

16. **Continuous branches of the logarithm**

Let $D \subset \mathbb{C}^\bullet$ be a domain which does not contain the origin. A continuous function $l : D \to \mathbb{C}$ with $\exp l(z) = z$ for all $z \in D$ is called *a continuous branch of the logarithm.*

Show:

(a) Any other continuous branch of the logarithm $\widetilde{l}$ has the form $\widetilde{l} = l + 2\pi\mathrm{i}k$, $k \in \mathbb{Z}$.

(b) Any continuous branch of the logarithm l is in fact analytic, and $l'(z) = 1/z$.

(c) On D there exists a unique continuous branch of the logarithm only if the function $1/z$ has a primitive on D.

(d) Construct two domains D_1 and D_2 and continuous branches $l_1 : D_1 \to \mathbb{C}$, $l_2 : D_2 \to \mathbb{C}$ of the logarithm, such that their difference is not constant on $D_1 \cap D_2$.

17. **Fresnel Integrals**

Show:

$$\int_0^{\infty} \cos\left(t^2\right)\,dt = \int_0^{\infty} \sin\left(t^2\right)\,dt = \frac{1}{4}\sqrt{2\pi}\ .$$

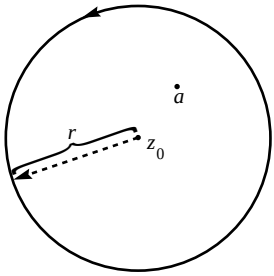
Hint. Compare the function $f(z) := \exp(\mathrm{i}z^2)$ on the real axis and on the first bisector. The value of the integral $\int_0^{\infty}\exp(-t^2)\,dt = \sqrt{\pi}/2$ can be used. Use also the inequality in Exercise 8, Sect. II.1.

II.3 The Cauchy Integral Formulas

The following lemma is a special case of the CAUCHY integral formula:

Lemma II.3.1 *One has*

$$\oint_\alpha \frac{d\zeta}{\zeta - a} = 2\pi\mathrm{i} \ ,$$



where integration is performed along the circle

$$\alpha(t) = z_0 + re^{\mathrm{i}t} \ , \qquad z_0 \in \mathbb{C} \ , \ 0 \le t \le 2\pi \ , \ r > 0 \ ,$$

and a is an arbitrary point in the interior of the disc ($|a - z_0| < r$).

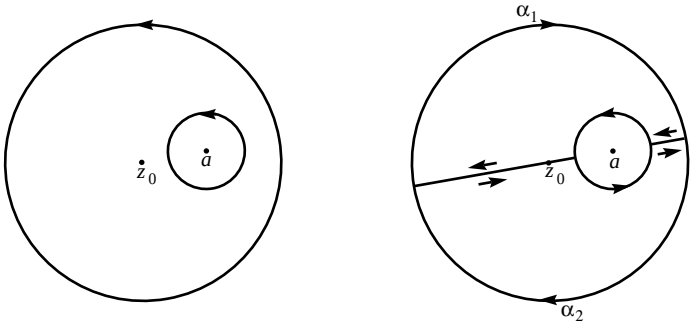
In the case $a = z_0 (= 0)$ we have already formulated this in II.1.7, and we can reduce II.3.1 to this case by using the CAUCHY integral theorem; in fact we will show

$$\oint_{|\zeta - z_0| = r} \frac{d\zeta}{\zeta - a} = \oint_{|\zeta - a| = \varrho} \frac{d\zeta}{\zeta - a} \ ,$$

where $\varrho \le r - |z_0 - a|$.

Remark. We use a suggestive way of writing integrals along circles, which is self understanding.

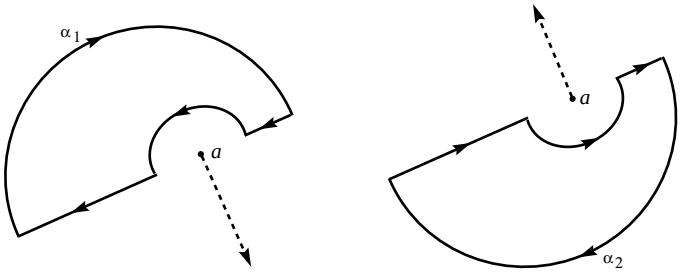
Proof.



So it is claimed that the integrals along both of the circles drawn above agree. We shall limit ourselves to make the proof intuitively clear from the figure. It is easy but a little wearisome, to translate it into precise formulas. We introduce two additional curves α_1 and α_2 (see the above figure on the right and the next on the left). Slit the plane along the dashed lines, and get, in this way, a star domain in which the function $z \mapsto \frac{1}{z-a}$ is analytic. The integral "along the closed curve we have drawn", which is composed by a (small) circular arc, line segments and a (large) circular arc, vanishes by the CAUCHY integral theorem II.2.7 for star domains. The same argument holds for the figure reflected along the line connecting a and z_0, and the curve α_2 sketched on the right. Therefore

$$\int_{\alpha_1} \frac{1}{\zeta - a}\, d\zeta = 0 \text{ and } \int_{\alpha_2} \frac{1}{\zeta - a}\, d\zeta = 0 \ .$$

If one adds the two integrals, the contributions of the straight segments cancel, since the lines are traversed in reverse directions:



Therefore it follows that (taking into account the orientation!)

$$2\pi\mathrm{i} = \oint_{|\zeta - a| = \varrho} \frac{1}{\zeta - a}\, d\zeta = \oint_{|\zeta - z_0| = r} \frac{1}{\zeta - a}\, d\zeta \ .$$

□

From now on we shall use the notations

$$U_r(z_0) = \{\ z \in \mathbb{C}\ ; \quad |z - z_0| < r\ \}$$
$$\overline{U}_r(z_0) = \{\ z \in \mathbb{C}\ ; \quad |z - z_0| \leq r\ \}$$

for the respectively open and closed disks of radius $r > 0$ around $z_0 \in \mathbb{C}$.

Theorem II.3.2 (Cauchy Integral Formula, A.L. Cauchy, 1831) *Let*

$$f : D \longrightarrow \mathbb{C}\ ,\ D \subset \mathbb{C} \text{ open }\ ,$$

be an analytic function. Assume that the closed disk $\overline{U}_r(z_0)$ is contained completely in D. Then for each point $z \in U_r(z_0)$

$$\boxed{f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\alpha \frac{f(\zeta)}{\zeta - z}\, d\zeta\ ,}$$

where the integral is taken "around the circle α", i.e. along the closed curve

$$\alpha(t) = z_0 + re^{\mathrm{i}t}\ ,\ 0 \leq t \leq 2\pi\ .$$

We emphasize that the point z needs not to be the center of the disk. It only has to lie in the interior of the disk!

Using the compactness of $\overline{U}_r(z_0)$ one can easily show that there exists an $R > r$ such that

$$D \supset U_R(z_0) \supset \overline{U}_r(z_0) \ .$$

We can thus assume that D is a disk. The function

$$g(w) := \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{for } w \neq z \ , \\ f'(z) & \text{for } w = z \ , \end{cases}$$

is continuous in D and away from z is, in fact, analytic. We can therefore apply the CAUCHY integral theorem II.2.7$_1$ and obtain

$$\oint \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = 0 \ .$$

The assertion now follows from II.3.1. □

In particular, the CAUCHY integral formula holds for $z = z_0$:

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\big(z_0 + r \exp(\mathrm{i}t)\big) \, dt$$

(this is the so-called *mean value equation*).

The essence of the Cauchy integral formula is that it computes the values of analytic functions in the interior of a disk from their values on the boundary.

From the LEIBNIZ rule one gets analogous formulas for the derivatives.

Lemma II.3.3 (Leibniz rule) *Let*

$$f : [a, b] \times D \longrightarrow \mathbb{C} \ , \quad D \subset \mathbb{C} \text{ open} \ ,$$

be a continuous function, which is analytic in D for any fixed $t \in [a, b]$. The derivative

$$\frac{\partial f}{\partial z} : [a, b] \times D \longrightarrow \mathbb{C}$$

is also assumed to be continuous. Then the function

$$g(z) := \int_a^b f(t, z) \, dt$$

is analytic in D, and

$$g'(z) = \int_a^b \frac{\partial f(t, z)}{\partial z} \, dt \ .$$

Proof. One can reduce II.3.3 to the analogous result for the real case, since complex differentiability can be expressed using partial derivatives (Theorem I.5.3). Thus one uses the real form of the LEIBNIZ criterion to verify the CAUCHY-RIEMANN equations and the formula for the derivative of g.

For the sake of completeness we shall formulate and prove the real form of the LEIBNIZ rule that we need.

Let $f : [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be a continuous function. Suppose that the partial derivative

$$(t,x) \mapsto \frac{\partial}{\partial x} f(t,x)$$

exists and is continuous. Then

$$g(x) = \int_a^b f(t,x)\, dt$$

is also differentiable, and one has

$$g'(x) = \int_a^b \frac{\partial}{\partial x} f(t,x)\, dt \ .$$

Proof. We take the difference quotient at $x_0 \in D$:

$$\frac{g(x) - g(x_0)}{x - x_0} = \int_a^b \frac{f(t,x) - f(t,x_0)}{x - x_0}\, dt \ .$$

By the *mean value theorem of differential calculus*

$$\frac{f(t,x) - f(t,x_0)}{x - x_0} = \frac{\partial}{\partial x} f(t,\xi)$$

with a t-dependent point ξ between x_0 and x. By the *theorem of uniform continuity* (cf. Exercise 7 from I.3) for any given $\varepsilon > 0$ there exists a $\delta > 0$ with the property

$$\left| \frac{\partial}{\partial x} f(t_1,x_1) - \frac{\partial}{\partial x} f(t_2,x_2) \right| < \varepsilon \qquad \text{if } |x_1 - x_2| < \delta \ , \quad |t_1 - t_2| < \delta \ .$$

In particular,

$$\left| \frac{\partial}{\partial x} f(t,\xi) - \frac{\partial}{\partial x} f(t,x_0) \right| < \varepsilon \qquad \text{if } |x - x_0| < \delta \ .$$

It is decisive here that δ does not depend on t! Now we obtain

$$\left| \frac{g(x) - g(x_0)}{x - x_0} - \int_a^b \frac{\partial}{\partial x} f(t,x_0)\, dt \right| \ \leq \ \varepsilon(b-a) \qquad \text{if } |x - x_0| < \delta \ .$$

□

Theorem II.3.4 (Generalized Cauchy Integral Formula) *With the assumptions and notation of II.3.2 we have: Every analytic function is arbitrarily often complex differentiable. Each derivative is again analytic. For $n \in \mathbb{N}_0$ and all z with $|z - z_0| < r$ we have*

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_\alpha \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \ ,}$$

where $\alpha(t) = z_0 + re^{\mathrm{i}t}$, $0 \le t \le 2\pi$.

The proof follows by induction on n with the help of II.3.2 and II.3.3. □

For another proof see Exercise 10 in II.3.

Remark. Therefore it has also been proved that the assumptions of continuity of the derivative f', resp. of analyticity of f', we have previously made, were superfluous as they are automatically fulfilled. Moreover it follows that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are in fact $\mathcal{C}^\infty$-functions.

It was not necessary to use Lemma II.3.3 in its full generality for the proof of II.3.4. It would be possible just to check the required special case directly. Then one can get back II.3.3 from II.3.4 in full generality by using the FUBINI *theorem: If $f : [a, b] \times [c, d] \to \mathbb{C}$ is a continuous function, then*

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \ = \ \int_c^d \int_a^b f(x, y)\, dx\, dy \ .$$

The following theorem gives a kind of partial converse to the CAUCHY integral theorem.

Theorem II.3.5 (Morera's Theorem, (G. Morera, 1886)) *Let $D \subset \mathbb{C}$ be open and*

$$f : D \longrightarrow \mathbb{C}$$

be continuous. For every triangular path $\langle z_1, z_2, z_3 \rangle$ whose triangle is entirely contained in D assume

$$\int_{\langle z_1, z_2, z_3 \rangle} f(\zeta)\, d\zeta = 0 \ .$$

Then f is analytic.

Proof. For each point $z_0 \in D$ there is an open neighborhood $U_\varepsilon(z_0) \subset D$. It is enough to show that f is analytic in $U_\varepsilon(z_0)$. For $z \in U_\varepsilon(z_0)$ let

$$F(z) := \int_{\sigma(z_0, z)} f(\zeta)\, d\zeta \ ,$$

where $\sigma(z_0, z)$ is the line segment connecting z_0 and z. As in II.2.4 (c) $\Rightarrow$ (a) one shows that F is a primitive of f in $U_\varepsilon(z_0)$, i. e. $F'(z) = f(z)$ for $z \in U_\varepsilon(z_0)$. In particular, f is analytic itself as the derivative of an analytic function. □

Definition II.3.6 *An analytic function $f : \mathbb{C} \to \mathbb{C}$ is said to be* ***entire****.*

An entire function is thus an *analytic* function defined on the *entire complex plane* $\mathbb{C}$.

Examples. Polynomials $P : \mathbb{C} \to \mathbb{C}$, and $\exp, \cos, \sin : \mathbb{C} \to \mathbb{C}$ are entire functions.

Theorem II.3.7 (Liouville's Theorem, J. Liouville, 1847) *Every bounded entire function is constant.*
Equivalently: A nonconstant entire function cannot be bounded.

(In particular, for instance, cos cannot be bounded. In fact

$$\cos \mathrm{i}x = \frac{e^x + e^{-x}}{2} \to \infty \quad \text{for} \quad x \to \infty \ .$$

LIOUVILLE actually only treated the special case of elliptic functions (cf. Chapter V and Exercise 7 in II.3).

Proof. We show $f'(z) = 0$ for every point $z \in \mathbb{C}$. From the CAUCHY integral formula

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta - z| = r} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \ ,$$

which holds for *every* $r > 0$, it follows that

$$|f'(z)| \ \leq \ \frac{1}{2\pi} \underbrace{2\pi r}_{\substack{\text{arc} \\ \text{length}}} \frac{C}{r^2} \ = \ \frac{C}{r} \ .$$

The assertion can now be obtained by passing to the limit $r \to \infty$. □

From LIOUVILLE's Theorem follows easily:

Theorem II.3.8 (Fundamental Theorem of Algebra) *Each nonconstant complex polynomial has a root.*

Proof. Let

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \ , \qquad a_\nu \in \mathbb{C} \ , \ 0 \leq \nu \leq n \ , \ n \geq 1 \ , \ a_n \neq 0 \ .$$

be a polynomial of degree ≥ 1. Then

$$|P(z)| \to \infty \text{ for } |z| \to \infty$$

i.e. for each $C > 0$ there exists an $R > 0$ such that

$$|z| \geq R \implies |P(z)| \geq C \ ,$$

(Note:[2] One has $z^{-n} P(z) \to a_n$ for $|z| \to \infty$.) We assume that P has no complex root. Then $1/P$ is a bounded entire function and so $1/P$ is a constant by LIOUVILLE's theorem. □

[2] Cf. also Exercise 12 in II.2.

Corollary II.3.9 *Every polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \ , \quad a_\nu \in \mathbb{C},\ 0 \leq \nu \leq n \ ,$$

of degree $n \geq 1$ can be written as a product of n linear factors and a constant $C \in \mathbb{C}^\bullet$

$$P(z) = C(z - \alpha_1) \cdots (z - \alpha_n) \ .$$

The numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are uniquely determined up to their order, and $C = a_n$.

Proof. If $n \geq 1$, there exists a zero α_1. We reorder the polynomial by powers of $(z - \alpha_1)$

$$P(z) = b_0 + b_1(z - \alpha_1) + \cdots \quad .$$

From $P(\alpha_1) = 0$ it follows that $b_0 = 0$ and therefore

$$P(z) = (z - \alpha_1)Q(z) \ , \quad \text{degree}\, Q = n - 1 \ .$$

The assertion then follows by induction on n. □

If one collects equal α_ν, then one gets for P a formula

$$P(z) = C(z - \beta_1)^{\nu_1} \cdots (z - \beta_r)^{\nu_r}$$

with pairwise different $\beta_j \in \mathbb{C}$ and integers ν_j, for which we then have $\nu_1 + \cdots + \nu_r = n$.

We shall obtain other function-theoretic proofs of the fundamental theorem of algebra later (cf. also Exercise 13 in II.2 of this Chapter and application of the Residue Theorem III.6.3).

Exercises for II.3

We shall denote by $\alpha_{a;r}$ the curve whose image is the circle with center a and radius $r > 0$, i.e. with

$$\alpha_{a;r} : [0, 2\pi] \longrightarrow \mathbb{C}, \quad \alpha_{a,r}(t) = a + re^{\mathrm{i}t} \ .$$

1. Compute, using the CAUCHY integral theorem and the CAUCHY integral formula, the following integrals:

(a) $\displaystyle\int_{\alpha_{2;1}} \frac{z^7 + 1}{z^2(z^4 + 1)}\, dz \ ,$ (b) $\displaystyle\int_{\alpha_{1;3/2}} \frac{z^7 + 1}{z^2(z^4 + 1)}\, dz \ ,$

(c) $\displaystyle\int_{\alpha_{0;3}} \frac{e^{-z}}{(z + 2)^3}\, dz \ ,$ (d) $\displaystyle\int_{\alpha_{0;3}} \frac{\cos \pi z}{z^2 - 1}\, dz \ ,$

(e) $\displaystyle\int_{\alpha_{0;r}} \frac{\sin z}{z - b}\, dz \ , \qquad (b \in \mathbb{C} \ ,\ |b| \neq r) \ .$

2. Compute, using the CAUCHY integral theorem and the CAUCHY integral formula, the following integrals:

$$\text{(a)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{\mathrm{i};1}} \frac{e^z}{z^2+1}\, dz\ , \qquad \text{(b)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{-\mathrm{i};1}} \frac{e^z}{z^2+1}\, dz\ ,$$

$$\text{(c)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{0;3}} \frac{e^z}{z^2+1}\, dz\ , \qquad \text{(d)} \quad \frac{1}{2\pi\mathrm{i}} \int_{\alpha_{1+2\mathrm{i};5}} \frac{4z}{z^2+9}\, dz\ .$$

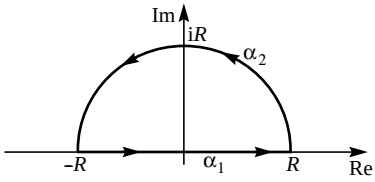
3. Compute

$$\text{(a)} \quad \int_{\alpha_{1;1}} \left(\frac{z}{z-1} \right)^n dz\ , \quad n \in \mathbb{N}\ ,$$

$$\text{(b)} \quad \int_{\alpha_{0;r}} \frac{1}{(z-a)^n (z-b)^m}\, dz\ , \quad |a| < r < |b|\ ,\ n, m \in \mathbb{N}\ .$$

4. Let $\alpha = \alpha_1 \oplus \alpha_2$ be the curve sketched in the figure with $R > 1$ and

$$f(z) := \frac{1}{1+z^2}\ .$$



Show:

$$\int_\alpha f(z)\, dz = \int_{\alpha_1} f(z)\, dz + \int_{\alpha_2} f(z)\, dz = \pi$$

and

$$\lim_{R\to\infty} \left| \int_{\alpha_2} f(z)\, dz \right| = 0\ .$$

Deduce that:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\, dx = \pi\ .$$

These indefinite integrals could have been calculated more easily (arctan is a primitive!). However, this gives a first indication of how one can compute real integrals using complex methods. We shall return to this when applying the residue theorem cf. III.7).

5. Let α be the closed curve considered in Exercise 4 of II.1 ("figure eight"). Compute the integral

$$\int_\alpha \frac{1}{1-z^2}\, dz\ .$$

6. *Show:* If $f : \mathbb{C} \to \mathbb{C}$ is analytic and if there is a real number M such that for all $z \in \mathbb{C}$

$$\operatorname{Re} f(z) \le M\ ,$$

then f is constant.

Hint. Consider $g := \exp \circ f$ and apply LIOUVILLE's theorem to g.

7. Let ω and ω' be complex numbers which are linearly independent over $\mathbb{R}$.

   Show: If $f : \mathbb{C} \to \mathbb{C}$ is analytic and

   $$f(z+\omega) = f(z) = f(z+\omega') \text{ for all } z \in \mathbb{C} ,$$

   then f is constant (J. LIOUVILLE, 1847).

8. **Gauss-Lucas Theorem** (C.F. GAUSS, 1816; F. LUCAS, 1879)

   Let P be a complex polynomial of degree $n \geq 1$, with n not necessarily different zeros $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$. Show that for all $z \in \mathbb{C} \setminus \{\zeta_1, \ldots, \zeta_n\}$

   $$\frac{P'(z)}{P(z)} = \frac{1}{z-\zeta_1} + \frac{1}{z-\zeta_2} + \cdots + \frac{1}{z-\zeta_n} = \sum_{\nu=1}^{n} \frac{\overline{z-\zeta_\nu}}{|z-\zeta_\nu|^2} .$$

   Deduce from this the GAUSS-LUCAS theorem:
   For each zero ζ of P' there are n real numbers $\lambda_1, \ldots, \lambda_n$ with

   $$\lambda_1 \geq 0, \ldots, \lambda_n \geq 0, \quad \sum_{j=1}^{n} \lambda_j = 1 \quad \textit{and} \quad \zeta = \sum_{\nu=1}^{n} \lambda_\nu \zeta_\nu .$$

   Thus one can say: The zeros of P' lie in the "convex hull" of the zero set of P.

9. Show that every rational function R (i.e. $R(z) = P(z)/Q(z)$, P, Q polynomials, $Q \neq 0$) can be written as the sum of a polynomial and a finite linear combination, with complex coefficients, of "simple functions" of the form

   $$z \mapsto \frac{1}{(z-s)^n} , \quad n \in \mathbb{N} , \ s \in \mathbb{C} ,$$

   the so-called "partial fractions" (Partial fraction decomposition theorem), see also Chapter III, Appendix A to Sections III.4 and III.5, Proposition A.7).

   Deduce: If the coefficients of P and Q are real, then f has "a *real* partial fraction decomposition" (by putting together pairs of complex conjugate zeros, or rather by putting together the corresponding partial fractions (see also Exercise 10 in I.1).

10. A somewhat more direct proof of the generalized CAUCHY integral formula (Theorem II.3.4) is obtained with the following *Lemma:*

    Let $\alpha : [a,b] \to \mathbb{C}$ be a piecewise smooth curve and let $\varphi :$ Image $\alpha \to \mathbb{C}$ be continuous. For $z \in D := \mathbb{C} \setminus$ Image α and $m \in \mathbb{N}$ let

    $$F_m(z) := \frac{1}{2\pi \mathrm{i}} \int_\alpha \frac{\varphi(\zeta)}{(\zeta - z)^m} \, d\zeta .$$

    Then F_m is analytic in D and for all $z \in D$

    $$F_m'(z) = m \, F_{m+1}(z) .$$

    Prove this by direct estimate (without using the LEIBNIZ rule).

11. Let $D \subset \mathbb{C}$ be open, and $L \subset \mathbb{C}$ a line. If $f : D \to \mathbb{C}$ is a continuous function, which is analytic at all points $z \in D$, $z \notin L$, then f is analytic on the whole D.

12. **The Schwarz Reflection principle** (H.A. SCHWARZ, 1867)

Let $D \neq \emptyset$ be a domain which is symmetric with respect to the real axis (i.e. $z \in D \implies \bar{z} \in D$). We consider the subsets

$$\begin{aligned} D_+ &:= \{\, z \in D\,; \quad \operatorname{Im} z > 0 \,\}\ , \\ D_- &:= \{\, z \in D\,; \quad \operatorname{Im} z < 0 \,\}\ , \\ D_0 &:= \{\, z \in D\,; \quad \operatorname{Im} z = 0 \,\} = D \cap \mathbb{R}\ . \end{aligned}$$

If $f : D_+ \cup D_0 \to \mathbb{C}$ is continuous, $f \mid D_+$ analytic and $f(D_0) \subset \mathbb{R}$, then the function defined by

$$\tilde{f}(z) := \begin{cases} f(z) & \text{for } z \in D_+ \cup D_0\ , \\ \overline{f(\bar{z})} & \text{for } z \in D_+\ , \end{cases}$$

is analytic.

13. Let f be a continuous function on the compact interval $[a, b]$.

Show: The function defined by

$$F(z) = \int_a^b \exp(-zt)\, f(t)\, dt$$

is analytic on the whole $\mathbb{C}$, and

$$F'(z) = -\int_a^b \exp(-zt) t f(t)\, dt\ .$$

14. Let $D \subset \mathbb{C}$ be a domain and

$$f : D \longrightarrow \mathbb{C}$$

be an analytic function.

Show: The function

$$\varphi : D \times D \longrightarrow \mathbb{C}$$

with

$$\varphi(\zeta, z) := \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z\ , \\ f'(\zeta) & \text{if } \zeta = z\ , \end{cases}$$

is a continuous function of two variables.

For each given $z \in D$ the function

$$\zeta \longmapsto \varphi(\zeta, z)$$

is analytic in D.

15. Determine all pairs (f, g) of entire functions with the property

$$f^2 + g^2 = 1\ .$$

Result:

$f = \cos \circ h$ and $g = \sin \circ h$, where h is an arbitrary entire function.

16. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant, entire function. Then $f(\mathbb{C})$ is dense in $\mathbb{C}$.