

Preface

A fundamental object of study in the theory of groups is the lower central series of groups whose terms are defined for a group G inductively by setting

$$\gamma_1(G) = G, \quad \gamma_{n+1}(G) = [G, \gamma_n(G)] \quad (n \geq 1),$$

where, for subsets H, K of G , $[H, K]$ denotes the subgroup of G generated by the commutators $[h, k] := h^{-1}k^{-1}hk$ for $h \in H$ and $k \in K$. The lower central series of free groups was first investigated by Magnus [Mag35]. To recall Magnus's work, let F be a free group with basis $\{x_i\}_{i \in I}$ and $\mathcal{A} = \mathbb{Z}[[X_i \mid i \in I]]$ the ring of formal power series in the non-commuting variables $\{X_i\}_{i \in I}$ over the ring \mathbb{Z} of integers. Let $\mathcal{U}(\mathcal{A})$ be the group of units of \mathcal{A} . The map $x_i \mapsto 1 + X_i$, $i \in I$, extends to a homomorphism

$$\theta : F \rightarrow \mathcal{U}(\mathcal{A}), \tag{1}$$

since $1 + X_i$ is invertible in \mathcal{A} with $1 - X_i + X_i^2 - \dots$ as its inverse. The homomorphism θ is, in fact, a monomorphism (Theorem 5.6 in [Mag66]). For $a \in \mathcal{A}$, let a_n denote its homogeneous component of degree n , so that

$$a = a_0 + a_1 + \dots + a_n + \dots .$$

Define

$$\mathcal{D}_n(F) := \{f \in F \mid \theta(f)_i = 0, 1 \leq i < n\}, \quad n \geq 1.$$

It is easy to see that $\mathcal{D}_n(F)$ is a normal subgroup of F and the series $\{\mathcal{D}_n(F)\}_{n \geq 1}$ is a central series in F , i.e., $[F, \mathcal{D}_n(F)] \subseteq \mathcal{D}_{n+1}(F)$ for all $n \geq 1$. Clearly, the intersection of the series $\{\mathcal{D}_n(F)\}_{n \geq 1}$ is trivial. Since $\{\mathcal{D}_n(F)\}_{n \geq 1}$

is a central series, we have $\gamma_n(F) \subseteq \mathcal{D}_n(F)$ for all $n \geq 1$. Thus, it follows that the intersection $\bigcap_{n \geq 1} \gamma_n(F)$ is trivial, i.e., F is residually nilpotent.

Let G be an arbitrary group and R a commutative ring with identity. The *group ring* of G over R , denoted by $R[G]$, is the R -algebra whose elements are the formal sums $\sum \alpha(g)g$, $g \in G$, $\alpha(g) \in R$, with only finitely many coefficients $\alpha(g)$ being non zero. The addition and multiplication in $R[G]$ are defined as follows:

$$\sum_{g \in G} \alpha(g)g + \sum_{g \in G} \beta(g)g = \sum_{g \in G} (\alpha(g) + \beta(g))g,$$

$$\sum_{g \in G} \alpha(g)g \sum_{h \in G} \beta(h)h = \sum_{x \in G} \left(\sum_{gh=x} \alpha(g)\beta(h) \right) x.$$

The group G can be identified with a subgroup of the group of units of $R[G]$, by identifying $g \in G$ with $1_R g$, where 1_R is the identity element of R , and it then constitutes an R -basis for $R[G]$. The map

$$\epsilon : R[G] \rightarrow R, \quad \sum \alpha(g)g \mapsto \sum \alpha(g),$$

is an algebra homomorphism and is called the *augmentation map*; its kernel is called the *augmentation ideal* of $R[G]$; we denote it by $\Delta_R(G)$. In the case when R is the ring \mathbb{Z} of integers, we refer to $\mathbb{Z}[G]$ as the integral group ring of G and denote the augmentation ideal also by \mathfrak{g} , the corresponding Euler fraktur lowercase letter.

The augmentation ideal $\Delta_R(G)$ leads to the following filtration of $R[G]$:

$$R[G] \supseteq \Delta_R(G) \supseteq \Delta_R^2(G) \supseteq \dots \supseteq \Delta_R^n(G) \supseteq \dots \quad (2)$$

Note that the subset $G \cap (1 + \Delta_R^n(G))$, $n \geq 1$, is a normal subgroup of G ; this subgroup is called the *n th dimension subgroup* of G over R and is denoted by $D_{n,R}(G)$. It is easy to see that $\{D_{n,R}(G)\}_{n \geq 1}$ is a central series in G , and therefore $\gamma_n(G) \subseteq D_{n,R}(G)$ for all $n \geq 1$. In the case when R is the ring \mathbb{Z} of integers, we drop the suffix \mathbb{Z} and write $D_n(G)$ for $D_{n,\mathbb{Z}}(G)$. The quotients $\Delta_R^n(G)/\Delta_R^{n+1}(G)$, $n \geq 1$, are $R[G]$ -modules with trivial G -action. There then naturally arise the following problems about dimension subgroups and augmentation powers.

Problem 0.1 Identify the subgroups $D_{n,R}(G) = G \cap (1 + \Delta_R^n(G))$, $n \geq 1$.

Problem 0.2 Describe the structure of the quotients $\Delta_R^n(G)/\Delta_R^{n+1}(G)$, $n \geq 1$.

Problem 0.3 Describe the intersection $\bigcap_{n \geq 1} \Delta_R^n(G)$; in particular, characterize the case when this intersection is trivial.

In the case when F is a free group, then, for all $n \geq 1$, $\gamma_n(F) \subseteq D_n(F) \subseteq \mathcal{D}_n(F)$. The homomorphism $\theta : F \rightarrow \mathcal{A}$, defined in (1), extends by linearity

to the integral group ring $\mathbb{Z}[F]$ of the free group F ; we continue to denote the extended map by θ :

$$\theta : \mathbb{Z}[F] \rightarrow \mathcal{A}.$$

Let \mathfrak{f} be the augmentation ideal of $\mathbb{Z}[F]$; then, for $\alpha \in \mathfrak{f}^n$, $\theta(\alpha)_i = 0$, $i \leq n-1$. With the help of free differential calculus, it can be seen that the intersection of the ideals \mathfrak{f}^n , $n \geq 1$, is zero and the homomorphism $\theta : \mathbb{Z}[F] \rightarrow \mathcal{A}$ is a monomorphism (see Chap. 4 in [Gru70]). A fundamental result about free groups ([Mag37], [Gru36], [Wit37]; see also [Röh85]) is that the inclusions $\gamma_n(F) \subseteq D_n(F) \subseteq \mathcal{D}_n(F)$ are equalities:

$$\gamma_n(F) = D_n(F) = \mathcal{D}_n(F), \text{ for all } n \geq 1. \quad (3)$$

This result exhibits a close relationship among the lower central series, the dimension series, and the powers of the augmentation ideal of the integral group ring of a free group. Thus, for free groups, Problems 1 and 3 have a definitive answer in the integral case. Problem 2 also has a simple answer in this case: for every $n \geq 1$, the quotient $\mathfrak{f}^n/\mathfrak{f}^{n+1}$ is a free abelian group with the set of elements $(x_{i_1} - 1) \dots (x_{i_n} - 1) + \mathfrak{f}^{n+1}$ as basis, where x_{i_j} range over a basis of F (see p. 116 in [Pas79]).

The foregoing results about free groups naturally raise the problem of investigation of the relationship among the lower central series $\{\gamma_n(G)\}_{n \geq 1}$, dimension series $\{D_{n,R}(G)\}_{n \geq 1}$, and augmentation quotients $\Delta_R^n(G)/\Delta_R^{n+1}(G)$, $n \geq 1$, of an arbitrary group G over the commutative ring R . While these series have been extensively studied by various authors over the last several decades (see [Pas79], [Gup87c]), we are still far from a definitive theory. The most challenging case here is that when R is the ring \mathbb{Z} of integers, where a striking feature is that, unlike the case of free groups, the lower central series can differ from the dimension series, as first shown by Rips [Rip72].

Besides being purely of algebraic interest, lower central series and augmentation powers occur naturally in several other contexts, notably in algebraic K-theory, number theory, and topology. For example, the lower central series is the main ingredient of the theory of Milnor's $\bar{\mu}$ -invariants of classical links [Mil57]; the lower central series and augmentation powers come naturally in [Cur71], [Gru80], [Qui69], and in the works of many other authors.

The main object of this monograph is to present an exposition of different methods related to the theory of the lower central series of groups, the dimension subgroups, and the augmentation powers. We will also be concerned with another important related series, namely, the derived series whose terms are defined, for a given group G , inductively by setting

$$\delta_0(G) = G, \quad \delta_{n+1}(G) = [\delta_n(G), \delta_n(G)] \text{ for } n \geq 0.$$

Our focus will be primarily on homological, homotopical, and combinatorial methods for the study of group rings. Simplicial methods, in fact, provide

new possibilities for the theory of groups, Lie algebras, and group rings. For example, the derived functors of endofunctors on the category of groups come into play. Thus, working with simplicial objects and homotopy theory suggests new approaches for studying invariants of group presentations, a point of view which may be termed as “homotopical group theory.” By homological group theory one normally means the study of properties of groups based on the properties of projective resolutions over their group rings. In contrast to this theory, by homotopical group theory we may understand the study of groups with the help of simplicial resolutions. From this point of view, homological group theory then appears as an abelianization of the homotopical one.

We now briefly describe the contents of this monograph.

Chapter 1: Lower central series We discuss examples and methods for investigating the lower central series of groups with a view to examining residual nilpotence, i.e., the property that this series intersects in the identity subgroup. We begin with Magnus’s theorem [Mag35] on residual nilpotence of free groups and Gruenberg’s [Gru57] result on free polynilpotent groups. Mal’cev’s observation [Mal68] on the adjoint group of an algebra provides a method for constructing residually nilpotent groups. We consider next free products and describe Lichtman’s characterization [Lic78] for the residual nilpotence of a free product of groups. If a group G is such that the augmentation ideal \mathfrak{g} of its integral group ring $\mathbb{Z}[G]$ is residually nilpotent, i.e., $\bigcap_{n \geq 0} \mathfrak{g}^n = 0$, then it is easily seen that G is residually nilpotent. This property, namely, the residual nilpotence of \mathfrak{g} , has been characterized by Lichtman [Lic77]. Our next object is to discuss residual nilpotence of wreath products. We give a detailed account of Hartley’s work, along with Shmelkin’s theorem [Shm73] on verbal wreath products. For HNN-extensions we discuss a method introduced by Raptis and Varsos [Rap89]. Turning to linear groups, we give an exposition of recent work of Mikhailov and Bardakov [Bar07]. An interesting class of groups arising from geometric considerations is that of braid groups; we discuss the result of Falk and Randell [Fal88] on pure braid groups.

If $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ is a free presentation of a group G , then the quotient group $R/[R, R]$ of R can be viewed as a G -module, called a “relation module.” The relationship between the properties of $F/[R, R]$ and those of F/R has been investigated by many authors [Gru70], [Gru], [Gup87c]. We discuss a generalization of this notion. Let R and S be normal subgroups of the free group F . Then the quotient group $(R \cap S)/[R, S]$ is abelian and it can be viewed, in a natural way, as a module over F/RS . Clearly the relation modules, and more generally, the higher relation modules $\gamma_n(R)/\gamma_{n+1}(R)$, $n \geq 2$, are all special cases of this construction. Such modules are related to the second homotopy module $\pi_2(X)$ of the standard complex X associated to the free presentation $G \simeq F/R$. We discuss here the work in [Mik06b]. The main point of investigation here is the faithfulness of the F/RS -module $(R \cap S)/[R, S]$.

We next turn to k -central extensions, namely, the extensions

$$1 \rightarrow N \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

where N is contained in the k th central subgroup $\zeta_k(\tilde{G})$ of \tilde{G} . We examine the connection between residual nilpotence of G and that of \tilde{G} . In general, neither implies the other.

The construction of the lower central series $\{\gamma_n(G)\}_{n \geq 1}$ of a group G can be extended in an obvious way to define the transfinite terms $\gamma_\tau(G)$ of the lower central series for infinite ordinals τ . Let ω denote the first non-finite ordinal. The groups G whose lower central series has the property that $\gamma_\omega(G) \neq \gamma_{\omega+1}(G)$, called groups with long lower central series, are of topological interest. We discuss methods for constructing such groups.

It has long been known that the Schur multiplier is related to the study of the lower central series. We explore a similar relationship with generalized multipliers, better known as Baer invariants of free presentations of groups. In particular, we discuss generalized Dwyer filtration of Baer invariants and its relation with the residual nilpotence of groups. Using a generalization of the Magnus embedding, we shall see that every 2-central extension of a one-relator residually nilpotent group is itself residually nilpotent [Mik07a].

The residually nilpotent groups with the same lower central quotients as some free group are known as para-free groups. Non-free para-free groups were first discovered by Baumslag [Bau67]. We make some remarks related to the para-free conjecture, namely, the statement that a finitely generated para-free group has trivial Schur multiplier.

Next we study the nilpotent completion $Z_\infty(G)$, which is the inverse limit of the system of epimorphisms $G \rightarrow G/\gamma_n(G)$, $n \geq 1$, and certain subgroups of this completion. We study the Bousfield–Kan completion $R_\infty X$ of a simplicial set X over a commutative ring R , and homological localization (called HZ-localization) functor $L : G \mapsto L(G)$ on the category of groups, due to Bousfield, in particular, the uncountability of the Schur multiplier of the free nilpotent completion of a non abelian free group. After discussing the homology of the nilpotent completion, we go on to study transfinite para-free groups. Given an ordinal number τ , a group G is defined to be τ -para-free if there exists a homomorphism $F \rightarrow G$, with F free, such that

$$L(F)/\gamma_\tau(L(F)) \simeq L(G)/\gamma_\tau(L(G)).$$

Our final topic of discussion in this chapter is the study of the lower central series and the homology of crossed modules. These modules were first defined by Whitehead [Whi41]. With added structure to take into account, the computations naturally become rather more complicated. It has recently been shown that the cokernel of a non aspherical projective crossed module with a free base group acts faithfully on its kernel [Mik06a]; we give an exposition of this and some other related results.

Chapter 2: Dimension subgroups In this chapter we study various problems concerning the dimension subgroups. The relationship between the lower central and the dimension series of groups is highly intriguing.

For every group G and integer $n \geq 1$, we have

$$\gamma_n(G) \subseteq D_n(G) := G \cap (1 + \mathfrak{g}^n).$$

As first shown by Rips [Rip72], equality does not hold in general. We pursue the counter example of Rips and the subsequent counterexamples constructed by N. Gupta, and construct several further examples of groups without the dimension property, i.e., groups where the lower central series does not coincide with the integral dimension series. We construct a 4-generator and 3-relator example of a group G with $D_4(G) \neq \gamma_4(G)$ and show further that, in a sense, this is a minimal counter example by proving that every 2-relator group G has the property that $D_4(G) = \gamma_4(G)$. At the moment it seems to be an intractable problem to compute the length of the dimension series of a finite 2-group of class 3. However, we show that for the group without the dimension property considered by Gupta and Passi (see p.76 in [Gup87c]), the fifth dimension subgroup is trivial. Examples of groups with $D_n(G) \neq \gamma_n(G)$ with $n \geq 5$ were first constructed by Gupta [Gup90]. In this direction, for each $n \geq 5$, we construct a 5-generator 5-relator group \mathfrak{G}_n such that $D_n(\mathfrak{G}_n) \neq \gamma_n(\mathfrak{G}_n)$. We also construct a nilpotent group of class 4 with non trivial sixth dimension subgroup. We hope that these examples will lead to a closer understanding of groups without the dimension property.

For each $n \geq 4$, in view of the existence of groups with $\gamma_n(G) \neq D_n(G)$, the class \mathcal{D}_n of groups with trivial n th dimension subgroup is not a variety. This class is, however, a quasi-variety [Plo71]. We present an account of our work [Mik06c] showing that the quasi-variety \mathcal{D}_4 is not finitely based, thus answering a problem of Plotkin.

We next review the progress on the identification of integral dimension subgroups and on Plotkin's problems about the length of the dimension series of nilpotent groups.

Related to the dimension subgroups are the Lie dimension subgroups $D_{[n]}(G)$ and $D_{(n)}(G)$, $n \geq 1$, with $D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G)$ (see Sect. 2.10 for definitions). We show that, for every natural number s , there exists a group of class n such that $D_{[n+s]}(G) \neq \gamma_{n+s}(G) \neq 1$. In contrast with the integral case, many more definitive results are known about the dimension subgroups and the Lie dimension subgroups over fields. We review these results and their applications, in particular, to the study of Lie nilpotency indices of the augmentation ideals.

Chapter 3: Derived series We study the derived series of free nilpotent groups. Combinatorial methods developed for the study of the dimension subgroups can be employed for this purpose. We give an exposition of the work of Gupta and Passi [Gup07].

Homological methods were first effectively applied by Strebel [Str74] for the investigation of the derived series of groups. We give an exposition of Strebel's work and then apply it to study properties analogous to those studied in Chap. 1 for the lower central series of groups; in particular, we study the behaviour of the transfinite terms of the derived series. We show that this study has an impact on Whitehead's asphericity question which asks whether every sub complex of an aspherical two-dimensional complex is itself aspherical.

Chapter 4: Augmentation powers The structure of the augmentation powers \mathfrak{g}^n and the quotients $\mathfrak{g}^n/\mathfrak{g}^{n+1}$ for the group G is of algebraic and number-theoretic interest. While the augmentation powers have been investigated by several authors, a solution to this problem, when G is torsion free or torsion abelian group, has recently been given by Bak and Tang [Bak04]. We give an account of the main features of this work. We then discuss transfinite augmentation powers \mathfrak{g}^τ , where τ is any ordinal number. We next give an exposition of some of Hartley's work on the augmentation powers.

A filtration $\{P_n H^2(G, \mathbb{T})\}_{n \geq 0}$ of the Schur multiplier $H^2(G, \mathbb{T})$ arising from the notion of the polynomial 2-cocycles is a useful tool for the investigation of the dimension subgroups. This approach to dimension subgroups naturally leads to relative dimension subgroups $D_n(E, N) := E \cap (1 + \mathfrak{e}^n + \mathfrak{n}\mathfrak{e})$, where N is a normal subgroup of the group E . The relative dimension subgroups provide a generalization of dimension subgroups, and have been extensively studied by Hartl [Har08]. Using the above-mentioned filtration of the Schur multiplier, Passi and Stammbach [Pas74] have given a characterization of para-free groups. These ideas were further developed in [Mik04, Mik05b]. We give here an account of this (co)homological approach for the study of subgroups determined by two-sided ideals in group rings.

In analogy with the notion of HZ-localization of groups, we study the Bousfield HZ localization of modules. Given a group G , a G -module homomorphism $f : M \rightarrow N$ is said to be an HZ-map if the maps $f_0 : H_0(G, M) \rightarrow H_0(G, N)$ and $f_1 : H_1(G, M) \rightarrow H_1(G, N)$ induced on the homology groups are such that f_0 is an epimorphism and f_1 is an isomorphism. In the category of G -modules, we examine the localization of G -modules with respect to the class \mathcal{HZ} of HZ maps. We discuss the work of Brown and Dror [Bro75] and of Dwyer [Dwy75] where the relation between the HZ localization of a module M and its \mathfrak{g} -adic completion $\varprojlim M/\mathfrak{g}^n M$ has been investigated.

Chapter 5: Homotopical aspects After recalling the construction of certain functors, we give an exposition of the work of Curtis [Cur63] on the lower central series of simplicial groups. Of particular interest to us are his two spectral sequences of homotopy exact couples arising from the application of the lower central and the augmentation power functors, γ_n and Δ^n , respectively, to a simplicial group. Our study is motivated by the work Stallings [Sta75] where he suggested a program and pointed out the main problem in its pursuit:

Finally, Rips has shown that there is a difference between the “dimension subgroups” and the terms of the lower central series The problem would be, how to compute with the Curtis spectral sequence, at least to the point of going through the Rips example in detail?

To some extent, this program was realized by Sjögren [Sjo79]. However, in general, it is an open question whether there exists any homotopical role of the groups without the dimension property. The simplicial approach for the investigation of the relationship between the lower central series and the augmentation powers of a group has been studied by Gruenfelder [Gru80]. We compute certain initial terms of the Curtis spectral sequences and, following [Har08], discuss applications to the identification of dimension subgroups. We show that it is possible to place some of the known results on the dimension subgroups in a categorical setting, and thus hope that this point of view might lead to a deeper insight. One of the interesting features of the simplicial approach is the connection that it provides to the derived functors of non-additive functors in the sense of Dold and Puppe [Dol61]. Finally, we present a number of homotopical applications; in particular, we give an algebraic proof of the computation of the low-dimensional homotopy groups of the 2-sphere. Another application that we give is an algebraic proof of the well-known theorem, due to Serre [Ser51], about the p -torsion of the homotopy groups of the 2-sphere.

Chapter 6: Miscellanea In this concluding chapter we present some applications of the group ring construction in different, rather unexpected, contexts. As examples, we may mention here (1) the solution, due to Lam and Leung [Lam00], of a problem in number theory which asks, for a given natural number m , the computation of the set $W(m)$ of all possible integers n for which there exist m th roots of unity $\alpha_1, \dots, \alpha_n$ in the field \mathbb{C} of complex numbers such that $\alpha_1 + \dots + \alpha_n = 0$, and (2) application of dimension subgroups by Massuyeau [Mas07] in low-dimensional topology.

Appendix At several places in the text we need results about simplicial objects. Rather than interrupt the discussion at each such point, we have preferred to collect the results needed in an appendix to which the reader may refer as and when necessary. The material in the appendix is mainly that which is needed for the group-theoretic problems in hand. For a more detailed exposition of simplicial methods, the reader is referred to the books of Goerss and Jardine [Goe99] and May [May67].

To conclude, we may mention that a crucial point that emerges from the study of the various series carried out in this monograph is that the least transfinite step in all cases is the one which at the moment deserves most to be understood from the point of view of applications.

Chapter 2

Dimension Subgroups

Let $\mathbb{Z}[G]$ be the integral group ring of a group G and let \mathfrak{g} be its augmentation ideal. For each natural number $n \geq 1$, $D_n(G) = G \cap (1 + \mathfrak{g}^n)$ is a normal subgroup of G called the n th *integral dimension subgroup* of G . It is easy to see that the decreasing series

$$G = D_1(G) \supseteq D_2(G) \supseteq \dots \supseteq D_n(G) \supseteq \dots$$

is a central series in G , i.e., $[G, D_n(G)] \subseteq D_{n+1}(G)$ for all $n \geq 1$. Therefore, $\gamma_n(G) \subseteq D_n(G)$ for all $n \geq 1$, where $\gamma_n(G)$ is the n th term in the lower central series of G . The identification of dimension subgroups, and, in particular, whether $\gamma_n(G) = D_n(G)$, has been a subject of intensive investigation for the last over fifty years. It is now known that, whereas $D_n(G) = \gamma_n(G)$ for $n = 1, 2, 3$ for every group G (see [Pas79]), there exist groups G whose series $\{D_n(G)\}_{n \geq 1}$ of dimension subgroups differs from the lower central series $\{\gamma_n(G)\}_{n \geq 1}$ ([Rip72], [Tah77b], [Tah78b], [Gup90]). The various developments in this area have been reported in [Pas79] and [Gup87c]. In the present exposition, we will primarily concentrate on the results that have appeared since the publication of [Gup87c]. We particularly focus attention on the fourth and the fifth dimension subgroups. We recall the description of the fifth dimension subgroup due to Tahara (Theorem 2.29) and give a proof of one of his theorems which states that, for every group G , $D_5(G)^6 \subseteq \gamma_5(G)$ (see Theorem 2.27). The proof here is, hopefully, shorter than the original one.

2.1 Groups Without Dimension Property

Given a group G , we call the quotient $D_n(G)/\gamma_n(G)$, $n \geq 1$, the n th *dimension quotient* of G . In case all dimension quotients are trivial, we say that the group G has the *dimension property*. We begin with examples of groups which do *not* have the dimension property.

Example 2.1 (Rips [Rip72]).

The first example of a group without dimension property was given by E. Rips. Following the notation from [Rip72], consider the group G with generators

$$a_0, a_1, a_2, a_3, b_1, b_2, b_3, c$$

and defining relations

$$\begin{aligned} b_1^{64} &= b_2^{16} = b_3^4 = c^{256} = 1, \\ [b_2, b_1] &= [b_3, b_1] = [b_3, b_2] = [c, b_1] = [c, b_2] = [c, b_3] = 1, \\ [a_0^{64} &= b_1^{32}, a_1^{64} = b_2^{-4}b_3^{-2}, a_2^{16} = b_1^4b_3^{-1}, a_3^4 = b_1^2b_2, \\ [a_1, a_0] &= b_1c^2, [a_2, a_0] = b_2c^8, [a_3, a_0] = b_3c^{32}, \\ [a_2, a_1] &= c, [a_3, a_1] = c^2, [a_3, a_2] = c^4, \\ [b_1, a_1] &= c^4, [b_2, a_2] = c^{16}, [b_3, a_3] = c^{64}, \\ [b_i, a_j] &= 1 \text{ if } i \neq j, [c, a_i] = 1 \text{ for } i = 0, 1, 2, 3. \end{aligned}$$

Then $\gamma_4(G) = 1$ while the element

$$[a_1, a_2]^{128}[a_1, a_3]^{64}[a_2, a_3]^{32} = c^{128}$$

is a non-identity element in $D_4(G)$.

Example 2.2 (Tahara [Tah78a]).

The above example was generalized by Ken-Ichi Tahara as follows:

Let $G_{k,l}$ ($k \geq 2$, $l \geq 0$) be a group with generators x_1, x_2, x_3, x_4 and defining relations

$$\begin{aligned} x_1^{2^{k+l+4}} &= [x_2, x_3]^{-2^{k+l+4}} [x_2, x_1]^{-2^{k+l+3}}, \\ x_2^{2^{k+4}} &= [x_1, x_3]^{2^k} [x_1, x_4]^{-2^{k-1}} [x_2, x_3]^{3 \cdot 2^{k+3}}, \\ x_3^{2^{k+2}} &= [x_1, x_2]^{2^k} [x_1, x_4]^{2^{k-2}} [x_3, x_2]^{5 \cdot 2^{k+1}}, \\ x_4^{2^k} &= [x_1, x_2]^{2^{k-1}} [x_2, x_3]^{2^k} [x_1, x_3]^{2^{k-2}}, \\ [x_3, x_2]^4 &= [x_1, x_2, x_2], [x_2, x_3]^{16} = [x_1, x_3, x_3], [x_3, x_2]^{64} = [x_1, x_4, x_4], \\ [x_2, x_3, x_1] &= [x_2, x_3, x_2] = [x_2, x_3, x_3] = [x_2, x_3, x_4] = 1, \\ [x_1, x_2, x_1] &= [x_1, x_2, x_3] = [x_1, x_2, x_4] = 1, \\ [x_1, x_3, x_1] &= [x_1, x_3, x_2] = [x_1, x_3, x_4] = 1, \\ [x_1, x_4, x_1] &= [x_1, x_4, x_2] = [x_1, x_4, x_3] = 1. \end{aligned}$$

Then

$$w = [x_2, x_3]^{2^{k+5}} \in D_4(G_{k,l}) \setminus \gamma_4(G_{k,l}).$$

The case $k = 2$, $l = 0$ is exactly the example due to Rips.

We continue the above constructions of groups without dimension property by constructing a 4-generator and 3-relator example of a group G with $D_4(G) \neq \gamma_4(G)$ and, for each $n \geq 5$, a 5-generator 5-relator example of a group G with $D_n(G) \neq \gamma_n(G)$. Our motivation for constructing these examples is to develop a closer understanding of groups without dimension property and also to look for simpler, and in a sense minimal, examples of such groups.

Example 2.3 Let G be the group defined by the presentation

$$\langle x_1, x_2, x_3, x_4 \mid x_1^4[x_4, x_3]^2[x_4, x_2] = 1, \\ x_2^{16}[x_4, x_3]^4[x_4, x_1]^{-1} = 1, x_3^{64}[x_4, x_2]^{-4}[x_4, x_1]^{-2} = 1 \rangle. \quad (2.1)$$

Then $w = [x_1, x_2^{32}][x_1, x_3^{64}][x_2, x_3^{128}] \in D_4(G) \setminus \gamma_4(G)$.

To prove the above statement, we need the following lemma:

Lemma 2.4 Let Π be a group. If $x_1, x_2, x_3 \in \Pi$ and there exist $\xi_j \in \gamma_2(\Pi)$, $j = 1, \dots, 6$ and $\eta_i \in \gamma_3(\Pi)$, such that

$$x_1^4 = \xi_1, x_2^{16} = \xi_2, x_2^{32}x_3^{64} = \xi_4\eta_1, x_1^{-32}x_3^{128} = \xi_5^{16}\eta_2, x_1^{-64}x_2^{-128} = \xi_6^{64}\eta_3$$

then

$$w = [x_1, x_2^{32}][x_1, x_3^{64}][x_2, x_3^{128}] \in D_4(\Pi).$$

Proof. Since $\gamma_2(\Pi) \subseteq 1 + \Delta^2(\Pi)$, we have

$$1 - w \equiv \alpha_1 + \alpha_2 + \alpha_3 \pmod{\Delta^4(\Pi)},$$

where $\alpha_1 = (1 - [x_1, x_2^{32}])$, $\alpha_2 = (1 - [x_1, x_3^{64}])$, $\alpha_3 = (1 - [x_2, x_3^{128}])$. Now, working modulo $\Delta^4(\Pi)$, we have

$$\begin{aligned} \alpha_1 &\equiv (1 - x_2^{32})(1 - x_1) - (1 - x_1)(1 - x_2^{32}) \\ &\equiv (1 - x_2^{32})(1 - x_1) - 32(1 - x_1)(1 - x_2) + \binom{32}{2}(1 - x_1)(1 - x_2)^2 \\ &\equiv (1 - x_2^{32})(1 - x_1) - (1 - x_1^{32})(1 - x_2) \quad (\text{since } x_1^4 \in \gamma_2(\Pi)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \alpha_2 &\equiv (1 - x_3^{64})(1 - x_1) - (1 - x_1^{64})(1 - x_3), \\ \alpha_3 &\equiv (1 - x_3^{128})(1 - x_2) - (1 - x_2^{128})(1 - x_3). \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &\equiv (2 - x_2^{32} - x_3^{64})(1 - x_1) + (x_1^{32} - x_3^{128})(1 - x_2) + \\ &(x_1^{64} + x_2^{128} - 2)(1 - x_3) \equiv (1 - \xi_4\eta_1)(1 - x_1) + (1 - \xi_5^{16}\eta_2)(1 - x_2) + (1 - \xi_6^{64}\eta_3)(1 - x_3) \equiv \\ &(1 - \eta_1)(1 - x_1) + (1 - \eta_2)(1 - x_2) + (1 - \eta_3)(1 - x_3) + \\ &(1 - \xi_4)(1 - \xi_1) + (1 - \xi_5)(1 - \xi_2) + (1 - \xi_6)(1 - \xi_2^{-2}\xi_4^4) \equiv 0, \end{aligned}$$

and hence $w \in D_4(\Pi)$. \square

Proof of Example 2.3. Modulo $\gamma_4(G)$, we have

$$x_2^{32}x_3^{64} = [x_4, x_1]^2[x_3, x_4]^8[x_4, x_1]^2[x_4, x_2]^4 = [x_3, x_4]^8[x_4, x_1]^4[x_4, x_2]^4 = \xi_4^4\eta_1,$$

with $\xi_4 = [x_3, x_4]^2[x_4, x_2] \in \gamma_2(G)$, $\eta_1 = [x_4, x_1]^4 \in \gamma_3(G)$;

$$\begin{aligned} x_1^{-32}x_3^{128} &= [x_4, x_3]^{16}[x_4, x_2]^8[x_4, x_1]^4[x_4, x_2]^8 = \\ &= [x_4, x_3]^{16}[x_4, x_2]^{16}[x_4, x_1]^4 = \xi_5^{16}\eta_2, \end{aligned}$$

with $\xi_5 = [x_4, x_3] \in \gamma_2(G)$, $\eta_2 = [x_4, x_2]^{16}[x_4, x_1]^4 \in \gamma_3(G)$;

$$\begin{aligned} x_1^{-64}x_2^{-128} &= [x_4, x_2]^{16}[x_4, x_3]^{32}[x_4, x_1]^{-8}[x_4, x_3]^{32} = \\ &= [x_4, x_2]^{16}[x_4, x_3]^{64}[x_4, x_1]^{-8} = \eta_3 \in \gamma_3(G). \end{aligned}$$

By Lemma 2.4, $w \in D_4(G)$. It remains to show that $w \notin \gamma_4(G)$. We shall construct a nilpotent group H of class 3, which is an epimorphic image of G with nontrivial image of w .

The construction of H is a slight simplification of the construction of Passi and Gupta (see [Gup87c], Example 2.1, p. 76). Let F be a free group with generators $\langle x_1, x_2, x_3, x_4 \rangle$. Define R_1 to be the fourth term of the lower central series of F , i.e., $R_1 = \gamma_4(F)$. Define

$$R_2 = \langle R_1, [x_i, x_j, x_k] \notin \langle \alpha, \beta, \gamma \rangle R_1 \text{ for all } i, j, k, \alpha^4\beta^{-1}, \beta^4\gamma^{-1}, \gamma^4 \rangle,$$

where $\alpha = [x_4, x_3, x_3]$, $\beta = [x_4, x_2, x_2]$, $\gamma = [x_4, x_1, x_1]$;

$$\begin{aligned} R_3 = \langle R_2, [x_4, x_3]^{64}\alpha^{32}, [x_4, x_2]^{16}\beta^8, [x_4, x_1]^4\gamma^2, \\ [x_3, x_2]^{16}\beta^{-1}, [x_3, x_1]^4\alpha^{-2}, [x_2, x_1]^4\beta^{-1} \rangle, \end{aligned}$$

$$R_4 = \langle R_3, c_1, c_2, c_3 \rangle,$$

where

$$\begin{aligned} c_1 &= x_1^4[x_4, x_3]^2[x_4, x_2], \\ c_2 &= x_2^{16}[x_4, x_3]^4[x_4, x_1]^{-1}, \\ c_3 &= x_3^{64}[x_4, x_2]^{-4}[x_4, x_1]^{-2}. \end{aligned}$$

We set $H = F/R_4$.

Clearly, the group H is a natural epimorphic image of G . Hence it remains to show that the element

$$w_0 = [x_1, x_2^{32}][x_1, x_3^{64}][x_2, x_3^{128}]$$

is nontrivial in H .

We claim that $[R_{i+1}, F] \subseteq R_i$, $i = 1, 2, 3$. This is obvious for $i = 1, 2$. We show it for $i = 3$. Working modulo R_3 , we have:

$$\begin{aligned}
[c_1, x_1] &= 1, \\
[c_1, x_2] &= [x_1, x_2]^4 [x_4, x_2, x_2] = [x_1, x_2]^4 \beta = 1, \\
[c_1, x_3] &= [x_1, x_3]^4 [x_4, x_3, x_3]^2 = [x_1, x_3]^4 \alpha^2 = 1, \\
[c_1, x_4] &= [x_1, x_4]^4 [x_1, x_4, x_1]^2 = [x_1, x_4]^4 \gamma^{-2} = 1, \\
[c_2, x_1] &= [x_2, x_1]^{16} [x_4, x_1, x_1]^{-1} = \beta^4 \gamma^{-1} = 1, \\
[c_2, x_2] &= 1, \\
[c_2, x_3] &= [x_2, x_3]^{16} [x_4, x_3, x_3]^4 = \beta^{-1} \alpha^4 = 1, \\
[c_2, x_4] &= [x_2, x_4]^{16} [x_2, x_4, x_2]^8 = [x_2, x_4]^{16} \beta^{-8} = 1, \\
[c_3, x_1] &= [x_3, x_1]^{64} [x_4, x_1, x_1]^{-2} = \alpha^{32} \gamma^{-2} = 1, \\
[c_3, x_2] &= [x_3, x_2]^{64} [x_4, x_2, x_2]^{-4} = [x_3, x_2]^{64} \beta^{-4} = 1, \\
[c_3, x_3] &= 1, \\
[c_3, x_4] &= [x_3, x_4]^{64} [x_3, x_4, x_3]^{32} = [x_3, x_4]^{64} \alpha^{-32} = 1.
\end{aligned}$$

Clearly, R_2/R_1 is cyclic of order 64, generated by the element α . We claim that the element α has order exactly 64 in H . Suppose $\alpha^s \in R_4$, $s > 0$ and s is not divisible by 64. Then R_4/R_2 has a torsion element α^s , since $\alpha^{64} \in R_2$. We have the following group extension:

$$1 \rightarrow R_3/R_2 \rightarrow R_4/R_2 \rightarrow R_4/R_3 \rightarrow 1.$$

Hence at least one of two groups: R_3/R_2 or R_4/R_3 has a torsion. Since $[R_4, F] \subseteq R_3$, every element of R_4/R_3 can be written as $c_1^{h_1} c_2^{h_2} c_3^{h_3}$ for some integers h_1, h_2, h_3 . Clearly it is a free abelian group of rank 3, since $R_4/R_4 \cap \gamma_2(F)$ is free abelian, which is an epimorphic image of R_4/R_3 . The same argument works for the quotient R_3/R_2 , since all commutators which we added to R_2 to get R_3 are of the form $[x_i, x_j]^{h_{ij}} q_{ij}$, $q_{ij} \in \gamma_3(F)$, $h_{ij} \in \mathbb{Z}$, but commutators $[x_i, x_j]$ are basic commutators in F , i.e., they are linearly independent modulo $\gamma_3(F)$. Hence both R_4/R_3 and R_3/R_2 are free abelian and the element α has order exactly 64 in H .

Finally, note that $w_0 = \alpha^{32}$ is nontrivial in H ; hence the element w does not lie in $\gamma_4(G)$. \square

Example 2.5 Let G be the group defined by the presentation

$$\langle x_1, x_2, x_3, x_4 \mid x_1^4 = \xi_1, x_2^{16} = \xi_2, x_2^{32} x_3^{64} = \xi_4^4, x_1^{-32} x_3^{128} = \xi_5^{16}, \xi_1^{16} \xi_2^8 = 1 \rangle, \quad (2.2)$$

where

$$\begin{aligned}
\xi_1 &= [x_2, x_4][x_3, x_4]^2, \quad \xi_2 = [x_4, x_1][x_3, x_4]^4, \\
\xi_4 &= [x_3, x_4]^2 [x_4, x_2][x_4, x_1], \quad \xi_5 = [x_4, x_3].
\end{aligned}$$

Then $w = [x_1, x_2^{32}][x_1, x_3^{64}][x_2, x_3^{128}] \in D_4(G) \setminus \gamma_4(G)$.

Proof. By Lemma 2.4, $w \in D_4(G)$. Note that the group H occurring in the proof of Example 2.3 is a natural epimorphic image of G . Indeed, the first two relations of G are also among the defining relations of H (due to relators c_1, c_2), and therefore we only need to check the other three. In H ,

$$x_2^{32}x_3^{64} = [x_4, x_1]^2[x_3, x_4]^8[x_4, x_1]^2[x_4, x_2]^4 = [x_3, x_4]^8[x_4, x_1]^4[x_4, x_2]^4 = \xi_4^4;$$

$$\begin{aligned} x_1^{-32}x_3^{128} &= [x_4, x_3]^{16}[x_4, x_2]^8[x_4, x_1]^4[x_4, x_2]^8 = \\ &= [x_4, x_3]^{16}[x_4, x_2, x_2]^{-8}[x_4, x_1, x_1]^{-2} = [x_4, x_3]^{16}[x_4, x_1, x_1]^{-4} = \\ &= [x_4, x_3]^{16} = \xi_5^{16}; \end{aligned}$$

$$\begin{aligned} \xi_1^{16}\xi_2^8 &= [x_2, x_4]^{16}[x_3, x_4]^{32}[x_4, x_1]^8[x_3, x_4]^{32} = \\ &= [x_4, x_2, x_2]^8[x_4, x_1, x_1]^4[x_4, x_3, x_3]^{32} = [x_4, x_3, x_3]^{64} = 1. \end{aligned}$$

Thus we have an epimorphism $\theta : G \rightarrow H$, $x_i \mapsto x_i, 1 \leq i \leq 4$. It is shown in the proof of Example 2.3, that $w_0 = \theta(w)$ is nontrivial in H , which is nilpotent of class 3. Hence $w \notin \gamma_4(G)$. \square

Example 2.6 *Let*

$$\begin{aligned} \Pi &= \langle x_1, x_2, x_3, y_1, \dots, y_{26} \mid x_1^4 = \prod_{i=0}^2 [y_{2i+1}, y_{2i+2}], x_2^{16} = \prod_{i=3}^7 [y_{2i+1}, y_{2i+2}], \\ x_2^{32}x_3^{64} &= \left(\prod_{i=8}^{11} [y_{2i+1}, y_{2i+2}] \right)^4, x_1^{-32}x_3^{128} = [y_{25}, y_{26}]^{16}, x_1^{-64}x_2^{-128} = 1 \rangle. \quad (2.3) \end{aligned}$$

Then $w = [x_1, x_2^{32}][x_1, x_3^{64}][x_2, x_3^{128}] \in D_4(\Pi) \setminus \gamma_4(\Pi)$.

Proof. By Lemma 2.4, $w \in D_4(\Pi)$. Consider the group $\Pi' = \Pi * Z$, where Z is an infinite cyclic group with generator x , say. It is easy to see that there exists an epimorphism $\theta : \Pi' \rightarrow G$, where G is the group considered in Example 2.5 and θ maps $x_i \mapsto x_i, i = 1, 2, 3, x \mapsto x_4$. Clearly, for such an epimorphism θ , $\theta(w) \notin \gamma_4(G)$ by Example 2.5, and therefore $w \notin \gamma_4(\Pi)$. \square

Example 2.7 *For $n \geq 5$, let*

$$\begin{aligned} G(n) &= \langle x_1, x_2, x_3, y_1, \dots, y_{10n} \mid x_1^4 = \xi_{1,(n)}, x_2^{16} = \xi_{2,(n)}, \\ x_2^{32}x_3^{64} &= \xi_{3,(n)}^4, x_1^{-32}x_3^{128} = \xi_{4,(n)}^{16}, x_1^{-64}x_2^{-128} = \xi_{5,(n)}^{64} \rangle, \quad (2.4) \end{aligned}$$

where

$$\xi_{i,(n)} = [y_{(2i-2)n+1}, y_{(2i-2)n+2}] \cdots [y_{2in-1}, y_{2in}], \quad 1 \leq i \leq 5.$$

Then $w = [x_1, x_2^{32}][x_1, x_3^{64}][x_2, x_3^{128}] \in D_4(G(n)) \setminus \gamma_4(G(n))$.

Proof. Observe that there exists an epimorphism $G(n) \rightarrow \Pi$, Π being the group considered in Example 2.6, which maps $x_i \mapsto x_i, i = 1, 2, 3$. The assertion thus follows from Lemma 2.4. \square

The same principle can be used to construct more examples of groups without dimension property. The following example is a base for a later construction in Theorem 2.14.

Example 2.8 Let $k \geq 9$, and G the group given by the following presentation:

$$\langle x_1, x_2, x_3, x_4 \mid x_1^8[x_4, x_3]^4[x_4, x_2], \\ x_2^{64}[x_4, x_3]^{-16}[x_4, x_1]^{-1}, x_3^{2k}[x_4, x_2]^{16}[x_4, x_1]^{-4} \rangle. \quad (2.5)$$

Then $[x_1, x_2^{256}][x_1, x_3^{2k}][x_2, x_3^{k+1}] \in D_4(G) \setminus \gamma_4(G)$.

Example 2.9 Let

$$r \geq t \geq 2, \quad k \geq q + r, \\ s \geq l + 3, \quad q \geq s + r + 2$$

and G the group with generators x_1, x_2, x_3, x_4, x_5 and relators

$$x_1^{2^l} = [x_4, x_2]^{-2^{l-r-2}}[x_4, x_3]^{2^{l-1}}[x_4, x_5]^{-2^{l-t}}; \\ x_2^{2^s} = [x_4, x_1]^{2^{l-r-2}}[x_4, x_3]^{-2^{s-3}}[x_4, x_5]^{2^{s-t-1}}; \\ x_3^{2^k} = [x_4, x_1]^{-2^{l-1}}[x_4, x_2]^{2^{s-3}}[x_4, x_5]^{2^{q-5}}; \\ x_5^{2^q} = [x_4, x_1]^{2^{l-t}}[x_4, x_2]^{-2^{s-t-1}}[x_4, x_3]^{-2^{q-5}}.$$

Then, for

$$w = [x_1, x_2^{2^{s+r}}][x_1, x_3^{2^k}][x_1, x_5^{2^{q+t-2}}][x_2, x_3^{2^{k+1}}][x_2, x_5^{2^{q+t}}][x_5, x_3^{2^{k+3}}],$$

we have

$$w \in D_4(G) \setminus \gamma_4(G).$$

Proof. Since $x_1^{2^l}, x_2^{2^s}, x_3^{2^k}, x_5^{2^q} \in \gamma_3(G)$, we have

$$1 - w \equiv \sum_{i=1}^6 (1 - \alpha_i) \pmod{\mathfrak{g}^6},$$

where

$$\begin{aligned}\alpha_1 &= [x_1, x_2^{2^{s+r}}], \quad \alpha_2 = [x_1, x_3^{2^k}], \quad \alpha_3 = [x_1, x_5^{2^{q+t-2}}], \\ \alpha_4 &= [x_2, x_3^{2^{k+1}}], \quad \alpha_5 = [x_2, x_5^{2^{q+t}}], \quad \alpha_6 = [x_5, x_3^{2^{k+3}}].\end{aligned}$$

Clearly, we have

$$\begin{aligned}1 - \alpha_1 &\equiv (1 - x_2^{2^{s+r}})(1 - x_1) - (1 - x_1)(1 - x_2^{2^{s+r}}) \\ &\equiv (1 - x_2^{2^{s+r}})(1 - x_1) - 2^{s+r}(1 - x_1)(1 - x_2) + \binom{2^{s+r}}{2}(1 - x_1)(1 - x_2)^2 \\ &\equiv (1 - x_2^{2^{s+r}})(1 - x_1) - (1 - x_1^{2^{s+r}})(1 - x_2) \pmod{\mathfrak{g}^4}; \\ 1 - \alpha_2 &\equiv (1 - x_3^{2^k})(1 - x_1) - (1 - x_1^{2^k})(1 - x_3) \pmod{\mathfrak{g}^4}; \\ 1 - \alpha_3 &\equiv (1 - x_5^{2^{q+t-2}})(1 - x_1) - (1 - x_1^{2^{q+t-2}})(1 - x_5) \pmod{\mathfrak{g}^4}; \\ 1 - \alpha_4 &\equiv (1 - x_3^{2^{k+1}})(1 - x_2) - (1 - x_2^{2^{k+1}})(1 - x_3) \pmod{\mathfrak{g}^4}; \\ 1 - \alpha_5 &\equiv (1 - x_5^{2^{q+t}})(1 - x_2) - (1 - x_2^{2^{q+t}})(1 - x_5) \pmod{\mathfrak{g}^4}; \\ 1 - \alpha_6 &\equiv (1 - x_3^{2^{k+3}})(1 - x_5) - (1 - x_5^{2^{k+3}})(1 - x_3) \pmod{\mathfrak{g}^4}.\end{aligned}$$

Hence

$$\begin{aligned}1 - w &\equiv (1 - x_2^{2^{s+r}} x_3^{2^k} x_5^{2^{q+t-2}})(1 - x_1) + (1 - x_1^{-2^{s+r}} x_3^{2^{k+1}} x_5^{2^{q+t}})(1 - x_2) + \\ &\quad (1 - x_1^{-2^k} x_2^{-2^{k+1}} x_5^{-2^{k+3}})(1 - x_3) + (1 - x_1^{-2^{q+t-2}} x_2^{-2^{q+t}} x_3^{2^{k+3}})(1 - x_5) \pmod{\mathfrak{g}^4}\end{aligned}$$

In the group G , we have:

$$\begin{aligned}x_2^{2^{s+r}} x_3^{2^k} x_5^{2^{q+t-2}} &= [x_4, x_1]^{2^{l-2}} [x_4, x_3]^{-2^{s+r-3}} [x_4, x_5]^{2^{s-t-1+r}} \\ &\quad [x_4, x_1]^{-2^{l-1}} [x_4, x_2]^{2^{s-3}} [x_4, x_5]^{2^{q-5}} [x_4, x_1]^{2^{l-2}} [x_4, x_2]^{-2^{s-3}} [x_4, x_3]^{-2^{q+t-7}} = \\ &\quad [x_4, x_2]^{2^{s-t-1+r}} [x_4, x_2]^{-2^{s-3}} [x_4, x_3]^{-2^{s+r-3}} [x_4, x_3]^{-2^{q+t-7}} [x_4, x_5]^{2^{q-5}} [x_4, x_5]^{2^{s-t-1+r}}.\end{aligned}$$

$$\begin{aligned}x_1^{-2^{s+r}} x_3^{2^{k+1}} x_5^{2^{q+t}} &= [x_4, x_2]^{2^{s-2}} [x_4, x_3]^{-2^{s+r-1}} [x_4, x_5]^{2^{s+r-t}} \\ &\quad [x_4, x_1]^{-2^l} [x_4, x_2]^{2^{s-2}} [x_4, x_5]^{2^{q-4}} [x_4, x_1]^{2^l} [x_4, x_2]^{-2^{s-1}} [x_4, x_3]^{-2^{q-5+t}} = \\ &\quad [x_4, x_3]^{-2^{s+r-1}} [x_4, x_3]^{-2^{q+t-5}} [x_4, x_5]^{2^{s+r-t}} [x_4, x_5]^{2^{q-4}}.\end{aligned}$$

$$\begin{aligned}x_1^{-2^k} x_2^{-2^{k+1}} x_5^{-2^{k+3}} &= [x_4, x_2]^{2^{k-r-2}} [x_4, x_3]^{-2^{k-1}} [x_4, x_5]^{2^{k-t}} [x_4, x_1]^{-2^{l-r-1+k-s}} \\ &\quad [x_4, x_3]^{2^{k-2}} [x_4, x_5]^{-2^{k-t}} [x_4, x_1]^{-2^{l-t+k-q+3}} [x_4, x_2]^{2^{s-t+k-q+2}} [x_4, x_3]^{2^{k-2}} = \\ &\quad [x_4, x_1]^{-2^{l-r+k-s}} [x_4, x_1]^{-2^{l-t+k-q+3}} [x_4, x_2]^{2^{k-r-2}} [x_4, x_2]^{2^{s-t+k-q+2}}.\end{aligned}$$

$$\begin{aligned}
x_1^{-2^{q+t-2}} x_2^{-2^{q+t}} x_3^{2^{k+3}} &= [x_4, x_2]^{2^{q+t-r-4}} [x_4, x_3]^{-2^{q+t-3}} [x_4, x_5]^{2^{q-2}} \\
[x_4, x_1]^{-2^{l-r-2+q+t-s}} [x_4, x_3]^{2^{q+t-3}} [x_4, x_5]^{-2^{q-1}} [x_4, x_1]^{2^{l+2}} [x_4, x_2]^{2^s} [x_4, x_5]^{2^{q-2}} &= \\
[x_4, x_2]^{2^{q+t-r-4}} [x_4, x_2]^{2^s} [x_4, x_1]^{q+l+t-s-r-2} [x_4, x_1]^{2^{l+2}}. &
\end{aligned}$$

Hence,

$$\begin{aligned}
1 - w &\equiv (1 - \eta_1^{2^l})(1 - x_1) + (1 - \eta_2^{2^s})(1 - x_2) + \\
&\quad (1 - \eta_3)(1 - x_3) + (1 - \eta_5)(1 - x_5) \pmod{\mathfrak{g}^4},
\end{aligned}$$

where

$$\begin{aligned}
\eta_1 &= [x_4, x_2]^{2^{s-t-1+r-l}} [x_4, x_2]^{-2^{s-3-l}} [x_4, x_3]^{-2^{s+r-l-3}} [x_4, x_3]^{-2^{q+t-l-7}} \\
&\quad [x_4, x_5]^{2^{q-l-5}} [x_4, x_5]^{2^{s-t-1+r-l}}, \\
\eta_2 &= [x_4, x_3]^{-2^{r-1}} [x_4, x_3]^{-2^{q+t-s-5}} [x_4, x_5]^{2^{r-t}} [x_4, x_5]^{2^{q-s-4}}, \\
\eta_3 &= [x_4, x_1]^{-2^{l-r+k-s}} [x_4, x_1]^{-2^{l-t+k-q+3}} [x_4, x_2]^{2^{k-r-2}} [x_4, x_2]^{2^{s-t+k-q+2}} \in \gamma_3(G), \\
\eta_4 &= [x_4, x_2]^{2^{q+t-r-4}} [x_4, x_2]^{2^s} [x_4, x_1]^{q+l+t-s-r-2} [x_4, x_1]^{2^{l+2}} \in \gamma_3(G).
\end{aligned}$$

Therefore,

$$1 - w \equiv (1 - \eta_1)(1 - x_1^{2^l}) + (1 - \eta_2)(1 - x_2^{2^s}) \equiv 0 \pmod{\mathfrak{g}^4}.$$

The proof that $w \notin \gamma_4(G)$ is by the same principle as that in the proof of Example 2.3.

Let F be a free group with generators x_1, x_2, x_3, x_4, x_5 . Define R_1 to be the fourth term of the lower central series of F , i.e., $R_1 = \gamma_4(F)$. Define

$$\begin{aligned}
R_2 &= \langle R_1, [x_i, x_j, x_k] \notin \langle \alpha, \beta, \gamma, \delta \rangle R_1 \text{ for all } i, j, k, \\
&\quad \alpha^{2^{k-q}} \beta^{-1}, \beta^{2^{q-s}} \gamma^{-1}, \gamma^{2^{s-l}} \delta^{-1}, \delta^{2^l} \rangle,
\end{aligned}$$

where $\alpha = [x_4, x_3, x_3], \beta = [x_4, x_5, x_5], \gamma = [x_4, x_2, x_2], \delta = [x_4, x_1, x_1]$;

$$\begin{aligned}
R_3 &= \langle R_2, [x_4, x_3]^{2^k} \alpha^{2^{k-1}}, [x_4, x_5]^{2^q} \beta^{2^q}, [x_4, x_2]^{2^s} \gamma^{2^{s-1}}, \\
&\quad [x_4, x_1]^{2^l} \delta^{2^{l-1}}, [x_3, x_5]^{-2^q} \alpha^{2^{q-5}}, [x_3, x_2]^{-2^s} \alpha^{2^{s-3}}, \\
&\quad [x_3, x_1]^{2^l} \alpha^{2^{l-1}}, [x_5, x_2]^{2^s} \beta^{2^{s-t-1}}, [x_5, x_1]^{-2^l} \beta^{-2^{l-t}}, [x_2, x_1]^{-2^l} \gamma^{2^{l-r-2}} \rangle,
\end{aligned}$$

$$R_4 = \langle R_3, c_1, c_2, c_3, c_4 \rangle,$$

where

$$\begin{aligned} c_1 &= x_1^{-2^l} [x_4, x_2]^{-2^{l-r-2}} [x_4, x_3]^{2^{l-1}} [x_4, x_5]^{-2^{l-t}}; \\ c_2 &= x_2^{-2^s} [x_4, x_1]^{2^{l-r-2}} [x_4, x_3]^{-2^{s-3}} [x_4, x_5]^{2^{s-t-1}}; \\ c_3 &= x_3^{-2^k} [x_4, x_1]^{-2^{l-1}} [x_4, x_2]^{2^{s-3}} [x_4, x_5]^{2^{q-5}}; \\ c_4 &= x_5^{-2^q} [x_4, x_1]^{2^{l-t}} [x_4, x_2]^{-2^{s-t-1}} [x_4, x_3]^{-2^{q-5}}. \end{aligned}$$

We set $H = F/R_4$.

Clearly, the group H is a natural epimorphic image of G . Hence it remains to show that the element

$$w_0 = [x_1, x_2^{2^{s+r}}][x_1, x_3^{2^k}][x_1, x_5^{2^{q+t-2}}][x_2, x_3^{2^{k+1}}][x_2, x_5^{2^{q+t}}][x_5, x_3^{2^{k+3}}]$$

is nontrivial in H . We claim that $[R_{i+1}, F] \subseteq R_i$, $i = 1, 2, 3$. The proof is straightforward. In analogy with Example 2.3, one can show that $\gamma_3(G)$ is cyclic of order 2^k with generator α , but $w_0 = \alpha^{2^{k-1}}$. Therefore, $w_0 \neq 1$ and $w \notin \gamma_4(G)$. \square

We next discuss examples of groups without dimension property in arbitrary dimension. First examples of groups without dimension property in higher dimensions were constructed by N. Gupta [Gup90].

Example 2.10 (Gupta [Gup90], [Gup91a]). *Let $n \geq 4$ be fixed and let F be a free group of rank 4 with basis $\{r, a, b, c\}$. Set $x_0 = y_0 = z_0 = r$, and define commutators $x_i = [x_{i-1}, a]$, $y_i = [y_{i-1}, b]$, $z_i = [z_{i-1}, c]$, $i = 1, 2, \dots$. Let \mathfrak{G}_n be the quotient of F with the following defining relations:*

$$(i) \ r^{2^{2n-1}} = 1, \ a^{2^{n+2}} = y_{n-3}^4 z_{n-3}^2, \ b^{2^n} = x_{n-3}^{-4} z_{n-3}, \ c^{2^{n-2}} = x_{n-3}^{-2} y_{n-3}^{-1};$$

$$(ii) \ z_{n-2} = y_{n-2}^4, \ y_{n-2} = x_{n-2}^4;$$

$$(iii) \ x_{n-1} = 1, \ y_{n-1} = 1, \ z_{n-1} = 1;$$

$$(iv) \ [a, b, g] = 1, \ [b, c, g] = 1, \ [a, c, g] = 1, \ \text{for all } g \in F;$$

$$(v) \ [x_i, b] = 1, \ [x_i, c] = 1, \ [y_i, a] = 1, \ [y_i, c] = 1, \ [z_i, a] = 1, \ [z_i, b] = 1, \\ i \geq 1;$$

$$(vi) \ [x_i, x_j] = 1, \ [x_i, y_j] = 1, \ [x_i, z_j] = 1, \ [y_i, y_j] = 1, \ [y_i, z_j] = 1, \ [z_i, z_j] = 1, \\ i, j \geq 0.$$

Let

$$g = [a, b]^{2^{2n-1}} [a, c]^{2^{2n-2}} [b, c]^{2^{2n-3}}.$$

Then $g \in D_n(\mathfrak{G}_n) \setminus \gamma_n(\mathfrak{G}_n)$.

Example 2.11 For every integer $n \geq 0$, there exist integers $k > l$, such that for the group \mathfrak{G}_n defined by the presentation

$$\langle x_1, x_2, x_3, x_4, x_5 \mid x_1^4 \xi_1 = 1, x_2^{2^l} \xi_2 = 1, x_3^{2^k} \xi_3 = 1, \\ [[x_5, nx_4], x_1]^4 [[x_5, nx_4], x_3, x_3]^{2^{k-1}} = 1, \xi_1^{2^{k-2}} \xi_2^{2^{k-l+1}} = 1 \rangle,$$

where

$$\xi_1 = [[x_5, nx_4], x_3]^2 [[x_5, nx_4], x_2] [x_5, x_{n+1}x_4]^2, \\ \xi_2 = [[x_5, nx_4], x_3]^{2^{l-2}} [[x_5, nx_4], x_1]^{-1} [x_5, x_{n+1}x_4]^2, \\ \xi_3 = [[x_5, nx_4], x_2]^{-2^{l-2}} [[x_5, nx_4], x_1]^{-2},$$

the element $w_n = [x_1, x_2^{2^{l+1}}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}] \in D_{4+n}(\mathfrak{G}_n) \setminus \gamma_{4+n}(\mathfrak{G}_n)$.

To prove the above assertion we need some technical lemmas. The following lemma is a generalization of Lemma 2.4.

Lemma 2.12 Let Π be a group and $n \geq 4$ an integer. If $x_1, x_2, x_3 \in \Pi$ are such that there exist $\xi_i \in \gamma_{n-2}(\Pi)$, $i = 1, \dots, 4$, satisfying

$$x_1^4 = \xi_1, x_2^{2^l} = \xi_2, x_2^{2^{l+1}} x_3^{2^k} = \xi_3^4, x_1^{-2^{l+1}} x_3^{2^{k+1}} = \xi_4^2, x_1^{2^k} x_2^{2^{k+1}} = 1$$

then

$$w = [x_1, x_2^{2^{l+1}}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}] \in D_n(\Pi),$$

provided k, l are sufficiently large integers.

Proof. Since $1 - x \in \Delta^{n-2}(\Pi)$ for $x \in \gamma_{n-2}(\Pi)$, we have

$$1 - w \equiv \alpha_1 + \alpha_2 + \alpha_3 \pmod{\Delta^n(\Pi)},$$

where $\alpha_1 = (1 - [x_1, x_2^{2^{l+1}}])$, $\alpha_2 = (1 - [x_1, x_3^{2^k}])$, $\alpha_3 = (1 - [x_2, x_3^{2^{k+1}}])$. Now, working modulo $\Delta^n(\Pi)$, we have

$$\alpha_1 \equiv (1 - x_2^{2^{l+1}})(1 - x_1) - (1 - x_1)(1 - x_2^{2^{l+1}}) \\ \equiv (1 - x_2^{2^{l+1}})(1 - x_1) - 2^{l+1}(1 - x_1)(1 - x_2) + \\ \sum_{i=2}^{n-1} (-1)^i \binom{2^{l+1}}{i} (1 - x_1)(1 - x_2)^i.$$

Note that, for sufficiently large l and $i \leq n$, the integer $\binom{2^{l+1}}{i}$ is divisible by 4^n . Hence, for such an integer l , we have

$$\sum_{i=2}^{n-1} (-1)^i \binom{2^{l+1}}{i} (1-x_1)(1-x_2)^i \in \Delta^n(\Pi),$$

and

$$2^{l+1}(1-x_1)(1-x_2) \equiv (1-x_1^{2^{l+1}})(1-x_2) \pmod{\Delta^n(\Pi)}.$$

Therefore,

$$\alpha_1 \equiv (1-x_2^{2^{l+1}})(1-x_1) - (1-x_1^{2^{l+1}})(1-x_2) \pmod{\Delta^n(\Pi)}.$$

Assuming k to be large enough so that $\binom{2^{k+1}}{i}$ is divisible by 2^{ln} for $i \leq n$, we have

$$\begin{aligned} \alpha_2 &\equiv (1-x_3^{2^k})(1-x_1) - (1-x_1^{2^k})(1-x_3), \\ \alpha_3 &\equiv (1-x_3^{2^{k+1}})(1-x_2) - (1-x_2^{2^{k+1}})(1-x_3). \end{aligned}$$

Therefore, $\pmod{\Delta^n(\Pi)}$,

$$\begin{aligned} &\alpha_1 + \alpha_2 + \alpha_3 \\ &\equiv (2-x_2^{2^{l+1}}-x_3^{2^k})(1-x_1) + (x_1^{2^{l+1}}-x_3^{2^{k+1}})(1-x_2) + \\ &\quad (x_1^{2^k}+x_2^{2^{k+1}}-2)(1-x_3) \\ &\equiv (1-\xi_3^4)(1-x_1) + (1-\xi_4^{2^l})(1-x_2) + (1-x_1^{2^k}x_2^{2^{k+1}})(1-x_3) \\ &\equiv (1-\xi_3)(1-\xi_1) + (1-\xi_4)(1-\xi_2) \\ &\equiv 0, \end{aligned}$$

and hence $w \in D_n(\Pi)$. \square

Lemma 2.13 *Let $k \geq l+2$, $l \geq 4$, be integers and G the group defined by the presentation*

$$\begin{aligned} \langle x_1, x_2, x_3, x_4, x_5 \mid &x_1^4 \xi_1 = 1, x_2^{2^l} \xi_2 = 1, x_3^{2^k} \xi_3 = 1 \\ &[x_4, x_1]^4 [x_4, x_3, x_3]^{2^{k-1}} = 1, \xi_1^{2^{k-2}} \xi_2^{2^{k-l+1}} = 1, \\ &[x_4, x_i, x_4] = 1, i = 1, \dots, 4 \rangle, \end{aligned}$$

where

$$\begin{aligned} \xi_1 &= [x_4, x_3]^2 [x_4, x_2] [x_4, x_5]^2, \\ \xi_2 &= [x_4, x_3]^{2^{l-2}} [x_4, x_1]^{-1} [x_4, x_5]^2, \\ \xi_3 &= [x_4, x_2]^{-2^{l-2}} [x_4, x_1]^{-2}. \end{aligned}$$

Then the element $w = [x_1, x_2^{2^{l+1}}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}]$ does not lie in $\gamma_4(G)$.

Proof. We shall construct a nilpotent group of class 3 which is an epimorphic image of the given group G and is such that the image of the element w is nontrivial.

Let F be a free group with basis $\{x_1, \dots, x_5\}$. Consider the following four types of relations:

$$R_1 = \gamma_4(F),$$

$R_2 = \langle R_1 \cup \{[x_i, x_j, x_k] : (i, j, k) \neq (4, 1, 1), (4, 2, 2), (4, 3, 3)\}, \alpha^{2^{k-1}}\beta^{-1}, \beta^{2^{l-2}}\gamma^{-1}, \gamma^A \rangle$, where $\alpha = [x_4, x_3, x_3]$, $\beta = [x_4, x_2, x_2]$, $\gamma = [x_4, x_1, x_1]$. Now define R_3 to be the product of R_2 and the normal closure in F of the following words:

$$\begin{aligned} & [x_4, x_3]^{2^k} \alpha^{2^{k-1}}, [x_4, x_2]^{2^l} \beta^{2^{l-1}}, \\ & [x_4, x_1]^4 \gamma^2, [x_3, x_2]^{2^l} \alpha^{-2^{l-2}}, \\ & [x_3, x_1]^4 \alpha^{-2}, [x_2, x_1]^4 \beta^{-1}, \\ & [x_4, x_5]^{2^{k-l+2}} \alpha^{-2^{k-1}} \\ & [x_5, x_i], i \neq 1. \end{aligned}$$

Finally, let R_4 be the product of R_3 and the normal closure in F of the following words:

$$\begin{aligned} c_1 &= x_1^4 [x_4, x_3]^2 [x_4, x_2] [x_4, x_5]^2, \\ c_2 &= x_2^{2^l} [x_4, x_3]^{2^{l-2}} [x_4, x_1]^{-1} [x_4, x_5]^2, \\ c_3 &= x_3^{2^k} [x_4, x_2]^{-2^{l-2}} [x_4, x_1]^{-2}. \end{aligned}$$

We claim that

$$[R_{i+1}, F] \subseteq R_i, \text{ for } i = 1, 2, 3.$$

This is obvious for $i = 1$, and 2 and it remains only to check for $i = 3$.

We note that, modulo R_3 , we have:

$$\begin{aligned} [c_1, x_1] &= 1, \\ [c_1, x_2] &= [x_1, x_2]^4 [x_4, x_3, x_3] = [x_1, x_2]^4 \beta = 1, \\ [c_1, x_3] &= [x_1, x_3]^4 [x_4, x_3, x_3]^2 = [x_1, x_3]^4 \alpha^2 = 1, \\ [c_1, x_4] &= [x_1, x_4]^4 [x_1, x_4, x_1]^2 = [x_1, x_4]^4 \gamma^2 = 1, \\ [c_1, x_5] &= 1, \\ [c_2, x_1] &= [x_2, x_1]^{2^l} [x_4, x_1, x_1]^{-1} = \beta^{2^{l-2}} \gamma^{-1} = 1, \\ [c_2, x_2] &= 1, \\ [c_2, x_3] &= [x_2, x_3]^{2^l} [x_4, x_3, x_3]^{2^{l-2}} = [x_2, x_3]^{2^l} \alpha^{2^{l-2}} = 1, \\ [c_2, x_4] &= [x_2, x_4]^{2^l} [x_2, x_4, x_2]^{2^{l-1}} = [x_2, x_4]^{2^l} \beta^{-2^{l-1}} = 1, \\ [c_2, x_5] &= 1, \\ [c_3, x_1] &= [x_3, x_1]^{2^k} [x_4, x_1, x_1]^{-2} = \alpha^{2^{k-1}} \gamma^{-2} = 1, \end{aligned}$$

$$\begin{aligned}
[c_3, x_2] &= [x_3, x_2]^{2^k} [x_4, x_2, x_2]^{-2^{l-2}} = \alpha^{2^{k-2}} \beta^{-2^{l-2}} = 1, \\
[c_3, x_3] &= 1, \\
[c_3, x_4] &= [x_3, x_4]^{2^k} [x_3, x_4, x_3]^{2^{k-1}} = [x_3, x_4]^{2^k} \alpha^{2^{k-1}} = 1, \\
[c_3, x_5] &= 1.
\end{aligned}$$

Clearly, $\gamma_3(F)/R_2$ is a cyclic group of order 2^k generated by the element α . To see that α has order exactly 2^k in the group F/R_4 , as in the case of the proof of Theorem 2.5, we note that the groups R_3/R_2 and R_4/R_3 are free abelian. Hence, the relation $\alpha^s \in R_4$, $s > 0$ implies that s is divisible by 2^k . As a consequence we get that α has order exactly 2^k in F/R_4 . Hence, modulo R_4 , the word $w = [x_1, x_2^{2^{l+1}}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}] \equiv \alpha^{2^{k-1}} \equiv \beta^8 \equiv \gamma^2 \neq 1$.

We claim that F/R_4 is a natural epimorphic image of the given group G . The first three relations of G hold in F/R_4 by construction. The relation $[x_4, x_1]^4 [x_4, x_3, x_3]^{2^{k-1}} = [x_4, x_1]^4 \gamma^2$ holds modulo R_3 . Now, modulo R_4 , we have

$$\begin{aligned}
&\xi_1^{2^{k-2}} \xi_2^{2^{k-l+1}} \\
&= [x_4, x_3]^{2^{k-1}} [x_4, x_2]^{2^{k-2}} [x_4, x_5]^{2^{k-1}} [x_4, x_3]^{2^{k-1}} [x_4, x_1]^{-2^{k-l+1}} [x_4, x_5]^{2^{k-l+2}} \\
&= [x_4, x_3]^{2^k} \alpha^{2^{k-1}} \alpha^{2^{k+l-4}} [x_4, x_1]^{-2^{k-l+1}} [x_4, x_2]^{2^{k-2}} \\
&= \alpha^{2^{k+l-4}} \beta^{-2^{k-3}} \gamma^{2^{k-l-2}} = 1.
\end{aligned}$$

The relations $[x_4, x_i, x_4]$, $i \in \{1, 2, 3, 4\}$, clearly lie in R_2 . Hence F/R_4 is a natural epimorphic image of G and the image of w is nontrivial in F/R_4 . Therefore, $w \notin \gamma_4(G)$. \square

Proof of Example 2.11. The case $n = 0$ is exactly Lemma 2.13. Assume that the result holds for some $n \geq 0$, i.e., $w_n \notin \gamma_{4+n}(\mathfrak{G}_n)$. We shall prove it for $n + 1$, i.e., that $w_{n+1} \notin \gamma_{5+n}(\mathfrak{G}_{n+1})$.

Consider the quotient $\mathfrak{G}'_n = \mathfrak{G}_n / \gamma_{4+n}(\mathfrak{G}_n) N_n$, where N_n is the normal subgroup in \mathfrak{G}_n , generated by all left-normed commutators $[y_1, \dots, y_s]$, $s \geq 3$, such that there are at least two entries with $y_i = x_4$. The automorphism of the free group of rank 5, given by

$$\begin{aligned}
x_1 &\mapsto x_1, \\
x_2 &\mapsto x_2, \\
x_3 &\mapsto x_3, \\
x_4 &\mapsto x_4, \\
x_5 &\mapsto x_5 x_4
\end{aligned}$$

can be extended to an automorphism of \mathfrak{G}'_n ; this follows from the fact that this automorphism preserves all relations. This automorphism defines a semidirect product $H_n = \mathfrak{G}'_n \rtimes \langle x \rangle$, where x acts as the described automorphism. Clearly, we have $[x, x_i] = 1$, $i = 1, 2, 3, 4$ and $[x_5, x] = x_4$ in H_n and H_n is nilpotent:

$\gamma_{5+n}(H_n) = 1$. Evidently the natural map $f : \mathfrak{G}'_n \rightarrow H_n$ is a monomorphism. However, it is easy to see that H_n is an epimorphic image of \mathfrak{G}_{n+1} , which sends w_{n+1} to $f(w_n)$. Hence, w_{n+1} can not lie in $\gamma_{5+n}(\mathfrak{G}_{n+1})$. \square

We next make somewhat more complicated constructions, working on the same principles as above, and show that there exists a nilpotent group of class 4 with nontrivial sixth dimension subgroup.

Theorem 2.14 *There exists a nilpotent group G of class 3 with*

$$G \cap (1 + \Delta(\gamma_2(G))^2 \mathbb{Z}[G] + \mathfrak{g}^5) \neq 1.$$

Proof. Let F be a free group with basis $\{x_1, x_2, x_3, x_4, x_5\}$. Let $R_1 := \gamma_4(F)$. Define

$R_2 = \langle R_1, [x_i, x_j, x_k] \notin \langle \alpha, \beta, \gamma, \delta \rangle R_1 \text{ for all } i, j, k, \delta^{16}\beta, \alpha^{2^{k-10}}\delta, \beta^8\gamma^{-1}, \gamma^8 \rangle$, where $\delta = [x_4, x_5, x_5]$, $\alpha = [x_4, x_3, x_3]$, $\beta = [x_4, x_2, x_2]$, $\gamma = [x_4, x_1, x_1]$; Let R_3 be R_2 together with the following set of words:

$$\begin{aligned} & [x_1, x_2]^8 [x_4, x_2, x_2], \\ & [x_1, x_3]^8 [x_4, x_3, x_3]^4, \\ & [x_2, x_3]^{64} [x_4, x_3, x_3]^{-16}, \\ & [x_1, x_4]^8 [x_4, x_1, x_1]^4, \\ & [x_2, x_4]^{64} [x_4, x_2, x_2]^{32}, \\ & [x_3, x_4]^{2^k} [x_4, x_3, x_3]^{2^{k-1}}, \\ & [x_2, x_5]^{64} [x_4, x_5, x_5]^{16}, \\ & [x_5, x_4]^{1024} [x_4, x_5, x_5]^{512}, \\ & [x_1, x_5]^8, [x_3, x_5]^{1024}. \end{aligned}$$

Let R_4 be R_3 together with the following set of words:

$$\begin{aligned} c_1 &= x_1^8 [x_4, x_3]^4 [x_4, x_2]; \\ c_2 &= x_2^{64} [x_4, x_3]^{-16} [x_4, x_1]^{-1} [x_4, x_5]^{16}; \\ c_3 &= x_3^{2^k} [x_4, x_2]^{16} [x_4, x_1]^{-4} \\ c_4 &= x_5^{1024} [x_4, x_2]^{16}; \end{aligned}$$

For any $i = 1, 2, 3$, $[F, R_{i+1}] \subseteq R_i$ and $k \geq 12$. The case $i = 1$ is obvious. The case $i = 2$ easily can be checked. We shall consider the most difficult case $i = 3$. Working modulo R_3 , we shall show that $[c_i, x_j] = 1$ for all i, j :

$$\begin{aligned} [c_1, x_1] &= 1; \\ [c_1, x_2] &= [x_1, x_2]^8 [x_4, x_2, x_2] = 1; \\ [c_1, x_3] &= [x_1, x_3]^8 [x_4, x_3, x_3]^4 = 1; \end{aligned}$$

$$\begin{aligned}
[c_1, x_4] &= [x_1, x_4]^8 [x_4, x_1, x_1]^4 = 1, \text{ since } \gamma^8 \in R_3; \\
[c_1, x_5] &= [x_1, x_5]^8 = 1; \\
[c_2, x_1] &= [x_2, x_1]^{64} [x_4, x_1, x_1]^{-1} = [x_4, x_2, x_2]^8 [x_4, x_1, x_1]^{-1} = 1; \\
[c_2, x_2] &= 1; \\
[c_2, x_3] &= [x_2, x_3]^{64} [x_4, x_3, x_3]^{-16}; \\
[c_2, x_4] &= [x_2, x_4]^{64} [x_4, x_2, x_2]^{32} = 1; \\
[c_2, x_5] &= [x_2, x_5]^{64} [x_4, x_5, x_5]^{16} = 1; \\
[c_3, x_1] &= [x_3, x_1]^{2^k} [x_4, x_1, x_1]^{-4} = [x_4, x_3, x_3]^{2^{k-1}} [x_4, x_1, x_1]^{-1} = 1; \\
[c_3, x_2] &= [x_3, x_2]^{2^k} [x_4, x_2, x_2]^{16} = [x_4, x_3, x_3]^{-2^{k-2}} [x_4, x_2, x_2]^{16} = 1; \\
[c_3, x_3] &= 1; \\
[c_3, x_4] &= [x_3, x_4]^{2^k} [x_4, x_3, x_3]^{2^{k-1}} = 1; \\
[c_3, x_5] &= [x_3, x_5]^{2^k} = 1, \text{ since } k > 10; \\
[c_4, x_1] &= [x_5, x_1]^{1024} = 1; \\
[c_4, x_2] &= [x_5, x_2]^{1024} [x_4, x_2, x_2]^{16} = [x_4, x_5, x_5]^{256} [x_4, x_2, x_2]^{16} = 1; \\
[c_4, x_3] &= [x_5, x_3]^{1024} = 1; \\
[c_4, x_4] &= [x_5, x_4]^{1024} [x_4, x_5, x_5]^{512} = 1; \\
[c_4, x_5] &= 1;
\end{aligned}$$

Clearly, $\gamma_3(F/R_2)$ is a cyclic group of order 2^k generated by element α . To see that α has order exactly 2^k in the group F/R_4 , as in the case of the proof of Theorem 2.3, we note that the groups R_3/R_2 and R_4/R_3 are free abelian. Hence, the relation $\alpha^s \in R_4$, $s > 0$ implies that s divides 2^k . As a consequence we get the fact that α has order exactly 2^k in F/R_4 . And therefore, our element

$$w = [x_1, x_2^{256}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}]$$

is equal to $\alpha^{2^{k-1}} = \delta^{512} = \beta^{32} = \gamma^4 \neq 1$.

Since $x_1^8, x_2^{64}, x_3^{2^k} \in \gamma_2(G)$, modulo \mathfrak{g}^6 , we have the following equivalences:

$$1 - w \equiv (1 - [x_1, x_2^{256}]) + (1 - [x_1, x_3^{2^k}]) + (1 - [x_2, x_3^{2^{k+1}}]).$$

Since $64(1 - x_1)^2, 2^k(1 - x_1)^2, 2^{k+1}(1 - x_2)^2 \in \mathfrak{g}^4, 64(1 - x_2) \in \mathfrak{g}^2$, modulo \mathfrak{g}^5 we have

$$\begin{aligned}
&1 - [x_1, x_2^{256}] \\
&\equiv (1 - x_2^{256})(1 - x_1) - (1 - x_1)(1 - x_2^{256}) \\
&\equiv (1 - x_2^{256})(1 - x_1) - (1 - x_1^{256})(1 - x_2) + \binom{256}{2}(1 - x_1)(1 - x_2)^2 \\
&\equiv (1 - x_2^{256})(1 - x_1) - (1 - x_1^{256})(1 - x_2) + (1 - x_1^{128})(1 - x_2)^2.
\end{aligned}$$

Note that modulo \mathfrak{g}^5 :

$$\begin{aligned}
& (1 - x_1^{128})(1 - x_2)^2 \\
& \equiv (1 - [x_4, x_3]^{-64}[x_4, x_2]^{-16})(1 - x_2)^2 \\
& \equiv (1 - x_2) + (1 - x_5^{1024})(1 - x_2)^2 \\
& \equiv 1024(1 - x_5)(1 - x_2)^2 \\
& \equiv (1 - x_5)(1 - x_2)(1 - x_2^{1024}) \\
& \equiv (1 - x_5)(1 - x_2)(1 - [x_4, x_3]^{256}[x_4, x_1]^{16}[x_4, x_5]^{-256}) \\
& \equiv 0,
\end{aligned}$$

therefore, modulo \mathfrak{g}^5 ,

$$1 - [x_1, x_2^{256}] \equiv (1 - x_2^{256})(1 - x_1) - (1 - x_1^{256})(1 - x_2). \quad (2.6)$$

Analogically, it is easy to show that modulo \mathfrak{g}^5 ,

$$1 - [x_1, x_3^{2^k}] \equiv (1 - x_3^{2^k})(1 - x_1) - (1 - x_1^{2^k})(1 - x_3) + (1 - x_1^{2^{k-1}})(1 - x_3)^2, \quad (2.7)$$

$$1 - [x_2, x_3^{2^{k+1}}] \equiv (1 - x_3^{2^{k+1}})(1 - x_2) - (1 - x_2^{2^{k+1}})(1 - x_3) + (1 - x_2^{2^k})(1 - x_3)^2. \quad (2.8)$$

Note that

$$\begin{aligned}
x_1^{2^{k-1}} x_2^{2^k} &= [x_4, x_3]^{-2^{k-2}} [x_4, x_2]^{-2^{k-4}} [x_4, x_3]^{2^{k-2}} [x_4, x_1]^{2^{k-6}} [x_4, x_5]^{-2^{k-2}} = \\
& [x_4, x_2]^{-2^{k-4}} [x_4, x_1]^{2^{k-6}} [x_4, x_5]^{-2^{k-2}}.
\end{aligned}$$

Hence, for $k \geq 13$, we have $x_1^{2^{k-1}} x_2^{2^k} = 1$; therefore, modulo \mathfrak{g}^5 , we have

$$\begin{aligned}
& (1 - x_1^{2^k})(1 - x_3) + (1 - x_1^{2^{k-1}})(1 - x_3)^2 + (1 - x_2^{2^{k+1}})(1 - x_3) + (1 - x_2^{2^k})(1 - x_3)^2 \equiv \\
& (1 - x_1^{2^k} x_2^{2^{k+1}})(1 - x_3) + (1 - x_1^{2^{k-1}} x_2^{2^k})(1 - x_3)^2 \equiv 0. \quad (2.9)
\end{aligned}$$

Equivalences (2.6) - (2.9) imply that, modulo \mathfrak{g}^5 ,

$$\begin{aligned}
1 - w &\equiv (1 - x_2^{256})(1 - x_1) - (1 - x_1^{256})(1 - x_2) + \\
& (1 - x_3^{2^{k+1}})(1 - x_2) + (1 - x_3^{2^k})(1 - x_1) \\
& \equiv (1 - x_2^{256} x_3^{2^k})(1 - x_1) + (1 - x_1^{-256} x_3^{2^{k+1}})(1 - x_2) \\
& \equiv (1 - \zeta_1^{16})(1 - x_1) + (1 - \zeta_2^{128})(1 - x_2),
\end{aligned}$$

where

$$\begin{aligned}
\zeta_1 &= [x_4, x_3]^4 [x_4, x_5]^{-4} [x_4, x_2]^{-1} [x_4, x_2, x_2]^2, \\
\zeta_2 &= [x_4, x_3][x_4, x_5, x_5]^{-4}.
\end{aligned}$$

Hence, modulo \mathfrak{g}^5 ,

$$\begin{aligned} 1 - w &\equiv 16(1 - \zeta_1)(1 - x_1) + 128(1 - \zeta_2)(1 - x_2) \\ &\equiv (1 - \zeta_1)(1 - x_1^{16}) - 8(1 - \zeta_1)(1 - x_1)^2 + (1 - \zeta_2)(1 - x_2^{128}) \\ &\quad - 64(1 - \zeta_2)(1 - x_2)^2 \\ &\equiv (1 - \zeta_1)(1 - x_1^{16}) + (1 - \zeta_2)(1 - x_2^{128}). \end{aligned}$$

Since $x_1^8, x_2^{64} \in \gamma_2(G)$, we conclude

$$1 - w \in \Delta(\gamma_2(G))^2 \mathbb{Z}[G] + \mathfrak{g}^5.$$

Furthermore, the detailed analysis of the above construction shows that

$$1 - w \in \Delta([\langle x_4 \rangle^G, G])^2 \mathbb{Z}[G] + \Delta([\langle x_4 \rangle^G, {}_4G]) \mathbb{Z}[G], \quad (2.10)$$

since all commutators used in the words c_i , $i = 1, \dots, 4$, have a nontrivial entry of the generator x_4 . \square

Theorem 2.15 *There exists a nilpotent group Π of class 4 with $D_6(\Pi) \neq 1$.*

Proof. Consider the 5-generated group G of Theorem 2.14 which is nilpotent of class 3. Let $G_1 = G * \langle t \rangle / \gamma_4(G * \langle t \rangle)$, the quotient of the free product of G with infinite cyclic group with generator t modulo its fourth lower central subgroup. Clearly (2.10) implies that, for the image in G_1 of the element w (we retain the notation of elements of G when naturally viewed as elements of G_1), we have

$$1 - w \in \Delta([\langle x_4 \rangle^{G_1}, G_1])^2 \mathbb{Z}[G_1] + \Delta([\langle x_4 \rangle^{G_1}, {}_4G_1]) \mathbb{Z}[G_1]. \quad (2.11)$$

Clearly, $w \notin \langle t \rangle^{G_1}$. Define the quotient $G_2 = G_1 / \langle [x_4, t, x_4] \rangle^{G_1}$. Let f be an automorphism of the free group with basis $\{x_1, x_2, x_3, x_4, x_5, t\}$ defined by

$$\begin{aligned} x_i &\mapsto x_i, \quad i = 1, \dots, 5 \\ t &\mapsto tx_4. \end{aligned}$$

It is easy to see that f can be extended to an automorphism of the group G_2 . Thus we can consider the semi-direct product $\Pi = G_2 \rtimes \langle x \rangle$. We have the following relations in the group Π :

$$[x_i, x] = 1, \quad i = 1, \dots, 5, \quad [t, x] = x_4.$$

Since, in G_2 , we have the relations $[x_4, x_i, x_4] = [x_4, t, x_4] = 1$ for all i , the group Π is nilpotent of class 4. The natural map $G_2 \rightarrow \Pi$ is a monomorphism; hence the image of the element

$$w = [x_1, x_2^{256}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}]$$

is nontrivial in Π . However, (2.11) implies that

$$1 - w \in \Delta(\langle [t, x]^\Pi, \Pi \rangle^2 \mathbb{Z}[\Pi] + \Delta(\langle [t, x]^\Pi, {}_4\Pi \rangle) \mathbb{Z}[\Pi]) \subseteq \Delta^6(\Pi).$$

Therefore, $1 \neq w \in D_6(\Pi)$. \square

Example 2.16

The reader can check that the constructions given in the proofs of Theorems 2.14 and 2.15 show that for the group Γ given by the following presentation:

$$\begin{aligned} \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid & x_1^8 [x_4, x_6, x_3]^4 [x_4, x_6, x_2], \\ & x_2^{64} [x_4, x_6, x_3]^{-16} [x_4, x_6, x_1]^{-1} [x_4, x_6, x_5]^{16}, \\ & x_3^{2^k} [x_4, x_7, x_2]^{16} [x_4, x_6, x_1]^{-4}, x_5^{1024} [x_4, x_6, x_2]^{16}, \\ & [x_4, x_6, x_5]^{2048}, [x_4, x_6, x_1]^{16}, [x_4, x_6, x_2]^{128}, \\ & [x_4, x_6, x_1, x_1] [x_4, x_6, x_2, x_2]^{-8}, \\ & [x_4, x_6, x_2, x_2]^{-8} [x_4, x_6, x_3, x_3]^{2^{k-3}} \rangle, \end{aligned}$$

for $k \geq 13$,

$$D_6(\Gamma) \not\subseteq \gamma_5(\Gamma).$$

The arguments from the proof of Theorem 2.14 imply that the relations of Γ are enough for the element

$$w = [x_1, x_2^{256}] [x_1, x_3^{2^k}] [x_2, x_3^{2^{k+1}}]$$

to lie in $D_6(\Gamma)$. However, the group Π , constructed in Theorem 2.15 is the natural epimorphic image of Γ , and consequently $w \notin \gamma_5(\Gamma)$.

2.2 Sjögren's Theorem

For every natural number k , let

$$b(k) = \text{the least common multiple of } 1, 2, \dots, k,$$

and let

$$c(1) = c(2) = 1, \quad c(n) = b(1)^{\binom{n-2}{1}} \dots b(n-2)^{\binom{n-2}{n-2}}, \quad n \geq 3.$$

The most general result known about dimension quotients is the following:

Theorem 2.17 (Sjögren [Sjo79]). *For every group G ,*

$$D_n(G)^{c(n)} \subseteq \gamma_n(G), \quad n \geq 1.$$

Alternate proofs of Sjögren's theorem have been given by Gupta [Gup87c] and Cliff-Hartley [Cli87]. In case G is a metabelian group, Gupta [Gup87d] has given the following sharper bound for the exponents of dimension quotients:

Theorem 2.18 (Gupta [Gup87d]). *If G is a metabelian group, then*

$$D_n(G)^{2b(1)\dots b(n-2)} \subseteq \gamma_n(G), \quad n \geq 3.$$

Let F be a free group and R a normal subgroup of F . For $k \geq 1$, let

$$R(k) = [\dots [R, \underbrace{F, F, \dots, F}_{k-1}],$$

and

$$\mathfrak{r}(k) = \sum \mathbb{Z}[F] \mathfrak{r}_1 \mathfrak{r}_2 \dots \mathfrak{r}_k,$$

where $R_i \in \{R, F\}$ and exactly one $R_i = R$.

The following two lemmas are the key results in the proof of Sjögren's theorem.

Lemma 2.19 *Let $w \in \gamma_n(F)$, $n \geq 2$, be such that $w - 1 \in \mathfrak{f}^{n+1} + \mathfrak{r}(k)$ for some k , $1 \leq k \leq n$. Then $w^{b(k)} - 1 \equiv f_k - 1 \pmod{\mathfrak{f}^{n+1} + \mathfrak{r}(k+1)}$ for some $f_i \in R(k)$.*

Lemma 2.20 *For $n \geq 1$, $F \cap (1 + \mathfrak{f}^{n+1} + \mathfrak{r}(n)) = \gamma_{n+1}(F)R(n)$.*

From Lemmas 2.19 and 2.20 Sjögren's theorem follows by using a process of descent:

Let $H_1 \supseteq H_2 \supseteq \dots$ and $K_1 \supseteq K_2 \supseteq \dots$ be two series, and $\{N_{m,l} : 1 \leq m \leq l\}$ a family of normal subgroups of a group G satisfying

$$\left. \begin{aligned} N_{m,m+1} &= H_m K_{m+1}, \\ H_m K_l &\subseteq N_{m,l}, \\ N_{m,l+1} &\subseteq N_{m,l} \text{ for all } m < l. \end{aligned} \right\} \quad (2.12)$$

Lemma 2.21 ([Gup87c], [Har82a]). *If n is a positive integer and there exist positive integers $a(l)$ such that*

$$(K_{l+m} \cap N_{l,l+m+1})^{a(l)} \subseteq N_{l+1,l+m+1} H_l, \quad l + m \leq n + 1,$$

then

$$N_{1,n+2}^{a(1,n+2)} \subseteq H_1 K_{n+2},$$

where

$$a(1, n+2) = \prod_{i=1}^n a(i) \binom{n}{i}.$$

2.3 Fourth Dimension Subgroup

An identification of the fourth dimension subgroup is known.

Theorem 2.22 (see [Gup87c], [Tah77b]).

Let G be a nilpotent group of class 3 given by its pre-abelian presentation:

$$\langle x_1, \dots, x_m \mid x_1^{d(1)}\xi_1, \dots, x_k^{d(k)}\xi_k, \xi_{k+1}, \dots, \gamma_4(\langle x_1, \dots, x_m \rangle) \rangle$$

with $k \leq m$, $d(i) > 0$, $d(k) \mid \dots \mid d(2) \mid d(1)$ and $\xi_i \in \gamma_2(\langle x_1, \dots, x_m \rangle)$. Then, the group $D_4(G)$ consists of all elements

$$w = \prod_{1 \leq i < j \leq k} [x_i^{d(i)}, x_j]^{a_{ij}}, \quad a_{ij} \in \mathbb{Z}, \quad (2.13)$$

such that

$$d(j) \mid \binom{d(i)}{2} a_{ij} \quad (1 \leq i < j \leq m), \quad (2.14)$$

and

$$y_l = \prod_{1 \leq i < l} x_i^{-d(i)a_{il}} \prod_{l < j \leq k} x_j^{d(l)a_{lj}} \in \gamma_2(G)^{d(l)}\gamma_3(G) \quad \text{for } 1 \leq l \leq k. \quad (2.15)$$

Theorem 2.23 (Losey [Los74], Tahara [Tah77a], Sjögren [Sjo79], Passi ([Pas68a], [Pas79])). For any group G , $D_4(G)/\gamma_4(G)$ has exponent 2.

It may be noted that every 3-generator group G has the property that $D_4(G) = \gamma_4(G)$ (see [Gup87c]). In Example 2.3 we have a 4-generator group G with 3 relators such that $D_4(G) \neq \gamma_4(G)$. We now show that every 2-relator group G has the property that $D_4(G) = \gamma_4(G)$. Thus, in a sense, Example 2.3 is a minimal example of a group G with $D_4(G) \neq \gamma_4(G)$.

Theorem 2.24 Let $G = \langle X \mid r_1, r_2 \rangle$ be a 2-relator group. Then $D_4(G) = \gamma_4(G)$.

Proof. Observe that G has a pre-abelian presentation of the form

$$G = \langle x_1, \dots, x_n, \dots \mid x_1^{d(1)}\xi_1, x_2^{d(2)}\xi_2, \xi_3, \dots \rangle$$

with $\xi_i \in \gamma_2(\langle x_1, \dots \rangle)$ and $d(2) \mid d(1)$. Then, modulo $\gamma_4(G)$, the group $D_4(G)$ consists of the elements

$$w = [x_1^{d(1)}, x_2]^{a_{12}},$$

such that

$$d(2) \mid \binom{d(1)}{2} a_{12},$$

and

$$y_2 = x_1^{-d(1)a_{12}} \in \gamma_2(G)^{d(2)} \gamma_3(G).$$

Therefore, modulo $\gamma_4(G)$, for some $z \in \gamma_2(G)$, we have

$$w = [x_1^{d(1)a_{12}}, x_2] = [y_2, x_2] = [z^{d(2)}, x_2] = [z, x_2^{d(2)}] = 1.$$

□

Theorem 2.25 [Gup92] *For any group G , $[D_4(G), G] = \gamma_5(G)$.*

Proof. In view of Theorem 2.23, it suffices to prove the statement for finite 2-groups. Let G be a finite 2-group, generated by elements x_1, \dots, x_k such that $x_i^{d(i)} \in \gamma_2(G)$ for some $d(i) = 2^{\alpha_i}$, with ordering $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k \geq 1$. Let $w \in D_4(G)$. Theorem 2.22 implies that modulo $\gamma_4(G)$, w can be expressed in the form (2.13), such that the conditions (2.14) and (2.15) are satisfied. Let h be arbitrary element of G . Then we have the following equivalences modulo $\gamma_5(G)$:

$$\begin{aligned} [w, h] &= \left[\prod_{1 \leq i < j \leq k} [x_i^{d(i)}, x_j]^{a_{ij}}, h \right] \equiv \prod_{1 \leq i < j \leq k} [x_i^{d(i)}, x_j, h]^{a_{ij}} \equiv \\ &\prod_{1 \leq i < j \leq k} [x_i^{d(i)}, h, x_j]^{a_{ij}} \prod_{1 \leq i < j \leq k} [x_j, h, x_i^{d(i)}]^{-a_{ij}} \pmod{\gamma_5(G)}. \end{aligned} \quad (2.16)$$

Condition (2.14) implies that

$$\begin{aligned} \prod_{1 \leq i < j \leq k} [x_i^{d(i)}, h, x_j]^{a_{ij}} &\equiv \prod_{1 \leq i < j \leq k} [x_i, h, x_j^{d(i)}]^{a_{ij}} \equiv \prod_{1 \leq i < j \leq k} [x_i, h, x_j^{d(i)a_{ij}}] \pmod{\gamma_5(G);} \\ \prod_{1 \leq i < j \leq k} [x_j, h, x_i^{d(i)}]^{-a_{ij}} &\equiv \prod_{1 \leq i < j \leq k} [x_j, h, x_i^{-d(i)a_{ij}}] \pmod{\gamma_5(G)}. \end{aligned}$$

Therefore, by condition (2.15), we have

$$\begin{aligned} [w, h] &\equiv \prod_{1 \leq i < j \leq k} [x_i, h, x_j^{d(i)a_{ij}}] \prod_{1 \leq i < j \leq k} [x_j, h, x_i^{-d(i)a_{ij}}] \equiv \\ &\prod_{1 \leq t \leq k} [x_t, h, \prod_{1 \leq s < t} x_s^{-d(s)a_{st}} \prod_{t < r \leq k} x_r^{-d(t)a_{tr}}] \equiv \\ &\prod_{1 \leq t \leq k} [x_t, h, y_t] \equiv 1 \pmod{\gamma_5(G)}. \quad \square \end{aligned}$$

An extensive analysis of the counter-examples to the equality of the fourth dimension subgroup with the fourth lower central subgroup has been carried out by M. Hartl [Har].

Theorem 2.26 (Hartl [Har]; see also [Har98, Theorem 7.2.6, p. 72]). *Let $A = \mathbb{Z}_{2^{\beta_1}} \oplus \mathbb{Z}_{2^{\beta_2}} \oplus \mathbb{Z}_{2^{\beta_3}} \oplus \mathbb{Z}_{2^{\beta_4}}$ with $\beta_1 \leq \beta_2 \leq \beta_3 \leq \beta_4$ and $n \geq 1$. Then there exists a finite nilpotent group G of class 3 with $G_{ab} \simeq A$, such that $D_4(G) \neq 1$ and $[v, x] = 1$ for every $v \in \gamma_2(G)$, $x \in G$, such that $x\gamma_2(G)$ is a generator of the summand $\mathbb{Z}_{2^{\beta_4}}$ in G_{ab} if and only if the following conditions hold:*

- (i) $\beta_1, \beta_2 - \beta_1, \beta_3 - \beta_2 \geq 2$,
- (ii) $\beta_3 > n > \max\{\beta_2, \beta_3 - \beta_1\}$.

Moreover, under conditions (i) and (ii), the group G can be chosen to be of order $2^{4\beta_1+3\beta_2+2\beta_3+\beta_4+n+1}$.

2.4 Fifth Dimension Subgroup

The structure of the fifth dimension subgroup has been described by Tahara [Tah81], and it has been further shown that $D_5^6(G) \subseteq \gamma_5(G)$:

Theorem 2.27 (Tahara [Tah81]). *For every group G , $D_5(G)^6 \subseteq \gamma_5(G)$.*

Analysis of Tahara's description of the fifth dimension subgroup leads us to the following result.

Theorem 2.28 *For every group G , $D_5(G)^2 \subseteq \delta_2(G)\gamma_5(G)$.*

Let G be a finite group of class 4. Choose the elements

$$\begin{aligned} &\{x_{1i} \in G \setminus \gamma_2(G)\}_{i=1, \dots, s}, \\ &\{x_{2i} \in \gamma_2(G) \setminus \gamma_3(G)\}_{i=1, \dots, t}, \\ &\{x_{3i} \in \gamma_3(G) \setminus \gamma_4(G)\}_{i=1, \dots, u}, \\ &\{x_{4i} \in \gamma_4(G)\}_{i=1, \dots, v} \end{aligned}$$

to be such that $\{x_{li}\gamma_{l+1}(G)\}$ forms a basis of $\gamma_l(G)/\gamma_{l+1}(G)$. Let $d(i)$ be the order of $x_{1i}\gamma_2(G)$ in $G/\gamma_2(G)$, $e(i)$ the order of $x_{2i}\gamma_3(G)$ in $\gamma_2(G)/\gamma_3(G)$, $f(i)$ the order of $x_{3i}\gamma_4(G)$ in $\gamma_3(G)/\gamma_4(G)$. We then have

$$\begin{aligned} x_{1i}^{d(i)} &= \prod_{1 \leq j \leq t} x_{2j}^{b_{ij}} \prod_{1 \leq j \leq u} x_{3j}^{c_{ij}} y_{4i}, \quad y_{4i} \in \gamma_4(G), \quad 1 \leq i \leq s; \\ x_{2i}^{e(i)} &= \prod_{1 \leq j \leq t} x_{3i}^{d_{ij}} y'_{4i}, \quad y'_{4i} \in \gamma_4(G), \quad 1 \leq i \leq t; \\ x_{3i}^{f(i)} &= \prod_{1 \leq j \leq v} x_{4j}^{f_{ij}}, \quad 1 \leq i \leq u; \\ [x_{1i}^{d(i)}, x_{1j}] &= \prod_{1 \leq k \leq u} x_{3k}^{\alpha_k^{(ij)}} y''_{ij}, \quad y''_{ij} \in \gamma_4(G), \quad 1 \leq i < j \leq s. \end{aligned}$$

We choose the element x_{ij} in such a way that $d(i)|d(i+1)$, $e(i)|e(i+1)$, $f(i)|f(i+1)$.

Theorem 2.29 (Tahara [Tah81]). *With the above notations, the subgroup $D_5(G)$ is equal to the subgroup generated by the elements*

$$\prod_{1 \leq i \leq j \leq s} [x_{1i}^{u_{ij}d(j)}, x_{1j}] \prod_{1 \leq i \leq s, 1 \leq k \leq t, k < l} [x_{2l}, x_{2k}]^{b_{il}v_{ik}} \prod_{1 \leq i \leq j \leq k \leq s} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}}, \quad (2.17)$$

where

$$\begin{aligned} u_{ij}, & \quad 1 \leq i < j \leq s, \\ v_{ik}, & \quad 1 \leq i \leq s, 1 \leq k \leq t, \\ v'_{ik}, & \quad 1 \leq i \leq s, 1 \leq k \leq t, \\ w_{ijk}, & \quad 1 \leq i \leq j \leq k \leq s, \\ w'_{ijk}, & \quad 1 \leq i < j \leq k \leq s, \\ w''_{ijk}, & \quad 1 \leq i \leq j < k \leq s, \end{aligned}$$

are integers satisfying the following conditions:

$$w_{iii} = 0, \quad 1 \leq i \leq s; \quad (2.18)$$

$$u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{2} + w_{iij}d(i) + w''_{iij}d(j) = 0, \quad 1 \leq i < j \leq s; \quad (2.19)$$

$$-u_{ij} \binom{d(j)}{2} + w_{ijj}d(i) + w'_{ijj}d(j) = 0, \quad 1 \leq i < j \leq s; \quad (2.20)$$

$$w_{ijk}d(i) + w'_{ijk}d(j) + w''_{ijk}d(k) = 0, \quad 1 \leq i < j < k \leq s; \quad (2.21)$$

$$\sum_{i < h} u_{ih}b_{hk} - \sum_{h < i} u_{hi} \frac{d(i)}{d(h)} b_{hk} + v_{ik}d(i) + v'_{ik}e(k) = 0, \quad 1 \leq i \leq s, 1 \leq k \leq t; \quad (2.22)$$

$$u_{ij} \frac{d(j)}{d(i)} \binom{d(i)}{3} + w_{iij} \binom{d(i)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j \leq s; \quad (2.23)$$

$$w_{iij} \binom{d(i)}{2} + w''_{iij} \binom{d(j)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j \leq s; \quad (2.24)$$

$$-u_{ij} \binom{d(j)}{3} + w'_{ijj} \binom{d(j)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j \leq s; \quad (2.25)$$

$$w_{ijk} \binom{d(i)}{2}, w'_{ijk} \binom{d(j)}{2}, w''_{ijk} \binom{d(k)}{2} \equiv 0 \pmod{d(i)}, \quad 1 \leq i < j < k \leq t; \quad (2.26)$$

$$v_{ik} \binom{d(i)}{2} - \sum_{h \leq i} w_{hii} b_{hk} - \sum_{i < h} w''_{iih} b_{hk} \equiv 0 \pmod{(d(i), e(k))},$$

$$1 \leq i \leq s, 1 \leq k \leq t; \quad (2.27)$$

$$\sum_{h \leq i} w_{hij} b_{hk} + \sum_{i < h \leq j} w'_{ihj} b_{hk} + \sum_{j < h} w''_{ijh} b_{hk} \equiv 0 \pmod{(d(i), e(k))},$$

$$1 \leq i < j \leq s, 1 \leq k \leq t; \quad (2.28)$$

$$- \sum_{h < i} u_{hi} \frac{d(i)}{d(h)} \alpha_l^{(hi)} + \sum_{i < h} u_{ih} c_{hl} - \sum_{h < i} u_{hi} \frac{d(i)}{d(h)} c_{hl} - \sum_k v'_{ik} d_{kl} -$$

$$\sum_{g \leq i \leq h} w_{gih} \alpha_l^{(gh)} - \sum_{g \leq h \leq i} w_{ghi} \alpha_l^{(gh)} - \sum_{i < g \leq h} w'_{igh} \alpha_l^{(gh)} \equiv 0 \pmod{(d(i), f(l))},$$

$$1 \leq i \leq s, 1 \leq l \leq s; \quad (2.29)$$

$$\sum_i v_{ik} b_{ik} \equiv 0 \pmod{e(k)}, 1 \leq k \leq t; \quad (2.30)$$

$$\sum_i v_{ik} b_{il} + \sum_i v_{il} b_{ik} \equiv 0 \pmod{e(k)}, 1 \leq k < l \leq t. \quad (2.31)$$

Proof of Theorem 2.28. Standard reduction argument shows that it is enough to consider finite groups. Commutator identities (see Chapter 1, 1.1) and condition (2.25) imply

$$[x_{1i}^{u_{ij}d(j)}, x_{1j}] = [x_{1i}, x_{1j}^{u_{ij}d(j)}][x_{1i}, x_{1j}, x_{1j}]^{-u_{ij} \binom{d(j)}{2}} [x_{1j}, x_{1i}, x_{1i}]^{-u_{ij} \binom{d(j)}{2}}.$$

$$[x_{1i}, x_{1j}, x_{1j}, x_{1j}]^{-u_{ij} \binom{d(j)}{3}} [x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{-u_{ij} \binom{d(j)}{3}} =$$

$$[x_{1i}, x_{1j}^{u_{ij}d(j)}][x_{1i}, x_{1j}, x_{1j}]^{-u_{ij} \binom{d(j)}{2}} [x_{1j}, x_{1i}, x_{1i}]^{-u_{ij} \binom{d(j)}{2}}.$$

$$[x_{1i}, x_{1j}, x_{1j}, x_{1j}]^{-w'_{ijj} \binom{d(j)}{2}} [x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{-w'_{ijj} \binom{d(j)}{2}}. \quad (2.32)$$

Observe that

$$[x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{-2w'_{ijj} \binom{d(j)}{2}} = 1,$$

$$[x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{-3w'_{ijj} \binom{d(j)}{2}} = [x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{-3u_{ij} \binom{d(j)}{3}}$$

$$= [x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{-u_{ij} d(j) \frac{(d(j)-1)(d(j)-2)}{2}} = 1$$

Therefore,

$$[x_{1j}, x_{1i}, x_{1i}, x_{1i}]^{w'_{ijj} \binom{d(j)}{2}} = 1.$$

Analogously,

$$[x_{1i}, x_{1j}, x_{1j}, x_{1j}]^{w'_{ijj} \binom{d(j)}{2}} = 1.$$

$$\begin{aligned} \prod_{i<j} [x_{1i}^{u_{ij}d(j)}, x_{1j}]^2 &= \prod_j \left(\prod_{i<j} [x_{1i}^{u_{ij}d(j)}, x_{1j}] \prod_{j<k} [x_{1j}^{u_{jk}d(k)}, x_{1k}] \right) = \\ &= \prod_j \left(\left(\prod_{i<j} x_{1i}^{u_{ij}d(j)} \prod_{j<k} x_{1k}^{-u_{jk}d(k)}, x_{1j} \right) \right) \cdot B, \end{aligned} \quad (2.33)$$

where

$$B := \prod_{j<k} \left([x_{1j}, x_{1k}, x_{1k}]^{-u_{jk} \binom{d(k)}{2}} [x_{1k}, x_{1j}, x_{1j}]^{-u_{jk} \binom{d(k)}{2}} \right).$$

Also we have

$$\begin{aligned} & \left(\prod_{i<j} x_{1i}^{u_{ij}d(j)} \prod_{j<k} x_{1k}^{-u_{jk}d(k)}, x_{1j} \right) = \\ & \left(\prod_l x_{2l}^{\sum_{i<j} u_{ij} \frac{d(j)}{d(i)} b_{il} - \sum_{j<k} u_{jk} b_{kl}} \prod_q x_{3q}^{\sum_{i<j} u_{ij} \frac{d(j)}{d(i)} c_{iq} - \sum_{j<k} u_{jk} c_{kq}}, x_{1j} \right) = \\ & \left(\prod_l x_{2l}^{v_{jl}d(j)+v'_{jl}e(l)}, x_{1j} \right) \left[\prod_q x_{3q}^{A_{qj}}, x_{1j} \right] = \quad (2.34) \\ & \prod_l [x_{2l}, x_{1j}, x_{1j}]^{-v_{jl} \binom{d(j)}{2}} \left[\prod_l x_{2l}^{v_{jl}}, x_{1j}^{d(j)} \right] \left[\prod_l x_{2l}^{v'_{jl}e(l)}, x_{1j} \right] \left[\prod_q x_{3q}^{A_{qj}}, x_{1j} \right], \end{aligned}$$

where $A_{qj} = \sum_{i<j} u_{ij} \frac{d(j)}{d(i)} c_{iq} - \sum_{j<k} u_{jk} c_{kq}$.

The condition (2.27) implies that

$$\begin{aligned} & \prod_l [x_{2l}, x_{1j}, x_{1j}]^{-v_{jl} \binom{d(j)}{2}} = \\ & \prod_l [x_{2l}, x_{1j}, x_{1j}]^{-\sum_{i \leq j} w_{ijj} b_{il} - \sum_{j \leq k} w'_{jjk} b_{kl}} = \quad (2.35) \\ & \prod_{i \leq j} [x_{1i}, x_{1j}, x_{1j}]^{-w_{ijj} d(i)} [x_{1j}, x_{1i}, x_{1i}]^{-w'_{ijj} d(j)} \end{aligned}$$

The condition (2.31) implies that

$$\begin{aligned} C &:= \prod_j \left[\prod_l [x_{2l}^{v_{jl}}, x_{1j}^{d(j)}] \right] = \\ & \prod_j \left(\prod_{l'>l} [x_{2l}, x_{2l'}]^{v_{jl} b_{jl'}} \prod_{l'<l} [x_{2l}, x_{2l'}]^{v_{jl} b_{jl'}} \right) = \quad (2.36) \\ & \prod_{l'>l} [x_{2l}, x_{2l'}]^{\sum_j v_{jl} b_{jl'}} \prod_{l>l'} [x_{2l}, x_{2l'}]^{\sum_j v_{jl} b_{jl'}} = \\ & \prod_{l'>l} [x_{2l}, x_{2l'}]^{\sum_j v_{jl} b_{jl'}} \prod_{l>l'} [x_{2l}, x_{2l'}]^{-\sum_j v_{jl} b_{jl'}} = \prod_{l'>l} [x_{2l}, x_{2l'}]^{2 \sum_j v_{jl} b_{jl'}}. \end{aligned}$$

The condition (2.29) implies that

$$\begin{aligned}
D &:= \prod_j \left[\prod_l [x_{2l}^{v'_j e(l)}, x_{1j}] \left[\prod_q [x_{3q}^{A_{qj}}, x_{1j}] \right] \right] = \prod_{1 \leq q \leq u, 1 \leq j \leq s} [x_{3q}, x_{1j}]^{\sum_i v'_j d_{lq} + A_{qj}} = \\
&\prod_{\substack{1 \leq j \leq s, \\ 1 \leq q \leq u}} [x_{3q}, x_{1j}]^{-\sum_{h < j} u_{hj} \frac{d(j)}{d(h)} \alpha_q^{(hj)} - \sum_{g \leq j \leq h} w_{gjh} \alpha_q^{(gh)} - \sum_{g \leq h \leq j} w_{ghj} \alpha_q^{(gh)} - \sum_{j < g \leq h} w'_{jgh} \alpha_q^{(gh)}} = \\
&\prod_{h < j} [x_{1h}^{d(j)}, x_{1j}, x_{1j}]^{-u_{hj}} \prod_{g \leq j \leq h} [x_{1g}^{d(g)}, x_{1h}, x_{1j}]^{-w_{gjh}}. \quad (2.37) \\
&\prod_{g < h \leq j} [x_{1g}^{d(g)}, x_{1h}, x_{1j}]^{-w_{ghj}} \prod_{j < g \leq h} [x_{1g}^{d(g)}, x_{1h}, x_{1j}]^{-w'_{jgh}}.
\end{aligned}$$

Since $[x_{1i}^{d(i)}, x_{1k}, x_{1j}] [x_{1k}, x_{1j}, x_{1i}^{d(i)}] [x_{1j}, x_{1i}^{d(i)}, x_{1k}] = 1$, we have

$$\prod_{i \leq j \leq k} [x_{1i}^{d(i)}, x_{1k}, x_{1j}]^{-w_{ijk}} = \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}} \prod_{i \leq j \leq k} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{-w_{ijk}}.$$

We change the subscripts g, h in (2.37) by appropriate subscripts i, j, k . The conditions (2.20) and (2.26) then imply

$$\begin{aligned}
D &= \prod_{i < j} [x_{1i}, x_{1j}, x_{1j}]^{-u_{ij} d(j)} \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}}. \\
&\prod_{i \leq j \leq k} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{-2w_{ijk}} \prod_{j < i \leq k} [x_{1i}^{d(i)}, x_{1k}, x_{1j}]^{-w'_{jik}} = \quad (2.38) \\
&\prod_{i < j} [x_{1i}, x_{1j}, x_{1j}^{d(j)}]^{-u_{ij}} \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}}. \\
&\prod_{i \leq j \leq k} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{-2w_{ijk}} \prod_{j < i \leq k} [x_{1k}, x_{1i}, x_{1j}^{d(i)}]^{w'_{jik}}.
\end{aligned}$$

Hence

$$\begin{aligned}
g^2 &= B \cdot C \cdot D \cdot \prod_{i \leq j} [x_{1i}, x_{1j}, x_{1j}]^{-w_{ijj} d(i)} [x_{1j}, x_{1i}, x_{1i}]^{-w''_{ijj} d(j)}. \\
&\prod_{1 \leq i \leq s, 1 \leq k \leq t, k < l} [x_{2l}, x_{2k}]^{2b_{il} v_{ik}} \prod_{1 \leq i \leq j \leq k \leq s} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{2w_{ijk}} \\
&= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}]^{-u_{jk} \binom{d(k)}{2}} [x_{1k}, x_{1j}, x_{1j}]^{-u_{jk} \binom{d(k)}{2}}). \\
&\prod_{i \leq j} [x_{1i}, x_{1j}, x_{1j}]^{-w_{ijj} d(i)} [x_{1j}, x_{1i}, x_{1i}]^{-w''_{ijj} d(j)}. \\
&\prod_{i < j} [x_{1i}, x_{1j}, x_{1j}^{d(j)}]^{-u_{ij}} \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}} \prod_{i < j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(j)}]^{w'_{ijk}} \quad (2.39)
\end{aligned}$$

$$\begin{aligned}
&= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}]^{w'_{jkk}d(k)} [x_{1k}, x_{1j}, x_{1j}]^{w_{jjk}d(j)}). \\
&\quad \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}} \prod_{i < j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(j)}]^{w'_{ijk}} \\
&= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}^{d(k)}]^{w'_{jkk}} [x_{1k}, x_{1j}, x_{1j}^{d(j)}]^{w_{jjk}} [x_{1k}, x_{1j}, x_{1j}, x_{1j}]^{-w_{jjk} \binom{d(j)}{2}}). \\
&\quad \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}} \prod_{i < j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(j)}]^{w'_{ijk}} \\
&= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}^{d(k)}]^{w'_{jkk}} [x_{1k}, x_{1j}, x_{1j}^{d(j)}]^{w_{jjk}}) \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}}. \\
&\quad \prod_{i < j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(j)}]^{w'_{ijk}},
\end{aligned}$$

since $[x_{1k}, x_{1j}, x_{1j}, x_{1j}]^{2w_{jjk} \binom{d(j)}{2}} = 1$, and

$$[x_{1k}, x_{1j}, x_{1j}, x_{1j}]^{3w_{jjk} \binom{d(j)}{2}} = [x_{1k}, x_{1j}, x_{1j}, x_{1j}]^{-3u_{jk} \frac{d(k)}{d(j)} \binom{d(j)}{3}} = 1$$

by (2.23). Consequently, $g^2 \in \delta_2(G)$, and the proof is complete. \square

Proof of Theorem 2.27. Multiplying $[x_{1i}, x_{1j}, x_{2k}]$ by left hand side of (2.28) and taking the product over all $i < j$ and k , we obtain the following:

$$\begin{aligned}
1 &= \prod_{i \leq j < k} [x_{1j}, x_{1k}, x_{1i}^{d(i)}]^{w_{ijk}} \prod_{i < j \leq k} [x_{1i}, x_{1k}, x_{1j}^{d(j)}]^{w'_{ijk}} \prod_{i < j < k} [x_{1i}, x_{1j}, x_{1k}^{d(k)}]^{w'_{ijk}} \\
&= \left(\prod_{i < j} [x_{1i}, x_{1j}, x_{1i}]^{w_{iij}d(i)} [x_{1i}, x_{1j}, x_{1j}]^{w'_{iij}d(j)} \right). \\
&\quad \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1i}, x_{1k}, x_{1j}^{d(j)}]^{w'_{ijk}}. \\
&\quad \prod_{i < j < k} [x_{1j}, x_{1i}, x_{1k}]^{w'_{ijk}d(j) + w_{ijk}d(i)} \\
&= \left(\prod_{i < j} [x_{1i}, x_{1j}, x_{1i}]^{w_{iij}d(i)} [x_{1i}, x_{1j}, x_{1j}]^{w'_{iij}d(j)} \right). \\
&\quad \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1j}, x_{1i}, x_{1k}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w'_{ijk}d(j)}.
\end{aligned}$$

Therefore,

$$\begin{aligned} & \prod_{i < j < k} [x_{1k}, x_{1j}, x_{1i}]^{w'_{ijk}d(j)} \\ &= \left(\prod_{i < j} [x_{1i}, x_{1j}, x_{1i}]^{w_{ijj}d(i)} [x_{1i}, x_{1j}, x_{1j}]^{w'_{ijj}d(j)} \right) \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)}. \\ & \prod_{i < j < k} [x_{1j}, x_{1i}, x_{1k}]^{w_{ijk}d(i)} \end{aligned}$$

Now consider the element g^2 given in (2.39):

$$\begin{aligned} g^2 &= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}^{d(k)}]^{w'_{jkk}} [x_{1k}, x_{1j}, x_{1j}^{d(j)}]^{w_{jjk}}). \\ & \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}} \prod_{i < j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(j)}]^{w'_{ijk}} \\ &= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}^{d(k)}]^{w'_{jkk}} [x_{1k}, x_{1j}, x_{1j}^{d(j)}]^{w_{jjk}}) \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{w_{ijk}}. \\ & \left(\prod_{i < j} [x_{1i}, x_{1j}, x_{1i}]^{w_{ijj}d(i)} [x_{1i}, x_{1j}, x_{1j}]^{w'_{ijj}d(j)} \right). \tag{2.40} \\ & \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1j}, x_{1i}, x_{1k}]^{w_{ijk}d(i)} \\ &= \left(\prod_{i < j} [x_{1j}, x_{1i}, x_{1i}]^{w_{ijj}d(i)} [x_{1i}, x_{1j}, x_{1j}]^{2w'_{ijj}d(j)} \right) \prod_{i < j < k} [x_{1j}, x_{1i}, x_{1k}]^{w_{ijk}d(i)} \end{aligned}$$

Analogously, multiplying $[x_{2k}, x_{1j}, x_{1i}]$ by left hand side of (2.28) and taking the product over all $i < j$ and k , we obtain the following:

$$\begin{aligned} 1 &= \prod_{i \leq j < k} [x_{1i}^{d(i)}, x_{1k}, x_{1j}]^{w_{ijk}} \prod_{i < j < k} [x_{1j}^{d(j)}, x_{1k}, x_{1i}]^{w'_{ijk}} \prod_{i < j < k} [x_{1k}^{d(k)}, x_{1j}, x_{1i}]^{w''_{ijk}} \\ &= \prod_{i \leq j < k} [x_{1i}^{d(i)}, x_{1k}, x_{1j}]^{w_{ijk}} \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w'_{ijk}d(j)}. \\ & \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i) + w'_{ijk}d(j)} \\ &= \prod_{i < j} [x_{1i}^{d(i)}, x_{1j}, x_{1i}]^{w_{ijj}} \prod_{i < j < k} [x_{1i}, x_{1k}, x_{1j}]^{w_{ijk}d(i)}. \\ & \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{2w'_{ijk}d(j)} \end{aligned}$$

Hence

$$\begin{aligned} & \prod_{i < j < k} [x_{1k}, x_{1j}, x_{1i}]^{2w'_{ijk}d(j)} \\ &= \prod_{i < j} [x_{1i}^{d(i)}, x_{1j}, x_{1i}]^{w_{ijj}} \prod_{i < j < k} [x_{1i}, x_{1k}, x_{1j}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)}. \end{aligned}$$

Now consider the element g^4 obtained by squaring the element given in (2.39):

$$\begin{aligned} g^4 &= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}^{d(k)}]^{2w'_{jkk}} [x_{1k}, x_{1j}, x_{1j}^{d(j)}]^{2w_{jjk}}) \\ & \quad \prod_{i \leq j \leq k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{2w_{ijk}} \prod_{i < j < k} [x_{1k}, x_{1j}, x_{1i}^{d(j)}]^{2w'_{ijk}} \\ &= \prod_{j < k} ([x_{1j}, x_{1k}, x_{1k}^{d(k)}]^{2w'_{jkk}} [x_{1k}, x_{1j}, x_{1j}^{d(j)}]^{4w_{jjk}}) \prod_{i < j < k} [x_{1k}, x_{1j}, x_{1i}^{d(i)}]^{2w_{ijk}} \\ & \quad \prod_{i < j} [x_{1i}^{d(i)}, x_{1j}, x_{1i}]^{w_{ijj}} \prod_{i < j < k} [x_{1i}, x_{1k}, x_{1j}]^{w_{ijk}d(i)} \prod_{i < j < k} [x_{1j}, x_{1k}, x_{1i}]^{w_{ijk}d(i)} \\ &= \prod_{i < j} ([x_{1i}, x_{1j}, x_{1j}^{d(j)}]^{2w'_{ijj}} [x_{1j}, x_{1i}, x_{1i}^{d(i)}]^{3w_{ijj}}) \prod_{i < j < k} [x_{1i}, x_{1j}, x_{1k}]^{w_{ijk}d(i)}. \end{aligned} \tag{2.41}$$

Multiplying (2.40) and (2.41), we obtain

$$g^6 = \prod_{i < j} ([x_{1i}, x_{1j}, x_{1j}^{d(j)}]^{4w'_{ijj}} [x_{1j}, x_{1i}, x_{1i}^{d(i)}]^{4w_{ijj}}).$$

The condition (2.27) implies that

$$\prod_{i < j} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{ijj}} [x_{1j}^{d(j)}, x_{1i}, x_{1i}]^{w'_{ijj}} = \prod_{i, k} v_{ik} \binom{d(i)}{2} [x_{2k}, x_{1i}, x_{1i}].$$

Conditions (2.19), (2.20), (2.22) imply that

$$\begin{aligned} 1 &= \prod_{i < j} [x_{1i}, x_{1j}, x_{1j}]^{2w_{ijj}d(i)} [x_{1j}, x_{1i}, x_{1i}]^{2w'_{ijj}d(j)} \\ &= \prod_{i < j} [x_{1i}, x_{1j}, x_{1j}]^{-2w'_{ijj}d(j) - u_{ij}d(j)} [x_{1j}, x_{1i}, x_{1i}]^{-2w_{ijj}d(i) + u_{ij}d(j)} \\ &= \left(\prod_{i < j} [x_{1i}, x_{1j}, x_{1j}]^{-2w'_{ijj}d(j)} [x_{1j}, x_{1i}, x_{1i}]^{-2w_{ijj}d(i)} \right) \\ & \quad \prod_{i < j} [x_{1i}, x_{1j}, x_{1j}]^{-u_{ij}d(j)} [x_{1i}, x_{1j}, x_{1i}]^{-u_{ij}d(j)} \\ &= \prod_{i < j} [x_{1i}, x_{1j}, x_{1j}]^{-2w'_{ijj}d(j)} [x_{1j}, x_{1i}, x_{1i}]^{-2w_{ijj}d(i)}. \end{aligned}$$

Hence $g^6 = 1$. \square

Problem 2.30 *If G is a nilpotent group of class three, then must $D_5(G)$ be trivial?*

We illustrate the complexity of the above problem by verifying it for a group, without dimension property, considered by Gupta-Passi ([Gup87c], p. 76). Let us recall the construction of this group.

Let F be the free group with basis x_1, x_2, x_3, x_4 and let R be the normal subgroup generated by

$$r_1 = x_4^{64}[x_4, x_3]^{32}, r_2 = x_3^{64}[x_4, x_2]^{-4}[x_4, x_1]^{-2}, r_3 = x_2^{16}[x_4, x_3]^4[x_4, x_1]^{-1},$$

$$r_4 = x_1^4[x_4, x_3]^2[x_4, x_2], r_5 = [x_4, x_3]^{64}[x_4, x_3, x_3]^{32},$$

$$r_6 = [x_4, x_2]^{16}[x_4, x_2, x_2]^8, r_7 = [x_4, x_1]^4[x_4, x_1, x_1]^2,$$

$$r_8 = [x_3, x_2]^{16}[x_4, x_2, x_2]^{-1}, r_9 = [x_3, x_1]^4[x_4, x_3, x_3]^{-2},$$

$$r_{10} = [x_2, x_1]^4[x_4, x_2, x_2]^{-1}, r_{11} = [x_4, x_3, x_3]^4[x_4, x_2, x_2]^{-1},$$

$$r_{12} = [x_4, x_2, x_2]^4[x_4, x_1, x_1]^{-1}, r_{13} = [x_4, x_1, x_1]^4,$$

$\gamma_4(F)$, and all commutators $[x_i, x_j, x_k]$ ($1 \leq i, j, k \leq 4$) which do not belong to

$$\langle [x_4, x_1, x_1], [x_4, x_2, x_2], [x_4, x_3, x_3] \rangle \gamma_4(F).$$

Then the group

$$G := F/R \tag{2.42}$$

is a finite 2-group of class 3 with the non-identity element

$$w_0 = [x_3^{64}, x_2]^2[x_3^{64}, x_1][x_2^{16}, x_1]^2 R$$

in $D_4(G)$.

With the notations of Theorem 2.29, we choose

$$x_{11} = x_1, x_{12} = x_2, x_{13} = x_3, x_{14} = x_4,$$

$$x_{21} = [x_1, x_2], x_{22} = [x_1, x_3], x_{23} = [x_1, x_4],$$

$$x_{24} = [x_2, x_3], x_{25} = [x_2, x_4], x_{26} = [x_3, x_4],$$

$$x_{31} = [x_4, x_3, x_3].$$

For this group we have the following constants:

$$d(1) = 4, d(2) = 16, d(3) = 64, d(4) = 64,$$

$$e(1) = 4, e(2) = 4, e(3) = 4, e(4) = 16, e(5) = 16, e(6) = 64,$$

$$b_{15} = 1, b_{16} = 2, b_{23} = -1, b_{26} = 4,$$

$$b_{33} = 2, b_{35} = -4, b_{46} = 32, \text{ all other } b_{ij} \text{ are zero,}$$

$$d_{11} = -4, d_{21} = -2, d_{31} = 32, d_{41} = -4, d_{51} = 32, d_{61} = 32, \\ \text{all other } d_{ij} \text{ are zero,}$$

$$\alpha_1^{(12)} = -4, \alpha_1^{(23)} = -4, \alpha_1^{(13)} = -2, \text{ all other } \alpha_1^{(ij)} \text{ are zero,}$$

$$f(1) = 64.$$

Theorem 2.31 *For the group G defined by the presentation (2.42),*

$$D_5(G) = 1.$$

Proof. With the constants $d(i)$, $e(i)$, $f(i)$, d_{ij} , described above, let

$$u_{ij}, v_{ik}, v'_{ik}, w_{ijk}, w'_{ijk}, w''_{ijk},$$

be constants satisfying the conditions (2.18)-(2.31), and let g be the corresponding element, defined by (2.17). Since the group G is nilpotent of class 3, the element g can be written as

$$g = \prod_{1 \leq i \leq j \leq s} [x_{1i}^{u_{ij} d(j)}, x_{1j}];$$

by Theorem 2.29, the fifth dimension subgroup $D_5(G)$ is generated by such elements. From the defining relations of the group G , it follows that $[x_i^{d(i)}, x_4] = 1$, $i = 1, 2, 3$; therefore,

$$g = [x_1, x_2]^{16u_{12}} [x_1, x_3]^{64u_{13}} [x_2, x_3]^{64u_{23}}.$$

Consider the condition (2.22) for the case $i = 1$, $k = 6$:

$$u_{12}b_{26} + u_{14}b_{46} + v_{16}d(1) + v'_{16}e(6) = 4u_{12} + 32u_{14} + 4v_{16} + 64v'_{16} = 0.$$

It follows that

$$u_{12} + v_{16} \equiv 0 \pmod{4}. \quad (2.43)$$

Next, consider the condition (2.30) for the case $k = 6$, we have:

$$2v_{16} + 4v_{26} + 32v_{46} \equiv 0 \pmod{64},$$

and we have

$$v_{16} + 2v_{26} \equiv 0 \pmod{16}. \quad (2.44)$$

From the condition (2.31) for the case $k = 3, l = 6$, we have:

$$2v_{13} + 4v_{23} + 32v_{43} - v_{26} + 2v_{36} \equiv 0 \pmod{4},$$

and thus we conclude that

$$v_{26} \equiv 0 \pmod{2}. \quad (2.45)$$

The conditions (2.43), (2.44), (2.45) then imply that

$$u_{12} \equiv 0 \pmod{4}. \quad (2.46)$$

It is clear from the defining relations of the group G that

$$[x_1, x_2]^{64} = [x_4, x_3, x_3]^{64} = 1.$$

Therefore,

$$g = [x_1, x_3]^{64u_{13}} [x_2, x_3]^{64u_{23}} = [x_1^{64u_{13}} x_2^{64u_{23}}, x_3] = [x_4, x_3, x_3]^{-32u_{13}-16u_{23}} = x_{31}^{-32u_{13}-16u_{23}}.$$

Now consider the condition (2.22) for the case $i = 3, k = 6$. We have

$$32u_{34} - 32u_{13} - 16u_{23} + 64v_{36} + 64v'_{36} = 0.$$

Hence,

$$32u_{34} - 32u_{13} - 16u_{23} \equiv 0 \pmod{64}. \quad (2.47)$$

Note that the condition (2.20) for the case $i = 3, j = 4$, implies that

$$u_{34} \binom{64}{2} \equiv 0 \pmod{64};$$

hence

$$u_{34} \equiv 0 \pmod{2}. \quad (2.48)$$

Congruences (2.47) and (2.48) imply that

$$32u_{13} + 16u_{23} \equiv 0 \pmod{64}.$$

Therefore, we have

$$g = [x_4, x_3, x_3]^{-32u_{13}-16u_{23}} = 1. \quad \square$$

Problem 2.32 *Is it true that $[D_5(G), G, G] = \gamma_7(G)$ for every group G ?*

2.5 Quasi-varieties of Groups

Our discussion in this and the next section follows [Mik06c].

Recall that a variety \mathcal{V} of groups is a class of groups defined by a set of identities. Let \mathcal{D}_n ($n \geq 2$) denote the class of groups with trivial n th dimension subgroup. The existence of groups without dimension property shows that \mathcal{D}_n is not a variety of groups for $n \geq 4$, since a variety of groups is always quotient closed. The classes \mathcal{D}_n , however, are quasi-varieties (Theorem 2.35). We recall in this section some of the basic notions about quasi-varieties.

Let F_∞ be a free group of countable rank with basis $\{x_1, x_2, \dots\}$ and w_1, \dots, w_k, v some words in F_∞ . A *quasi-identity* is a formal implication:

$$(w_1 = 1 \ \& \ \dots \ \& \ w_n = 1) \implies (v = 1). \quad (2.49)$$

A quasi-identity (2.49) is said to hold in a given group G if it is a true implication for every substitution $x_i = g_i$, $g_i \in G$.

A *quasi-variety* \mathcal{V}_S is a class of groups defined by a set S of quasi-identities, i.e., \mathcal{V}_S is the class of all groups in which every quasi-identity from S holds.

Example 2.33

The class \mathcal{T}_0 of all torsion-free groups is a quasi-variety; it is defined by the infinite set of quasi-identities

$$x^p = 1 \implies x = 1,$$

where p runs over the set of all primes. Trivially, \mathcal{T}_0 is not a variety.

Recall that a non-empty class \mathcal{F} of subsets of a given set I is called a *filter* on I if the following conditions are satisfied:

- (i) $\emptyset \notin \mathcal{F}$;
- (ii) $A \in \mathcal{F}, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$;
- (iii) $A \in \mathcal{F}, A \subseteq B \implies B \in \mathcal{F}$.

Let $\{A_i\}_{i \in I}$ be a family of groups indexed by the elements of a set I , and \mathcal{F} a filter on I . Let A be the Cartesian product

$$A = \prod_{i \in I} A_i.$$

For a given $a \in A$, denote by a_i the i th component of a in A . Consider the relation $\sim_{\mathcal{F}}$ on A defined by setting

$$a \sim_{\mathcal{F}} b \text{ if and only if } \{i \mid a_i = b_i\} \in \mathcal{F}, \quad a, b \in A.$$

It follows directly from the properties of a filter that this relation is, in fact, an equivalence relation. The *filtered product of the family* $\{A_i\}_{i \in I}$ *of groups, with respect to the filter* \mathcal{F} , is, by definition, the quotient group

$$\prod_{\mathcal{F}} A_i := A / \sim_{\mathcal{F}}.$$

The following result of A. I. Mal'cev gives a characterization of quasi-varieties of groups.

Theorem 2.34 (Mal'cev [Mal70]). *A class \mathcal{X} of groups is a quasi-variety if and only if it contains the trivial group and is closed under subgroups and filtered products.*

Recall that \mathcal{D}_n ($n \geq 2$) denotes the class of groups with trivial n th dimension subgroup. For $n = 2$, and 3, the class \mathcal{D}_n coincides with the variety \mathfrak{N}_n of nilpotent groups of nilpotency class $\leq n$. On the other hand, for all $n \geq 4$, as already mentioned, the existence of groups without dimension property shows that the class \mathcal{D}_n is not a variety of groups. However, there is the following result:

Theorem 2.35 (Plotkin [Plö71]). *For all $n \geq 1$, the class \mathcal{D}_n is a quasi-variety of groups.*

Proof. The fact that the class \mathcal{D}_n , $n \geq 1$, is nonempty and closed under subgroups is obvious.

Let $\{A_i\}_{i \in I}$ be a family of groups in the class \mathcal{D}_n , and let \mathcal{F} be a filter on I . Consider the Cartesian product $A = \prod_{i \in I} A_i$. Let N be the normal subgroup of A consisting of elements $(g_i)_{i \in I}$ with $J := \{i \in I \mid g_i = 1\} \in \mathcal{F}$. If $\prod_{\mathcal{F}} A_i \notin \mathcal{D}_n$, then there exists an element $g \in A$ such that

$$g - 1 \in \mathfrak{a}^n + \sum_{s \in S} (y_s - 1)\alpha_s, \quad (2.50)$$

where the sum is finite, $\alpha_s \in \mathbb{Z}[A]$, and $y_s \in N$. Define

$$J_s := \{i \in I \mid \text{the } i\text{th component of } y_s \text{ is } 1\}.$$

By definition, $J_s \in \mathcal{F}$. Since the set S in the sum (2.50) is finite, we have

$$\bar{J} = \bigcap_{s \in S} J_s \in \mathcal{F}.$$

For $j \in \bar{J}$, projecting g to the j -th component, we get $g_j \in D_n(A_j)$ and hence $g_j = 1$, $j \in J$. Consider the set

$$K := \{i \in I \mid g_i = 1\}.$$

Since $\bar{J} \subseteq K$, we conclude that $K \in \mathcal{F}$. Hence $g \in N$ and therefore, $\prod_{\mathcal{F}} A_i \in \mathcal{D}_n$. Consequently, the class \mathcal{D}_n is closed under filtered products. Hence, by Mal'cev's criterion (Theorem 2.34), the class \mathcal{D}_n is a quasi-variety. \square

In view of Theorem 2.22 the quasi-variety \mathcal{D}_4 is defined by the following implications:

Given integers k , c_i , d_{ij} ($1 \leq i, j \leq k$) and elements g_1, \dots, g_k of the group G , if the following conditions hold

- (1) $2^{c_i} d_{ij} + 2^{c_j} d_{ji} = 0$ ($1 \leq i, j \leq k$),
- (2) if $c_i = c_j$, then d_{ij} is even,
- (3) $g_i^{2^{c_i}} \in \gamma_2(G)$ ($1 \leq i \leq k$),
- (4) $\prod_{i=1}^k g_i^{2^{c_i}} d_{ij} \in \gamma_2(G)^{2^{c_j}} \gamma_3(G)$ ($1 \leq j \leq k$),

then

$$\prod_{i=1}^k \prod_{j=i+1}^k [g_i, g_j]^{2^{c_i} d_{ij}} = 1.$$

Clearly, this set of implications is equivalent to a suitable set of quasi-identities.

A quasi-variety \mathcal{Q} is said to be *finitely based* if it can be defined by a finite number of quasi-identities.

Let \mathcal{Q} be a quasi-variety of groups. Then the rank $rk(\mathcal{Q})$ of \mathcal{Q} is the minimal number n (which may be infinite) such that there exists a system of quasi-identities

$$(w_1^i = 1 \ \& \ \dots \ \& \ w_{n_i}^i = 1) \implies (v_i = 1), \quad i = 1, 2, \dots \quad (2.51)$$

such that all words w_i^j , v_i are from a free group F_n of rank n .

Example 2.36

- (i) For the quasi-variety \mathcal{T}_0 of torsion-free groups, $rk(\mathcal{T}_0) = 1$.
- (ii) The quasi-variety defined by the quasi-identity

$$([x, y]^2 = 1) \implies ([x, y] = 1)$$

clearly has rank 2.

Proposition 2.37 *Let \mathcal{Q} be a quasi-variety and G a group. Then $G \in \mathcal{Q}$ if and only if all $rk(\mathcal{Q})$ -generated subgroups of G lie in \mathcal{Q} .*

Proof. One side is clear, due to the fact that quasi-varieties are closed under the operation of taking subgroups.

Suppose G is a group such that all its $rk(\mathcal{Q})$ -generated subgroups lie in \mathcal{Q} . Consider the quasi-identity system (2.51) which defines \mathcal{Q} and the total number of variables entering in (2.51) is $rk(\mathcal{Q})$, i.e., all words w_i^j, v_i in (2.51) are from a free group of rank $rk(\mathcal{Q})$. Then (2.51) holds for any choice of elements $g_1, \dots, g_{rk(\mathcal{Q})}$ from G , since it holds for any elements from the subgroup in G generated by $g_1, \dots, g_{rk(\mathcal{Q})}$ (which is at most $rk(\mathcal{Q})$ -generated. Hence (2.51) holds for all possible substitutions of elements from G and $G \in \mathcal{Q}$ by definition. \square

The following observation is immediate:

Proposition 2.38 *If \mathcal{Q} is finitely based, then $rk(\mathcal{Q})$ is finite.*

The next result provides a method for showing that a given quasi-variety is not finitely based.

Proposition 2.39 *Let \mathcal{Q} be a quasi-variety such that there exists a sequence of finitely-generated groups $G_i, i = 1, 2, \dots$, such that the following conditions are satisfied:*

- (i) $G_i \notin \mathcal{Q}$.
- (ii) For any i there exists $f(i)$ such that all $f(i)$ -generated subgroups of G_i lie in \mathcal{Q} .
- (iii) The function $f(i)$ is not bounded, i.e., $f(i) \rightarrow \infty$ for $i \rightarrow \infty$.

Then $rk(\mathcal{Q}) = \infty$ and hence \mathcal{Q} is not finitely based.

Proof. Suppose $rk(\mathcal{Q}) < \infty$. Then, by (iii), there exists an integer i that $f(i) > rk(\mathcal{Q})$. Since every $f(i)$ -generated subgroup of G_i lies in \mathcal{Q} , every $rk(\mathcal{Q})$ -generated subgroup also lies in \mathcal{Q} . Therefore, $G_i \in \mathcal{Q}$ by Proposition 2.37; but this contradicts (i). Hence $rk(\mathcal{Q}) = \infty$, and \mathcal{Q} is not finitely based. \square

2.6 The Quasi-variety \mathcal{D}_4

For the study of the quasi-variety \mathcal{D}_4 , recall that the precise structure of the fourth dimension subgroup for finitely generated nilpotent groups of class 3 is given by Theorem 2.22. It has been shown by Mikhailov-Passi [Mik06c] that

the quasi-variety \mathcal{D}_4 is not finitely based, thus answering a problem of Plotkin ([Plo83], p. 144, Problem 12.3.2). The proof requires a technical result about certain finite groups of class 2.

Lemma 2.40 *Let n, s be natural numbers,*

$$G = \langle x_1, \dots, x_{2n} \mid x_i^s = 1 \ (1 \leq i \leq 2n) \rangle$$

and $\Pi = G/\gamma_3(G)$. *If*

$$[x_1, x_2]^k \dots [x_{2n-1}, x_{2n}]^k = [h_1, h_2] \dots [h_{2l-1}, h_{2l}], \quad (2.52)$$

with $0 < k < s$, $h_1, \dots, h_{2l} \in \Pi$, then $l \geq n$.

In particular, if H be an m -generator subgroup of Π and

$$[x_1, x_2]^k \dots [x_{2n-1}, x_{2n}]^k \in \gamma_2(H),$$

then $\binom{m}{2} \geq n$.

Proof. Suppose

$$h_i \equiv x_1^{a_{i,1}} \dots x_{2n}^{a_{i,2n}} \pmod{\gamma_2(\Pi)},$$

where $0 \leq a_{i,j} < s$, $1 \leq i \leq 2l$, $1 \leq j \leq 2n$. Substituting in the equation (2.52), we have the following equation in Π :

$$[x_1, x_2]^k \dots [x_{2n-1}, x_{2n}]^k = \prod_{1 \leq i < j \leq 2n} [x_i, x_j]^{b_{ij}}, \quad (2.53)$$

where

$$b_{ij} = \sum_{r=1}^l (a_{2r-1,i} a_{2r,j} - a_{2r-1,j} a_{2r,i}).$$

Observe that $\gamma_2(\Pi) = \prod_{1 \leq i < j \leq 2n} \langle [x_i, x_j] \rangle$ and $\langle [x_i, x_j] \rangle$ is a cyclic group of order s . Therefore, from equation (2.53), comparing the exponents of the generators $[x_i, x_j]$, $1 \leq i < j \leq 2n$ of the summands, we have:

$$b_{2t-1, 2t} \equiv k \pmod{s}, \quad 1 \leq t \leq n, \quad (2.54)$$

$$b_{i,j} \equiv 0 \pmod{s}, \quad 1 \leq i < j \leq 2n, \ (i, j) \neq (2t-1, 2t). \quad (2.55)$$

Let $M_{p,q}(\mathbb{Z}_s)$ denote the set of $p \times q$ matrices over the ring \mathbb{Z}_s of integers mod s . Let $A = (a_{i,j})_{1 \leq i \leq 2l, 1 \leq j \leq 2n} \in M_{2l, 2n}(\mathbb{Z}_s)$ and define a matrix $D \in M_{2n, 2l}(\mathbb{Z}_s)$ as follows:

$$D = (D_{p,q})_{1 \leq p \leq n, 1 \leq q \leq l},$$

where

$$D_{p,q} = \begin{pmatrix} a_{2q,2p} & -a_{2q-1,2p} \\ -a_{2q,2p-1} & a_{2q-1,2p-1} \end{pmatrix} \in M_{2,2}(\mathbb{Z}_s).$$

A straightforward verification shows that

$$DA = kI_{2n,2n},$$

where $I_{2n,2n} \in M_{2n,2n}(\mathbb{Z}_s)$ is the identity matrix, and it follows that $l \geq n$.

Next let H be an m -generator subgroup of Π . It is easy to see that every element of $\gamma_2(H)$ can be expressed as a product of at most $\binom{m}{2}$ commutators of elements in H , since H is nilpotent of class 2. The second assertion in Lemma thus follows from the preceding result. \square

Theorem 2.41 *The quasi-variety \mathcal{D}_4 is not finitely based.*

Proof. For $n \geq 5$, let $\Pi = G(n)/\gamma_4(G(n))$ be the lower central quotient of the group considered in Example 2.7. We assert that every m -generator subgroup H of Π , with $\binom{m}{2} < n$, has the property that $D_4(H) = 1$. Clearly then $rk(\mathcal{D}_4) = \infty$ (by Proposition 2.39) and the assertion in Theorem 2.41 is an immediate consequence. We continue to denote by x_i, y_i the set of generators of Π .

Let H be an m -generator subgroup of Π and h_1, \dots, h_m a set of generators of H . Assume that, modulo $\gamma_2(H)$, h_1, \dots, h_k ($k \leq m$) are of finite order and h_{k+1}, \dots, h_m are of infinite order.

For $g \in \Pi$, let \bar{g} denote the image of g in $\Pi/\gamma_2(\Pi)$ under the natural projection. Observe from the structure of Π that the torsion subgroup of $\Pi/\gamma_2(\Pi)$ is equal to

$$\langle \bar{x}_1 \rangle \oplus \langle \bar{x}_2 \rangle \oplus \langle \bar{x}_3 \rangle \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{64}.$$

By suitably replacing h_1, \dots, h_k , if necessary, we can assume that

$$h_1 = x_1^{l_{1,1}} x_2^{l_{1,2}} x_3^{l_{1,3}} \lambda_1, \quad h_2 = x_1^{l_{2,1}} x_2^{l_{2,2}} \lambda_2, \quad h_3 = x_1^{l_{3,1}} \lambda_3,$$

$$h_j = \lambda_j \quad (4 \leq j \leq k),$$

where $l_{i,j} \in \mathbb{Z}$, $\lambda_i \in H \cap \gamma_2(\Pi)$ ($1 \leq i \leq k$).

Let $d(i)$ be the order of h_i modulo $\gamma_2(H)$. Then, in particular,

$$l_{3,1}d(3) \equiv 0 \pmod{4}, \tag{2.56}$$

$$l_{2,2}d(2) \equiv 0 \pmod{16}. \tag{2.57}$$

We can assume also that

$$d(k)|d(k-1)|\dots|d(2)|d(1).$$

Therefore, by Theorem 2.22, the group $D_4(H)$ consists of the following elements:

$$w = \prod_{1 \leq i < j \leq k} [h_i^{d(i)}, h_j]^{a_{ij}},$$

where the integers a_{ij} satisfy the conditions (2.14) and (2.15).

We have, for $j \geq 4$, $[h_i^{d(i)}, h_j] = [h_i^{d(i)}, \lambda_j] = 1$; therefore,

$$w = [h_1^{d(1)}, h_2]^{a_{12}} [h_1^{d(1)}, h_3]^{a_{13}} [h_2^{d(2)}, h_3]^{a_{23}} = [h_1, h_2^{d(1)a_{12}} h_3^{d(1)a_{13}}] [h_2^{d(2)}, h_3]^{a_{23}}. \quad (2.58)$$

Since

$$y_1 = \prod_{1 < j \leq k} h_j^{d(1)a_{1j}} \in \gamma_2(\Pi)^{d(1)} \gamma_3(\Pi) \quad \text{by (2.15),}$$

we have,

$$w = [h_1, y_1] [h_2^{d(2)}, h_3]^{a_{23}} = [h_2^{d(2)}, h_3]^{a_{23}}.$$

We claim that $[h_2^{d(2)}, h_3]^{a_{23}} = 1$.

Consider the element $h_3 = x_1^{l_{3,1}} \lambda_3$. We have

$$x_1^{l_{3,1}d(3)} \lambda_3^{d(3)} = \prod_{1 \leq i < j \leq m} [h_i, h_j]^{u_{ij}} \gamma, \quad (2.59)$$

for some $\gamma \in \gamma_3(\Pi)$ and $u_{ij} \in \mathbb{Z}$.

Let E be the normal subgroup in Π generated by $x_2, x_3, Y_2, Y_3, [x_1, Y_j]$ ($j \in \{1, 4, 5\}$), $[Y_i, Y_j]$ ($i, j \in \{1, 4, 5\}, i \neq j$) and $\gamma_3(\Pi)$, where

$$Y_i = \{y_{(2i-2)n+1}, \dots, y_{2in}\}, \quad i = 1, \dots, 5.$$

Let

$$S = \langle x_1, Y_1, Y_4, Y_5 \mid x_1^4 = \xi_{1,(n)}, x_1^{-32} = \xi_{4,(n)}^{16}, x_1^{-64} = \xi_{5,(n)}^{64}, [x_1, Y_i] = 1 \ (i \in \{1, 4, 5\}), [Y_i, Y_j] = 1 \ (i, j \in \{1, 4, 5\}, i \neq j) \rangle, \quad (2.60)$$

We note that

$$\Pi/E \simeq S/\gamma_3(S).$$

Let $p : \Pi \rightarrow S/\gamma_3(S)$ be the composition of the projections $\Pi \rightarrow \Pi/E$ and $\Pi/E \rightarrow S/\gamma_3(S)$. Applying the projection p to the equation (2.59) in Π , we have the following equation in $S/\gamma_3(S)$:

$$x_1^{l_{3,1}d(3)} p(\lambda_1)^{d(3)} = \prod_{i < j} [p_1(h_i), p_1(h_j)]^{u_{ij}}, \quad (2.61)$$

Note that

$$S/\gamma_3(S) = (\langle x_1 \rangle \oplus \mathcal{Y}_1/\gamma_3(\mathcal{Y}_1) \oplus \mathcal{Y}_4/\gamma_3(\mathcal{Y}_4) \oplus \mathcal{Y}_5/\gamma_3(\mathcal{Y}_5)) / N,$$

where \mathcal{Y}_i , $1 \leq i \leq 5$ is a free group with basis Y_i ,

$$N = \langle x_{1,(n)}^4 \xi_{1,(n)}^{-1}, \xi_{1,(n)}^8 \xi_{4,(n)}^{16}, \xi_{1,(n)}^{16} \xi_{5,(n)}^{64} \rangle.$$

Therefore (2.61) implies that in the direct product

$$\mathcal{Y} := \mathcal{Y}_1/\gamma_3(\mathcal{Y}_1) \oplus \mathcal{Y}_4/\gamma_3(\mathcal{Y}_4) \oplus \mathcal{Y}_5/\gamma_3(\mathcal{Y}_5).$$

We have

$$l_{3,1}d(3) \equiv 0 \pmod{4}, \quad (2.62)$$

and

$$\xi_{1,(n)}^{\frac{l_{3,1}d(3)}{4}} \mu_1^{d(3)} (\xi_{1,(n)}^8 \xi_{4,(n)}^{16})^{k_1} (\xi_{1,(n)}^{16} \xi_{5,(n)}^{64})^{k_2} = \prod_{1 \leq i < j \leq m} [z_i, z_j]^{u_{ij}}, \quad (2.63)$$

for some integers k_1, k_2 and elements $\mu_1 \in \gamma_2(\mathcal{Y})$, $z_i \in \mathcal{Y}$, $1 \leq i \leq m$. Projecting (2.63) to each of the three summands of \mathcal{Y} we have the following three equations:

$$\xi_{1,(n)}^{d_1} \mu_{1,1}^{d(3)} = \prod_{1 \leq i < j \leq m} [z_{i,1}, z_{j,1}]^{u_{ij}}, \text{ in } \mathcal{Y}_1/\gamma_3(\mathcal{Y}_1), \quad d_1 = \frac{l_{3,1}d(3)}{4} + 8k_1 + 16k_2, \quad (2.64)$$

$$\xi_{4,(n)}^{d_4} \mu_{1,4}^{d(3)} = \prod_{1 \leq i < j \leq m} [z_{i,4}, z_{j,4}]^{u_{ij}}, \text{ in } \mathcal{Y}_4/\gamma_3(\mathcal{Y}_4), \quad d_4 = 16k_1, \quad (2.65)$$

$$\xi_{5,(n)}^{d_5} \mu_{1,5}^{d(3)} = \prod_{1 \leq i < j \leq m} [z_{i,5}, z_{j,5}]^{u_{ij}}, \text{ in } \mathcal{Y}_5/\gamma_3(\mathcal{Y}_5), \quad d_5 = 64k_2, \quad (2.66)$$

for some $\mu_{1,l} \in \gamma_2(\mathcal{Y}_l)/\gamma_3(\mathcal{Y}_l)$, $z_{i,l} \in \mathcal{Y}_l/\gamma_3(\mathcal{Y}_l)$, $1 \leq i \leq m$, $l \in \{1, 4, 5\}$.

Case (a): $l_{3,1}$ is odd.

In view of (2.62), we have $d(3) = 4s$ for some integer s . Let

$$Z_i = \langle Y_i \mid y_i^{4s} = 1 \ (y_i \in Y_i), \gamma_3(\mathcal{Y}_i) \rangle,$$

and $p_i : \mathcal{Y}_i/\gamma_3(\mathcal{Y}_i) \rightarrow Z_i$ be the natural projection, $i \in \{1, 4, 5\}$. Projecting the equations (2.64), (2.65) and (2.66) into Z_1, Z_4, Z_5 respectively, we conclude, by an application of Lemma 2.40, that

$$d_i \equiv 0 \pmod{4s} \ (i \in \{1, 4, 5\}).$$

From (2.64) and (2.65), we therefore have

$$l_{3,1}s + 8k_1 + 16k_2 \equiv 0 \pmod{4s}, \quad (2.67)$$

$$16k_1 \equiv 0 \pmod{4s}. \quad (2.68)$$

It follows easily that $s \equiv 0 \pmod{16}$, and consequently,

$$d(3) \equiv 0 \pmod{64}.$$

Let $d(3) = 64f$, $f \in \mathbb{Z}$, and suppose $d(2) = d(3)c$ ($c \in \mathbb{Z}$) (recall that $d(3)|d(2)$). Then we have

$$\begin{aligned} w &= [h_2^{d(2)}, h_3]^{a_{23}} = [h_2, h_3^{d(3)}]^{ca_{23}} = \\ &= [h_2, x_1^{64l_{3,1}f} \lambda_3^{64f}]^{ca_{23}} = [h_2^{16}, x_1^{4l_{3,1}f} \lambda_3^{4f}]^{ca_{23}}. \end{aligned}$$

Since $h_2^{16} \in \gamma_2(\Pi)$, it follows that $w = 1$.

Case (b): $l_{3,1} = 2l$ and l is odd. We assert that in this case

$$d(3) \equiv 0 \pmod{16}. \quad (2.69)$$

Since $x_1^{2ld(3)} \in \gamma_2(\Pi)$, we have $d(3) = 2r$ for some $r > 0$. Projecting the square of the equation (2.64) to Z_1 under the map p_1 , we conclude, by an application of Lemma 2.40, that $2d_1 \equiv 0 \pmod{4r}$.

Therefore we have

$$2d_1 = 2lr + 16k_1 + 32k_2 \equiv 0 \pmod{4r},$$

which implies that $r \equiv 0 \pmod{8}$, and consequently, we have (2.69).

Now consider the element $h_2 = x_1^{l_2,1} x_2^{l_2,2} \lambda_2$. We have

$$h_2^{d(2)} = (x_1^{l_2,1} x_2^{l_2,2})^{d(2)} \lambda_2^{d(2)} = \prod_{1 \leq i < j \leq m} [h_i, h_j]^{v_{ij}} \gamma, \quad (2.70)$$

for some $\gamma \in \gamma_3(\Pi)$ and $v_{ij} \in \mathbb{Z}$.

Let I be the normal subgroup in Π generated by $x_1, x_3, Y_1, Y_4, [x_2, Y_j], j \in \{2, 3, 5\}, [Y_i, Y_j] (i, j \in \{2, 3, 5\}, i \neq j)$ and $\gamma_3(\Pi)$. Let

$$Q = \langle x_2, Y_2, Y_3, Y_5 \mid x_2^{16} = \xi_{2,(n)}, \xi_{2,(n)}^2 = \xi_{3,(n)}^4, \xi_{2,(n)}^8 = \xi_{5,(n)}^{64} \rangle.$$

Note that $\Pi/I \simeq Q/\gamma_3(Q)$ and

$$Q/\gamma_3(Q) \simeq (\langle x_2 \rangle \oplus \mathcal{Y}_2/\gamma_3(\mathcal{Y}_2) \oplus \mathcal{Y}_3/\gamma_3(\mathcal{Y}_3) \oplus \mathcal{Y}_5/\gamma_3(\mathcal{Y}_5))/M,$$

where $M = \langle x_2^{16} \xi_{2,(n)}^{-1}, \xi_{2,(n)}^2 \xi_{3,(n)}^{-4}, \xi_{2,(n)}^8 \xi_{5,(n)}^{-64} \rangle$. Let $q : \Pi \rightarrow Q$ be the natural projection. Applying q to the equation (2.70), we have the following equation

$$q(x_2)^{l_{2,2}d(2)}q(\lambda_2)^{d(2)} = \prod_{1 \leq i < j \leq m} [q(h_i), q(h_j)]^{v_{ij}} \quad (2.71)$$

in the group $Q/\gamma_3(Q)$. This equation implies that, in the direct product

$$\mathcal{V} := \mathcal{Y}_2/\gamma_3(\mathcal{Y}_2) \oplus \mathcal{Y}_3/\gamma_3(\mathcal{Y}_3) \oplus \mathcal{Y}_5/\gamma_3(\mathcal{Y}_5),$$

we have (using (2.57))

$$\xi_{2,(n)}^{\frac{l_{2,2}d(2)}{16}} \mu_2^{d(2)} (\xi_{2,(n)}^2 \xi_{3,(n)}^{-4})^{m_1} (\xi_{2,(n)}^8 \xi_{5,(n)}^{64})^{m_2} = \prod_{1 \leq i < j \leq m} [v_i, v_j]^{v_{ij}}, \quad (2.72)$$

for some integers m_1, m_2 and the elements $\mu_2 \in \gamma_2(\mathcal{V})$, $v_i \in \mathcal{V}$, $1 \leq i \leq m$. Projecting (2.72) to the first summand of \mathcal{V} , we have the following equation:

$$\xi_{2,(n)}^{e_1} \mu_{2,1}^{d(2)} = \prod_{1 \leq i < j \leq m} [v_{i,1}, v_{j,1}],$$

where

$$e_1 = \frac{l_{2,2}d(2)}{16} + 2m_1 + 8m_2,$$

and $\mu_{2,1} \in \gamma_2(\mathcal{Y}_2)/\gamma_3(\mathcal{Y}_2)$, $v_{i,1} \in \mathcal{Y}_2/\gamma_3(\mathcal{Y}_2)$, $1 \leq i \leq m$. Since $d(3)|d(2)$, therefore $d(2) = 16t$ for some t . An application of Lemma 2.40 once again shows that $e_1 \equiv 0 \pmod{16t}$; consequently, $l_{2,2}t$ is even and so $l_{2,2}d(2) = 32f$ for some f . Hence

$$\begin{aligned} w &= [h_2^{d(2)}, h_3]^{a_{23}} = [(x_1^{l_{2,1}} x_2^{l_{2,2}} \lambda_2)^{d(2)}, x_1^{2l}]^{a_{23}} = [x_1^{l_{2,1}} x_2^{l_{2,2}} \lambda_2, x_1^{l_{3,1}d(2)}]^{a_{23}} \\ &= [x_2^{l_{2,2}}, x_1^{l_{3,1}d(2)}]^{a_{23}} = [x_2, x_1^{l_{3,1}l_{2,2}d(2)}]^{a_{23}} = [x_2, x_1^{64lf}]^{a_{23}} \\ &= [x_2, \xi_{1,(n)}^{16lf}]^{a_{23}} = [x_2^{16}, \xi_{1,(n)}^{lf}]^{a_{23}} = 1. \end{aligned}$$

Case (c): $l_{3,1} \equiv 0 \pmod{4}$. In this case $h_3 \in \gamma_2(\Pi)$, since $x_1^4 \in \gamma_2(\Pi)$; therefore, $w = 1$.

Thus, in all cases, $w = 1$, and consequently, $D_4(H) = 1$. This completes the proof. \square

2.7 Dimension Quotients

If G is a finite p -group, p odd, then $D_4(G) = \gamma_4(G)$ [Pas68a]. Refuting the long standing *dimension conjecture* that $D_n(G) = \gamma_n(G)$ always, Rips [Rip72] constructed a 2-group (Example 2.1) with $D_4(G) \neq \gamma_4(G) = 1$. Extending

these results N. Gupta has shown that odd prime power groups have the dimension property [Gup02] and, for every $n \geq 4$, there exist 2-groups with $D_n(G) \neq \gamma_n(G)$ [Gup90]. For odd prime p , the dimension property was earlier shown to hold for metabelian p -groups by Gupta [Gup91b] and for centre-by-metabelian p -groups by Gupta-Gupta-Passi [Gup94]. The result for odd prime power groups is an immediate consequence of the following result.

Theorem 2.42 (N. Gupta [Gup02]). *The n th dimension quotient of a finite nilpotent group has exponent dividing 2^l , where l is the least natural number such that $2^l \geq n$.*

Let $n \geq 3$ be an arbitrary but *fixed* integer and let G be a finite nilpotent group with $\gamma_n(G) = 1$. Choose a non-cyclic free presentation (see [Mag66], Theorem 3.5, p. 140)

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1,$$

where F is the free group with basis $\{x_1, \dots, x_m\}$, $m \geq 2$, and R is the normal closure in F of the set of relators $\{x_1^{e(1)}\xi_1, \dots, x_m^{e(m)}\xi_m\} \cup T$ such that $e(i) > 1$, $\xi_i \in [F, F]$ and T is a finite subset of $[F, F]$.

Let l be the least positive integer such that $2^l \geq n$. Let

$$G = \delta_0(G) \supseteq \delta_1(G) \dots \supseteq \delta_{l-1}(G) \supseteq \delta_l(G) = 1$$

be the derived series of G . Then $\delta_k(G) \simeq \delta_k(F)R/R$, $0 \leq k \leq l-1$, and therefore we can have a presentation

$$1 \rightarrow R^{(k)} \rightarrow F^{(k)} \rightarrow \delta_k(G) \rightarrow 1$$

where $F^{(k)}$ is a free subgroup of the k th derived subgroup $\delta_k(F)$ of F with ordered basis $B(k) = \{x_{k,1}, \dots, x_{k,m_k}\}$, $m_k \geq 2$, $R^{(k)}$ is the normal closure in $F^{(k)}$ of the set of relators $\{x_{k,1}^{e(k,1)}\xi_{k,1}, \dots, x_{k,m_k}^{e(k,m_k)}\xi_{k,m_k}\} \cup T_k$ with $e(k,i) > 1$, $\xi_{k,i} \in [F^{(k)}, F^{(k)}]$ and $T_k \subset [F^{(k)}, F^{(k)}]$ a finite subset. Furthermore, it is possible to define a weight function and a weight-preserving order on the set $\cup_k B(k)$. To this end, we need the following basic results.

Lemma 2.43 *If S is a set of generators of a free group F which is linearly independent modulo $[F, F]$, then S is a basis of F .*

Proof. Let X be a set equinumerous with S and $\alpha : X \rightarrow S$ a bijection. Let \mathfrak{F} be the free group on X . Then the map α extends to a homomorphism $\bar{\alpha} : \mathfrak{F} \rightarrow F$. Since S generates F and is linearly independent modulo $[F, F]$, the homomorphism $\bar{\alpha}$ is an epimorphism and the induced homomorphism $\mathfrak{F}/[\mathfrak{F}, \mathfrak{F}] \rightarrow F/[F, F]$ is an isomorphism. By Theorem 1.76, the induced homomorphisms $\mathfrak{F}/\gamma_m(\mathfrak{F}) \rightarrow F/\gamma_m(F)$, $m \geq 2$, are all isomorphisms, since, both F and \mathfrak{F} being free, $H_2(\mathfrak{F}) = H_2(F) = 0$. Hence $\ker(\bar{\alpha}) \subseteq \gamma_\omega(\mathfrak{F}) = 1$. It thus follows that $\bar{\alpha}$ is an isomorphism, and so S is a free set of generators of F . \square

Lemma 2.44 *Let B be an ordered basis of a free group F . Then the basic commutators*

$$C(t) = [y_1, y_2, \dots, y_t], \quad y_i \in B, \quad t \geq 2,$$

satisfying $y_1 > y_2 \leq y_3 \leq \dots \leq y_t$ are linearly independent modulo $\delta_2(F)$. \square

Proof. Let $\mathfrak{a} = \mathbb{Z}[F]\Delta(\delta_1(F))$. Consider the Magnus embedding

$$\theta : \delta_1(F)/\delta_2(F) \rightarrow \mathfrak{f}/\mathfrak{fa}, \quad x\delta_2(F) \mapsto (x-1) + \mathfrak{fa}, \quad x \in \delta_1(F). \quad (2.73)$$

Suppose we have an inclusion

$$\prod_{i=1}^m \mathbf{y}_i^{n(\mathbf{y}_i)} \in \delta_2(F), \quad (2.74)$$

where \mathbf{y}_i , $i = 1, 2, \dots, m$, are left-normed commutators $[y_{i1}, y_{i2}, \dots, y_{it_i}]$ satisfying $y_{i1} > y_{i2} \leq \dots \leq y_{it_i}$. On applying θ , we then have

$$\sum_{i=1}^m ([y_{i1}, y_{i2}] - 1)(y_{i3} - 1) \dots (y_{it_i} - 1) \equiv 0 \pmod{\mathfrak{fa}}. \quad (2.75)$$

Since \mathfrak{f} is a free right $\mathbb{Z}[F]$ -module with basis $B - 1$, it follows that

$$\sum n(\mathbf{y}_i)(y_{i2} - 1) \dots (y_{im} - 1) \equiv 0 \pmod{\mathfrak{a}},$$

where the sum is taken over all i for which the first entry y_{i1} in \mathbf{y}_i is the same. Since the elements $(y_1 - 1)(y_2 - 1) \dots (y_r - 1)$, $y_1 \leq y_2 \leq \dots \leq y_r$ with y_i 's in B are linearly independent modulo \mathfrak{a} , it follows that $n(\mathbf{y}_i) = 0$ for all $i = 1, 2, \dots, m$. \square

The chain

$$F = F^{(0)} \supset F^{(1)} \supset \dots \supset F^{(l)} = \{1\}, \quad (2.76)$$

can be constructed inductively as follows. Let the basis $\{x_1, \dots, x_m\}$ of $F = F^{(0)}$ be renamed as $B(0) = \{x_{0,1}, \dots, x_{0,m_0}\}$ by defining $m_0 = m$ and setting $x_{0,1} = x_1, \dots, x_{0,m_0} = x_m$. To each basis element $x_{0,i}$ in $B(0)$, we assign weight 1:

$$\text{wt}(x_{0,i}) = 1 \text{ for } i = 1, \dots, m_0.$$

Having defined, for $k \geq 1$, the subgroup $F^{(k-1)}$ with an ordered basis

$$B(k-1) = \{x_{k-1,1}, \dots, x_{k-1,m_{k-1}}\}$$

satisfying $x_{k-1,i} < x_{k-1,i+1}$ and $\text{wt}(x_{k-1,i}) < n$ for $i = 1, \dots, m_{k-1}$, to define the subgroup $F^{(k)}$ with a *weight preserving* ordered basis, list the finite set $B(k)$ of *all* left-normed basic commutators of the form

$$C(t) = [y_1, y_2, \dots, y_t], y_i \in B(k-1), t \geq 2, \quad (2.77)$$

satisfying $y_1 > y_2 \leq \dots \leq y_t$ and $\text{wt}(y_1) + \dots + \text{wt}(y_t) < n$. Let $F^{(k)}$ be the subgroup generated by $B(k)$. By Lemmas 2.43 and 2.44 the commutators $C(t)$ constitute a free basis of $F^{(k)}$. Now define

$$\text{wt}(C(t)) = \text{wt}(y_1) + \dots + \text{wt}(y_t).$$

Define any weight-preserving order relation on the set $B(k)$ and relabel its elements following this order to obtain the basis

$$B(k) = \{x_{k,1}, \dots, x_{k,m(k)}\} \quad (2.78)$$

of the subgroup $F^{(k)}$.

Let $k \in \{0, 1, \dots, l-1\}$ be arbitrary but fixed. In the free group rings $\mathbb{Z}[F^{(k)}]$ set

$$\begin{aligned} \mathbf{r}^{(k)} &= \mathbb{Z}[F^{(k)}](R^{(k)} - 1), \\ \mathbf{f}^{(n,k)} &= \mathbb{Z}\text{-span}\{(y_1^{\pm 1} - 1) \dots (y_t^{\pm 1} - 1) \mid t \geq 2\} \\ &\text{with } y_i \in B(k) \text{ satisfying } \text{wt}(y_1) + \dots + \text{wt}(y_t) \geq n. \end{aligned} \quad (2.79)$$

Next, define the k th *partial dimension subgroup* by

$$D_{(n)}(R^{(k)}) = F^{(k)} \cap (1 + \mathbf{r}^{(k)} + \mathbf{f}^{(n,k)}) \quad (2.80)$$

and the k th *partial lower central subgroup* $\gamma_{(n)}(F^{(k)})$ to be the normal closure of the set

$$\{[y_1, \dots, y_t], y_i \in B(k), t \geq 2, y_1 > y_2 \leq \dots \leq y_t\},$$

where $\text{wt}(y_1) + \dots + \text{wt}(y_t) \geq n$ and $\text{wt}(y_1) + \dots + \text{wt}(y_{t-1}) < n$. We thus have the following subnormal chain of subgroups:

$$D_{(n)}(R^{(0)}) \supseteq D_{(n)}(R^{(1)}) \supseteq \dots \supseteq D_{(n)}(R^{(l)}) = 1 \quad (2.81)$$

where clearly $R^{(k)}\gamma_{(n)}(F^{(k)}) \leq D_{(n)}(R^{(k)})$.

The main result in [Gup02] is the following

Theorem 2.45 *For each $k \in \{0, 1, \dots, l-1\}$,*

$$D_{(n)}(R^{(k)})^2 \subseteq R^{(k)}\gamma_{(n)}(F^{(k)})D_{(n)}(R^{(k+1)}).$$

Theorem 2.42 is an immediate consequence of the above result. For, let $w \in F \cap (1 + \mathbf{r} + \mathbf{f}^n)$. Then $w - 1 \in \mathbf{r}^{(0)} + \mathbf{f}^{(n,0)}$ and $w \in D_{(n)}(R^{(0)})$.

Theorem 2.45 implies that there exist elements

$$g_0 \in R^{(0)}\gamma_{(n)}(F^{(0)}), g_1 \in R^{(1)}\gamma_{(n)}(F^{(1)}), \dots, g_{l-1} \in R^{(l-1)}\gamma_{(n)}(F^{(l-1)})$$

such that

$$(\dots((w^2 g_0)^2 g_1)^2 \dots)^2 g_{l-1} = 1$$

and, since $R^{(k)}\gamma_{(n)}(F^{(k)}) \subseteq R\gamma_n(F)$ for each k , Theorem 2.42 follows.

If G is a group whose lower central factors $\gamma_n(G)/\gamma_{n+1}(G)$ are all torsion-free, then G has the dimension property (see [Pas79], p. 48). Thus, in particular, free nilpotent groups and the free poly-nilpotent groups have the dimension property.

Theorem 2.46 (Kuz'min [Kuz96]). *If G is an extension of a group whose lower central quotients are torsion-free by an abelian group, then G has the dimension property.*

It is known [Gup73] that the lower central factors of the free centre-by-metabelian group are, in general, not torsion-free. However, we have the following

Theorem 2.47 (Gupta-Levin [Gup86]). *Free centre-by-metabelian groups have the dimension property.*

Let \mathfrak{f} be the augmentation ideal of the free group ring $\mathbb{Z}[F]$. For $c \geq 1$, let \mathfrak{a}_c be the ideal $\mathbb{Z}[F](\gamma_c(F) - 1)$.

Theorem 2.48 (Gupta-Gupta-Levin [Gup87b]). *For all $n, c \geq 1$,*

$$F \cap (1 + \mathfrak{f}\mathfrak{a}_c + \mathfrak{f}^{n+1}) = [\gamma_c(F), \gamma_c(F)]\gamma_{n+1}(F).$$

In particular, the groups $F/[\gamma_c(F), \gamma_c(F)]$, $c \geq 1$, have the dimension property.

For $c = 2$, the above result was proved earlier by Gupta [Gup82].

Theorem 2.49 (Gupta-Kuz'min). *For any $n \geq 1$ and a group G , the subquotient group $D_n(G)/\gamma_{n+1}(G)$ is abelian.*

Proof. Let G be a nilpotent of class n . We have to show that $D_n(G)$ is abelian. Let A be a maximal abelian normal subgroup of G . It is easy to show that A coincides with its centralizer $C_G(A)$. We can view A as a G -module via conjugation. Then for any $k \geq 1$, we have

$$a \circ (g - 1) \subseteq \gamma_{k+1}(G), \quad g \in D_k(G).$$

In particular, any $g \in D_n(G)$ lies in $C_G(A)$. Therefore $D_n(G) \subseteq C_G(A)$ and hence $D_n(G)$ is an abelian group. \square

2.8 Plotkin's Problems

The following problems have been raised and discussed by Plotkin in [Plo73] (see also Hartley [Har84]).

Problem 2.50 *For every group G , is it true that $D_\omega(G) = \gamma_\omega(G)$.*

Problem 2.51 *Is it true that for every nilpotent group G , there exists an integer $n(G)$ such that $D_{n(G)}(G) = 1$? In other words, does every nilpotent group have finite dimension series?*

Plotkin conjectures that problem 2.50 has an affirmative answer.

Theorem 2.52 (Hartley [Har82c]). *If G is a nilpotent group in which the torsion subgroup has finite dimension series, then G itself has finite dimension series.*

For a group G , let $s(G)$ denote the least natural number n , if it exists, such that $D_n(G) = 1$, and infinity otherwise. Let \mathfrak{N}_c denote the variety of nilpotent groups of class $\leq c$. It is easy to see that *finitely generated nilpotent groups and torsion-free nilpotent groups have finite dimension series.*

Let c be a natural number and suppose that every group in \mathfrak{N}_c has finite dimension series. Then there exists a natural number $r = r(c)$ such that $D_r(G) = 1$ for every $G \in \mathfrak{N}_c$. For, if not, then we can find groups in \mathfrak{N}_c having arbitrarily long dimension series. Choose groups G_1, G_2, \dots in \mathfrak{N}_c so that G_i has dimension series of length $\geq i$. Then the group $\Gamma = \bigoplus_{i=1}^{\infty} G_i$, is in \mathfrak{N}_c , but its dimension series does not terminate with identity in a finite number of steps. A standard reduction argument (see [Pas68a]) shows that if $s = s(c)$ is a number such that, for every finite p -group $G \in \mathfrak{N}_c$, $D_s(G) = 1$, then, for every group $\Gamma \in \mathfrak{N}_c$, $D_s(\Gamma) = 1$.

Lemma 2.53 *Let $H \triangleleft G$ and suppose that*

$$[H, {}_m G] := [\dots [H, \underbrace{G, G, \dots, G}_{m \text{ terms}}], G] = 1.$$

Let M be a right G -module such that $M \cdot \mathfrak{g}^r \subseteq M \cdot \mathfrak{h}$ for some integer $r \geq 1$. Then $M \cdot \mathfrak{g}^{rn} \subseteq M \cdot \mathfrak{h}^n$ for all $n \geq 1$.

Proof. We proceed by induction on $m \geq 1$. If $m = 1$, then H is a central subgroup. Therefore, repeated use of $M \cdot \mathfrak{g}^r \subseteq M \cdot \mathfrak{h}$ gives the required inclusion:

$$M \cdot \mathfrak{g}^{rn} \subseteq M \cdot \mathfrak{h}^n.$$

Now suppose $m > 1$ and the result holds for $m - 1$. Let $K = [H, {}_{m-1} G]$ and consider the groups $\bar{H} = H/K$ and $\bar{G} = G/K$. Note that $\bar{H} \triangleleft \bar{G}$ and

$[\bar{H}, {}_{m-1}\bar{G}] = 1$. The quotient $\bar{M} = M/M.\mathfrak{k}$ is a \bar{G} -module under the action induced by that of M as a G -module and

$$\bar{M}.\bar{\mathfrak{g}}^r \subseteq \bar{M}.\bar{\mathfrak{h}}.$$

Therefore, by induction hypothesis,

$$\bar{M}.\bar{\mathfrak{g}}^{r.n^{m-1}} \subseteq \bar{M}.\bar{\mathfrak{h}}^n,$$

for all $n \geq 1$. This implies that

$$M.\mathfrak{g}^{rn^{m-1}} \subseteq M.\mathfrak{h}^n + M.\mathfrak{k}.$$

Since K is a central subgroup of G , iteration gives

$$M.\mathfrak{g}^{rn^m} \subseteq M.\mathfrak{h}^n,$$

and the proof is complete. \square

Lemma 2.54 *Let G be a group, and suppose that $H \triangleleft G$, $G = HF$ for some finite p -group $F \subseteq G$, $[H, {}_m G] = 1$ for some integer $m \geq 1$. Then for every $r \geq 1$, there exists $u = u(r)$ such that*

$$\mathbb{Z}[H] \cap \mathfrak{g}^u \subseteq \mathfrak{h}^r.$$

Proof. Let $D = H \cap F$. Then D is a finite p -group. Let $r \geq 1$ be given. Choose $s \geq 1$ such that $p^s \mathfrak{d} \subseteq \mathfrak{d}^r \subseteq \mathfrak{h}^r$. Observe that $\mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{f} \mathbb{Z}[H]$ is a right ideal of $\mathbb{Z}[G]$. Consider the right G -module $M = \mathbb{Z}[G]/(\mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{h} \mathbb{Z}[H])$. Since G/H is a finite p -group, there exists $n \geq 1$ such that $\mathfrak{g}^n \subseteq \mathfrak{h} \mathbb{Z}[G] + p^s \mathfrak{f} \mathbb{Z}[H]$. Hence, by Lemma 2.53, we can conclude that there exists an integer $u = u(r) \geq 1$ such that $M.\mathfrak{g}^u \subseteq M.\mathfrak{h}^r$, i.e.,

$$\mathfrak{g}^u \subseteq \mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{f} \mathbb{Z}[H].$$

Intersecting with $\mathbb{Z}[H]$ we get

$$\mathbb{Z}[H] \cap \mathfrak{g}^u \subseteq \mathbb{Z}[H] \cap (\mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{f} \mathbb{Z}[H]). \quad (2.82)$$

If T is a transversal for D in F including 1, then by the choice of s , we have

$$\mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{f} \mathbb{Z}[H] = \mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{t} \mathbb{Z}[H],$$

where \mathfrak{t} is the additive subgroup of $\mathbb{Z}[G]$ generated by $t - 1$, $t \in T$. Let $\theta : \mathbb{Z}[G] \rightarrow \mathbb{Z}[H]$ be the linear extension of the map $G \rightarrow H$ given by $g = th \mapsto h$ ($t \in T$, $h \in H$). Applying θ to the inclusion (2.82) we get

$$\mathbb{Z}[H] \cap (\mathfrak{h}^r \mathbb{Z}[G] + p^s \mathfrak{f} \mathbb{Z}[H]) = \mathfrak{h}^r.$$

Hence $\mathbb{Z}[H] \cap \mathfrak{g}^u \subseteq \mathfrak{h}^r$. \square

Theorem 2.55 (Kuskulei, see [Plo73]). *If G is a nilpotent group having a subgroup H of finite index whose dimension series is finite, then G has finite dimension series.*

Proof. It clearly suffices to consider the case when $H \triangleleft G$ and G/H is a cyclic group of prime order, p say. If the torsion subgroup T of G lies in H , then T has finite dimension series and therefore, by Theorem 2.52, G has finite dimension series. If $T \not\subseteq H$, then H has a supplement of p -power order in G , and Lemma 2.54 implies that G has finite dimension series. \square

Theorem 2.56 (Tokarenko and Rips [Plo73]). *If a semi-direct product $G = H \rtimes K$ is nilpotent and both H and K have finite dimension series, then G has finite dimension series and $s(G) \leq \max(s(H)^c, s(K))$.*

Proof. Regard $\mathbb{Z}[H]$ as a right G -module as follows. For $\alpha \in \mathbb{Z}[H]$, $g = hk \in G$, $h \in H$, $k \in K$, define

$$\alpha.g = \alpha^k h,$$

where α^k stands for the element of $\mathbb{Z}[H]$ obtained on conjugating by k each element in the support of α . Then, as can be seen by induction on the class of G ,

$$\mathbb{Z}[H].\mathfrak{g}^{m^c} \subseteq \mathfrak{h}^m.$$

Since K has finite dimension series, $D_n(G) \subseteq H$ for $n \geq s(K)$. Let $n \geq \max(s(H)^c, s(K))$ and $x \in D_n(G)$. Then $x - 1 \in \mathbb{Z}[H] \cap \mathfrak{g}^{s(H)^c}$. Hence

$$1.(x - 1) \in \mathfrak{h}^{s(H)}.$$

However, under the G -module action we are considering, $1.(x - 1) = x - 1$. Therefore, it follows that $x - 1 \in \mathfrak{h}^{s(H)}$, and consequently $x = 1$, showing that G has finite dimension series with $s(G) \leq \max(s(H)^c, s(K))$. \square

Corollary 2.57 (Valenza [Val80]). *If G is a nilpotent group and $G = H \rtimes K$ with K abelian, then $s(G)$ is bounded by a function of $s(H)$ and the class of G .*

A group G is said to satisfy the minimal condition on subgroups if each nonempty collection of subgroups contains a minimal element; or, equivalently, each descending chain of subgroups stabilizes after a finite number of steps. A solvable group satisfies the minimum condition on subgroups if and only if it is an extension of a direct product of finitely many quasicyclic groups by a finite group (see [Rob95, p. 156]). Thus, in view of Theorem 2.55, we have:

Proposition 2.58 *Every nilpotent group which satisfies minimum condition on subgroups has finite dimension series.*

An A_3 -group, in the notation of Mal'cev [Mal56], is an abelian group G whose periodic part P satisfies the minimum condition on subgroups and the quotient G/P has finite rank. A nilpotent A_3 -group is a nilpotent group having a finite normal series in which the factor groups are A_3 -groups. Clearly, the torsion subgroup of a nilpotent A_3 -group satisfies the minimum condition on subgroups and therefore, by Proposition 2.58, the torsion part, and hence by Theorem 2.52, the group itself has finite dimension series:

Theorem 2.59 (Plotkin [Pl73]). *A nilpotent A_3 -group has finite dimension series.*

2.9 Modular Dimension Subgroups

In contrast to the case of integral dimension subgroups, definitive answer for the identification of dimension subgroups over fields has long been known. To state the result we need the following definitions, given a group G and a prime p :

(i) Define the series $\{M_{n,p}(G)\}_{n \geq 1}$ by setting

$$M_{1,p}(G) = G, \quad M_{2,p}(G) = \gamma_2(G), \quad M_{n+1,p}(G) = [G, M_{n,p}(G)]M_{\binom{n}{p},p}^p(G) \quad (2.83)$$

for $n \geq 2$, where $\binom{r}{s}$ denotes the least integer $\geq \frac{r}{s}$.

(ii) Define the series $\{G_{n,p}\}_{n \geq 1}$ by setting

$$G_{n,p} = \prod_{ip^j \geq n} \gamma_i(G)^{p^j}. \quad (2.84)$$

If H is a subset of a group G , we denote by \sqrt{H} the radical of H :

$$\sqrt{H} = \{x \in G \mid x^m \in H, \text{ for some } m > 0\}. \quad (2.85)$$

Theorem 2.60 (Jennings, [Jen41], [Jen55]). *Let F be a field and G a group.*

(i) *If $\text{char}(F) = 0$, then $D_{n,F}(G) = \sqrt{\gamma_n(G)}$ for all $n \geq 1$.*

(ii) *If $\text{char}(F) = p > 0$, then $D_{n,F}(G) = M_{(n),p}(G) = G_{n,p}$ for all $n \geq 1$.*

Over general rings, it is known that the dimension subgroups of groups depend only on the ones over the rings \mathbb{Z}_n , $n \geq 0$ (see Passi [Pas79], p. 16 for details). We mention here a few results in low dimensions.

Theorem 2.61 (Moran [Mor70]; see also Tasić [Tas93]). *For every group G , prime p and integer $e \geq 1$,*

$$D_{n, \mathbb{Z}_{p^e}}(G) = G^{p^e} \gamma_n(G) \text{ for } 1 \leq n \leq p.$$

Let n be a non-negative integer; if n is even, let $n = 2qm$, where q is a power of 2 and m is odd. Let

$$K_n(G) = \begin{cases} G^m \gamma_3(G), & \text{if } n \text{ is odd or } 0, \\ (G^m \gamma_3(G)) \cap \langle x^{2q} \mid x^q \in G^{2q} \gamma_2(G) \rangle \gamma_3(G), & \text{if } n \text{ is even.} \end{cases}$$

Let $N/K_n(G)$ be the subgroup of the centre of $G/K_n(G)$ consisting of the elements of order dividing n .

Theorem 2.62 (Passi-Sharma [Pas74]).

- (i) $G \cap (1 + \Delta_{\mathbb{Z}_n}^3(G) + \Delta_{\mathbb{Z}_n}(G)\Delta_{\mathbb{Z}_n}(N)) = K_n(G)$ if n is odd or 0.
- (ii) $G \cap (1 + \Delta_{\mathbb{Z}_n}^3(G) + \Delta_{\mathbb{Z}_n}(G)\Delta_{\mathbb{Z}_n}(N)) = K_n(G)\langle x^n \mid x^{qm} \in N \rangle$ if n is even.
- (iii) $G \cap (1 + \Delta_{\mathbb{Z}_n}^3(G)) = K_n(G)$ for all n .

2.10 Lie Dimension Subgroups

Given a multiplicative group G and a commutative ring R with identity, define ideals $\Delta_R^{(n)}(G)$, $n \geq 1$, inductively by setting $\Delta_R^{(1)}(G) = \Delta_R(G)$, the augmentation ideal of the group ring $R[G]$, and

$$\Delta_R^{(n)}(G) = [\Delta_R^{(n-1)}(G), \Delta_R(G)]R[G], \quad n > 1, \quad (2.86)$$

the two-sided ideal of $R[G]$ generated by $[\alpha, \beta] = \alpha\beta - \beta\alpha$, $\alpha \in \Delta_R^{(n-1)}(G)$, $\beta \in \Delta_R(G)$. We then have a decreasing series

$$\Delta_R(G) = \Delta_R^{(1)}(G) \supseteq \Delta_R^{(2)}(G) \supseteq \dots \supseteq \dots \Delta_R^{(n)}(G) \supseteq \dots$$

of two-sided ideals in $R[G]$; this series has the property that

$$\Delta_R^{(m)}(G) \cdot \Delta_R^{(n)}(G) \subseteq \Delta_R^{(n+m-1)}(G) \quad (2.87)$$

for all $m, n \geq 1$ (see [Pas79], Prop. 1.7 (iii), p.4). Let

$$D_{(n), R}(G) = G \cap (1 + \Delta_R^{(n)}(G)), \quad n \geq 1.$$

We call $D_{(n),R}(G)$ the n th upper Lie dimension subgroup of G over R . In view of (2.87), $\{D_{(n),R}(G)\}_{n \geq 1}$ is a central series in G . When $R = \mathbb{Z}$, we drop the suffix and write simply $D_{(n)}(G)$ instead of $D_{(n),\mathbb{Z}}(G)$.

Let L be a Lie ring. For subsets H, K of L , we denote by $[H, K]$ the additive subgroup of L spanned by the commutators $[h, k] = hk - kh$, $h \in H, k \in K$. Recall that the lower central series $\{L_n\}_{n \geq 1}$ of L is defined inductively by setting $L_1 = L$, and $L_{n+1} = [L, L_n]$ for $n \geq 1$. The Lie ring L is said to be *nilpotent* if $L_n = 0$ for some $n \geq 1$.

Let A be an associative ring. We can view A as a Lie ring with the bracket operation defined by

$$[\alpha, \beta] = \alpha\beta - \beta\alpha, \quad \alpha, \beta \in A.$$

Define a series of two-sided ideals $\{A^{[n]}\}_{n \geq 1}$ of A by setting $A^{[1]} = A$ and $A^{[n]}$, $n > 1$, to be the two-sided ideal of A generated by the n th term A_n in the lower central series of A viewed as a Lie ring. We say that A is *Lie nilpotent* (resp. *residually Lie nilpotent*) if $A^{[n]} = 0$ for some $n \geq 1$ (resp. $\bigcap A^{[n]} = 0$).

Theorem 2.63 (Gupta-Levin [Gup83]). *Let A be an associative ring with identity and let $\mathcal{U} = \mathcal{U}(A)$ be its group of units. Then*

$$A^{[m]}.A^{[n]} \subseteq A^{[m+n-2]} \text{ for all } m, n \geq 2.$$

Let G be a multiplicative group and R a commutative ring with identity. Consider the series $\{\Delta_R^{[n]}(G)\}_{n \geq 1}$ of two-sided ideals in $R[G]$. Clearly

$$\Delta_R^{[n]}(G) \subseteq \Delta^{(n)}(G) \text{ for all } n \geq 1,$$

and, by Theorem 2.63, we have

$$\Delta^{[n]}(G)\Delta^{[m]}(G) \subseteq \Delta^{[n+m-2]}(G)\mathbb{Z}[G] \quad (2.88)$$

for every group G . The filtration $\{\Delta_R^{[n]}(G)\}_{n \geq 1}$ of $\Delta_R(G)$ defines a series of normal subgroups $\{D_{[n],R}(G)\}_{n \geq 1}$ in G :

$$D_{[n],R}(G) = G \cap (1 + \Delta_R^{[n]}(G)). \quad (2.89)$$

We call $D_{[n],R}(G)$, $n \geq 1$, the n th lower Lie dimension subgroup of G over R . As usual, when the ring R is \mathbb{Z} , we drop the suffix R and write $D_{[n]}(G)$ instead of $D_{[n],\mathbb{Z}}(G)$.

From definitions, and in view of Theorems 1.6 and 2.63, it is then clear that for any group G and integer $n \geq 1$, we have the following inclusions:

$$\gamma_n(G) \subseteq D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G). \quad (2.90)$$

In general, not only the inclusion $\gamma_n(G) \subseteq D_n(G)$ can be strict, but even the inclusion $\gamma_n(G) \subseteq D_{[n]}(G)$ can be strict. To this end, we have

Theorem 2.64 *Let s be an arbitrary natural number. Then there exists a natural number n and a nilpotent group G of class n , such that $D_{[n+s]}(G) \neq 1$.*

We first prove two lemmas.

Lemma 2.65 *Let Π be a group, $k \gg l \gg 4$. If $x_1, x_2, x_3 \in \gamma_m(\Pi)$ and there exist $\xi_i \in \gamma_n(\Pi)$, $i = 1, \dots, 5$, $n \geq 2m$, $m \geq 3$, such that*

$$x_1^4 = \xi_1, x_2^{2^l} = \xi_2, x_2^{2^{l+1}} x_3^{2^k} = \xi_3, x_1^{-2^{l+1}} x_3^{2^{k+1}} = \xi_4, x_1^{2^k} x_2^{2^{k+1}} = \xi_5,$$

then

$$w = [x_1, x_2^{2^{l+1}}][x_1, x_3^{2^k}][x_2, x_3^{2^{k+1}}] \in D_{[n+2m-6]}(\Pi).$$

Proof. Since $1 - x \in \Delta^{[n]}(\Pi)\mathbb{Z}[\Pi]$ for $x \in \gamma_n(\Pi)$, we have

$$1 - w \equiv \alpha_1 + \alpha_2 + \alpha_3 \pmod{\Delta^{[2n]}(\Pi)\mathbb{Z}[\Pi]},$$

where $\alpha_1 = (1 - [x_1, x_2^{2^{l+1}}])$, $\alpha_2 = (1 - [x_1, x_3^{2^k}])$, $\alpha_3 = (1 - [x_2, x_3^{2^{k+1}}])$. Now, working modulo $\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]$, we have

$$\begin{aligned} \alpha_1 &\equiv (1 + (x_1^{-1}x_2^{-2^{l+1}} - 1))((1 - x_2^{2^{l+1}})(1 - x_1) - (1 - x_1)(1 - x_2^{2^{l+1}})) \\ &\equiv (1 - x_2^{2^{l+1}})(1 - x_1) - (1 - x_1)(1 - x_2^{2^{l+1}}), \end{aligned}$$

since $x_1^{-1}x_2^{-2^{l+1}} \in \gamma_m(\Pi)$ and

$$(x_1^{-1}x_2^{-2^{l+1}} - 1)((1 - x_2^{2^{l+1}})(1 - x_1) - (1 - x_1)(1 - x_2^{2^{l+1}})) \in \Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]$$

by (2.88). Modulo $\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]$, we have:

$$\begin{aligned} \alpha_1 &\equiv (1 - x_2^{2^{l+1}})(1 - x_1) - 2^{l+1}(1 - x_1)(1 - x_2) \\ &\quad + \sum_{i=2}^{n-1} (-1)^i \binom{2^{l+1}}{i} (1 - x_1)(1 - x_2)^i \end{aligned}$$

Note that $\binom{2^{l+1}}{i}$ is divisible by 4^n for sufficiently large l and $i \leq n$. Hence, for such a large l ,

$$\sum_{i=2}^{n-1} (-1)^i \binom{2^{l+1}}{i} (1 - x_1)(1 - x_2)^i \in \Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi].$$

By the same principle, we get

$$2^{l+1}(1 - x_1)(1 - x_2) \equiv (1 - x_1^{2^{l+1}})(1 - x_2) \pmod{\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]}.$$

Therefore,

$$\alpha_1 \equiv (1 - x_2^{2^{l+1}})(1 - x_1) - (1 - x_1^{2^{l+1}})(1 - x_2) \pmod{\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]}.$$

Choosing k to be such that $\binom{2^{k+1}}{i}$ is divisible by 2^{ln} for any $i \leq n$, we get

$$\begin{aligned} \alpha_2 &\equiv (1 - x_3^{2^k})(1 - x_1) - (1 - x_1^{2^k})(1 - x_3) \pmod{\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]}, \\ \alpha_3 &\equiv (1 - x_3^{2^{k+1}})(1 - x_2) - (1 - x_2^{2^{k+1}})(1 - x_3) \pmod{\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]}. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &\equiv (2 - x_2^{2^{l+1}} - x_3^{2^k})(1 - x_1) + (x_1^{2^{l+1}} - x_3^{2^{k+1}})(1 - x_2) + \\ &(x_1^{2^k} + x_2^{2^{k+1}} - 2)(1 - x_3) \equiv (1 - \xi_3^4)(1 - x_1) + (1 - \xi_4^{2^l})(1 - x_2) + (1 - \xi_5^{2^k})(1 - x_3) \equiv \\ &(1 - \xi_3)(1 - \xi_1) + (1 - \xi_4)(1 - \xi_2) + (1 - \xi_5)(1 - \xi_3) \equiv 0 \pmod{\Delta^{[n+2m-6]}(\Pi)\mathbb{Z}[\Pi]}, \end{aligned}$$

and hence $w \in D_{[n+2m-6]}(\Pi)$. \square

Lemma 2.66 *Let $W_{m,n}$ be the group given by the following presentation:*

$$\begin{aligned} \langle x_1, \dots, x_{14} \mid [x_1, mx_{11}]^4 \xi_1, [x_2, mx_{12}]^{2^l} \xi_2, [x_3, mx_{13}]^{2^k} \xi_3, \\ [x_4, nx_{14}, x_{10}]^4 [x_4, nx_{14}, x_3, mx_{13}, x_7]^{2^{k-1}}, \xi_1^{2^{k-2}} \xi_2^{2^{k-l+1}} \rangle, \end{aligned}$$

where

$$\begin{aligned} \xi_1 &= [x_4, nx_{14}, x_7]^2 [x_4, nx_{14}, x_6] [x_4, nx_{14}, x_5]^2, \\ \xi_2 &= [x_4, nx_{14}, x_7]^{2^{l-2}} [x_4, nx_{14}, x_{10}]^{-1} [x_4, nx_{14}, x_5]^2, \\ \xi_3 &= [x_4, nx_{14}, x_6]^{-2^{l-2}} [x_4, nx_{14}, x_{10}]^{-2}. \end{aligned}$$

Then the element

$$w_{n,m} = [x_1, mx_{11}, [x_2, mx_{12}]^{2^{l+1}}] [x_1, mx_{11}, [x_3, mx_{13}]^{2^k}] [x_2, mx_{12}, [x_3, mx_{13}]^{2^{k+1}}]$$

does not lie in $\gamma_{n+m+4}(W_{n,m})$, $n \geq m \geq 0$.

Proof. Let F be a free group with basis $\{x_1, \dots, x_{10}\}$. Consider four types of relations:

$$R_1 = \gamma_4(F),$$

$$\begin{aligned} R_2 = \langle R_1, [x_i, x_j, x_k] \notin \langle \alpha, \beta, \gamma, \delta, \epsilon, \theta \rangle R_1 \text{ for all } i, j, k, \\ \alpha^{2^{k-l}} \beta^{-1}, \beta^{2^{l-2}} \gamma^{-1}, \gamma^4, \beta\epsilon, \alpha\delta, \theta\gamma \rangle, \end{aligned}$$

where

$$\begin{aligned}\alpha &= [x_4, x_3, x_7], \beta = [x_4, x_2, x_6], \gamma = [x_4, x_{10}, x_1], \delta = [x_4, x_7, x_3], \\ \epsilon &= [x_4, x_6, x_2], \theta = [x_4, x_1, x_{10}];\end{aligned}$$

Now define R_3 to be the subgroup generated by R_2 together with the normal closure of the following words:

$$\begin{aligned}[x_3, x_4]^{2^k}, [x_2, x_4]^{2^l}, \\ [x_4, x_1]^4, [x_2, x_3]^{2^l} \alpha^{-2^{l-2}}, \\ [x_3, x_1]^4 \alpha^{-2}, [x_2, x_1]^4 \beta^{-1}, \\ [x_4, x_5]^{2^{k-l+2}} \alpha^{-2^{k-1}}, [x_4, x_7]^{2^k} \gamma^2, \\ [x_4, x_6]^{2^{k-2}}, [x_4, x_{10}]^4 \gamma^2, \\ [x_5, x_i], i \neq 1, [x_1, x_i], [x_2, x_i], [x_3, x_i], i > 4.\end{aligned}$$

Let R_4 be the subgroup generated by R_3 and the normal closure of words

$$\begin{aligned}c_1 &= x_1^4 [x_4, x_7]^2 [x_4, x_6] [x_4, x_5]^2, \\ c_2 &= x_2^{2^l} [x_4, x_7]^{2^{l-2}} [x_4, x_{10}]^{-1} [x_4, x_5]^2, \\ c_3 &= x_3^{2^k} [x_4, x_6]^{-2^{l-2}} [x_4, x_{10}]^{-2}.\end{aligned}$$

Set $H = F/R_4$. We claim that $[R_{i+1}, F] \subseteq R_i$, $i = 1, 2, 3$. This is obvious for $i = 1, 2$. We show it for $i = 3$. Working modulo R_3 , we have:

$$\begin{aligned}[c_1, x_1] &= 1, \\ [c_1, x_2] &= [x_1, x_2]^4 [x_4, x_6, x_2] = [x_1, x_2]^4 \beta = 1, \\ [c_1, x_3] &= [x_1, x_3]^4 [x_4, x_7, x_3]^2 = [x_1, x_3]^4 \alpha^2 = 1, \\ [c_1, x_4] &= [x_1, x_4]^4 = 1, \\ [c_2, x_1] &= [x_2, x_1]^{2^l} [x_4, x_{10}, x_1]^{-1} = \beta^{2^{l-2}} \gamma^{-1} = 1, \\ [c_2, x_2] &= 1, \\ [c_2, x_3] &= [x_2, x_3]^{2^l} [x_4, x_7, x_3]^{2^{l-2}} = [x_2, x_3]^{2^l} \alpha^{-2^{l-2}} = 1, \\ [c_2, x_4] &= [x_2, x_4]^{2^l} = 1, \\ [c_3, x_1] &= [x_3, x_1]^{2^k} [x_4, x_{10}, x_1]^{-2} = \alpha^{2^{k-1}} \gamma^{-2} = 1, \\ [c_3, x_2] &= [x_3, x_2]^{2^k} [x_4, x_6, x_2]^{-2^{l-2}} = \alpha^{2^{k-2}} \beta^{-2^{l-2}} = 1, \\ [c_3, x_3] &= 1, \\ [c_3, x_4] &= [x_3, x_4]^{2^k} = 1.\end{aligned}$$

By standard arguments, one can show that the element

$$w = [x_1, x_2^{2^{l+1}}] [x_1, x_3^{2^k}] [x_2, x_3^{2^{k+1}}]$$

is nontrivial in H . Note that all brackets $[x_j, x_i, x_j]$ are trivial in H .

Let W be a group given by the following presentation:

$$\langle x_1, \dots, x_{10} \mid x_1^4 \xi_1, x_2^{2^l} \xi_2, x_3^{2^k} \xi_3, \\ [x_4, x_{10}]^4 [x_4, x_3, x_7]^{2^{k-1}}, \xi_1^{2^{k-2}} \xi_2^{2^{k-l+1}} \rangle,$$

where

$$\xi_1 = [x_4, x_7]^2 [x_4, x_6] [x_4, x_5]^2, \\ \xi_2 = [x_4, x_7]^{2^{l-2}} [x_4, x_{10}]^{-1} [x_4, x_5]^2, \\ \xi_3 = [x_4, x_6]^{-2^{l-2}} [x_4, x_{10}]^{-2}.$$

It is easy to see that the group $W_{0,0}$ is a free product of W with a free group of rank 5. The group W naturally maps onto H , and $W_{0,0}$ maps onto G_2 . The image of $w_{0,0}$ is exactly the element w which is nontrivial, hence $w_{0,0} \notin \gamma_4(W_{0,0})$.

We shall prove first that $w_{m,m} \notin \gamma_{2m+4}(W_{m,m})$, i.e., the case $n = m$. For any m consider the quotient $W'_{m,m} = W_{m,m} / \gamma_{2m+4}(W_{m,m}) N_m$, where N_m is the normal closure in $W_{m,m}$ of brackets $[y_1, \dots, y_t]$, $t \geq 3$, such that there are at least two occurrences of $y_i = x_1$ or $y_i = x_2$, or $y_i = x_3$, or $y_i = x_4$ in this bracket, or at least three occurrences of elements from $\{x_1, x_2, x_3, x_4\}$ simultaneously. We see that all such brackets are trivial in H , hence $w_{0,0}$ is nontrivial in $W'_{0,0}$.

We assume that the element $w_{m,m}$ is nontrivial in $W'_{m,m}$ for a given m and we shall prove the statement for $m + 1$.

Consider the following automorphism f of the free group of rank 14:

$$x_i \mapsto x_i, \quad i \neq 11, 12, 13, 14, \\ x_{11} \mapsto x_{11} x_1, \\ x_{12} \mapsto x_{12} x_2, \\ x_{13} \mapsto x_{13} x_3, \\ x_{14} \mapsto x_{14} x_4.$$

Clearly, this automorphism can be extended to get an automorphism f' of a group $W'_{m,m}$. This automorphism defines the semi-direct product

$$W''_{m,m} = W'_{m,m} \rtimes \langle x \rangle,$$

where x acts as f' . Clearly, we have in $W''_{m,m}$:

$$[x, x_i] = 1, \quad i \neq 11, 12, 13, 14, \\ [x_{11}, x] = x_1, [x_{12}, x] = x_2, [x_{13}, x] = x_3, [x_{14}, x] = x_4.$$

It is easy to see that $W''_{m,m}$ is nilpotent with $\gamma_{2m+6}(W''_{m,m}) = 1$. Note that $W''_{m,m}$ is an epimorphic image of $W_{m+1,m+1}$; thus the image of the element $w_{m+1,m+1}$ is nontrivial in $W''_{m,m}$, since it is the same as the element $w_{m,m}$ in $W''_{m,m}$. Thus we have proved that $w_{m+1,m+1} \notin \gamma_{2m+6}(W'_{m+1,m+1})$. The induction is thus complete and we have

$$w_{m,m} \notin \gamma_{2m+4}(W'_{m,m})$$

for any $m \geq 0$.

Now we shall prove the needed result for general case $n \geq m \geq 0$. We fix m and make an induction on $t = n - m$. For the case $t = 0$ we already proved the needed result. We consider the quotient $W'_{n,m} = W_{n,m} / \gamma_{n+m+4}(W_{n,m})N_m$. Consider the following automorphism f' of a free group on generators x_i :

$$\begin{aligned} x_i &\mapsto x_i, \quad i \neq 14, \\ x_{14} &\mapsto x_{14}x_4. \end{aligned}$$

Clearly, it extends to an automorphism of the group $W'_{n,m}$. Then the corresponding semi-direct product $W''_{n,m} = W_{n,m} \rtimes x$ is nilpotent with $\gamma_{n+m+5}(W''_{n,m}) = 1$. Observe that $W'_{n+1,m}$ naturally maps onto $W''_{n,m}$, sending non-trivially the element $w_{n+1,m}$. Hence $w_{n+1,m} \notin \gamma_{n+m+5}(W'_{n+1,m})$ and we have thus completed the induction. \square

Proof of Theorem 2.64. By Lemma 2.65, for $k \gg l \gg 4$, we have

$$w_{n,m} \in D_{[n+2m-2]}(W_{n,m}) \setminus \gamma_{n+m+4}(W_{n,m}).$$

Since the difference $(n+2m-2) - (n+m+4) = m-6$ can be taken arbitrarily, the statement of the Theorem 2.64 follows. \square

When R is a field, upper Lie dimension subgroups have been identified in [Pas75b] (see also [Pas79]). To state the result we need the following definitions, given a group G and a prime p :

(i) Define the series $\{M_{(n),p}(G)\}_{n \geq 1}$ by setting $M_{(1),p}(G) = G$, $M_{(2),p}(G) = \gamma_2(G)$, $M_{(n+1),p}(G) = [G, M_{(n),p}(G)]M_{(\frac{n+p}{p},p)}^p(G)$ for $n \geq 2$, where $(\frac{r}{s})$ denotes the least integer $\geq \frac{r}{s}$.

(ii) Define the series $\{G_{(n),p}\}_{n \geq 1}$ by setting

$$G_{(n),p} = \prod_{(i-1)p^j \geq n} \gamma_i(G)^{p^j}.$$

Theorem 2.67 (Passi-Sehgal [Pas75b]). *Let G be a group and F a field. Then, for $n \geq 2$,*

$$D_{(n), F}(G) = \begin{cases} \sqrt{\gamma_n(G)} \cap \gamma_2(G), & \text{if } \text{char}(F) = 0, \\ G_{(n-1), p} = M_{(n), p}(G), & \text{if } \text{char}(F) = p > 0. \end{cases}$$

Some very interesting properties of lower central and dimension subgroups have been observed by A. Shalev [Sha90a]. To mention a sample, let us adopt the following

Notation. For integers $n \geq 1$, $k \geq 0$, write

$$D_{n, k}(G) = \prod_{ip^j \geq n} \gamma_{i+k}^{p^j}(G).$$

Proposition 2.68 (Shalev [Sha90a].) *For integers $n \geq 1$, $k \geq 0$,*

$$[D_{n, k}(G), G] = D_{n, k+1}(G).$$

We next consider the lower Lie dimension subgroups in characteristic $p > 0$. An identification of these subgroups is known when $p \neq 2, 3$. First note the following

Proposition 2.69 *The series $\{D_{[n], \mathbb{F}_p}(G)\}_{n \geq 1}$ is a central series of G satisfying*

- (i) $[D_{[m], \mathbb{F}_p}(G), D_{[n], \mathbb{F}_p}(G)] \subseteq D_{[m+n-2], \mathbb{F}_p}(G)$, $m, n \geq 2$.
- (ii) $(D_{[n], \mathbb{F}_p}(G))^p \subseteq D_{[p(n-2)+2], \mathbb{F}_p}(G)$, $n \geq 2$.

Theorem 2.70 (Bhandari-Passi [Bha92b]), Riley [Ril91]. *For every group G and field F with $\text{char}(F) \neq 2, 3$,*

$$D_{[n], F}(G) = D_{(n), F}(G) \text{ for all } n \geq 1.$$

Theorem 2.71 (Bhandari-Passi [Bha92b]). *Let G be a group. Then for all $n \geq 0$*

- (i) $D_{[2^{n+2}], \mathbb{F}_2}(G) = D_{(2^{n+2}), \mathbb{F}_2}(G)$;
- (ii) $D_{[a3^{n+2}], \mathbb{F}_3}(G) = D_{(a3^{n+2}), \mathbb{F}_3}(G)$, $0 \leq a \leq 2$.

As a result of Theorems 2.70 and 2.71, we have

Corollary 2.72 *The following statements for a group algebra $F[G]$ are equivalent:*

- (i) $F[G]$ is residually Lie nilpotent.
- (ii) $\bigcap_{n \geq 1} D_{[n], F}(G) = 1$.
- (iii) Either $\text{char}(F)$ is zero and G is residually “nilpotent with derived group torsion free”, or $\text{char}(F) = p > 0$ and G is residually “nilpotent with derived group a p -group of bounded exponent”.

We end this section with a review of the results on integral Lie dimension subgroups.

Theorem 2.73 $D_{(n)}(G) = \gamma_n(G)$ for $1 \leq n \leq 8$.

The above result for $1 \leq n \leq 6$ is due to Sandling [San72a] and the cases $n = 7, 8$ are due to Gupta-Tahara [Gup93].

Theorem 2.74 (Gupta-Srivastava [Gup91c]). *In general,*

$$D_{[n], \mathbb{Z}}(G) \neq \gamma_n(G) \text{ for } 9 \leq n \leq 13.$$

Theorem 2.75 (Hurley-Sehgal [Har91b]). *In general,*

$$D_{[n], \mathbb{Z}}(G) \neq \gamma_n(G) \text{ for } n \geq 14,$$

and

$$D_{(n), \mathbb{Z}}(G) \neq \gamma_n(G) \text{ for } n \geq 9.$$

2.11 Lie Nilpotency Indices

Theorem 2.76 (Passi, Passman and Sehgal [Pas73]). *The group algebra $F[G]$ of a group G over a field F is Lie nilpotent if and only if either the characteristic of F is zero and G is abelian, or the characteristic of F is a prime p , G is nilpotent and G' , the derived subgroup of G , is a finite p -group.*

As a consequence of the above theorem, we have

Corollary 2.77 *The following two statements are equivalent:*

- (i) $F(G)^{(m)} = 0$ for some $m \geq 1$.
- (ii) $F(G)^{[n]} = 0$ for some $n \geq 1$.

For a Lie nilpotent group algebra $F[G]$, define the *upper and lower Lie nilpotency indices* $t^L(F[G])$ and $t_L(F[G])$ as follows:

$$t^L(F[G]) = \min\{m \mid F[G]^{(m)} = 0\},$$

$$t_L(F[G]) = \min\{m \mid F[G]^{[m]} = 0\}.$$

Clearly $t_L(F[G]) \leq t^L(F[G])$, and by Theorem 2.63, the unit group $\mathcal{U}(F[G])$ is nilpotent of class c , say, with $c + 1 \leq t_L(F[G])$. In fact, in view of a result of Du [Du,92], $c + 1 = t_L(F[G])$ (see Theorems 2.79, 2.80 below).

Recall that a ring R is said to be a Jacobson radical ring if, for every $r \in R$, there exists $s \in R$ such that

$$r + s - rs = 0 = r + s - sr.$$

Let R be a Jacobson radical ring. Define a binary operation on R by setting

$$a \circ b = a + b - ab, \quad a, b \in R.$$

With this binary operation, R is a group, called the *adjoint group* of R ; we denote this group by (R, \circ) .

Example 2.78

Let G be a finite p -group and F a field of characteristic p . Then the augmentation ideal $\Delta_F(G)$ is nilpotent; therefore $\Delta_F(G)$ is a Jacobson radical ring. Observe that the group $(\Delta_F(G), \circ)$ is isomorphic to the group $\mathcal{U}_1(F[G])$ of units of augmentation 1 under the map

$$\alpha \mapsto 1 - \alpha, \quad \alpha \in \Delta_F(G).$$

Theorem 2.79 *If G is a finite p -group, F a field of characteristic p , then*

$$\text{nilpotency class of } \mathcal{U}(F[G]) = t_L(F[G]).$$

This result is an immediate consequence of the following

Theorem 2.80 (Du [Du,92]). *The associated Lie ring of a Jacobson radical ring is nilpotent of class n if and only if its adjoint group is nilpotent of class n .*

Theorem 2.81 (Bhandari-Passi [Bha92a]). *Let F be a field of characteristic $p > 3$ and let G be a group such that $F[G]$ is Lie nilpotent. Then*

$$t_L(F[G]) = t^L(F[G]) = 2 + (p-1) \sum_{m \geq 1} md_{(m+1)},$$

where, for $m \geq 2$, $p^{d(m)} = [D_{(m), F}(G) : D_{(m+1), F}(G)]$.

The proof of the above theorem requires the following results of Sharma-Srivastava. Following their notation, let $L_n(R)$ denote the n th term in the lower central series of the ring R when viewed as a Lie ring under commutation.

Theorem 2.82 (Sharma-Srivastava [Sha90b], Theorem 2.8). *Let R be a ring in which both 2, 3 are invertible. If m and n are any two positive integers such that one of them is odd, then*

$$L_m(R)RL_n(R)R \subseteq L_{m+n-1}(R)R.$$

Lemma 2.83 (Sharma-Srivastava [Sha90b], Lemma 2.11). *Let R be a ring in which both 2, 3 are invertible. Then for any positive integers m and n and for all $g_1, g_2, \dots, g_m \in \mathcal{U}(R)$,*

$$([g_1, g_2, \dots, g_{m+1}] - 1)^n \in L_{mn+1}(R)R.$$

Proof of Theorem 2.81. Since $F[G]$ is Lie nilpotent, by Theorem 2.76, G is nilpotent and its derived subgroup G' is a finite p -group. If G is abelian, then the assertion is obviously true; thus we assume that $G' \neq 1$.

Let $H_i = D_{(i+1), F}(G)$, $i \geq 1$, and $p^{e_i} = [H_i : H_{i+1}]$ so that $e_i = d_{(i+1)}$. The series

$$G = H_1 \supset H_2 \supset \dots \supset H_d \supset H_{d+1} = 1$$

is a restricted N -series in G' , i.e.,

$$[H_i, H_j] \subseteq H_{i+j}, \quad H_i \subseteq H_{ip}, \quad \text{for all } i, j \geq 1.$$

By Theorem 2.67,

$$H_n = \prod_{(i-1)p^j \geq n} \gamma_i(G)^{p^j}.$$

Now observe that H_n is generated, modulo H_{n+1} , by the elements of the type x^{p^j} , where x is a left-normed group commutator of weight i and $(i-1)p^j = n$. Thus it is possible to choose a canonical basis (see [Pas79], p. 23) $\{x_{11}, x_{12}, \dots, x_{1e_1}, x_{21}, x_{22}, \dots, x_{2e_2}, \dots, x_{d1}, x_{d2}, \dots, x_{de_d}\}$ of G' , where for $1 \leq r \leq d$, $1 \leq k \leq e_r$, x_{rk} is an element of the type $\xi_i^{p^j}$, where ξ_i is a left-normed group commutator of weight i and $(i-1)p^j = r$. It then follows that the element

$$\alpha = (x_{11} - 1)^{(p-1)}(x_{12} - 1)^{(p-1)} \dots (x_{1e_1} - 1)^{(p-1)} \dots \\ (x_{d1} - 1)^{(p-1)} \dots (x_{de_d} - 1)^{(p-1)} \quad (2.91)$$

is a non-zero element of $F[G]$. For $1 \leq r \leq d$, $1 \leq k \leq e_r$, $x_{rk} = \xi_i^{p^j}$, by Lemma 2.83, we have

$$(x_{rk} - 1)^{(p-1)} = (\xi_i - 1)^{p^j(p-1)} \in F[G]^{[(i-1)p^j(p-1)+1]} = F[G]^{[r(p-1)+1]}.$$

Moreover, by Theorem 2.82

$$(x_{r1} - 1)^{(p-1)}(x_{r2} - 1)^{(p-1)} \dots (x_{re_r} - 1)^{(p-1)} \in F[G]^{[re_r(p-1)+1]},$$

which in turn yields that $\alpha \in F[G]^{[1+(p-1)\sum_{r=1}^d re_r]}$. Since $\alpha \neq 0$, it follows that

$$t_L(F[G]) \geq 2 + (p-1) \sum_{r=1}^d re_r = 2 + (p-1) \sum_{m \geq 1} md_{(m+1)},$$

as $e_r = d_{(r+1)}$ for $r \geq 1$.

Since, $t^L(F[G]) = 2 + (p-1) \sum_{r=1}^d re_r = 2 + (p-1) \sum_{m \geq 1} md_{(m+1)}$ (see [Pas79], p. 47) and $t^L(F[G]) \geq t_L(F[G])$ always, the proof is complete. \square

Let p be a prime and R a ring of characteristic p . Consider the Lie powers $R^{(m)}$, $m \geq 1$, of R defined inductively by setting $R^{(1)} = R$, and, for $m \geq 1$,

$R^{(m+1)}$ to be the two-sided ideal of R generated by the ring commutators $xy - yx$, $x \in R^{(m)}$, $y \in R$.

Theorem 2.84 (Shalev [Sha91]). *If $R^{(1+(p-1)p^{i-1})} = 0$ for some $i \geq 1$, then R satisfies the identity*

$$(X + Y)^{p^i} = X^{p^i} + Y^{p^i}. \quad (P_i)$$

Corollary 2.85 *Let G be a finite group of exponent p^e , and K a field of characteristic p . If $K[G]^{(1+(p-1)p^{e-1})} = 0$, then $\exp(\mathcal{U}_1(K[G])) = \exp(G)$, where $\mathcal{U}_1(K[G])$ is the group of units of $K[G]$ having augmentation 1.*

Proof. Since $G \subseteq \mathcal{U}_1(K[G])$, it only needs to be checked that $\exp(\mathcal{U}_1(K[G])) \leq \exp(G)$.

Let $u = \sum a_j g_j \in \mathcal{U}_1(K[G])$, $a_j \in K$, $g_j \in G$. Since, by Theorem 2.84, $K[G]$ satisfies the identity (P_i) , we have

$$u^{p^e} = \sum (a_j g_j)^{p^e} = \sum a_j^{p^e} \cdot 1 = 1. \quad \square$$

In [Sha91] Shalev shows that the hypothesis of the above Corollary is satisfied if $p \geq 7$ and $\exp(G)^3 > |G|$, and thus we have

Theorem 2.86 (Shalev [Sha91]). *Let K be a field of prime characteristic p , and G a finite p -group. Then G and $\mathcal{U}_1(K[G])$ have the same exponent if $p \geq 7$ and $\exp(G)^3 > |G|$.*

2.12 Subgroups Dual to Dimension Subgroups

Let G be a group and R a commutative ring with identity. Define a series $\{Z_n(R[G])\}_{n \geq 0}$ of two-sided ideals in $R[G]$ inductively by setting

$$Z_0(R[G]) = 0$$

and

$$Z_{n+1}(R[G]) = \{\alpha \in R[G] \mid \alpha(g-1) \in Z_n(R[G]), (g-1)\alpha \in Z_n(R[G])\}$$

for $n \geq 0$. This ascending series $\{Z_n(R[G])\}_{n \geq 0}$ of two-sided ideals of $R[G]$ is the most rapidly ascending among all ascending series stabilized by G ; it defines an ascending series $\{\mathfrak{z}_{n,R}(G)\}_{n \geq 0}$ of normal subgroups of G , by setting

$$\mathfrak{z}_{n,R}(G) := G \cap (1 + Z_n(R[G])).$$

The series $\{\mathfrak{z}_{n,R}(G)\}_{n \geq 0}$ has been investigated by R. Sandling [San72b]; it is in a sense dual to the dimension series $\{D_{n,R}(G)\}_{n \geq 1}$ defined by the series of

augmentation powers $\Delta_R^n(G)$, $n \geq 1$, which is the most rapidly descending among the descending series stabilized by G . The subgroups $\mathfrak{z}_{n,R}(G)$, $n \geq 1$, are rarely non-trivial. More precisely, we have

Theorem 2.87 (Sandling [San72b]). *The normal subgroup $\mathfrak{z}_{n,R}(G)$ is non-trivial for some $n \geq 1$ if and only if the group G is a finite p -group and the ring R is of characteristic p^e for some $e \geq 1$.*

Proof. Suppose $N := \mathfrak{z}_{n,R}(G)$ has a non-trivial element g , say, for some $n \geq 1$. Then, by definition of $Z_n(R[G])$, $(g-1)\Delta_R^n(G) = 0$. This is not possible if G is infinite. Hence G must be finite. Now $\Delta_R^{n+1}(N) = 0$. Therefore, there exists a prime p such that N a finite p -group and R has characteristic p^e for some $e \geq 1$. If $1 \neq h \in G$ is a p' -element, then $(g-1).(h-1)^n = 0$. It then follows that $(g-1)(h-1) = 0$ and hence $h = 1$, a contradiction. Hence G is a finite p -group. \square

Let R be a commutative ring with identity and G a group. Let $W = R \wr G$, the standard wreath product of the abelian group R and the group G . Let

$$J_n(R, G) := \{\alpha \in R[G] \mid \alpha.\Delta_R^n(G) = 0\},$$

i.e., the left annihilator of $\Delta_R^n(G)$. The subgroups $\mathfrak{z}_n(R, G)$ are related to the upper central series $\{\zeta_n(W)\}_{n \geq 0}$ of W . More precisely, there is the following result which is easily proved by induction.

Theorem 2.88 (Sandling [San72b]). *For all natural numbers n ,*

$$\zeta_n(W) = \mathfrak{z}_{n-1}(R, G)J_n(R, G).$$