Preface

The principal aim of the book is to give a comprehensive account of the variety of approaches to such an important and complex concept as Integrability. Developing mathematical models, physicists often raise the following questions: whether the model obtained is integrable or close in some sense to an integrable one and whether it can be studied in depth analytically. In this book we have tried to create a mathematical framework to address these issues, and we give descriptions of methods and review results.

In the Introduction we give a historical account of the birth and development of the theory of integrable equations, focusing on the main issue of the book – the concept of integrability itself. A universal definition of *Integrability* is proving to be elusive despite more than 40 years of its development. Often such notions as "exact solvability" or "regular behaviour" of solutions are associated with integrable systems. Unfortunately these notions do not lead to any rigorous mathematical definition. A constructive approach could be based upon the study of hidden and rich algebraic or analytic structures associated with integrable equations. The requirement of existence of elements of these structures could, in principle, be taken as a definition for integrability. It is astonishing that the final result is not sensitive to the choice of the structure taken; eventually we arrive at the same pattern of equations. The relationship between the different approaches is often far from obvious and needs to be understood better.

Integrable equations possess hidden symmetries and actually possess infinite hierarchies of local symmetries. This property is taken as a definition of integrability in the symmetry approach. A detailed introduction and review of the modern state of the symmetry approach is given in Chap. 1, written by A.V. Mikhailov and V. Sokolov. The symmetry approach provides powerful necessary conditions for the existence of local higher symmetries and/or conservation laws for systems of differential equations. For a given system of equations these conditions are easily verifiable and eventually can serve as a criterion of integrability. Chapter 1 also contains an account of classification results obtained and an extensive bibliography.

For evolutionary equations whose right-hand side is a homogeneous differential polynomial, the symbolic representation and powerful results of number theory allow us to achieve global classification results (Chap. 2, written by J. Sanders and J.P. Wang). One of the most spectacular results of this theory can be formulated as follows: any scalar integrable evolutionary equation whose right-hand side is a homogeneous differential polynomial (with a positive weight) belongs to one of the infinite hierarchies of equations of order $2, 3$ or 5 and all these integrable hierarchies are explicitly listed. It is shown that for a scalar evolutionary equation the existence of one higher symmetry implies the existence of an infinite hierarchy of hidden symmetries and therefore the integrability of the equation. For systems of equations a similar statement is not valid: there are examples of systems which have only a finite number of higher local symmetries. Chapter 2 is an excellent introduction to the symbolic method and contains relevant number theory results in applications to the theory of integrable equations.

In Chap. 3, written by S.P. Novikov, the phenomenon of integrability is associated with hidden symmetries of linear spectral problems. Darboux and Laplas transformations for one- and two-dimensional Schrödinger operators are famous examples of the spectral symmetries. The proper discretisation of these operators, the corresponding discrete Darboux and Laplas transformations and their relation to integrable equations and finite gap solutions are discussed. Chapter 4, written by A. Shabat is devoted to a detailed study of continuous and discrete spectral symmetries in the one-dimensional case. A connection of these symmetries with the famous list of Painlevé equations and with dressing chains is discussed.

Chapters 5 and 6 explore perturbative and asymptotic aspects of integrable equations. The concept of approximate integrability, approximate symmetries and conservation laws are discussed in Chap. 5, written by Y. Hiraoka and Y. Kodama. It is an attempt to extend the classical theory of normal forms to the case of partial differential equations. If the main approximation is given by an integrable equation, the higher order corrections often violate integrability and give rise to new effects, such as inelasticity in soliton interaction, creation of new solitons as a result of soliton collisions, etc. Chapter 6, written by A. Degasperis, addresses multiscaling expansion and universal equations, i.e. nonlinear equations which determine the leading term in the asymptotic expansion. Francesco Calogero gave a simple explanation for why integrable equations, which are rather exceptional, are widely applicable. Universal equations have a good chance to be integrable, since the multiscaling expansion preserves the main attributes of integrability, such as symmetries, local conservation laws, etc. The analysis of higher order corrections in a multiscale expansion of a given system provides necessary conditions for integrability of the system.

In the analytical theory of differential equations we study the structure of singularities of the solutions. The absence of movable critical singularities can be taken as a criteria for isolation of integrable systems. This is at the heart of the Painlevé approach and its generalisations described in Chap. 7 written by A. Hone.

Chapter 8, written by J. Hietarinta, describes the modern development of the Hirota approach and bi-linear representation of integrable equations. This kind of representation proved to be very useful for construction and analysis of explicit multi-soliton solutions. It can also be used for a classification of integrable equations of special form.

Quantum integrability is a separate and well-developed subject. It deserves a separate volume. We include lectures of T. Miwa (Chap. 9) in order to give a flavour of quantum integrability and to highlight the symmetry aspects of quantum integrable systems in the example of XXZ model.

This book is a unique collection of articles which could serve as the core material for a number of graduate lecture courses. The chapters in the book are independent and self-contained. They can be read in any order. Chapter 1 is probably more pedagogical than others and can be recommended for those wishing to become acquainted with the subject. The book was specifically designed to be accessible to graduate students and post-docs.

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Chapter 2
Number Theory and the Symmetry Classification of Integrable Systems

J.A. Sanders and J.P. Wang

2.1 Introduction

The theory of integrable systems has developed in many directions, and although the interconnections between the different subjects are clearly suggested by the similarity of the results, they are not always so easy to prove or even formulate. Of the various methods used to characterize integrable differential equations, including existence of infinitely many symmetries and/or conservation laws, soliton solutions, linearization by inverse scattering or differential substitution, Bäcklund transformation, Painlevé property, bi-Hamiltonian structure, recursion operator, formal symmetry of infinite rank, etc. [35], the most fruitful for systematic classification and discovery of new systems has been the characterization of integrable systems by the existence of a sufficient number of higher order symmetries. The main questions in this respect are the following:

- Can we decide, given an equation, whether there exists a generalized symmetry (the recognition problem)?
- And if so, can we answer the question whether this leads to infinitely many symmetries (the symmetry-integrability problem)?
- Given a class of equations with arbitrary parameters, possibly functions of given type, can we completely classify this class with respect to the existence of symmetries (The classification problem)?

As it turns out, these three questions are strongly related. In certain cases, they can be effectively and completely analyzed by an adaptation of the symbolic method of classical invariant theory [22], after which powerful number-theoretic results on factorizability of polynomials based on Diophantine approximation theory [2] are applied to complete the classification.

The history of the subject experienced two developmental periods. In the first, following the discovery of the Korteweg–de Vries (KdV) equation, a surprisingly large number of other integrable hierarchies, including mKdV, Sawada–Kotera,

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Kaup–Kupershmidt, were soon found. However, the second period was more disappointing in this respect, as the integrable well quickly dried up, at least in the most basic case that scalar, polynomial evolution equations are linear in the highest order derivative. This led to the conjecture that all integrable systems of this particular form had been found. In this chapter, we describe rigorous classification results for both commutative and noncommutative systems [24, 27, 28, 33], including a proof of this particular conjecture and a discussion of the general methods by which such complete classification results are established, cf. Sect. 2.4.

To do so, we prove that symmetry-integrability of an equation of the form

$$u_t = u_n + f(u, \cdots, u_{n-1}), \quad \textit{where} \quad u_n = D_x^n u \tag{2.1}$$

with f a formal power series starting with terms that are at least quadratic, is determined by

- the existence of one generalized symmetry,
- the existence of approximate symmetries.

This led to the proof of the remark made in [7]

> Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalized symmetry implies the existence of infinitely many. (However, this has not been proved in general.)

under fairly relaxed conditions. In particular, for homogeneous scalar evolution equations, to prove the integrability of an equation of order 2 we need a symmetry of order 3; for an equation of order 3 we need a symmetry of order 5; for an equation of order 5 we need a symmetry of order 7; and for an equation of order 7 we need a symmetry of order 13; this enables us to give the complete list of integrable homogeneous equations. The result also confirms the remark made in [10]:

> It turns out from practice that if the first integrability conditions [...] are fulfilled, then often all the others are fulfilled as well.

However, the conjecture

> the existence of one symmetry implies the existence of (infinitely many) others

has been disproved using the example in [1]. This example does not contradict our theorem, since it proves the nonexistence of certain quadratic terms, the existence of which is one of the conditions in our theorem. In this chapter, we give the strict proof that Bakirov's example has only one symmetry using p-adic analysis, cf. Sect. 2.6.

We should remark that the modified conjecture made in [8]

> ... Similarly for n-component equations one needs n symmetries

has also been disproved in [11, 12], where the authors found a two-component system that has only two symmetries.

This theory was soon successfully applied to noncommutative evolution equations of the form (2.1) in which the field variable u takes its values in an associative, non-commutative algebra [24]. In this manner, it was rigorously proved that the list of integrable evolution equation in [23] is complete. These equations can be regarded as quantizing classical integrable systems; see [6], where the authors treated the Korteweg–de Vries equation.

The classification problem has been noticed and studied since the 1960s. The group consisting of A.B. Shabat, A.V. Mikhailov, V.V. Sokolov, S.I. Svinolupov, R.I. Yamilov and co-workers, cf. [19, 25], was successful in giving the complete classification for equations of fixed order, allowing for much bigger equivalence classes. We only work with homogeneous equations and transformations that do not change the weight of the dependent variables, but this restriction enables us, at least in the scalar case, to obtain results for all orders of the evolution equation.

2.2 The Symbolic Method

2.2.1 Basic Definitions

The symbolic method was first introduced by Gel'fand-Dikiĭ [9] and used in [32] to show (as an example) that the symmetries of the Sawada–Kotera equation have to be of order 1 or 5 $\pmod 6$. The basic idea of the symbolic method is simply to replace u_i, where i is an index – in our case counting the number of derivatives – by $u\xi^i$, where ξ is now a symbol. We see that the basic operation of differentiation, i.e. replacing u_i by u_{i+1}, is now replaced by multiplication with ξ, as is the case in Fourier transform theory. For higher degree terms with multiple us, one uses different symbols to denote differentiation; for example, the noncommutative binomial $u_i u_j$ has symbolic form $u^2\xi_1^i\xi_2^j$. In the commutative case, one needs to average over permutations of the differentiation symbols so that $u_i u_j$ and $u_j u_i$ have the same symbolic form. However, in the noncommutative case, this is no longer necessary. In other words, the noncommutative symbolic method works with general tensors, while in the commutative case one restricts to (multi)-symmetric tensors, or polynomials for short.

Usually one replaces u_i by ξ^i, but this leads to confusion for the expressions like u^n since the distinction between the powers disappears.

With this method one can readily translate solvability questions into divisibility questions and we can use generating functions to handle infinitely many orders at once. While this does not mean that the questions are much easier to answer, we do now have the whole machinery which has been developed in number theory available, and this makes a crucial difference.

For simplicity, we restrict our attention to the case of a single independent variable x and a single dependent field variable u. Extensions of the basic ideas to several (noncommutative) dependent variables are immediate, see Sect. 2.5, and to several commutative independent variables can be found in [34].

A differential monomial takes the form $u_I = u_{i_1}u_{i_2}\cdots u_{i_k}$. We call k the *degree* of the monomial, $\#I = i_1 + \cdots + i_k$ the *index*, and $\max(i_j,\ j = 1,\cdots,k)$ the *order*. For brevity, $[u]$ is used to denote the set of arguments $u, u_1, u_2, \ldots$.

We denote by $\mathscr{U}_n^k$ the set of differential polynomials in $[u]$ of degree $k+1$ and index n. Let $\mathscr{U}^k = \bigoplus_n \mathscr{U}_n^k$, and $\mathscr{U} = \bigoplus_{k\geq 0} \mathscr{U}^k$, the algebra of all differential polynomials. Notice that we consider $k \geq 0$ that excludes the constant case, i.e. $1 \notin \mathscr{U}$. The *order* of a differential polynomial is the maximum of the orders of its constituent monomials.

The *symbolic transform* defines a linear isomorphism between the space $\mathscr{U}^k$ of (non)-commutative differential polynomials of degree $k+1$ and the space $\mathscr{A}^k = \mathrm{R}[\xi_1,\ldots,\xi_{k+1}]$ of algebraic polynomials in $k+1$ variables. It is uniquely defined by its action on monomials.

Definition 1. The symbolic form of a differential monomial is defined as

$$\begin{aligned} u_{i_1}u_{i_2}\cdots u_{i_k} \quad &\longmapsto \quad u^k < \xi_1^{i_1}\xi_2^{i_2}\cdots\xi_k^{i_k} > \\ &= \begin{cases} u^k \xi_1^{i_1}\xi_2^{i_2}\cdots\xi_k^{i_k} & \text{(noncommutative)}; \\ \dfrac{u^k}{k!}\displaystyle\sum_{\pi\in\mathbb{S}^k} \xi_{\pi(1)}^{i_1}\xi_{\pi(2)}^{i_2}\cdots\xi_{\pi(k)}^{i_k} & \text{(commutative)}, \end{cases} \end{aligned}$$

where $\mathbb{S}^k$ is the permutation group of k elements.

In general, in analogy with Fourier transforms, we denote the symbolic form of $P \in \mathscr{U}^k$, whether it is commutative or not, by $\widehat{P}$. The transform has two basic properties:

$$\begin{aligned} \widehat{D_x P}(\xi_1,\ldots,\xi_{k+1}) &= (\xi_1+\cdots+\xi_{k+1})\,\widehat{P}(\xi_1,\ldots,\xi_{k+1}), \\ \widehat{\frac{\partial P}{\partial u_i}}(\xi_1,\cdots,\xi_k) &= \frac{1}{i!}\frac{1}{k+1}\sum_{j=1}^{k+1}\frac{\partial^{i+1}\widehat{P}}{\partial u(\partial\xi_j)^i}(\xi_1,\cdots,\xi_{j-1},0,\xi_j,\cdots,\xi_k). \end{aligned} \tag{2.2}$$

The following key result is a consequence of these formulae.

Proposition 2. *Let $K \in \mathscr{U}^m$ and $Q \in \mathscr{U}^n$. Then $D_K(Q) \in \mathscr{U}^{m+n}$, where D_K is the Fréchet derivative of K, and*

$$\begin{aligned} \widehat{D_K[Q]} = \frac{1}{m+1}\sum_{\tau=1}^{m+1} \\ \times\left\langle \frac{\partial\widehat{K}}{\partial u}\left(\xi_1,\ldots,\xi_{\tau-1},\sum_{\kappa=0}^{n}\xi_{\tau+\kappa},\xi_{\tau+n+1},\ldots,\xi_{m+n+1}\right)\widehat{Q}(\xi_\tau,\ldots,\xi_{\tau+n})\right\rangle. \end{aligned}$$

Proof. Using (2.2), we compute

$$\begin{aligned} \widehat{D_K(Q)} &= \left\langle \sum_i \widehat{\frac{\partial K}{\partial u_i}}\widehat{D_x^i Q}\right\rangle \\ &= \left\langle \sum_i \frac{1}{i!}\frac{1}{m+1}\sum_{\tau=1}^{m+1}\frac{\partial^{i+1}\widehat{K}}{\partial u(\partial\xi_\tau)^i}(\xi_1,\cdots,\xi_{\tau-1},0,\xi_\tau,\cdots,\xi_m)\right. \\ &\quad\left.\times(\zeta_1+\cdots+\zeta_{n+1})^i\hat{Q}(\zeta_1,\cdots,\zeta_{n+1})\right\rangle \end{aligned}$$

$$= \frac{1}{m+1}\sum_{\tau=1}^{m+1}\left\langle \frac{\partial \widehat{K}}{\partial u}\left(\xi_1,\cdots,\xi_{\tau-1},\sum_{\kappa=1}^{n+1}\zeta_\kappa,\xi_\tau,\cdots,\xi_m\right)\hat{Q}(\zeta_1,\cdots,\zeta_{n+1})\right\rangle$$

and the conclusion follows. □

For any $K,Q\in\mathscr{U}$, we define $[K,Q]=D_Q(K)-D_K(Q)$. This bracket makes $\mathscr{U}$ into a graded Lie algebra.

The following polynomials play a critical role in the analysis.

Definition 3. The G-functions are the (commutative) polynomials

$$G_k^{(m)}=\xi_1^k+\cdots+\xi_{m+1}^k-\left(\xi_1+\cdots+\xi_{m+1}\right)^k.$$

The key fact is the following formula for the bracket of a differential polynomial with a linear differential polynomial:

$$\widehat{[u_k,Q]}=G_k^{(m)}\widehat{Q},\qquad \text{whenever}\qquad Q\in\mathscr{U}^m. \tag{2.3}$$

This follows directly from Proposition 2 and the fact that u_k has symbolic form $\widehat{u_k}=u\,\xi_1^k$. An immediate application is the known result that the space of the symmetries of linear evolution equations $u_t=u_n$ with $n>1$ is $\mathscr{U}^0$, as shown in the following:

Proposition 4. *Consider the linear evolution equation $u_t=\sum_{j=1}^p\lambda_j u_j$, where the λ_j are constants and $\lambda_p\neq 0$. The space of its symmetries is*

- $\mathscr{U}$ *iff* $p=1$*;*
- $\mathscr{U}^0$ *iff* $p>1$.

Proof. Let $Q\in\mathscr{U}$ and $Q=\sum Q^i$, where $Q^i\in\mathscr{U}^i$. Since $\mathscr{U}$ is a graded Lie algebra, Q is a symmetry of this equation iff $[\sum_{j=1}^p\lambda_j u_j,\ Q^i]=0$ for any $i\geq 0$. Formula (2.3) leads to

$$\sum_{j=1}^p\lambda_j G_j^{(i)}=0.$$

Under the assumption, this holds iff either $p=1$ or $p\neq 1$ and $i=0$. □

The crucial step is the following result [2] on the divisibility properties of the G-functions. The proof relies on sophisticated techniques from diophantine analysis.

Proposition 5. *The symmetric polynomials $G_n^{(k)}$ can be factorized as*

$$G_n^{(k)}=t_n^k g_n^{(k)},\textit{where}\quad (g_n^{(k)},g_m^{(k)})=1,\quad \textit{for all } n<m,$$

and t_n^k is one of the following polynomials:

- $k=1:$
  - $m=0\pmod 2):\quad \xi_1\xi_2$
  - $m=3\pmod 6):\quad \xi_1\xi_2(\xi_1+\xi_2)$

– $m = 5 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)$
– $m = 1 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)^2$

- $k = 2$:

  – $m = 0 \pmod 2$: 1
  – $m = 1 \pmod 2$: $(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)$

- $k > 2 : 1$

Proof. For $k = 1$, this was proved by F. Beukers using diophantine approximation theory [2]; for $k = 2$, see Appendix 2.8; and $k > 2$ is a special case of Theorem 16. □

Despite the innocent look of the polynomials involved, we have not been able to find a simpler proof for $k = 1$. It is conjectured that the $g_m^{(1)}$ are $\mathbb{Q}[\xi]$-irreducible.

2.2.2 Computational Example: Fifth-Order Symmetry of KdV

To illustrate how the symbolic method works, we give the symbolic calculation for the fifth-order symmetry of the Korteweg–de Vries equation. When one computes a symmetry, the natural approach is to do this degree by degree. So for instance, if we have the equation

$$u_t = K = K_3^0 + K_1^1 = u_3 + uu_1 \qquad \text{(KdV)}$$

then we try a symmetry

$$S_5 = S_5^0 + S_3^1 + \cdots = u_5 + a_1uu_3 + a_2u_1u_2 + \cdots ,$$

where $K_i^j, S_i^j \in \mathscr{U}_i^j$. We have to solve $[K_3^0, S_3^1] + [K_1^1, S_5^0] = 0$, i.e.

$$\begin{aligned} &D_x^3S_3^1 + uD_xS_5^0 + u_1S_5^0 \\ &= D_x^5K_1^1 + a_1uD_x^3K_3^0 + a_1u_3K_3^0 + a_2u_1D_x^2K_3^0 + a_2u_2D_xK_3^0. \end{aligned}$$

Translating this to the symbols, we have

$$\begin{aligned} &(\xi_1+\xi_2)^3\,\hat{S}_3^1 + \left(\xi_1^5+\xi_2^5\right)\hat{K}_1^1 \\ &= (\xi_1+\xi_2)^5\,\hat{K}_1^1 + \left(\xi_1^3+\xi_2^3\right)\hat{S}_3^1, \end{aligned}$$

where $\hat{K}_1^1 = \frac{u^2}{2}(\xi_1+\xi_2)$. We can now (formally) express $\hat{S}_3^1$ in terms of $\hat{K}_1^1$:

$$\hat{S}_3^1 = \frac{(\xi_1+\xi_2)^5 - \xi_1^5 - \xi_2^5}{(\xi_1+\xi_2)^3 - \xi_1^3 - \xi_2^3}\hat{K}_1^1.$$

By Definition 3, this can be rewritten as $\hat{S}_3^1 = \frac{G_5^{(1)}}{G_3^{(1)}}\hat{K}_1^1$. This is a real solution if $\hat{S}_3^1$ turns out to be a polynomial. Thus we have translated our problem into the question whether the polynomials $G_5^{(1)}$ and $G_3^{(1)}$ have common factors.

The symbolic method brings the possibility to apply the invariant theory of the permutation group to attack the classification problem.

Let us introduce ξ_0 by requiring that $\xi_0 + \xi_1 + \xi_2 = 0$. For odd n, we have

$$G_n^{(1)} = \sum_{i=0}^{2} \xi_i^n,$$

that is, the $G_n^{(1)}$ are $\mathbb{S}^3$-invariants, where $\mathbb{S}^3$ permutes the ξ-indices. Let

$$c_n = \sum_{i=0}^{2} \xi_i^n, \qquad n = 2,3. \tag{2.4}$$

The invariants of $\mathbb{S}^3$ are generated by c_2 and c_3. This implies that $G_3^{(1)} \equiv c_3$ and $G_5^{(1)} \equiv c_2 c_3$ up to multiplication by constants, since there is only one way in which we can write 5 as an additive combination of 2 and 3. Therefore $\hat{S}_3^1 \equiv c_2\hat{K}_1^1$. To be explicit,

$$\hat{S}_3^1 = \frac{5}{3}\left(\xi_1^2 + \xi_1\xi_2 + \xi_2^2\right)\hat{K}_1^1 = \frac{5}{6}\left(\xi_1^3 + 2\xi_1^2\xi_2 + 2\xi_1\xi_2^2 + \xi_2^3\right)u^2.$$

Let us compute S_1^2 by solving $[S_3^1, K_1^1] + [S_1^2, K_3^0] = 0$. By Proposition 2, this leads to

$$\hat{S}_1^2 = \frac{5}{6}\frac{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)(\xi_1+\xi_2+\xi_3)}{(\xi_1+\xi_2+\xi_3)^3 - \xi_1^3 - \xi_2^3 - \xi_3^3}u^3 = \frac{5}{18}(\xi_1+\xi_2+\xi_3)u^3.$$

Note that $[S_1^2, K_1^1] = 0$ in the next degree. Thus, the fifth-order symmetry is

$$S_5 = S_5^0 + S_3^1 + S_1^2 = u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1,$$

the well-known Lax equation.

This illustrates both the simplification induced by the symbolic method as well as the role of the G-functions in the whole analysis. The fact that the fifth-order integrable equations like Kaup–Kupershmidt and Sawada–Kotera have hierarchies with period 6 can now be explained by the invariant group $\mathbb{S}^3$.

2.2.3 The Higher Order Symmetries of KdV

What do we need in order to show that there exists a symmetry at every odd order for the Korteweg–de Vries equation? Let us sketch the computation for a higher order symmetry

$$S_{2k+1} = S^0_{2k+1} + S^1_{2k-1} + \cdots = u_{2k+1} + a_1 u u_{2k-1} + a_2 u_1 u_{2k-2} + \cdots .$$

First we have to solve $[K^0_3, S^1_{2k-1}] + [K^1_1, S^0_{2k+1}] = 0$. If we translate this to the symbols, by Definition 3 we obtain

$$G^{(1)}_3 \hat{S}^1_{2k-1} - G^{(1)}_{2k+1} \hat{K}^1_1 = 0.$$

We can now (formally) express $\hat{S}^1_{2k-1}$ in terms of $\hat{K}^1_1$ as

$$\hat{S}^1_{2k-1} = \frac{G^{(1)}_{2k+1}}{G^{(1)}_3} \hat{K}^1_1,$$

and this is a real solution if $\hat{S}^1_{2k-1}$ turns out to be a polynomial. Since the invariants of $\mathbb{S}^3$ are generated by c_2 and c_3, cf. (2.4), that is, $G^{(1)}_{2k+1}$ is a polynomial in these two, we must have $c_3 | G^{(1)}_{2k+1}$. Therefore, $\hat{S}^1_{2k-1}$ is polynomial. Note that the whole argument is completely independent from the fact that we started with the Korteweg–de Vries equation; it only depends on the equation being third order. This means in general that there are no obstructions to be expected in computing the quadratic terms of an odd-order symmetry for third-order equation. The first obstructions do occur in the computation of the cubic terms.

2.3 An Implicit Function Theorem

In this section we formulate a theorem that leads to the proof that the existence of one generalized symmetry implies infinitely many under fairly relaxed condition. The theorem itself, stated in the context of graded (or filtered) Lie algebras, is not difficult to prove. Its difficulty lies in formulating and checking some technical conditions, which derive immediately from the symbolic formulation. Here we give the theorem in graded Lie algebra version so that the reader can understand it better. The filtered Lie algebra version is put in Appendix 2.9.

Consider a graded Lie algebra $\mathfrak{g} = \prod_{i=0}^{\infty} \mathfrak{g}^i$ and let V be a graded $\mathfrak{g}$-module $\prod_{i=0}^{\infty} V^i$, where the action of $\mathfrak{g}$ on V is such that if $X^i \in \mathfrak{g}^i$ and $v^j \in V^j$, then $X^i \cdot v^j \in V^{i+j}$.

Example 6. A typical example is $\mathfrak{g}^j = \mathscr{V}^j = \mathscr{U}^j$, the set of differential polynomials of degree $j+1$, and the action of $\mathfrak{g}$ on $\mathscr{V}$ is the usual adjoint action given by the Lie bracket.

Definition 7. We call $K^0 \in \mathfrak{g}^0$ **nonlinear injective** if $\mathrm{Ker}\, K^0 \cdot \subset V^0$.

For Example 6, any element in $\mathfrak{g}^0$ is nonlinear injective unless it is a multiple of u_1, cf. Proposition 4.

Definition 8. We call $S^0 \in \mathfrak{g}^0$ **relatively l-prime** with respect to $K^0 \in \mathfrak{g}^0$ if $S^0 \cdot X^j \in \mathrm{Im}\, K^0 \Rightarrow X^j \in \mathrm{Im}\, K^0$ for all $j \geq l$.

We know from formula (2.3) that the Lie bracket of a differential polynomial with an element in $\mathfrak{g}^0$ equals the multiplication with a G-function, cf. Definition 3. In this case, this definition can be checked by answering whether the corresponding G-functions of S^0 and K^0 have common factors.

Theorem 9. *Let* $K = \sum_{i=0}^{k} K^i$ *and* $S = \sum_{i=0}^{s} S^i$, *where* $K^i, S^i \in \mathfrak{g}^i$ *and* $0 < k, s \in \mathbb{N}$. *Suppose there exists* $Q^j \in V^j$, $j = 0, \cdots, l-1$ *such that*

- $[K, S] = 0$,
- K^0 *is nonlinear injective,*
- S^0 *is relatively l-prime with respect to* K^0,
- $\sum_{i=0}^{p} K^i \cdot Q^{p-i} = 0$ *for* $p = 0, \cdots, l-1$ *and* $S^0 \cdot Q^0 = 0$.

Then there exists a unique $Q = \sum_{i=0}^{\infty} Q^i$, $Q^i \in V^i$, *such that* $K \cdot Q = S \cdot Q = 0$.

Proof. First we prove that $\sum_{i=0}^{j} S^{j-i} \cdot Q^i = 0$ for all $0 \leq p < l$ by induction.

For $p = 0$ this is true by assumption. Suppose it is true for all $j \leq p < l-1$. Now we show it is also true for $p+1$. We know that the action of a Lie algebra on a module is $[K, S] \cdot = K \cdot S \cdot - S \cdot K \cdot$ and that the assumption that $[K, S] = 0$ implies that $\sum_{j=0}^{q} [K^j, S^{q-j}] = 0$, for any $q \in \mathbb{N}$. It follows

$$\begin{aligned} K^0 \cdot \sum_{i=0}^{p+1} S^{p+1-i} \cdot Q^i &= \sum_{i=0}^{p+1} \left[K^0, S^{p+1-i}\right] \cdot Q^i + \sum_{i=0}^{p+1} S^{p+1-i} \cdot K^0 \cdot Q^i \\ &= -\sum_{i=0}^{p+1} \sum_{j=1}^{p+1-i} \left[K^j, S^{p+1-i-j}\right] \cdot Q^i - \sum_{i=0}^{p+1} S^{p+1-i} \cdot \sum_{j=1}^{i} K^j \cdot Q^{i-j} \\ &= -\sum_{j=1}^{p+1} \sum_{i=0}^{p+1-j} \left[K^j, S^{p+1-i-j}\right] \cdot Q^i - \sum_{j=1}^{p+1} \sum_{i=j}^{p+1} S^{p+1-i} \cdot K^j \cdot Q^{i-j} \\ &= -\sum_{j=1}^{p+1} \sum_{i=0}^{p+1-j} \left([K^j, S^{p+1-i-j}] \cdot Q^i + S^{p+1-i-j} \cdot K^j \cdot Q^i\right) \\ &= -\sum_{j=1}^{p+1} K^j \cdot \sum_{i=0}^{p+1-j} S^{p+1-i-j} \cdot Q^i = -\sum_{j=0}^{p} K^{p-j+1} \cdot \sum_{i=0}^{j} S^{j-i} \cdot Q^i = 0. \end{aligned}$$

By the nonlinear injectiveness of K^0, we obtain that $\sum_{i=0}^{p+1} S^{p+1-i} \cdot Q^i = 0$.

Next we suppose that there exists $\sum_{j=0}^{p-1} Q^j$ satisfying $\sum_{i=0}^{j} K^{j-i} \cdot Q^i = 0$ and $\sum_{i=0}^{j} S^{j-i} \cdot Q^i = 0$ for $j = 0, \ldots, p-1$.

For $p = l$, this follows from the previous.

$$\begin{aligned} K^0 \cdot \sum_{i=0}^{p-1} S^{p-i} \cdot Q^i &= \sum_{i=0}^{p-1} \left[K^0, S^{p-i}\right] \cdot Q^i + \sum_{i=0}^{p-1} S^{p-i} \cdot K^0 \cdot Q^i \\ &= -\sum_{i=0}^{p-1} \sum_{j=1}^{p-i} \left[K^j, S^{p-i-j}\right] \cdot Q^i - \sum_{i=0}^{p-1} S^{p-i} \cdot \sum_{j=1}^{i} K^j \cdot Q^{i-j} \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{p}\sum_{i=0}^{p-j}\left[K^j,S^{p-i-j}\right]\cdot Q^i - \sum_{j=1}^{p}\sum_{i=j}^{p-1} S^{p-i}\cdot K^j\cdot Q^{i-j}\\
&= -\sum_{j=1}^{p}\left(\sum_{i=0}^{p-j}\left[K^j,S^{p-i-j}\right]\cdot Q^i + \sum_{i=0}^{p-1-j} S^{p-i-j}\cdot K^j\cdot Q^i\right)\\
&= -\sum_{j=1}^{p}K^j\cdot\sum_{i=0}^{p-j}S^{p-i-j}\cdot Q^i + S^0\cdot\sum_{j=1}^{p}K^j\cdot Q^{p-j} = S^0\cdot\sum_{j=1}^{p}K^j\cdot Q^{p-j}.
\end{aligned}
$$

We have $\sum_{j=1}^{p}K^j\cdot Q^{p-j}\in \mathrm{Im}\,K^0$ since S^0 is relatively l-prime with respect to K^0. So we can uniquely define Q^p by

$$
K^0\cdot Q^p = -\sum_{j=1}^{p}K^j\cdot Q^{p-j}\ .
$$

We then automatically have $\sum_{i=0}^{p}K^{p-i}Q^i=0$. That $\sum_{i=0}^{p}S^{p-i}Q^i=0$ follows from the first part of the proof. Again by induction on p, we prove that Q can always be extended such that all graded parts of $K\cdot Q$ and $S\cdot Q$ vanish. □

If one thinks of the application of this theorem to the computation of symmetries of evolution equations, cf. Example 6, then this proves (at least up till the existence of $\sum_{i=0}^{l}Q^i$) the long-held belief that one nontrivial symmetry S of the equation K is enough for integrability. With such a strong result one has to inspect the conditions. The strangest of them seems to be the relative prime condition. In the next sections, however, we show that for scalar equations with linear part $u_t=u_k$ any symmetry S starting with $u_s, s\notin\{1,k\}$ satisfies the conditions of the theorem with $l=2$ when $K^1\neq 0$ and $l=3$ when $K^1=0$ and $K^2\neq 0$.

2.4 Symmetry-Integrable Evolution Equations

2.4.1 Symmetries of λ-Homogeneous Equations

In this section we give the complete classification for homogeneous scalar commutative and noncommutative evolution equations. A key result is that it suffices to compute the linear and quadratic terms, or cubic if the quadratic terms are zero, of a nontrivial odd-order symmetry in order to guarantee its existence. This speeds up the classification process, since any obstructions to the existence of symmetries have to show up early in the computation.

The differential equation (2.1) is said to be **λ-homogeneous** of **weight** μ if it admits the one-parameter group of scaling symmetries

$$
(x,t,u)\longmapsto(a^{-1}x,a^{-\mu}t,a^{\lambda}u),\qquad a\in\mathbb{R}^+.
$$

For example, the Korteweg–de Vries equation $u_t = u_{xxx} + uu_x$ is homogeneous of weight 3 for $\lambda = 2$.

Two evolution equations $u_t = K$ and $u_t = Q$ are symmetries of each other if and only if [21]

$$[K,\ Q] = 0. \tag{2.5}$$

An equation is called (symmetry-)**integrable** if it has infinitely many linearly independent higher order symmetries.

Any λ-homogeneous evolution equation of order n can be broken up into its homogeneous components, and so it takes the form

$$u_t = K = \sum_{i\geq 0} K^i_{n-\lambda i},\ \ \left(K^i_{n-\lambda i} \in \mathscr{U}^i_{n-\lambda i}\right). \tag{2.6}$$

We assume that $K^0_n = u_n$, $n \geq 2$, and $0 < \lambda \in \mathbb{Q}$. When $i\lambda \notin \mathbb{N}$, $K^i_{n-i\lambda} = 0$. This reduces the number of relevant λ to a finite set.

For $\lambda = 1$, this describes the family of Burgers-like equations and for $\lambda = 2$ the family of KdV-like equations.

Let $S \in \mathscr{U}$ be a symmetry of order m of the evolution equation (2.6). We break up the bracket condition $[S,K] = 0$ into its homogeneous summands, leading to the series of successive symmetry equations

$$\sum_{i+j=r} \left[S^j_{m-\lambda j}, K^i_{n-\lambda i}\right] = 0, \qquad \text{for} \qquad r = 0,1,2,\dots\ . \tag{2.7}$$

According to Proposition 4, S must have nontrivial linear term, $S^0_m \neq 0$, and we can set $S^0_m = u_m$ without loss of generality. Clearly we have $[S^0_m, K^0_n] = 0$. The next equation to be solved is

$$\left[S^0_m, K^1_{n-\lambda}\right] + \left[S^1_{m-\lambda}, K^0_n\right] = 0. \tag{2.8}$$

Condition (2.8) is trivially satisfied if K has no quadratic terms: $K^1_{n-\lambda} = 0$. Let us concentrate on the case $K^1_{n-\lambda} \neq 0$. We use (2.3) and Proposition 5 to rewrite (2.8) in symbolic form:

$$\widehat{K}^1_{n-\lambda} = \frac{\widehat{S}^1_{m-\lambda}}{G^{(1)}_m} G^{(1)}_n = u^2 \frac{p(\xi_1,\xi_2)}{\xi_1\xi_2(\xi_1+\xi_2)} G^{(1)}_n, \tag{2.9}$$

where $\lim_{\xi_1+\xi_2\to 0} p(\xi_1,\xi_2)$ exists. We next set $r = 2$ in (2.7) and find

$$\widehat{S}^2_{m-2\lambda} = \frac{\widehat{K}^2_{n-2\lambda} G^{(2)}_m + \widehat{M}}{G^{(2)}_n}, \tag{2.10}$$

where $\widehat{M}$ is the symbolic form of the commutator

$$M = \left[S^1_{m-\lambda}, K^1_{n-\lambda}\right] \tag{2.11}$$

between the quadratic terms.

We use the notation $q|p$ to indicate that the polynomial q divides the polynomial p. Consider the set

$$\mathscr{I} = \{ \ p(\xi_1,\xi_2) \ : \ (\xi_1+\xi_2)|p(\xi_1,\xi_2) \ \ or \ \ \xi_1\xi_2|p(\xi_1,\xi_2) \ \}$$

consisting of bivariate polynomials $p(\xi_1,\xi_2)$ that have either $\xi_1+\xi_2$ or $\xi_1\xi_2$ as a factor.

Proposition 10. *Suppose m and n are both odd. Let $\widehat{M}$ and p be given by* (2.11) *and* (2.9), *respectively. Then $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ divides $\widehat{M}$ iff $p\in\mathscr{I}$.*

Proof. Using formula (2.9), we compute $\widehat{M}$ to be

$$\begin{aligned}\widehat{M} = u^3 &\left\langle \frac{p(\xi_1+\xi_2,\xi_3)p(\xi_1,\xi_2)F_{\xi_2,\xi_3}(\xi_1+\xi_2)}{\xi_1\xi_2\xi_3(\xi_1+\xi_2)^2(\xi_1+\xi_2+\xi_3)} \right\rangle \\ &+ u^3 \left\langle \frac{p(\xi_1,\xi_2+\xi_3)p(\xi_2,\xi_3)F_{\xi_2,\xi_1}(\xi_2+\xi_3)}{\xi_1\xi_2\xi_3(\xi_2+\xi_3)^2(\xi_1+\xi_2+\xi_3)} \right\rangle,\end{aligned}$$

where

$$F_{\xi_i,\xi_j}(\eta) = G_n^{(1)}(\eta,\xi_j)\,G_m^{(1)}(\eta-\xi_i,\xi_i) - G_m^{(1)}(\eta,\xi_j)\,G_n^{(1)}(\eta-\xi_i,\xi_i)\,.$$

Here we only write out the analysis for noncommutative case. For the commutative case, the expression of $\widehat{M}$ needs to be symmetrized. However, the proof is quite similar, cf. [27].

Notice that $\xi_1+\xi_3$ is a factor of $\widehat{M}$. We now prove that $\lim_{\xi_1+\xi_2\to 0}\widehat{M}=0$. The second summand has

$$\begin{aligned}&\lim_{\xi_1+\xi_2\to 0} F_{\xi_2,\xi_1}(\xi_2+\xi_3) \\ &= G_n^{(1)}(-\xi_2,\xi_2+\xi_3)\,G_m^{(1)}(\xi_2,\xi_3) - G_m^{(1)}(-\xi_2,\xi_2+\xi_3)\,G_n^{(1)}(\xi_2,\xi_3) \\ &= -G_n^{(1)}(\xi_2,\xi_3)\,G_m^{(1)}(\xi_2,\xi_3) + G_m^{(1)}(\xi_2,\xi_3)\,G_n^{(1)}(\xi_2,\xi_3) = 0.\end{aligned}$$

As for the first part, a straightforward computation shows that

$$F_{\xi_2,\xi_3}(0) = 0 = \frac{d}{d\eta}F_{\xi_2,\xi_3}(0).$$

Moreover,

$$\begin{aligned}\frac{d^2}{d\eta^2}F_{\xi_2,\xi_3}(0) &= 2\left(\frac{d}{d\eta}G_n^{(1)}(\xi_3,\eta)\frac{d}{d\eta}G_m^{(1)}(\eta-\xi_2,\xi_2)\right. \\ &\qquad \left. - \frac{d}{d\eta}G_n^{(1)}(\eta-\xi_2,\xi_2)\frac{d}{d\eta}G_m^{(1)}(\xi_3,\eta)\right) \Big|_{\eta=0} \\ &= 2nm\left(\xi_3^{m-1}\xi_2^{n-1} - \xi_3^{n-1}\xi_2^{m-1}\right) \neq 0.\end{aligned}$$

This implies that

$$\lim_{\xi_1+\xi_2\to 0}\frac{F_{\xi_2,\xi_3}(\xi_1+\xi_2)}{(\xi_1+\xi_2)^2}\neq 0$$

and therefore $(\xi_1+\xi_2)\not|\widehat{M}$ unless $(\xi_1+\xi_2)|p(\xi_1+\xi_2,\xi_3)p(\xi_1,\xi_2)$ or, equivalently, $(\xi_1+\xi_2)|p(\xi_1,\xi_2)$ or $\xi_1|p(\xi_1,\xi_2)$. Similarly, when we deal with factor $\xi_2+\xi_3$, we obtain $(\xi_2+\xi_3)\not|\widehat{M}$ unless $(\xi_1+\xi_2)|p(\xi_1,\xi_2)$ or $\xi_2|p(\xi_1,\xi_2)$. Therefore, the statement of the proposition follows. □

Corollary 11. *Assume m and n are odd. Then* $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ *divides* $\widehat{K}^2_{n-2\lambda}G^{(2)}_m+\widehat{M}$ *if and only if* $\widehat{K}^1_{n-\lambda}(\xi_1,\xi_2)\in\mathscr{I}$.

We next state a result that says the symmetry algebra of a commutative or non-commutative polynomial evolution equation is commutative. Moreover, every symmetry is uniquely determined by its quadratic terms.

Theorem 12. *Suppose the evolution equation* (2.6) *has a nonzero symmetry S of order* $m\geq 2$ *. Suppose* $Q^1_{q-\lambda}$ *is a nonzero quadratic differential polynomial* ($q\geq\lambda$), *where* $q\notin\{m,n\}$, *and q is odd if n is odd, which satisfies the leading order symmetry condition* $[K^0_n,Q^1_{q-\lambda}]+[K^1_{n-\lambda},Q^0_q]=0$, *cf.* (2.8). *Then there exists a unique symmetry of the form* $Q=\sum_{i\geq 0}Q^i_{q-i\lambda}$. *Moreover, the symmetries Q and S commute.*

Proof. For even n or m, this follows from Theorem 9, since S^0_m is relatively 2-prime with respect to K^0_n.

We conclude from the existence of S that $(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)$ divides

$$\widehat{K}^2_{n-2\lambda}G^{(2)}_m+\left[\widehat{S}^1_{m-\lambda},\widehat{K}^1_{n-\lambda}\right] \tag{2.12}$$

for odd n and m. In other words, $\widehat{K}^1_{n-\lambda}(\xi_1,\xi_2)\in\mathscr{I}$.

Since S is a symmetry, i.e. $[K,\ S]=0$, we have

$$[K,\ [S,\ Q]]=[S,\ [K,\ Q]]$$

from Jacobi identity. We break it up into its homogeneous summands leading to

$$g^{(2)}_n\left(\left[\widehat{S}^1,\widehat{Q}^1\right]+\left[\widehat{S}^2,\widehat{Q}^0\right]\right)=g^{(2)}_m\left(\left[\widehat{K}^1,\widehat{Q}^1\right]+\left[\widehat{K}^2,\widehat{Q}^0\right]\right).$$

We know that $(g^{(2)}_m,g^{(2)}_n)=1$, and (by exactly the same argument as for S)

$$(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)\,|\left(\left[\widehat{K}^1,\widehat{Q}^1\right]+\left[\widehat{K}^2,\widehat{Q}^0\right]\right).$$

This implies that $G^{(2)}_n$ divides $[\widehat{K}^1,\widehat{Q}^1]+[\widehat{K}^2,\widehat{Q}^0]$. Therefore,

$$\widehat{Q}^2_{q-2\lambda}=\frac{[\widehat{Q}^1,\widehat{K}^1]+[\widehat{Q}^0,\widehat{K}^2]}{G^{(2)}_n}$$

is well defined. Since the $G_n^{(k)}$ are relative prime for $k > 2$, this means that K_m^0 is relatively 2-prime and we can apply Theorem 9 to draw the conclusion that there indeed exists a symmetry Q commuting with S. □

We make a very interesting observation. Suppose Q is a nontrivial qth odd-order symmetry of (2.6) with odd n, whose quadratic terms, cf. (2.9), have the following symbolic expression:

$$\widehat{Q}^1_{q-\lambda} = \frac{\widehat{K}^1_{n-\lambda}\ \left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right)^{s-s'}\ g_q^{(1)}}{g_n^{(1)}}.$$

Proposition 5 implies that $\lambda \leq 3+2\min(s,s')$, where $s' = \frac{n+3}{2} \pmod 3$ and $s = \frac{q+3}{2} \pmod 3$. Then Theorem 12 implies that

$$\widehat{Q}^1_{2s+3-\lambda} = \frac{\widehat{K}^1_{n-\lambda}\ \left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right)^{s-s'}\ g_{2s+3}^{(1)}}{g_n^{(1)}}$$

gives rise to a symmetry $Q = Q^0_{2s+3} + Q^1_{2s+3-\lambda} + \cdots$ of the original equation. (Of course, one can use this argument to generate an entire hierarchy of symmetries.) This implies that the evolution equations defined by Q and K have the same symmetries, so instead of considering K we may consider the equation given by Q, which is of order $q = 2s+3$ for $s = 0,1,2$. It follows that we only need to find the symmetries of λ-homogeneous equations (with $\lambda \leq 7$) of order ≤ 7 in order to obtain the complete classification of symmetries of λ-homogeneous scalar polynomial equations starting with linear terms.

A similar observation can be made for even $n > 2$. Suppose we have found a nontrivial symmetry with quadratic term

$$\widehat{Q}^1_{q-\lambda} = \frac{\widehat{K}^1_{n-\lambda} G_q^{(1)}}{\xi_1\xi_2\, g_n^{(1)}}.$$

This immediately implies $\lambda \leq 2$. Then $\widehat{Q}^1_{2-\lambda} = 2\widehat{K}^1_{n-\lambda}/g_n^{(1)}$ gives rise to a symmetry $Q = Q^0_2 + Q^1_{2-\lambda} + \cdots$ of the original equation. Therefore, we only need to find the symmetries of equations of order 2 to get the complete classification of symmetries of λ-homogeneous scalar polynomial equations (with $\lambda \leq 2$) starting with an even linear term.

Finally, we must analyze the case when K has no quadratic terms. Assume that $K^i_{n-\lambda i} = 0$ for $i = 1,\ldots,j-1$, and $K^j_{n-\lambda j} \neq 0$ for some $j > 1$. In place of (2.8), we now need to solve the leading order equation

$$[S^0_m, K^j_{n-j\lambda}] + [S^j_{m-j\lambda}, K^0_n] = 0.$$

Using (2.3), the symbolic form of this condition is

$$\widehat{S}^j_{m-j\lambda} = \frac{\widehat{K}^j_{n-j\lambda} G^{(j)}_m}{G^{(j)}_n}. \tag{2.13}$$

Proposition 5 implies that this polynomial identity has no solutions when $j \geq 3$, or when $j = 2$ and n is even, since $G^{(j)}_m$ and $G^{(j)}_n$ have no common factors, and the degree of $K^j_{n-j\lambda}$ is $n - j\lambda < n$, which is the degree of $G^{(j)}_n$. Thus there are no symmetries for such equations. When $j = 2$ and n is odd, the equation can only have odd-order symmetries. If Eq. (2.13) can be solved for any m, it can also be solved for $m = 3$.

By now, we have proved the following

Theorem 13. *A nontrivial symmetry of a λ-homogeneous equation with $\lambda > 0$ is part of a hierarchy starting at order $3, 5$ or 7 in the odd case, and at order 2 in the even case.*

2.4.2 The List of Symmetry-Integrable Equations

Only an equation with nonzero quadratic or cubic terms can have a nontrivial symmetry. For each possible $\lambda > 0$, we must find a third-order symmetry for a second-order equation, a fifth-order symmetry for a third-order equation, a seventh-order symmetry for a fifth-order equation with quadratic terms and the thirteenth-order symmetry for a seventh-order equation with quadratic terms. The last case can be easily reduced to the case of fifth-order equations by determining the quadratic terms of the equation. The details of this final symbolic computation are completed as in the commutative case described in [26].

2.4.2.1 Commutative Case

We list all integrable hierarchies which are λ-homogeneous, with $\lambda \geq 0$. For $\lambda = 0$, details can be found in [28]. For $\lambda > 0$ the equivalence transformations are just scalings $u \mapsto \alpha u$, while for $\lambda = 0$ we allow arbitrary change of variables $u \mapsto h(u)$. The classification theorem states that every λ-homogeneous evolution equation with linear leading term is equivalent, modulo homogeneous transformations in u, to an equation lying in one of the following hierarchies.

Korteweg–de Vries

$$u_t = u_3 + uu_1$$

Kaup–Kupershmidt

$$u_t = u_5 + 10uu_3 + 25u_1u_2 + 20u^2u_1$$

Sawada–Kotera

$$u_t = u_5 + 10uu_3 + 10u_1u_2 + 20u^2u_1$$

Burgers

$$u_t = u_2 + uu_1$$

Potential Korteweg–de Vries

$$u_t = u_3 + u_1^2$$

Modified Korteweg–de Vries

$$u_t = u_3 + u^2u_1$$

Potential Kaup–Kupershmidt

$$u_t = u_5 + 10u_1u_3 + \frac{15}{2}u_2^2 + \frac{20}{3}u_1^3$$

Potential Sawada–Kotera

$$u_t = u_5 + 10u_1u_3 + \frac{20}{3}u_1^3$$

Kupershmidt Equation [19, 4.2.6]

$$u_t = u_5 + 5u_1u_3 + 5u_2^2 - 5u^2u_3 - 20\,uu_1u_2 - 5u_1^3 + 5u^4u_1$$

Ibragimov–Shabat [5]

$$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$$

Potential Burgers/Heat Equation

$$u_t = u_2 \quad \sim \quad u_t = u_2 + u_1^2$$

Potential modified Korteweg–de Vries

$$u_t = u_3 + u_1^3$$

Potential Kupershmidt Equation

$$u_t = u_5 + 5u_2u_3 - 5u_1^2u_3 - 5u_1u_2^2 + u_1^5$$

2.4.2.2 Noncommutative Case

Recently, the analysis of integrable evolution equations in which the field variable u takes its values in an associative, noncommutative algebra, such as matrix, operator, Clifford and group algebras, has attracted attention. A complete classification for $\lambda > 0$ homogeneous equations with linear leading term was established in

[24]. (The case $\lambda = 0$ poses considerable technical difficulties.) There are only five noncommutative hierarchies, each generalizing one of the preceding commutative hierarchies.

Korteweg–de Vries

$$u_t = u_3 + uu_1 + u_1 u$$

Burgers

$$u_t = u_2 + uu_1, \qquad u_t = u_2 + u_1 u$$

Potential Korteweg–de Vries

$$u_t = u_3 + u_1^2$$

Modified Korteweg–de Vries I

$$u_t = u_3 + u^2 u_1 + u_1 u^2$$

Modified Korteweg–de Vries II

$$u_t = u_3 + uu_2 - u_2 u - \frac{2}{3} uu_1 u$$

Interestingly, whereas the mKdV has two inequivalent noncommutative versions, there is no noncommutative generalization of the Sawada–Kotera, Kaup-Kupershmidt, Kupershmidt, or Ibragimov–Shabat hierarchies.

2.5 Evolution Systems with *k* Components

In this section, we use a simple geometric fact to prove that homogeneous evolution systems with positive weights of order larger than 1 and their linear parts with distinct nonzero eigenvalues are not symmetry-integrable without quadratic and cubic terms.

As we mentioned in Sect. 2.2, the generalization of the symbolic method to more dependent variables is straightforward. We introduce a symbol for each of dependent variables, like u and v, for instance ξ and η. Thus the symbolic expression for $u_1 u_2 v_3$ is $\frac{1}{2}\xi_1\xi_2\eta_1^3(\xi_1+\xi_2)u^2 v$, symmetric with respect to ξ_1 and ξ_2, the symbols from us, and with respect to η_1, the symbol from v. If we would not carry along the u's and v's, information would be lost: consider the expressions uv and u^2. The alternative would be to keep the zeroth power of any symbol, so that uv would go to $\xi^0\eta^0$, but this is very awkward in actual polynomial computations.

Consider evolutionary vectorfields with two components u and v. Let $\mathscr{U}_m^{(i,j)}$ denote a set of differential polynomial vectorfields with index m, total number of x-derivatives, and degree i in u and j in v. This degree can be -1: $\frac{\partial}{\partial u} \in \mathscr{U}_0^{(-1,0)}$.

Assume the weights of u and v are λ_1 and λ_2, respectively, and $\lambda_2 \geq \lambda_1 > 0$. So any nth-order homogeneous system can be written:

$$u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} = K = \sum_{i,j} K^{(i,j)}_{n-i\lambda_1 - j\lambda_2}, \quad K^{(i,j)}_l \in \mathscr{U}^{(i,j)}_l, \quad i,j \geq -1. \tag{2.14}$$

Only when $n - i\lambda_1 - j\lambda_2 < n \in \mathbb{N}$ does the term $K^{(i,j)}$ make sense and can appear in the system. The linear part of the system can be written as $K^{(0,0)}_n + K^{(-1,1)}_{n-\lambda_2+\lambda_1}$, where $K^{(0,0)}_n = a_1 u_n \frac{\partial}{\partial u} + a_2 v_n \frac{\partial}{\partial v}$, $K^{(-1,1)}_{n-\lambda_2+\lambda_1} = a_3 v_{n-\lambda_2+\lambda_1} \frac{\partial}{\partial u}$, and $a_i \in \mathbb{C}$.

Assumption 14. *We assume that the linear part of the system equals*

$$K^{(0,0)}_n = a_1 u_n \frac{\partial}{\partial u} + a_2 v_n \frac{\partial}{\partial v}, \quad a_1 a_2 \neq 0, \quad a_1 \neq a_2, \quad n \geq 2. \tag{2.15}$$

Since the linear part is diagonal, it will act semisimply on polynomial vectorfields. This simplifies the analysis considerably. Let us compute the action of the diagonal linear part on vectorfields of $Q^{(i,j)}$ using the symbolic method:

$$\left[Q^{(i,j)}, \widehat{\begin{pmatrix} a_1 u_n \\ a_2 v_n \end{pmatrix}} \right] = \begin{pmatrix} f^{(i,j)}_{u;n}(a_1,a_2;\xi;\eta) & 0 \\ 0 & f^{(i,j)}_{v;n}(a_1,a_2;\xi;\eta) \end{pmatrix} \hat{Q}^{(i,j)}(\xi;\eta),$$

where $\hat{Q}^{(i,j)}(\xi;\eta)$ is the symbolic expression of $Q^{(i,j)}$ and

$$\begin{aligned} f^{(i,j)}_{u;n}(a_1,a_2;\xi;\eta) &= a_1 \left(\sum_{l=1}^{i+1} \xi_l + \sum_{k=1}^{j} \eta_k \right)^n - a_1 \sum_{l=1}^{i+1} \xi_l^n - a_2 \sum_{k=1}^{j} \eta_k^n; \\ f^{(i,j)}_{v;n}(a_1,a_2;\xi;\eta) &= a_2 \left(\sum_{l=1}^{i} \xi_l + \sum_{k=1}^{j+1} \eta_k \right)^n - a_1 \sum_{l=1}^{i} \xi_l^n - a_2 \sum_{k=1}^{j+1} \eta_k^n. \end{aligned} \tag{2.16}$$

These are two important polynomials corresponding to the G-functions in scalar case, cf. Definition 3, and related by

$$f^{(i,j)}_{u;n}(a_1,a_2;\xi;\eta) = f^{(j,i)}_{v;n}(a_2,a_1;\eta;\xi). \tag{2.17}$$

This calculation immediately leads to the following result (cf. Proposition 4):

Proposition 15. *The space of the symmetries of a linear system of the form of* (2.15) *is* $\mathscr{U}^{(0,0)} = \bigoplus_m \mathscr{U}^{(0,0)}_m$.

We are now in the position to do the same analysis as in Sect. 2.4. However, since we do not have the neat results on functions (2.16) as in Proposition 5 for the G-functions, the analysis is more complicated and difficult, for details see [31], where we did complete classification for second-order evolution equations with two components.

Let S be a symmetry of order m of system (2.14). Its linear part is in $\mathscr{U}^{(0,0)}$. Without loss of generality, we set $S^{(0,0)}_m = b_1 u_m \frac{\partial}{\partial u} + b_2 v_m \frac{\partial}{\partial v}$. The next equation to be solved is

$$\left[S_m^{(0,0)}, K_{n-i\lambda_1-j\lambda_2}^{(i,j)}\right] = \left[K_n^{(0,0)}, S_{m-i\lambda_1-j\lambda_2}^{(i,j)}\right], \quad i+j=1. \tag{2.18}$$

Assume that system (2.14) has no quadratic and cubic terms, that is,

$$K_{n-i\lambda_1-j\lambda_2}^{(i,j)} = 0, \quad 1 \le i+j \le 2.$$

We then need to solve (2.18) for $i+j=3$. Translating this to the symbolic language, we need to study

$$\left(f_{u;n}^{(i,j)}(a_1,a_2;\xi;\eta), f_{u;m}^{(i,j)}(b_1,b_2;\xi;\eta)\right), \quad i+j=3.$$

If they have no common factors, system (2.14) has no such symmetry.

The following theorem is due to Frits Beukers.

Theorem 16. *For any positive integer m the polynomial*

$$h_{c,m} = (\xi_1+\xi_2+\xi_3+\xi_4)^m - c_1^{m-1}\xi_1^m - c_2^{m-1}\xi_2^m - c_3^{m-1}\xi_3^m - c_4^{m-1}\xi_4^m,$$

where $\Pi_{i=1}^4 c_i \neq 0$*, is irreducible over* $\mathbb{C}$.

Proof. Suppose that $h_{a,m} = A \cdot B$ with A, B polynomial of positive degree. Then the projective hypersurface Σ given by $h_{a,m} = 0$ consists of two components Σ_A, Σ_B given by $A=0$ and $B=0$, respectively. $\Sigma_A \bigcap \Sigma_B$ consists of an infinite number of points, which should be singularities of Σ since

$$\frac{dh_{a,m}}{d\xi_i} = \frac{dA}{d\xi_i} \cdot B + A \cdot \frac{dB}{d\xi_i}|_{\Sigma_A \bigcap \Sigma_B} = 0.$$

Thus it suffices to show that Σ has finitely many singular points.

We compute the singular points by setting the partial derivatives of $h_{c,m}$ equal to zero, i.e.

$$\begin{aligned}
(\xi_1+\xi_2+\xi_3+\xi_4)^{m-1} - (c_1\xi_1)^{m-1} &= 0,\\
(\xi_1+\xi_2+\xi_3+\xi_4)^{m-1} - (c_2\xi_2)^{m-1} &= 0,\\
(\xi_1+\xi_2+\xi_3+\xi_4)^{m-1} - (c_3\xi_3)^{m-1} &= 0,\\
(\xi_1+\xi_2+\xi_3+\xi_4)^{m-1} - (c_4\xi_4)^{m-1} &= 0.
\end{aligned}$$

From these equations it follows in particular that

$$\xi_1 = \zeta_1/c_1,\ \xi_2 = \zeta_2/c_2,\ \xi_3 = \zeta_3/c_3,\ \xi_4 = \zeta_4/c_4,$$

where $\zeta_i^{m-1} = 1$ and $\zeta_1/c_1 + \zeta_2/c_2 + \zeta_3/c_3 + \zeta_4/c_4 = 1$. For given $c_i, i = 1, \cdots, 4$, we get finitely many singular points. □

In two-component case, the c_i are determined by a_1 and a_2. The condition $\Pi_{i=1}^4 c_i \neq 0$ is automatically satisfied due to the assumption that $a_1 a_2 \neq 0$. This implies that when system (2.14) has no quadratic and cubic terms, i.e. $K^{(i,j)} = 0$

($1 \leq i+j \leq 2$), it is not integrable. One can even make the stronger statement that it has no nontrivial generalized symmetries at all!

We can draw the similar conclusion to k-component systems from this theorem that homogeneous evolution systems with positive weights of order large than 1 and their linear parts with distinct nonzero eigenvalues cannot have nontrivial generalized symmetries without quadratic and cubic terms.

2.6 One Symmetry Does not Imply Integrability

As we proved in Sect. 2.4, scalar evolution equations are integrable once one nontrivial generalized symmetry exists. However, this cannot be generalized to multi-component systems. The first example was found by Bakirov [1] (see also [21, p. 381], exercise 5.15 and [3]) that the system

$$\begin{cases} u_t = u_4 + v^2 \\ v_t = \frac{1}{5} v_4 \end{cases} \tag{2.19}$$

has one symmetry of order 6, but no others were found up till order 53. In this section, we prove that indeed no other symmetries exist for this system. Further classification and recognition of integrable such type of equations can be found in [13].

2.6.1 The Symbolic Interpretation of Bakirov's Example

We rewrite system (2.19) as $(u_4 + v^2)\frac{\partial}{\partial u} + \frac{1}{5} v_4 \frac{\partial}{\partial v}$. Its symbolic form is

$$\left(\xi_1^4 u + v^2\right) \frac{\partial}{\partial u} + \frac{1}{5} \eta_1^4 v \frac{\partial}{\partial v}$$

Since the system satisfies Assumption 14, from Proposition 15, its symmetry of a given order m has to start with $a u_m \frac{\partial}{\partial u} + b v_m \frac{\partial}{\partial v}$, i.e.

$$a \xi_1^m u \frac{\partial}{\partial u} + b \eta_1^m v \frac{\partial}{\partial v}.$$

At first sight we are losing some candidates (for being a symmetry) here, since we implicitly assume the vectorfield to be polynomial. As is shown in [1], however, this is not a restriction.

Computing the commutator of the quadratic part of system (2.19) with this linear part of the (potential) symmetry, we have

$$\left[\frac{\eta_1^0 + \eta_2^0}{2} v^2 \frac{\partial}{\partial u}, a \xi_1^m u \frac{\partial}{\partial u} + b \eta_1^m v \frac{\partial}{\partial v}\right] = \left(a(\eta_1 + \eta_2)^m - b(\eta_1^m + \eta_2^m)\right) v^2 \frac{\partial}{\partial u}.$$

Notation 17. *Let* $F_a^{(n)} = a(\eta_1+\eta_2)^n - (\eta_1^n+\eta_2^n)$ *and* $\overline{F}_a^{(n)} = a(x+1)^n - (x^n+1)$.

We now construct the quadratic terms of the symmetry. Provided $b \neq 0$, we compute

$$\left[\left(\xi_1^4 u + v^2\right)\frac{\partial}{\partial u} + \frac{1}{5}\eta_1^4 v\frac{\partial}{\partial v}, \left(a\xi_1^m u + \hat{A}v^2\right)\frac{\partial}{\partial u} + b\eta_1^m v\frac{\partial}{\partial v}\right]$$
$$= \left(bF_{a/b}^{(m)} - \frac{1}{5}\hat{A}F_5^{(4)}\right)v^2\frac{\partial}{\partial u}.$$

Let $\hat{A} = 5\, b\, F_{a/b}^{(m)}/F_5^{(4)}$. If $\hat{A}$ is polynomial in η_1, η_2, then

$$\left(a\xi_1^p u + \hat{A}v^2\right)\frac{\partial}{\partial u} + b\eta_1^p v\frac{\partial}{\partial v}$$

is a symmetry of system (2.19).

Therefore, the question about the existence of symmetries of an evolution system of the form (2.19) is translated into:

Question 18. Given a, n, for which $b \in \mathbb{C}$ and $m \in \mathbb{N}$ does $F_a^{(n)}$ divide $F_b^{(m)}$?

This can be answered by the following results.

Theorem 19. *Let* $a \in \mathbb{C}\backslash\{0,1\}$ *and* $n \in \mathbb{N}_{\geq 2}$. *We consider* $\overline{F}_a^{(n)}$. *Suppose that at least one of the following conditions holds:*

1. $n \geq 6$,
2. $n = 4, 5$ *and* $\overline{F}_a^{(n)}$ *has two zeros* $\alpha, \beta \neq 0, -1$ *such that* $\alpha/\beta, (1+\alpha)/(1+\beta)$ *or* $\alpha\beta, (1+\alpha)/(1+1/\beta)$ *are not simultaneously roots of unity.*

Then there exist at most finitely many pairs $b \in \mathbb{C}, m \in \mathbb{N}$ *such that* $\overline{F}_a^{(n)}$ *divides* $\overline{F}_b^{(m)}$.

This theorem will be proved in Sect. 2.6.2.

Remark 20. • For $n = 2$ or 3, it is easy to check that there are infinitely many such pairs. Condition 2 in the theorem is violated only in seven cases including $a = 1$, see [4] for details.

• Since there is one-to-one correspondence between $F_a^{(n)}$ and $\overline{F}_a^{(n)}$, we can translate the results on $\overline{F}_a^{(n)}$ to those on $F_a^{(n)}$, and further back on symmetries of the evolution systems.

In particular given a, n it is often possible to compute the complete set of b, m explicitly. This will be done for the example $a = 5, n = 4$ in Sect. 2.6.3, which is precisely Bakirov's example. Here we only give the result.

Theorem 21. *Suppose* $F_5^{(4)}$ *divides* $F_b^{(m)}$. *Then* (b, m) *equals* $(5, 4)$ *or* $(11, 6)$.

In the first case, it leads to the system itself. For $(b, n) = (11, 6)$, we find $\hat{A} = \frac{25}{22}\eta_2^2 + \frac{20}{11}\eta_1\eta_2 + \frac{25}{22}\eta_1^2$. We now translate these results back to results on symmetries of system (2.19).

Corollary 22. *The system*

$$\begin{cases} u_t = u_4 + v^2, \\ v_t = \frac{1}{5} v_4 \end{cases}$$

has one and only one nontrivial symmetry:

$$\left(u_6 + \frac{5}{11}\left(5vv_2 + 4v_1^2\right)\right)\frac{\partial}{\partial u} + \frac{1}{11} v_6 \frac{\partial}{\partial v}.$$

2.6.2 The Lech–Mahler Theorem

In this section we prove Theorem 19 by using the Lech–Mahler theorem from number theory.

First we realize that $\overline{F}_a^{(n)}$ has double zeros for some values of a, which is important for our analysis later on.

Lemma 23. *Suppose that $\overline{F}_a^{(n)}$ has a multiple zero. Then this is given by an $(n-1)$th root of unity ζ and $a = 1/(\zeta+1)^{n-1}$. Together with $1/\zeta$ these are the only multiple zeros and they have multiplicity two.*

Proof. We solve the simultaneous equations $\overline{F}_a^{(n)} = d\overline{F}_a^{(n)}/dx = 0$. Explicitly, $a(x+1)^n = x^n + 1$ and $a(x+1)^{n-1} = x^{n-1}$. Multiply the second by $x+1$ and subtract the equations. We obtain $0 = 1 - x^{n-1}$. Hence the roots of $\overline{F}_a^{(n)}$, denoted by X, are an $n-1$th root of unity and from the second equation we get $a = 1/(1+X)^{n-1}$. Since

$$\frac{d^2\overline{F}_a^{(n)}}{dx^2}\Big|_X = n(n-1)\left(a(x+1)^{n-2} - x^{n-2}\right)\Big|_X = n(n+1)\left(\frac{1}{X+1} - \frac{1}{X}\right) \neq 0,$$

the root X is a double zero. Suppose we have a second $(n-1)$th root of unity Y such that $a(1+Y)^{n-1} = 1$. In particular we find that $|1+Y| = |1+X|$ and $|X| = |Y|$. This implies that either $X = Y$ or $X = \overline{Y} = 1/Y$. This proves our lemma. □

For the proof of Theorem 19 we shall use the following theorem from number theory [14].

Theorem 24 (Lech, Mahler). *Let $A_1, A_2, \ldots, A_n \in \mathbb{C}$ be nonzero complex numbers and similarly for $a_1, a_2, \ldots, a_n$. Suppose that none of the ratios A_i/A_j with $i \neq j$ is a root of unity. Then the equation*

$$a_1 A_1^k + a_2 A_2^k + \cdots + a_n A_n^k = 0$$

in the unknown integer k has finitely many solutions.

Repeatedly applying this theorem, we obtain the following corollary:

Corollary 25. *Let $A,B,C,D \in \mathbb{C}$ be nonzero complex numbers. Suppose that the equation*

$$A^k + B^k = C^k + D^k$$

has infinitely many integers k with $A^k + B^k \neq 0$ as solution. Then at least one of the pairs $A/C, B/D$ or $A/D, B/C$ consists of roots of unity.

Proof of Theorem 19. Let α, β be complex zeros of $\overline{F}_a^{(n)}$ not equal to $0, -1$ such that condition (2) of Theorem 19 is satisfied.

For $n = 4$ or 5 such zeros exist by assumption. For $n \geq 6$ we shall prove that such zeros also exist.

Suppose that $\alpha/\beta, (1+\alpha)/(1+\beta)$ are roots of unity. Then we have $|\alpha| = |\beta|$ and $|1+\alpha| = |1+\beta|$. Hence β lies on the intersection of the circles $|z| = |\alpha|$ and $|z+1| = |1+\alpha|$ which implies $\beta = \alpha$ or $\beta = \overline{\alpha}$. Similarly if $\alpha\beta$ and $(1+\alpha)/(1+1/\beta)$ are roots of unity then $\beta = 1/\alpha$ or $\beta = 1/\overline{\alpha}$. As a consequence of the statement, we need to prove there exists a root of $\overline{F}_a^{(n)}$ such that it is not in a set of the form $V_\alpha = \{0, -1, \alpha, 1/\alpha, \overline{\alpha}, 1/\overline{\alpha}\}$. If $\overline{F}_a^{(n)}$ has multiple zeros then, according to Lemma 23, the multiple zero is an $(n-1)$th root of unity, which we may assume to be equal to α. Together with $1/\alpha$ these are the only multiple zeros and they have multiplicity two. Whether $G_a^{(m)}$ has multiple zeros or not, it is clear that if $a \neq 1$ and $m \geq 6$, $\overline{F}_a^{(m)}$ has a zero outside V_α.

Note that α, β being zeros of $\overline{F}_a^{(n)}$ implies

$$(\alpha^n + 1)/(\alpha + 1)^n = (\beta^n + 1)/(\beta + 1)^n = a,$$

that is,

$$\left(\frac{1}{1+1/\alpha}\right)^n + \left(\frac{1}{\alpha+1}\right)^n = \left(\frac{1}{1+1/\beta}\right)^n + \left(\frac{1}{\beta+1}\right)^n.$$

Suppose $\overline{F}_a^{(n)}$ divides $\overline{F}_b^{(m)}$ for some $b \in \mathbb{C}, m \in \mathbb{N}$. Then we also have

$$\left(\frac{1}{1+1/\alpha}\right)^m + \left(\frac{1}{\alpha+1}\right)^m = \left(\frac{1}{1+1/\beta}\right)^m + \left(\frac{1}{\beta+1}\right)^m.$$

Suppose there are infinitely many such pairs (b,m). Then, according to Corollary 25, the ratios

$$\frac{1+1/\alpha}{1+1/\beta}, \frac{1+\alpha}{1+\beta} \quad \textit{or} \quad \frac{1+1/\alpha}{1+\beta}, \frac{1+\alpha}{1+1/\beta}$$

are roots of unity. Let us assume the first. Then we see that the ratios α/β and $(1+\alpha)/(1+\beta)$ are roots of unity. This was excluded by our assumptions. We deal similarly with the second case. □

2.6.3 Skolem's Method

In this section we prove Theorem 21. We assume that the reader is familiar with the concept of p-adic numbers. The set of p-adic numbers is denoted by $\mathbb{Q}_p$ and the set of p-adic integers by $\mathbb{Z}_p$.

Lemma 26. *(Skolem's method) Suppose p is an odd prime. Let $A,B,C,D \in \mathbb{Z}_p$ and suppose they are not zero modulo p. Write*

$$A^{p-1} = 1 + p\alpha,\ B^{p-1} = 1 + p\beta,\ C^{p-1} = 1 + p\gamma,\ D^{p-1} = 1 + p\delta,$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{Z}_p$. Denote for every $k \in \mathbb{Z}$, $H_k = A^k + B^k - C^k - D^k$.

Suppose that $H_k \not\equiv 0 \pmod p$. Then $H_{k+r(p-1)} \neq 0$ for all $r \in \mathbb{Z}$.

Suppose $H_k = 0$ and $\alpha A^k + \beta B^k - \gamma C^k - \delta D^k \not\equiv 0 \pmod p$. Then, for $r \in \mathbb{Z}$, $H_{k+r(p-1)} = 0$ implies $r = 0$.

Proof. Note that by Fermat's little theorem,

$$\begin{aligned} H_{k+r(p-1)} &= A^{k+r(p-1)} + B^{k+r(p-1)} - C^{k+r(p-1)} - D^{k+r(p-1)} \\ &\equiv A^k + B^k - C^k - D^k \equiv H_k \pmod p. \end{aligned}$$

Since $H_k \not\equiv 0 \pmod p$ we conclude that $H_{k+r(p-1)} \not\equiv 0 \pmod p$ for all $r \in \mathbb{Z}$ and our first statement follows.

Suppose $H_{k+r(p-1)} = 0$ and assume $r \geq 0$. Then

$$\begin{aligned} 0 &= A^{k+r(p-1)} + B^{k+r(p-1)} - C^{k+r(p-1)} - D^{k+r(p-1)} \\ &= A^k(1+p\alpha)^r + B^k(1+p\beta)^r - C^k(1+p\gamma)^r - D^k(1+p\delta)^r \\ &= \sum_{i=1}^{r} \binom{r}{i} p^i \left(A^k\alpha^i + B^k\beta^i - C^k\gamma^i - D^k\delta^i\right). \end{aligned}$$

Suppose that $r \neq 0$. Dividing by pr and using the fact that

$$\frac{1}{r}\binom{r}{i} = \frac{1}{i}\binom{r-1}{i-1},$$

we obtain

$$0 = A^k\alpha + B^k\beta - C^k\gamma - D^k\delta + \sum_{i=2}^{r} \binom{r-1}{i-1} \frac{p^{i-1}}{i} \left(A^k\alpha^i + +B^k\beta^i - C^k\gamma^i - D^k\delta^i\right).$$

The summation is of course empty when $r = 1$. Since $p \geq 3$ the number $\frac{p^{i-1}}{i}$ has p-adic valuation less than $1/p$. So after reduction modulo p we obtain

$$0 \equiv A^k\alpha + B^k\beta - C^k\gamma - D^k\delta \pmod p$$

which contradicts our assumption. Hence we conclude $r = 0$. When $r < 0$ we can repeat the above proof with $A^{-1}, B^{-1}, C^{-1}, D^{-1}$ instead of A, B, C, D. □

Proof of Theorem 21. When $F_5^{(4)}$ divides $F_b^{(m)}$ this means in particular that the zeros of $f = \overline{F}_5^{(4)}$ are a subset of the zeros of $\overline{F}_b^{(m)}$. This holds true in any field, also p-adic fields. Let r, s be two zeros of f. Then clearly, $\frac{(r+1)^4}{r^4+1} = \frac{(s+1)^4}{s^4+1}$. Suppose f divides $\overline{F}_b^{(m)}$ for some b, m. Then we also have $\frac{(r+1)^m}{r^m+1} = \frac{(s+1)^m}{s^m+1}$ and hence

$$((r+1)s)^m + (r+1)^m - ((s+1)r)^m - (s+1)^m = 0.$$

Note that when modulo 181 we have the factorisation

$$f \equiv 4(x-66)(x-139)(x-96)(x-56) \pmod{181}.$$

Since 181 does not divide the discriminant of f, this implies that f has four roots in $\mathbb{Q}_{181}$. They are

$$66 + 13 \cdot 181,\ 139 + 29 \cdot 181,\ 96 + 93 \cdot 181,\ 56 + 44 \cdot 181 \pmod{181^2}.$$

We now apply Lemma 26 with $p = 181$ and $A = (r+1)s, B = r+1, C = r(s+1), D = s+1$. We take r, s to be the first two roots. Then, using modulo 181^2, we get

$$A \equiv 67 + 13 \cdot 181,\ B \equiv 82,\ C \equiv 140 + 29 \cdot 181,\ D \equiv 9 + 165 \cdot 181 \pmod{181^2}.$$

We also compute modulo 181,

$$\alpha \equiv 33,\ \beta \equiv 46,\ \gamma \equiv 40,\ \delta \equiv 140 \pmod{181}.$$

A straightforward computation shows that $H_k \equiv 0 \pmod{181}$ and $0 \le k < 180$ yields $k = 0, 1, 4, 6$. Lemma 26 now implies that $H_{k+180r} \neq 0$ for all r when $k \neq 0, 1, 4, 6$. When $k = 0, 1, 4$ or 6 we easily check that $H_k = 0$ and

$$\alpha A^k + \beta B^k - \gamma C^k - \delta D^k \not\equiv 0 \pmod{181}.$$

Again, application of Lemma 26 shows that $H_k = 0 \Rightarrow k = 0, 1, 4, 6$. When $k = 6$ we check that $b = \frac{r^6+1}{(r+1)^6} = 11$ and f divides indeed $11(x+1)^6 - x^6 - 1$. □

We finally remark that the method sketched in this section works also for other cases. When $(a, b, n, m) = (29, 3599, 4, 10)$ we can take $p = 491$. When $(a, b, n, m) = (11, 14867171, 4, 28)$ or $(a, b, n, m) = (17/3, 78719/81, 4, 16)$ we can take $p = 101$.

2.7 Concluding Remarks, Open Problems and Further Development

We have shown in this chapter that the symbolic method, combined with the implicit function theorem for filtered Lie algebras, gives us a powerful technique, which translates our classification questions into questions about divisibility. To attack these, we have at our disposal the results of centuries of mathematics, ranging from number theoretical methods as diophantine approximation theory and p-adic methods, to algebraic geometry. Still not all problems have been solved, and the two- and three-variable version of Theorem 16 would be very welcome, even in some restricted form with relations between the parameters. Nevertheless, all this seems to be within range, and we may hope that further results along these lines will enable us to completely classify evolution systems under certain conditions.

We have not discussed here the application of these methods to for instance the classification of co-symmetries. In principle the same techniques apply, but there are two difficulties. First of all, the G-functions do not belong to the same class now, and we have to look at the quotient of a regular G-function and a dual G-function. This complicates the analysis and makes the results less regular than for symmetries. The second problem arises when the system does not have a symmetry. In this case we cannot apply the implicit function theorem for filtered Lie algebras and we have to go back to ad hoc techniques. These issues are discussed in [26, 29]. Similar remarks apply to the classification of other objects like recursion operators or formal symmetries.

One can also start, once partial classification results are available, to apply larger transformation 'groups' to the integrable equations, to see which can be transformed into one another. The introduction of canonical densities as new coordinates can lead to remarkable simplification of the results, and smaller lists, as was pointed out to us by Prof. V.V. Sokolov and A. Meshkov.

Further development has shown that symbolic representation can be extended to differential [30] and pseudo-differential operators [15]. It has been a suitable tool to study integrability of nonevolutionary [15, 17, 18, 20], nonlocal (integro-differential) [16] and multi-dimensional equations [34].

2.8 Some Irreducibility Results by F. Beukers

The results in this appendix are obtained by F. Beukers, Mathematical Department, University of Utrecht and are published here with his kind permission.

Theorem 27. *Consider the polynomial* $G_k^{(2)} = \xi_1^k + \xi_2^k + \xi_3^k + (-\xi_1 - \xi_2 - \xi_3)^k$. *Then* $G_k^{(2)}$ *is absolutely irreducible if* k *is even. When* k *is odd it factors as* $(\xi_1+\xi_2)(\xi_1+\xi_3)(\xi_2+\xi_3)g_k^{(2)}$, *where* $g_k^{(2)}$ *is absolutely irreducible.*

Proof. Consider the projective curve $\mathscr{C}$ defined by $G_k^{(2)} = 0$. Suppose that $G_k^{(2)} = A \cdot B$, where A and B are two polynomials of positive degree. Geometrically the curve $\mathscr{C}$ now consists of two components $\mathscr{C}_1, \mathscr{C}_2$ given by $A = 0, B = 0$, respectively. The curves $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect in at least one point, which implies that the curve $\mathscr{C}$ has a singularity.

Let us now determine the singularities of $\mathscr{C}$, i.e. the projective points (ξ_1, ξ_2, ξ_3) where all partial derivatives of $G_k^{(2)}$ vanish. Hence

$$\begin{aligned} k\xi_1^{k-1} - k(-\xi_1 - \xi_2 - \xi_3)^{k-1} &= 0, \\ k\xi_2^{k-1} - k(-\xi_1 - \xi_2 - \xi_3)^{k-1} &= 0, \\ k\xi_3^{k-1} - k(-\xi_1 - \xi_2 - \xi_3)^{k-1} &= 0. \end{aligned}$$

We see that $\xi_1^{k-1} = \xi_2^{k-1} = \xi_3^{k-1} = \xi_0^{k-1}$ where $\xi_0 = -\xi_1 - \xi_2 - \xi_3$. By taking $\xi_3 = 1$, say, we can assume that ξ_1, ξ_2, ξ_0 are $(k-1)$th roots of unity such that $\xi_0 + \xi_1 + \xi_2 + 1 = 0$. Note that four complex numbers of the same absolute value can only add up to zero if they form the sides of a parallelogram with equal sides. Hence one of the ξ_1, ξ_2, ξ_3 is -1 and the others are opposite. Suppose without loss of generality that $\xi_0 = -1$ and $\xi_1 = -\xi_2$. If k is even we see that $1 = \xi_3^{k-1} = -(-1) = -\xi_0^{k-1}$, contradicting $\xi_3^{k-1} = \xi_0^{k-1}$. Hence $\mathscr{C}$ is nonsingular if k is even. In particular $\mathscr{C}$ is irreducible in this case.

Now suppose that k is odd. Then we have $3k-6$ singular points, namely $(\zeta, -\zeta, 1)$, $(\zeta, -1, 1)$, $(-1, \zeta, 1)$ where $\zeta^{k-1} = 1$. Note that we have a priori $3k-3$ singular points, but some of them coincide. Consider such a singular point, say $(\zeta, -\zeta, 1)$. We study the singular point locally by introducing the coordinates $\xi_1 = \zeta + u, \xi_2 = -\zeta + v$. Up to third-order terms we find the local equation $(\zeta(u+v) - (u-v))(u+v) + \cdots$. Since the quadratic part factors in two distinct factors the singularity is simple, i.e. there are two distinct tangent lines through the point. Consider now the curves $(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3) = 0$ and $g_k^{(2)} = 0$. These curves intersect in $3(k-3)$ points. Moreover, the first curve has three singularities. This accounts for the $3k-6$ singular points we found. Hence $g_k^{(2)} = 0$ cannot have any singular points and in particular it is irreducible. □

2.9 The Filtered Lie Algebra Version of the Implicit Function Theorem

We give a filtered Lie algebra version of the implicit function theorem in Sect. 2.3. The proof is quite neat, but more abstract.

Consider a filtered Lie algebra $\mathscr{F} = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \cdots \supset \mathscr{F}^n \supset \cdots$ and let $\mathscr{V}$ be a filtered $\mathscr{F}$-module $\mathscr{V} = \mathscr{V}^0 \supset \mathscr{V}^1 \supset \cdots \supset \mathscr{V}^n \supset \cdots$ (with $\bigcap_{i=0}^{\infty} \mathscr{V}^j = 0$), where the action of $\mathscr{F}$ on $\mathscr{V}$ is such that if $X^i \in \mathscr{F}^i$ and $v^j \in \mathscr{V}^j$, then $X^i \cdot v^j \in \mathscr{V}^{i+j}$.

Definition 28. We call $K \in \mathscr{F}$ **nonlinear injective** if for all $X^l \in \mathscr{V}^l, l > 0$, $K \cdot X^l \in \mathscr{V}^{l+1} \Rightarrow X^l \in \mathscr{V}^{l+1}$.

The nonlinear injectiveness of $K \in \mathscr{F}$ implies that $K \pmod{\mathscr{F}^1} \neq 0$.

Definition 29. We call $S \in \mathscr{F}$ **relatively l-prime** with respect to $K \in \mathscr{F}$ if $S \cdot X^j \in \operatorname{Im} K \pmod{\mathscr{V}^{j+1}} \Rightarrow X^j \in \operatorname{Im} K|_{\mathscr{V}^j} \pmod{\mathscr{V}^{j+1}}$ for all $j \geq l$ and $X^j \in \mathscr{V}^j$.

Theorem 30. *Let $K,S \in \mathscr{F}$ be linearly independent. Suppose there exists some $\overline{Q} \in \mathscr{V}$ such that*

- $[K,S] = 0$,
- *K is nonlinear injective,*
- *S is relatively l-prime with respect to K*

and there exists some $\overline{Q} \in \mathscr{V}$ such that

- *$K \cdot \overline{Q} \in \mathscr{V}^l$ and $S \cdot \overline{Q} \in \mathscr{V}^1$.*

Then there exists a unique $Q = \overline{Q} + Q^l, Q^l \in \mathscr{V}^l$ such that

$$K \cdot Q = S \cdot Q = 0.$$

Proof. We use the fact that we have an action of a Lie algebra on a module, i.e. $[K,S]\cdot = K \cdot S \cdot - S \cdot K \cdot$. It follows that

$$K \cdot S \cdot \overline{Q} = S \cdot K \cdot \overline{Q}$$

By the nonlinear injectiveness of K it follows that $S \cdot \overline{Q} \in \mathscr{V}^l$.

Now we prove by induction on p that there exists $\tilde{Q}$ satisfies that $K \cdot \tilde{Q} \in \mathscr{V}^p$ and $S \cdot \tilde{Q} \in \mathscr{V}^p$, $p \geq l$. For $p = l$ we can take $\tilde{Q} = \overline{Q}$. We have

$$K \cdot S \cdot \tilde{Q} = S \cdot K \cdot \tilde{Q}$$

and therefore $S \cdot K \cdot \tilde{Q} \in im\, K \pmod{\mathscr{V}^{p+1}}$. It follows from the relatively l-primeness that $K \cdot \tilde{Q} \in \operatorname{Im} K|_{\mathscr{V}^p} \pmod{\mathscr{V}^{p+1}}$. So we can define $Q^p \in \mathscr{V}^p$ by

$$K \cdot Q^p = -K \cdot \tilde{Q}.$$

By construction $\hat{Q} = \tilde{Q} + Q^p$ obeys $K \cdot \hat{Q} = 0 \pmod{\mathscr{V}^{p+1}}$. It then follows from the nonlinear injectiveness of K that $S \cdot \hat{Q} \in \mathscr{V}^{p+1}$. Therefore there exists a convergent (in the filtration topology) sequence with limit $Q = \tilde{Q} + \sum_{p=l+1}^{\infty} Q^p$ such that $K \cdot Q$ and $S \cdot Q$ vanish. Uniqueness follows from the assumption that $\bigcap_{p=0}^{\infty} \mathscr{V}^p = 0$. This proves the statement. □

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