Chapter 2
Problem Setting

In this chapter we introduce the notation and the preliminaries to rigorously set the problem of optimal networks. The formulation in the sense of L. Kantorovich, by using *transport plans*, i.e. measures on the product space $\Omega \times \Omega$, will be presented together with a second equivalent formulation where the main tools are the so-called *transport path measures* that are measures on the family of curves in Ω. This seems to be a very natural formulation that has already been used in previous papers (see for instance [24, 65, 6, 58]) and that allows to obtain in a rather simple way existence results and necessary conditions of optimality.

2.1 Notation and Preliminaries

In this monograph the ambient space Ω is assumed to be a bounded, closed, $N-$dimensional convex subset of $\mathbb{R}^N$, $N \geq 2$, equipped with the Euclidean distance; the convexity assumption is made here only for simplicity of presentation; in fact, all the results are still valid in the more general case of bounded Lipschitz domains. For any pair of Lipschitz paths θ_1, $\theta_2 : [0,1] \to \Omega$, we introduce the distance

$$\begin{aligned} d_\Theta(\theta_1, \theta_2) := \inf \Big\{ & \max_{t\in[0,1]} |\theta_1(t) - \theta_2(\varphi(t))| \,, \\ & \varphi : [0,1] \to [0,1] \text{ increasing and bijective} \Big\}, \end{aligned} \tag{2.1}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$. We define then Θ as the set of the equivalence classes of Lipschitz paths in Ω parametrized over $[0,1]$, where two paths θ_1 and θ_2 are considered equivalent whenever $d_\Theta(\theta_1, \theta_2) = 0$: it is easily noticed that Θ is a separable metric space equipped with the distance d_Θ. Moreover, simple examples show that the infimum in (2.1) might not be attained. It will be often useful to remind that, given any sequence $\{\theta_n\}$

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of paths in Θ with uniformly bounded Euclidean lengths, by Ascoli–Arzelà Theorem one can find a $\theta \in \Theta$ such that (possibly up to a subsequence) $\theta_n \xrightarrow{d_\Theta} \theta$. This implies, in particular, that the corresponding curves $\theta_n([0,1])$ converge in the Hausdorff distance to $\theta([0,1])$, while the converse implication is not true. Notice that

$$\theta_n \xrightarrow{d_\Theta} \theta \quad \Longrightarrow \quad \mathscr{H}^1\big(\theta([0,1])\big) \leq \liminf_{n\to\infty} \mathscr{H}^1\big(\theta_n([0,1])\big)\,,$$

where $\mathscr{H}^1$ denotes the one-dimensional Hausdorff measure.

In the sequel, for the sake of brevity we will abuse the notation calling θ also the set $\theta([0,1]) \subseteq \Omega$, when not misleading. We call *endpoints* of the path θ the points $\theta(0)$ and $\theta(1)$, and, given two paths $\theta_1, \theta_2 \in \Theta$ such that $\theta_1(1) = \theta_2(0)$, the *composition* $\theta_1 \cdot \theta_2$ is defined by the formula

$$\theta_1 \cdot \theta_2(t) := \begin{cases} \theta_1(2t) & \text{for } 0 \leq t \leq 1/2\,, \\ \theta_2(2t-1) & \text{for } 1/2 \leq t \leq 1\,. \end{cases}$$

As already introduced in Chapter 1, we let now $A,\ B : \mathbb{R}^+ \to \mathbb{R}^+$ be the costs of moving by own means and by using the network, i.e. $A(s)$ (resp. $B(s)$) is the cost corresponding to a part of the itinerary of length s covered by own means (resp. with the use of the network). This means that, if the urban network is a Borel set $\Sigma \subseteq \Omega$ of finite length, the total cost of covering a path $\theta \in \Theta$ is given by

$$\delta_\Sigma(\theta) := A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big)\,, \tag{2.2}$$

since the length $\mathscr{H}^1(\theta\setminus\Sigma)$ is covered by own means and the length $\mathscr{H}^1(\theta\cap\Sigma)$ is covered by the use of the network. Concerning the functions A and B, we make from now on the following assumptions:

$$A \text{ is nondecreasing, continuous and } A(0) = 0\,; \tag{2.3}$$

$$B \text{ is nondecreasing, l.s.c. and } B(0) = 0\,. \tag{2.4}$$

Note that these hypotheses follow the intuition: the meaning of the assumptions $A(0) = 0$, $B(0) = 0$ and of the monotonicity are obvious, while the continuity of the function A means that a slightly longer path cannot have a much higher cost, and it is a natural assumption once one moves by own means. On the contrary, a continuity assumption on the function B would rule out some of the most common pricing policies which occur in many real life urban transportation networks: for instance, often such a pricing policy is given by a fixed price (the price of a single ticket) for any positive distance, or is a piecewise constant function.

We define now a "distance" on Ω which depends on Σ and is given by the least cost of the paths connecting two points: in short,

$$d_\Sigma(x,y) := \inf \left\{ \delta_\Sigma(\theta) :\ \theta \in \Theta,\, \theta(0) = x,\, \theta(1) = y \right\}\,. \tag{2.5}$$

The infimum in the above definition is not always attained, as we will see in Example 2.8. Moreover, it has to be pointed out that in general the function d_Σ is not a distance; for instance, with $A(s) = B(s) = s^2$ it is easy to see that the triangle inequality does not hold. However, when A and B are subadditive functions, i.e.

$$A(s_1 + s_2) \leq A(s_1) + A(s_2) \text{ for all } s_1,\ s_2\ \in \mathbb{R}^+ ,$$
$$B(s_1 + s_2) \leq B(s_1) + B(s_2) \text{ for all } s_1,\ s_2\ \in \mathbb{R}^+ ,$$

and they are strictly positive on $(0, +\infty)$, then an easy computation shows that d_Σ is in fact a distance (the strict positivity is needed to ensure that $d_\Sigma(x, y) = 0$ implies $x = y$). Nevertheless, with an abuse of notation, we will call d_Σ a distance in any case.

Lemma 2.1. *For any $\theta \in \Theta$ and any $\varepsilon > 0$, there is a path $\theta_\varepsilon \in \Theta$ such that*

$$\theta_\varepsilon(0) = \theta(0)\,, \qquad \theta_\varepsilon(1) = \theta(1)\,, \qquad d_\Theta(\theta, \theta_\varepsilon) < \varepsilon\,,$$
$$\mathscr{H}^1(\theta_\varepsilon) < \mathscr{H}^1(\theta) + \varepsilon\,, \qquad \mathscr{H}^1(\theta_\varepsilon \cap \Sigma) = 0\,.$$

Proof. Since $\Omega \subseteq \mathbb{R}^N$ and $N \geq 2$, we can take a more than countable family $\{\theta_i\}_{i\in I}$ of elements of Θ such that

- $\theta_i(0) = \theta(0)$ and $\theta_i(1) = \theta(1)$ for each $i \in I$;
- $d_\Theta(\theta, \theta_i) < \varepsilon$ and $\mathscr{H}^1(\theta_\varepsilon) < \mathscr{H}^1(\theta) + \varepsilon$ for each $i \in I$;
- for all i, $j \in I$ with $i \neq j$, $\theta_i \cap \theta_j$ consists of finitely many points.

The proof of this assertion is trivial if the curve θ is given by a finite union of segments, as Figure 2.1 shows. The general case is now easily achieved approximating any path θ by a finite union of segments as needed.

The thesis can be then proved making use of the paths θ_i: since $\mathscr{H}^1(\Sigma) < \infty$, the condition $\mathscr{H}^1(\theta_i \cap \Sigma) > 0$ may occur at most for a countable set of indices $i \in I$; one then concludes just by taking one of the remaining paths θ_i. ☐

Corollary 2.2. *For any $\theta \in \Theta$, $\varepsilon > 0$ and $l \leq \mathscr{H}^1(\theta \cap \Sigma)$, there is a path $\theta_{l,\varepsilon} \in \Theta$ such that*

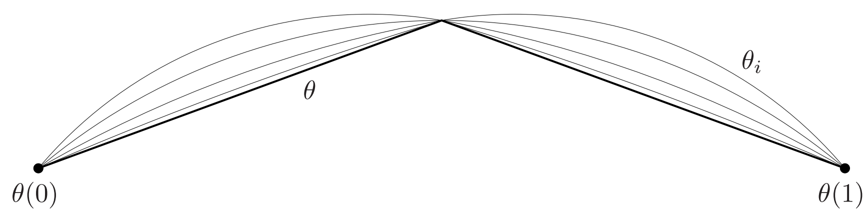

Fig. 2.1 The path θ and some paths θ_i

$$\theta_{l,\varepsilon}(0) = \theta(0)\,, \qquad \theta_{l,\varepsilon}(1) = \theta(1)\,, \qquad d_\Theta(\theta,\theta_{l,\varepsilon}) < \varepsilon\,,$$
$$\mathscr{H}^1(\theta_{l,\varepsilon}) < \mathscr{H}^1(\theta) + \varepsilon\,, \qquad \mathscr{H}^1(\theta_{l,\varepsilon} \cap \Sigma) = l\,.$$

Proof. This follows easily by Lemma 2.1: let $t \in [0,1]$ be such that

$$\mathscr{H}^1\big(\theta([0,t])\big) = l\,,$$

and define θ_1 to be the restriction of θ to $[0,t]$ and θ_2 to be the restriction of θ to $[t,1]$, so that

$$\theta = \theta_1 \cdot \theta_2\,, \qquad \mathscr{H}^1(\theta_1 \cap \Sigma) = l\,.$$

It suffices then to apply Lemma 2.1 to θ_2 and to compose θ_1 with the resulting path. □

Proposition 2.3. *The function $d_\Sigma : \Omega \times \Omega \to \mathbb{R}^+$ is continuous.*

Proof. This is a consequence of (2.3): take $(x,y) \in \Omega \times \Omega$ and a path θ between x and y with

$$\delta_\Sigma(\theta) < d_\Sigma(x,y) + \varepsilon\,.$$

Then, given any pair $(\tilde{x},\tilde{y}) \in \Omega \times \Omega$, we can define a path between $\tilde{x}$ and $\tilde{y}$ by setting $\tilde{\theta} := \alpha \cdot \theta \cdot \beta$ for any choice of paths α and β connecting $\tilde{x}$ to x and y to $\tilde{y}$ respectively. Thanks to Lemma 2.1, we may choose α and β having $\mathscr{H}^1$-negligible intersection with Σ and length less than $|x-\tilde{x}|+\varepsilon$ and $|y-\tilde{y}|+\varepsilon$ respectively. We infer thus

$$\begin{aligned} d_\Sigma(\tilde{x},\tilde{y}) &\le \delta_\Sigma(\tilde{\theta}) \le A\big(\mathscr{H}^1(\theta \setminus \Sigma) + |x-\tilde{x}| + |y-\tilde{y}| + 2\varepsilon\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big) \\ &= \delta_\Sigma(\theta) + A\big(\mathscr{H}^1(\theta \setminus \Sigma) + |x-\tilde{x}| + |y-\tilde{y}| + 2\varepsilon\big) - A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) \\ &\le d_\Sigma(x,y) + \varepsilon + A\big(\mathscr{H}^1(\theta \setminus \Sigma) + |x-\tilde{x}| + |y-\tilde{y}| + 2\varepsilon\big) \\ &\qquad\qquad - A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big)\,, \end{aligned}$$

and the upper semicontinuity of d_Σ follows since $\varepsilon > 0$ is arbitrary and A is continuous.

Concerning the lower semicontinuity of d_Σ, suppose that $x_n \to x$, $y_n \to y$ and that $d_\Sigma(x_n,y_n) \to d$ as $n \to \infty$. This means that there exist paths θ_n connecting x_n to y_n and satisfying $\delta_\Sigma(\theta_n) \to d$. Composing as before θ_n with short paths α_n and β_n connecting x to x_n and y_n to y respectively, and having

$$\mathscr{H}^1(\alpha_n \cap \Sigma) = \mathscr{H}^1(\beta_n \cap \Sigma) = 0\,,$$

we find the paths $\tilde{\theta}_n$ between x and y satisfying

$$\delta_\Sigma(\tilde{\theta}_n) = \delta_\Sigma(\theta_n) + A\big(\mathscr{H}^1(\theta_n \setminus \Sigma) + \mathscr{H}^1(\alpha_n) + \mathscr{H}^1(\beta_n)\big) - A\big(\mathscr{H}^1(\theta_n \setminus \Sigma)\big)\,.$$

Since $\delta_\Sigma(\theta_n) \to d$ and since

$$\mathscr{H}^1(\alpha_n) + \mathscr{H}^1(\beta_n) \to 0\,,$$

the conclusion follows if $\mathscr{H}^1(\theta_n \setminus \Sigma)$ is uniformly bounded, because A is continuous hence uniformly continuous on compact sets. At last, if $\mathscr{H}^1(\theta_n \setminus \Sigma)$ is not uniformly bounded, then

$$\mathscr{H}^1(\theta_n \setminus \Sigma) > |x-y| + 1$$

for n arbitrarily large; in this case, we could directly take a path θ close to the segment connecting x to y and having negligible intersection with Σ, so that

$$\delta_\Sigma(\theta) = A\big(\mathscr{H}^1(\theta)\big) \le A\big(|x-y|+1\big) \le A\big(\mathscr{H}^1(\theta_n \setminus \Sigma)\big) \le \delta_\Sigma(\theta_n)\,,$$

and hence, the thesis follows in this case too. □

The problem we want to study is to find the best transportation network Σ to move the population from their "homes" to their "workplaces". To set the problem, we consider two probability measures f^+, f^- on Ω describing the distributions of homes and workplaces respectively. The following notion is often used in transportation theory; throughout the monograph, $\pi_i : \Omega\times\Omega \to \Omega$, $i = 1, 2$, stands for the $i-$th projection, and for a Borel map $g : X \to Y$ the push-forward $g_\# : \mathcal{M}^+(X) \to \mathcal{M}^+(Y)$ is defined by

$$g_\#\mu(A) := \mu\big(g^{-1}(A)\big) \qquad\qquad \text{for any Borel set } A \subseteq Y\,,$$

where $\mathcal{M}^+(Z)$ is the space of the finite positive measures on a generic space Z (see Appendix B.1).

Definition 2.4. A *transport plan* is a positive measure $\gamma \in \mathcal{M}^+(\Omega \times \Omega)$, the marginals of which are f^+ and f^-, i.e.

$$\pi_{1\#}\gamma = f^+\,, \qquad\qquad \pi_{2\#}\gamma = f^-\,.$$

One can intuitively think that $\gamma(x, y)$ is the number of people moving from x to y, or, more precisely, that $\gamma(C \times D)$ is the number of people living in $C \subseteq \Omega$ and working in $D \subseteq \Omega$. To each transport plan γ we associate the total cost of transportation according to the formula

$$I_\Sigma(\gamma) := \iint_{\Omega\times\Omega} d_\Sigma(x, y)\, d\gamma(x, y)\,. \tag{2.6}$$

The Monge-Kantorovich optimal transport problem consists in finding a transport plan $\bar\gamma \in \mathcal{M}^+(\Omega \times \Omega)$ (which is usually called *optimal transport plan*) minimizing I_Σ.

It is important to notice that the transport plan γ gives no precise information on *how* the mass is moving (i.e. which trajectories are chosen for transportation). To be able to recover such an information we will make use of the following definition, already used in [58] (a quite similar idea was already used elsewhere, for instance in [24, 65, 6]).

Definition 2.5. A *transport path measure* (shortly t.p.m. in the sequel) is a measure $\eta \in \mathcal{M}^+(\Theta)$ with the property that its first and last projections are f^+ and f^-, i.e.

$$p_{0\#}\eta = f^+ \qquad\qquad p_{1\#}\eta = f^- \,, \tag{2.7}$$

where for $t = 0, 1$ we denote by $p_t : \Theta \to \Omega$ the function $p_t(\theta) := \theta(t)$.

It is important to understand the meaning of the above definition: roughly speaking, if η is a t.p.m., then $\eta(\theta)$ indicates the amount of mass to be moved along the path θ; more precisely, $\eta(E)$ is the mass following the paths contained in $E \subseteq \Theta$. The meaning of condition (2.7) is then clear, since $p_{0\#}\eta$ and $p_{1\#}\eta$ are respectively the measure from which η starts and the measure to which it is transported.

We are now able to define the total cost of transportation associated to any t.p.m. by the formula

$$C_\Sigma(\eta) := \int_\Theta \delta_\Sigma(\theta)\, d\eta(\theta)\,. \tag{2.8}$$

Finally, we denote by $MK(\Sigma)$ the infimum of the above costs, namely,

$$MK(\Sigma) := \inf \{C_\Sigma(\eta) \;:\; \eta \text{ is a t.p.m.}\}\,. \tag{2.9}$$

The purpose of this monograph is to study the problem of finding the best possible network Σ: in other words, we want to find a set Σ having minimal total cost of usage (defined below). To do that, as already discussed in Chapter 1, we consider a function $H : \mathbb{R}^+ \to \overline{R}^+$, where $H(l)$ represents the maintenance cost of a network Σ of length $\mathscr{H}^1(\Sigma) = l$. We assume on H the natural conditions

$$H \text{ is nondecreasing, l.s.c., } H(0) = 0 \text{ and } H(l) \to \infty \text{ as } l \to \infty\,. \tag{2.10}$$

Finally, the *total cost* of usage of Σ is defined by the formula

$$\mathfrak{F}(\Sigma) := MK(\Sigma) + H\big(\mathscr{H}^1(\Sigma)\big)\,. \tag{2.11}$$

Our goal is to study the problem of minimizing the functional $\mathfrak{F}$.

2.2 Properties of Optimal Paths and Relaxed Costs

In (2.5) we defined a distance in Ω as the infimum of the costs of the paths connecting two given points. We show now the possibility to choose a Borel selection of paths which have almost minimal costs in the sense of proposition below.

Proposition 2.6. *For any $\varepsilon > 0$ there is a Borel function $q_\varepsilon : \Omega \times \Omega \to \Theta$ such that $q_\varepsilon(x, y)$ is a path connecting x to y with*

$$\delta_\Sigma\big(q_\varepsilon(x, y)\big) < d_\Sigma(x, y) + \varepsilon\,. \tag{2.12}$$

Proof. Fix a $\rho > 0$ and let $\{x_i\}$ be a finite set of points in Ω such that

$$\bigcup B(x_i, \rho) \supseteq \Omega\,.$$

Let then $C_{ij} \subseteq \Omega \times \Omega$ be pairwise disjoint Borel sets covering $\Omega \times \Omega$, each contained in $B\big((x_i, x_j), 2\rho\big)$. Now, given i, j, let $\theta_{ij} \in \Theta$ be a path connecting x_i to x_j and having a cost minimal up to an error ρ, that is

$$\delta_\Sigma(\theta_{ij}) < d_\Sigma(x_i, x_j) + \rho\,.$$

We claim that the conclusion follows if for every $x \in \Omega$ there is a Borel map

$$\alpha_x : B(x, 2\rho) \to \Theta$$

such that $\alpha_x(y)$ is a path between x and y with length less than 4ρ and having $\mathscr{H}^1$−negligible intersection with Σ. Indeed, defining on each C_{ij} the function q_ε by the formula

$$q_\varepsilon(x, y) := \widehat{\alpha_{x_i}(x)} \cdot \theta_{ij} \cdot \alpha_{x_j}(y)$$

(where $\hat\theta(t) := \theta(1 - t)$), one has that q_ε is a Borel function; moreover, if ρ is sufficiently small, one gets (2.12) by the continuity of A. It suffices therefore to prove the existence of such an α_x (observe that Lemma 2.1 already provides a map satisfying all the required conditions except for the Borel property). For this purpose, we begin defining $\alpha_x(y)$ as the line segment between x and y. Since Σ is rectifiable, such a segment has $\mathscr{H}^1$-negligible intersection with Σ unless y is contained in one of countably many radii $\{R_k\}_{k\in\mathbb{N}}$ of the ball $B(x, 2\rho)$. For each $k \in \mathbb{N}$, choose arbitrarily a two-dimensional halfplane Π_k containing R_k on its boundary; then, for $y \in R_k$, define $\alpha_x(y)$ as the half circle joining x to y and lying on Π_k. Arguing as before, it is clear that such a path has $\mathscr{H}^1$-negligible intersection with Σ except for countably many points $y \in R_k$. Finally, for each of these latter y, by Lemma 2.1 we may arbitrarily select a path $\alpha_x(y)$ connecting x to y which is shorter than 4ρ and has $\mathscr{H}^1-$negligible intersection with Σ. The resulting function α_x has the required properties and so the proof is completed. □

Corollary 2.7. *For any $\varepsilon > 0$ there is a Borel function $q'_\varepsilon : \Omega \times \Omega \to \Theta$ such that $q'_\varepsilon(x,y)$ is a path connecting x with y and satisfying*

$$\mathscr{H}^1(q'_\varepsilon(x,y)) \leq |y-x| + \varepsilon\,, \qquad \mathscr{H}^1(q'_\varepsilon(x,y) \cap \Sigma) = 0\,.$$

Proof. Consider the case when

$$A(s) = s\,, \qquad B(s) = \operatorname{diam}\Omega + 2\varepsilon$$

for every $s > 0$. By Lemma 2.1 it is clear that $d_\Sigma(x,y) = |y-x|$ and that $\delta_\Sigma(\theta) = \mathscr{H}^1(\theta)$ whenever $\mathscr{H}^1(\theta \cap \Sigma) = 0$. Apply now Proposition 2.6 to find a map q'_ε such that

$$\delta_\Sigma(q'_\varepsilon(x,y)) < d_\Sigma(x,y) + \varepsilon = |y-x| + \varepsilon\,.$$

If

$$\mathscr{H}^1(q'_\varepsilon(x,y) \cap \Sigma) > 0\,,$$

then

$$\delta_\Sigma(q'_\varepsilon(x,y)) \geq \operatorname{diam}\Omega + 2\varepsilon > |y-x| + \varepsilon\,,$$

and this gives a contradiction. Thus,

$$\mathscr{H}^1(q'_\varepsilon(x,y) \cap \Sigma) = 0$$

and, as a consequence,

$$\mathscr{H}^1(q'_\varepsilon(x,y)) = \delta_\Sigma(q'_\varepsilon(x,y)) < |y-x| + \varepsilon\,;$$

hence the thesis follows. □

We see now an example, showing that the infimum in (2.5) may be not a minimum, and that δ_Σ may be not lower semicontinuous.

Example 2.8. Let Ω be the ball in $\mathbb{R}^2$ centered at the origin and with radius 2, let $\Sigma = [0,1] \times \{0\}$, $A(t) = t$ and $B(t) = 2t$; let moreover θ and θ_n be the paths connecting $(0,0)$ to $(1,0)$ given by

$$\theta(t) := (t,0)\,, \qquad \theta_n(t) := \left(t, \frac{1-|2t-1|}{n}\right)\,.$$

Then one has that θ_n converges to θ in (Θ, d_Θ), $\delta_\Sigma(\theta) = 2$, while $\delta_\Sigma(\theta_n) \to 1$: therefore, δ_Σ is not l.s.c. Moreover, it is clear that

$$d_\Sigma\big((0,0),(1,0)\big) = 1\,,$$

but $\delta_\Sigma(\sigma) > 1$ for each path $\sigma \in \Theta$ connecting $(0,0)$ and $(1,0)$. Hence, the infimum in (2.5) is not a minimum.

Since δ_Σ is not, in general, l.s.c., we compute now its relaxed envelope with fixed endpoints,

$$\bar{\delta}_\Sigma(\theta) := \inf \left\{ \liminf_{n\to\infty} \delta_\Sigma(\theta_n) : \theta_n(0) = \theta(0),\, \theta_n(1) = \theta(1),\ \theta_n \xrightarrow{\Theta} \theta \right\} . \tag{2.13}$$

Notice that $\bar{\delta}_\Sigma \le \delta_\Sigma$, and that the infimum in (2.13) is a minimum. Thanks to the standard properties of relaxed envelopes (see [16]), we are allowed to rewrite (2.5) obtaining

$$d_\Sigma(x,y) = \inf \left\{ \bar{\delta}_\Sigma(\theta) :\ \theta \in \Theta,\, \theta(0) = x,\, \theta(1) = y \right\} . \tag{2.14}$$

Proposition 2.9. *The function $\bar{\delta}_\Sigma : \Theta \to \mathbb{R}^+$ is l.s.c.*

Proof. Let us take $\theta_n \to \theta$ in Θ: then, without loss of generality, we may assume

$$|\theta_n(0) - \theta(0)| \le \frac{1}{n}\,, \qquad\qquad |\theta_n(1) - \theta(1)| \le \frac{1}{n}\,.$$

Following (2.13), we choose $\hat{\theta}_n$ having the same endpoints as θ_n and such that

$$d_\Theta(\theta_n, \hat{\theta}_n) \le \frac{1}{n}\,, \qquad\qquad \delta_\Sigma(\hat{\theta}_n) \le \bar{\delta}_\Sigma(\theta_n) + \frac{1}{n}\,. \tag{2.15}$$

Take now, according to Lemma 2.1, two paths α_n and β_n connecting $\theta(0)$ with $\hat{\theta}_n(0)$ and $\hat{\theta}_n(1)$ with $\theta(1)$ respectively, with the properties

$$\begin{aligned} &\mathscr{H}^1(\alpha_n \setminus \Sigma) \le \frac{2}{n}\,, &&\mathscr{H}^1(\beta_n \setminus \Sigma) \le \frac{2}{n}\,, \\ &\mathscr{H}^1(\alpha_n \cap \Sigma) = 0\,, &&\mathscr{H}^1(\beta_n \cap \Sigma) = 0\,. \end{aligned} \tag{2.16}$$

Define then $\overline{\theta}_n := \alpha_n \cdot \hat{\theta}_n \cdot \beta_n$, so that $\{\overline{\theta}_n\}_{n\in\mathbb{N}}$ is a sequence of paths connecting $\theta(0)$ to $\theta(1)$ which still converges to θ. For any $n \in \mathbb{N}$, by (2.15) and (2.16) we have

$$\begin{aligned} \delta_\Sigma(\overline{\theta}_n) &= A\Big(\mathscr{H}^1\big(\overline{\theta}_n \setminus \Sigma\big)\Big) + B\Big(\mathscr{H}^1\big(\overline{\theta}_n \cap \Sigma\big)\Big) \\ &= A\Big(\mathscr{H}^1\big((\hat{\theta}_n \cup \alpha_n \cup \beta_n) \setminus \Sigma\big)\Big) + B\Big(\mathscr{H}^1\big(\hat{\theta}_n \cap \Sigma\big)\Big) \\ &\le \delta_\Sigma(\hat{\theta}_n) + A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big) + 4/n\Big) - A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big)\Big) \\ &\le \bar{\delta}_\Sigma(\theta_n) + 1/n + A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big) + 4/n\Big) - A\Big(\mathscr{H}^1\big(\hat{\theta}_n \setminus \Sigma\big)\Big). \end{aligned}$$

Since the paths $\{\theta_n\}$ have uniformly bounded lengths, by the uniform continuity of A in the bounded intervals and by (2.13) we infer

$$\bar{\delta}_\Sigma(\theta) \leq \liminf_{n\to\infty} \delta_\Sigma(\overline{\theta}_n) \leq \liminf_{n\to\infty} \bar{\delta}_\Sigma(\theta_n)\,,$$

so the proof is completed. □

Corollary 2.10. *The l.s.c. envelope of δ_Σ in (Θ, d_Θ) is $\bar{\delta}_\Sigma$.*

Proof. The l.s.c. envelope of δ_Σ in (Θ, d_Θ) is lower than $\bar{\delta}_\Sigma$, as a direct consequence of the definition (2.13). On the other hand, it is the greatest l.s.c. function lower than δ_Σ, thus it is also greater than $\bar{\delta}_\Sigma$ by Proposition 2.9. □

Corollary 2.11. *The infimum in (2.14) is actually a minimum.*

Proof. Let us choose x and y and take a minimizing sequence θ_n for (2.14): if the Euclidean lengths of θ_n (possibly up to a subsequence) are bounded, then the result immediately follows from the lower semicontinuity of $\bar{\delta}_\Sigma$ and by Ascoli–Arzelà Theorem. Otherwise, since Σ has finite length, it would follow that

$$\limsup \mathscr{H}^1(\theta_n \setminus \Sigma) = \infty\,;$$

in this case, take a path θ joining x to y with $\mathscr{H}^1-$negligible intersection with Σ and with finite length: since A is nondecreasing, this path provides the minimum in (2.14). □

More precisely, we see that one can somehow "pass to the limit" in Proposition 2.6. Throughout the monograph, we will call *geodesics* the paths θ such that

$$\bar{\delta}_\Sigma(\theta) = d_\Sigma\big(\theta(0), \theta(1)\big)\,.$$

Corollary 2.12. *There is a Borel function $q : \Omega \times \Omega \to \Theta$ such that $q(x, y)$ is a path connecting x to y with cost $\bar{\delta}_\Sigma\big(q(x, y)\big) = d_\Sigma(x, y)$.*

Proof. Using the classical results in [28], it is sufficient to show that the subset G of Θ given by the geodesics is closed and there is at least one element of G connecting any couple of points in $\Omega \times \Omega$. The second fact follows from Corollary 2.11, while the closedness of G is a direct consequence of the lower semicontinuity of $\bar{\delta}_\Sigma$ and of the continuity of d_Σ. □

Lemma 2.13. *For any $\varepsilon > 0$, there is a Borel function $\alpha_\varepsilon : \Theta \to \Theta$ such that for any $\theta \in \Theta$ one has*

$$\begin{aligned} &\big(\alpha_\varepsilon(\theta)\big)(0) = \theta(0)\,, &&\big(\alpha_\varepsilon(\theta)\big)(1) = \theta(1)\,, &&d_\Theta(\alpha_\varepsilon(\theta), \theta) \leq \varepsilon\,,\\ &\mathscr{H}^1(\alpha_\varepsilon(\theta)) \leq \mathscr{H}^1(\theta) + \varepsilon\,, &&\delta_\Sigma(\alpha_\varepsilon(\theta)) \leq \bar{\delta}_\Sigma(\theta) + \varepsilon\,. \end{aligned}$$

Proof. Our argument is quite similar to the one in Proposition 2.6: fixed $L > 0$ and fixed arbitrarily a path $\theta \in \Theta$ with $\mathscr{H}^1(\theta) \le L$, we know by definition of $\bar\delta_\Sigma$ the existence of a path $\tilde\theta$ with

$$\begin{aligned} &\tilde\theta(0) = \theta(0)\,, && \tilde\theta(1) = \theta(1)\,, && d_\Theta(\tilde\theta,\theta) \le \frac{\varepsilon}{4}\,, \\ &\mathscr{H}^1(\tilde\theta) \le \mathscr{H}^1(\theta) + \frac{\varepsilon}{4}\,, && \delta_\Sigma(\tilde\theta) \le \bar\delta_\Sigma(\theta) + \frac{\varepsilon}{4}\,. \end{aligned} \tag{2.17}$$

We take now a number $\delta \le \varepsilon/8$ such that

$$A(s+4\delta) - A(s) \le \frac{\varepsilon}{2}$$

for any $0 \le s \le L$, which is possible by the continuity of A; moreover, since the Euclidean length and the map $\bar\delta_\Sigma$ are l.s.c., we can also assume that δ is so small that

$$\begin{cases} \mathscr{H}^1(\sigma) \ge \mathscr{H}^1(\theta) - \dfrac{\varepsilon}{4}\,, \\ \bar\delta_\Sigma(\sigma) \ge \bar\delta_\Sigma(\theta) - \dfrac{\varepsilon}{4}\,, \end{cases} \qquad \text{whenever} \quad d_\Theta(\theta,\sigma) \le \delta\,. \tag{2.18}$$

If we can define a Borel function $\alpha_\varepsilon : B_\Theta(\theta,\delta) \to \Theta$ as in the claim of this corollary, this will show the thesis: indeed, since the subset Θ_L of Θ made by the paths of Euclidean length bounded by L is compact, it can be covered by a finite number of balls $B_\Theta(\theta_i,\delta_i)$, so that we infer the existence of a Borel function $\alpha_\varepsilon : \Theta_L \to \Theta$ as in the claim; finally, it is immediate to conclude the thesis covering Θ with countably many sets Θ_{L_i} for a sequence $L_i \to \infty$. Summarizing, we can restrict our attention to a ball $B_\Theta(\theta,\delta)$.

Define now the Borel function $\beta_1 : B_\Theta(\theta,\delta) \to \Theta$ as

$$\beta_1(\sigma) := q'_\delta\big(\sigma(0),\theta(0)\big)\,,$$

where q'_δ is as in Corollary 2.7: then $\beta_1(\sigma)$ is a path connecting $\sigma(0)$ with $\theta(0)$ such that

$$\mathscr{H}^1(\beta_1(\sigma)\cap\Sigma) = 0\,, \qquad \mathscr{H}^1(\beta_1(\sigma)) \le |\sigma(0)-\theta(0)| + \delta \le 2\delta\,. \tag{2.19}$$

Similarly, we let $\beta_2 : B_\Theta(\theta,\delta) \to \Theta$ to be a Borel function such that $\beta_2(\sigma)$ is a path connecting $\theta(1)$ with $\sigma(1)$ satisfying

$$\mathscr{H}^1(\beta_2(\sigma)\cap\Sigma) = 0\,, \qquad \mathscr{H}^1(\beta_2(\sigma)) \le 2\delta\,; \tag{2.20}$$

We finally define $\alpha_\varepsilon(\sigma) := \beta_1(\sigma)\cdot\tilde\theta\cdot\beta_2(\sigma)$: by construction, the map

$$B_\Theta(\theta,\delta) \ni \sigma \mapsto \alpha_\varepsilon(\sigma) \in \Theta$$

is Borel; moreover,

$$\alpha_\varepsilon(\sigma(0)) = \sigma(0)\,, \qquad\qquad \alpha_\varepsilon(\sigma(1)) = \sigma(1)\,.$$

In addition, minding (2.19), (2.20) and (2.17), we get

$$d_\Theta(\alpha_\varepsilon(\sigma),\sigma) \le d_\Theta(\alpha_\varepsilon(\sigma),\tilde\theta) + d_\Theta(\tilde\theta,\theta) + d_\Theta(\theta,\sigma) \le 2\delta + \frac{\varepsilon}{4} + \delta < \varepsilon\,.$$

Again by (2.19), (2.20), (2.17) and (2.18) one has

$$\mathscr{H}^1(\alpha_\varepsilon(\sigma)) \le 4\delta + \mathscr{H}^1(\tilde\theta) \le 4\delta + \frac{\varepsilon}{4} + \mathscr{H}^1(\theta) \le \mathscr{H}^1(\sigma) + \varepsilon\,.$$

Finally, by (2.19) and (2.20) we know that

$$\mathscr{H}^1(\alpha_\varepsilon(\sigma)\cap\Sigma) = \mathscr{H}^1(\tilde\theta\cap\Sigma)$$

so that again (2.19) and (2.20), together with (2.17) and (2.18), yield

$$\begin{aligned}\delta_\Sigma(\alpha_\varepsilon(\sigma)) &\le A\big(\mathscr{H}^1(\tilde\theta\setminus\Sigma)+4\delta\big) + B\big(\mathscr{H}^1(\tilde\theta\cap\Sigma)\big)\\ &\le \delta_\Sigma(\tilde\theta) + A\big(\mathscr{H}^1(\tilde\theta\setminus\Sigma)+4\delta\big) - A\big(\mathscr{H}^1(\tilde\theta\setminus\Sigma)\big)\\ &\le \delta_\Sigma(\tilde\theta) + \frac{\varepsilon}{2} \le \bar\delta_\Sigma(\theta) + \frac{3}{4}\varepsilon \le \bar\delta_\Sigma(\sigma)+\varepsilon\,:\end{aligned}$$

hence, the proof is complete. □

Now, generalizing (2.8), set

$$\overline{C}_\Sigma(\eta) := \int_\Theta \bar\delta_\Sigma(\theta)\,d\eta(\theta)\,. \tag{2.21}$$

Proposition 2.14. *The following equalities hold*

$$\begin{aligned}\inf\big\{C_\Sigma(\eta):\ \eta \text{ is a t.p.m.}\big\} &= \min\{I_\Sigma(\gamma):\ \gamma \text{ is a transport plan}\}\\ &= \min\big\{\overline{C}_\Sigma(\eta):\ \eta \text{ is a t.p.m.}\big\} \qquad \big(=MK(\Sigma)\big)\,.\end{aligned} \tag{2.22}$$

Before giving the proof, we point out the following remark.

Remark 2.15. The equality (2.22) ensures the existence of at least one optimal transport plan $\gamma_{\rm opt}$ and one t.p.m. $\eta_{\rm opt}$ optimal with respect to $\overline{C}_\Sigma$, which satisfy the equality $I_\Sigma(\gamma_{\rm opt}) = \overline{C}_\Sigma(\eta_{\rm opt})$. On the other hand, the infimum in (2.22) needs not to be achieved: for instance, just consider the situation of Example 2.8 with $f^+ := \delta_{(0,0)}$ and $f^- := \delta_{(1,0)}$.

Concerning the equality between the two minima in (2.22), in particular, if $\gamma_{\rm opt}$ is an optimal transport plan then $q_\#\gamma_{\rm opt}$ is an optimal t.p.m., where q is defined in Corollary 2.12. Conversely, if $\eta_{\rm opt}$ is an optimal t.p.m. then $(p_0,p_1)_\#\eta_{\rm opt}$ is an optimal transport plan, where p_0 and p_1 are as in Definition 2.5.

Proof (of Proposition 2.14). First of all, note that the set of all transport plans is a bounded and weakly* closed subset of $\mathcal{M}^+(\Omega \times \Omega)$; hence, it is weakly* compact by tightness (see Appendix B.1). Moreover, I_Σ is a continuous function on $\mathcal{M}(\Omega \times \Omega)$ with respect to the weak* topology thanks to Proposition 2.3. Therefore, the existence of some optimal transport plan is straightforward.

Given now a t.p.m. η, one can construct the associated transport plan $\gamma = (p_0, p_1)_\# \eta$, and from (2.5) we get $I_\Sigma(\gamma) \leq C_\Sigma(\eta)$. On the other hand, given any transport plan γ and $\varepsilon > 0$, we can define $\eta := q_{\varepsilon\#}\gamma$ where q_ε is as in Proposition 2.6; we obtain $C_\Sigma(\eta) \leq I_\Sigma(\gamma) + \varepsilon$, thus the first equality in (2.22) is established.

Concerning the second one, using (2.14) in place of (2.5) in the previous argument one gets

$$\min\{I_\Sigma(\gamma)\} \leq \inf\{\overline{C}_\Sigma(\eta)\}\,.$$

But since $\overline{C}_\Sigma \leq C_\Sigma$ (because $\bar{\delta}_\Sigma \leq \delta_\Sigma$), it is also true that

$$\inf\{\overline{C}_\Sigma(\eta)\} \leq \inf\{C_\Sigma(\eta)\}\,.$$

We derive $\min\{I_\Sigma(\gamma)\} = \inf\{\overline{C}_\Sigma(\eta)\}$, so to conclude we need only to prove that the last inf is a minimum. To this aim, it suffices to take an optimal transport plan γ_{opt} and to define $\eta := q_\# \gamma_{\mathrm{opt}}$ where q is as in Corollary 2.12: by definition of q, one has $\overline{C}_\Sigma(\eta) = I_\Sigma(\gamma_{\mathrm{opt}})$, so η minimizes $\overline{C}_\Sigma$ and the proof is achieved. □

From now on we will often say that a set $\Delta \subseteq \Theta$ is *bounded in Θ by L* if for any $\theta \in \Delta$ we have $\mathscr{H}^1(\theta) \leq L$; we will also say that Δ is a *bounded subset of Θ* if it is bounded in Θ by some constant L. Notice that this last definition does not coincide with the usual boundedness in Θ with respect to the distance d_Θ, which we will never consider; in fact, this last notion of boundedness would be useless, since the whole set Θ is clearly bounded with respect to d_Θ by the diameter of Ω. We recall that, as already mentioned at the beginning of Section 2.1, the bounded subsets of Θ are sequentially compact with respect to d_Θ; this becomes particularly helpful once we know that a t.p.m. is concentrated on a bounded subset of Θ, which is the argument of Corollary 2.17 below.

Lemma 2.16. *If $A(s)$ is not constant for large s (for instance, if $A(s) \to \infty$ as $s \to \infty$), then there is a constant $L \in \mathbb{R}$ such that the Euclidean length $\mathscr{H}^1(\theta)$ of any geodesic θ is bounded by L. Otherwise, if $A(s)$ is constant for large s, it is still true that for any pair (x, y) of points in Ω there exists some geodesic of length bounded by L. In both cases, the constant L depends only on A, Ω and $\mathscr{H}^1(\Sigma)$ (but not on Σ).*

Proof. Suppose first that $A(s)$ is not constant for large s, and let L be a sufficiently large number such that

$$A\big(L - \mathscr{H}^1(\Sigma)\big) > A\big(\operatorname{diam}\Omega + 1\big)\,.$$

Take now a path $\theta \in \Theta$ with $\mathscr{H}^1(\theta) > L$ and let $\hat{\theta}$, according to Lemma 2.1, be a path with length less than

$$|\theta(1) - \theta(0)| + 1 \leq \operatorname{diam} \Omega + 1$$

connecting $\theta(0)$ to $\theta(1)$ and having $\mathscr{H}^1-$negligible intersection with Σ. Since

$$\mathscr{H}^1(\theta \setminus \Sigma) \geq \mathscr{H}^1(\theta) - \mathscr{H}^1(\Sigma) > L - \mathscr{H}^1(\Sigma)\,,$$

we immediately get $\bar{\delta}_\Sigma(\theta) > \bar{\delta}_\Sigma(\hat{\theta})$, so that θ is not a geodesic and the first part of the proof is achieved.

Consider now the case when $A(s)$ is constant for large s, and let

$$L := \mathscr{H}^1(\Sigma) + \operatorname{diam} \Omega + 1\,.$$

Arguing exactly as in the first part of the proof, we see that for any path θ there is a path $\hat{\theta}$ with $\mathscr{H}^1(\hat{\theta}) \leq \operatorname{diam} \Omega + 1$ and with $\bar{\delta}_\Sigma(\hat{\theta}) \leq \bar{\delta}_\Sigma(\theta)$ (the only difference is that this time the strict inequality $\bar{\delta}_\Sigma(\hat{\theta}) < \bar{\delta}_\Sigma(\theta)$ in the case $\mathscr{H}^1(\theta) > L$ may be false). Hence, it is not true that *all* the geodesics have Euclidean length less than L, but that for any pair $(x, y) \in \Omega \times \Omega$ there is *at least one* geodesic between x and y of Euclidean length less than L. □

Corollary 2.17. *If $A(s)$ is not constant for large s then the support of any t.p.m. η which is optimal with respect to $\overline{C}_\Sigma$ is bounded in Θ by L, where L is as in the previous Lemma. Otherwise, if $A(s)$ is constant for large s, it is still true that there exists some optimal t.p.m. η the support of which is bounded in Θ by L.*

Proof. Recall that, thanks to (2.22), any t.p.m. optimal with respect to $\overline{C}_\Sigma$ is concentrated in the set of all geodesics; this set is closed, as already noticed in Corollary 2.12, hence the whole support of any optimal t.p.m. is made by geodesics and the first part of the proof is trivial.

Concerning the second claim, we recall that Corollary 2.12 implies that the set G of all geodesics is a closed subset of Θ containing at least one path which connects any given pair of points in $\Omega \times \Omega$. The same property is true for the set

$$G_L := G \cap \left\{\theta \in \Theta :\, \mathscr{H}^1(\theta) \leq L\right\},$$

by the above lemma and since the Euclidean length is l.s.c. with respect to the distance in Θ. Therefore, arguing as in Corollary 2.12, we find a Borel map $\tilde{q} : \Omega \times \Omega \to \Theta$ such that $\tilde{q}(x, y)$ is a geodesic between x and y of Euclidean length less than L. This easily gives also the second part of the thesis: arguing as in Proposition 2.14, taken any optimal transport plan γ, one has that the t.p.m. $\tilde{q}_\# \gamma$ is as required. □

We present now a useful exact formula for $\bar{\delta}_\Sigma$.

Proposition 2.18. *The following equality holds:*

$$\bar{\delta}_\Sigma(\theta) = J\big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta \cap \Sigma)\big)\,, \tag{2.23}$$

where the function $J\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is given by*

$$J(a,b) := \inf\big\{A\big(a+l\big) + B\big(b-l\big) :\ 0 \le l \le b\big\}\ . \tag{2.24}$$

Before giving the proof, we shortly discuss the above formula.

Remark 2.19. The meaning of (2.23), as one can understand comparing with (2.2), is that, roughly speaking, one can "walk on the railway": in other words, the cost $\bar{\delta}_\Sigma$ of some path θ is not necessarily given by the cost of moving by own means out of the network and by train along it, but moving by own means out of the network and possibly in some part of it, and by train along the remaining part. The basic idea of the proof is then easily imagined: instead of walking on the network, one can just walk very close to it, which is possible since the dimension N is larger than 1.

Proof (of Proposition 2.18). Set $a := \mathscr{H}^1(\theta \setminus \Sigma)$ and $b := \mathscr{H}^1(\theta \cap \Sigma)$, then take an arbitrary sequence θ_n of paths having the same endpoints as θ and converging to θ. It is known that

$$\mathscr{H}^1(\theta) \le \liminf_{n\to\infty} \mathscr{H}^1(\theta_n)\,, \tag{2.25}$$

$$\mathscr{H}^1(\theta \setminus \Sigma) \le \liminf_{n\to\infty} \mathscr{H}^1(\theta_n \setminus \Sigma)\,; \tag{2.26}$$

the first inequality is the classical lower semicontinuity of the length, the second is a recent generalization of the Gołąb theorem that we state in Theorem 3.6 (see also for instance [14] and [30]).

For a given $n \in \mathbb{N}$, assume that

$$\mathscr{H}^1(\theta_n \cap \Sigma) \ge \mathscr{H}^1(\theta \cap \Sigma)\ :$$

then, taking $l = 0$ in (2.24), we obtain

$$\begin{aligned} J(a,b) &\le A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta \cap \Sigma)\big) \\ &\le A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) + B\big(\mathscr{H}^1(\theta_n \cap \Sigma)\big) \\ &\le \delta_\Sigma(\theta_n) + A\big(\mathscr{H}^1(\theta \setminus \Sigma)\big) - A\big(\mathscr{H}^1(\theta_n \setminus \Sigma)\big)\,. \end{aligned} \tag{2.27}$$

On the other hand, if

$$\mathscr{H}^1(\theta_n \cap \Sigma) < \mathscr{H}^1(\theta \cap \Sigma)$$

then, taking

$$l := \mathscr{H}^1(\theta \cap \Sigma) - \mathscr{H}^1(\theta_n \cap \Sigma)$$

in (2.24), we obtain

$$\begin{aligned} J(a,b) &\le A\big(\mathscr{H}^1(\theta) - \mathscr{H}^1(\theta_n \cap \Sigma)\big) + B\big(\mathscr{H}^1(\theta_n \cap \Sigma)\big) \\ &\le \delta_\Sigma(\theta_n) + A\big(\mathscr{H}^1(\theta) - \mathscr{H}^1(\theta_n \cap \Sigma)\big) \\ &\qquad\qquad - A\big(\mathscr{H}^1(\theta_n) - \mathscr{H}^1(\theta_n \cap \Sigma)\big)\,. \end{aligned} \tag{2.28}$$

Recalling now that A is nondecreasing and continuous, combining (2.27) with (2.26) and (2.28) with (2.25) gives $J(a,b) \le \liminf \delta_\Sigma(\theta_n)$, and therefore $J(a,b) \le \bar{\delta}_\Sigma(\theta)$.

To prove the opposite inequality take $0 \le l \le \mathscr{H}^1(\theta \cap \Sigma)$ and let $\{\theta_n\}$ be, according to Corollary 2.2, a sequence of paths connecting $\theta(0)$ and $\theta(1)$ such that

$$\theta_n \to \theta\,, \quad \mathscr{H}^1(\theta_n) \to \mathscr{H}^1(\theta)\,, \quad \mathscr{H}^1(\theta_n \cap \Sigma) = \mathscr{H}^1(\theta \cap \Sigma) - l \;\forall\, n \in \mathbb{N}\,.$$

Hence, making use of the continuity of A, one gets

$$\delta_\Sigma(\theta_n) = A\big(\mathscr{H}^1(\theta_n) - (\mathscr{H}^1(\theta \cap \Sigma) - l)\big) + B\big(b - l\big) \xrightarrow[n\to\infty]{} A(a+l) + B(b-l)\,.$$

Thus for every $0 \le l \le b$ one has

$$\bar{\delta}_\Sigma(\theta) \le A(a+l) + B(b-l)\,,$$

so the inequality $J(a,b) \ge \bar{\delta}_\Sigma(\theta)$ follows taking the infimum on l. □

It is also convenient to introduce an auxiliary function, namely

$$D(a,b) := J(a, b-a)\,; \tag{2.29}$$

indeed, the above proposition tells us that

$$\bar{\delta}_\Sigma(\theta) = J\big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta \cap \Sigma)\big)\,,$$

or equivalently that

$$\bar{\delta}_\Sigma(\theta) = D\big(\mathscr{H}^1(\theta \setminus \Sigma), \mathscr{H}^1(\theta)\big)\,. \tag{2.30}$$

In other words, we can express $\bar{\delta}_\Sigma(\theta)$ in terms of the length $\mathscr{H}^1(\theta \setminus \Sigma)$ outside of the network and of the length $\mathscr{H}^1(\theta \cap \Sigma)$ inside the network if we make use of J, or in terms of the length $\mathscr{H}^1(\theta \setminus \Sigma)$ out of the network and of the total length $\mathscr{H}^1(\theta)$ if we make use of D. The advantage of the second possibility, i.e. the advantage of (2.30) with respect to (2.23), is that the variables $\mathscr{H}^1(\theta \setminus \Sigma)$ and $\mathscr{H}^1(\theta)$ satisfy the useful lim inf inequalities (2.25)–(2.26), while the same is not true for $\mathscr{H}^1(\theta \cap \Sigma)$; on the contrary, for $\mathscr{H}^1(\theta \cap \Sigma)$ the lim sup inequality is true, as one can immediately deduce by Lemma 4.1. Another easy interesting property of both D and J is the following one.

Proposition 2.20. *The functions J and D are nondecreasing in each of their variables.*

Proof. Consider first J: take $b \geq 0$ and $a' \geq a \geq 0$; for $0 \leq l \leq b$ one has $A(a+l) \leq A(a'+l)$, so by (2.24) one gets $J(a',b) \geq J(a,b)$ and thus J is nondecreasing in its first variable. Concerning the second one, take $a \geq 0$ and $b' \geq b \geq 0$: one has

$$A(a+l) + B(b'-l) \geq A(a+l) + B(b-l) \geq J(a,b) \qquad \forall\, 0 \leq l \leq b\,;$$

on the other hand, one has

$$A(a+l) + B(b'-l) \geq A(a+b) + B(0) \geq J(a,b) \qquad \forall\, b \leq l \leq b'\,.$$

It follows that $J(a,b') \geq J(a,b)$, so J is nondecreasing also in its second variable.

Consider now D: first of all, we rewrite (2.29) in a more convenient way as

$$\begin{aligned} D(a,b) &= \inf\left\{A(a+l) + B(b-a-l) :\ 0 \leq l \leq b-a\right\} \\ &= \inf\left\{A(l) + B(b-l) :\ a \leq l \leq b\right\}\,. \end{aligned} \tag{2.31}$$

Then, take $b \geq 0$ and $a' \geq a \geq 0$: if $a' \leq l \leq b$ then a fortiori $a \leq l \leq b$, hence one gets $D(a,b) \leq D(a',b)$ directly by (2.31), and consequently D is nondecreasing in its first variable. Finally, concerning the second one, take $a \geq 0$ and $b' \geq b \geq 0$: if $a \leq l \leq b$ then

$$A(l) + B(b'-l) \geq A(l) + B(b-l) \geq D(a,b)\,;$$

on the other hand, if $b \leq l \leq b'$ then

$$A(l) + B(b'-l) \geq A(b) + B(0) \geq D(a,b)\,.$$

It follows that $D(a,b') \geq D(a,b)$, so D is nondecreasing also in its second variable and the proof is completed. □