Preface to the Fourth Edition

We are pleased by the positive resonance of our book which now necessitates a fourth edition. We have used this opportunity to implement corrections of misprints and amendments at several places, and to extend and improve the discussion of many of the exercises and examples. We hope that our presentation of the method of equivalent photons (Example 3.17), the form factor of the electron (Example 5.7), the infrared catastrophe (Example 5.8) and the energy shift of atomic levels (Example 5.9) are now even better to understand. The new Exercise 5.10 shows in detail how to arrive at the non-relativistic limit for the calculation of form factors. Moreover, we have brought up-to-date the Biographical Notes about physicists who have contributed to the development of quantum electrodynamics, and references to experimental tests of the theory. For example, there has been recent progress in the determination of the electric and magnetic form factors of the proton (discussed in Exercise 3.5 on the Rosenbluth formula) and the Lamb shift of high-Z atoms (discussed in Example 5.9 on the energy shift of atomic levels), while the experimental verification of the birefringence of the QED vacuum in a strong magnetic field (Example 7.8) remains unsettled and is a topic of active ongoing research.

Again, we thank all colleagues and readers for their comments and information about misprints in the book, and are grateful to the team at Springer-Verlag and especially to Dr. Stefan Scherer for smoothly handling the preparation of this fourth edition.

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October 2008   *Joachim Reinhardt*

The Propagators for Electrons and Positrons 2

In the following we will generalize the nonrelativistic propagator theory developed in the previous chapter to the relativistic theory of electrons and positrons. We will be guided by the picture of the nonrelativistic theory where the propagator $G^+(x'; x)$ is interpreted as the *probability amplitude for a particle wave originating at the space–time point x to propagate to the space–time point x'*. This amplitude can be decomposed as in (1.28) into a sum of partial amplitudes, the nth such partial amplitude being a product of factors illustrated in Fig. 2.1. According to (1.28), the probability amplitude consists of factors that describe the propagation of the particle between the particular scattering events (caused by the interaction $V(x)$) and when integrated over the space–time coordinates of the points of interaction represent the nth-order scattering process of the particle.

Each line in Fig. 2.1 represents a Green's function; e.g. the line $\overline{x_{i-1} \quad x_i}$ signifies the Green's function $G_0^+(x_i, x_{i-1})$, i.e. the amplitude that a particle wave originating at the space–time point x_{i-1} propagates freely to the space–time point x_i. The space–time points where an interaction occurs (vertices) are represented by small circles (●). At the point x_i the particle wave is scattered with the probability amplitude $V(x_i)$ per unit space–time volume. The resulting scattered wave then again propagates freely forward in time from the space–time point x_i towards the point x_{i+1} with the amplitude $G_0^+(x_{i+1}, x_i)$ where the next interaction happens, and so on. The total amplitude is then given by the sum over contributions from all space–time points at which an interaction occurs. The particular space–time points at which the particle wave experiences an interaction are termed vertices. One may also describe the individual scattering processes by saying that *the interaction at the i'th vertex annihilates the particle that has propagated freely up to x_i, and creates a particle that propagates on to x_{i+1}, with $t_{i+1} \geq t_i$*.

This latter interpretation of scattering events is well suited for a generalization to relativistic hole theory since it contains the overall space–time structure of the scattering process and the interaction.

Our aim is now to develop, by analogy with the nonrelativistic propagator theory, methods to describe and calculate scattering processes mathematically within the framework of the Dirac hole theory. We need to focus on the new feature of *pair creation and annihilation processes* that are now contained in our relativistic picture of scattering processes. We shall adopt many of the calculation rules intuitively by requiring them to be consistent with the dynamics of the Dirac equation. A more rigorous mathematical justification of these rules can be given using the methods of quantum field theory. Some references on this subject are given in the appendix. In the following we shall use mainly heuristic arguments.
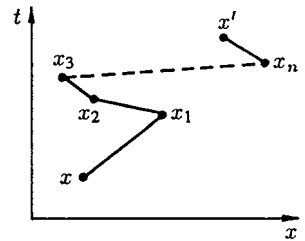

Fig. 2.1. Illustration of the nth-order contribution to the Green's function $G^+(x'; x)$ which describes the probability amplitude for multiple scattering of a particle

Fig. 2.2. Some examples of processes encountered within the electron–positron theory. The diagrams represent: (**a**) electron scattering; (**b**) positron scattering; (**c**) electron–positron pair creation; (**d**) pair annihilation; (**e**) electron scattering that in addition includes an electron–positron pair creation process; and (**f**) a closed loop describing vacuum polarization
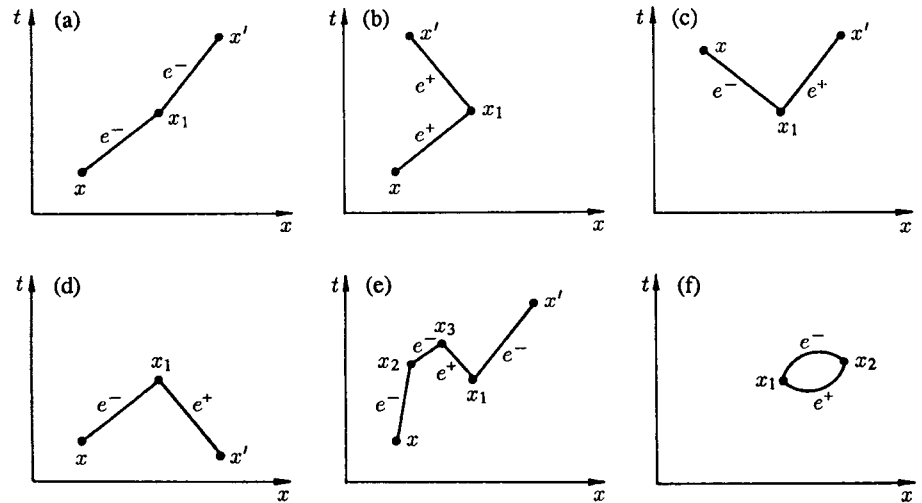
Let us now take a look at the typical processes that must be described within the relativistic theory. These are collected in Fig. 2.2, illustrated by diagrams that we shall learn to understand in the following.

In addition to the ordinary scattering processes of an electron (Fig. 2.2a) or a positron (Fig. 2.2b) there are also pair production and annihilation processes (Fig. 2.2c–f). Let us first take a look at the pair production illustrated in Fig. 2.2c: The electron–positron pair is created by a potential acting at space–time point x_1. The two particles then propagate freely forward in time, the positron to x' and the electron to x. Similarly, Fig. 2.2d shows the trajectories of an electron and a positron which start from the points x and x', respectively, and meet at the point x_1 where they annihilate.

Diagram 2.2e represents the scattering of an electron originating at x moving forward in time, experiencing several scatterings, and ending up at x'. Along its way from x to x' a pair is produced by a potential acting at x_1; the two created particles propagate forward in time. The positron of this pair and the initial electron converge at x_3 and are annihilated. The surviving electron of the pair then propagates to x'.

Diagram 2.2f shows a pair produced at x_1, propagating up to x_2, and being annihilated in the field there. It was only "virtually" present for a short intermediate time interval. Below we will recognize this process as the polarization of the vacuum.

These simple considerations already show that the relativistic electron–positron theory contains more ingredients than its nonrelativistic counterpart: we need to describe not only the amplitude for a particle (electron) to propagate from x_1 to x_2 but also the amplitude for the creation of a positron that propagates from one space–time point to another, where it is destroyed again. It is this *positron amplitude* we have to construct in the first place, enabling us then to find the total amplitude for the various processes illustrated in Fig. 2.2 by summing, or integrating, over all intermediate points (interaction events) that can contribute to the total process. In a scattering event (e.g. Fig. 2.2e) in general both electron and positron amplitudes will contribute.

The Dirac hole theory (see Theoretical Physics, by W. Greiner: *Relativistic Quantum Mechanics – Wave Equations*, hereon referred to as *RQM*) interprets a positron as a hole in the Dirac sea, i.e. the absence of an electron with negative energy from the filled sea. Thus we may view the destruction of a positron at some space–time point as equivalent to the creation of an electron with negative energy at this point. This suggests the possibility, e.g. in Fig. 2.2e, that the amplitude for creating the positron at x_1

and destroying it at x_3 is related to the amplitude for creating a negative-energy electron at x_3 and destroying it at x_1 where $t_1 < t_3$. In this picture a pair creation process such as in Fig. 2.2e,c therefore leads to the following definition of positrons: *Positrons with positive energy moving forward in space–time are viewed within the propagator theory as electrons with negative energy travelling backward in space–time.*

This is the Stückelberg–Feynman definition of positrons, which we already encountered in the discussion of the time reversal and PCT symmetries (see *RQM*, Chap. 12). *Electrons are represented by particle waves with positive energy propagating forward in space–time.* A process such as in Fig. 2.3 can therefore be interpreted using two different but equivalent languages as follows.

An electron originating at x propagates forward in time, is scattered into a state of negative energy at x_2 by the interaction $V(x_2)$, propagates backward in time to x_1, where it is scattered again into a state of positive energy, and finally propagates forward in time to x'. Alternatively one may say that an electron originating at x moves forward in time up to x_2, where it is destroyed by the interaction $V(x_2)$ together with the positron of the $e^+ - e^-$ pair that has been created earlier at x_1 by $V(x_1)$. The electron of this pair propagates forward in time to x'.

Processes that are represented by closed loops as illustrated in Fig. 2.4 are interpreted in terms of an $e^+ - e^-$ pair being produced at x_1 by $V(x_1)$ that propagates forward in time to x_2, where it is destroyed again by $V(x_2)$. Equivalently, within the picture of the hole theory, we can say that the potential $V(x_1)$ at x_1 scatters an electron from the sea of negative-energy states into a state of positive energy leaving a hole behind; it is then scattered back into the sea, recombining with the hole at x_2 under the action of $V(x_2)$. Or, in propagator language, the electron created at x_1 is scattered back in time at x_2 to destroy itself at x_1.

Our next aim is to find a unified mathematical description for the various processes making use of the relativistic propagator formalism. The first step is to construct the Green's function for electrons and positrons. It is known as the relativistic propagator[1]

$$S_F(x', x; A) \tag{2.1}$$

and is required in analogy to the nonrelativistic propagator (1.64), to satisfy the following differential equation:

$$\sum_{\lambda=1}^{4} \left[\gamma_\mu \left(i\hbar \frac{\partial}{\partial x'_\mu} - \frac{e}{c} A^\mu(x') \right) - m_0 c \right]_{\alpha\lambda} (S_F)_{\lambda\beta}(x', x; A) = \hbar \delta_{\alpha\beta} \delta^4(x' - x) \ , \tag{2.2}$$

which is the Dirac equation with a pointlike inhomogeneous term. By means of this definition the propagator $S_F(x', x; A)$ is a 4×4 matrix corresponding to the dimension of the γ matrices. The third argument of S_F serves as a reminder that the propagator defined by (2.2) depends on the electromagnetic field A_μ.
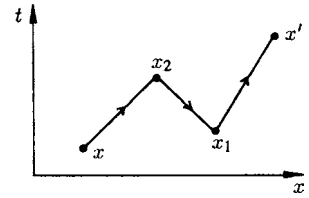

Fig. 2.3. Electron scattering with involving an intermediate pair creation process
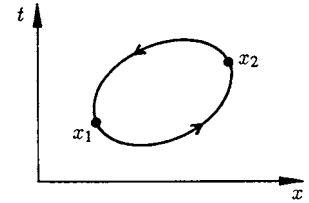

Fig. 2.4. A loop diagram

[1] The symbol S_F has been aptly chosen, bearing in mind that the originators of the relativistic propagator formalism were Stückelberg and Feynman: the propagator is commonly called the *Feynman propagator*. The original references are E.C.G. Stückelberg and D. Rivier: Helv. Phys. Acta **22**, 215 (1949) and R.P. Feynman: Phys. Rev. **76**, 749 (1949).

It is useful to remember from relativistic quantum mechanics the standard representation of the γ^μ matrices and their commutation relations:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \hat{\sigma}^i \\ -\hat{\sigma}^i & 0 \end{pmatrix} , \tag{2.3a}$$

and

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{1}_{4\times 4} . \tag{2.3b}$$

Here, $\hat{\sigma}^i$ are the 2×2 Pauli matrices, obeying

$$\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2\delta_{ij} \mathbb{1}_{2\times 2} . \tag{2.3c}$$

In relativistic quantum theory usually one employs "natural units" and sets $\hbar = c = 1$, implying the substitutions

$$\frac{e}{\hbar c} \to e \quad , \quad \frac{m_0 c}{\hbar} \to m_0 . \tag{2.3}$$

Thus in matrix notation with indices suppressed (2.2) becomes

$$(i\slashed{\nabla}' - e\slashed{A}' - m_0)S_F(x', x; A) = \delta^4(x' - x)\mathbb{1} . \tag{2.4}$$

Note that the definition of the relativistic propagator (2.2, 2.4) differs from the nonrelativistic counterpart (1.64): the differential operator $i\partial/\partial t' - \hat{H}(x')$ occurring in (1.64) has been multiplied by γ^0 in (2.2, 2.4) in order to form the covariant operator $(i\slashed{\nabla}' - e\slashed{A}' - m_0)$. The unit matrix in spinor space on the right-hand side of (2.4) is most commonly suppressed, i.e.

$$(i\slashed{\nabla}' - e\slashed{A}' - m_0)S_F(x', x; A) = \delta^4(x' - x) . \tag{2.5}$$

However, it must be kept in mind that (2.5) is a matrix equation so that the delta function in (2.5) is meant to be multiplied by $\mathbb{1}$.

The *free-particle propagator* must satisfy (2.5) with the interaction term $e\slashed{A}'$ absent, i.e.

$$(i\slashed{\nabla}' - m_0)S_F(x', x) = \delta^4(x' - x) . \tag{2.6}$$

As in the nonrelativistic case we compute $S_F(x', x)$ in momentum space, using the fact that $S_F(x', x)$ depends only on the distance vector $x' - x$. This property is a manifestation of the homogeneity of space and time and in general would not be valid for the interacting propagator $S_F(x', x; A)$. Fourier transformation to four-dimensional momentum space then yields for the free propagator

$$S_F(x', x) = S_F(x' - x) = \int \frac{d^4 p}{(2\pi)^4} \exp\left[-ip \cdot (x' - x)\right] S_F(p) . \tag{2.7}$$

Inserting (2.7) into (2.6) we obtain an equation that determines the Fourier amplitude $S_F(p)$, namely

$$\int \frac{d^4 p}{(2\pi)^4} (\slashed{p} - m_0) S_F(p) \exp\left[-ip \cdot (x' - x)\right] = \int \frac{d^4 p}{(2\pi)^4} \exp\left[-ip \cdot (x' - x)\right] ,$$

which implies that

$$(\not{p} - m_0)S_F(p) = \mathbb{1} \tag{2.8}$$

or, in detail, restoring the indices,

$$\sum_{\lambda=1}^{4}(\not{p} - m_0)_{\alpha\lambda}(S_F(p))_{\lambda\beta} = \delta_{\alpha\beta} \ . \tag{2.9}$$

Equation (2.8) can be solved for the Fourier amplitude $S_F(p)$ by multiplying with $(\not{p} + m_0)$ from the left:

$$(\not{p} + m_0)(\not{p} - m_0)S_F(p) = (\not{p} + m_0) \ . \tag{2.10}$$

Since

$$\not{p}\not{p} = \gamma_\mu\gamma_\nu p^\mu p^\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)p^\mu p^\nu = g_{\mu\nu}p^\mu p^\nu = p_\mu p^\mu = p^2 \ , \tag{2.11}$$

(2.10) becomes

$$(p^2 - m_0^2)S_F(p) = (\not{p} + m_0) \tag{2.12}$$

or

$$S_F(p) = \frac{\not{p} + m_0}{p^2 - m_0^2} \quad \text{for} \quad p^2 \neq m_0^2 \ . \tag{2.13}$$

In order to complete the definition of $S_F(p)$ we must give a prescription to handle the singularities at $p^2 = m_0^2$ which is just the mass-shell condition $p_0^2 - \boldsymbol{p}^2 = m_0^2$ or $p_0 = \pm\sqrt{m_0^2 + \boldsymbol{p}^2} = \pm E_p$. From the foregoing discussion of the nonrelativistic prop-agator formalism we know that this additional information comes from the boundary conditions that are imposed on $S_F(x' - x)$. We will now put into practice the previous interpretation of positrons as negative-energy electrons moving backwards in time. In order to implement this concept we return to the Fourier representation (2.7) and the Fourier amplitude (2.13) and perform the energy integration ($\mathrm{d}p_0$ integration) along the special contour $\boldsymbol{C}_F$ shown in Fig. 2.5.
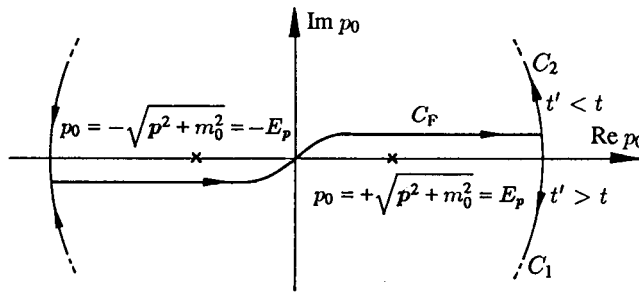

Fig. 2.5. Integration contour C_F that defines the Feynman propagator. The singularities are located on the real p_0 axis at $p_0 = -E_p$ and $p_0 = +E_p$

We obtain

$$
\begin{aligned}
S_{\mathrm{F}}(x' - x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} S_{\mathrm{F}}(p) \exp\left[-\mathrm{i} p \cdot (x' - x)\right] \\
&= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} S_{\mathrm{F}}(p) \exp\left\{-\mathrm{i}\left[p_0(t' - t) - \boldsymbol{p} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right]\right\} \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp\left[\mathrm{i} \boldsymbol{p} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right] \\
&\qquad \times \int_{C_{\mathrm{F}}} \frac{\mathrm{d} p_0}{2\pi} \frac{\exp\left[-\mathrm{i} p_0(t' - t)\right]}{p^2 - m_0^2}(\not{p} + m_0) \ .
\end{aligned}
\tag{2.14}
$$

For $t' > t$ we close the integration contour in the lower half plane, since in this case the integral along the lower semicircle, parametrized by $p_0 = \varrho e^{\mathrm{i}\phi}$, does not contribute for $\varrho \to \infty$. By means of the residue theorem then only the positive energy pole at

$$
p_0 = E_p = +\sqrt{\boldsymbol{p}^2 + m_0^2}
$$

contributes to the p_0 integration. Hence, we obtain

$$
\begin{aligned}
&\int_{C_{\mathrm{F}}} \frac{\mathrm{d} p_0}{2\pi} \frac{\exp\left[-\mathrm{i} p_0(t' - t)\right]}{p_0^2 - \boldsymbol{p}^2 - m_0^2}(\not{p} + m_0) \\
&\qquad = \int_{C_{\mathrm{F}}+C_1} \frac{\mathrm{d} p_0}{2\pi} \frac{\exp\left[-\mathrm{i} p_0(t' - t)\right]}{(p_0 - E_p)(p_0 + E_p)}(p_0 \gamma^0 + p_i \gamma^i + m_0) \\
&\qquad = -2\pi \mathrm{i} \frac{\exp\left[-\mathrm{i} E_p(t' - t)\right]}{2\pi 2 E_p}(E_p \gamma^0 - \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0) \ ,
\end{aligned}
\tag{2.15}
$$

so that (2.14) yields

$$
\begin{aligned}
S_{\mathrm{F}}(x' - x) &= -\mathrm{i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp\left[\mathrm{i} \boldsymbol{p} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right] \exp\left[-\mathrm{i} E_p(t' - t)\right] \\
&\qquad \times \frac{(E_p \gamma^0 - \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0)}{2 E_p} \quad \text{for} \quad t' > t \ .
\end{aligned}
\tag{2.16}
$$

The minus sign in (2.15) results from integrating along the contour in a mathematically negative (clockwise) sense. This propagator describes particle motion from x to x' *forward in time* $(t' > t)$. At $x' = (\boldsymbol{x}', t')$ S_{F} contains positive-energy components only, since the energy factor occurring in the exponent of $\exp(-\mathrm{i} E_p(t' - t))$ is defined to be positive, $E_p = +\sqrt{\boldsymbol{p}^2 + m_0^2}$.

On the other hand, considering the particle propagation backward in time implies that $t' - t$ is negative so that the p_0 integration must be performed along the contour closed in the upper half plane in order to give a zero contribution along the semicircle for $\varrho \to \infty$. Then only the negative-energy pole at

$$
p_0 = -E_p = -\sqrt{\boldsymbol{p}^2 + m_0^2}
$$

contributes (Fig. 2.5). This yields

$$\int\limits_{C_F+C_2} \frac{dp_0}{2\pi} \frac{\exp\left[-ip_0(t'-t)\right]}{(p_0-E_p)(p_0+E_p)}(p_0\gamma^0 + p_i\gamma^i + m_0)$$

$$= 2\pi i \frac{\exp\left[-i(-E_p)(t'-t)\right]}{2\pi(-2E_p)}(-E_p\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m_0) \ . \tag{2.17}$$

Thus the propagator (2.14) for the case $t' < t$ reads

$$S_F(x'-x) = -i\int \frac{d^3p}{(2\pi)^3} \exp\left[i\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})\right]\exp\left[+iE_p(t'-t)\right]$$

$$\times \frac{(-E_p\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m_0)}{2E_p} \quad \text{for} \quad t' < t \ . \tag{2.18}$$

This propagator describes the propagation of negative-energy particle waves backward in time, as can be read off the factor $\exp\left(-i(-E_p)(t'-t)\right)$. These negative-energy waves are absent in the nonrelativistic theory, since no solution of the energy–momentum relation at $p_0 = -E_p = -\sqrt{\boldsymbol{p}^2 + m_0^2}$ exists. Here, in the relativistic case, they are unavoidable owing to the quadratic form of the energy–momentum dispersion relation.

We note that other choices of the integration contour C_F, e.g. as in Fig. 2.6, would lead to contributions from negative-energy waves propagating into the future (case a) or positive-energy waves into the past (case b). As we can see, the choice of the contour C_F according to Fig. 2.5 results in positive-energy waves moving forward in time and negative-energy waves backward in time, just as we required. *These negative-energy waves propagating backward in time we identify with positrons.*
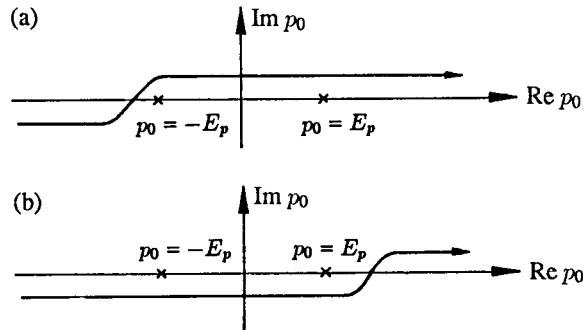

Fig. 2.6. Possible alternative choices for the integration contour that lead to propagators with the wrong asymptotic behaviour

As we recall from hole theory it is the definition of the vacuum (specified by the position of the Fermi surface E_F) that prescribes which of the particle-wave states are to be interpreted as electrons and which as positrons. It is assumed that particle states with $E < E_F$ are occupied and that the absence of a particle in such a state is interpreted in terms of a positron. The choice of the propagator is based on this definition of the vacuum, which determines the choice of the integration contour C, i.e. the transition of C from the lower to the upper complex p_0 half plane. For example, in supercritical fields (see Chap. 7) the vacuum carries charge. Consequently, some of the negative-energy states are to be interpreted as electrons propagating forward in time. For an atom the Fermi surface is usually located at a bound state. Hence, in this

case the propagator is required to be evaluated along an integration contour that passes over from the lower half plane to the upper half plane at an energy slightly above the highest occupied bound state. Another example of a modified ground state (and thus a modified propagator) is the Fermi gas of electrons, which will be introduced in Exercise 2.2 below.

The integration contour C_F determining the propagator $S_F(x' - x)$ may be alternatively characterized by adding a small positive imaginary part $+i\varepsilon$ to the denominator in (2.14), where the limit $\varepsilon \to 0$ is to be taken at the end of the calculation:

$$S_F(x' - x) = \int \frac{d^4 p}{(2\pi)^4} \frac{\exp\left[-ip \cdot (x' - x)\right]}{p^2 - m_0^2 + i\varepsilon} (\not{p} + m_0) \ . \tag{2.19}$$

Then the singularities corresponding to positive-energy states,

$$p_0 = +\sqrt{\boldsymbol{p}^2 + m_0^2 - i\varepsilon} = +\sqrt{\boldsymbol{p}^2 + m_0^2} - i\eta(\varepsilon) \ , \tag{2.20a}$$

lie below the real p_0 axis while the poles corresponding to negative-energy states,

$$p_0 = -\sqrt{\boldsymbol{p}^2 + m_0^2 - i\varepsilon} = -\sqrt{\boldsymbol{p}^2 + m_0^2} + i\eta(\varepsilon) \ , \tag{2.20b}$$

are located above the p_0 axis, just as required for the contour C_F. The prescription of (2.19) is most easily remembered in the form of a rule: To ensure the correct boundary conditions, *the mass has to be given a small negative imaginary part*. The two propagators describing positive-energy particle waves (2.16) and negative-energy particle waves (2.18) moving forward and backward in time, respectively, may be combined by introducing the energy projection operators $\hat{\Lambda}_{\pm}(p)$ (see *RQM*, Chap. 7)

$$\hat{\Lambda}_r(p) = \frac{\varepsilon_r \not{p} + m_0}{2m_0} \ ,$$

$$\varepsilon_r = \begin{cases} +1 & \text{for waves of positive energy} \\ -1 & \text{for waves of negative energy} \end{cases} \ . \tag{2.21}$$

Then, by changing the three–momentum $\boldsymbol{p}$ to $-\boldsymbol{p}$ in the propagator for negative-energy waves (2.18), which does not alter the result since the integral $\int d^3 p$ includes all directions of the three–momentum, we can write

$$\begin{aligned} S_F(x' - x) = -i \int \frac{d^3 p}{(2\pi)^3} \Bigg\{ &\exp\left[-i(+E_p)(t' - t)\right] \exp\left[+i\boldsymbol{p} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right] \\ &\times \frac{(+E_p \gamma^0 - \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0)}{2E_p} \Theta(t' - t) \\ &+ \exp\left[-i(-E_p)(t' - t)\right] \exp\left[-i\boldsymbol{p} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right] \\ &\times \frac{(-E_p \gamma^0 + \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0)}{2E_p} \Theta(t - t') \Bigg\} \\ = -i \int &\frac{d^3 p}{(2\pi)^3} \frac{m_0}{E_p} \end{aligned}$$

$$\times \left(\frac{p_0\gamma^0 + p_i\gamma^i + m_0}{2m_0} \exp\left\{-\mathrm{i}\left[p_0(t'-t) - \boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})\right]\right\} \Theta(t'-t) \right.$$

$$\left. + \frac{-p_0\gamma^0 - p_i\gamma^i + m_0}{2m_0} \exp\left\{+\mathrm{i}\left[p_0(t'-t) - \boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})\right]\right\} \Theta(t-t') \right)$$

$$= -\mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{m_0}{E_p} \left\{ \frac{\not{p}+m_0}{2m_0} \exp\left[-\mathrm{i}p\cdot(x'-x)\right]\Theta(t'-t) \right.$$

$$\left. + \frac{-\not{p}+m_0}{2m_0} \exp\left[+\mathrm{i}p\cdot(x'-x)\right]\Theta(t-t') \right\}$$

$$= -\mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{m_0}{E_p} \left\{ \hat{\Lambda}_+(p)\exp\left[-\mathrm{i}p\cdot(x'-x)\right]\Theta(t'-t) \right.$$

$$\left. + \hat{\Lambda}_-(p)\exp\left[+\mathrm{i}p\cdot(x'-x)\right]\Theta(t-t') \right\} \; . \tag{2.22}$$

Equivalently, by means of the normalized Dirac plane waves (see *RQM*, Chap. 6)

$$\psi_p^r(x) = \sqrt{\frac{m_0}{E_p}} \frac{1}{\sqrt{2\pi}^3} \, \omega^r(\boldsymbol{p})\exp\left(-\mathrm{i}\varepsilon_r\, p\cdot x\right) \;, \tag{2.23}$$

with the normalization

$$\int \mathrm{d}^3x\, \psi_p^{r\dagger}(x)\psi_{p'}^{r'}(x) = \delta_{rr'}\delta^3(\boldsymbol{p}-\boldsymbol{p}') . \tag{2.23a}$$

$S_\mathrm{F}(x'-x)$ can be transcribed to the following form (cf. Exercise 2.1):

$$S_\mathrm{F}(x'-x) = -\mathrm{i}\Theta(t'-t)\int \mathrm{d}^3p \sum_{r=1}^{2} \psi_p^r(x')\bar{\psi}_p^r(x)$$

$$+ \mathrm{i}\Theta(t-t')\int \mathrm{d}^3p \sum_{r=3}^{4} \psi_p^r(x')\bar{\psi}_p^r(x) \; . \tag{2.24}$$

This result is the relativistic generalization of the nonrelativistic Green's function (1.77). The propagator S_F now consists of two parts: the first describes the propagation of positive-energy states forward in time, the latter the propagation of negative-energy states backward in time. With the aid of (2.24) the following relations for positive-energy solutions ($\psi^{(+E)}$) and negative-energy solutions ($\psi^{(-E)}$) are easily verified:

$$\Theta(t'-t)\psi^{(+E)}(x') = \mathrm{i}\int \mathrm{d}^3x\, S_\mathrm{F}(x'-x)\gamma_0\psi^{(+E)}(x) \;, \tag{2.25}$$

$$\Theta(t-t')\psi^{(-E)}(x') = -\mathrm{i}\int \mathrm{d}^3x\, S_\mathrm{F}(x'-x)\gamma_0\psi^{(-E)}(x) \; . \tag{2.26}$$

In analogy to the nonrelativistic propagator theory (cf. (1.7) and (1.9)) the occurrence of an additional minus sign in (2.26) results from the difference of the direction of propagation in time between (2.25) and (2.26) corresponding to propagation of positive-energy solutions forward in time and negative-energy solutions backward in

time, respectively. The validity of (2.25) can be seen by writing

$$
\mathrm{i} \int \mathrm{d}^3 x \, S_{\mathrm{F}}(x' - x) \gamma_0 \psi^{(+E)}(x)
$$

$$
= \Theta(t' - t) \int \mathrm{d}^3 p \sum_{r=1}^{2} \psi_p^r(x') \int \mathrm{d}^3 x \, \psi_p^{r\dagger}(x) \psi^{(+E)}(x)
$$

$$
- \Theta(t - t') \int \mathrm{d}^3 p \sum_{r=3}^{4} \psi_p^r(x') \int \mathrm{d}^3 x \, \psi_p^{r\dagger}(x) \psi^{(+E)}(x) \tag{2.27}
$$

and expanding the general positive-energy solution $\psi^{(+E)}(x)$ in terms of Dirac plane waves

$$
\psi^{(+E)}(x) = \int \mathrm{d}^3 p \sum_{r=1}^{4} a_r(p) \psi_p^r(x) \; . \tag{2.28}
$$

The coefficients $a_r(p)$ vanish except for $r = 1, 2$, since by definition $\psi^{(+E_p)}$ describes a wave packet containing only positive energies or "frequencies". By means of the orthonormality relations of the $\psi_p^r(x)$

$$
a_r(p) = \int \mathrm{d}^3 x \, \psi^{(+E)}(x) \psi_p^{r\dagger}(x) \neq 0 \qquad (r = 1, 2) \; , \tag{2.29}
$$

$$
a_r(p) = \int \mathrm{d}^3 x \, \psi^{(+E)}(x) \psi_p^{r\dagger}(x) = 0 \qquad (r = 3, 4) \; , \tag{2.30}
$$

the second term in (2.27) vanishes, while the first term gives

$$
\Theta(t' - t) \int \mathrm{d}^3 p \sum_{r=1}^{2} a_r(p) \psi_p^r(x') = \Theta(t' - t) \psi^{(+E)}(x') \; . \tag{2.31}
$$

Thus we have proved the relation (2.25).

Equation (2.26) can be verified in similar manner. Equations (2.25) and (2.26) explicitly express our interpretation of electrons and positrons in terms of positive-energy solutions propagating forward in time and negative-energy solutions moving backward in time, respectively.

EXERCISE

2.1 Plane-Wave Decomposition of the Feynman Propagator

Problem. Prove that the Stückelberg–Feynman propagator

$$
S_{\mathrm{F}}(x' - x) = -\mathrm{i} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{m_0}{E_p} \left\{ \hat{\Lambda}_+(p) \exp\left[-\mathrm{i} p \cdot (x' - x)\right] \Theta(t' - t) \right.
$$

$$
\left. + \, \hat{\Lambda}_-(p) \exp\left[\mathrm{i} p \cdot (x' - x)\right] \Theta(t - t') \right\}
$$

may equivalently be represented as

$$S_F(x' - x) = -i\Theta(t' - t) \int d^3p \sum_{r=1}^{2} \psi_p^r(x') \bar{\psi}_p^r(x)$$

$$+ i\Theta(t - t') \int d^3p \sum_{r=3}^{4} \psi_p^r(x') \bar{\psi}_p^r(x) \; .$$

Solution. As we recall from *RQM*, Chap. 6, the Dirac plane waves (2.23) satisfy the following relations ($\hbar = c = 1$!):

$$(\hat{\not{p}} - m_0)\psi_p^r(x) = 0 \; , \tag{1}$$

$$(\varepsilon_r \not{p} - m_0)\omega^r(\boldsymbol{p}) = 0 \quad \text{or} \quad (\not{p} - \varepsilon_r m_0)\omega^r(\boldsymbol{p}) = 0 \; , \tag{2}$$

$$\bar{\omega}^r(\boldsymbol{p})(\not{p} - \varepsilon_r m_0) = 0 \quad \text{where} \quad \bar{\omega}^r(\boldsymbol{p}) = \omega^r(\boldsymbol{p})^\dagger \gamma^0 \; , \tag{3}$$

$$\bar{\omega}^r(\boldsymbol{p})\omega^{r'}(\boldsymbol{p}) = \delta_{rr'}\varepsilon_r \; , \tag{4}$$

$$\sum_{r=1}^{4} \varepsilon_r \omega_\alpha^r(\boldsymbol{p}) \bar{\omega}_\beta^r(\boldsymbol{p}) = \delta_{\alpha\beta} \; , \tag{5}$$

$$\omega^{r\dagger}(\varepsilon_r \boldsymbol{p})\omega^{r'}(\varepsilon_{r'}\boldsymbol{p}) = \frac{E_p}{m_0}\delta_{rr'} \; . \tag{6}$$

Remember also that $\gamma^0 \gamma^{\mu\dagger} = \gamma^\mu \gamma^0$. Here r and r' can take on the values $1, 2, 3, 4$. With the aid of (2.23) we find that (bearing in mind that $\varepsilon_1 = \varepsilon_2 = +1$, $\varepsilon_3 = \varepsilon_4 = -1$)

$$\sum_{r=1}^{2} \psi_p^r(x') \bar{\psi}_p^r(x) = \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[-ip \cdot (x' - x)\right] \sum_{r=1}^{2} \omega^r(\boldsymbol{p})\bar{\omega}^r(\boldsymbol{p})$$

$$= \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[-ip \cdot (x' - x)\right] \sum_{r=1}^{4} \omega^r(\boldsymbol{p})\bar{\omega}^r(\boldsymbol{p})\frac{\not{p} + m_0}{2m_0}$$

$$= \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[-ip \cdot (x' - x)\right] \underbrace{\sum_{r=1}^{4} \varepsilon_r \omega^r(\boldsymbol{p})\bar{\omega}^r(\boldsymbol{p})}_{=\mathbf{1} \text{ (because of (5))}} \frac{\not{p} + m_0}{2m_0}$$

$$= \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[-ip \cdot (x' - x)\right] \frac{\not{p} + m_0}{2m_0}$$

$$= \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[-ip \cdot (x' - x)\right] \hat{\Lambda}_+(p) \; .$$

This result is just the first term of the propagator $S_F(x' - x)$ in its representation in terms of the projection operators $\hat{\Lambda}_\pm$. Similarly, for the second part one obtains

$$\sum_{r=3}^{4} \psi_p^r(x') \bar{\psi}_p^r(x) = \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[ip \cdot (x' - x)\right] \sum_{r=3}^{4} \omega^r(\boldsymbol{p})\bar{\omega}^r(\boldsymbol{p})$$

$$= \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[+\mathrm{i}p \cdot (x'-x)\right] \sum_{r=1}^{4} \frac{-(\not{p}-m_0)}{2m_0} \omega^r(\boldsymbol{p}) \bar{\omega}^r(\boldsymbol{p})$$

$$= \frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[+\mathrm{i}p \cdot (x'-x)\right] \sum_{r=1}^{4} \frac{(-\not{p}+m_0)}{2m_0} (-\varepsilon_r) \omega^r(\boldsymbol{p}) \bar{\omega}^r(\boldsymbol{p})$$

$$= -\frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[\mathrm{i}p \cdot (x'-x)\right] \frac{(-\not{p}+m_0)}{2m_0}$$

$$= -\frac{1}{(2\pi)^3} \frac{m_0}{E_p} \exp\left[\mathrm{i}p \cdot (x'-x)\right] \hat{\Lambda}_-(p) \ .$$

Thus we have verified the proposed equivalence between the two representations of $S_\mathrm{F}(x'-x)$.

Equations (2.22) or (2.24) determine the free-particle propagator of the electron–positron theory. In analogy to (1.83) and (1.86), respectively, we may now formally construct the complete Green's function and the S matrix for the electron–positron field interacting with an electromagnetic potential A. This will then enable us to calculate various scattering processes of electrons and positrons in the presence of external fields, as will be demonstrated in the following chapter. To accomplish the aim of constructing the exact propagator $S_\mathrm{F}(x', x; A)$ we start from the differential equation (2.5) that determines $S_\mathrm{F}(x', x; A)$ and transcribe it, paraphrasing the nonrelativistic treatment (cf. (1.80)), to the following form:

$$(\mathrm{i}\not{\nabla}' - m_0) S_\mathrm{F}(x', x; A) = \delta^4(x'-x) + e\not{A}(x') S_\mathrm{F}(x', x; A) \ . \tag{2.32}$$

This can be viewed as an inhomogeneous Dirac equation of the form

$$(\mathrm{i}\not{\nabla} - m_0) \Psi(x) = \varrho(x) \ , \tag{2.33}$$

which is solved by the Green's function technique as follows

$$\Psi(x) = \Psi_0(x) + \int \mathrm{d}^4 y \, S_\mathrm{F}(x-y) \varrho(y) \ , \tag{2.34}$$

$\Psi_0(x)$ solves the homogeneous equation. In this way (2.32) leads to an integral equation for the Stückelberg–Feynman propagator

$$S_\mathrm{F}(x', x; A) = \int \mathrm{d}^4 y \, S_\mathrm{F}(x'-y) \left[\delta^4(y-x) + e\not{A}(y) S_\mathrm{F}(y, x; A)\right]$$

$$= S_\mathrm{F}(x'-x) + e \int \mathrm{d}^4 y \, S_\mathrm{F}(x', y) \not{A}(y) S_\mathrm{F}(y, x; A) \ . \tag{2.35}$$

Note that the homogeneous solution of (2.32) is a superposition of plane waves with an arbitrary constant factor which is set to zero because the solution of the homogeneous equation is not a Green's function. Equation (2.35) is the relativistic counterpart of the Lippmann–Schwinger equation (1.83). This integral equation determines the complete propagator $S_\mathrm{F}(x', x; A)$ in terms of the free-particle propagator $S_\mathrm{F}(x', x)$.

Proceeding in analogy to the nonrelativistic treatment (cf. (1.28)) the iteration of the integral equation yields the following multiple scattering expansion:

$$S_F(x', x; A) = S_F(x' - x) + e \int d^4x_1 S_F(x' - x_1) \slashed{A}(x_1) S_F(x_1 - x)$$

$$+ e^2 \int d^4x_1 d^4x_2 S_F(x' - x_1) \slashed{A}(x_1) S_F(x_1 - x_2) \slashed{A}(x_2) S_F(x_2 - x)$$

$$+ \dots . \tag{2.36}$$

In analogy to (1.31) the exact solution of the Dirac equation

$$(i \slashed{\nabla}_x - m_0) \Psi(x) = e \slashed{A}(x) \Psi(x) \tag{2.37}$$

is completely determined in terms of S_F if one imposes the boundary condition of Feynman and Stückelberg, namely

$$\Psi(x) = \psi(x) + \int d^4y\, S_F(x - y) e \slashed{A}(y) \Psi(y) . \tag{2.38}$$

Here $\psi(x)$ is a solution of the free Dirac equation, i.e. of the homogeneous version of (2.37). The potential $V(x)$ occurring in (1.31) is now replaced by $e \slashed{A}(x)$. The second term on the right-hand side represents the scattered wave. In accordance with the properties of the Stückelberg–Feynman propagator (2.24) this scattered wave contains only positive frequencies in the distant future and only negative frequencies in the distant past, since

$$\Psi(x) - \psi(x) \Rightarrow \int d^3p \sum_{r=1}^{2} \psi_p^r(x) \left(-ie \int d^4y\, \bar{\psi}_p^r(y) \slashed{A}(y) \Psi(y) \right) \text{ for } t \to +\infty \tag{2.39}$$

and

$$\Psi(x) - \psi(x) \Rightarrow \int d^3p \sum_{r=3}^{4} \psi_p^r(x) \left(+ie \int d^4y\, \bar{\psi}_p^r(y) \slashed{A}(y) \Psi(y) \right) \text{ for } t \to -\infty . \tag{2.40}$$

Notice that here x and y are to be identified with x' and x, respectively, in (2.24), and t in (2.39, 2.40) corresponds to t' in (2.24).

The result (2.39) expresses our formulation of the relativistic scattering problem, which is consistent with the requirements of hole theory. These requirements have been essentially built into the Stückelberg–Feynman propagator by the special choice of the integration contour and thus take into account the location of the Fermi border (cf. Fig. 2.5 and Exercise 2.2). Furthermore, according to (2.39), an electron cannot "fall into the sea" of (occupied) negative-energy states after scattering by an external field $\slashed{A}(y)$, since only the unoccupied positive-energy states are available. In contrast, positrons interpreted in terms of negative-energy electrons travelling backward in time are scattered back to earlier times into other negative-energy states according to (2.40).

The *S-matrix elements* are defined in the same manner as in the nonrelativistic case (1.37). Terming $\psi_f(x)$ the final free wave with the quantum numbers f that is

observed at the end of the scattering process, we infer from (2.38)–(2.40) with the aid of (2.24) that

$$S_{fi} = \lim_{t \to \pm\infty} \langle \psi_f(x) | \Psi_i(x) \rangle$$

$$= \lim_{t \to \pm\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4 y \, S_F(x-y) e A(y) \Psi_i(y) \right\rangle . \qquad (2.41)$$

Here the limit $t \to +\infty$ is understood if $\psi_f(x)$ describes an electron and $t \to -\infty$ if $\psi_f(x)$ means a positron, since the latter is considered a negative-energy electron moving backward in time. For electron scattering we have

$$S_{fi} = \delta_{fi} - ie \lim_{t \to +\infty} \left\langle \psi_f(x) \middle| \int d^3 p \sum_{r=1}^{2} \psi_p^r(x) \int d^4 y \, \bar{\psi}_p^r(y) A(y) \Psi_i(y) \right\rangle ,$$

$$(2.41a)$$

while positron scattering is described by

$$S_{fi} = \delta_{fi} + ie \lim_{t \to -\infty} \left\langle \psi_f(x) \middle| \int d^3 p \sum_{r=3}^{4} \psi_p^r(x) \int d^4 y \, \bar{\psi}_p^r(y) A(y) \Psi_i(y) \right\rangle .$$

$$(2.41b)$$

The $\int d^3 x$ integral implied by the brackets projects out just that state $\psi_p^r(x)$ whose quantum numbers agree with $\psi_f(x)$. All other terms of the integral-sum $\int d^3 p \sum_r$ do not contribute. This yields for (2.41a)

$$S_{fi} = \delta_{fi} - ie \int d^4 y \, \bar{\psi}_f(y) A(y) \Psi_i(y)$$

and a similar expression for positron scattering. Both results can be combined by writing ($\varepsilon_f = +1$ for positive-energy waves in the future and $\varepsilon_f = -1$ for negative-energy waves in the past)

$$S_{fi} = \delta_{fi} - ie\varepsilon_f \int d^4 y \, \bar{\psi}_f(y) A(y) \Psi_i(y) . \qquad (2.42)$$

Depending on whether $\psi_f(x)$ represents an electron or a positron, the first or the second term, respectively, is nonzero. In (2.42) $\Psi_i(x)$ stands for the incoming wave, which either reduces at $y_0 \to -\infty$ to an incident positive-frequency wave $\psi_i(x)$ carrying the quantum numbers i or at $y_0 \to +\infty$ to an incident negative-frequency wave propagating into the past with quantum numbers i, according to the Stückelberg–Feynman boundary conditions.

To elucidate how the various scattering processes are contained in (2.42) we first consider the "ordinary" scattering of electrons. In this case

$$\Psi_i(y) \quad \overset{y_0 \to -\infty}{\Longrightarrow} \quad \psi_i^{(+E)}(y) = \sqrt{\frac{m_0}{E_-}} \frac{1}{(2\pi)^{3/2}} u(p_-, s_-) \exp(-ip_- \cdot y), \qquad (2.43)$$

with $u(p_-, 1/2) \equiv w^1(p_-)$, $u(p_-, -1/2) \equiv w^2(p_-)$ reduces to an incoming electron wave with positive energy E_-, momentum p_- and spin s_-. The minus sign here designates the negative charge of the electron. The nth order contribution to the perturbation

expansion of the S-matrix element (2.42) is then

$$S_{fi}^{(n)} = -\mathrm{i}e^n \int \mathrm{d}^4 y_1 \ldots \mathrm{d}^4 y_n\, \bar{\psi}_f^{(+E)}(y_n)\,\slashed{A}(y_n)\,S_F(y_n - y_{n-1})\,\slashed{A}(y_{n-1})\ldots$$
$$\times\, S_F(y_2 - y_1)\,\slashed{A}(y_1)\,\psi_i^{(+E)}(y_1)\ . \tag{2.44}$$

This expression contains both types of graphs shown in Fig. 2.7: That is, in addition to ordinary scattering intermediate pair creation and pair annihilation are included in the series, since the various $\mathrm{d}^4 y$ integrations also allow for a reverse time ordering, $y_{n+1}^0 < y_n^0$. We therefore recognize that, inevitably, the second part of the propagator (2.24) also contributes.

Next we consider the *pair production* process. In accordance with the developed formalism, $\Psi_i(y)$ in this case at $y_0 \to +\infty$ reduces to a plane wave with negative energy. This particle state propagating backward in time then represents a positron. We use the notation $\boldsymbol{p}_-, s_-$ for three–momentum and spin corresponding to the physical electron and $\boldsymbol{p}_+, s_+$ for the physical positron where $p_\pm^0 > 0$. The physical positron state at $t \to \infty$ is described by a plane wave of negative energy with quantum numbers $-\boldsymbol{p}_+, -s_+, \varepsilon = -1$. This wave propagating backward in time enters into the vertex. That is,

$$\Psi_i(y) \overset{y_0 \to \infty}{\Longrightarrow} \psi_i^{(-E)}(y) = \sqrt{\frac{m_0}{E_+}}\frac{1}{(2\pi)^{3/2}} v(p_+, s_+)\exp{(\mathrm{i}p_+ \cdot y)}\ . \tag{2.45}$$

This form of the wave function explicitly exhibits the negative energy and negative three–momentum of the particle wave. The positive sign in the exponent in (2.45) obviously expresses this property since a wave with positive energy and positive three–momentum carries a phase factor $\exp(-\mathrm{i}p_- \cdot y)$. The fact that the spin direction is reversed, i.e. $-s_+$, is taken into account by the definition of the spinor $v(p_+, s_+)$. As we recall from *RQM*, Chap. 6, the spinors have been defined according to

$$v(p_+, +1/2) = \omega^4(p_+) \quad \text{and} \quad v(p_+, -1/2) = \omega^3(p_+)\ ,$$

where ω^4 is the spinor corresponding to a negative-energy electron with spin up and ω^3 a negative-energy electron with spin down.

The final wave function ψ_f in the case of the pair creation process is a positive-energy solution carrying the quantum numbers $\boldsymbol{p}_-, s_-, \varepsilon = +1$ and describes the electron.

To resume our previous considerations, from hole theory (see *RQM*, Chap. 12) we know that the absence of a negative-energy electron with four–momentum $-\boldsymbol{p}_+$ and spin $-s_+$ is interpreted in terms of a positron with four–momentum $+\boldsymbol{p}_+$ and polarization $+s_+$. Within the framework of the propagator formalism the probability amplitude for the creation of a positron at x propagating forward in space–time and emerging out of the interaction region into the final free state $(\boldsymbol{p}_+, s_+)$ at x' is calculated by the probability amplitude for the propagation of a negative-energy electron (four–momentum $-p_+$, spin $-s_+$) backward in time entering into the interaction region. Then, being scattered by the force field, it emerges out of the interaction volume as a positive-energy state propagating forward in time. The diagrams for the pair creation are illustrated in Fig. 2.8. We emphasize that the second-order amplitude consists of two diagrams corresponding to the second scattering of the positron. These two second-order diagrams are said to differ in the time ordering of the two scattering processes.
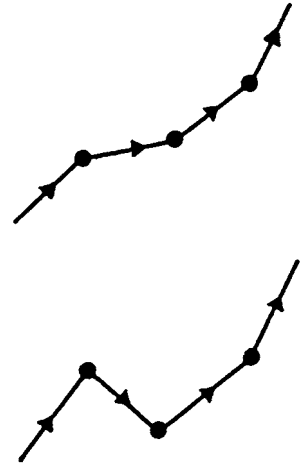

Fig. 2.7. Two graphs for third-order electron scattering. The lower graph involves an intermediate electron–positron pair
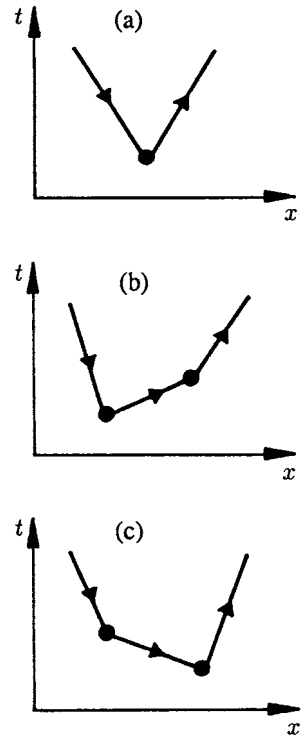

Fig. 2.8. First- and second-order Feynman diagrams for electron–positron pair creation

Since the Feynman propagator according to (2.24) consists of two parts there is no need to deal explicitly with time orderings when calculating any process. The formula for the S matrix automatically contains them all.

Now let us consider *pair annihilation*. This process in lowest order is represented by the graph of Fig. 2.9. In this case we insert for $\Psi_i(y)$ a solution of (2.38) that reduces to $\psi_i^{(+E)}(y)$ at $t \to -\infty$. This positive-energy solution represents an electron that propagates forward in time into the interaction volume, to be scattered backward in time and emerges into a negative energy state. According to (2.42) the nth-order amplitude that the electron scatters into a given final state $\psi_f^{(-E)}$, labelled by the physical quantum numbers $\boldsymbol{p}_+, \boldsymbol{s}_+, \varepsilon_f = -1$ (the corresponding formal quantum numbers entering the wave function, however, are $-\boldsymbol{p}_+, -\boldsymbol{s}_+$; cf. the discussion following (2.45)), is given by

$$S_{fi}^{(n)} = \mathrm{i}e^n \int \mathrm{d}^4 y_1 \dots \int \mathrm{d}^4 y_n \bar{\psi}_f^{(-E)}(y_n) \slashed{A}(y_n) S_{\mathrm{F}}(y_n - y_{n-1}) \dots \slashed{A}(y_1) \psi_i^{(+E)}(y_1) \ .$$

(2.46)

Fig. 2.9. The graph for pair annihilation

In the language of hole theory this is the nth-order amplitude that a positive-energy electron is scattered into an electron state of negative energy, negative three–momentum $-\boldsymbol{p}_+$, and spin $-\boldsymbol{s}_+$. This state must of course have been empty at $t \to -\infty$. That is, there must have been a hole or positron present with four–momentum p_+ and spin or polarization s_+.

Finally let us turn to *positron scattering*, which (in lowest order) is represented by either of the two equivalent graphs of Fig. 2.10. The incident wave is an electron of negative frequency (negative energy) labelled by the quantum numbers $-\boldsymbol{p}_+, -\boldsymbol{s}_+, \varepsilon_f = -1$. The final state (outgoing wave) is represented as a negative-energy electron too. Notice that the incoming electron of negative energy characterizes the outgoing positron of positive energy, and similarly the incoming positron is represented as an outgoing negative-energy electron. In Sect. 3.4 we will elaborate this explicitly.
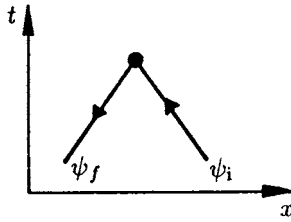

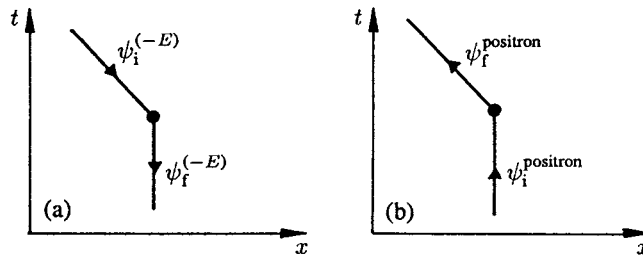**Fig. 2.10.** Positron scattering in lowest order. The emerging positron ($\psi_f^{\mathrm{positron}}$, in (**b**)) corresponds to an incoming negative-energy electron ($\psi_i^{(-E)}$ in (**a**)). Similarly, the incident positron ($\psi_i^{\mathrm{positron}}$ in (**b**)) is represented in terms of an outcoming negative-energy electron ($\psi_f^{(-E)}$ in (**a**)). In other words, (**a**) describes the scattering process in accordance with our calculational techniques, whereas (**b**) illustrates the real physical picture of positron scattering

EXERCISE

2.2 Feynman Propagator for a Fermi Gas

Problem. Suppose in our formalism we replace the vacuum by a noninteracting Fermi gas of electrons with Fermi momentum k_F. How is the Stückelberg–Feynman propagator modified? Evaluate S_F in the low-density limit.

Solution. In a degenerate Fermi gas the levels in the positive-energy electron continuum are occupied up to the Fermi momentum k_F. These occupied states have to be treated like the negative-energy states of the Dirac sea. That is, the Feynman propagator is modified according to

$$
\begin{aligned}
\mathrm{i}S_F^G(x'-x) = {} & \Theta(t'-t) \sum_{\substack{k \\ r=1,2}} \psi_k^r(x')\bar{\psi}_k^r(x)\Theta(k-k_F) \\
& - \Theta(t-t') \sum_k \left(\sum_{r=3,4} \psi_k^r(x')\bar{\psi}_k^r(x) \right. \\
& \left. + \sum_{r=1,2} \psi_k^r(x')\bar{\psi}_k^r(x)\Theta(k_F-k) \right) ,
\end{aligned}
\tag{1}
$$

where

$$
\psi_k^r(x) = (m_0/E_k)^{1/2} (2\pi)^{-3/2} \omega^r(k) \exp(-\mathrm{i}\varepsilon_r k \cdot x) ,
\tag{2}
$$

with $k_0 = E_k = \sqrt{k^2 + m_0^2}$ are the normalized Dirac plane waves. For the special case $k_F = 0$ this expression reduces to the ordinary Feynman propagator. We recall the following representations of the Θ function:

$$
\Theta(t'-t) = \mathrm{i} \int\limits_{-\infty}^{+\infty} \frac{dp_0'}{2\pi} \exp\left[-\mathrm{i}p_0'(t'-t)\right] \frac{1}{p_0' + \mathrm{i}\varepsilon} ,
\tag{3a}
$$

$$
\Theta(t-t') = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \frac{dp_0'}{2\pi} \exp\left[-\mathrm{i}p_0'(t'-t)\right] \frac{1}{p_0' - \mathrm{i}\varepsilon} ,
\tag{3b}
$$

where the second expression is obtained from the first by complex conjugation. Furthermore we need the relations

$$
\sum_{r=1,2} \omega^r(k)\bar{\omega}^r(k) = \frac{\not{k} + m_0}{2m_0} = \hat{\Lambda}_+(k) ,
$$

$$
\sum_{r=3,4} \omega^r(k)\bar{\omega}^r(k) = \frac{\not{k} - m_0}{2m_0} = -\hat{\Lambda}_-(k) .
\tag{4}
$$

Exercise 2.2   With the aid of (4), (1) yields

$$i S_F^G(x' - x) = \Theta(t' - t) \int \frac{d^3k}{(2\pi)^3} \frac{m_0}{E_k} \hat{\Lambda}_+(k) \exp\left[-ik \cdot (x' - x)\right] \Theta(k - k_F)$$

$$+ \Theta(t - t') \int \frac{d^3k}{(2\pi)^3} \frac{m_0}{E_k} \hat{\Lambda}_-(k) \exp\left[ik \cdot (x' - x)\right]$$

$$- \Theta(t - t') \int \frac{d^3k}{(2\pi)^3} \frac{m_0}{E_k} \hat{\Lambda}_+(k) \exp\left[-ik \cdot (x' - x)\right] \Theta(k_F - k)$$

$$\equiv I_1 + I_2 + I_3 \ . \tag{5}$$

Substituting the representation (3a) of the Θ function we find that

$$I_1 = i \int \frac{d^3k}{(2\pi)^3} \frac{m_0}{E_k} \hat{\Lambda}_+(k) \exp\left\{-i\left[E_k(t' - t) - \boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right]\right\}$$

$$\times \int \frac{dk_0'}{2\pi} \exp\left\{-i\left[k_0'(t' - t)\right]\right\} \frac{1}{k_0' + i\varepsilon} \Theta(k - k_F)$$

$$= i \int \frac{d^3k \, dk_0'}{(2\pi)^4} \frac{1}{2E_k} \exp\left\{-i\left[(E_k + k_0')(t' - t) - \boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right]\right\}$$

$$\times \frac{E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0' + i\varepsilon} \Theta(k - k_F) \ . \tag{6a}$$

Similarly, using (3b), we get

$$I_2 = -i \int \frac{d^3k \, dk_0'}{(2\pi)^4} \frac{1}{2E_k} \exp\left\{i\left[(E_k - k_0')(t' - t) - \boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right]\right\}$$

$$\times \frac{-E_k \gamma_0 + \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0' - i\varepsilon} \ , \tag{6b}$$

$$I_3 = i \int \frac{d^3k \, dk_0'}{(2\pi)^4} \frac{1}{2E_k} \exp\left\{-i\left[(E_k + k_0')(t' - t) - \boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right]\right\}$$

$$\times \frac{E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0' - i\varepsilon} \Theta(k_F - k) \ . \tag{6c}$$

In order to evaluate these integrals, we introduce the following substitutions:

$$k_0 = k_0' + E_k \quad \text{in} \quad I_1 \text{ and } I_3 \ , \tag{7a}$$

$$k_0 = k_0' - E_k \quad \text{and} \quad \boldsymbol{k} \to -\boldsymbol{k} \quad \text{in} \quad I_2 \ . \tag{7b}$$

In addition, in the integral I_2 we make use of the identity

$$1 = \Theta(k - k_F) + \Theta(k_F - k) \tag{8}$$

so that (6) becomes

$$I_1 = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{2E_k} \exp\left[-ik \cdot (x' - x)\right] \frac{E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 - E_k + i\varepsilon} \Theta(k - k_F) \ ,$$

$$I_2 = -\mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{2E_k} \exp\left[-\mathrm{i}k \cdot (x' - x)\right] \frac{-E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 + E_k - \mathrm{i}\varepsilon}$$

$$\times \left[\Theta(k - k_{\mathrm{F}}) + \Theta(k_{\mathrm{F}} - k)\right] \ ,$$

$$I_3 = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{2E_k} \exp\left[-\mathrm{i}k \cdot (x' - x)\right] \frac{E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 - E_k - \mathrm{i}\varepsilon} \Theta(k_{\mathrm{F}} - k) \ . \tag{9}$$

In the next step we add I_1 to that part of I_2 which contains $\Theta(k - k_{\mathrm{F}})$. The combined denominator of the two integrands is

$$(k_0 - E_k + \mathrm{i}\varepsilon)(k_0 + E_k - \mathrm{i}\varepsilon) = k_0^2 - E_k^2 + 2\mathrm{i}\varepsilon E_k + \varepsilon^2$$

$$= k_0^2 - E_k^2 + \mathrm{i}\varepsilon' = k^2 - m_0^2 + \mathrm{i}\varepsilon' \ , \tag{10}$$

since ε is an infinitesimal quantity and $E_k > 0$. This results in

$$\frac{E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 - E_k + \mathrm{i}\varepsilon} - \frac{-E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 + E_k - \mathrm{i}\varepsilon}$$

$$= \frac{(E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0)(k_0 + E_k) - (-E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0)(k_0 - E_k)}{k^2 - m_0^2 + \mathrm{i}\varepsilon'}$$

$$= \frac{2E_k(k_0 \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0)}{k^2 - m_0^2 + \mathrm{i}\varepsilon'}$$

$$= \frac{2E_k(k \cdot \gamma + m_0)}{k^2 - m_0^2 + \mathrm{i}\varepsilon'} \ . \tag{11}$$

Similarly the second part of I_2 is added to I_3. The combined denominator in this case is

$$(k_0 + E_k - \mathrm{i}\varepsilon)(k_0 - E_k - \mathrm{i}\varepsilon) = k_0^2 - E_k^2 - 2\mathrm{i}\varepsilon k_0$$

$$= k^2 - m_0^2 - \mathrm{i}\varepsilon' k_0 \ . \tag{12}$$

Proceeding as in (11), we find that

$$\frac{E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 - E_k - \mathrm{i}\varepsilon} - \frac{-E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 + E_k - \mathrm{i}\varepsilon}$$

$$= \frac{(E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0)(k_0 + E_k) - (-E_k \gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0)(k_0 - E_k)}{k^2 - m_0^2 - \mathrm{i}\varepsilon' k_0}$$

$$= \frac{2E_k(k \cdot \gamma + m_0)}{k^2 - m_0^2 - \mathrm{i}\varepsilon' k_0} \ . \tag{13}$$

We insert these expressions into (5) and obtain

$$S_{\mathrm{F}}^G(x' - x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma \cdot k + m_0}{k^2 - m_0^2 + \mathrm{i}\varepsilon} \exp\left[-\mathrm{i}k \cdot (x' - x)\right] \Theta(k - k_{\mathrm{F}})$$

$$+ \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\gamma \cdot k + m_0}{k^2 - m_0^2 - \mathrm{i}\varepsilon k_0} \exp\left[-\mathrm{i}k \cdot (x' - x)\right] \Theta(k_{\mathrm{F}} - k) \ . \tag{14}$$

Instead of adding an infinitesimal $\mathrm{i}\varepsilon$ to the denominator of the propagators (14), one may alternatively perform the integrations along the contours in the complex k_0

plane as shown in Fig. 2.11:

$$S_F^G(x' - x) = \int\limits_{C_F} \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-ik \cdot (x' - x)\right]}{k \cdot \gamma - m_0} \Theta(k - k_F)$$

$$+ \int\limits_{C_A} \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-ik \cdot (x' - x)\right]}{k \cdot \gamma - m_0} \Theta(k_F - k) \ , \tag{15}$$

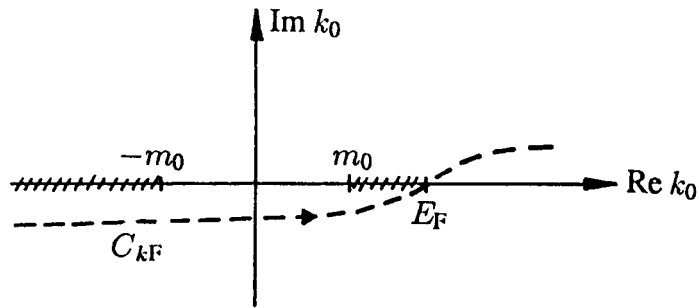where we have introduced the symbolic notation

$$\frac{\gamma \cdot k + m_0}{k^2 - m_0^2} = \frac{\gamma \cdot k + m_0}{(\gamma \cdot k + m_0)(\gamma \cdot k - m_0)} = \frac{1}{\gamma \cdot k - m_0} \ . \tag{16}$$

For $t' > t$ the second integral in (15),

$$\int\limits_{C_A} \frac{d^4k}{(2\pi)^4} \frac{\exp\left\{-i\left[k_0(t' - t) - \boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x})\right]\right\}}{k \cdot \gamma - m_0} \Theta(k_F - k) \tag{17}$$

is evaluated along the contour C_A closed in the lower half plane so that it vanishes. This procedure yields the *advanced* propagator that transforms all solutions below the Fermi surface $(k < k_F)$ backward in time. The integration contour C_F in Fig. 2.11 is the ordinary contour in the vacuum, since the old vacuum remains unchanged above the Fermi momentum k_F. The corresponding "causal" propagator transforms particles (positive-energy solutions) to propagate forward in time. In Fig. 2.12 we have illustrated these properties:

Solutions with a momentum $k < k_F$, i.e. with an energy below the corresponding Fermi energy $E_F = \sqrt{k_F^2 + m_0^2}$, propagate backward in time and are pictured as holes

(hatched region). Particles, on the other hand, have energies larger than E_F and propagate forward in time.

We summarize the steps that led to this result. For particles of a Fermi gas the integration contour cuts the real k_0 axis just above the Fermi energy E_F. In the ordinary vacuum only the negative energy states are occupied. In this case one chooses $k_F = 0$, that is, $E_F \leq |E_k|$ for all k, and the point where the contour cuts the real k_0 axis lies somewhere in the interval $[-E_k, E_k]$, the precise position being irrelevant. In the case of the Fermi gas ($k_F > 0$) we have to distinguish between two alternatives. For $k > k_F$ the integration contour passes the same interval because $E_F < |E_k|$. In both cases the contour agrees with C_F. On the other hand, at low momenta $k < k_F$, implying $E_F > |E_k|$, i.e. $E_F > E_k > -E_k$, the integration has to be performed along the dashed contour, which is equivalent to C_A! This prescription is symbolically expressed as

$$S_F^G(x' - x) = \int\limits_{C_{kF}} \frac{d^4k}{(2\pi)^4} \frac{\exp\left[-ik \cdot (x' - x)\right]}{k \cdot \gamma - m_0} \ , \tag{18}$$

where the contour C_{kF} crosses the real axis at $k_0 = E_F$.

The extension of this prescription to the case of the Feynman propagator in the presence of an external field $A_\mu(x)$ is straightforward. For example, consider an atom with bound states (located within the interval $-m_0 < E < m_0$). In this case the integration contour in the complex k_0 plane has to be chosen such that it passes below the occupied and above the empty states.

For practical purposes it is convenient to split the propagator into a free and a density-dependent part. In momentum space the result takes the simple form

$$S_F^G(k) = S_F(k) + (\gamma \cdot k + m_0)\delta(k_0 - E_k)\Theta(k_F - k) \ . \tag{19}$$

This can be easily derived from (14) by using the identity

$$\frac{1}{z - i\epsilon} = \frac{1}{z + i\epsilon} + 2\pi i\delta(z) \ . \tag{20}$$

In the low-density limit the Fermi momentum k_F is directly related to the density of the electron gas. That is, with the normalization condition for a box of volume V the particle number is given by

$$N = \sum_{r=1}^{2} \sum_{\mathbf{k}} \Theta(k_F - k) \quad \rightarrow 2V \int \frac{d^3k}{(2\pi)^3} \Theta(k_F - k) = \frac{V}{3\pi^2}k_F^3 \ , \tag{21}$$

where the factor 2 accounts for the spin degeneracy. Thus in the low-density limit, $\varrho = N/V = k_F^3/3\pi^3 \rightarrow 0$, the Fermi momentum k_F approaches 0, so that the propagator S_F^G reduces to S_F.

Supplement. *Finite Temperatures.* The result (14) can be generalized to the case of a free-electron gas at finite temperature T. From statistical mechanics it is well known that a quantum-mechanical state with an energy E cannot definitely be said to be occupied or empty. Instead an *occupation probability* function $f(E)$ is introduced. The explicit form of this function depends on the type of particle considered; for particles with half-integer spin, Fermi–Dirac statistics requires $f(E)$ to be of the form

$$f(E) \equiv f(E, T, \mu) = \frac{1}{\exp\left[(E - \mu)/k_B T + 1\right]} \ , \tag{22}$$

where k_B is the Boltzmann constant $k_B = 8.62 \times 10^{-11}$ MeV/K. The Fermi function contains two free parameters, the temperature T and the chemical potential μ. The latter is a generalization of the Fermi energy $E_F = \sqrt{k_F^2 + m_0^2}$, as becomes obvious in the limit $T \to 0$, when the Fermi function approaches the Θ function,

$$f(E, T, \mu) \to \Theta(\mu - E) \ . \tag{23}$$

That is, below the chemical potential μ all states are occupied, whereas above μ all states are empty.

To generalize (1) to the case of finite temperature, we therefore replace the Θ functions $\Theta(k - k_F) = \Theta(E_k - E_F)$ and $\Theta(k_F - k) = \Theta(E_F - E_k)$ by the occupation function (23). However, we must be careful to distinguish four different contributions: free-electron states ($r = 1, 2$) and occupied positron states ($r = 3, 4$) propagating forward in time, as well as occupied electron states and free-positron states, propagating backward in time. In contrast to (1), where all positron states were assumed empty, the electron gas also contains positrons owing to thermal excitation, as expressed by the Fermi function (22). This is depicted in Fig. 2.13. However, the temperature at which these contributions become important, i.e. where $k_B T \approx 2 m_0 c^2$, is quite large, $T \approx 10^{-10}$ K.

Fig. 2.13. The occupation probability for a hot electron gas. The hatched regions mark the occupied electron and positron states
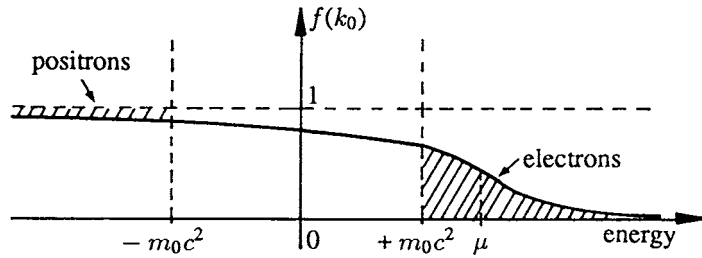
According to these considerations the temperature-dependent Feynman propagator must be of the following form:

$$
\begin{aligned}
\mathrm{i}S_F^G(x' - x) = &\left[\sum_{\boldsymbol{k}} \sum_{r=1,2} (1 - f(E_k))\, \psi_{\boldsymbol{k}}^r(x')\bar{\psi}_{\boldsymbol{k}}^r(x) \right. \\
&\left. + \sum_{\boldsymbol{k}} \sum_{r=3,4} (1 - f(-E_k))\, \psi_{\boldsymbol{k}}^r(x')\bar{\psi}_{\boldsymbol{k}}^r(x) \right] \Theta(t' - t) \\
&- \left[\sum_{\boldsymbol{k}} \sum_{r=3,4} f(-E_k)\psi_{\boldsymbol{k}}^r(x')\bar{\psi}_{\boldsymbol{k}}^r(x) \right. \\
&\left. + \sum_{\boldsymbol{k}} \sum_{r=1,2} f(E_k)\psi_{\boldsymbol{k}}^r(x')\bar{\psi}_{\boldsymbol{k}}^r(x) \right] \Theta(t - t') \ .
\end{aligned}
\tag{24}
$$

With the aid of the integral representation of the Θ function (3) and by employing (2) and (4), (24) yields

$$
iS_F^G(x' - x) = i \int \frac{dk_0'}{(2\pi)} \frac{1}{k_0' + i\varepsilon} \exp\left[-ik_0'(t' - t)\right]
$$

$$
\times \left\{ \int \frac{d^3k}{(2\pi)^3} (1 - f(E_k)) \frac{m_0}{E_k} \frac{\slashed{k} + m_0}{2m_0} \exp\left[-ik \cdot (x' - x)\right] \right.
$$

$$
\left. - \int \frac{d^3k}{(2\pi)^3} (1 - f(-E_k)) \frac{m_0}{E_k} \frac{-\slashed{k} + m_0}{2m_0} \exp\left[+ik \cdot (x' - x)\right] \right\}
$$

$$
+ i \int \frac{dk_0'}{(2\pi)} \frac{1}{k_0' - i\varepsilon} \exp\left[-ik_0'(t' - t)\right]
$$

$$
\times \left\{ -\int \frac{d^3k}{(2\pi)^3} f(-E_k) \frac{m_0}{E_k} \frac{-\slashed{k} + m_0}{2m_0} \exp\left[-ik \cdot (x' - x)\right] \right.
$$

$$
\left. + \int \frac{d^3k}{(2\pi)^3} f(E_k) \frac{m_0}{E_k} \frac{\slashed{k} + m_0}{2m_0} \exp\left[-ik \cdot (x' - x)\right] \right\} \ . \tag{25}
$$

To evaluate the four integrals we proceed as before by shifting the frequency variables and inverting the momentum variables, i.e. by carrying out the substitution (7a) in the first and last integrals and the substitution (7b) in the second and third integral. This gives

$$
S_F^G(x' - x) = \int \frac{d^4k}{(2\pi)^4} S_F^G(k) \exp\left[-ik \cdot (x' - x)\right] \ , \tag{26}
$$

where

$$
S_F^G(k) = \frac{1}{2E_k} \left[(1 - f(E_k)) \frac{E_k\gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 - E_k + i\varepsilon} \right.
$$

$$
- (1 - f(-E_k)) \frac{-E_k\gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 + E_k + i\varepsilon}
$$

$$
\left. - f(-E_k) \frac{-E_k\gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 + E_k - i\varepsilon} + f(E_k) \frac{E_k\gamma_0 - \boldsymbol{k} \cdot \boldsymbol{\gamma} + m_0}{k_0 - E_k + i\varepsilon} \right] \ . \tag{27}
$$

We combine the four terms into two using the identity

$$
\frac{1}{x - i\varepsilon} = \frac{1}{x + i\varepsilon} + 2\pi i\delta(x) \tag{28}
$$

and obtain

$$
S_F^G(k) = \frac{\slashed{k} + m_0}{k^2 - m_0^2 + i\varepsilon k_0} + \frac{1}{2E_k}(\slashed{k} + m_0)2\pi i\left[f(E_k)\delta(k_0 - E_k) \right.
$$

$$
\left. - f(-E_k)\delta(k_0 + E_k) \right] \ . \tag{29}
$$

Finally, employing the relations

$$\Theta(k_0)\delta(k^2 - m_0^2) = \frac{1}{2E_k}\delta(k_0 - E_k) \ ,$$

$$\Theta(-k_0)\delta(k^2 - m_0^2) = \frac{1}{2E_k}\delta(k_0 + E_k) \tag{30}$$

we find that

$$S_F^G(k) = \frac{\not{k} + m_0}{k^2 - m_0^2 + i\varepsilon k_0} + 2\pi i(\not{k} + m_0)\delta(k^2 - m_0^2)$$

$$\times \left[f(E_k)\Theta(k_0) - f(-E_k)\Theta(-k_0) \right] \ . \tag{31}$$

This expression may be transformed to a more symmetric form by separating off the free Feynman propagator according to the following identity:

$$\frac{\not{k} + m_0}{k^2 - m_0^2 + i\varepsilon k_0} = \frac{\not{k} + m_0}{k^2 - m_0^2 + i\varepsilon} + 2\pi i(\not{k} + m_0)\delta(k^2 - m_0^2)\Theta(-k_0) \ . \tag{32}$$

Hence, we have the final result

$$S_F^G(k) = S_F(k) + 2\pi i(\not{k} + m_0)\delta(k^2 - m_0^2)$$

$$\times \left\{ \frac{\Theta(k_0)}{\exp\left[(E_k - \mu)/k_B T + 1\right]} + \frac{\Theta(-k_0)}{\exp\left[(E_k + \mu)/k_B T\right] + 1} \right\} \ . \tag{33}$$

In the low-temperature limit $T \to 0$ and $\mu = E_F > 0$, (33) reduces to the previous expression (14) for the electron gas. This is easily proved by inserting (32) into (14).

EXERCISE ▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮▮

2.3 Nonrelativistic Limit of the Feynman Propagator

Problem.   Show that $S_F(x', x)$ reduces to the free retarded propagator for the Schrödinger equation in the nonrelativistic limit.

Solution.   To solve the problem it is advantageous to change to momentum space. The representation of the propagators in configuration space is then obtained by Fourier transformation. However, to determine the propagators uniquely we need to give a prescription how the singularities have to be treated.

The Feynman propagator is

$$S_F(x' - x) = \int \frac{d^4 p}{(2\pi)^4} \exp\left[-ip(x' - x)\right] S_F(p) \tag{1}$$

and the nonrelativistic retarded propagator is

$$
G_0^+ (x' - x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \exp\left[\mathrm{i} \boldsymbol{p} \cdot (x' - x)\right]
$$
$$
\times \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{2\pi} \exp\left[-\mathrm{i}\omega(t' - t)\right] G_0^+ (\boldsymbol{p}, \omega) \ . \tag{2}
$$

From the previous discussion of the Feynman propagator we have learnt that the appropriate boundary conditions correspond to shifting the poles by adding an infinitesimal imaginary constant, such that

$$
S_{\mathrm{F}}(p) = \frac{\not{p} + m_0}{p^2 - m_0^2 + \mathrm{i}\varepsilon} \ . \tag{3}
$$

This form implies positive-energy solutions propagating forward in time and negative-energy solutions backward in time. In order to find the nonrelativistic limit of S_{F} we consider (3) in the approximation $|\boldsymbol{p}|/m_0 \ll 1$ and investigate the vicinity of the poles. We write

$$
\frac{\not{p} + m_0}{p_0^2 - \boldsymbol{p}^2 - m_0^2 + \mathrm{i}\varepsilon} = \frac{p_0 \gamma_0 - \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0}{\left(p_0 - \sqrt{\boldsymbol{p}^2 + m_0^2}\right)\left(p_0 + \sqrt{\boldsymbol{p}^2 + m_0^2}\right) + \mathrm{i}\varepsilon} \tag{4}
$$

and obtain, using the approximation $\sqrt{\boldsymbol{p}^2 + m_0} = m_0 + \boldsymbol{p}^2/2m_0 + O(\boldsymbol{p}^4/m_0^4)$,

$$
S_{\mathrm{F}}(p) \approx \frac{p_0 \gamma_0 - \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0}{\left(p_0 - m_0 - \frac{\boldsymbol{p}^2}{2m_0}\right)\left(p_0 + m_0 + \frac{\boldsymbol{p}^2}{2m_0}\right) + \mathrm{i}\varepsilon} \ . \tag{5}
$$

Now we study the behaviour of the propagator in the vicinity of its positive-frequency pole. Introducing $\omega = p_0 - m_0$ we can reduce (5) to

$$
S_{\mathrm{F}}(p) \approx \frac{(\omega + m_0)\gamma_0 - \boldsymbol{p} \cdot \boldsymbol{\gamma} + m_0}{\left(\omega - \frac{\boldsymbol{p}^2}{2m_0}\right)\left(\omega + 2m_0 + \frac{\boldsymbol{p}^2}{2m_0}\right) + \mathrm{i}\varepsilon} \ . \tag{6}
$$

For the positive-frequency pole, ω lies in the vicinity of $\boldsymbol{p}^2/2m_0$. Therefore we have $\omega > 0$ and $(\omega + 2m_0 + \boldsymbol{p}^2/2m_0) \approx 2m_0 > 0$. Thus, within the approximation of small momenta, (5) can be transformed into

$$
S_{\mathrm{F}}(p) \approx \frac{1}{2m_0} \frac{m_0(\gamma_0 + 1) - \boldsymbol{p} \cdot \boldsymbol{\gamma}}{\left(\omega - \frac{\boldsymbol{p}^2}{2m_0}\right) + \frac{\mathrm{i}\varepsilon}{2m_0}}
$$
$$
= \frac{\frac{1}{2}(\gamma_0 + 1) - \frac{\boldsymbol{p} \cdot \boldsymbol{\gamma}}{2m_0}}{\left(\omega - \frac{\boldsymbol{p}^2}{2m_0}\right) + \mathrm{i}\varepsilon'} \ , \tag{7}
$$

where also ε' is a small imaginary constant. The first term

$$
\frac{1}{2}(\gamma_0 + 1) = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}
$$

selects the two upper components of a given bispinor. Since we have restricted our consideration to positive energy solutions by choosing the positive-energy pole, the two large spinor components are extracted. The second matrix

$$-\frac{\boldsymbol{p} \cdot \boldsymbol{\gamma}}{2m_0} = \begin{pmatrix} 0 & -\frac{\boldsymbol{p} \cdot \boldsymbol{\sigma}}{2m_0} \\ \frac{\boldsymbol{p} \cdot \boldsymbol{\sigma}}{2m_0} & 0 \end{pmatrix} \tag{8}$$

mixes the upper and lower components of the bispinor $\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$. Since $|\chi| \ll |\varphi|$ the contribution of this term is quadratic in $\boldsymbol{p}/m_0$, however, and can therefore be neglected within our small–momentum approximation. Thus the numerator of (7) reduces to unity (or, more precisely, to the unit matrix in spin space). We therefore have the result

$$S_{\mathrm{F}}(p) \to \frac{1}{\omega - \boldsymbol{p}^2/2m_0 + \mathrm{i}\varepsilon} = G_0^+(\boldsymbol{p}, \omega) \ . \tag{9}$$

Fourier transforming (8) back to coordinate space then yields the retarded propagator of the Schrödinger theory.

Remark. In the vicinity of the pole $p_0 = -\sqrt{\boldsymbol{p}^2 + m_0^2}$ the procedure outlined above, but with the substitution $\omega = -p_0 - m_0$, would yield the same result (8). However, when Fourier transforming back to configuration space the energy-dependent exponent $\exp\left(\mathrm{i} p_0(t' - t)\right) = \exp\left(\mathrm{i} m(t' - t)\right) \exp\left(+\mathrm{i}\omega(t' - t)\right)$ produces a time dependence $\Theta(t - t')$. Thus, for antiparticles the Feynman propagator reduces to the advanced Green's function in the nonrelativistic limit.

EXERCISE �

2.4 Time-Evolution of Dirac Wave Functions

Problem.   Prove the following identities:

$$\Theta(t' - t)\psi^{(+E)}(x') = \mathrm{i} \int \mathrm{d}^3x\, S_{\mathrm{F}}(x' - x)\gamma_0 \psi^{(+E)}(x) \ , \tag{1}$$

$$\Theta(t - t')\psi^{(-E)}(x') = -\mathrm{i} \int \mathrm{d}^3x\, S_{\mathrm{F}}(x' - x)\gamma_0 \psi^{(-E)}(x) \ , \tag{2}$$

and deduce similar relations for the adjoint solutions $\bar{\psi}^{(+E)}$ and $\bar{\psi}^{(-E)}$.

Solution.   A wave packet of positive energy may be expressed in terms of a superposition of normalized plane waves:

$$\psi^{(+E)}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b(p, r)\omega^r(\boldsymbol{p}) \exp\left(-\mathrm{i}\varepsilon_r\, p \cdot x\right) \ , \tag{3}$$

where $E_p = \sqrt{p^2 + m_0^2}$ and $\varepsilon_1 = \varepsilon_2 = +1$. Similarly, for negative-energy wave packets we write $\varepsilon_3 = \varepsilon_4 = -1$.

$$\psi^{(-E)}(x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=3}^{4} d^*(p, r) \omega^r(\boldsymbol{p}) \exp(-i\varepsilon_r p \cdot x) \ . \tag{4}$$

In order to make use of the orthogonality condition for spinors

$$\omega^{r\dagger}(\varepsilon_r \boldsymbol{p}) \omega^{r'}(\varepsilon_{r'} \boldsymbol{p}) = \frac{E_p}{m_0} \delta_{rr'} \tag{5}$$

we employ the plane-wave representation of the Feynman propagator

$$S_F(x' - x) = -i\Theta(t' - t) \int d^3 p \sum_{r=1}^{2} \psi_p^r(x') \bar{\psi}_p^r(x)$$

$$+ i\Theta(t - t') \int d^3 p \sum_{r=3}^{4} \psi_p^r(x') \bar{\psi}_p^r(x) \ , \tag{6}$$

where the $\psi_p^r(x)$ are given by

$$\psi_p^r = \sqrt{\frac{m_0}{E_p}} \frac{1}{(2\pi)^{3/2}} \omega^r(\boldsymbol{p}) \exp(-i\varepsilon_r p \cdot x) \ . \tag{7}$$

Inserting (3), (6) and (7) into the right-hand side of (1) we then obtain

$$i \int d^3 x \, S_F(x' - x) \gamma_0 \psi^{(+E)}$$

$$= \Theta(t' - t) \int d^3 x \int \frac{d^3 p}{(2\pi)^3} \frac{m_0}{E_p}$$

$$\times \sum_{r=1}^{2} \omega^r(\boldsymbol{p}) \bar{\omega}^r(\boldsymbol{p}) \gamma_0 \exp[-i\varepsilon_r p \cdot (x' - x)] \int \frac{d^3 p'}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_{p'}}}$$

$$\times \sum_{r'=1}^{2} b(p', r') \omega^{r'}(\boldsymbol{p}') \exp(-i\varepsilon_{r'} p' \cdot x)$$

$$- \Theta(t - t') \int d^3 x \int \frac{d^3 p}{(2\pi)^3} \frac{m_0}{E_p} \sum_{r=3}^{4} \omega^r(\boldsymbol{p}) \bar{\omega}^r(\boldsymbol{p}) \gamma_0 \exp[-i\varepsilon_r p \cdot (x' - x)]$$

$$\times \int \frac{d^3 p'}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_{p'}}} \sum_{r'=1}^{2} b(p', r') \omega^{r'}(\boldsymbol{p}') \exp(-i\varepsilon_r p' \cdot x)$$

$$= \Theta(t' - t) \int \frac{d^3 p \, d^3 p'}{(2\pi)^{3/2}} \frac{m_0}{E_p} \sqrt{\frac{m_0}{E_{p'}}}$$

$$\times \sum_{\substack{r=1,2 \\ r'=1,2}} \omega^r(\boldsymbol{p})\omega^{r\dagger}(\boldsymbol{p})\omega^{r'}(\boldsymbol{p}')b(\boldsymbol{p}',r')\exp\left(-\mathrm{i}\varepsilon_r\,p\cdot x'\right)$$

$$\times \int \frac{\mathrm{d}^3x}{(2\pi)^3}\exp\left[\mathrm{i}(\varepsilon_r\,p-\varepsilon_{r'}\,p')\cdot x\right]-\Theta(t-t')\int\frac{\mathrm{d}^3p\,\mathrm{d}^3p'}{(2\pi)^{3/2}}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_{p'}}}$$

$$\times \sum_{\substack{r=3,4 \\ r'=1,2}} \omega^r(\boldsymbol{p})\omega^{r\dagger}(\boldsymbol{p})\omega^{r'}(\boldsymbol{p}')b(\boldsymbol{p}',r')\exp\left(-\mathrm{i}\varepsilon_r\,p\cdot x'\right)$$

$$\times \int \frac{\mathrm{d}^3x}{(2\pi)^3}\exp\left[\mathrm{i}(\varepsilon_r\,p-\varepsilon_{r'}\,p')\cdot x\right]\ . \tag{8}$$

Performing the x integration in the $\Theta(t'-t)$ term yields

$$\exp\left[\mathrm{i}(E_p-E_{p'})t\right]\delta^3(\boldsymbol{p}-\boldsymbol{p}')\ , \tag{9}$$

where we have used $\varepsilon_r=\varepsilon_{r'}=1$, since $r,r'=1,2$. The $\Theta(t-t')$ term in the last line of (8) on the other hand produces a factor

$$\exp\left[-\mathrm{i}(E_p+E_{p'})t\right]\delta^3(\boldsymbol{p}+\boldsymbol{p}')\ , \tag{10}$$

since in this case $\varepsilon_r=-1$ ($r=3,4$) and $\varepsilon_{r'}=+1$ ($r'=1,2$). Integrating over d^3p and relabelling $\boldsymbol{p}'$ as $\boldsymbol{p}$ we find that

$$\mathrm{i}\int\mathrm{d}^3x\,S_{\mathrm{F}}(x'-x)\gamma_0\psi^{(+E)}(x)$$

$$=\Theta(t'-t)\int\frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2}$$

$$\times\sum_{\substack{r=1,2 \\ r'=1,2}}\omega^r(\boldsymbol{p})\omega^{r\dagger}(\boldsymbol{p})\omega^{r'}(\boldsymbol{p})b(p,r')\exp\left(-\mathrm{i}\varepsilon_r\,p\cdot x'\right)$$

$$-\Theta(t-t')\int\frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2}\sum_{\substack{r=3,4 \\ r'=1,2}}\omega^r(-\boldsymbol{p})\omega^{r\dagger}(-\boldsymbol{p})\omega^{r'}(+\boldsymbol{p})b(p,r')$$

$$\times\exp\left(-\mathrm{i}\varepsilon_r\,\tilde{p}\cdot x'\right)\exp\left(-\mathrm{i}2E_pt\right) \tag{11}$$

where $\tilde{p}=(p_0,-\boldsymbol{p})$. Now we make use of the orthogonality relation (5), which reads, for $r,r'=1,2$,

$$\omega^{r\dagger}(\boldsymbol{p})\omega^{r'}(\boldsymbol{p})=\omega^{r\dagger}(\varepsilon_r\,\boldsymbol{p})\omega^{r'}(\varepsilon_{r'}\,\boldsymbol{p})=\frac{E_p}{m_0}\delta_{rr'} \tag{12}$$

and, for $r=3,4$ and $r'=1,2$,

$$\omega^{r\dagger}(-\boldsymbol{p})\omega^{r'}(\boldsymbol{p})=\omega^{r\dagger}(\varepsilon_r\,\boldsymbol{p})\omega^{r'}(\varepsilon_{r'}\,\boldsymbol{p})=0\ , \tag{13}$$

i.e. the second term in (11) vanishes. The remaining first term gives

$$
\mathrm{i} \int \mathrm{d}^3 x \, S_{\mathrm{F}}(x' - x) \gamma_0 \psi^{(+E)}(x)
$$

$$
= \Theta(t' - t) \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b(p, r) \omega^r(\boldsymbol{p}) \exp\left(-\mathrm{i}\varepsilon_r \, p \cdot x'\right)
$$

$$
= \Theta(t' - t) \psi^{(+E)}(x') \;, \tag{14}
$$

completing the proof of (1). The relation (2) is verified in an analogous manner. In this case, since $\psi^{(-E)}$ consists of spinors with $r = 3, 4$ only, the second part of $S_{\mathrm{F}}(x' - x)$ contributes while the first term vanishes, thus yielding $-\Theta(t - t') \psi^{(-E)}(x')$.

Very similar relations can also be deduced for the propagation of the adjoint spinors $\bar{\psi}^{(+E)}(x)$, $\bar{\psi}^{(-E)}(x)$. Since the ordering of operators is inverted when performing Hermitian conjugation, the propagator S_{F} should now stand to the right of $\bar{\psi}$. Therefore we study the following integral

$$
\mathrm{i} \int \mathrm{d}^3 x \, \bar{\psi}^{(+E)}(x) \gamma_0 S_{\mathrm{F}}(x - x')
$$

$$
= \mathrm{i} \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p'}{(2\pi)^{3/2}} \frac{m_0}{E_p} \sqrt{\frac{m_0}{E_{p'}}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3}
$$

$$
\times \sum_{r=1}^{2} b^*(p', r') \bar{\omega}^{r'}(\boldsymbol{p}') \exp\left(\mathrm{i}p' x\right) \gamma_0
$$

$$
\times \left\{ -\mathrm{i}\Theta(t - t') \sum_{r=1}^{2} \omega^r(\boldsymbol{p}) \bar{\omega}^r(\boldsymbol{p}) \exp\left[-\mathrm{i}p \cdot (x - x')\right] \right.
$$

$$
\left. + \mathrm{i}\Theta(t' - t) \sum_{r=3}^{4} \omega^r(\boldsymbol{p}) \bar{\omega}^r(\boldsymbol{p}) \exp\left[+\mathrm{i}p \cdot (x - x')\right] \right\} \;. \tag{15}
$$

Now the calculation that led from (8) to (14) can be repeated, i.e. the x integration can be performed and the orthogonality relations for the unit spinors used. Then (15) reduces to the simple expression

$$
\Theta(t - t') \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \sqrt{\frac{m_0}{E_p}} \sum_{r=1}^{2} b^*(p, r) \bar{\omega}^r(\boldsymbol{p}) \exp\left(\mathrm{i}p \cdot x\right) \;. \tag{16}
$$

This is the expansion of the adjoint spinor $\bar{\psi}^{(+E)}(x')$. Thus the ansatz (15) has indeed led to a propagation equation for the adjoint wave function, namely

$$
\Theta(t - t') \bar{\psi}^{(+E)}(x') = \mathrm{i} \int \mathrm{d}^3 x \, \bar{\psi}^{(+E)}(x) \gamma_0 S_{\mathrm{F}}(x - x') \;. \tag{17}
$$

In a similar manner one derives the relation

$$
\Theta(t' - t) \bar{\psi}^{(-E)}(x') = -\mathrm{i} \int d^3 x \, \bar{\psi}^{(-E)}(x) \gamma_0 S_{\mathrm{F}}(x - x') \;. \tag{18}
$$

A comparison of (17, 18) with (1, 2) reveals that the order of the time arguments t and t' is interchanged. This is not surprising, since $\psi(x)$ describes an incoming wave and $\bar{\psi}(x)$ an outgoing wave.

EXERCISE

2.5 The Explicit Form of $S_F(x)$ in Coordinate Space

Problem. Derive a closed expression for the Feynman propagator in configuration space. How does it behave on the light cone, $x^2 \to 0$, and at large spacelike or timelike separations $x^2 \to \pm\infty$?

Solution. Our starting point is the integral representation of the Feynman propagator of the Dirac equation. The integral can be simplified by factorizing out the Dirac differential operator:

$$
S_F(x) = \int\limits_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2} (\not{p} + m)
$$

$$
= \int \frac{d^4 p}{(2\pi)^4} \frac{i\partial_\mu \gamma^\mu + m}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot x}
$$

$$
= (i\gamma \cdot \partial + m) \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon}
$$

$$
= (i\gamma \cdot \partial + m)\Delta_F(x) \ . \tag{1}
$$

Fig. 2.14. The integration contour C_F
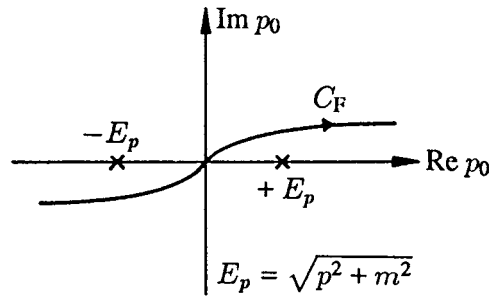

$$E_p = \sqrt{p^2 + m^2}$$

Figure 2.14 illustrates the integration contour C_F. Alternatively, the integration may be performed by shifting the poles by an infinitesimal constant $i\varepsilon$. The integral

$$
\Delta_F(x) \equiv \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2 + i\varepsilon} = \int\limits_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 - m^2} \ , \tag{2}
$$

which we introduced for the sake of mathematical simplification, also has a physical meaning. It is the *Feynman propagator of the Klein–Gordon field!*
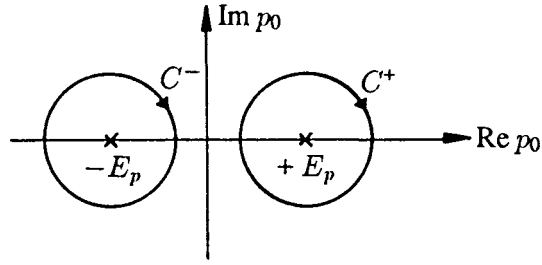
Fig. 2.15. Definition of the integration contours C^- and C^+

The p_0 integration in $\Delta_F(x)$ can be evaluated by using the residue theorem, which determines the values of integrals along *closed* contours in the complex plane. Since the integrand carries a factor $\exp(-\mathrm{i}p_0x_0 + \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})$, it is obvious that for $x_0 > 0$ the integrand vanishes asymptotically for large $|p_0|$ in the lower half plane. Thus, for $x_0 > 0$ an "infinite" semicircle in the lower half plane can be appended to the contour C_F without affecting the value of the integral. Since the integrand is regular everywhere except for the two poles, the path of integration can be contracted to a contour C^+ which encircles the point $p_0 = +E_p$, as shown in the Fig. 2.15. Conversely, for $x_0 < 0$ the contour C_F needs to be closed in the upper half plane and we may integrate along the contour $-C^-$ (the direction of integration is essential). Thus, we obtain

$$\Delta_F(x) = \Theta(x_0)\Delta^+(x) - \Theta(-x_0)\Delta^-(x) \ , \tag{3}$$

where

$$\Delta^\pm(x) = \oint_{C^\pm} \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{e}^{-\mathrm{i}p\cdot x}}{p^2 - m^2} \ . \tag{4}$$

We proceed by rewriting the denominator as

$$\frac{1}{p^2 - m^2} = \frac{1}{2E_p}\left(\frac{1}{p_0 - E_p} - \frac{1}{p_0 + E_p}\right) \tag{5}$$

where $E_p = +\sqrt{\boldsymbol{p}^2 + m^2}$ to isolate the two poles and obtain

$$\Delta^\pm(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})$$

$$\times \frac{1}{2E_p} \oint_{C^\pm} \frac{\mathrm{d}p_0}{2\pi} \exp(-\mathrm{i}p_0x_0)\left(\frac{1}{p_0 - E_p} - \frac{1}{p_0 + E_p}\right)$$

$$= \mp\mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p} \exp\left[-\mathrm{i}(\pm E_px_0 - \boldsymbol{p}\cdot\boldsymbol{x})\right] \ . \tag{6}$$

Notice that the contours $C_\pm$ are directed in a negative mathematical sense.

Using this result both contributions to $\Delta_F(x)$ in (3) can be combined into a single expression

$$\Delta_F(x) = -\mathrm{i}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_p} \exp\left(-\mathrm{i}E_p|x_0| + \mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}\right) \ . \tag{7}$$

In order to evaluate this three-dimensional integral we introduce spherical polar coordinates. The angular integrations can be carried out immediately

$$\Delta_F(x) = -\frac{i}{(2\pi)^3} \int\limits_0^\infty dp \int\limits_{-1}^{+1} d\cos\theta \int\limits_0^{2\pi} d\phi \, \frac{p^2}{2E_p} \exp\left(-iE_p|x_0| + ipr\cos\theta\right)$$

$$= -\frac{i}{(2\pi)^3} 2\pi \int\limits_0^\infty dp \, \frac{p^2}{2E_p} \frac{1}{ipr} \exp\left(-iE_p|x_0|\right) \left(\exp\left(ipr\right) - \exp\left(-ipr\right)\right)$$

$$= -\frac{1}{8\pi^2 r} \int\limits_0^\infty dp \, \frac{p}{E_p} \exp\left(-iE_p|x_0|\right) \left(\exp\left(ipr\right) - \exp\left(-ipr\right)\right) \, , \tag{8}$$

where we have written $|\boldsymbol{p}| = p$ and $|\boldsymbol{x}| = r$. Substituting $p \to -p$ in the second term the two contributions in (8) can be combined into a single expression. Furthermore, the factor p under the integral can be replaced by a differentiation with respect to the parameter r

$$\Delta_F(x) = -\frac{1}{8\pi^2 r} \int\limits_{-\infty}^\infty dp \, \frac{p}{E_p} \exp\left(-iE_p|x_0|\right) \exp\left(ipr\right)$$

$$= \frac{i}{8\pi^2 r} \frac{\partial}{\partial r} \int\limits_{-\infty}^\infty dp \, \frac{\exp\left[-i(E_p|x_0| - pr)\right]}{E_p} \, . \tag{9}$$

This integral can be brought into a more convenient form using the substitution

$$E_p = m\cosh\eta \, , \quad p = m\sinh\eta \, , \tag{10}$$

which obviously satisfies the relativistic energy momentum relation $E_p^2 - p^2 = m^2$. Now (9) takes the form

$$\Delta_F(x) = \frac{i}{8\pi^2 r} \frac{\partial}{\partial r} \int\limits_{-\infty}^\infty d\eta \, \frac{dp}{d\eta} \frac{\exp\left[-im(\cosh\eta|x_0| - \sinh\eta \, r)\right]}{m\cosh\eta}$$

$$= \frac{i}{8\pi^2 r} \frac{\partial}{\partial r} \int\limits_{-\infty}^\infty d\eta \, \exp\left[-im(|x_0|\cosh\eta - r\sinh\eta)\right] \, . \tag{11}$$

The further evaluation of this integral depends on the relative size of the time and space arguments, $|x_0|$ and r. We will separately discuss the three possible cases.

Case 1: Timelike separation $x^2 > 0$, i.e. $|x_0| > r$. We substitute

$$|x_0| = \sqrt{x_0^2 - r^2}\cosh\theta \, ,$$

$$r = \sqrt{x_0^2 - r^2}\sinh\theta \, , \tag{12}$$

and use one of the addition theorems for the hyperbolic functions

$$|x_0| \cosh \eta - r \sinh \eta = \sqrt{x_0^2 - r^2}(\cosh \theta \cosh \eta - \sinh \theta \sinh \eta)$$

$$= \sqrt{x_0^2 - r^2} \cosh(\eta - \theta) \ . \tag{13}$$

Thus we have

$$\Delta_F(x) = \frac{i}{8\pi^2 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} d\eta \exp\left(-im\sqrt{x_0^2 - r^2}\cosh(\eta - \theta)\right)$$

$$= \frac{i}{8\pi^2 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} d\eta \exp\left(-im\sqrt{x_0^2 - r^2}\cosh\eta\right) \ . \tag{14}$$

This integral can be solved in terms of Bessel functions[2] of zeroth order:

$$\int_{-\infty}^{\infty} d\eta \exp(-iz\cosh\eta) = 2\int_{0}^{\infty} d\eta \cos(z\cosh\eta) - 2i\int_{0}^{\infty} d\eta \sin(z\cosh\eta)$$

$$= -i\pi J_0(z) - \pi N_0(z) = -i\pi H_0^{(2)}(z) \ . \tag{15}$$

J_0 and N_0 are the Bessel functions of first kind (often simply called the Bessel function) and of second kind (also known as the Neumann function). Both can be combined to yield the complex Hankel function $H_0^{(2)}(z)$ (Bessel function of third kind). The functions $J_0(z)$ and $N_0(z)$ are sketched in the Fig. 2.16. At $z \to 0$ $J_0(z)$ approaches 1 while $N_0(z)$ has a logarithmic singularity. Using the identity

$$\frac{d}{dz} H_0^{(2)}(z) = -H_1^{(2)}(z) \tag{16}$$

we obtain the scalar Feynman propagator for $|x_0| > r$

Fig. 2.16. Bessel function J_0 and Neumann function N_0 of zeroth order

$$\Delta_F(x) = \frac{1}{8\pi r} \frac{d\left(m\sqrt{x_0^2 - r^2}\right)}{dr}\left[-H_1^{(2)}\left(m\sqrt{x_0^2 - r^2}\right)\right]$$

$$= \frac{m}{8\pi\sqrt{x_0^2 - r^2}} H_1^{(2)}\left(m\sqrt{x_0^2 - r^2}\right) \ . \tag{17}$$

Case 2: Spacelike separation $x^2 < 0$, i.e. $|x_0| < r$. Here we substitute

$$|x_0| = \sqrt{r^2 - x_0^2} \sinh\theta \ ,$$

$$r = \sqrt{r^2 - x_0^2} \cosh\theta \ , \tag{18}$$

and use the addition theorem

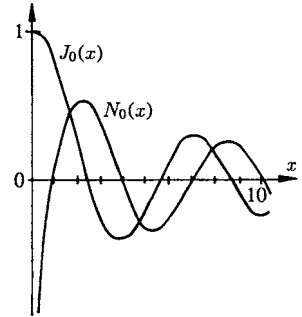$$\sinh\theta \cosh\eta - \cosh\theta \sinh\eta = -\sinh(\eta - \theta) \ . \tag{19}$$

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[2] See e.g. M. Abramowitz, I.A. Stegun: *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 9.

This leads to

$$\Delta_{\mathrm{F}}(x) = \frac{\mathrm{i}}{8\pi^2 r} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \mathrm{d}\eta \exp\left(\mathrm{i}m\sqrt{r^2 - x_0^2}\sinh\eta\right)$$

$$= \frac{\mathrm{i}}{8\pi^2 r} \frac{\partial}{\partial r} 2 \int_0^{\infty} \mathrm{d}\eta \cos\left(m\sqrt{r^2 - x_0^2}\sinh\eta\right) \ . \tag{20}$$

The sin term does not contribute, being an odd function in η. Here we encounter the integral representation of the modified Bessel function $K_0(z)$ (also known as the MacDonald function) which is related to the Hankel function of imaginary argument

$$2 \int_0^{\infty} \mathrm{d}\eta \cos(z\sinh\eta) = 2K_0(z)$$

$$= -\mathrm{i}\pi H_0^{(2)}(-\mathrm{i}z) \ . \tag{21}$$

The MacDonald function has a logarithmic singularity at $z \to 0$ and falls off like $\sqrt{\pi/2z}\exp(-z)$ at $z \to \infty$, see Fig. 2.17. Using (21) we obtain for $r > |x_0|$

$$\Delta_{\mathrm{F}}(x) = \frac{1}{8\pi r} \frac{\mathrm{d}\left(-m\mathrm{i}\sqrt{r^2 - x_0^2}\right)}{\mathrm{d}r} \left[-H_1^{(2)}\left(-\mathrm{i}m\sqrt{r^2 - x_0^2}\right)\right]$$

$$= \frac{\mathrm{i}m}{8\pi\sqrt{r^2 - x_0^2}} H_1^{(2)}\left(-\mathrm{i}m\sqrt{r^2 - x_0^2}\right) \ . \tag{22}$$

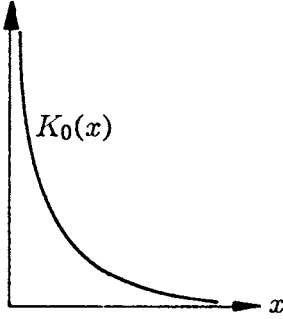**Fig. 2.17.** Modified Bessel function of zeroth order $K_0(x)$

Obviously this is the analytical continuation of the result of case 1, (17).

Case 3: Lightlike separation $x^2 = 0$, i.e. $|x_0| = r$. This case has to be treated with special care since here the integral (8) is divergent. For large values of p the integrand approaches

$$\lim_{p\to\infty} \frac{p}{E_p} \exp\left(-\mathrm{i}E_p r\right)\left(\exp\left(\mathrm{i}pr\right) - \exp\left(\mathrm{i}pr\right)\right)$$

$$= \lim_{p\to\infty}\left(1 - \exp\left(-2\mathrm{i}pr\right)\right) \ . \tag{23}$$

Since the first term approaches a constant (instead of oscillating, which would be the case for $|x_0| \neq r$) the integral will diverge. A certain singular behaviour of $S_{\mathrm{F}}(x)$ is already apparent when the results (17) or (22) are continued to the argument $|x_0| \to r$. In addition, however, also a singular distribution might contribute which has its support solely on the light cone $|x_0| = r$ and thus does not emerge when one studies the limit just mentioned.

It is easy to see that this indeed is the case. Let us study the divergent part of the integral (8) explicitly. For this it is justified to replace $E_p \to p$. Then we find

$$\Delta_{\mathrm{F}}(x)|_{x_0 \to r} \simeq -\frac{1}{8\pi^2 r} \int_0^{\infty} \mathrm{d}p\left\{\exp\left[-\mathrm{i}p(|x_0| - r)\right] - \exp\left[-\mathrm{i}p(|x_0| + r)\right]\right\}$$

$$\simeq -\frac{1}{8\pi^2 r}\left(\frac{1}{2} 2\pi\delta(|x_0| - r) - \frac{1}{2}2\pi\delta(|x_0| + r)\right) \ . \tag{24}$$

This calculation has taken into account only the delta-function contribution. Both terms in (24) can be combined to yield

$$\Delta_F(x)|_{x_0 \to r} \simeq -\frac{1}{4\pi}\delta(x_0^2 - r^2) \ . \tag{25}$$

We have to add this singular contribution to our earlier result. The final result for the Feynman propagator Δ_F for the Klein–Gordon field then reads (using $x^2 = x_0^2 - r^2$)

$$\Delta_F(x) = -\frac{1}{4\pi}\delta(x^2) + \frac{m\Theta(x^2)}{8\pi\sqrt{x^2}}H_1^{(2)}\left(m\sqrt{x^2}\right)$$

$$+ \frac{im\Theta(-x^2)}{8\pi\sqrt{-x^2}}H_1^{(2)}\left(-im\sqrt{-x^2}\right) \ . \tag{26}$$

As an important special case of $\Delta_F(x)$, let us consider the limit $m \to 0$. Since $H_1^{(2)}(z) \sim 2i/\pi z$ for $z \to 0$ (Abramowitz, Stegun, p. 360, No. 9.1.9) it follows that

$$D_F(x) \equiv \lim_{m \to 0}\Delta_F(x)$$

$$= -\frac{1}{4\pi}\delta(x^2) + \lim_{m \to 0}\left(\frac{\Theta(x^2)2im}{8\pi^2mx^2} + \frac{\Theta(-x^2)2imi}{8\pi^2imx^2}\right)$$

$$= -\frac{1}{4\pi}\delta(x^2) + \frac{i}{4\pi^2}\frac{1}{x^2}$$

$$= \frac{i}{4\pi^2}\frac{1}{x^2 - i\varepsilon} \ . \tag{27}$$

Up to a constant factor this agrees with the *photon propagator*, which will be discussed in Sect. 3.2.

Let us return to the Feynman propagator of the Dirac equation $S_F(x)$ which is related to $\Delta_F(x)$ by (1)

$$S_F(x) = (i\gamma \cdot \partial + m)\Delta_F(x) = m\Delta_F(x) + i\gamma \cdot \partial\Delta_F(x) \ , \tag{28}$$

where the first term tacitly contains the unit matrix in spinor space. Often it is sufficient to work with this representation of the propagator. For completeness, however, we will derive the explicit form of $S_F(x)$ which calls for an evaluation of the derivative of $\Delta_F(x)$ given in (26). We proceed by employing the following identities:

$$\partial_\mu\Theta(x^2) = 2x_\mu\delta(x^2) = -\partial_\mu\Theta(-x^2) \ , \tag{29}$$

$$\partial_\mu(x^2)^{1/2} = x_\mu(x^2)^{-1/2} \ ,$$

$$\partial_\mu(x^2)^{-1/2} = -x_\mu(x^2)^{-3/2} \ , \tag{30}$$

and also (Abramowitz, Stegun, p. 361, No. 9.1.27)

$$\frac{d}{dz}H_1^{(2)}(z) = \frac{1}{2}\left(H_0^{(2)}(z) - H_2^{(2)}(z)\right) \ . \tag{31}$$

Exercise 2.5   The last term in (28) has the form

$$
i\gamma \cdot \partial \Delta_F(x) = -i\frac{1}{4\pi}\gamma \cdot \partial \delta(x^2) + \frac{m}{8\pi}i\gamma \cdot \partial\left[\frac{\Theta(x^2)}{\sqrt{x^2}}H_1^{(2)}\left(m\sqrt{x^2}\right)\right.
$$
$$
\left. + \frac{i\Theta(-x^2)}{\sqrt{-x^2}}H_1^{(2)}\left(-im\sqrt{-x^2}\right)\right] . \tag{32}
$$

We evaluate the derivative of the term in square brackets by using (29)–(31) and obtain

$$
\frac{m}{8\pi}i\gamma^\mu x_\mu \left\{2\frac{\delta(x^2)}{(x^2)^{1/2}}H_1^{(2)}\left(m\sqrt{x^2}\right) - \frac{\Theta(x^2)}{(x^2)^{3/2}}H_1^{(2)}\left(m\sqrt{x^2}\right)\right.
$$
$$
+ \frac{m\Theta(x^2)}{2x^2}\left[H_0^{(2)}\left(m\sqrt{x^2}\right) - H_2^{(2)}\left(m\sqrt{x^2}\right)\right]
$$
$$
- 2i\frac{\delta(x^2)}{(-x^2)^{1/2}}H_1^{(2)}\left(-im\sqrt{-x^2}\right) + \frac{i\Theta(-x^2)}{(-x^2)^{3/2}}H_1^{(2)}\left(-im\sqrt{-x^2}\right)
$$
$$
\left. + \frac{m\Theta(-x^2)}{2x^2}\left[H_0^{(2)}\left(-im\sqrt{-x^2}\right) - H_2^{(2)}\left(-im\sqrt{-x^2}\right)\right]\right\} . \tag{33}
$$

The two factors that are multiplied by $\delta(x^2)$ can be combined. Then we have

$$
\lim_{|x^2|\to 0}\left[\frac{H_1^{(2)}\left(m\sqrt{|x^2|}\right)}{\sqrt{|x^2|}} - i\frac{H_1^{(2)}\left(-im\sqrt{|x^2|}\right)}{\sqrt{|x^2|}}\right]
$$
$$
= \lim_{|x^2|\to 0}\left[\frac{i}{\pi}\frac{2}{m\left(\sqrt{|x^2|}\right)^2} - \frac{i}{\pi}\frac{2i}{(-im)\left(\sqrt{|x^2|}\right)^2}\right]
$$
$$
= \lim_{|x^2|\to 0}\left[\frac{4i}{\pi m}\frac{1}{|x^2|}\right] , \tag{34}
$$

where we have used the asymptotic expansion of the Hankel functions for small arguments

$$
H_\nu^{(2)}(z) \sim \frac{i}{\pi}\Gamma(\nu)\left(\frac{2}{z}\right)^\nu , \quad \nu > 0 . \tag{35}
$$

Thus the explicit expression for the Feynman propagator in coordinate space reads

$$
S_{F_{\alpha\beta}}(x) = m\delta_{\alpha\beta}\Delta_F(x) - \frac{i}{4\pi}\gamma^\mu_{\alpha\beta}\partial_\mu\delta(x^2) - \frac{1}{\pi^2}\gamma^\mu_{\alpha\beta}x_\mu\frac{\delta(x^2)}{|x^2|}
$$
$$
+ \frac{im}{8\pi}\gamma^\mu_{\alpha\beta}x_\mu\left\{\Theta(x^2)\left[-\frac{1}{(x^2)^{3/2}}H_1^{(2)}\left(m\sqrt{x^2}\right)\right.\right.
$$
$$
+ \frac{m}{2x^2}\left(H_0^{(2)}\left(m\sqrt{x^2}\right) - H_2^{(2)}\left(m\sqrt{x^2}\right)\right)\bigg]
$$
$$
+ i\Theta(-x^2)\left[\frac{1}{(-x^2)^{(3/2)}}H_1^{(2)}\left(-im\sqrt{-x^2}\right)\right.
$$
$$
\left.\left. - i\frac{m}{2x^2}\left(H_0^{(2)}\left(-im\sqrt{-x^2}\right) - H_2^{(2)}\left(-im\sqrt{-x^2}\right)\right)\right]\right\} , \tag{36}
$$

where $\Delta_F(x)$ is given by (26). We emphasize that propagators like $\Delta_F(x)$ and $S_F(x)$, looked upon mathematically are *distributions*, that is, they only make sense in integrals when multiplied with suitable "well-behaved" test functions.

Asymptotic Behaviour.

1) x^2 *small*: $\Delta_F(x)$ and – even more so – $S_F(x)$ exhibit several kinds of *singularities on the light cone* $x^2 \to 0$. A study of the asymptotic behaviour of the scalar Feynman propagator, (26), leads to

$$\Delta_F(x) \approx -\frac{1}{4\pi}\delta(x^2) + \frac{i}{4\pi^2}\frac{1}{x^2} - \frac{im^2}{8\pi^2}\ln\left(\frac{m\sqrt{|x^2|}}{2}\right) + \frac{m^2}{16\pi}\Theta(x^2) \ . \tag{37}$$

The leading singularity is contained in the first two terms, namely

$$\Delta_F(x) \approx \frac{i}{4\pi^2}\frac{1}{x^2 - i\varepsilon} + O(m^2) \ . \tag{38}$$

Note that this result agrees with the massless propagator $\Delta_F(x)$ given in (27). This coincidence is quite reasonable since the singularity at $x^2 \to 0$ in momentum space is related to the divergence of integrals at $p \to \infty$. In this region the mass can be neglected. The singular nature of the propagators is the cause of great concern when integrals involving the product of several propagators have to be evaluated. In general the "collision" of singularities will render the integral divergent. The elaborate formalism of renormalization theory is required to extract meaningful results from these infinite quantities, see Chap. 5. These calculations, however, are more easily performed using momentum space propagators $\Delta_F(p)$.

2) x^2 *large*: The Hankel function behaves for *large* arguments $|z|$ as

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\exp\left[-i\left(z - \frac{\pi\nu}{2} - \frac{1}{4}\pi\right)\right] \quad \text{for} \quad |z| \to \infty \ . \tag{39}$$

Applying this relation to (26) we deduce the following asymptotic behaviour of the scalar Feynman propagator

$$\Delta_F(x) \to \text{const.}\left(x^2\right)^{-\frac{3}{4}}\exp\left(-im\sqrt{x^2}\right) \quad \text{for} \quad x^2 \to \infty \ , \tag{40a}$$

$$\Delta_F(x) \to \text{const.}\,|x^2|^{-\frac{3}{4}}\exp\left(-m\sqrt{|x^2|}\right) \quad \text{for} \quad x^2 \to -\infty \ . \tag{40b}$$

Thus for large *timelike* distances ($x^2 \to +\infty$) the propagator is an oscillating function slowly decreasing in amplitude owing to the power-law factor. On the other hand, for large spacelike distances ($x^2 \to -\infty$) the propagator rapidly falls to zero according to the exponential function in (40b). The scale is set by the inverse mass of the particle,

Fig. 2.18. The propagators $\Delta_F(x)$ and $S_F(x)$ are oscillating functions inside the light cone and fall off outside the light cone. On the light cone they are singular distributions
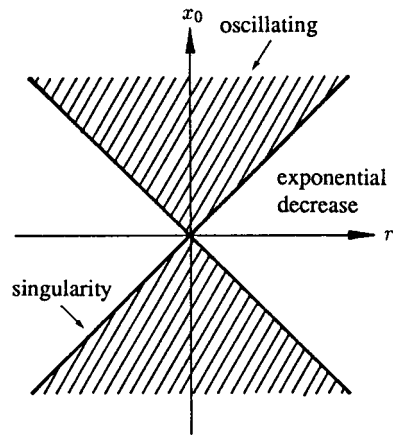
i.e. by its Compton wavelength. These conclusions remain valid also for the spin-1/2 Feynman propagator $S_F(x)$ given in (36).

Figure 2.18 illustrates the qualitative behaviour of the propagators. This result can be understood quite easily if one thinks of the propagation of a wave $\Psi(x) \rightarrow \Psi(x')$ in terms of Huygens' principle. Classically, from each point x elementary waves emanate which can propagate with velocities up to the velocity of light, i.e. inside the forward light cone $(x' - x)^2 \geq 0$. The fact that the propagator is nonzero (albeit rapidly decreasing) also in the region of spacelike distances is a quantum mechanical tunnelling phenomenon caused by the difficulty to localize a particle on a scale smaller than its Compton wavelength. This apparent violation of causality vanishes in the classical limit $m \rightarrow \infty$ (or, formally, $\hbar \rightarrow 0$).