Preface

The main goal of this book is to construct a theory of weights for the log crystalline cohomologies of families of open smooth varieties in characteristic $p > 0$. This is a p-adic analogue of the theory of the mixed Hodge structure on the cohomologies of open smooth varieties over $\mathbb{C}$ developed by Deligne in [23]. We also prove the fundamental properties of the weight-filtered log crystalline cohomologies such as the p-adic purity, the functoriality, the weight-filtered base change theorem, the weight-filtered Künneth formula, the convergence of the weight filtration, the weight-filtered Poincaré duality and the E_2-degeneration of p-adic weight spectral sequences. One can regard some of these results as the logarithmic and weight-filtered version of the corresponding results of Berthelot in [3] and K. Kato in [54].

Following the suggestion of one of the referees, we have decided to state some theorems on the weight filtration and the slope filtration on the rigid cohomology of separated schemes of finite type over a perfect field of characteristic $p > 0$. This is a p-adic analogue of the mixed Hodge structure on the cohomologies of separated schemes of finite type over $\mathbb{C}$ developped by Deligne in [24]. The detailed proof for them is given in another book [70] by the first-named author.

We have to assume that the reader is familiar with the basic premises and properties of log schemes ([54], [55]) and (log) crystalline cohomologies ([3], [11], [54]). We hope that the findings in this book will serve as a role as a first step to understanding the rich structures which p-adic cohomology theory should have.

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Chapter 2
Weight Filtrations on Log Crystalline Cohomologies

In this chapter, we construct a theory of weights of the log crystalline cohomologies of families of open smooth varieties in characteristic $p > 0$, by constructing four filtered complexes. We prove fundamental properties of these filtered complexes. Especially we prove the p-adic purity, the functoriality of three filtered complexes, the convergence of the weight filtration, the weight-filtered Künneth formula, the weight-filtered Poincaré duality and the E_2-degeneration of p-adic weight spectral sequences. We also prove that our weight filtration on log crystalline cohomology coincides with the one defined by Mokrane in the case where the base scheme is the spectrum of a perfect field of characteristic $p > 0$.

2.1 Exact Closed Immersions, SNCD's and Admissible Immersions

In this section we give some results on exact closed immersions. After that, we define a relative simple normal crossing divisor (=:relative SNCD) and a key notion *admissible immersion* of a smooth scheme with a relative SNCD.

(1) Let the notations be as in §1.6. Consider triples

$$(V, \mathfrak{D}_V(\mathcal{V}), [\ ])\text{'s}, \tag{2.1.0.1}$$

where V is an open log subscheme of Y, $\iota \colon V \overset{\subset}{\longrightarrow} \mathcal{V}$ is an exact immersion into a log smooth scheme over S and $\mathfrak{D}_V(\mathcal{V})$ is the log PD-envelope of ι over $(S, \mathcal{I}, \gamma)$. Let $(Y/S)^{\log}_{\mathrm{ERcrys}}$ be a full subcategory of $(Y/S)^{\log}_{\mathrm{crys}}$ whose objects are the triples (2.1.0.1). We define the topology of $(Y/S)^{\log}_{\mathrm{ERcrys}}$ as the induced topology by that of $(Y/S)^{\log}_{\mathrm{crys}}$.

Definition 2.1.1. We call the site $(Y/S)^{\log}_{\mathrm{ERcrys}}$ (resp. the topos $(\widetilde{Y/S})^{\log}_{\mathrm{ERcrys}}$) the *exact restricted log crystalline site* (resp. *exact restricted log crystalline topos*) of $Y/(S,\mathcal{I},\gamma)$.

Let

$$Q^{\mathrm{ER}}_{Y/S}\colon (\widetilde{Y/S})^{\log}_{\mathrm{ERcrys}} \longrightarrow (\widetilde{Y/S})^{\log}_{\mathrm{Rcrys}} \tag{2.1.1.1}$$

be a natural morphism of topoi: $Q^{\mathrm{ER}*}_{Y/S}(E)$ for an object $E \in (\widetilde{Y/S})^{\log}_{\mathrm{Rcrys}}$ is the natural restriction of E and $Q^{\mathrm{ER}*}_{Y/S}$ commutes with inverse limits. We also have a morphism

(2.1.1.2)
$$Q^{\mathrm{ER}}_{Y/S}\colon ((\widetilde{Y/S})^{\log}_{\mathrm{ERcrys}}, Q^{\mathrm{ER}*}_{Y/S}Q^{*}_{Y/S}(\mathcal{O}_{Y/S})) \longrightarrow ((\widetilde{Y/S})^{\log}_{\mathrm{Rcrys}}, Q^{*}_{Y/S}(\mathcal{O}_{Y/S}))$$

of ringed topoi.

Proposition 2.1.2. *The morphism* (2.1.1.1) (*resp.* (2.1.1.2)) *gives an equivalence of topoi* (*resp. ringed topoi*)*.*

Proof. One can check easily the isomorphism $F \xrightarrow{=} Q^{\mathrm{ER}*}_{Y/S}Q^{\mathrm{ER}}_{Y/S*}F$ for any $F \in (\widetilde{Y/S})^{\log}_{\mathrm{ERcrys}}$.

On the other hand, let $\mathfrak{D} := (V, \mathfrak{D}_V(\mathcal{V}), [\ ])$ be an object of $(Y/S)^{\log}_{\mathrm{Rcrys}}$. By [54, (5.6)], $\mathfrak{D}_V(\mathcal{V})$ is constructed locally by a local exactification $V \overset{\subset}{\longrightarrow} \mathcal{V}^{\mathrm{ex}}$ of $V \overset{\subset}{\longrightarrow} \mathcal{V}$. Hence there exists a covering $\mathfrak{D} = \bigcup_i \mathfrak{D}_i$ such that each $\mathfrak{D}_i$ is an object in $(Y/S)^{\log}_{\mathrm{ERcrys}}$. Note that $\mathfrak{D}_i \times_{\mathfrak{D}} \mathfrak{D}_{i'}$ is also an object in $(Y/S)^{\log}_{\mathrm{ERcrys}}$. Then, for any $F \in (\widetilde{Y/S})^{\log}_{\mathrm{Rcrys}}$, we have

$$\begin{aligned}
F(\mathfrak{D}) &= \mathrm{Ker}\Big(\prod_i F(\mathfrak{D}_i) \longrightarrow \prod_{i,i'} F(\mathfrak{D}_i \times_{\mathfrak{D}} \mathfrak{D}_{i'})\Big)\\
&= \mathrm{Ker}\Big(\prod_i Q^{\mathrm{ER}}_{Y/S*}Q^{\mathrm{ER}*}_{Y/S}F(\mathfrak{D}_i) \longrightarrow \prod_{i,i'} Q^{\mathrm{ER}}_{Y/S*}Q^{\mathrm{ER}*}_{Y/S}F(\mathfrak{D}_i \times_{\mathfrak{D}} \mathfrak{D}_{i'})\Big)\\
&= Q^{\mathrm{ER}}_{Y/S*}Q^{\mathrm{ER}*}_{Y/S}F(\mathfrak{D}).
\end{aligned}$$

Hence we have $F = Q^{\mathrm{ER}}_{Y/S*}Q^{\mathrm{ER}*}_{Y/S}F$. Thus the equivalences follow. □

Next we prove the second fundamental exact sequence for exact closed immersions of fine log schemes and using this, we give a local description of exact closed immersions of fine log schemes under certain assumption.

Lemma 2.1.3 (Second fundamental exact sequence).

Let $\iota\colon Z \overset{\subset}{\longrightarrow} Y$ be an exact closed immersion of fine log schemes over a fine log scheme S defined by a coherent ideal $\mathcal{J}$ of $\mathcal{O}_Y$. Then the following sequence

(2.1.3.1) $$\mathcal{J}/\mathcal{J}^2 \xrightarrow{\Delta} \iota^*(\Lambda^1_{Y/S}) \longrightarrow \Lambda^1_{Z/S} \longrightarrow 0$$

is exact. Here Δ *is the composite morphism* $\Delta\colon \mathcal{J}/\mathcal{J}^2 \longrightarrow \iota^*(\Omega^1_{Y/S}) \longrightarrow \iota^*(\Lambda^1_{Y/S})$. *If* Z/S *is log smooth, then* Δ *is injective. If* Z/S *is log smooth and if* $\overset{\circ}{Y}$ *is affine, then* (2.1.3.1) *is split.*

Proof. Let M_Y and M_Z be the log structures of Y and Z with structural morphisms $\alpha_Y\colon M_Y \longrightarrow \mathcal{O}_Y$ and $\alpha_Z\colon M_Z \longrightarrow \mathcal{O}_Z$, respectively. Let M_S be the log structure of S. Because the natural morphisms $\iota^*(\Omega^1_{\overset{\circ}{Y}/\overset{\circ}{S}}) \longrightarrow \Omega^1_{\overset{\circ}{Z}/\overset{\circ}{S}}$ and $\iota^{-1}(M_Y/\mathcal{O}_Y^*) \longrightarrow M_Z/\mathcal{O}_Z^*$ are surjective, so is $\iota^*(\Lambda^1_{Y/S}) \longrightarrow \Lambda^1_{Z/S}$. To prove the exactness of the middle term of (2.1.3.1), it suffices to prove that the following sequence

(2.1.3.2) $$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{J}/\mathcal{J}^2, \mathcal{E}) \longleftarrow \mathrm{Hom}_{\mathcal{O}_Z}(\iota^*(\Lambda^1_{Y/S}), \mathcal{E}) \longleftarrow \mathrm{Hom}_{\mathcal{O}_Z}(\Lambda^1_{Z/S}, \mathcal{E})$$

is exact for any $\mathcal{O}_Z$-module $\mathcal{E}$. The question is local. Assume that the restriction of an element of $g \in \mathrm{Hom}_{\mathcal{O}_Z}(\iota^*(\Lambda^1_{Y/S}), \mathcal{E})$ to $\Delta(\mathcal{J})$ is the zero. Let t be a section of $\mathcal{J}$ such that $1+t \in \mathcal{O}_Y^*$. Then $g(d\log(1+t)) = g(dt/(1+t)) = g(dt) = 0$. Let $\beta\colon \iota^{-1}(M_Y) \longrightarrow \iota^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_Y/\mathcal{J}$ be the natural morphism. Since M_Z is the push-out of the following diagram

$$\begin{array}{ccc} \beta^{-1}((\mathcal{O}_Y/\mathcal{J})^*) & \longrightarrow & \iota^{-1}(M_Y) \\ \big\downarrow & & \\ (\mathcal{O}_Y/\mathcal{J})^* & & , \end{array}$$

we may assume that a local section of M_Z is represented by (u, m) ($u \in (\mathcal{O}_Y/\mathcal{J})^*, m \in \iota^{-1}(M_Y)$). Let $g'\colon \Lambda^1_{Z/S} \longrightarrow \mathcal{E}$ be a morphism defined by $g'(\omega) = g(\widetilde{\omega})$ ($\omega \in \Omega^1_{\overset{\circ}{Z}/\overset{\circ}{S}}$) and $g'([(u,m)]) = g(d\log u) + g(d\log m)$ ($[(u,m)] \in M_Z$), where $\widetilde{\omega}$ denotes any lift of ω to $\iota^*(\Omega^1_{\overset{\circ}{Y}/\overset{\circ}{S}})$. It is straightforward to check that g' is well-defined and that g' induces g. Thus (2.1.3.2) is exact.

Next assume that Z/S is log smooth and that $\overset{\circ}{Y}$ is affine. Let Y^1 be the first log infinitesimal neighborhood of the exact closed immersion $Z \overset{\subset}{\longrightarrow} Y$. For two sections $m \in \iota^{-1}(M_Y)$ and $a \in \iota^{-1}(\mathcal{O}_Y)$, let $[m]$ and $[a]$ be the images in M_Z and $\mathcal{O}_Z$, respectively. Because Z/S is log smooth and $\overset{\circ}{Y}$ is affine, there exists a section $s\colon Y^1 \longrightarrow Z$ of the exact closed immersion $Z \overset{\subset}{\longrightarrow} Y^1$ induced by ι. In particular, there exist morphisms $s_{\mathrm{mo}}\colon s^{-1}(M_Z) \longrightarrow M_Y|_{Y^1}$ and $s_{\mathrm{ri}}\colon s^{-1}(\mathcal{O}_Z) \longrightarrow \mathcal{O}_{Y^1}$ such that $s_{\mathrm{mo}}([m]) = m(1+t)$ ($\exists t \in \mathcal{J}/\mathcal{J}^2, 1+t \in \mathcal{O}_{Y^1}^*$) and $s_{\mathrm{ri}}([a]) = a + t'$ ($\exists t' \in \mathcal{J}/\mathcal{J}^2$); moreover, s_{mo} and s_{ri} fit into the following commutative diagram:

$$\begin{array}{ccc} s^{-1}(M_Z) & \xrightarrow{s_{\mathrm{mo}}} & M_Y|_{Y^1} \\ {\scriptstyle s^{-1}(\alpha_Z)}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_Y|_{Y^1}} \\ s^{-1}(\mathcal{O}_Z) & \xrightarrow{s_{\mathrm{ri}}} & \mathcal{O}_{Y^1}, \end{array}$$

where the vertical morphisms above are structural morphisms.

To prove the existence of the local splitting of (2.1.3.1), we need the module of the log derivations, e.g., in [53, (5.1)].

Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Let $f\colon Y \longrightarrow S$ be the structural morphism. Let $\mathrm{Der}_S(Y,\mathcal{F})$ be a $\Gamma(S,\mathcal{O}_S)$-module whose elements are the pairs $(\overset{\circ}{\delta},\delta)$'s satisfying the following conditions:

(1) $\overset{\circ}{\delta}$ is a derivation $\mathcal{O}_Y \longrightarrow \mathcal{F}$ over S,
(2) δ is a morphism $M_Y \longrightarrow \mathcal{F}$ of monoids,
(3) $\alpha_Y(m)\delta(m) = \overset{\circ}{\delta}(\alpha_Y(m))$ $(m \in M_Y)$,
(4) $\delta(f^{-1}(n)) = 0$ $(n \in M_S)$.

Then, by [53, (5.3)], we have an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(\Lambda^1_{Y/S},\mathcal{F}) \ni h \longmapsto (h\circ d, h\circ d\log) \in \mathrm{Der}_S(Y,\mathcal{F}).$$

In particular,

$$\mathrm{Hom}_{\mathcal{O}_Z}(\iota^*(\Lambda^1_{Y/S}),\mathcal{J}/\mathcal{J}^2) = \mathrm{Hom}_{\mathcal{O}_Y}(\Lambda^1_{Y/S},\mathcal{J}/\mathcal{J}^2) \xrightarrow{\sim} \mathrm{Der}_S(Y,\mathcal{J}/\mathcal{J}^2).$$

Let β be the isomorphism $(1+\mathcal{J})/(1+\mathcal{J}^2) \ni 1+t \longmapsto t \in \mathcal{J}/\mathcal{J}^2$ of abelian sheaves. It is easy to check that the morphisms $\overset{\circ}{\delta}\colon \mathcal{O}_Y \ni a \longmapsto a - s_{\mathrm{ri}}([a]) \in \mathcal{J}/\mathcal{J}^2$ and $\delta\colon M_Y \ni m \longmapsto \beta(m/s_{\mathrm{mo}}([m])) \in \mathcal{J}/\mathcal{J}^2$ satisfy (1) $\sim$ (4) and give a local splitting of (2.1.3.1). $\square$

Lemma 2.1.4. *Let the notations be as in* (2.1.3) *with Y, Z log smooth over S. Let $\mathbb{A}^n_S$ $(n \in \mathbb{N})$ be a log scheme whose underlying scheme is $\mathbb{A}^n_{\overset{\circ}{S}}$ and whose log structure is the pull-back of that of S by the natural projection $\mathbb{A}^n_{\overset{\circ}{S}} \longrightarrow \overset{\circ}{S}$. Let z be a point of $\overset{\circ}{Z}$ and assume that there exists a chart $(Q \longrightarrow M_S, P \longrightarrow M_Z, Q \xrightarrow{\rho} P)$ of $Z \longrightarrow S$ on a neighborhood of z such that ρ is injective, such that* $\mathrm{Coker}(\rho^{\mathrm{gp}})$ *is torsion free and that the natural homomorphism $\mathcal{O}_{Z,z} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \longrightarrow \Lambda^1_{Z/S,z}$ is an isomorphism. Then, on a neighborhood of z, there exist a nonnegative integer c and the following cartesian diagram*:

(2.1.4.1)
$$\begin{array}{ccc} Z & \xrightarrow{\iota} & Y \\ \big\downarrow & & \big\downarrow \\ (S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], P^a) & \xrightarrow{\subset} & (S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], P^a) \times_S \mathbb{A}^c_S. \end{array}$$

Here the vertical morphisms are strict etale and the lower horizontal morphism is the base change of the zero section $S \longrightarrow \mathbb{A}^c_S$.

Proof. Assume that $\overset{\circ}{Y}$ is affine. By (2.1.3) we have the following split exact sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 \xrightarrow{\Delta} \iota^*(\Lambda^1_{Y/S}) \longrightarrow \Lambda^1_{Z/S} \longrightarrow 0. \tag{2.1.4.2}$$

Let s be the image of z in S. Since $\mathrm{Coker}(\rho^{\mathrm{gp}})$ is torsion free, there exists a homomorphism $P^{\mathrm{gp}} \longrightarrow M^{\mathrm{gp}}_{Y,\iota(z)}$ which is compatible with the monoid homomorphisms $Q \longrightarrow M_{S,s} \longrightarrow M_{Y,\iota(z)}$ and $P \longrightarrow M_{Z,z}$. Since we have $(M_Z/\mathcal{O}^*_Z)_z = (M_Y/\mathcal{O}^*_Y)_{\iota(z)}$, the homomorphism $P^{\mathrm{gp}} \longrightarrow M^{\mathrm{gp}}_{Y,\iota(z)}$ induces the homomorphism $P \longrightarrow M_{Y,\iota(z)}$, which induces a chart of $Y \longrightarrow S$ on a neighborhood of $\iota(z)$. By the exact sequence (2.1.4.2), there exist local sections $x_{r+1}, \ldots, x_{r+c} \in \mathcal{J}$ and elements $m_1, \ldots, m_r \in P$ such that $\{d\log m_i\}_{i=1}^r$ is a basis of $\Lambda^1_{Z/S,z}$ and $\{\{d\log m_i\}_{i=1}^r, \{dx_j\}_{j=r+1}^{d+c}\}$ is a basis of $\Lambda^1_{Y/S,\iota(z)}$. By the same argument as that in [54, p. 205], we have compatible etale morphisms $\overset{\circ}{Z} \longrightarrow \overset{\circ}{S} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ and $\overset{\circ}{Y} \longrightarrow (\overset{\circ}{S} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) \times_{\overset{\circ}{S}} \underline{\mathrm{Spec}}_{\overset{\circ}{S}}(\mathcal{O}_{\overset{\circ}{S}}[x_{d+1}, \ldots, x_{d+c}])$ in the classical sense. $\square$

Corollary 2.1.5. *Let $S_0 \overset{\subset}{\longrightarrow} S$ be a closed immersion of fine log schemes. Let Z_0 (resp. Y) be a log smooth scheme over S_0 (resp. S), which can be considered as a log scheme over S. Let $\iota\colon Z_0 \overset{\subset}{\longrightarrow} Y$ be an exact closed immersion over S. Let z be a point of $\overset{\circ}{Z}_0$ and assume that there exists a chart $(Q \longrightarrow M_S, P \longrightarrow M_{Z_0}, Q \overset{\rho}{\longrightarrow} P)$ of $Z_0 \longrightarrow S_0 \overset{\subset}{\longrightarrow} S$ on a neighborhood of z such that ρ is injective, such that $\mathrm{Coker}(\rho^{\mathrm{gp}})$ is torsion free and that the natural homomorphism $\mathcal{O}_{Z_0,z} \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \longrightarrow \Lambda^1_{Z/S_0,z}$ is an isomorphism. Then, on a neighborhood of z, there exist a nonnegative integer c and the following cartesian diagram*

(2.1.5.1)
$$\begin{array}{ccccc}
Z_0 & \longrightarrow & Y' & \longrightarrow & Y \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
(S_0 \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], P^a) & \overset{\subset}{\longrightarrow} & (S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], P^a) & \overset{\subset}{\longrightarrow} & (S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], P^a) \times_S \mathbb{A}^c_S,
\end{array}$$

where the vertical morphisms are strict etale and the lower second horizontal morphism is the base change of the zero section $S \overset{\subset}{\longrightarrow} \mathbb{A}^c_S$ and $Y' := Y \times_{\mathbb{A}^c_S} S$.

Proof. Set $Y_0 := Y \times_S S_0$ and let $\iota_0\colon Z_0 \overset{\subset}{\longrightarrow} Y_0$ be the closed immersion induced by ι. Apply (2.1.4) for ι_0. Then we have a cartesian diagram (2.1.4.1) for Z_0/S_0 and Y_0/S_0 around any point $z \in \overset{\circ}{Z}_0$. By the same argument as in the proof of (2.1.4) using the isomorphism $(M_Y/\mathcal{O}^*_Y)_{\iota(z)} \simeq (M_{Y_0}/\mathcal{O}^*_{Y_0})_{\iota_0(z)}$, we see that the chart $P \longrightarrow M_{Y_0}$ extends to a chart

$P \longrightarrow M_Y$ around $\iota(z)$. Let $\mathcal{J}_0$ (resp. $\mathcal{J}$) be the ideal sheaf of ι_0 (resp. ι). Let $\{\{d\log m_i\}_{i=1}^r, \{dx_j^{(0)}\}_{j=r+1}^{r+c}\}$ $(m_i \in P, x_j^{(0)} \in \mathcal{J}_0)$ be a basis of $\Lambda^1_{Y_0/S_0}$. Let x_j be any lift of $x_j^{(0)}$ in $\mathcal{J}$. Then, using [13, Corollaire to II §3 Proposition 6], we see that $\{\{d\log m_i\}_{i=1}^r, \{dx_j\}_{j=r+1}^{r+c}\}$ is a basis of $\Lambda^1_{Y/S,\iota(z)}$ (cf.[40, 4 (17.12.2)]). Hence we have a strict etale morphism $Y \longrightarrow (S \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P], P^a) \times_S \mathbb{A}^c_S$. Now we obtain the diagram (2.1.5.1). $\square$

Remark 2.1.6. By a similar argument to the proof of (2.1.4) and (2.1.5) and using [54, (3.5), (3.13)], we see that the diagrams as in (2.1.4.1), (2.1.5.1) always exist etale locally (for some $Q \longrightarrow P$) even if we drop the condition on the existence of a nice chart which we assumed in (2.1.4), (2.1.5).

(2) Let Y be a scheme over a scheme T. Let $\mathrm{Div}(Y/T)_{\geq 0}$ be the integral monoid of effective Cartier divisors on Y over T (e.g., [56, (1.1.1)]). We say that a family $\{E_\lambda\}_{\lambda\in\Lambda}$ of non-zero elements in $\mathrm{Div}(Y/T)_{\geq 0}$ has a *locally finite intersection* if, for any point $z \in Y$, there exists a Zariski open neighborhood V of z such that $\Lambda_V := \{\lambda \in \Lambda \mid E_\lambda|_V \neq 0\}$ is a finite set. If $\{E_\lambda\}_{\lambda\in\Lambda}$ has a locally finite intersection, then we can define a sum $\sum_{\lambda\in\Lambda} n_\lambda E_\lambda$ $(n_\lambda \in \mathbb{N})$ in $\mathrm{Div}(Y/T)_{\geq 0}$.

Let $f\colon X \longrightarrow S_0$ be a smooth morphism of schemes.

Definition 2.1.7. We call an effective Cartier divisor D on X/S_0 is a *relative simple normal crossing divisor* (=:*relative SNCD*) on X/S_0 if there exists a family $\Delta := \{D_\lambda\}_{\lambda\in\Lambda}$ of non-zero effective Cartier divisors on X/S_0 of locally finite intersection which are smooth schemes over S_0 such that

(2.1.7.1) $$D = \sum_{\lambda\in\Lambda} D_\lambda \quad \text{in} \quad \mathrm{Div}(X/S_0)_{\geq 0}$$

and, for any point z of D, there exist a Zariski open neighborhood V of z in X and the following cartesian diagram:

(2.1.7.2)
$$\begin{tikzcd} D|_V \arrow[r, "\subset"] \arrow[d] & V \arrow[d, "g"] \\ \underline{\mathrm{Spec}}_{S_0}(\mathcal{O}_{S_0}[y_1, \ldots, y_d]/(y_1 \cdots y_s)) \arrow[r] & \underline{\mathrm{Spec}}_{S_0}(\mathcal{O}_{S_0}[y_1, \ldots, y_d]) \end{tikzcd}$$

(for some positive integers s and d such that $s \leq d$), where the morphism g is etale.

Note that we do not require a relation a priori between $\{D_\lambda|_V\}_{\lambda\in\Lambda_V}$ and the family $\{y_i = 0\}_{i=1}^s$ of closed subschemes in V in the diagram (2.1.7.2). However, by (A.0.1) below, we obtain $\{D_\lambda|_V\}_{\lambda\in\Lambda_V} = \{\{y_i = 0\}\}_{i=1}^s$ in the diagram (2.1.7.2) if V is small.

Definition 2.1.8. We call a smooth divisor on X/S_0 contained in D a *smooth component* of D. We call $\Delta = \{D_\lambda\}_{\lambda\in\Lambda}$ a *decomposition of D by smooth components of D over S_0*.

Note that the decomposition of a relative SNCD by smooth components is not unique.

Let $\mathrm{Div}_D(X/S_0)_{\geq 0}$ be a submonoid of $\mathrm{Div}(X/S_0)_{\geq 0}$ consisting of effective Cartier divisors E's on X/S_0 such that there exists an open covering $X = \bigcup_{i\in I} V_i$ (depending on E) of X such that $E|_{V_i}$ is contained in the submonoid of $\mathrm{Div}(V_i/S_0)_{\geq 0}$ generated by $D_\lambda|_{V_i}$ ($\lambda\in\Lambda$). By (A.0.1) below, we see that the definition of $\mathrm{Div}_D(X/S_0)_{\geq 0}$ is independent of the choice of Δ.

The pair (X,D) gives a natural fs(=fine and saturated) log structure in $\widetilde{X}_{\mathrm{zar}}$ as follows (cf. [54, p. 222–223], [29, §2]).

Let $M(D)'$ be a presheaf of monoids in $\widetilde{X}_{\mathrm{zar}}$ defined as follows: for an open subscheme V of X,

$$(2.1.8.1)\qquad \Gamma(V,M(D)') := \{(E,a) \in \mathrm{Div}_{D|_V}(V/S_0)_{\geq 0}\times \Gamma(V,\mathcal{O}_X) \mid a \text{ is a generator of } \Gamma(V,\mathcal{O}_X(-E))\}$$

with a monoid structure defined by $(E,a)\cdot(E',a') := (E+E',aa')$. The natural morphism $M(D)' \longrightarrow \mathcal{O}_X$ defined by the second projection $(E,a)\mapsto a$ induces a morphism $M(D)' \longrightarrow (\mathcal{O}_X,*)$ of presheaves of monoids in $\widetilde{X}_{\mathrm{zar}}$. The log structure $M(D)$ is, by definition, the associated log structure to the sheafification of $M(D)'$. Because $\mathrm{Div}_{D|_V}(V/S_0)_{\geq 0}$ is independent of the choice of the decomposition of $D|_V$ by smooth components, $M(D)$ is independent of the choice of the decomposition of D by smooth components of D.

Proposition 2.1.9. *Let the notations be as above. Let z be a point of D and let V be an open neighborhood of z in X which admits the diagram (2.1.7.2). Assume that $z\in\bigcap_{i=1}^{s}\{y_i=0\}$. If V is small, then the log structure $M(D)|_V \longrightarrow \mathcal{O}_V$ is isomorphic to $\mathcal{O}_V^* y_1^{\mathbb{N}}\cdots y_s^{\mathbb{N}} \overset{\subset}{\longrightarrow} \mathcal{O}_V$. Consequently $M(D)|_V$ is associated to the homomorphism $\mathbb{N}_V^s \ni e_i \longmapsto y_i \in M(D)|_V$ $(1\leq i\leq s)$ of sheaves of monoids on V, where $\{e_i\}_{i=1}^s$ is the canonical basis of $\mathbb{N}^s$. In particular, $M(D)$ is fs.*

Proof. By the definition of $M'(D)$ and by (A.0.1) below, the homomorphism $M'(D)|_V \longrightarrow \mathcal{O}_V$ factors through $\mathcal{O}_V^* y_1^{\mathbb{N}}\cdots y_s^{\mathbb{N}}$ if V is small. Hence there exists a natural morphism $M(D)|_V \longrightarrow \mathcal{O}_V^* y_1^{\mathbb{N}}\cdots y_s^{\mathbb{N}}$ of log structures on V. By taking the stalks, one can easily check that the morphism above is an isomorphism. □

By abuse of notation, we denote the log scheme $(X,M(D))$ by (X,D).

Set $U := X\setminus D$ and let $j\colon U \overset{\subset}{\longrightarrow} X$ be the natural open immersion. Set $N(D) := \mathcal{O}_X\cap j_*(\mathcal{O}_U^*)$. We remark that $M(D)\subsetneqq N(D)$ in general; indeed, the stalks of $N(D)/\mathcal{O}_X^*$ are not even finitely generated in general (see (A.0.9) below).

Let $S_0 \overset{\subset}{\longrightarrow} S$ be a closed immersion of schemes defined by a quasi-coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_S$. We can consider the scheme X as a scheme over S by the closed immersion $S_0 \overset{\subset}{\longrightarrow} S$. Let $(\mathcal{X}, \mathcal{D})(= (\mathcal{X}, M(\mathcal{D})))$ be a smooth scheme with a relative SNCD over S. Let $\iota \colon X \longrightarrow \mathcal{X}$ be a closed immersion over S defined by a quasi-coherent ideal sheaf of $\mathcal{O}_{\mathcal{X}}$.

Definition 2.1.10. Let $\Delta := \{D_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of D by smooth components of D. Let $\iota \colon (X, D) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D})$ be an exact (closed) immersion into a smooth scheme with a relative SNCD over S. Then we call ι (or a pair $(\mathcal{X}, \mathcal{D})/S$ by abuse of terminology) an *admissible* (*closed*) *immersion* over S with respect to Δ if $\mathcal{D}$ admits a decomposition $\widetilde{\Delta} := \{\mathcal{D}_\lambda\}_{\lambda \in \Lambda}$ by smooth components of $\mathcal{D}$ such that ι induces an isomorphism $D_\lambda \overset{\sim}{\longrightarrow} \mathcal{D}_\lambda \times_{\mathcal{X}} X$ of schemes over S_0 for all $\lambda \in \Lambda$. We say that $\widetilde{\Delta}$ is *compatible* with Δ. We sometimes denote the admissible (closed) immersion by $\iota \colon (X, D; \Delta) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D}; \widetilde{\Delta})$.

Remark 2.1.11. If the underlying topological spaces of $\overset{\circ}{S}_0$ and $\overset{\circ}{S}$ are the same and if $(\mathcal{X}, \mathcal{D})$ is a lift of (X, D) with a decomposition Δ of D by smooth components of D, we obtain the decomposition $\widetilde{\Delta}$ of $\mathcal{D}$ by smooth components of $\mathcal{D}$ canonically.

Let $\iota \colon (X, D; \Delta) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D}; \widetilde{\Delta})$ be an admissible immersion. Let V be an open subscheme of X. If we set $\mathcal{V} := \mathcal{X} \setminus (\overline{X} \setminus V)$ (here $\overline{X}$ is the closure of X in $\mathcal{X}$), the restriction of ι to $(V, D \cap V)$

$$\iota_V \colon (V, D \cap V) \overset{\subset}{\longrightarrow} (\mathcal{V}, (\bigcup_{\lambda \in \Lambda_V} \mathcal{D}_\lambda) \cap \mathcal{V}) \tag{2.1.11.1}$$

is an admissible immersion with respect to $\{D_\lambda\}_{\lambda \in \Lambda_V}$.

Definition 2.1.12. We call the admissible immersion ι_V the *restriction* of ι to V, and $\Delta|_V := \{D_\lambda\}_{\lambda \in \Lambda_V}$ the *restriction* of Δ to V.

By (2.1.5) and (A.0.1) below, we have the following:

Lemma 2.1.13. *Let $\iota \colon (X, D; \Delta) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D}; \widetilde{\Delta})$ be an admissible immersion. Then, for any point z of X, there exist Zariski open neighborhoods V of z and $\mathcal{V}$ of $\iota(z)$, positive integers $s \leq d \leq d'$ and the following two cartesian diagrams*:

(2.1.13.1)

$$\begin{array}{ccc} \mathcal{D}|_{\mathcal{V}} & \overset{\subset}{\longrightarrow} & \mathcal{V} \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ \underline{\mathrm{Spec}}_S(\mathcal{O}_S[x_1, \ldots, x_{d'}]/(x_1 \cdots x_s)) & \longrightarrow & \underline{\mathrm{Spec}}_S(\mathcal{O}_S[x_1, \ldots, x_{d'}]), \end{array}$$

(2.1.13.2)

$$\begin{array}{ccc} V & \xrightarrow{\subset} & \mathcal{V} \\ \big\downarrow & & \big\downarrow g \\ \underline{\mathrm{Spec}}_{S_0}(\mathcal{O}_{S_0}[x_1,\ldots,x_{d'}]/(x_{d+1},\ldots,x_{d'})) & \longrightarrow & \underline{\mathrm{Spec}}_S(\mathcal{O}_S[x_1,\ldots,x_{d'}]), \end{array}$$

where g is etale and $\{\mathcal{D}_\lambda|_{\mathcal{V}}\}_{\lambda\in\Lambda_{\mathcal{V}}} = \{\{x_i = 0\}\}_{i=1}^{s}$ *in the diagram* (2.1.13.1).

Let $(S,\mathcal{I},\gamma)$ be a PD-scheme and let (X,D) be a smooth scheme with a relative SNCD over $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I})$. Let Δ be a decomposition of D by smooth components of D. Consider triples

(2.1.13.3) $$((U,D|_U),\mathfrak{D}_{(U,D|_U)}((\mathcal{U},\mathcal{D})),[\ ])\text{'s},$$

where U is an open subscheme of X, $(U,D|_U) \xrightarrow{\subset} (\mathcal{U},\mathcal{D})$ is an admissible immersion over S with respect to Δ_U and $\mathfrak{D}_{(U,D|_U)}((\mathcal{U},\mathcal{D}))$ is the log PD-envelope of the immersion above over $(S,\mathcal{I},\gamma)$. Let $((X,D)/S)^{\log}_{\mathrm{ARcrys}}$ be a full subcategory of $((X,D)/S)^{\log}_{\mathrm{crys}}$ whose objects are the triples (2.1.13.3). We define the topology of $((X,D)/S)^{\log}_{\mathrm{ARcrys}}$ as the induced topology by that of $((X,D)/S)^{\log}_{\mathrm{crys}}$. Let $(\widetilde{(X,D)/S})^{\log}_{\mathrm{ARcrys}}$ be the topos associated to $((X,D)/S)^{\log}_{\mathrm{ARcrys}}$.

Definition 2.1.14. We call the site $((X,D)/S)^{\log}_{\mathrm{ARcrys}}$ (resp. the topos $(\widetilde{(X,D)/S})^{\log}_{\mathrm{ARcrys}}$) the *admissible restricted log crystalline site* (resp. *admissible restricted log crystalline topos*) of $(X,D)/(S,\mathcal{I},\gamma)$.

Let

(2.1.14.1) $$Q^{\mathrm{AR}}_{(X,D)/S}\colon (\widetilde{(X,D)/S})^{\log}_{\mathrm{ARcrys}} \longrightarrow (\widetilde{(X,D)/S})^{\log}_{\mathrm{Rcrys}}$$

be a natural morphism of topoi: For an object $E \in (\widetilde{(X,D)/S})^{\log}_{\mathrm{Rcrys}}$, $Q^{\mathrm{AR}*}_{(X,D)/S}(E)$ is the natural restriction of E and $Q^{\mathrm{AR}*}_{(X,D)/S}$ commutes with inverse limits. We also have a morphism

(2.1.14.2) $$Q^{\mathrm{AR}}_{(X,D)/S}\colon ((\widetilde{(X,D)/S})^{\log}_{\mathrm{ARcrys}}, Q^{\mathrm{AR}*}_{(X,D)/S}Q^*_{(X,D)/S}(\mathcal{O}_{(X,D)/S})) \longrightarrow ((\widetilde{(X,D)/S})^{\log}_{\mathrm{Rcrys}}, Q^*_{(X,D)/S}(\mathcal{O}_{(X,D)/S}))$$

of ringed topoi.

Proposition 2.1.15. *The morphism* (2.1.14.1) (*resp.* (2.1.14.2)) *gives an equivalence of topoi* (*resp. ringed topoi*).

Proof. Let $\iota\colon (X,D) \xrightarrow{\subset} \mathcal{P}$ be an exact closed immersion into a log smooth scheme over S. Let $\mathcal{P}'$ be an exact closed log subscheme of $\mathcal{P}$ locally obtained in (2.1.5) for ι. Then ι is locally an admissible immersion with respect to the restriction of Δ to an open subscheme of X since $\mathcal{P}'$ is a local lift of (X, D). Hence we obtain (2.1.15) by (2.1.2) and by the proof of (2.1.2). $\square$

2.2 The Log Linearization Functor

In this section we recall the log version of the linearization functor in [11, §6] (cf. [54, (6.9)]) and the log HPD differential operators. After that, we show some properties of the log linearization functor for a smooth scheme with a relative SNCD.

(1) Let $(S, \mathcal{I}, \gamma)$ and $f\colon Y \longrightarrow S$ be as in §1.6. For an object $(V, T, M_T, \iota, \delta)$ of the log crystalline site $(Y/S)^{\log}_{\rm crys}$, we sometimes denote it simply by (V, T, M_T, δ), (V, T, δ) or even T as usual. We also denote by T the representable sheaf in $(\widetilde{Y/S})^{\log}_{\rm crys}$ defined by T. Let F be an object of $(\widetilde{Y/S})^{\log}_{\rm crys}$. Let $(\widetilde{Y/S})^{\log}_{\rm crys}|_F$ be the localization of the topos $(\widetilde{Y/S})^{\log}_{\rm crys}$ at F: the objects in $(\widetilde{Y/S})^{\log}_{\rm crys}|_F$ are the pairs (E, ϕ)'s, where E is an object in $(\widetilde{Y/S})^{\log}_{\rm crys}$ and ϕ is a morphism $E \longrightarrow F$ in $(\widetilde{Y/S})^{\log}_{\rm crys}$. As usual, let

$$j_F\colon (\widetilde{Y/S})^{\log}_{\rm crys}|_F \longrightarrow (\widetilde{Y/S})^{\log}_{\rm crys} \tag{2.2.0.1}$$

be a morphism of topoi defined by the following: for an object E in $(\widetilde{Y/S})^{\log}_{\rm crys}$, $j_F^*(E)$ is a pair $(E \times F, E \times F \overset{\rm proj.}{\longrightarrow} F)$; for an object (E, ϕ) in $(\widetilde{Y/S})^{\log}_{\rm crys}|_F$, $j_{F*}((E, \phi))$ is the sheaf of the sections of ϕ.

Let (V, T, M_T, δ) be an object of the log crystalline site $(Y/S)^{\log}_{\rm crys}$. Let

$$j_T\colon (\widetilde{Y/S})^{\log}_{\rm crys}|_T \longrightarrow (\widetilde{Y/S})^{\log}_{\rm crys}$$

be the localization morphism in (2.2.0.1) for $F = T$. Let

$$\varphi\colon ((\widetilde{Y/S})^{\log}_{\rm crys}|_T, \mathcal{O}_{Y/S}|_T) \longrightarrow (\widetilde{T}_{\rm zar}, \mathcal{O}_T) \tag{2.2.0.2}$$

be a morphism of ringed topoi defined by the following (cf. [11, 5.26 Proposition]): for an $\mathcal{O}_T$-module $\mathcal{E}$, the sections of $\varphi^*(\mathcal{E})$ at (T', ϕ) is $\Gamma(T', \phi^*(\mathcal{E}))$; for an $\mathcal{O}_{Y/S}$-module E in $(\widetilde{Y/S})^{\log}_{\rm crys}|_T$, $\varphi_*(E)$ is defined as follows: let T' be an open log subscheme of T. Let T' also denote the object $(T' \times_T V \overset{\subset}{\longrightarrow} (T' \times_T T = T'))$ in $(Y/S)^{\log}_{\rm crys}$. Then we have a natural morphism $\iota\colon T' \longrightarrow T$ in $(Y/S)^{\log}_{\rm crys}$; the section of $\varphi_*(E)$ is, by definition, $\Gamma(T', \varphi_*(E)) := E((T', \iota))$. By the log version of the ringed topos version of [11, 5.26 Proposition], we have the following diagram of ringed topoi

(2.2.0.3)

$$\begin{array}{ccccc} ((\widetilde{Y/S})^{\log}_{\rm crys}|_T, \mathcal{O}_{Y/S}|_T) & \xrightarrow{j_T} & ((\widetilde{Y/S})^{\log}_{\rm crys}, \mathcal{O}_{Y/S}) & \xrightarrow{u_{Y/S}} & (\widetilde{Y}_{\rm zar}, f^{-1}(\mathcal{O}_S)) \\ {\scriptstyle\varphi}\big\downarrow & & & & \big\uparrow \\ (\widetilde{T}_{\rm zar}, \mathcal{O}_T) & \longleftarrow & (\widetilde{V}_{\rm zar}, \mathcal{O}_V) & \longrightarrow & (\widetilde{V}_{\rm zar}, f^{-1}(\mathcal{O}_S)|_V) \end{array}$$

and the following commutative diagram of topoi

(2.2.0.4)
$$\begin{array}{ccc} (\widetilde{Y/S})^{\log}_{\rm crys}|_T & \xrightarrow{j_T} & (\widetilde{Y/S})^{\log}_{\rm crys} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle u_{Y/S}} \\ \widetilde{T}_{\rm zar} = \widetilde{V}_{\rm zar} & \longrightarrow & \widetilde{Y}_{\rm zar}, \end{array}$$

where φ is defined as follows: $\Gamma((T',\phi),\varphi^{-1}(\mathcal{E})) := \Gamma(T',\phi^{-1}(\mathcal{E}))$ for $\mathcal{E} \in \widetilde{T}_{\rm zar}$ and $(T',\phi) \in (\widetilde{Y/S})^{\log}_{\rm crys}|_T$.

By the log version of [11, 5.27 Corollary], we have the following:

Proposition 2.2.1. *Let the notations be as above. Assume that $V = Y$. Then the following hold*:

(1) *The functors j_{T*} is exact.*

(2) *For an abelian sheaf E in $(\widetilde{Y/S})^{\log}_{\rm crys}$, $j_{T*}(E)$ is $u_{Y/S*}$-acyclic.*

Now let us recall the log linearization functor briefly (cf. [11, 6.10 Proposition], [54, (6.9)]).

Let $\iota\colon Y \overset{\subset}{\longrightarrow} \mathcal{Y}$ be a closed immersion into a log smooth scheme over S such that γ extends to $\overset{\circ}{\mathcal{Y}}$. Let $\mathfrak{D}_Y(\mathcal{Y})$ be the log PD-envelope of ι over $(S,\mathcal{I},\gamma)$. Let

(2.2.1.1)
$$\varphi\colon ((\widetilde{Y/S})^{\log}_{\rm crys}|_{\mathfrak{D}_Y(\mathcal{Y})}, \mathcal{O}_{Y/S}|_{\mathfrak{D}_Y(\mathcal{Y})}) \longrightarrow (\widetilde{\overset{\circ}{\mathfrak{D}}_Y(\mathcal{Y})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})})$$

be the morphism (2.2.0.2) for $T = \mathfrak{D}_Y(\mathcal{Y})$. For an $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-module $\mathcal{E}$, we define $L(\mathcal{E})$ as follows:

(2.2.1.2)
$$L(\mathcal{E}) := j_{\mathfrak{D}_Y(\mathcal{Y})*}\varphi^*(\mathcal{E}) \in (\widetilde{Y/S})^{\log}_{\rm crys}.$$

As in the classical crystalline case, L defines a functor:

(2.2.1.3)
$$\{\text{the category of } \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\text{-modules and } \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\text{-linear morphisms}\} \longrightarrow \{\mathcal{O}_{Y/S}\text{-modules}\}.$$

For $(U,T,\delta) \in (Y/S)^{\log}_{\rm crys}$, let $\mathfrak{D}_U(T\times_S\mathcal{Y})$ be the PD-envelope of $U \overset{\subset}{\longrightarrow} T\times_S\mathcal{Y}$ compatible with γ and δ and let $p_T : \mathfrak{D}_U(T\times_S\mathcal{Y}) \longrightarrow T$, $p_Y : \mathfrak{D}_U(T\times_S\mathcal{Y}) \longrightarrow \mathfrak{D}_Y(\mathcal{Y})$ be natural morphisms. Then the sheaf $L(\mathcal{E})_{(U,T,\delta)}$ on $T_{\rm zar}$ induced by $L(\mathcal{E})$ is given by $L(\mathcal{E})_{(U,T,\delta)} = p_{T*}p_Y^*\mathcal{E} = \mathcal{O}_{\mathfrak{D}_U(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E}$.

As in the classical crystalline case, another definition of the log linearization functor is known. To state it, we need to recall the definition of a log HPD stratification (cf. [11, 4.3H Definition]; however there is a mistype in [loc.cit., 1)]: "$\mathcal{D}_{X/S}$-linear" should be replaced by "$\mathcal{D}_{X/S}(1)$-linear").

Let $\mathfrak{D}_Y(\mathcal{Y}^2)$ be the log PD-envelope of the locally closed immersion $Y \overset{\subset}{\longrightarrow} \mathcal{Y} \times_S \mathcal{Y}$ over $(S, \mathcal{I}, \gamma)$. Let $\mathcal{J}$ be the PD-ideal sheaf defining the exact locally closed immersion $Y \overset{\subset}{\longrightarrow} \mathfrak{D}_Y(\mathcal{Y}^2)$.

Definition 2.2.2. Let $\mathcal{E}$ and $\mathcal{F}$ be two $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-modules.

(1) An isomorphism $\epsilon\colon \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}$ is called a *log HPD stratification* if ϵ is $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}$-linear, if $\epsilon \mod \mathcal{J}$ is the identity and if the cocycle condition holds.

(2) ([75, (1.1.3)]) An $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-linear morphism $u : \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \longrightarrow \mathcal{F}$ is called a *log HPD differential operator*.

(3) ([75, (1.1.3)]) For a positive integer n, an $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-linear morphism

$$u : (\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}/\mathcal{J}^{[n+1]}) \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \longrightarrow \mathcal{F}$$

is called a *log PD differential operator of order* $\leq n$.

Set $L'(\mathcal{E}) := \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E}$. Then, as in the classical crystalline case, there is a canonical log HPD stratification

$$\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} L'(\mathcal{E}) \overset{\sim}{\longrightarrow} L'(\mathcal{E}) \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}.$$

Hence $L'(\mathcal{E})$ defines a crystal of $\mathcal{O}_{Y/S}$-modules (cf. [54, (6.7)]), which we denote by the same symbol $L'(\mathcal{E})$. L' defines a functor

$$\{\text{the category of } \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\text{-modules and log HPD differential operators}\} \longrightarrow$$

$$\{\text{the category of crystals of } \mathcal{O}_{Y/S}\text{-modules}\} :$$

For a log HPD differential operator $u : \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \longrightarrow \mathcal{F}$, $L'(u) : L'(\mathcal{E}) \longrightarrow L'(\mathcal{F})$ is given by the composite

$$\begin{aligned} (2.2.2.1) \quad \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} &\overset{\delta\otimes\mathrm{id}}{\longrightarrow} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \\ &\overset{\mathrm{id}\otimes u}{\longrightarrow} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{F}, \end{aligned}$$

where $\delta\colon \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \longrightarrow \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} = \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^3)}$ is the map induced by the projection $\mathcal{Y}^3 \longrightarrow \mathcal{Y}^2$ to the first and the third factors. By the log version of [11, 6.10 Proposition], the following holds:

Proposition 2.2.3. *For an $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-module $\mathcal{E}$, there exists a canonical isomorphism*

$$L'(\mathcal{E}) \overset{\sim}{\longrightarrow} L(\mathcal{E}).$$

Hence L also defines the functor

$$\{\text{the category of } \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\text{-modules and log HPD differential operators}\} \longrightarrow$$

$$\{\text{the category of crystals of } \mathcal{O}_{Y/S}\text{-modules}\}.$$

By (2.2.2.1) and (2.2.3), we see the following: For a log HPD differential operator $u : \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \longrightarrow \mathcal{F}$ and $(U, T, \delta) \in (Y/S)^{\log}_{\text{crys}}$, $L(u)_{(U,T,\delta)} : L(\mathcal{E})_{(U,T,\delta)} \longrightarrow L(\mathcal{F})_{(U,T,\delta)}$ is given by the composite

(2.2.3.1)

$$\mathcal{O}_{\mathfrak{D}_Y(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E} \xrightarrow{\delta_T\otimes \mathrm{id}} \mathcal{O}_{\mathfrak{D}_Y(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{E}$$
$$\xrightarrow{\mathrm{id}\otimes u} \mathcal{O}_{\mathfrak{D}_Y(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{F},$$

where $\delta_T : \mathcal{O}_{\mathfrak{D}_Y(T\times_S\mathcal{Y})} \longrightarrow \mathcal{O}_{\mathfrak{D}_Y(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} = \mathcal{O}_{\mathfrak{D}_Y(T\times_S\mathcal{Y}^2)}$ (the equality follows from the log version of [11, 6.3, proof of 6.10]) is the map induced by the projection $T \times_S \mathcal{Y}^2 \longrightarrow T \times_S \mathcal{Y}$ to the first and the third factors. It is easy to obtain the following lemma from the definition of L'.

Lemma 2.2.4. *The functor L, regarded as the functor*

$$\{\textit{the category of } \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\textit{-modules and } \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\textit{-linear morphisms}\} \longrightarrow$$

$$\{\textit{the category of crystals of } \mathcal{O}_{Y/S}\textit{-modules}\}$$

is exact.

Remark 2.2.5. (cf. [3, IV Remarque 1.7.8]) The functor L is not left exact as a functor (2.2.1.3) in general. Indeed, let κ be a perfect field of characteristic $p > 0$ and let W_n ($n \in \mathbb{Z}_{\geq 2}$) be the Witt ring of κ of length n. Set $S := (\mathrm{Spec}(W_n), W_n^*, pW_n, [\ ])$, $Y := (\mathrm{Spec}(\kappa), \kappa^*)$, $\mathcal{Y} := S$ and $\mathcal{E} := W_n$. Then, though a sequence

$$0 \longrightarrow p\mathcal{E} \longrightarrow \mathcal{E} \xrightarrow{p^{n-1}\times} \mathcal{E}$$

of W_n-modules is exact, the following sequence

$$0 \longrightarrow L(p\mathcal{E}) \longrightarrow L(\mathcal{E}) \xrightarrow{p^{n-1}\times} L(\mathcal{E})$$

in $\mathcal{O}_{Y/S}$-modules is not exact since the value of the sequence above at Y is

$$0 \longrightarrow pW_n/p^2W_n \xrightarrow{0} \kappa \xrightarrow{0} \kappa.$$

The following is analogous to [11, 6.2 Proposition].

Lemma 2.2.6. (1) *Let Y_i $(i = 1, 2)$ and $(S, \mathcal{I}, \gamma)$ be as in §1.6. Let $T_i = (U_i, T_i, \delta_i) = (U_i, T_i, M_{T_i}, \delta_i)$ $(i = 1, 2)$ be an object of the log crystalline site $(Y_i/S)^{\log}_{\text{crys}}$, which is considered as a representable sheaf in the topos $(\widetilde{Y_i/S})^{\log}_{\text{crys}}$.*

Let $\mathcal{J}_i$ be the defining ideal sheaf of the closed immersion $U_i \overset{\subset}{\longrightarrow} T_i$. Let $Y_1 \overset{\subset}{\longrightarrow} Y_2$ be an exact closed immersion which induces an exact closed immersion $U_1 \overset{\subset}{\longrightarrow} U_2$. Let $g\colon T_1 \longrightarrow T_2$ be an exact closed immersion of fine log PD-schemes over S fitting into the following commutative diagram

$$\begin{array}{ccc} U_1 & \overset{\subset}{\longrightarrow} & U_2 \\ \cap\downarrow & & \downarrow\cap \\ T_1 & \underset{\subset}{\overset{g}{\longrightarrow}} & T_2. \end{array}$$

Assume that g^* induces a surjective morphism $g^*\colon g^*(\mathcal{J}_2) \longrightarrow \mathcal{J}_1$. Let

$$\iota\colon (\widetilde{Y_1/S})^{\log}_{\rm crys}|_{T_1} \longrightarrow (\widetilde{Y_2/S})^{\log}_{\rm crys}|_{T_2}$$

be the induced morphism of topoi. Let $(U, T, \delta, \phi) = (U, T, M_T, \delta, \phi)$ be a representable object in $(\widetilde{Y_2/S})^{\log}_{\rm crys}|_{T_2}$. Let $\mathcal{J}$ be the defining ideal sheaf of the closed immersion $U \overset{\subset}{\longrightarrow} T$. Set $\overline{\mathcal{J}} := \mathcal{J} + \mathcal{I}\mathcal{O}_T$ and let $\overline{\delta}$ be the extension of δ and γ on $\overline{\mathcal{J}}$. Let $\mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1)$ be the log PD-envelope of the closed immersion $U \times_{U_2} U_1 \overset{\subset}{\longrightarrow} T \times_{T_2} T_1$ over $(T, \overline{\mathcal{J}}, \overline{\delta})$ with natural morphism $q\colon (U \times_{U_2} U_1, \mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1), [\ ]) \longrightarrow (U_1, T_1, \delta_1)$ in $(Y_1/S)^{\log}_{\rm crys}$. Then $\iota^*((U, T, \delta, \phi))$ is representable by an object $(U \times_{U_2} U_1, \mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1), [\ ], q) \in (Y_1/S)^{\log}_{\rm crys}|_{T_1}$; the functor ι_* is exact.

(2) *Let the notations and the assumptions be as in* (1). *Then $\mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1) = T \times_{T_2} T_1$.*

Proof. (1): We have to check that $(U \times_{U_2} U_1, \mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1), [\ ], q)$ is actually an object of $(Y_1/S)^{\log}_{\rm crys}|_{T_1}$.

Since $U \times_{U_2} U_1$ is an open subscheme of U_1, γ extends to $\mathcal{O}_{U \times_{U_2} U_1}$. Since the image $\overline{\mathcal{J}}$ in $\mathcal{O}_{U \times_{U_2} U_1}$ is $\mathcal{I}\mathcal{O}_{U \times_{U_2} U_1}$, $\overline{\delta}$ actually extends to $\mathcal{O}_{U \times_{U_2} U_1}$ (cf. [11, 6.2.1 Lemma]). Since $\overline{\delta}$ extends to $\mathcal{O}_{U \times_{U_2} U_1}$, the exact closed immersion $U \times_{U_2} U_1 \overset{\subset}{\longrightarrow} \mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1)$ is a PD closed immersion by [11, 3.20 Remarks 4)].

Set $\overline{\mathcal{J}}_i := \mathcal{J}_i + \mathcal{I}\mathcal{O}_{T_i}$ $(i = 1, 2)$ and let $\overline{\delta}_i$ be the extension of δ_i and γ on $\overline{\mathcal{J}}_i$. Set $\mathcal{J}_{12,T} := {\rm Ker}(\mathcal{O}_{T \times_{T_2} T_1} \longrightarrow \mathcal{O}_{U \times_{U_2} U_1})$. Let $\overline{\mathcal{J}}'_{12,T}$ be the PD-ideal sheaf of $\mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1)}$ obtained from $\mathcal{J}_{12,T}$. Set $\overline{\mathcal{J}}_{12,T} := \overline{\mathcal{J}}'_{12,T} + \mathcal{I}\mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1)}$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1)} & = & \mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T \times_{T_2} T_1)} \\ \uparrow & & \uparrow \\ \mathcal{O}_{T \times_{T_2} T_1} & & \mathcal{O}_T \\ \uparrow & & \uparrow \\ \mathcal{O}_{T_1} & \longleftarrow & \mathcal{O}_{T_2}. \end{array}$$

Here we omit to write the direct images. We claim that the left vertical composite morphism induces a PD-morphism $(\mathcal{O}_{T_1},\overline{\mathcal{J}}_1)\longrightarrow(\mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1)},\overline{\mathcal{J}}_{12,T})$. Indeed, by the definition of $\overline{\delta}$, the composite morphism $(\mathcal{O}_{T_2},\overline{\mathcal{J}}_2)\longrightarrow(\mathcal{O}_T,\overline{\mathcal{J}})\longrightarrow(\mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1)},\overline{\mathcal{J}}_{12,T})$ is a PD-morphism. Let s be a local section of $\mathrm{Ker}(g^*\colon\mathcal{J}_2\longrightarrow\mathcal{J}_1)$. Then the image of s in $\mathcal{O}_{\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1)}$ by the right vertical composite morphism is the zero. Hence the claim follows because $g^*\colon g^*(\overline{\mathcal{J}}_2)\longrightarrow\overline{\mathcal{J}}_1$ is surjective by the assumption. Consequently we actually have a natural morphism

$$q\colon(U\times_{U_2}U_1,\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1),[\ ])\longrightarrow(U_1,T_1,\delta_1)$$

of log PD-schemes over $(S,\mathcal{I},\gamma)$.

By using the universality of the log PD-envelope, it is straightforward to see that

$$\iota^*((U,T,\delta,\phi))=(U\times_{U_2}U_1,\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1),[\ ],q). \tag{2.2.6.1}$$

Therefore, for an object E in $(\widetilde{Y_2/S})^{\log}_{\mathrm{crys}}|_{T_2}$, we have

$$\begin{aligned}\iota_*E((U,T,\delta,\phi))&=\mathrm{Hom}_{(\widetilde{Y_1/S})^{\log}_{\mathrm{crys}}|_{T_1}}(\iota^*((U,T,\delta,\phi)),E)\\&=E((U\times_{U_2}U_1,\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1),[\ ],q)).\end{aligned} \tag{2.2.6.2}$$

Using the formula (2.2.6.2) and noting that $\mathfrak{D}_{\overline{\delta}}(T\times_{T_2}T_1)\approx T\times_{T_2}T_1$ is a closed set of T as a topological space, we can easily see that the functor ι_* is exact.

(2): Set $\mathcal{J}_{12}:=\mathrm{Ker}(\mathcal{O}_{T_2}\longrightarrow g_*(\mathcal{O}_{T_1}))$. The structure sheaf of $T\times_{T_2}T_1$ is equal to $\mathcal{O}_T/\mathcal{J}_{12}\mathcal{O}_T$. By the following commutative diagram

$$\begin{tikzcd}\mathcal{O}_T/\mathcal{J} \arrow[r, equal] & \mathcal{O}_U \\ \phi^{-1}(\mathcal{O}_{T_2}/\mathcal{J}_2) \arrow[u] \arrow[r, equal] & \phi^{-1}(\mathcal{O}_{U_2}), \arrow[u, equal]\end{tikzcd}$$

we have $\mathcal{J}\cap\phi^{-1}(\mathcal{J}_{12})\mathcal{O}_T=\phi^{-1}(\mathcal{J}_2\cap\mathcal{J}_{12})\mathcal{O}_T$. It is easy to see that the ideal sheaf $\mathcal{J}_{12}\cap\mathcal{J}_2$ is a sub PD-ideal sheaf of $\mathcal{J}_2$. Hence, by the same proof of [11, 3.5 Lemma], the PD-structure δ defines a unique PD-structure δ_{12} on $\mathcal{J}(\mathcal{O}_T/\mathcal{J}_{12}\mathcal{O}_T)$. Moreover, it is easy to see that γ extends to $\mathcal{O}_{T\times_{T_2}T_1}$. Hence $(\mathcal{O}_T/\mathcal{J}_{12}\mathcal{O}_T,\mathcal{J}(\mathcal{O}_T/\mathcal{J}_{12}\mathcal{O}_T),\delta_{12})$ is a sheaf of the universal PD-algebras of $(\mathcal{O}_{T\times_{T_2}T_1},\mathcal{J}_{12,T})$ over $(\mathcal{O}_T,\overline{\mathcal{J}},\overline{\delta})$, that is, we have (2). □

Following [31], let us denote by $\Lambda^i_{Y/S}$ the sheaf of log differential forms of Y/S of degree i ($i\in\mathbb{N}$). The following is a log version of [11, 6.12 Theorem]:

Proposition 2.2.7. *Let $\iota\colon Y\overset{\subset}{\longrightarrow}\mathcal{Y}$ be a closed immersion of fine log schemes over S. Assume that $\mathcal{Y}$ is log smooth over S and that γ extends* *to* $\overset{\circ}{Y}$. *Let* $\mathfrak{D}_Y(\mathcal{Y})$ *be the log PD-envelope of* ι *over* $(S, \mathcal{I}, \gamma)$. *Then the natural morphism*

$$\mathcal{O}_{Y/S} \longrightarrow L(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S}) \tag{2.2.7.1}$$

is a quasi-isomorphism.

Proof. Let $\mathfrak{D}_Y(\mathcal{Y}^i)$ $(i \in \mathbb{Z}_{>0})$ be the log PD-envelope of the composite immersion $Y \xrightarrow{\subset} \mathcal{Y} \xrightarrow{\subset} \mathcal{Y}^i$ over S, where $\mathcal{Y} \xrightarrow{\subset} \mathcal{Y}^i$ is the diagonal immersion. Let $p_i \colon \mathfrak{D}_{\mathcal{Y}}(\mathcal{Y}^2) \longrightarrow \mathcal{Y}^2 \xrightarrow{i\text{-th proj.}} \mathcal{Y}$ $(i = 1, 2)$ be a natural morphism and let $\mathcal{J}$ be the ideal sheaf of the locally exact closed immersion $\mathcal{Y} \longrightarrow \mathfrak{D}_{\mathcal{Y}}(\mathcal{Y}^2)$. The problem is local as in [11, 6.12 Theorem]; we may assume that $\Lambda^1_{\mathcal{Y}/S}$ has a basis $\{d\log t_j\}_{j=1}^n$, where t_j is a local section of the log structure of $\mathcal{Y}$. Let u_j be a local section of $\mathrm{Ker}(\mathcal{O}^*_{\mathfrak{D}_{\mathcal{Y}}(\mathcal{Y}^2)} \longrightarrow \mathcal{O}^*_{\mathcal{Y}})$ such that $p_2^*(t_j) = p_1^*(t_j)u_j$. Then, by [54, (6.5)], the following morphism

$$\mathcal{O}_{\mathcal{Y}}\langle s_1, \ldots, s_n\rangle \ni s_j^{[n]} \longmapsto (u_j - 1)^{[n]} \in \mathcal{O}_{\mathfrak{D}_{\mathcal{Y}}(\mathcal{Y}^2)}$$

is an isomorphism, where s_j's are independent indeterminates. We identify $\mathcal{O}_{\mathfrak{D}_{\mathcal{Y}}(\mathcal{Y}^2)}$ with $\mathcal{O}_{\mathcal{Y}}\langle s_1, \ldots, s_n\rangle$ by this isomorphism. By the log version of [11, 6.2 Proposition], $\iota^{\log}_{\mathrm{crys}*}(\mathcal{O}_{Y/S})$ is a crystal of $\mathcal{O}_{\mathcal{Y}/S}$-modules. Hence, as in [11, 6.3 Corollary], we obtain a canonical isomorphism $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathfrak{D}_{\mathcal{Y}}(\mathcal{Y}^2)} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}$. Consequently we can identify $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}$ with $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}\langle s_1, \ldots, s_n\rangle$ (cf. [54, (6.5)]). Moreover, by [54, (5.8.1)] and [81, Proposition 3.2.5], there exists an isomorphism $\Lambda^1_{\mathcal{Y}/S} \ni d\log t_j \overset{\sim}{\longmapsto} u_j - 1 \in \mathcal{J}/\mathcal{J}^2 = \mathcal{J}/\mathcal{J}^{[2]}$ of $\mathcal{O}_{\mathcal{Y}}$-modules.

Let $p_{13} \colon \mathfrak{D}_Y(\mathcal{Y}^3) \longrightarrow \mathfrak{D}_Y(\mathcal{Y}^2)$ be the induced morphism by the product of the first and the third projections $\mathcal{Y}^3 \longrightarrow \mathcal{Y}^2$. Let

$$\delta \colon \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \xrightarrow{p_{13}^*} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^3)} \xrightarrow{\sim} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)}$$

be the morphism in [75, p. 14]. Then, by the formula [75, (1.1.4.2)], $\delta(u_j) = u_j \otimes u_j$. Hence $\delta(s_j) = s_j \otimes s_j + s_j \otimes 1 + 1 \otimes s_j$ (the last formula in [75, p. 16]). Hence the natural connection

$$\nabla \colon \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^q_{\mathcal{Y}/S} \longrightarrow \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{q+1}_{\mathcal{Y}/S}$$

is given by

$$\nabla(as_1^{[i_1]} \cdots s_n^{[i_n]} \otimes \omega) = a(\sum_{j=1}^n s_1^{[i_1]} \cdots s_j^{[i_j-1]} \cdots s_n^{[i_n]}(s_j+1)d\log t_j \wedge \omega \tag{2.2.7.2}$$

$$+ s_1^{[i_1]} \cdots s_n^{[i_n]} \otimes d\omega) \quad (a \in \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})},\ i_1, \ldots, i_n \in \mathbb{N},\ \omega \in \Lambda^q_{\mathcal{Y}/S})$$

as in [11, 6.11 Lemma]. Let (U,T,δ) be an object of $(Y/S)^{\log}_{\mathrm{crys}}$. Because the problem is local, we may assume that there exists the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\subset} & T \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\iota} & \mathcal{Y}. \end{array}$$

Then we have a natural morphism $(U,T,\delta) \longrightarrow (Y,\mathfrak{D}_Y(\mathcal{Y}),[\ ])$ in $(Y/S)^{\log}_{\mathrm{crys}}$ and a natural complex $\mathcal{O}_T\langle s_1,\ldots,s_n\rangle \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S}$, which is equal to the complex $L(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S})_{(U,T,\delta)}$.

Now, consider the case $n=1$ and set $s_1 = s$ and $t_1 = t$. Then the complex $\mathcal{O}_T\langle s_1,\ldots,s_n\rangle \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S}$ is equal to $\mathcal{O}_T\langle s\rangle \xrightarrow{\nabla_T} \mathcal{O}_T\langle s\rangle d\log t$. Because $\nabla_T(s^{[n]}) = s^{[n-1]}(s+1)d\log t = (ns^{[n]} + s^{[n-1]})d\log t$ for a positive integer n, we have the following formula

(2.2.7.3)
$$\nabla_T\Big(\sum_{n=0}^{m} a_n s^{[n]}\Big) = \sum_{n=1}^{m}(a_n + (n-1)a_{n-1})s^{[n-1]}d\log t + m a_m s^{[m]} d\log t$$
$$(m \in \mathbb{N}, a_n \in \mathcal{O}_T\ (0 \le n \le m)).$$

Hence $\mathrm{Ker}(\nabla_T) = \mathcal{O}_T$. Because p is locally nilpotent on $\overset{\circ}{S}$, we may assume that $p^N a_{p^N} = 0$ if N is sufficiently large. Hence we see that $\mathrm{Coker}(\nabla_T) = 0$ by the formula (2.2.7.3). Therefore we have checked that the morphism (2.2.7.1) is a quasi-isomorphism for the case $n=1$.

The rest of the proof is the same as that of [11, 6.12 Theorem]. □

Proposition 2.2.8 ([54, the proof of (6.9)]).

Let $\iota\colon Y \xrightarrow{\subset} \mathcal{Y}$, $\mathcal{Y}$ and $\mathfrak{D}_Y(\mathcal{Y})$ be as in (2.2.7). *Let E be a crystal of $\mathcal{O}_{Y/S}$-modules. Let $(\mathcal{E},\nabla)$ be the corresponding $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-module with integrable connection. Then there exists a natural quasi-isomorphism*

(2.2.8.1)
$$E \longrightarrow L(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S}).$$

Proof. The proof is the same as that in [11, 6.14 Theorem]: we have the following equalities in $\mathrm{D}^+(\mathcal{O}_{Y/S})$:

$$\begin{aligned} E &= E \otimes_{\mathcal{O}_{Y/S}} L(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S}) \\ &= L(\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Lambda^{\bullet}_{\mathcal{Y}/S}). \end{aligned}$$

□

Let $\iota\colon Z \xrightarrow{\subset} Y$ be an exact closed immersion of fine log schemes over S to which γ extends. Assume that there exists the following cartesian diagram

(2.2.8.2)
$$\begin{tikzcd} Z \arrow[r, "\iota", "\subset"'] \arrow[d, "\cap"'] & Y \arrow[d, "\cap"] \\ \mathcal{Z} \arrow[r, "\iota_{\mathcal{Y},\mathcal{Z}}", "\subset"'] & \mathcal{Y}, \end{tikzcd}$$

where $\iota_{\mathcal{Y},\mathcal{Z}}$ is an exact closed immersion of fine log schemes over S and the vertical two morphisms are closed immersions. Let $\mathfrak{D}_Z(\mathcal{Z})$ and $\mathfrak{D}_Y(\mathcal{Y})$ be the log PD-envelopes of the closed immersions $Z \overset{\subset}{\longrightarrow} \mathcal{Z}$ and $Y \overset{\subset}{\longrightarrow} \mathcal{Y}$ over $(S, \mathcal{I}, \gamma)$, respectively. Then we have the following diagram of ringed topoi:

(2.2.8.3)
$$\begin{tikzcd} (\widetilde{\mathcal{Z}}_{\rm zar}, \mathcal{O}_{\mathcal{Z}}) \arrow[d, "\iota_{\mathcal{Y},\mathcal{Z}}"'] & (\widetilde{\mathfrak{D}_Z(\mathcal{Z})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}) \arrow[l, "g_{\mathcal{Z}}"'] \arrow[d, "\iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}}"'] & ((\widetilde{Z/S})^{\log}_{\rm crys}|_{\mathfrak{D}_Z(\mathcal{Z})}, \mathcal{O}_{Z/S}|_{\mathfrak{D}_Z(\mathcal{Z})}) \arrow[l, "\varphi_{\mathfrak{D}_Z(\mathcal{Z})}"'] \arrow[d, "\iota^{\log,\rm loc}_{\rm crys}"'] \\ (\widetilde{\mathcal{Y}}_{\rm zar}, \mathcal{O}_{\mathcal{Y}}) & (\widetilde{\mathfrak{D}_Y(\mathcal{Y})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}) \arrow[l, "g_{\mathcal{Y}}"'] & ((\widetilde{Y/S})^{\log}_{\rm crys}|_{\mathfrak{D}_Y(\mathcal{Y})}, \mathcal{O}_{Y/S}|_{\mathfrak{D}_Y(\mathcal{Y})}) \arrow[l, "\varphi_{\mathfrak{D}_Y(\mathcal{Y})}"'] \end{tikzcd}$$
$$\xrightarrow{j_{\mathfrak{D}_Z(\mathcal{Z})}} \quad ((\widetilde{Z/S})^{\log}_{\rm crys}, \mathcal{O}_{Z/S})$$
$$\downarrow \iota^{\log}_{\rm crys}$$
$$\xrightarrow{j_{\mathfrak{D}_Y(\mathcal{Y})}} \quad ((\widetilde{Y/S})^{\log}_{\rm crys}, \mathcal{O}_{Y/S}).$$

Let $\overline{\mathcal{J}}_{\mathcal{Z}}$ (resp. $\overline{\mathcal{J}}_{\mathcal{Y}}$) be the PD-ideal sheaf of $\mathfrak{D}_Z(\mathcal{Z})$ (resp. $\mathfrak{D}_Y(\mathcal{Y})$). Let $\mathcal{J}_{\mathcal{Y},\mathcal{Z}}$ be the ideal sheaf of the closed immersion $\iota_{\mathcal{Y},\mathcal{Z}}$.

Lemma 2.2.9. *Assume that $\mathfrak{D}_Z(\mathcal{Z}) = \mathcal{Z} \times_{\mathcal{Y}} \mathfrak{D}_Y(\mathcal{Y})$. Then the diagram*

(2.2.9.1)
$$\begin{tikzcd} (\widetilde{\mathcal{Z}}_{\rm zar}, \mathcal{O}_{\mathcal{Z}}) \arrow[r, "g^*_{\mathcal{Z}}"] \arrow[d, "\iota_{\mathcal{Y},\mathcal{Z}*}"'] & (\widetilde{\mathfrak{D}_Z(\mathcal{Z})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}) \arrow[d, "\iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}"] \\ (\widetilde{\mathcal{Y}}_{\rm zar}, \mathcal{O}_{\mathcal{Y}}) \arrow[r, "g^*_{\mathcal{Y}}"] & (\widetilde{\mathfrak{D}_Y(\mathcal{Y})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}). \end{tikzcd}$$

is commutative for a quasi-coherent $\mathcal{O}_{\mathcal{Z}}$-module $\mathcal{E}$, that is, the natural morphism $g^*_{\mathcal{Y}}\iota_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E}) \longrightarrow \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*} g^*_{\mathcal{Z}}(\mathcal{E})$ is an isomorphism.

Proof. Since $\overset{\circ}{\iota}_{\mathcal{Y},\mathcal{Z}}$ is affine, (2.2.9) immediately follows from the affine base change theorem ([39, (1.5.2)]). □

Lemma 2.2.10. *Assume that $\iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}}$ induces a surjection $\iota^{\rm PD*}_{\mathcal{Y},\mathcal{Z}}(\overline{\mathcal{J}}_{\mathcal{Y}}) \longrightarrow \overline{\mathcal{J}}_{\mathcal{Z}}$. Then the diagram*

(2.2.10.1)
$$\begin{tikzcd} (\widetilde{\mathfrak{D}_Z(\mathcal{Z})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}) \arrow[r, "\varphi^*_{\mathfrak{D}_Z(\mathcal{Z})}"] \arrow[d, "\iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}"'] & ((\widetilde{Z/S})^{\log}_{\rm crys}|_{\mathfrak{D}_Z(\mathcal{Z})}, \mathcal{O}_{Z/S}|_{\mathfrak{D}_Z(\mathcal{Z})}) \arrow[d, "\iota^{\log,\rm loc}_{\rm crys*}"] \\ (\widetilde{\mathfrak{D}_Y(\mathcal{Y})}_{\rm zar}, \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}) \arrow[r, "\varphi^*_{\mathfrak{D}_Y(\mathcal{Y})}"] & ((\widetilde{Y/S})^{\log}_{\rm crys}|_{\mathfrak{D}_Y(\mathcal{Y})}, \mathcal{O}_{Y/S}|_{\mathfrak{D}_Y(\mathcal{Y})}) \end{tikzcd}$$

is commutative for a quasi-coherent $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-module $\mathcal{E}$, that is, the natural morphism $\varphi^*_{\mathfrak{D}_Y(\mathcal{Y})}\iota^{\mathrm{PD}}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E})\longrightarrow\iota^{\mathrm{log,loc}}_{\mathrm{crys}*}\varphi^*_{\mathfrak{D}_Z(\mathcal{Z})}(\mathcal{E})$ is an isomorphism.

Proof. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-module. Let $(T,\phi)=(U,T,M_T,\delta,\phi)$ be an object of $(Y/S)^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_Y(\mathcal{Y})}$. Then, by (2.2.6) (1) and (2),

$$\iota^{\mathrm{log,loc}}_{\mathrm{crys}*}\varphi^*_{\mathfrak{D}_Z(\mathcal{Z})}(\mathcal{E})(T,\phi)=\Gamma(T\times_{\mathfrak{D}_Y(\mathcal{Y})}\mathfrak{D}_Z(\mathcal{Z}),p_2^*(\mathcal{E})), \tag{2.2.10.2}$$

where $p_2\colon T\times_{\mathfrak{D}_Y(\mathcal{Y})}\mathfrak{D}_Z(\mathcal{Z})\longrightarrow\mathfrak{D}_Z(\mathcal{Z})$ is the second projection. On the other hand,

$$\varphi^*_{\mathfrak{D}_Y(\mathcal{Y})}\iota^{\mathrm{PD}}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E})(T,\phi)=\Gamma(T,\phi^*\iota^{\mathrm{PD}}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E})). \tag{2.2.10.3}$$

Since $\mathfrak{D}_Z(\mathcal{Z})\longrightarrow\mathfrak{D}_Y(\mathcal{Y})$ is a closed immersion, in particular, an affine morphism, the affine base change theorem tells us that both right hand sides of (2.2.10.2) and (2.2.10.3) are the same. This completes the proof of (2.2.10). □

Lemma 2.2.11. *Assume that $\iota^{\mathrm{PD}}_{\mathcal{Y},\mathcal{Z}}$ induces a surjection $\iota^{\mathrm{PD}*}_{\mathcal{Y},\mathcal{Z}}(\overline{\mathcal{J}}_{\mathcal{Y}})\longrightarrow\overline{\mathcal{J}}_{\mathcal{Z}}$. Then the following diagram of topoi*

$$\begin{array}{ccc}(\widetilde{Z/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_Z(\mathcal{Z})} & \xrightarrow{j_{\mathfrak{D}_Z(\mathcal{Z})}} & (\widetilde{Z/S})^{\log}_{\mathrm{crys}}\\ {\scriptstyle\iota^{\mathrm{log,loc}}_{\mathrm{crys}}}\big\downarrow & & \big\downarrow{\scriptstyle\iota^{\log}_{\mathrm{crys}}}\\ (\widetilde{Y/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_Y(\mathcal{Y})} & \xrightarrow{j_{\mathfrak{D}_Y(\mathcal{Y})}} & (\widetilde{Y/S})^{\log}_{\mathrm{crys}}.\end{array} \tag{2.2.11.1}$$

is commutative.

Proof. Let $T=(U,T,M_T,\delta)$ be an object of $(Y/S)^{\log}_{\mathrm{crys}}$. Let $\overline{\delta}$ be the PD-structure of $\mathrm{Ker}(\mathcal{O}_T\longrightarrow\mathcal{O}_U)+\mathcal{I}\mathcal{O}_T$ which is an extension of δ and γ. Let $\mathfrak{D}(T):=\mathfrak{D}_{U\cap Z,\overline{\delta}}(T)$ be the log PD-envelope of the closed immersion $U\cap Z\longrightarrow T$ over $(T,M_T,\overline{\delta})$. By the log version of [11, 6.2.1 Lemma], $\iota^{\log*}_{\mathrm{crys}}(T)=(U\cap Z,\mathfrak{D}(T))$. Hence $j^*_{\mathfrak{D}_Z(\mathcal{Z})}\iota^{\log*}_{\mathrm{crys}}(T)=(\mathfrak{D}(T)\times\mathfrak{D}_Z(\mathcal{Z}),p_{2,\mathcal{Z}})$ as a sheaf, where $p_{2,\mathcal{Z}}\colon\mathfrak{D}(T)\times\mathfrak{D}_Z(\mathcal{Z})\longrightarrow\mathfrak{D}_Z(\mathcal{Z})$ is the second projection. Analogously, let $p_{2,\mathcal{Y}}\colon T\times\mathfrak{D}_Y(\mathcal{Y})\longrightarrow\mathfrak{D}_Y(\mathcal{Y})$ be the second projection. Let $\delta_{\mathcal{Z}}$ be the PD-structure of $\mathfrak{D}_Z(\mathcal{Z})$ and let $\overline{\delta}_{\mathcal{Z}}$ be the extension of the δ and γ on $\mathrm{Ker}(\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}\longrightarrow\mathcal{O}_Z)+\mathcal{I}\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$. Let $\mathfrak{D}(T\times_S\mathfrak{D}_Y(\mathcal{Y}))$ be the double log PD-envelope of T and $\mathfrak{D}_Y(\mathcal{Y})$ (cf. [11, 5.12 Lemma]) over $(S,\mathcal{I},\gamma)$. Let $\mathfrak{D}(\delta)$ be the PD-structure of $\mathfrak{D}(T\times_S\mathfrak{D}_Y(\mathcal{Y}))$ and $\overline{\mathfrak{D}}(\delta)$ the extension of $\mathfrak{D}(\delta)$ and γ. Then we have

$$\begin{aligned}\iota^{\mathrm{log,loc}*}_{\mathrm{crys}}j^*_{\mathfrak{D}_Y(\mathcal{Y})}(T)&=\iota^{\mathrm{log,loc}*}_{\mathrm{crys}}(T\times\mathfrak{D}_Y(\mathcal{Y}),p_{2,\mathcal{Y}})\\&=\iota^{\mathrm{log,loc}*}_{\mathrm{crys}}(\mathfrak{D}(T\times_S\mathfrak{D}_Y(\mathcal{Y})),p_{2,\mathcal{Y}})\\&=\mathfrak{D}_{\overline{\mathfrak{D}}(\delta)}(\mathfrak{D}(T\times_S\mathfrak{D}_Y(\mathcal{Y}))\times_{\mathfrak{D}_Y(\mathcal{Y})}\mathfrak{D}_Z(\mathcal{Z}))\quad(2.2.6)\\&=\mathfrak{D}(T)\times\mathfrak{D}_Z(\mathcal{Z})\quad(\text{the universality of }\mathfrak{D}(T\times_S\mathfrak{D}_Y(\mathcal{Y}))).\end{aligned}$$

Here we consider the last equality as sheaves in $(\widetilde{Z/S})^{\log}_{\rm crys}|_{\mathfrak{D}_Z(\mathcal{Z})}$. Hence (2.2.11.1) is commutative. □

Corollary 2.2.12. *Assume that $\mathfrak{D}_Z(\mathcal{Z}) = \mathcal{Z} \times_{\mathcal{Y}} \mathfrak{D}_Y(\mathcal{Y})$. Let $L^{\rm PD}_{Z/S}$ (resp. $L^{\rm PD}_{Y/S}$) be the linearization functor of $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-modules (resp. $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-modules). Then there exists a canonical isomorphism of functors*

$$L^{\rm PD}_{Y/S} \circ \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*} \longrightarrow \iota^{\log}_{{\rm crys}*} \circ L^{\rm PD}_{Z/S} \tag{2.2.12.1}$$

for quasi-coherent $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-modules. Set $L_{Y/S} := L^{\rm PD}_{Y/S} \circ g^*_{\mathcal{Y}}$ and $L_{Z/S} := L^{\rm PD}_{Z/S} \circ g^*_{\mathcal{Z}}$. Then there also exists a canonical isomorphism of functors

$$L_{Y/S} \circ \iota_{\mathcal{Y},\mathcal{Z}*} \longrightarrow \iota^{\log}_{{\rm crys}*} \circ L_{Z/S} \tag{2.2.12.2}$$

for quasi-coherent $\mathcal{O}_{\mathcal{Z}}$-modules. Moreover, the isomorphism (2.2.12.1) *is functorial with respect to log HPD differential operators of quasi-coherent $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-modules.*

Proof. Because $\mathfrak{D}_Z(\mathcal{Z}) = \mathcal{Z} \times_{\mathcal{Y}} \mathfrak{D}_Y(\mathcal{Y})$ and because the diagram (2.2.8.2) is cartesian, the natural morphism $\iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}} \colon \iota^{{\rm PD}*}_{\mathcal{Y},\mathcal{Z}}(\overline{\mathcal{J}}_{\mathcal{Y}}) \longrightarrow \overline{\mathcal{J}}_{\mathcal{Z}}$ is surjective. The first statement of (2.2.12) immediately follows from (2.2.1.2), (2.2.10) and (2.2.11). The second statement follows from the former and (2.2.9).

Let us prove the last statement. For a quasi-coherent $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-module $\mathcal{E}$ and $(U,T,\delta) \in (Y/S)^{\log}_{\rm crys}$, the isomorphism

$$L^{\rm PD}_{Y/S} \circ \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E})_T \longrightarrow \iota^{\log}_{{\rm crys}*} \circ L^{\rm PD}_{Z/S}(\mathcal{E})_T$$

induced by (2.2.12.1) is given by the natural homomorphism

$$\mathcal{O}_{\mathfrak{D}_U(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E}) \longrightarrow \mathcal{O}_{\mathfrak{D}_{U\times_Y Z}(T\times_S\mathcal{Z})} \otimes_{\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}} \mathcal{E}. \tag{2.2.12.3}$$

If we are given a log HPD differential operator $u : \mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}} \mathcal{E} \longrightarrow \mathcal{F}$ of $\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}$-modules, the composite morphism

$$\widetilde{u} : \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E}) \longrightarrow \mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z}^2)} \otimes_{\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}} \mathcal{E} \xrightarrow{u} \mathcal{F}$$

is a log HPD differential operator of $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-modules and we see easily that the diagram

$$\begin{array}{ccc}
\mathcal{O}_{\mathfrak{D}_U(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{E}) & \longrightarrow & \mathcal{O}_{\mathfrak{D}_U(T\times_S\mathcal{Y})} \otimes_{\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}} \iota^{\rm PD}_{\mathcal{Y},\mathcal{Z}*}(\mathcal{F}) \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}_{\mathfrak{D}_{U\times_Y Z}(T\times_S\mathcal{Z})} \otimes_{\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}} \mathcal{E} & \longrightarrow & \mathcal{O}_{\mathfrak{D}_{U\times_Y Z}(T\times_S\mathcal{Z})} \otimes_{\mathcal{O}_{\mathfrak{D}_Z(\mathcal{Z})}} \mathcal{F}
\end{array}$$

is commutative for any $T = (U,T,\delta) \in (Y/S)^{\log}_{\rm crys}$, where the upper horizontal morphism (resp. the lower horizontal morphism) is the homomorphism induced by $\widetilde{u}$ (resp. u) in the way described in (2.2.3.1) and the vertical morphisms are the homomorphism (2.2.12.3) for $\mathcal{E}$ and $\mathcal{F}$. Therefore we see the compatibility of (2.2.12.1) with log HPD differential operators. □

Remark 2.2.13. In the case where Y, Z are trivial log smooth schemes over a trivial log scheme S, we can also prove (2.2.12) by an analogous proof of [3, IV Proposition 3.1.7]. In the case where Y, Z are fine log (not necessarily smooth) schemes over a fine log scheme S, we can also prove (2.2.12) by the second fundamental exact sequence of log differential forms on fine log smooth schemes ((2.1.3)) and by the log version of an analogous proof of [3, IV Proposition 3.1.7].

(2) Now let us study some properties of log linearization functors for a smooth scheme with a relative SNCD.

Let $S_0 \overset{\subset}{\longrightarrow} S$ be a closed immersion of schemes(=trivial log schemes) defined by a quasi-coherent ideal sheaf. Let $f\colon X \longrightarrow S_0$ be a smooth scheme with a relative SNCD D on X over S_0. Let Z be a relative SNCD on X over S_0 which intersects D transversally over S_0. Let $\Delta_D := \{D_\lambda\}_\lambda$ (resp. $\Delta_Z := \{Z_\mu\}_\mu$) be a decomposition of D (resp. Z) by smooth components of D (resp. Z). Then $\Delta := \{D_\lambda, Z_\mu\}_{\lambda,\mu}$ is a decomposition of $D \cup Z$ by smooth components of $D \cup Z$. Let $(X, D \cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$ be an admissible closed immersion over S with respect to Δ. Let $\widetilde{\Delta} := \{\mathcal{D}_\lambda, \mathcal{Z}_\mu\}_{\lambda,\mu}$ be the decomposition of $\mathcal{D} \cup \mathcal{Z}$ which is compatible with Δ.

Set

$$D_{\{\lambda_1,\lambda_2,\ldots\lambda_k\}} := D_{\lambda_1} \cap D_{\lambda_2} \cap \cdots \cap D_{\lambda_k} \quad (\lambda_i \neq \lambda_j \text{ if } i \neq j) \tag{2.2.13.1}$$

for a positive integer k, and set

$$D^{(k)} = \begin{cases} X & (k = 0), \\ \displaystyle\coprod_{\{\lambda_1,\ldots,\lambda_k \;\mid\; \lambda_i \neq \lambda_j \;\; (i \neq j)\}} D_{\{\lambda_1,\lambda_2,\ldots,\lambda_k\}} & (k \geq 1) \end{cases} \tag{2.2.13.2}$$

for a nonnegative integer k. Set

$$D_\emptyset := X \tag{2.2.13.3}$$

for later convenience.

The following proposition says that a decomposition of a relative SNCD by smooth components is locally unique:

Proposition 2.2.14. *Let Δ and Δ' be decompositions of D by smooth components. Then, for any $z \in X$, there exists an open neighborhood V of z in X such that $\Delta_V = \Delta'_V$.*

Proof. If V is small enough, we can take the diagram (2.1.7.2) such that (A.0.1) below holds for both Δ and Δ'. Then $\Delta_V = \{y_i = 0\}_{i=1}^s = \Delta'_V$. □

Proposition 2.2.15. *$D^{(k)}$ is independent of the choice of the decomposition of D by smooth components of D.*

Proof. Obviously we may assume that k is positive.

First we prove (2.2.15) for the case $k = 1$. Let $\Delta_D = \{D_\lambda\}_\lambda$ and $\Delta'_D = \{D'_{\lambda'}\}_{\lambda'}$ be two decompositions of D by smooth components of D. By (2.2.14) there exists an open covering $\{X_i\}_i$ of X such that $\Delta_D|_{X_i} = \Delta'_D|_{X_i}$. Hence we have an isomorphism $(\coprod_\lambda D_\lambda) \times_X X_i \overset{\sim}{\longrightarrow} (\coprod_{\lambda'} D'_{\lambda'}) \times_X X_i$. This local isomorphism is compatible with the open immersions $X_i \cap X_{i'} \overset{\subset}{\longrightarrow} X_i$; therefore we have the global isomorphism $\coprod_\lambda D_\lambda \overset{\sim}{\longrightarrow} \coprod_{\lambda'} D'_{\lambda'}$.

Let $D^{[k]}$ be the k-fold fiber product of $D^{(1)}$ over X; $D^{[k]}$ admits the action of the symmetric group $\mathfrak{S}_k$ of degree k. For a positive integer k, denote the set $\{1, 2, \ldots, k\}$ by $[1, k]$. For a surjective map $\alpha : [1, k] \longrightarrow [1, l]$, we have the corresponding morphism $D^{[l]} \longrightarrow D^{[k]}$, which we denote by s_α. Let S_k be the set of surjective morphisms $[1, k] \longrightarrow [1, k-1]$. Set $D^{\{k\}} := D^{[k]} \setminus \bigcup_{\alpha \in S_k} s_\alpha(D^{[k-1]})$; $D^{\{k\}}$ is an open subscheme of $D^{[k]}$. The scheme $D^{\{k\}}$ also admits the action of $\mathfrak{S}_k$. Then we can check $D^{(k)} = D^{\{k\}}/\mathfrak{S}_k$ by the construction of $D^{\{k\}}$. Consequently $D^{(k)}$ is independent of the choice of the decomposition of D by smooth components of D. $\square$

Set

(2.2.15.1) $$Z|_{D^{(k)}} := Z \times_X D^{(k)}.$$

The scheme $Z|_{D^{(k)}}$ is a relative SNCD on $D^{(k)}$. We use analogous notations $\mathcal{D}^{(k)}$ and $\mathcal{Z}|_{\mathcal{D}^{(k)}}$ $(k \in \mathbb{N})$ for $\mathcal{D} \cup \mathcal{Z}$ with $\widetilde{\Delta}$. Let $a^{(k)} \colon (D^{(k)}, Z|_{D^{(k)}}) \longrightarrow (X, Z)$ and $b^{(k)} : (\mathcal{D}^{(k)}, \mathcal{Z}|_{\mathcal{D}^{(k)}}) \longrightarrow (\mathcal{X}, \mathcal{Z})$ be morphisms induced by natural closed immersions.

As usual, we define the *preweight filtration* $P^{\mathcal{D}}_{\bullet}$ on the sheaf of the log differential forms $\Omega^i_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z}))$ $(i \in \mathbb{N})$ in $\widetilde{\mathcal{X}}_{\rm zar}$ with respect to $\mathcal{D}$ as follows:

(2.2.15.2) $$P^{\mathcal{D}}_k \Omega^i_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})) =$$

$$\begin{cases} 0 & (k < 0), \\ \mathrm{Im}(\Omega^k_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})) \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{i-k}_{\mathcal{X}/S}(\log \mathcal{Z}) \longrightarrow \Omega^i_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z}))) & (0 \leq k \leq i), \\ \Omega^i_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})) & (k > i). \end{cases}$$

Now, assume that the defining ideal sheaf $\mathcal{I}$ of the closed immersion $S_0 \overset{\subset}{\longrightarrow} S$ is a PD-ideal sheaf with a PD-structure γ.

Let the right objects in the following table be the log PD-envelopes of the left exact closed immersions over $(S, \mathcal{I}, \gamma)$:

$(X, D \cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$	$\mathfrak{D}_{\mathcal{D}}$
$(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{Z})$	$\mathfrak{D}$
$(D^{(k)}, Z\vert_{D^{(k)}}) \overset{\subset}{\longrightarrow} (\mathcal{D}^{(k)}, \mathcal{Z}\vert_{\mathcal{D}^{(k)}})$	$\mathfrak{D}^{(k)}$

Let $g_{\mathcal{D}} \colon \mathfrak{D}_{\mathcal{D}} \longrightarrow (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$, $g \colon \mathfrak{D} \longrightarrow (\mathcal{X}, \mathcal{Z})$ and $g^{(k)} \colon \mathfrak{D}^{(k)} \longrightarrow (\mathcal{D}^{(k)}, \mathcal{Z}|_{\mathcal{D}^{(k)}})$ be natural morphisms. Note that the underlying schemes of the log schemes $\mathfrak{D}_{\mathcal{D}}$ and $\mathfrak{D}$ are the same. Let $c^{(k)} \colon \mathfrak{D}^{(k)} \longrightarrow \mathfrak{D}$ be a morphism induced by $b^{(k)} \colon (\mathcal{D}^{(k)}, \mathcal{Z}|_{\mathcal{D}^{(k)}}) \longrightarrow (\mathcal{X}, \mathcal{Z})$.

Lemma 2.2.16. (1) *The natural morphism $(D^{(k)}, Z|_{D^{(k)}}) \longrightarrow (\mathcal{D}^{(k)}, \mathcal{Z}|_{\mathcal{D}^{(k)}}) \times_{(\mathcal{X}, \mathcal{Z})} (X, Z)$ is an isomorphism.*

(2) *The natural morphism $\mathfrak{D}^{(k)} \longrightarrow \mathfrak{D} \times_{(\mathcal{X}, \mathcal{Z})} (\mathcal{D}^{(k)}, \mathcal{Z}|_{\mathcal{D}^{(k)}})$ is an isomorphism.*

(3) *Let $\overline{\mathcal{J}}$ (resp. $\overline{\mathcal{J}}^{(k)}$) be the PD-ideal sheaf of $\mathcal{O}_{\mathfrak{D}}$ (resp. $\mathcal{O}_{\mathfrak{D}^{(k)}}$). Then the natural morphism $c^{(k)*} \colon c^{(k)*}(\overline{\mathcal{J}}) \longrightarrow \overline{\mathcal{J}}^{(k)}$ is surjective.*

Proof. Apply (2.1.13) to the SNCD $\mathcal{D} \cup \mathcal{Z}$ and assume that $\mathcal{D}$ (resp. $\mathcal{Z}$) is defined by an equation $x_1 = \cdots = x_t = 0$ (resp. $x_{t+1} = \cdots = x_s = 0$) $(1 \leq t \leq s)$.

(1): (1) is obvious.

(2): By the universality of the log PD-envelope, this is a local question. We may have two cartesian diagrams in (2.1.13) for $\mathcal{D} \cup \mathcal{Z}$; we may assume that $k \leq t$. Let $\mathcal{D}_{1 \cdots k}$ be a closed subscheme defined by an equation $x_1 = \cdots = x_k = 0$. Then $\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{D}_{1 \cdots k}} = \mathcal{O}_{\mathcal{X}}\langle x_{d+1}, \ldots, x_{d'}\rangle \otimes_{\mathcal{O}_{\mathcal{X}}} (\mathcal{O}_{\mathcal{X}}/(x_1, \ldots, x_k)) = \mathcal{O}_{\mathcal{X}}\langle x_{d+1}, \ldots, x_{d'}\rangle/(x_1, \ldots, x_k)$.

Set $D_{1 \cdots k} := \mathcal{D}_{1 \cdots k} \times_{\mathcal{X}} X$. Then the structure sheaf of the PD-envelope of the closed immersion $D_{1 \cdots k} \overset{\subset}{\longrightarrow} \mathcal{D}_{1 \cdots k}$ is

$$\mathcal{O}_{\mathcal{D}_{1 \cdots k}}\langle x_{d+1}, \ldots, x_{d'}\rangle = \mathcal{O}_{\mathcal{X}}\langle x_{d+1}, \ldots, x_{d'}\rangle/(x_1, \ldots, x_k).$$

Furthermore it is immediate to see that there exists a natural isomorphism $\mathfrak{D}^{(k)} \simeq \mathfrak{D} \times_{(\mathcal{X}, \mathcal{Z})} (\mathcal{D}^{(k)}, \mathcal{Z}|_{\mathcal{D}^{(k)}})$ as log schemes. Thus (2) follows.

(3): The proof of (3) is evident by the local description of $\mathcal{O}_{\mathfrak{D}}$ and $\mathcal{O}_{\mathfrak{D}^{(k)}}$. $\square$

As usual, we denote the left objects in the following table by the right ones for simplicity of notation:

$((X, D \cup Z) \overset{\subset}{\longrightarrow} \mathfrak{D}_{\mathcal{D}}) \in ((X, D \cup Z)/S)^{\log}_{\mathrm{crys}}$	$\mathfrak{D}_{\mathcal{D}}$
$((X, Z) \overset{\subset}{\longrightarrow} \mathfrak{D}) \in ((X, Z)/S)^{\log}_{\mathrm{crys}}$	$\mathfrak{D}$
$((D^{(k)}, Z\vert_{D^{(k)}}) \overset{\subset}{\longrightarrow} \mathfrak{D}^{(k)}) \in ((D^{(k)}, Z\vert_{D^{(k)}})/S)^{\log}_{\mathrm{crys}}$	$\mathfrak{D}^{(k)}$

Furthermore, as usual, we identify the representable sheaf by $\mathfrak{D}_{\mathcal{D}}$ in $((X, \widetilde{D \cup Z})/S)^{\log}_{\mathrm{crys}}$ with $\mathfrak{D}_{\mathcal{D}}$. Let $((X, \widetilde{D \cup Z})/S)^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_{\mathcal{D}}}$ be the localization of

$(\widetilde{(X,D\cup Z)/S})^{\log}_{\mathrm{crys}}$ at $\mathfrak{D}_{\mathcal{D}}$. Let $(\widetilde{(X,Z)/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}}$ and $(\widetilde{(D^{(k)},Z|_{D^{(k)}})/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}^{(k)}}$ be obvious analogues. Let $a^{(k)\log}_{\mathrm{crys}} : (\widetilde{(D^{(k)},Z|_{D^{(k)}})/S})^{\log}_{\mathrm{crys}} \longrightarrow (\widetilde{(X,Z)/S})^{\log}_{\mathrm{crys}}$ be a morphism of topoi induced by the morphism $a^{(k)}$. By the log version of [11, 6.2 Proposition], the functor $a^{(k)\log}_{\mathrm{crys}*}$ is exact.

Let the right objects in the following table be the log PD-envelope of the locally closed immersion of the left ones:

$(X,D\cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X},\mathcal{D}\cup\mathcal{Z})\times_S(\mathcal{X},\mathcal{D}\cup\mathcal{Z})$	$\mathfrak{D}_{\mathcal{D}}(1)$
$(X,Z) \overset{\subset}{\longrightarrow} (\mathcal{X},\mathcal{Z})\times_S(\mathcal{X},\mathcal{Z})$	$\mathfrak{D}(1)$
$(D^{(k)},Z\vert_{D^{(k)}}) \overset{\subset}{\longrightarrow} (\mathcal{D}^{(k)},\mathcal{Z}\vert_{\mathcal{D}^{(k)}})\times_S(\mathcal{D}^{(k)},\mathcal{Z}\vert_{\mathcal{D}^{(k)}})$	$\mathfrak{D}^{(k)}(1)$

Let

$j_{\mathfrak{D}_{\mathcal{D}}} : (\widetilde{(X,D\cup Z)/S})^{\log}_{\mathrm{crys}}\vert_{\mathfrak{D}_{\mathcal{D}}} \longrightarrow (\widetilde{(X,D\cup Z)/S})^{\log}_{\mathrm{crys}}$
$j_{\mathfrak{D}} : (\widetilde{(X,Z)/S})^{\log}_{\mathrm{crys}}\vert_{\mathfrak{D}} \longrightarrow (\widetilde{(X,Z)/S})^{\log}_{\mathrm{crys}}$
$j_{\mathfrak{D}^{(k)}} : (\widetilde{(D^{(k)},Z\vert_{D^{(k)}})/S})^{\log}_{\mathrm{crys}}\vert_{\mathfrak{D}^{(k)}} \longrightarrow (\widetilde{(D^{(k)},Z\vert_{D^{(k)}})/S})^{\log}_{\mathrm{crys}}$

be localization functors (2.2.0.1) and let

$\varphi_{\mathcal{D}} : ((\widetilde{(X,D\cup Z)/S})^{\log}_{\mathrm{crys}}\vert_{\mathfrak{D}_{\mathcal{D}}}, \mathcal{O}_{(X,D\cup Z)/S}\vert_{\mathfrak{D}_{\mathcal{D}}}) \longrightarrow (\widetilde{\mathring{\mathfrak{D}}}_{\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}})$
$\varphi : ((\widetilde{(X,Z)/S})^{\log}_{\mathrm{crys}}\vert_{\mathfrak{D}}, \mathcal{O}_{(X,Z)/S}\vert_{\mathfrak{D}}) \longrightarrow (\widetilde{\mathring{\mathfrak{D}}}_{\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}})$
$\varphi^{(k)} : ((\widetilde{(D^{(k)},Z\vert_{D^{(k)}})/S})^{\log}_{\mathrm{crys}}\vert_{\mathfrak{D}^{(k)}}, \mathcal{O}_{(D^{(k)},Z\vert_{D^{(k)}})/S}\vert_{\mathfrak{D}^{(k)}}) \longrightarrow (\widetilde{\mathfrak{D}^{(k)}}_{\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}^{(k)}})$

be morphisms of ringed topoi defined in (2.2.1.1) and let

$g : (\widetilde{\mathring{\mathfrak{D}}}_{\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}}) \longrightarrow (\widetilde{\mathring{\mathcal{X}}}_{\mathrm{zar}}, \mathcal{O}_{\mathcal{X}})$
$g^{(k)} : (\widetilde{\mathring{\mathfrak{D}}^{(k)}}_{\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}}) \longrightarrow (\widetilde{\mathcal{D}^{(k)}}_{\mathrm{zar}}, \mathcal{O}_{\mathcal{D}^{(k)}})$

be natural morphisms.

For an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, set

$$L_{(X,D\cup Z)/S}(\mathcal{E}) := j_{\mathfrak{D}_{\mathcal{D}}*}\varphi^*_{\mathcal{D}}g^*(\mathcal{E}) \in (\widetilde{(X,D\cup Z)/S})^{\log}_{\mathrm{crys}}$$

and

$$L_{(X,Z)/S}(\mathcal{E}) := j_{\mathfrak{D}*}\varphi^*g^*(\mathcal{E}) \in (\widetilde{(X,Z)/S})^{\log}_{\mathrm{crys}}.$$

For an $\mathcal{O}_{\mathcal{D}^{(k)}}$-module $\mathcal{E}$, set also

$$L^{(k)}(\mathcal{E}) := j_{\mathfrak{D}^{(k)}*}\varphi^{(k)*}g^{(k)*}(\mathcal{E}) \in (\widetilde{(D^{(k)},Z|_{D^{(k)}})/S})^{\log}_{\mathrm{crys}}.$$

As usual, we have a complex $L_{(X,D\cup Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))$ of $\mathcal{O}_{(X,D\cup Z)/S}$-modules. By (2.2.7) we have a natural quasi-isomorphism

$$\mathcal{O}_{(X,D\cup Z)/S} \xrightarrow{\sim} L_{(X,D\cup Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))). \tag{2.2.16.1}$$

Similarly we have two quasi-isomorphisms:

$$\mathcal{O}_{(X,Z)/S} \xrightarrow{\sim} L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log\mathcal{Z})), \tag{2.2.16.2}$$

$$\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \xrightarrow{\sim} L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log\mathcal{Z}|_{\mathcal{D}^{(k)}})). \tag{2.2.16.3}$$

Let $\{P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))\}_{k\in\mathbb{Z}}$ be the filtration on $\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$ defined in (2.2.15.2). Then $P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$ forms a subcomplex of $\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$ and the boundary morphisms of $P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$ are log HPD differential operators of order ≤ 1 with respect to $(\mathcal{X},\mathcal{Z})/S$.

Set

$$P^{D}_kL_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))) := L_{(X,Z)/S}(P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))) \quad (k\in\mathbb{Z}).$$

Lemma 2.2.17. (1) *The natural morphism*

$$\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}} P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})) \longrightarrow \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})) \tag{2.2.17.1}$$

is injective.

(2) *The natural morphism*

(2.2.17.2)
$$Q^{*}_{(X,Z)/S}P^{D}_kL_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))) \longrightarrow Q^{*}_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))$$

is injective.

Proof. (1): The question is local. We may have cartesian diagrams (2.1.13.1) and (2.1.13.2) for SNCD $\mathcal{D}\cup\mathcal{Z}$ on $\mathcal{X}$; we assume that $\mathcal{D}$ (resp. $\mathcal{Z}$) is defined by an equation $x_1\cdots x_t=0$ (resp. $x_{t+1}\cdots x_s=0$). Set $\mathcal{J}:=(x_{d+1},\ldots,x_{d'})\mathcal{O}_{\mathcal{X}}$, $\mathcal{X}':=\underline{\mathrm{Spec}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}/\mathcal{J})$ and $\mathcal{X}'':=\underline{\mathrm{Spec}}_{S}(\mathcal{O}_S[x_{d+1},\ldots,x_{d'}])$. Then $\mathcal{X}'$ is smooth over S. Let $\mathcal{D}'$ (resp. $\mathcal{Z}'$) be a closed subscheme of $\mathcal{X}'$ defined by an equation $x_1\cdots x_t=0$ (resp. $x_{t+1}\cdots x_s=0$). Because p is locally nilpotent on S, we may assume that there exists a positive integer N such that $\mathcal{J}^N\mathcal{O}_{\mathfrak{D}}=0$. Since $\mathcal{X}'$ is smooth over S, there exists a section of the surjection $\mathcal{O}_{\mathcal{X}}/\mathcal{J}^N\longrightarrow\mathcal{O}_{\mathcal{X}'}$. Hence, as in [11, 3.32 Proposition], we have a morphism

$$\mathcal{O}_{\mathcal{X}'}[x_{d+1},\ldots,x_{d'}]\longrightarrow\mathcal{O}_{\mathcal{X}}/\mathcal{J}^N$$

such that the induced morphism $\mathcal{O}_{\mathcal{X}'}[x_{d+1},\ldots,x_{d'}]/\mathcal{J}_0^N \longrightarrow \mathcal{O}_{\mathcal{X}}/\mathcal{J}^N$ is an isomorphism, where $\mathcal{J}_0 := (x_{d+1},\ldots,x_{d'})$. By [11, 3.32 Proposition], $\mathcal{O}_{\mathfrak{D}}$ is isomorphic to the PD-polynomial algebra $\mathcal{O}_{\mathcal{X}'}\langle x_{d+1},\ldots,x_{d'}\rangle$. Hence we have the following isomorphisms

$$\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))\overset{\sim}{\longrightarrow}s(\Omega^{\bullet}_{\mathcal{X}'/S}(\log(\mathcal{D}'\cup\mathcal{Z}'))\otimes_{\mathcal{O}_S}\\ \mathcal{O}_S\langle x_{d+1},\ldots,x_{d'}\rangle\otimes_{\mathcal{O}_{\mathcal{X}''}}\Omega^{\bullet}_{\mathcal{X}''/S})$$

and

$$\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))\overset{\sim}{\longrightarrow}s(P^{\mathcal{D}'}_k\Omega^{\bullet}_{\mathcal{X}'/S}(\log(\mathcal{D}'\cup\mathcal{Z}'))\otimes_{\mathcal{O}_S}\\ \mathcal{O}_S\langle x_{d+1},\ldots,x_{d'}\rangle\otimes_{\mathcal{O}_{\mathcal{X}''}}\Omega^{\bullet}_{\mathcal{X}''/S}).$$

Since the complex $\mathcal{O}_S\langle x_{d+1},\ldots,x_{d'}\rangle\otimes_{\mathcal{O}_{\mathcal{X}''}}\Omega^{\bullet}_{\mathcal{X}''/S}$ consists of free $\mathcal{O}_S$-modules, we obtain the desired injectivity.

(2): By (1) and (2.2.4), the natural morphism

$$P^{D}_kL_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))\longrightarrow L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))) \tag{2.2.17.3}$$

is injective in the category of crystals of $\mathcal{O}_{(X,Z)/S}$-modules. As in [3, IV Proposition 2.1.3], the functor

$$\{\text{the category of crystals of } \mathcal{O}_{(X,Z)/S}\text{-modules}\}\longrightarrow$$

$$\{\text{the category of } Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})\text{-modules}\}$$

is exact. Hence (2.2.17.2) is injective. □

By (2.2.17) (2), a family $\{Q^*_{(X,Z)/S}P^D_kL_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))\}_{k\in\mathbb{Z}}$ of complexes of $Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})$-modules defines a filtration on the complex $Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))$. Hence we obtain an object

$$(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))),\\ \{Q^*_{(X,Z)/S}P^D_kL_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))\}_{k\in\mathbb{Z}})$$

in $\mathrm{C}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$.

Now we consider the Poincaré residue isomorphism with respect to D. Though a relative divisor in this book is a union of smooth divisors, we consider the orientation sheaf of it for showing that our theory in this book is independent of the choice of the numbering of the smooth components of a relative SNCD.

First, let us recall the orientation sheaf in [23, (3.1.4)].

Let E be a finite set with cardinality $k\geq 0$. Set $\varpi_E := \bigwedge^k\mathbb{Z}^E$ if $k\geq 1$ and $\varpi_E := \mathbb{Z}$ if $k=0$.

Let k be a positive integer. Let P be a point of $D^{(k)}$. Let $D_{\lambda_0}, \ldots, D_{\lambda_{k-1}}$ be different smooth components of D such that $D_{\lambda_0} \cap \cdots \cap D_{\lambda_{k-1}}$ contains P. Then the set $E := \{D_{\lambda_0}, \ldots, D_{\lambda_{k-1}}\}$ gives an abelian sheaf

$$\varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{zar}}(D/S_0) := \bigwedge^{k} \mathbb{Z}^{E}_{D_{\lambda_0} \cap \cdots \cap D_{\lambda_{k-1}}}$$

on a local neighborhood of P in $D^{(k)}$. The sheaf $\varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{zar}}(D/S_0)$ is globalized on $D^{(k)}$; we denote this globalized abelian sheaf by the same symbol $\varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{zar}}(D/S_0)$. We denote a local section of $\varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{zar}}(D/S_0)$ by the following way: $m(\lambda_0 \cdots \lambda_{k-1})$ ($m \in \mathbb{Z}$). Set

$$\varpi^{(k)}_{\mathrm{zar}}(D/S_0) := \bigoplus_{\{\lambda_0, \ldots \lambda_{k-1}\}} \varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{zar}}(D/S_0).$$

By abuse of notation, we often denote $a^{(k)}_{*} \varpi^{(k)}_{\mathrm{zar}}(D/S_0)$ simply by $\varpi^{(k)}_{\mathrm{zar}}(D/S_0)$. Set $\varpi^{(0)}_{\mathrm{zar}}(D/S_0) := \mathbb{Z}_X$. The sheaves $\varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{zar}}(D/S_0)$ and $\varpi^{(k)}_{\mathrm{zar}}(D/S_0)$ are extended to abelian sheaves $\varpi^{\log}_{\lambda_0 \cdots \lambda_{k-1} \mathrm{crys}}(D/S; Z)$ and $\varpi^{(k)\log}_{\mathrm{crys}}(D/S; Z)$, respectively, in the log crystalline topos $((D^{(k)}, \widetilde{Z|_{D^{(k)}}})/S)^{\log}_{\mathrm{crys}}$ since, for an object $(U, T, M_T, \iota, \delta) \in ((D^{(k)}, \widetilde{Z|_{D^{(k)}}})/S)^{\log}_{\mathrm{crys}}$, the closed immersion $\iota \colon U \xrightarrow{\subset} T$ is a homeomorphism of topological spaces. If $Z = \emptyset$, then denote $\varpi^{\log}_{\lambda_0 \cdots \lambda_{k-1} \mathrm{crys}}(D/S; Z)$ and $\varpi^{(k)\log}_{\mathrm{crys}}(D/S; Z)$ by $\varpi_{\lambda_0 \cdots \lambda_{k-1} \mathrm{crys}}(D/S)$ and $\varpi^{(k)}_{\mathrm{crys}}(D/S)$, respectively.

Definition 2.2.18. We call

$$\varpi^{(k)}_{\mathrm{zar}}(D/S_0) \ (\text{resp. } \varpi^{(k)}_{\mathrm{crys}}(D/S), \ \varpi^{(k)\log}_{\mathrm{crys}}(D/S; Z))$$

the *zariskian orientation sheaf* (resp. *crystalline orientation sheaf*, *log crystalline orientation sheaf*) of $D^{(k)}/S_0$ (resp. $D^{(k)}/(S, \mathcal{I}, \gamma)$, $(D^{(k)}, Z|_{D^{(k)}})/(S, \mathcal{I}, \gamma)$).

Remark 2.2.19. The sheaves $\varpi^{(k)}_{\mathrm{zar}}(D/S_0)$, $\varpi^{(k)}_{\mathrm{crys}}(D/S)$ and $\varpi^{(k)\log}_{\mathrm{crys}}(D/S; Z)$ are defined by the local nature of D; they are independent of the choice of the decomposition by smooth components of D.

Lemma 2.2.20. *Let $\mathcal{E}$ be an $\mathcal{O}_{\mathcal{D}^{(k)}}$-module. Then there exists a canonical isomorphism*

$$L^{(k)}(\mathcal{E} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S)) \xrightarrow{\sim} L^{(k)}(\mathcal{E}) \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S; Z). \tag{2.2.20.1}$$

Proof. (2.2.20) immediately follows from the definition of $\varpi^{(k)\log}_{\mathrm{crys}}(D/S; Z)$. $\square$

Proposition 2.2.21. (1) *There exists the following exact sequence*:

(2.2.21.1)

$$0 \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} P^{\mathcal{D}}_{k-1}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})) \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} P^{\mathcal{D}}_{k}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z}))$$
$$\longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} b^{(k)}_{*}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log \mathcal{Z}|_{\mathcal{D}^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S)\{-k\}) \longrightarrow 0.$$

(2) *There exist quasi-isomorphisms*

(2.2.21.2)

$$\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^D} Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})))$$
$$\xrightarrow{\sim} Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log \mathcal{Z}|_{\mathcal{D}^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S))\{-k\}$$
$$\xleftarrow{\sim} Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\}.$$

Proof. (1): By the Poincaré residue isomorphism with respect to $\mathcal{D}$ (cf. [21, 3.6]), we have the following isomorphism

(2.2.21.3)

$$\mathrm{Res}^{\mathcal{D}} : \mathrm{gr}_k^{P^{\mathcal{D}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z}))$$
$$\xrightarrow{\sim} b^{(k)}_{*}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log \mathcal{Z}|_{\mathcal{D}^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S)\{-k\}).$$

Hence (1) follows from (2.2.17) (1).

(2): By the isomorphism (2.2.21.3), (2.2.17) (1) and (2.2.4), we have

$$\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^D} Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})))$$
$$= Q^*_{(X,Z)/S}L_{(X,Z)/S}(\mathrm{gr}_k^{P^{\mathcal{D}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D} \cup \mathcal{Z})))$$
$$\overset{Q^*_{(X,Z)/S}L_{(X,Z)/S}(\mathrm{Res}^{\mathcal{D}})}{=} Q^*_{(X,Z)/S}L_{(X,Z)/S}(b^{(k)}_{*}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log \mathcal{Z}|_{\mathcal{D}^{(k)}})$$
$$\otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S)))\{-k\}.$$

By (2.2.12) and (2.2.16) (1), (2), this complex is equal to

$$Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log \mathcal{Z}|_{\mathcal{D}^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S))\{-k\},$$

which is equal to

$$Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log \mathcal{Z}|_{\mathcal{D}^{(k)}})) \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\}$$

by (2.2.20).

By (2.2.7) we obtain the second quasi-isomorphism in (2.2.21.2). $\square$

For simplicity of notation, set

$$(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))), Q^*_{(X,Z)/S}P^D) :=$$

$$(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))), \{Q^*_{(X,Z)/S}P^D_k L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))\}_{k\in\mathbb{Z}})$$

and

$$(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})), P^D) :=$$

$$(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})), \{\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))\}_{k\in\mathbb{Z}}).$$

Proposition 2.2.22. *Let*

(2.2.22.1)

$$\overline{u}_{(X,Z)/S}\colon (((\widetilde{X,Z})/S)^{\log}_{\mathrm{Rcrys}}, Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})) \longrightarrow (\widetilde{X}_{\mathrm{zar}}, f^{-1}(\mathcal{O}_S))$$

be the morphism in (1.6.1.2). *Then*

(2.2.22.2)

$$R\overline{u}_{(X,Z)/S*}(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))), Q^*_{(X,Z)/S}P^D)$$
$$= (\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})), P^D)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$.

Proof. By (1.6.3.1), by (2.2.1) (2) and by (1.3.1), the left hand side of (2.2.22.2) is equal to

$$(u_{(X,Z)/S*}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))), u_{(X,Z)/S*}L_{(X,Z)/S}(P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))).$$

For an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, we have

$$u_{(X,Z)/S*}L_{(X,Z)/S}(\mathcal{F}) = u_{(X,Z)/S*}j_{\mathfrak{D}*}\varphi^*g^*(\mathcal{F}) = \varphi_*\varphi^*g^*(\mathcal{F}) = \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{F}$$

by (2.2.0.4). Hence

$$u_{(X,Z)/S*}(L_{(X,Z)/S}(P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))) = \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})).$$

Thus (2.2.22) follows. □

Remark 2.2.23. For simplicity, we assume that $Z=\emptyset$ in this remark. By the proof of (2.2.7), the differential operator of $\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{X}}P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log\mathcal{D})$ is not a log HPD differential operator in general since the log HPD differential operator $\mathcal{O}_{\mathfrak{D}_{\mathcal{D}}(1)}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^{\bullet}_{\mathcal{X}/S}(\log\mathcal{D}) \longrightarrow \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{X}}\Omega^{\bullet+1}_{\mathcal{X}/S}(\log\mathcal{D})$ induces a morphism $\mathcal{O}_{\mathfrak{D}_{\mathcal{D}}(1)}\otimes_{\mathcal{O}_{\mathcal{X}}}P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log\mathcal{D}) \longrightarrow \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{X}}P^{\mathcal{D}}_{k+1}\Omega^{\bullet+1}_{\mathcal{X}/S}(\log\mathcal{D})$, but does not induce a morphism $\mathcal{O}_{\mathfrak{D}_{\mathcal{D}}(1)}\otimes_{\mathcal{O}_{\mathcal{X}}}P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log\mathcal{D}) \longrightarrow \mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{X}}P^{\mathcal{D}}_k\Omega^{\bullet+1}_{\mathcal{X}/S}(\log\mathcal{D})$ in general; there does not exist a complex $L_{(X,D)/S}(P^{\mathcal{D}}_k\Omega^{\bullet}_{\mathcal{X}/S}(\log\mathcal{D}))$ in $\mathrm{C}^+(\mathcal{O}_{(X,D)/S})$ in general.

2.3 Forgetting Log Morphisms and Vanishing Cycle Sheaves

In this section we investigate some properties of the forgetting log morphism of log crystalline topoi.

Let the notations be as in §1.6. However, in this section, we denote the underlying scheme of the log scheme Y also by Y by abuse of notation. Let M be the log structure of Y. Let $N \subset M$ be also a fine log structure on $Y_{\rm zar}$. Then we have a natural morphism

(2.3.0.1) $$\epsilon = \epsilon_{(Y,M,N)/S} \colon (Y, M) \longrightarrow (Y, N)$$

of log schemes over S. The morphism ϵ induces a morphism of topoi which is denoted by the same notation:

(2.3.0.2) $$\epsilon = \epsilon_{(Y,M,N)/S} \colon ((\widetilde{Y, M)/S})^{\log}_{\rm crys} \longrightarrow ((\widetilde{Y, N)/S})^{\log}_{\rm crys}.$$

When N is trivial, we denote $\epsilon_{(Y,M,N)/S}$ by $\epsilon_{Y/S}$; the morphism $\epsilon_{Y/S}$ is a p-adic analogue of the l-adic forgetting log morphism in [30] and [67, (1.1.2)].

In this section, let us assume the following condition on the log structure N unless otherwise stated:

(2.3.0.3)
Locally on Y, there exists a chart $P \longrightarrow N$ such that $P^{\rm gp}$ has no p-torsion.

Then we have the following lemma:

Lemma 2.3.1. *Let the notation be as above and let* $(U, T, M_T, \iota, \delta)$ *be an object of* $((Y, M)/S)^{\log}_{\rm crys}$, *let* $N_T^{\rm inv}$ *be the inverse image of* $N|_U/\mathcal{O}_U^*$ *by the following morphism*: $M_T \overset{\rm proj.}{\longrightarrow} M_T/\mathcal{O}_T^* \overset{\iota^*}{\underset{\sim}{\longrightarrow}} M|_U/\mathcal{O}_U^*$. *Then* $N_T^{\rm inv}$ *is a fine log structure on* T (*under the assumption* (2.3.0.3)).

Proof. It is easy to see that $N_T^{\rm inv}$ is a log structure on T such that $N_T^{\rm inv}/\mathcal{O}_T^* = N|_U/\mathcal{O}_U^*$. Set $\mathcal{I}_T := {\rm Ker}(\mathcal{O}_T \longrightarrow \mathcal{O}_U)$. Then we have the exact sequence

$$0 \longrightarrow 1 + \mathcal{I}_T \longrightarrow N_T^{\rm inv,gp} \longrightarrow N^{\rm gp}|_U \longrightarrow 0.$$

Shrink U and take a chart $\alpha : P \longrightarrow N|_U$ such that $P^{\rm gp}$ has no p-torsion. Then, since any element of $1 + \mathcal{I}_T$ is killed by some power of p, we have $\mathcal{E}{\rm xt}^1(P^{\rm gp}, 1 + \mathcal{I}_T) = 0$. Hence we have a homomorphism $\widetilde{\alpha} : P^{\rm gp} \longrightarrow N_T^{\rm inv,gp}$ lifting $\alpha^{\rm gp}$ locally on T and it induces the homomorphism of monoids $P \longrightarrow N_T^{\rm inv}$, which we also denote by $\widetilde{\alpha}$. If we denote the log structure associated to $P \overset{\widetilde{\alpha}}{\longrightarrow} N_T^{\rm inv} \longrightarrow \mathcal{O}_T$ by P^a, $\widetilde{\alpha}$ induces a homomorphism of log structures $\widetilde{\alpha}^a : P^a \longrightarrow N_T^{\rm inv}$ such that the induced homomorphism $\widetilde{\alpha}^a : P^a/\mathcal{O}_T^* \longrightarrow N_T^{\rm inv}/\mathcal{O}_T^*$ is nothing but the identity on $N|_U/\mathcal{O}_U^*$. Hence $\widetilde{\alpha}^a$ is an isomorphism, that is, $\widetilde{\alpha}$ is a chart of $N_T^{\rm inv}$. Therefore $N_T^{\rm inv}$ is a fine log structure. □

Under the assumption (2.3.0.3), the explicit description of $\epsilon = (\epsilon^*, \epsilon_*)$ is given as follows: for an object F of $((\widetilde{Y,N})/S)^{\log}_{\rm crys}$ and an object $(U, T, M_T, \iota, \delta) \in ((Y,M)/S)^{\log}_{\rm crys}$,

$$\epsilon^*(F)((U, T, M_T, \iota, \delta)) = F((U, T, N_T^{\rm inv}, \iota, \delta));$$

for an object G of $((\widetilde{Y,M})/S)_{\rm crys}$ and an object $(U, T, N_T, \iota, \delta) \in ((Y,N)/S)^{\log}_{\rm crys}$,

$$\epsilon_*(G)((U, T, N_T, \iota, \delta)) = {\rm Hom}_{((\widetilde{Y,M})/S)^{\log}_{\rm crys}}(\epsilon^*(T), G).$$

Definition 2.3.2. We call the morphism $\epsilon_{(Y,M,N)/S}$ in (2.3.0.1) and the morphism $\epsilon_{(Y,M,N)/S}$ in (2.3.0.2) the *forgetting log morphism* of log schemes over S *along* $M \setminus N$ and the *forgetting log morphism* of log crystalline topoi *along* $M \setminus N$, respectively. When N is trivial, we call the two $\epsilon_{(Y,M,N)/S}$'s the *forgetting log morphisms* of Y/S. When Y is a smooth scheme X over $S_0 := \underline{\rm Spec}_S(\mathcal{O}_S/\mathcal{I})$, $M = M(D \cup Z)$ and $N = M(Z)$, where D and Z are transversal relative SNCD's on X/S_0, we call the two $\epsilon_{(Y,M,N)/S}$'s the *forgetting log morphisms along* D and denote them by $\epsilon_{(X,D\cup Z,Z)/S}$.

Let $\{Y_i\}_{i\in I}$ be an open covering of Y. Let M_i (resp. N_i) be the pull-back of M (resp. N) to Y_i. Then we also have an analogous morphism of topoi

$$\epsilon_\bullet\colon ((\widetilde{Y_\bullet, M_\bullet})/S)^{\log}_{\rm crys} \longrightarrow ((\widetilde{Y_\bullet, N_\bullet})/S)^{\log}_{\rm crys}, \tag{2.3.2.1}$$

and we have the following commutative diagram

$$\begin{tikzcd}
((\widetilde{Y_\bullet, M_\bullet})/S)^{\log}_{\rm crys} \arrow[r, "\epsilon_\bullet"] \arrow[d, "\pi^{\log}_{M{\rm crys}}"'] & ((\widetilde{Y_\bullet, N_\bullet})/S)^{\log}_{\rm crys} \arrow[d, "\pi^{\log}_{N{\rm crys}}"] \\
((\widetilde{Y,M})/S)^{\log}_{\rm crys} \arrow[r, "\epsilon"] & ((\widetilde{Y,N})/S)^{\log}_{\rm crys}.
\end{tikzcd} \tag{2.3.2.2}$$

Here $\pi^{\log}_{M{\rm crys}}$ and $\pi^{\log}_{N{\rm crys}}$ are morphisms of topoi defined in §1.6; we have written the symbols M and N in subscripts for clarity. Let $u_{(Y,L)/S}$, $u_{(Y_\bullet,L_\bullet)/S}$ and $u_{(Y_{\bullet\bullet},L_{\bullet\bullet})/S}$ $(L := M, N)$ be the projections in (1.6.0.8), (1.6.0.9) and (1.6.0.10) for (Y,L), respectively. Since $\epsilon^* \circ u^*_{(Y,N)/S} = u^*_{(Y,M)/S}$ and $\epsilon^*_\bullet \circ u^*_{(Y_\bullet,N_\bullet)/S} = u^*_{(Y_\bullet,M_\bullet)/S}$, we have the following two equations

$$u_{(Y,N)/S} \circ \epsilon = u_{(Y,M)/S}, \quad u_{(Y_\bullet,N_\bullet)/S} \circ \epsilon_\bullet = u_{(Y_\bullet,M_\bullet)/S} \tag{2.3.2.3}$$

as morphisms of topoi.

Let the notations be as in §1.6. Then we have the following commutative diagram:

(2.3.2.4)
$$\begin{CD} ((Y_{\bullet\bullet},\widetilde{M_{\bullet\bullet})/S})^{\log}_{\mathrm{crys}} @>{\epsilon_{\bullet\bullet}}>> ((Y_{\bullet\bullet},\widetilde{N_{\bullet\bullet})/S})^{\log}_{\mathrm{crys}} \\ @V{\eta^{\log}_{M\mathrm{crys}}}VV @VV{\eta^{\log}_{N\mathrm{crys}}}V \\ ((Y_{\bullet},\widetilde{M_{\bullet})/S})^{\log}_{\mathrm{crys}} @>{\epsilon_{\bullet}}>> ((Y_{\bullet},\widetilde{N_{\bullet})/S})^{\log}_{\mathrm{crys}}. \end{CD}$$

Let $\mathcal{O}_{(Y,L)/S}$ $(L := M, N)$ be the structure sheaf in $((\widetilde{Y,L)/S})^{\log}_{\mathrm{crys}}$. Since there is a morphism $\epsilon^*(\mathcal{O}_{(Y,N)/S}) \longrightarrow \mathcal{O}_{(Y,M)/S}$, there is a morphism

(2.3.2.5)
$$\mathcal{O}_{(Y,N)/S} \longrightarrow \epsilon_*(\mathcal{O}_{(Y,M)/S}).$$

The morphism ϵ also induces a morphism

(2.3.2.6)
$$\epsilon\colon (((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y,M)/S}) \longrightarrow (((\widetilde{Y,N)/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y,N)/S})$$

of ringed topoi. We have the analogues of the commutative diagrams (2.3.2.2) and (2.3.2.4) for the ringed topoi:

(2.3.2.7)
$$\begin{CD} (((Y_{\bullet},\widetilde{M_{\bullet})/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y_{\bullet},M_{\bullet})/S}) @>{\epsilon_{\bullet}}>> (((Y_{\bullet},\widetilde{N_{\bullet})/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y_{\bullet},N_{\bullet})/S}) \\ @V{\pi^{\log}_{M\mathrm{crys}}}VV @VV{\pi^{\log}_{N\mathrm{crys}}}V \\ (((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y,M)/S}) @>{\epsilon}>> (((\widetilde{Y,N)/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y,N)/S})), \end{CD}$$

(2.3.2.8)
$$\begin{CD} (((Y_{\bullet\bullet},\widetilde{M_{\bullet\bullet})/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y_{\bullet\bullet},M_{\bullet\bullet})/S}) @>{\epsilon_{\bullet\bullet}}>> (((Y_{\bullet\bullet},\widetilde{N_{\bullet\bullet})/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y_{\bullet\bullet},N_{\bullet\bullet})/S}) \\ @V{\eta^{\log}_{M\mathrm{crys}}}VV @VV{\eta^{\log}_{N\mathrm{crys}}}V \\ (((Y_{\bullet},\widetilde{M_{\bullet})/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y_{\bullet},M_{\bullet})/S}) @>{\epsilon_{\bullet}}>> (((Y_{\bullet},\widetilde{N_{\bullet})/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y_{\bullet},N_{\bullet})/S}). \end{CD}$$

The morphism (2.3.2.5) gives a morphism

(2.3.2.9)
$$\mathcal{O}_{(Y,N)/S} \longrightarrow R\epsilon_*(\mathcal{O}_{(Y,M)/S}).$$

Using (2.3.2.3), we have a morphism

(2.3.2.10)
$$Ru_{(Y,N)/S*}(\mathcal{O}_{(Y,N)/S}) \longrightarrow Ru_{(Y,M)/S*}(\mathcal{O}_{(Y,M)/S}).$$

Next we define the localization of ϵ. Let $F_M = (U_F, T_F, M_F, \iota_F, \delta_F)$ be a representable sheaf in $((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}$. Set $F_N := (U_F, T_F, N_F^{\mathrm{inv}}, \iota_F, \delta_F)$, where N_F^{inv} is the inverse image of $N|_{U_F}$ by the morphism $M_F \longrightarrow M|_{U_F}/\mathcal{O}^*_{U_F}$ as before. Then we have a morphism

(2.3.2.11)
$$\epsilon|_F\colon ((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}|_{F_M} \longrightarrow ((\widetilde{Y,N)/S})^{\log}_{\mathrm{crys}}|_{F_N}$$

of topoi and a morphism

(2.3.2.12)

$$\epsilon|_F\colon (((\widetilde{Y,M})/S)^{\log}_{\text{crys}}|_{F_M}, \mathcal{O}_{(Y,M)/S}|_{F_M}) \longrightarrow (((\widetilde{Y,N})/S)^{\log}_{\text{crys}}|_{F_N}, \mathcal{O}_{(Y,N)/S}|_{F_N}).$$

of ringed topoi.

Lemma 2.3.3. *Let the notations be as above. Then the functor $\epsilon|_{F*}$ is exact.*

Proof. Let $(U, T, N_T, \iota, \delta, \phi_N)$ be an object in $((Y,N)/S)^{\log}_{\text{crys}}|_{F_N}$. Let $\phi\colon T \longrightarrow T_F$ be the underlying morphism of schemes of ϕ_N. Set $M_T := \phi^*(M_F)$. Let

$$\phi_M\colon (U, T, M_T, \iota, \delta) \longrightarrow (U_F, T_F, M_F, \iota_F, \delta_F)$$

be the natural morphism. Then $(U, T, M_T, \iota, \delta, \phi_M)$ is an object in $((Y,M)/S)^{\log}_{\text{crys}}|_{F_M}$. Let $(T_F, M_F)\times_{(T_F, N_F^{\text{inv}})}(T, N_T)$ be the fiber product of (T_F, M_F) and (T, N_T) over (T_F, N_F^{inv}) in the category of fine log schemes. We claim that

$$(T_F, M_F)\times_{(T_F,N_F^{\text{inv}})}(T, N_T) = (T, M_T). \tag{2.3.3.1}$$

Indeed, let $\phi_U\colon U \overset{\subset}{\longrightarrow} U_F$ be the open immersion. Then we have the following:

$$\begin{aligned}(\phi^*(M_F)\oplus_{\phi^*(N_F^{\text{inv}})} N_T)/\mathcal{O}_T^* &= \phi^*(M_F)/\mathcal{O}_T^* \oplus_{\phi^*(N_F^{\text{inv}})/\mathcal{O}_T^*} N_T/\mathcal{O}_T^* \\ &= \phi^{-1}(M_F/\mathcal{O}_{T_F}^*)\oplus_{\phi^{-1}(N_F^{\text{inv}}/\mathcal{O}_{T_F}^*)} N_T/\mathcal{O}_T^* \\ &\simeq \phi_U^{-1}(M|_{U_F}/\mathcal{O}_{U_F}^*)\oplus_{\phi_U^{-1}(N|_{U_F}/\mathcal{O}_{U_F}^*)} N|_U/\mathcal{O}_U^* \\ &= M|_U/\mathcal{O}_U^* \simeq \phi^*(M_F)/\mathcal{O}_T^*.\end{aligned}$$

Hence the natural morphism $\phi^*(M_F) \longrightarrow \phi^*(M_F)\oplus_{\phi^*(N_F^{\text{inv}})} N_T$ is an isomorphism and we have shown the claim. Denote $(U, T, L_T, \iota, \delta, \phi_L)$ $(L := M, N)$ by (T, L_T, ϕ_L) for simplicity of notation. By the formula (2.3.3.1), $(\epsilon|_F)^*((T, N_T, \phi_N))$ is represented by (T, M_T, ϕ_M). Therefore, for an object E in $((\widetilde{Y,M})/S)^{\log}_{\text{crys}}|_{F_M}$, we have

(2.3.3.2)

$$\begin{aligned}\varGamma((T, N_T, \phi_N), (\epsilon|_F)_*(E)) &= \operatorname{Hom}_{((\widetilde{Y,M})/S)^{\log}_{\text{crys}}|_{F_M}}((\epsilon|_F)^*((T, N_T, \phi_N)), E) \\ &= E((T, M_T, \phi_M)).\end{aligned}$$

Using this formula, we see that the functor $\epsilon|_{F*}$ is exact. □

Lemma 2.3.4. *Let the notations be as above. Then the following diagram of topoi is commutative*:

$$\begin{tikzcd} ((\widetilde{Y,M})/S)^{\log}_{\text{crys}}|_{F_M} \arrow[r, "j_{F_M}"] \arrow[d, "\epsilon|_F"'] & ((\widetilde{Y,M})/S)^{\log}_{\text{crys}} \arrow[d, "\epsilon"] \\ ((\widetilde{Y,N})/S)^{\log}_{\text{crys}}|_{F_N} \arrow[r, "j_{F_N}"] & ((\widetilde{Y,N})/S)^{\log}_{\text{crys}}. \end{tikzcd} \tag{2.3.4.1}$$

The obvious analogue of (2.3.4.1) *for ringed topoi also holds.*

Proof. Let G be an object of $((\widetilde{Y,N})/S)^{\log}_{\mathrm{crys}}$. By the proof of (2.3.3), $(\epsilon|_F)^*(F_N) = F_M$. Hence $(\epsilon|_F)^* j^*_{F_N}(G) = (\epsilon|_F)^*(G \times F_N) = \epsilon^*(G) \times F_M = j^*_{F_M}\epsilon^*(G)$. Hence the former statement follows.

The latter statement immediately follows. □

Lemma 2.3.5. *Let $F_M = (Y, T, M_T, \iota, \delta)$ be a representable sheaf in $((\widetilde{Y,M})/S)^{\log}_{\mathrm{crys}}$. Let $E \in ((\widetilde{Y,M})/S)^{\log}_{\mathrm{crys}}|_{F_M}$ be an $\mathcal{O}_{(Y,M)/S}|_{F_M}$-module. Then the canonical morphism*

$$\epsilon_* j_{F_M *}(E) \longrightarrow R\epsilon_* j_{F_M *}(E)$$

is an isomorphism in the derived category $\mathrm{D}^+(\mathcal{O}_{(Y,N)/S})$.

Proof. Indeed, we have

$$\begin{aligned}
\epsilon_* j_{F_M *}(E) &\overset{(2.3.4)}{=} j_{F_N *}(\epsilon|_F)_*(E) \overset{(2.3.3)}{=} j_{F_N *}R(\epsilon|_F)_*(E) \\
&\overset{(2.2.1)\ (1)}{=} Rj_{F_N *}R(\epsilon|_F)_*(E) = R(j_{F_N}\epsilon|_F)_*(E) \\
&\overset{(2.3.4)}{=} R(\epsilon j_{F_M})_*(E) = R\epsilon_* Rj_{F_M *}(E) \overset{(2.2.1)\ (1)}{=} R\epsilon_* j_{F_M *}(E).
\end{aligned}$$

□

Though ϵ_* is not exact in general (see (2.7.1) below), the following holds:

Corollary 2.3.6. *Let $\iota\colon (Y,M) \xrightarrow{\subset} (\mathcal{Y},\mathcal{M})$ be a closed immersion into a log smooth scheme over S. Let $\mathfrak{D}_Y(\mathcal{Y})$ be the log PD-envelope of ι over $(S,\mathcal{I},\gamma)$. Let $\mathcal{E}$ be an $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$-module. Let $L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E})$ be the linearization of $\mathcal{E}$ with respect to ι. Then the canonical morphism*

$$\epsilon_* L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E}) \longrightarrow R\epsilon_* L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E}) \tag{2.3.6.1}$$

is an isomorphism in the derived category $\mathrm{D}^+(\mathcal{O}_{(Y,N)/S})$.

Proof. (2.3.6) immediately follows from (2.2.1.2) and (2.3.5). □

Lemma 2.3.7. *Let $(\mathcal{Y},\mathcal{M})$ be a log smooth scheme over S. Let $\mathcal{N}$ be a fine sub-log structure of $\mathcal{M}$ on $\mathcal{Y}$ such that $(\mathcal{Y},\mathcal{N})$ is also log smooth over S. Let*

$$\begin{tikzcd}
(Y,M) \arrow[r, "\iota_{\mathcal{M}}"] \arrow[d, "\epsilon_{(Y,M,N)/S}"'] & (\mathcal{Y},\mathcal{M}) \arrow[d, "\epsilon_{(\mathcal{Y},\mathcal{M},\mathcal{N})/S}"] \\
(Y,N) \arrow[r, "\iota_{\mathcal{N}}"] & (\mathcal{Y},\mathcal{N})
\end{tikzcd} \tag{2.3.7.1}$$

be a commutative diagram whose horizontal morphisms are closed immersions. Let $\mathfrak{D}_{\mathcal{M}}$ and $\mathfrak{D}_{\mathcal{N}}$ be the log PD-envelopes of $\iota_{\mathcal{M}}$ and $\iota_{\mathcal{N}}$ over $(S,\mathcal{I},\gamma)$, respectively, with the natural following commutative diagram:

$$
\begin{tikzcd}
\mathfrak{D}_{\mathcal{M}} \arrow[r, "g_{\mathcal{M}}"] \arrow[d, "h"'] & (\mathcal{Y},\mathcal{M}) \arrow[d] \\
\mathfrak{D}_{\mathcal{N}} \arrow[r, "g_{\mathcal{N}}"] & (\mathcal{Y},\mathcal{N})
\end{tikzcd}
\tag{2.3.7.2}
$$

Assume that the underlying morphism $\overset{\circ}{h}$ *of schemes is the identity. Then there exist natural isomorphisms*

$$L^{\mathrm{PD}}_{(Y,N)/S} \xrightarrow{\sim} \epsilon_{(Y,M,N)/S*}L^{\mathrm{PD}}_{(Y,M)/S} \tag{2.3.7.3}$$

and

$$L^{\mathrm{PD}}_{(Y,N)/S}\overset{\circ}{g}{}^{*}_{\mathcal{N}} \xrightarrow{\sim} \epsilon_{(Y,M,N)/S*}L^{\mathrm{PD}}_{(Y,M)/S}\overset{\circ}{g}{}^{*}_{\mathcal{M}} \tag{2.3.7.4}$$

of functors. Moreover, the functor (2.3.7.3) *is functorial with respect to log HPD differential operators.*

Proof. Let $\varphi_{\mathcal{M}} \colon (((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_{\mathcal{M}}}, \mathcal{O}_{(Y,M)/S}|_{\mathfrak{D}_{\mathcal{M}}}) \longrightarrow (\widetilde{\overset{\circ}{\mathfrak{D}}}_{\mathcal{M}\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}_{\mathcal{M}}})$ be the morphism of ringed topoi in (2.2.1.1). Let $\varphi_{\mathcal{N}}$ be the analogue of $\varphi_{\mathcal{M}}$ for $(\mathcal{Y},\mathcal{N})$. Let

$$\epsilon|_{\mathfrak{D}} : (((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_{\mathcal{M}}}, \mathcal{O}_{(Y,M)/S}|_{\mathfrak{D}_{\mathcal{M}}}) \longrightarrow (((\widetilde{Y,N)/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_{\mathcal{N}}}, \mathcal{O}_{(Y,N)/S}|_{\mathfrak{D}_{\mathcal{N}}})$$

be the natural morphism. Then, using the formula (2.3.3.2), we can immediately check that $(\epsilon|_{\mathfrak{D}})_*\varphi^*_{\mathcal{M}} = \varphi^*_{\mathcal{N}}$. Hence we have the following commutative diagram

$$
\begin{tikzcd}
(\widetilde{\mathcal{Y}}_{\mathrm{zar}}, \mathcal{O}_{\mathcal{Y}}) \arrow[r, "\overset{\circ}{g}{}^{*}_{\mathcal{M}}"] \arrow[d, equal] & (\widetilde{\overset{\circ}{\mathfrak{D}}}_{\mathcal{M}\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}_{\mathcal{M}}}) \arrow[r, "\varphi^*_{\mathcal{M}}"] \arrow[d, equal] & {} \\
(\widetilde{\mathcal{Y}}_{\mathrm{zar}}, \mathcal{O}_{\mathcal{Y}}) \arrow[r, "\overset{\circ}{g}{}^{*}_{\mathcal{N}}"] & (\widetilde{\overset{\circ}{\mathfrak{D}}}_{\mathcal{N}\mathrm{zar}}, \mathcal{O}_{\mathfrak{D}_{\mathcal{N}}}) \arrow[r, "\varphi^*_{\mathcal{N}}"] & {}
\end{tikzcd}
\tag{2.3.7.5}
$$

$$
\begin{tikzcd}
(((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_{\mathcal{M}}}, \mathcal{O}_{(Y,M)/S}|_{\mathfrak{D}_{\mathcal{M}}}) \arrow[r, "j_{\mathfrak{D}_{\mathcal{M}}*}"] \arrow[d, "\epsilon|_{\mathfrak{D}*}"'] & (((\widetilde{Y,M)/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(Y,M)/S}) \arrow[d, "\epsilon_{(Y,M,N)/S*}"'] \\
(((\widetilde{Y,N)/S})^{\log}_{\mathrm{crys}}|_{\mathfrak{D}_{\mathcal{N}}}, \mathcal{O}_{(Y,N)/S}|_{\mathfrak{D}_{\mathcal{N}}}) \arrow[r, "j_{\mathfrak{D}_{\mathcal{N}}*}"] & (((\widetilde{Y,N)/S})^{\log}_{\mathrm{crys}}\mathcal{O}_{(Y,N)/S}),
\end{tikzcd}
$$

and this implies the isomorphisms (2.3.7.3), (2.3.7.4).

Finally we check the functoriality of the isomorphism (2.3.7.3) with respect to log HPD differential operators. To show this, it suffices to prove the required functoriality for the morphism

$$\epsilon^*_{(Y,M,N)/S}L^{\mathrm{PD}}_{(Y,N)/S} \longrightarrow L^{\mathrm{PD}}_{(Y,M)/S}. \tag{2.3.7.6}$$

For $T_M := (U, T, M_T, \iota, \delta)$ in $((Y, M)/S)^{\log}_{\mathrm{crys}}$, let $T_N := (U, T, N_T^{\mathrm{inv}}, \iota, \delta)$ be as above. Then, for an $\mathcal{O}_{\mathfrak{D}_N}$-module $\mathcal{E}$, the homomorphism

$$(\epsilon^*_{(Y,M,N)/S} L^{\mathrm{PD}}_{(Y,N)/S}(\mathcal{E}))_{T_M} \longrightarrow (L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E}))_{T_N}$$

induced by (2.3.7.6) is given by the canonical homomorphism

$$\mathcal{O}_{\mathfrak{D}_U(T_N \times_S (\mathcal{Y},\mathcal{N}))} \otimes_{\mathcal{O}_{\mathfrak{D}_N}} \mathcal{E} \longrightarrow \mathcal{O}_{\mathfrak{D}_U(T_M \times_S (\mathcal{Y},\mathcal{M}))} \otimes_{\mathcal{O}_{\mathfrak{D}_M}} \mathcal{E},$$

and it is easy to see that this homomorphism is functorial with respect to log HPD differential operators (see (2.2.3.1)). Hence we finish the proof of the lemma. □

Remark 2.3.8. In (2.3.7), we do not have to assume the condition (2.3.0.3) on the log structure N. The reason why we imposed the condition (2.3.0.3) was to assure that the log structure N_T^{inv} is always fine. However, in the situation in (2.3.7), the fineness of N_T^{inv} for any $T = (U, T, M_T)$ follows from the assumption. Indeed, we have a morphism $\psi : (T, M_T) \longrightarrow \mathfrak{D}_M$ etale locally on T and one can see that N_T^{inv} is isomorphic to the pull-back of the log structure of $\mathfrak{D}_N$ by ψ.

Definition 2.3.9. For an $\mathcal{O}_{Y/S}$-module E, we call $R\epsilon_{(Y,M,N)/S*}(E)$ the *vanishing cycle sheaf* of E *along* $M \setminus N$. We call $R\epsilon_{(Y,M,N)/S*}(\mathcal{O}_{(Y,M)/S})$ the *vanishing cycle sheaf* of $(Y, M)/(S, \mathcal{I}, \gamma)$ *along* $M \setminus N$. If N is trivial, we omit the word "along $M \setminus N$".

The following theorem is the crystalline Poincaré lemma of a vanishing cycle sheaf:

Theorem 2.3.10 (Poincaré lemma of a vanishing cycle sheaf). *Let $\mathcal{M}_S$ be the log structure of S. Let E be a crystal of $\mathcal{O}_{(Y,N)/S}$-modules and let $(\mathcal{E}, \nabla)$ be the $\mathcal{O}_{\mathfrak{D}_\mathcal{M}}$-module with integrable log connection corresponding to $\epsilon^*_{(Y,M,N)/S}(E)$. Assume that we are given the commutative diagram* (2.3.7.1) *and that $\overset{\circ}{h}$ in* (2.3.7) *is the identity. Then there exists a canonical isomorphism*

(2.3.10.1)
$$R\epsilon_{(Y,M,N)/S*}\epsilon^*_{(Y,M,N)/S}(E) \xrightarrow{\sim} L^{\mathrm{PD}}_{(Y,N)/S}(\mathcal{E} \otimes_{\mathcal{O}_\mathcal{Y}} \Omega^\bullet_{\mathcal{Y}/S}(\log \mathcal{M}/\mathcal{M}_S))$$

in $\mathrm{D}^+(\mathcal{O}_{(Y,N)/S})$.

Proof. By (2.2.8.1), we have an isomorphism

(2.3.10.2) $$\epsilon^*_{(Y,M,N)/S}(E) \xrightarrow{\sim} L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E} \otimes_{\mathcal{O}_\mathcal{Y}} \Omega^\bullet_{\mathcal{Y}/S}(\log \mathcal{M}/\mathcal{M}_S)).$$

Applying $R\epsilon_{(Y,M,N)/S*}$ to both hands of (2.3.10.2) and using (2.3.6) and (2.3.7), we obtain

$$
\begin{aligned}
&R\epsilon_{(Y,M,N)/S*}\epsilon^*_{(Y,M,N)/S}(E)\\
&\xrightarrow{\sim} R\epsilon_{(Y,M,N)/S*}L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/S}(\log\mathcal{M}/\mathcal{M}_S))\\
&\xleftarrow{\sim} \epsilon_{(Y,M,N)/S*}L^{\mathrm{PD}}_{(Y,M)/S}(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/S}(\log\mathcal{M}/\mathcal{M}_S))\\
&= L^{\mathrm{PD}}_{(Y,N)/S}(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^\bullet_{\mathcal{Y}/S}(\log\mathcal{M}/\mathcal{M}_S)).
\end{aligned}
$$

□

We prove the boundedness of log crystalline cohomology in a general situation.

Proposition 2.3.11. *Let $(S,\mathcal{I},\gamma)$ be the log PD-scheme in §1.6. Set $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I})$. Let $f : X \longrightarrow Y$ be a morphism of fine log schemes over S_0. Assume that $\overset{\circ}{X}$ and $\overset{\circ}{Y}$ are quasi-compact and that $\overset{\circ}{f} : \overset{\circ}{X} \longrightarrow \overset{\circ}{Y}$ is quasi-separated morphism of finite type. Let E be a quasi-coherent crystal of $\mathcal{O}_{Y/S}$-modules. Then $Rf^{\log}_{\mathrm{crys}*}(E)$ is bounded.*

Proof. For $(U,T,\delta)\in(Y/S)^{\log}_{\mathrm{crys}}$, put $X_U := X\times_Y U$ and denote the morphism of topoi $(f|_{X_U})\circ u_{X_U/T}$ by $f_{X_U/T}$. By the same argument as [3, V Théorème 3.2.4], we are reduced to proving the following claim: there exists a positive integer r such that, for any $(U,T,\delta)\in(Y/S)^{\log}_{\mathrm{crys}}$ and for any quasi-coherent crystal E on $(X_U/T)^{\log}_{\mathrm{crys}}$, we have $R^if_{X_U/T*}(E)=0$ for $i>r$. Again, by the same argument as [3, V Théorème 3.2.4, Proposition 3.2.5], we are reduced to showing the above claim in the case where $\overset{\circ}{X}$ and $\overset{\circ}{Y}$ are sufficiently small affine schemes. Hence we may assume that X admits a chart $\alpha : P \longrightarrow M_X$. (Note that, in this book, log structures are defined on a Zariski site.) Let us take surjections $\varphi_1 : \mathcal{O}_Y[\mathbb{N}^a] \longrightarrow \mathcal{O}_X$ and $\varphi_2 : \mathbb{N}^b \longrightarrow P$ $(a,b\in\mathbb{N})$. For $(U,T,M_T,\iota,\delta)\in(Y/S)^{\log}_{\mathrm{crys}}$, let us define $\widetilde{T} := (\overset{\circ}{\widetilde{T}}, M_{\widetilde{T}})$ by

$$\overset{\circ}{\widetilde{T}} := \underline{\mathrm{Spec}}_{\mathcal{O}_T}(\mathcal{O}_T[\mathbb{N}^a\oplus\mathbb{N}^b]),$$

$$M_{\widetilde{T}} := \text{the log structure associated to } M_T\oplus\mathbb{N}^b \longrightarrow \mathcal{O}_T[\mathbb{N}^a\oplus\mathbb{N}^b],$$

where the map $M_T\oplus\mathbb{N}^b \longrightarrow \mathcal{O}_T[\mathbb{N}^a\oplus\mathbb{N}^b]$ is induced by $M_T \longrightarrow \mathcal{O}_T$ and the natural inclusion $\mathbb{N}^b \overset{\subset}{\longrightarrow} \mathcal{O}_T[\mathbb{N}^a\oplus\mathbb{N}^b]$. Then we have the canonical affine log smooth morphism $g : \widetilde{T} \longrightarrow T$. Let ψ_1 be the morphism $\mathcal{O}_T[\mathbb{N}^a\oplus\mathbb{N}^b] \longrightarrow \mathcal{O}_{X_U}$ induced by $\mathbb{N}^a \overset{\subset}{\longrightarrow} \mathcal{O}_U[\mathbb{N}^a] \xrightarrow{\varphi_1|_U} \mathcal{O}_{X_U}$ and $\mathbb{N}^b \xrightarrow{\varphi_2} P \xrightarrow{\alpha|_{X_U}} M_{X_U} \longrightarrow \mathcal{O}_{X_U}$. Let ψ_2 be the morphism $M_T\oplus\mathbb{N}^b \longrightarrow M_{X_U}$ induced by $M_T \longrightarrow M_U \longrightarrow M_{X_U}$ and $\mathbb{N}^b \xrightarrow{\varphi_2} P \xrightarrow{\alpha|_{X_U}} M_{X_U}$. Then we have the closed immersion $\psi : X_U \overset{\subset}{\longrightarrow} \widetilde{T}$ of log schemes induced by ψ_1, ψ_2 and we have the commutative diagram of log schemes

$$\begin{array}{ccc} X_U & \xrightarrow{\psi} & \widetilde{T} \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ U & \xrightarrow{\iota} & T. \end{array}$$

Let $\mathfrak{D}$ be the log PD-envelope of ψ and let $h : \mathfrak{D} \longrightarrow T$ be the composite morphism $\mathfrak{D} \longrightarrow \widetilde{T} \longrightarrow T$. Then we have $R^i f_{X_U/T*}(E) = \mathcal{H}^i(h_*(E_{(X_U,\mathfrak{D})} \otimes_{\mathcal{O}_{\widetilde{T}}} \Lambda^\bullet_{\widetilde{T}/T})) = 0$ for $i > a + b$. Hence we have proved the claim and consequently we finish the proof of (2.3.11). □

Corollary 2.3.12. *Let* $(S, \mathcal{I}, \gamma)$ *and* S_0 *be as in* (2.3.11)*. Let* (Y, M) *be a fine log smooth scheme over* S_0 *such that* Y *is quasi-compact. Let* N *be a fine sub log structure of* M *on* Y*. Then, for a quasi-coherent crystal* E *of* $\mathcal{O}_{(Y,M)/S}$*-modules, the complex* $R\epsilon_{(Y,M,N)/S*}(E)$ *is bounded.*

Remark 2.3.13. In the proof of (2.3.11), we used the convention that the log structures in this book are defined on a Zariski site. However, if we assume that f is log smooth, we can prove the statement of (2.3.11) also in the case where the log structures are defined on an etale site. Indeed, in this case, by (2.3.14) below, if we assume that $\overset{\circ}{X}$ and $\overset{\circ}{Y}$ are affine, then we have always a log smooth lift $g : \widetilde{T} \longrightarrow T$ of $X_U \longrightarrow U$ for any $(U, T, \delta) \in (Y/S)^{\log}_{\rm crys}$ such that $\overset{\circ}{T}$ is affine. Then we have

$$R^i f_{X_U/T*}E = \mathcal{H}^i(g_*(E_{(X_U,\widetilde{T})} \otimes_{\mathcal{O}_{\widetilde{T}}} \Lambda^\bullet_{\widetilde{T}/T})) = 0$$

for $i > r$, where r is the maximum of the rank of $\Lambda^1_{X/Y,x}$ $(x \in X)$.

We give a proof of a lemma which has been used in (2.3.13), which is useful also in later sections.

Lemma 2.3.14. *Let* $\mathcal{S}$ *be a fine log scheme and let* $\mathcal{I}$ *be a quasi-coherent nil-ideal sheaf of* $\mathcal{O}_{\mathcal{S}}$*. Let* $\mathcal{S}_0$ *be an exact closed log subscheme of* $\mathcal{S}$ *defined by* $\mathcal{I}$*. Assume that* $\overset{\circ}{\mathcal{S}}$ *is affine. Let* Z *be a log smooth scheme over* $\mathcal{S}_0$*. Then, if* $\overset{\circ}{Z}$ *is affine, there exists a unique log smooth lift* $\mathcal{Z}$ (*up to an isomorphism*) *of* Z *over* $\mathcal{S}$ *and* $\mathcal{Z}$ *is also affine.*

Proof. Let (P) be a property of a scheme or a morphism of schemes. In this proof, for simplicity, we say that a log scheme W (resp. a morphism $f \colon W \longrightarrow W'$ of log schemes) has the property (P) if $\overset{\circ}{W}$ (resp. $\overset{\circ}{f}$) has the property (P). Though the unique existence of $\mathcal{Z}$ seems more or less well-known, we give a proof as follows (cf. [54, (3.14) (1)], [11, N.B. in 5.28]).

Express $\mathcal{I}$ as the inductive limit of the inductive system $\{\mathcal{I}_\lambda\}$ of quasi-coherent nilpotent ideal sheaves of $\mathcal{O}_{\mathcal{S}}$: $\mathcal{I} = \varinjlim_\lambda \mathcal{I}_\lambda$. Let $\mathcal{S}_\lambda$ be an exact closed log subscheme of $\mathcal{S}$ defined by $\mathcal{I}_\lambda$. Since $\mathcal{S}_0 = \varprojlim_\lambda \mathcal{S}_\lambda$ and since Z is of finite presentation over $\mathcal{S}_0$, there exists a fine log smooth scheme Z_λ over $\mathcal{S}_\lambda$

such that $Z = Z_\lambda \times_{\mathcal{S}_\lambda} \mathcal{S}_0$ for a large λ (cf. [40, 3 (8.8.2) (ii)], [40, 4 (17.7.8)], [86, 4.11]). By [40, 3 (8.10.5) (viii)], we may assume that Z_λ is affine. Since $\mathcal{I}_\lambda$ is nilpotent, the existence of $\mathcal{Z}$ follows from [54, (3.14) (1)].

Let $\mathcal{Z}'$ be another lift of Z over $\mathcal{S}$. Since the structural morphism $Z \longrightarrow \mathcal{S}_0$ is quasi-separated, the structural morphisms $\mathcal{Z} \longrightarrow \mathcal{S}$ and $\mathcal{Z}' \longrightarrow \mathcal{S}$ are quasi-separated by [40, 1 (1.2.5)]. Set $\mathcal{Z}_\lambda := \mathcal{Z} \times_{\mathcal{S}} \mathcal{S}_\lambda$ and $\mathcal{Z}'_\lambda := \mathcal{Z}' \times_{\mathcal{S}} \mathcal{S}_\lambda$. Then $\mathcal{Z}_\lambda$ and $\mathcal{Z}'_\lambda$ are quasi-compact, quasi-separated and of finite presentation over S_λ. Because $\varprojlim_\lambda \mathcal{Z}_\lambda = Z = \varprojlim_\lambda \mathcal{Z}'_\lambda$, there exists an isomorphism $\mathcal{Z}_\lambda \overset{\sim}{\longrightarrow} \mathcal{Z}'_\lambda$ over $\mathcal{S}_\lambda$ for a large λ which induces the identity of Z (cf. [40, 3 (8.8.2) (i)], [86, 4.11.3]). Since $\mathcal{I}_\lambda$ is nilpotent, there exists an isomorphism $\mathcal{Z} \overset{\sim}{\longrightarrow} \mathcal{Z}'$ over $\mathcal{S}$ which induces the isomorphism $\mathcal{Z}_\lambda \overset{\sim}{\longrightarrow} \mathcal{Z}'_\lambda$ ([54, (3.14) (1)]).

The rest we have to prove is that $\mathcal{Z}$ is affine. Let Z_λ be the affine fine log scheme above. Because $\mathcal{I}_\lambda$ is nilpotent, we may assume that $\mathcal{I}_\lambda^2 = 0$. Let $\mathcal{J}$ be a coherent ideal sheaf of $\mathcal{O}_{\mathcal{Z}}$. By the proof in [45, III (3.7)] of Serre's theorem on the criterion of the affineness of a scheme, we have only to prove that $H^1(\overset{\circ}{\mathcal{Z}}, \mathcal{J}) = 0$ (the assumption "noetherianness" in [loc. cit.] is unnecessary). Consider the following exact sequence

$$0 \longrightarrow \mathcal{I}_\lambda\mathcal{J} \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}/\mathcal{I}_\lambda\mathcal{J} \longrightarrow 0.$$

Because Z_λ is affine, $H^1(\overset{\circ}{\mathcal{Z}}, \mathcal{J}/\mathcal{I}_\lambda\mathcal{J}) = H^1(\overset{\circ}{Z}_\lambda, \mathcal{J}/\mathcal{I}_\lambda\mathcal{J}) = 0$. Similarly, $H^1(\overset{\circ}{\mathcal{Z}}, \mathcal{I}_\lambda\mathcal{J}) = 0$. Hence $H^1(\overset{\circ}{\mathcal{Z}}, \mathcal{J}) = 0$. Hence we finish the proof. □

Let X be a smooth scheme over S_0 and let D and Z be relative SNCD's on X/S_0 which meets transversally over S_0. In §2.7 below, we investigate important properties of $R\epsilon_{(Y,M,N)/S*}(\mathcal{O}_{(Y,M)/S})$ for the case where $(Y, M) = (X, D \cup Z)$ and $(Y, N) = (X, Z)$.

2.4 Preweight-Filtered Restricted Crystalline and Zariskian Complexes

Let $(S, \mathcal{I}, \gamma)$ be a PD-scheme such that $\mathcal{O}_S$ is killed by a power of a prime number p and such that $\mathcal{I}$ is quasi-coherent. Set $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I})$. Let $\overset{\circ}{f}\colon X \longrightarrow S_0$ be a smooth morphism and D a relative SNCD on X over S_0. Let $f\colon (X, D) \longrightarrow S_0$ be the natural morphism of log schemes. By abuse of notation, we also denote by f the composite morphism $(X, D) \longrightarrow S_0 \overset{\subset}{\longrightarrow} S$.

The aim in this section is to construct two fundamental objects in $\mathrm{D}^+\mathrm{F}(Q^*_{X/S}(\mathcal{O}_{X/S}))$ and in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$ which we call the *preweight-filtered restricted crystalline complex* of $(X, D)/(S, \mathcal{I}, \gamma)$ and *preweight-filtered zariskian complex* of $(X, D)/(S, \mathcal{I}, \gamma)$, respectively. In fact, we construct these complexes in a more general setting.

As explained in §2.1, X has the fs log structure $M(D)$ defined by D. As in §2.1, we denote this log scheme by (X, D). Let $\Delta = \{D_\lambda\}_{\lambda\in\Lambda}$ be a decomposition of D by smooth components of D over S_0. Let $X = \bigcup_{i_0\in I_0} X_{i_0}$ be an open covering, where I_0 is a set. Set $D_{i_0} := D \cap X_{i_0}$ and $D_{(\lambda;i_0)} := D_\lambda \cap X_{i_0}$. Fix a total order on I_0 and let I be the category defined in §1.5. For an object $i = (i_0, \ldots, i_r) \in I$, set $X_i := \bigcap_{s=0}^r X_{i_s}$, $D_i := \bigcap_{s=0}^r D_{i_s}$ and $D_{(\lambda;i)} := \bigcap_{s=0}^r D_{(\lambda;i_s)}$. As explained in §1.6, we have two ringed topoi $((\widetilde{(X_\bullet, D_\bullet)/S})^{\log}_{\mathrm{Rcrys}}, Q^*_{(X_\bullet,D_\bullet)/S}(\mathcal{O}_{(X_\bullet,D_\bullet)/S}))$ and $(\widetilde{X_{\bullet}}_{\mathrm{zar}}, f_\bullet^{-1}(\mathcal{O}_S))$.

Thus we have the following datum:

(2.4.0.1): An open covering $X = \bigcup_{i_0\in I_0} X_{i_0}$ and the family $\{(X_i, D_i)\}_{i\in I}$ of log schemes which form a diagram of log schemes over the log scheme (X, D), which we denote by $(X_\bullet, D_\bullet)$. That is, $(X_\bullet, D_\bullet)$ is nothing but a contravariant functor

$$I^o \longrightarrow \{\text{smooth schemes with relative SNCD's over } S_0 \\ \text{which are augmented to } (X, D)\}$$

Assume that, for any element i_0 of I_0, there exists a smooth scheme $\mathcal{X}_{i_0}$ with a relative SNCD $\mathcal{D}_{i_0}$ on $\mathcal{X}_{i_0}$ over S such that there exists an admissible immersion

$$(X_{i_0}, D_{i_0}) \xrightarrow{\subset} (\mathcal{X}_{i_0}, \mathcal{D}_{i_0})$$

with respect to $\Delta_{i_0} := \{D_{(\lambda;i_0)}\}_{\lambda\in\Lambda}$. By (2.3.14), if $\{X_{i_0}\}_{i_0\in I_0}$ is an affine open covering of X, we can assume that $(\mathcal{X}_{i_0}, \mathcal{D}_{i_0})$ is, in fact, a lift of (X_{i_0}, D_{i_0}): $(\mathcal{X}_{i_0}, \mathcal{D}_{i_0}) \times_S S_0 = (X_{i_0}, D_{i_0})$.

We wish to construct the following object:

(2.4.0.2): A diagram $(X_\bullet, D_\bullet) \xrightarrow{\subset} (\mathcal{X}_\bullet, \mathcal{D}_\bullet)$ $(\bullet \in I)$ of admissible immersions into a diagram of smooth schemes with relative SNCD's over S with respect to $\Delta_\bullet$, where $\Delta_i := \{D_{(\lambda;i)}\}_{\lambda\in\Lambda_{X_i}}$ $(i \in I)$.

Let $\widetilde{\Delta}_{i_0} = \{\mathcal{D}_{(\lambda;i_0)}\}_{\lambda\in\Lambda_{X_{i_0}}}$ be a decomposition of $\mathcal{D}_{i_0}$ which is compatible with Δ_{i_0}: $\mathcal{D}_{i_0} = \bigcup_{\lambda\in\Lambda_{X_{i_0}}} \mathcal{D}_{(\lambda;i_0)}$ and $\mathcal{D}_{(\lambda;i_0)} \times_{\mathcal{X}_{i_0}} X_{i_0} = D_{(\lambda;i_0)}$ $(\forall\lambda \in \Lambda_{X_{i_0}})$. Let $i = (i_0, \ldots, i_r)$ be an object of I. Set $\mathcal{X}_{(i_\alpha,i)} := \mathcal{X}_{i_\alpha} \setminus (\overline{X}_{i_\alpha} \setminus X_i)$ $(0 \leq \alpha \leq r)$, where $\overline{X}_{i_\alpha}$ denotes the closure of X_{i_α} in $\mathcal{X}_{i_\alpha}$. Since $\overline{X}_{i_\alpha} \setminus X_i$ is a closed subscheme of $\mathcal{X}_{i_\alpha}$, $\mathcal{X}_{(i_\alpha,i)}$ is an open subscheme of $\mathcal{X}_{i_\alpha}$. It is easy to see that the morphism $X_i \xrightarrow{\subset} \mathcal{X}_{(i_\alpha,i)}$ is a closed immersion. Denote by $\mathcal{D}_{(\lambda;i_\alpha,i)}$ (resp. $\mathcal{D}_{(i_\alpha,i)}$) the closed subscheme $\mathcal{D}_{(\lambda;i_\alpha)} \cap \mathcal{X}_{(i_\alpha,i)}$ (resp. $\mathcal{D}_{i_\alpha} \cap \mathcal{X}_{(i_\alpha,i)}$) of $\mathcal{X}_{(i_\alpha,i)}$. Set $\mathcal{X}'_i := \prod_{\alpha=0}^r{}_{S}\, \mathcal{X}_{(i_\alpha,i)}$. The closed immersions $X_i \xrightarrow{\subset} \mathcal{X}_{(i_\alpha,i)}$ $(\alpha = 0, \ldots, r)$ induce an immersion $X_i \xrightarrow{\subset} \mathcal{X}'_i$. Blow up $\mathcal{X}'_i$ along

$\bigcup_{\lambda\in\Lambda}\prod_{S}{}^{r}_{\alpha=0}\mathcal{D}_{(\lambda;i_\alpha,i)}$. Denote this scheme by $\mathcal{X}''_i$. We consider the complement $\mathcal{X}_i$ of the strict transform of

$$\bigcup_{\lambda\in\Lambda}\bigcup_{\beta=0}^{r}(\mathcal{X}_{(i_0,i)}\times_S\cdots\times_S\mathcal{X}_{(i_{\beta-1},i)}\times_S\mathcal{D}_{(\lambda;i_\beta,i)}\times_S\mathcal{X}_{(i_{\beta+1},i)}\times_S\cdots\times_S\mathcal{X}_{(i_r,i)})$$

in $\mathcal{X}''_i$. Let $\mathcal{D}_i$ be the exceptional divisor on $\mathcal{X}_i$. Then $\mathcal{D}_i$ is a relative SNCD on $\mathcal{X}_i$ by (2.4.2) below. Considering the strict transform of the image of X_i of the diagonal embedding in $\mathcal{X}'_i$, we have an immersion $X_i \overset{\subset}{\longrightarrow} \mathcal{X}_i$, in fact, an admissible immersion $(X_i, D_i) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{D}_i)$ with respect to Δ_i by (2.4.2) below. Let $\mathfrak{D}_i$ be the log PD-envelope of the immersion $(X_i, D_i) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{D}_i)$ over $(S, \mathcal{I}, \gamma)$ with structural morphism $f_i \colon \mathfrak{D}_i \longrightarrow S$.

First we give the local description of $\mathcal{O}_{\mathcal{X}_i}$ at a point of D_i (cf. [47, 2], [48, (1.7)], [64, 3.4]) for the warm up for the general description of $\mathcal{O}_{\mathcal{X}_i}$ in (2.4.2) below.

Lemma 2.4.1. *Let $i = (i_0, \ldots, i_r)$ be an element of I. Then, Zariski locally at the image of a point of D_i in $\mathcal{X}_i$, the structure sheaf $\mathcal{O}_{\mathcal{X}_i}$ of $\mathcal{X}_i$ is etale over the following sheaf of rings*

$$\mathcal{O}_S[x_1^{(i_\alpha)}, \ldots, x_{d_{i_\alpha}}^{(i_\alpha)} \mid 0 \le \alpha \le r][u_t^{(i_\alpha i_0)\pm 1} \mid 1 \le \alpha \le r, 1 \le t \le s]/$$

$$(x_t^{(i_\alpha)} - u_t^{(i_\alpha i_0)}x_t^{(i_0)} \mid 1 \le \alpha \le r, 1 \le t \le s),$$

where $x_1^{(i_\alpha)}, \ldots, x_{d_{i_\alpha}}^{(i_\alpha)}$ $(0 \le \alpha \le r)$ and $u_1^{(i_\alpha i_0)}, \ldots, u_s^{(i_\alpha i_0)}$ $(1 \le \alpha \le r)$ are independent variables over $\mathcal{O}_S$ and s is a positive integer. The exceptional divisor $\mathcal{D}_i$ is defined by an equation $x_1^{(i_0)}\cdots x_s^{(i_0)} = 0$.

Proof. The problem is etale local. We may assume that there exists an isomorphism $\mathcal{X}_{i_\alpha} \overset{\sim}{\longrightarrow} \underline{\mathrm{Spec}}_S(\mathcal{O}_S[x_1^{(i_\alpha)}, \ldots, x_{d_{i_\alpha}}^{(i_\alpha)}])$. Assume, furthermore, that $\mathcal{D}_{(i_\alpha,i)}$ is defined by an equation $x_1^{(i_\alpha)}\cdots x_s^{(i_\alpha)} = 0$ $(1 \le s \le \min\{d_{i_\alpha} \mid 0 \le \alpha \le r\})$. Here a positive integer s is independent of α.

Set $\mathcal{A} := \mathcal{O}_S[x_1^{(i_\alpha)}, \ldots, x_{d_{i_\alpha}}^{(i_\alpha)} \mid 0 \le \alpha \le r]$. Let $\mathcal{I}_t \subset \mathcal{A}$ $(1 \le t \le s)$ be the ideal sheaf of a closed subscheme

$$(x_t^{(i_0)} = 0) \cap (x_t^{(i_1)} = 0) \cap \cdots \cap (x_t^{(i_r)} = 0).$$

Set $\mathcal{U}_0 := \mathcal{X}'_i$ and let $\mathcal{U}_t$ $(1 \le t \le s)$ be a scheme defined inductively as follows: $\mathcal{U}_t$ is the blowing up of $\mathcal{U}_{t-1}$ with respect to the ideal sheaf $\mathcal{I}_t\mathcal{O}_{\mathcal{U}_{t-1}}$. Then, by [77, (5.1.2) (v)], $\mathcal{U}_s = \mathcal{X}''_i$. By the construction of $\mathcal{U}_s$, $\mathcal{U}_s$ is covered by the following spectrums over S of the following sheaves of rings:

$$\mathcal{A}[u_1^{(i_{\beta_1})}/u_1^{(i_{\alpha_1})}, \ldots, u_s^{(i_{\beta_s})}/u_s^{(i_{\alpha_s})} \mid 0 \le \beta_1, \ldots, \beta_s \le r]/$$

$$(x_1^{(i_{\beta_1})} - (u_1^{(i_{\beta_1})}/u_1^{(i_{\alpha_1})})x_1^{(i_{\alpha_1})}, \ldots, x_s^{(i_{\beta_s})} - (u_s^{(i_{\beta_1})}/u_s^{(i_{\alpha_s})})x_s^{(i_{\alpha_s})}) \quad (0 \le \alpha_1, \ldots, \alpha_s \le r).$$

Since the following equations

$$x_t^{(i_\beta)} \neq 0 \quad (1 \leq t \leq s, 0 \leq \forall\beta \leq r)$$

are equivalent to

$$u_t^{(i_\beta)} \neq 0, \quad x_t^{(i_\alpha)} \neq 0 \quad (1 \leq t \leq s, 0 \leq \forall\beta \leq r),$$

$\mathcal{O}_{\mathcal{X}_i}$ is isomorphic to

$$\mathcal{O}_S[x_1^{(i_\alpha)}, \ldots, x_{d_{i_\alpha}}^{(i_\alpha)} \mid 0 \leq \alpha \leq r][u_t^{(i_\beta i_\alpha)\pm 1} \mid 0 \leq \alpha \neq \beta \leq r,\ 1 \leq t \leq s]/$$

$$(x_t^{(i_\beta)} - u_t^{(i_\beta i_\alpha)} x_t^{(i_\alpha)}, u_t^{(i_\beta i_\alpha)} u_t^{(i_\alpha i_\beta)} - 1, u_t^{(i_\gamma i_\alpha)} - u_t^{(i_\gamma i_\beta)} u_t^{(i_\beta i_\alpha)}$$
$$\mid 0 \leq \alpha \neq \beta \neq \gamma \neq \alpha \leq r,\ 1 \leq t \leq s).$$

The last sheaf of rings is isomorphic to

$$\mathcal{O}_S[x_1^{(i_\alpha)}, \ldots, x_{d_{i_\alpha}}^{(i_\alpha)} \mid 0 \leq \alpha \leq r][u_t^{(i_\alpha i_0)\pm 1} \mid 1 \leq \alpha \leq r,\ 1 \leq t \leq s]/$$

$$(x_t^{(i_\alpha)} - u_t^{(i_\alpha i_0)} x_t^{(i_0)} \mid 1 \leq \alpha \leq r,\ 1 \leq t \leq s).$$

Now the claim on the exceptional divisor is obvious. □

We think that the reader is ready to read the following theorem which tells us that $(X_i, D_i) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{D}_i)$ is, indeed, an admissible immersion with respect to Δ_i.

Theorem 2.4.2. *Fix $i = (i_0, \ldots, i_r) \in I$. Let $\mathcal{A} := \boxtimes_{\alpha=0}^r \mathcal{O}_{\mathcal{X}_{(i_\alpha,i)}}$ be the structure sheaf of $\mathcal{X}_i'$. Set $\Lambda_i := \{\lambda \in \Lambda \mid \mathcal{D}_{(\lambda;i_\alpha,i)} \neq \emptyset\ (0 \leq \forall\alpha \leq r)\}$. (Then we have $\Lambda_i = \Lambda_{X_i}$.)*

Let $\mathcal{J}_{(\lambda;i_\alpha,i)}$ $(\lambda \in \Lambda_i)$ be the ideal sheaf of $\mathcal{O}_{\mathcal{X}_{(i_\alpha,i)}}$ defining the closed immersion $\mathcal{D}_{(\lambda;i_\alpha,i)} \overset{\subset}{\longrightarrow} \mathcal{X}_{(i_\alpha,i)}$. Let $\mathcal{X}_{(i_\alpha,i)} = \cup_{\mu_{(i_\alpha,i)}} \mathcal{X}_{\mu_{(i_\alpha,i)}}$ be an open covering of $\mathcal{X}_{(i_\alpha,i)}$ such that the restriction of $\mathcal{J}_{(\lambda;i_\alpha,i)}$ to $\mathcal{X}_{\mu_{(i_\alpha,i)}}$ is generated by a local section $x_\lambda^{(\mu_{(i_\alpha,i)})}$ for all $\lambda \in \Lambda_i$ (such open covering exists by the commutative diagram (2.1.7.2) *for $(\mathcal{X}_{(i_\alpha,i)}, \mathcal{D}_{(i_\alpha,i)})$). Set*

$$\begin{aligned}\Lambda_i^{(r)} &:= \Lambda_i^{(r)}(\mu_{(i_0,i)}, \ldots, \mu_{(i_r,i)}) \\ &:= \{\lambda \in \Lambda_i \mid \mathcal{D}_{(\lambda;i_\alpha,i)} \cap \mathcal{X}_{\mu_{(i_\alpha,i)}} \neq \emptyset,\ (0 \leq \forall\alpha \leq r)\}.\end{aligned}$$

Then $\mathcal{X}_i$ is covered by the spectrums over S of the following sheaves of rings

$$\mathcal{A}[(u_\lambda^{(\mu_{(i_\alpha,i)}\mu_{(i_0,i)})})^{\pm 1} \mid \lambda \in \Lambda_i^{(r)}, 1 \leq \alpha \leq r]/$$

$$(x_\lambda^{(\mu_{(i_\alpha,i)})} - u_\lambda^{(\mu_{(i_\alpha,i)}\mu_{(i_0,i)})} x_\lambda^{(\mu_{(i_0,i)})} \mid \lambda \in \Lambda_i^{(r)}) \quad (\mu_{(i_0,i)}, \ldots, \mu_{(i_r,i)}).$$

Here $u_{\lambda}^{(\mu_{(i_\alpha,i)}\mu_{(i_0,i)})}$*'s are independent variables. The exact locally closed immersion* $(X_i, D_i) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{D}_i)$ *is an admissible immersion with respect to* $\{D_{(\lambda;i)}\}_{\lambda\in\Lambda_i}$.

Proof. We have the restriction

$$(X_i, D_i) \overset{\subset}{\longrightarrow} (\mathcal{X}_{(i_\alpha,i)}, \bigcup_{\lambda\in\Lambda_i} \mathcal{D}_{(\lambda;i_\alpha,i)})$$

of the admissible immersion $(X_{i_\alpha}, D_{i_\alpha}) \overset{\subset}{\longrightarrow} (\mathcal{X}_{i_\alpha}, \mathcal{D}_{i_\alpha})$ with respect to Δ_{i_α} $(0 \leq \alpha \leq r)$.

Set

$$M(\lambda) := \{(\mu_{(i_0,i)}, \ldots, \mu_{(i_r,i)}) \mid \mathcal{D}_{(\lambda;i_\alpha)} \cap \mathcal{X}_{\mu_{(i_\alpha,i)}} \neq \emptyset \ (0 \leq \forall\alpha \leq r)\}$$

and let $M_1(\lambda)$ be the set of the $\mu_{(i_s,i)}$'s $(0 \leq s \leq r)$ appearing in an element of $M(\lambda)$. Then, by [77, (5.1.2) (v)], $\mathcal{X}''_i$ is covered by the spectrums over S of the following sheaves of rings:

$$\mathcal{A}[u_{\lambda}^{(\mu_{(i_{\beta_\lambda},i)})}/u_{\lambda}^{(\mu_{(i_{\alpha_\lambda},i)})} \mid 0 \leq \beta_\lambda \leq r, \lambda \in \Lambda_i^{(r)}, \mu_{(i_{\beta_\lambda},i)} \in M_1(\lambda)]/$$

$$(x_{\lambda}^{(\mu_{(i_{\beta_\lambda},i)})} - (u_{\lambda}^{(\mu_{(i_{\beta_\lambda},i)})}/u_{\lambda}^{(\mu_{(i_{\alpha_\lambda},i)})})x_{\lambda}^{(\mu_{(i_{\alpha_\lambda},i)})}) \quad (0 \leq \alpha_\lambda \leq r, \mu_{(i_{\alpha_\lambda},i)} \in M_1(\lambda)).$$

Since the following equations

$$x_{\lambda}^{(\mu_{(i_\alpha,i)})} \neq 0 \quad (0 \leq \forall\alpha \leq r)$$

are equivalent to

$$u_{\lambda}^{(\mu_{(i_\alpha,i)})} \neq 0, \quad x_{\lambda}^{(\mu_{(i_\alpha,i)})} \neq 0 \quad (0 \leq \forall\alpha \leq r),$$

$\mathcal{X}_i$ is covered by the spectrums of the quotient sheaves of

$$\mathcal{A}[(u_{\lambda}^{(\mu_{(i_\beta,i)}\mu_{(i_\alpha,i)})})^{\pm 1} \mid 0 \leq \alpha \neq \beta \leq r, \ \lambda \in \Lambda_i^{(r)}, \mu_{(i_\alpha,i)}, \mu_{(i_\beta,i)} \in M_1(\lambda)]$$

divided by ideal sheaves generated by

$$x_{\lambda}^{(\mu_{(i_\beta,i)})} - u_{\lambda}^{(\mu_{(i_\beta,i)}\mu_{(i_\alpha,i)})} x_{\lambda}^{(\mu_{(i_\alpha,i)})},$$

$$u_{\lambda}^{(\mu_{(i_\alpha,i)}\mu_{(i_\beta,i)})} u_{\lambda}^{(\mu_{(i_\beta,i)},\mu_{(i_\alpha,i)})} - 1,$$

and

$$u_{\lambda}^{(\mu_{(i_\gamma,i)}\mu_{(i_\alpha,i)})} - u_{\lambda}^{(\mu_{(i_\gamma,i)}\mu_{(i_\beta,i)})} u_{\lambda}^{(\mu_{(i_\beta,i)}\mu_{(i_\alpha,i)})}$$

$$(0 \leq \alpha \neq \beta \neq \gamma \neq \alpha \leq r, \ \lambda \in \Lambda_i^{(r)}, \mu_{(i_\alpha,i)}, \mu_{(i_\beta,i)}, \mu_{(i_\gamma,i)} \in M_1(\lambda)).$$

This quotient sheaf is isomorphic to

$$\mathcal{A}[(u_\lambda^{(\mu_{(i_\alpha,i)}\mu_{(i_0,i)})})^{\pm 1} \mid \lambda \in \Lambda_i^{(r)}]/$$

$$(x_\lambda^{(\mu_{(i_\alpha,i)})} - u_\lambda^{(\mu_{(i_\alpha,i)}\mu_{(i_0,i)})} x_\lambda^{(\mu_{(i_0,i)})} \mid \lambda \in \Lambda_i^{(r)}).$$

Let $\mathcal{D}_{(\lambda;i)}$ be the strict transform of $\prod_{\alpha=0}^r \mathcal{D}_{(\lambda;i_\alpha,i)}$ in $\mathcal{X}_i$. Now we see that, for $\lambda \in \Lambda_i^{(r)}$, the intersection of $\mathcal{D}_{(\lambda;i)}$ and the inverse image of $\prod_{\alpha=0}^{r}{}_{S}\mathcal{X}_{\mu_{(i_\alpha,i)}}$ in $\mathcal{X}_i$ is defined by an equation $x_\lambda^{(\mu_{(i_0,i)})} = 0$. Hence $\mathcal{D}_{(\lambda;i)}$ is a smooth divisor on $\mathcal{X}_i$ over S and $\mathcal{D}_i$ is a relative SNCD on $\mathcal{X}_i$ over S, and $\mathcal{D}_{(\lambda;i)} \times_{\mathcal{X}_i} X_i = D_{(\lambda;i)}$.

Therefore we obtain (2.4.2). □

Now we change notations. Let X be a smooth scheme and let D and Z be transversal relative SNCD's on X/S_0. Let $\Delta_D := \{D_\lambda\}_\lambda$ (resp. $\Delta_Z := \{Z_\mu\}_\mu$) be a decomposition of D (resp. Z) by smooth components of D (resp. Z). Then Δ_D and Δ_Z give a decomposition $\Delta := \{D_\lambda, Z_\mu\}_{\lambda,\mu}$ of $D \cup Z$ by smooth components of $D \cup Z$. We can construct the objects in (2.4.0.1) and (2.4.0.2) for $D \cup Z$ and Δ: $(X_\bullet, D_\bullet \cup Z_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)$. Let $\mathfrak{D}_i$ be the log PD-envelope of the admissible immersion $(X_i, Z_i) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{Z}_i)$ with respect to $\Delta_Z|_{X_i}$. Set $Z_i|_{D_i^{(k)}} := Z_i \cap D_i^{(k)}$ and $\mathcal{Z}_i|_{\mathcal{D}_i^{(k)}} := \mathcal{Z}_i \cap \mathcal{D}_i^{(k)}$ ($k \in \mathbb{N}$), where $\mathcal{D}_i^{(k)}$ is a scheme over S defined in (2.2.13.2) for $\mathcal{D}_i$.

Lemma 2.4.3. *The log scheme $\mathfrak{D}_i \times_{(\mathcal{X}_i,\mathcal{Z}_i)} (\mathcal{D}_i^{(k)}, \mathcal{Z}_i|_{\mathcal{D}_i^{(k)}})$ is the log PD-envelope of the locally closed immersion $(D_i^{(k)}, Z_i|_{D_i^{(k)}}) \longrightarrow (\mathcal{D}_i^{(k)}, \mathcal{Z}_i|_{\mathcal{D}_i^{(k)}})$.*

Proof. (2.4.3) is a special case of (2.2.16) (2). □

Let $\{P_k^{\mathcal{D}_i}\}_{k\in\mathbb{Z}}$ be the filtration on $\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i))$ defined in (2.2.15.2). As in §2.2, we set

$$P_k^{D_i} L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i))) := L_{(X_i,Z_i)/S}(P_k^{\mathcal{D}_i}\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i))),$$

$$P_k^{\mathcal{D}_i}(\mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}}\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{D}_i)) := \mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}} P_k^{\mathcal{D}_i}\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i)).$$

By (2.2.17) (1) and (2), we have two filtered complexes

$$(Q^*_{(X_i,Z_i)/S} L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i))), Q^*_{(X_i,Z_i)/S} P^{D_i}) \in \mathrm{C^+F}(\mathcal{O}_{(X_i,Z_i)/S}),$$

$$(\mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}}\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i)), P^{\mathcal{D}_i}) \in \mathrm{C^+F}(f_i^{-1}(\mathcal{O}_S)).$$

Lemma 2.4.4. *For a morphism $\alpha\colon i \longrightarrow j$ in I, let $\underline{\alpha}\colon (\mathcal{X}_j, \mathcal{D}_j \cup \mathcal{Z}_j) \longrightarrow (\mathcal{X}_i, \mathcal{D}_i \cup \mathcal{Z}_i)$ be the natural morphism. Then $\{(\mathcal{X}_i, \mathcal{D}_i \cup \mathcal{Z}_i), \underline{\alpha}\}_{i\in I,\alpha\in \mathrm{Mor}(I)}$ defines a diagram of smooth schemes with relative SNCD's over S :*

$$I^o \longrightarrow \{\textit{smooth schemes with relative SNCD's over } S\},$$

that is, for another morphism $\beta\colon j \longrightarrow l$ *in* I, $\underline{\alpha}\circ\underline{\beta} = \underline{\beta\circ\alpha}$, *and* $\underline{\mathrm{id}}_i = \mathrm{id}$. *Moreover,* $\{\mathfrak{D}_i\}_{i\in I}$ *is a diagram of log schemes. In particular, there are natural morphisms*

$$\begin{aligned}\rho_\alpha\colon \underline{\alpha}^{-1}(Q^*_{(X_i,Z_i)/S}P_k^{D_i}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i\cup\mathcal{Z}_i))))\\ \longrightarrow Q^*_{(X_j,Z_j)/S}P_k^{D_j}L_{(X_j,Z_j)/S}(\Omega^\bullet_{\mathcal{X}_j/S}(\log(\mathcal{D}_j\cup\mathcal{Z}_j))),\\ \rho_\alpha\colon \underline{\alpha}^{-1}(P_k^{\mathcal{D}_i}(\mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}}\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i\cup\mathcal{Z}_i))))\\ \longrightarrow P_k^{\mathcal{D}_j}(\mathcal{O}_{\mathfrak{D}_j}\otimes_{\mathcal{O}_{\mathcal{X}_j}}\Omega^\bullet_{\mathcal{X}_j/S}(\log(\mathcal{D}_j\cup\mathcal{Z}_j)))\end{aligned}$$

such that $\rho_{\mathrm{id}_i} = \mathrm{id}$ *and* $\rho_{\beta\circ\alpha} = \rho_\beta\circ\underline{\beta}^{-1}(\rho_\alpha)$.

Proof. The open immersion $X_j \overset{\subset}{\longrightarrow} X_i$ induces a morphism $\mathcal{X}'_j \longrightarrow \mathcal{X}'_i$. By the universality of the blow ups, we have a morphism $\mathcal{X}''_j \longrightarrow \mathcal{X}''_i$ and this morphism induces morphisms $(\mathcal{X}_j,\mathcal{D}_j\cup\mathcal{Z}_j) \longrightarrow (\mathcal{X}_i,\mathcal{D}_i\cup\mathcal{Z}_i)$, $(\mathcal{X}_j,\mathcal{D}_j) \longrightarrow (\mathcal{X}_i,\mathcal{D}_i)$ and $(\mathcal{X}_j,\mathcal{Z}_j) \longrightarrow (\mathcal{X}_i,\mathcal{Z}_i)$. The universality of the log PD-envelope induces a morphism $\mathfrak{D}_j \longrightarrow \mathfrak{D}_i$. Thus (2.4.4) follows. □

By (2.4.4), we obtain a complex

$$(Q^*_{(X_i,Z_i)/S}P_k^{D_i}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i\cup\mathcal{Z}_i))))_{i\in I} \in \mathrm{C}^+(Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S}))$$

of $Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S})$-modules and a complex

$$(P_k^{\mathcal{D}_i}(\mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}}\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i\cup\mathcal{Z}_i))))_{i\in I} \in \mathrm{C}^+(f_\bullet^{-1}(\mathcal{O}_S))$$

of $f_\bullet^{-1}(\mathcal{O}_S)$-modules. Now we have the following filtered complex of $Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S})$-modules and the following filtered complex of $f_\bullet^{-1}(\mathcal{O}_S)$-modules:

$$(C^{\log,Z_\bullet}_{\mathrm{Rcrys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}) :=$$

$$(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i\cup\mathcal{Z}_i))),Q^*_{(X_i,Z_i)/S}P^{D_i})_{i\in I}$$

and

$$(C^{\log,Z_\bullet}_{\mathrm{zar}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}) := (\mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}}\Omega^\bullet_{\mathcal{X}_i/S}(\log\mathcal{D}_i),P^{\mathcal{D}_i})_{i\in I}.$$

Remark 2.4.5. Once we are given the data (2.4.0.1) and (2.4.0.2) for $(X,D\cup Z)$ with respect to $\Delta = \{D_\lambda,Z_\mu\}_{\lambda,\mu}$, we can obtain two filtered complexes

$$(C^{\log,Z_\bullet}_{\mathrm{Rcrys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}) \text{ and } (C^{\log,Z_\bullet}_{\mathrm{zar}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}),P^{D_\bullet}).$$

Let

(2.4.5.1) $$\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}}\colon (((\widetilde{X_\bullet,Z_\bullet})/S)^{\log}_{\mathrm{Rcrys}},Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S})) \longrightarrow$$

$$(((\widetilde{X,Z)/S})^{\log}_{\mathrm{Rcrys}}, Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$$

and

$$\pi_{\mathrm{zar}} \colon (\widetilde{X_{\bullet}}_{\mathrm{zar}}, f_{\bullet}^{-1}(\mathcal{O}_S)) \longrightarrow (\widetilde{X}_{\mathrm{zar}}, f^{-1}(\mathcal{O}_S)) \tag{2.4.5.2}$$

be natural morphisms of ringed topoi defined in §1.5 and §1.6.

Definition 2.4.6. Assume that we are given the data (2.4.0.1) and (2.4.0.2) for $(X, D \cup Z)$ with respect to $\Delta = \{D_\lambda, Z_\mu\}_{\lambda,\mu}$.

(1) We call

(2.4.6.1)
$$R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}(C^{\log,Z_\bullet}_{\mathrm{Rcrys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}), P^{D_\bullet}) \in \mathrm{D}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$$

the *preweight-filtered restricted crystalline complex* of $\mathcal{O}_{(X,D\cup Z)/S}$ (or $(X, D\cup Z)/S$) with respect to D. We denote it by $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$. If $Z = \emptyset$, then we call $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ the *preweight-filtered restricted crystalline complex* of $\mathcal{O}_{(X,D)/S}$ or $(X, D)/S$ and we denote it by $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$.

(2) We call

$$R\pi_{\mathrm{zar}*}(C^{\log,Z_\bullet}_{\mathrm{zar}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}), P^D) \in \mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S)) \tag{2.4.6.2}$$

the *preweight-filtered zariskian complex* of $\mathcal{O}_{(X,D\cup Z)/S}$ (or $(X, D\cup Z)/S$) with respect to D. We denote it by $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$. If $Z = \emptyset$, then we call $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ the *preweight-filtered zariskian complex* of $\mathcal{O}_{(X,D)/S}$ or $(X, D)/S$ and we denote it by $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$.

Let

$$\begin{aligned}(2.4.6.3)\quad \epsilon_{(X,D\cup Z,Z)/S} \colon (((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}}, &\mathcal{O}_{(X,D\cup Z)/S}) \\ &\longrightarrow (((\widetilde{X,Z)/S})^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X,Z)/S})\end{aligned}$$

be the forgetting log morphism along D ((2.3.2)) and let

(2.4.6.4)
$$u_{(X,D\cup Z)/S} \colon (((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X,D\cup Z)/S}) \longrightarrow (\widetilde{X}_{\mathrm{zar}}, f^{-1}(\mathcal{O}_S))$$

be the canonical projection ((1.6.0.8)).

Proposition 2.4.7. *There exists the following canonical isomorphisms*

$$Q^*_{(X,Z)/S} R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) \xrightarrow{\sim} C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), \tag{2.4.7.1}$$

(2.4.7.2)
$$R\overline{u}_{(X,Z)/S*}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})) \xrightarrow{\sim} Ru_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}),$$

(2.4.7.3) $$Ru_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) \xrightarrow{\sim} C^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}).$$

Proof. Let

(2.4.7.4) $$\pi^{\log}_{(X,D\cup Z)/S\mathrm{crys}} \colon ((X_\bullet, \widetilde{D_\bullet \cup Z_\bullet})/S)^{\log}_{\rm crys} \longrightarrow ((X, \widetilde{D\cup Z})/S)^{\log}_{\rm crys}$$

and

(2.4.7.5) $$\pi^{\log}_{(X,D\cup Z)/S\mathrm{Rcrys}} \colon ((X_\bullet, \widetilde{D_\bullet \cup Z_\bullet})/S)^{\log}_{\rm Rcrys} \longrightarrow ((X, \widetilde{D\cup Z})/S)^{\log}_{\rm Rcrys}$$

be natural morphisms of topoi defined in §1.6.

The isomorphism (2.4.7.1) follows from the cohomological descent [42, $\mathrm{V}^{\rm bis}$], (2.3.2.2), (2.3.10.1), (1.6.4.1) and the definition of $C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S})$. Indeed, the left hand side of (2.4.7.1) is equal to

$$\begin{aligned}
&Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}R\pi^{\log}_{(X,D\cup Z)/S\mathrm{crys}*}\pi^{\log,-1}_{(X,D\cup Z)/S\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})\\
=&Q^*_{(X,Z)/S}R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}R\epsilon_{(X_\bullet,D_\bullet\cup Z_\bullet,Z_\bullet)/S*}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S})\\
=&Q^*_{(X,Z)/S}R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)))\\
=&R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)))\\
=&C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}).
\end{aligned}$$

By the trivially filtered case of (1.6.3.1) and by (2.4.7.1),

$$\begin{aligned}
R\overline{u}_{(X,Z)/S*}(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}) &= Ru_{(X,Z)/S*}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})\\
&= Ru_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}).
\end{aligned}$$

(2.4.7.3) is a special case of [46, (2.20)], which follows from the cohomological descent. □

Remark 2.4.8. (1) In the next section we shall prove that

$$(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \in \mathrm{D}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$$

and

$$(C^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \in \mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$$

are independent of the data (2.4.0.1) and (2.4.0.2) for $(X, D\cup Z)$ if we fix a decomposition of D and Z by their smooth components, and then, in §2.7, we shall prove that they are independent of the choice of the decompositions of D and Z by their smooth components. Once we know that the definitions of $(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ and $(C^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ are well-defined, we know that

(2.4.8.1)
$$R\overline{u}_{(X,Z)/S*}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) = (C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$$

by the constructions of them.

(2) The complex $C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})$ in (2.4.6) is different from that defined in [46, (2.19)]. Because the latter depends on an embedding system of (X, D), it should be called a *crystalline complex with respect to an embedding system*.

2.5 Well-Definedness of the Preweight-Filtered Restricted Crystalline and Zariskian Complexes

In this section we prove that the preweight-filtered restricted crystalline complex

$$(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \in \mathrm{D}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$$

in (2.4.6.1) and the preweight-filtered zariskian complex

$$(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \in \mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$$

in (2.4.6.2) are independent of the data (2.4.0.1) and (2.4.0.2). To prove this independence, we need not make local explicit calculations of PD-envelopes; the notion of the admissible immersion enables us to use the classical crystalline Poincaré lemma implicitly; see (2.5.1), (2.5.2) and (2.5.3) below for the detail.

Let $S_0 \overset{\subset}{\longrightarrow} S$ be a PD-closed immersion defined by a quasi-coherent ideal sheaf $\mathcal{I}$. Let $(X, D\cup Z)$, Δ_D, Δ_Z and Δ be as in the previous section. Consider the following commutative diagram

$$\begin{array}{ccc}
(X, D\cup Z) & \overset{\subset}{\longrightarrow} & (\mathcal{X}_1, \mathcal{D}_1 \cup \mathcal{Z}_1) \\
\| & & \downarrow \\
(X, D\cup Z) & \overset{\subset}{\longrightarrow} & (\mathcal{X}_2, \mathcal{D}_2 \cup \mathcal{Z}_2),
\end{array}$$

where the horizontal morphisms above are admissible immersions with respect to a decomposition Δ; assume that the horizontal morphisms induce admissible immersions $(X, D) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{D}_i)$ with respect to Δ_D and $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{Z}_i)$ with respect to Δ_Z $(i = 1, 2)$. Let $\mathfrak{D}_i$ $(i = 1, 2)$ be the log PD-envelope of the admissible immersion $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}_i, \mathcal{Z}_i)$. Then the following holds:

Lemma 2.5.1. *The induced morphisms*

(2.5.1.1)
$$(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^\bullet_{\mathcal{X}_2/S}(\log(\mathcal{D}_2 \cup \mathcal{Z}_2))), Q^*_{(X,Z)/S}P^D)$$
$$\longrightarrow (Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^\bullet_{\mathcal{X}_1/S}(\log(\mathcal{D}_1 \cup \mathcal{Z}_1))), Q^*_{(X,Z)/S}P^D),$$

(2.5.1.2)
$$(\mathcal{O}_{\mathfrak{D}_2}\otimes_{\mathcal{O}_{\mathcal{X}_2}}\Omega^{\bullet}_{\mathcal{X}_2/S}(\log(\mathcal{D}_2\cup\mathcal{Z}_2)),P^D)$$
$$\longrightarrow(\mathcal{O}_{\mathfrak{D}_1}\otimes_{\mathcal{O}_{\mathcal{X}_1}}\Omega^{\bullet}_{\mathcal{X}_1/S}(\log(\mathcal{D}_1\cup\mathcal{Z}_1)),P^D)$$

are filtered quasi-isomorphisms.

Proof. Apply the gr-functor $\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^D}$ ($k\in\mathbb{N}$) to (2.5.1.1). Then, by (2.2.21.2), we obtain the following morphism:

$$\begin{aligned}&\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^D}\{(2.5.1.1)\}:\\
&\quad Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_2/S}(\log\mathcal{Z}_2|_{\mathcal{D}^{(k)}_2})) \{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)\\
&\longrightarrow Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_1/S}(\log\mathcal{Z}_1|_{\mathcal{D}^{(k)}_1})) \{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z).\end{aligned}$$

Then $g_k:=\mathrm{gr}_k^{Q^*_{(X,Z)/S}P^D}\{(2.5.1.1)\}$ fits into the following commutative diagram:

$$\begin{aligned}&Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_2/S}(\log\mathcal{Z}_2|_{\mathcal{D}^{(k)}_2})) \{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)\\
&\qquad\qquad\Big\downarrow g_k\\
&Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}L^{(k)}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_1/S}(\log\mathcal{Z}_1|_{\mathcal{D}^{(k)}_1})) \{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)\\
&\longleftarrow\ Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)\\
&\qquad\qquad\Big\|\\
&\longleftarrow\ Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z),\end{aligned}$$

where the horizontal morphisms are quasi-isomorphisms. Hence g_k is also a quasi-isomorphism and so is (2.5.1.1).

Applying the filtered direct image $R\overline{u}_{(X,Z)/S*}$ to (2.5.1.1), we immediately see that (2.5.1.2) is a filtered quasi-isomorphism by the log version of [11, 5.27.2, (7.1.2)]. □

Remark 2.5.2. To compare our straight method with previous works, assume that $Z=\emptyset$ and consider two admissible immersions $(X,D)\xrightarrow{\subset}(\mathcal{X}_i,\mathcal{D}_i)$ ($i=1,2$) with respect to a decomposition $\Delta=\{D_\lambda\}_{\lambda\in\Lambda}$ of D by smooth components of D. As in §2.4, we make the following operation. Set $\mathcal{X}'_{12}:=\mathcal{X}_1\times_S\mathcal{X}_2$. Let $\mathcal{D}_i=\bigcup_{\lambda\in\Lambda}\mathcal{D}_{(\lambda;i)}$ ($i=1,2$) be the union of smooth components of $\mathcal{D}_i$. Blow up $\mathcal{X}'_{12}$ along $\bigcup_{\lambda\in\Lambda}(\mathcal{D}_{(\lambda;1)}\times_S\mathcal{D}_{(\lambda;2)})$. Let $\mathcal{X}_{12}$ be the complement of the strict transform of

$$\bigcup_{\lambda\in\Lambda}\{(\mathcal{D}_{(\lambda;1)}\times_S\mathcal{X}_2)\cup(\mathcal{X}_1\times_S\mathcal{D}_{(\lambda;2)})\}$$

in this blow up. Let $\mathcal{D}_{12}$ be the exceptional divisor on $\mathcal{X}_{12}$. By considering the strict transform of X in $\mathcal{X}_{12}$, we have an admissible immersion $(X,D) \xrightarrow{\subset} (\mathcal{X}_{12},\mathcal{D}_{12})$ with respect to Δ, and we have the following commutative diagram:

(2.5.2.1)
$$\begin{array}{ccc} (X,D) & \xrightarrow{\subset} & (\mathcal{X}_{12},\mathcal{D}_{12}) \\ \| & & \downarrow \\ (X,D) & \xrightarrow{\subset} & (\mathcal{X}_i,\mathcal{D}_i), \end{array}$$

Let $\mathfrak{D}_i$ $(i=1,2)$ and $\mathfrak{D}_{12}$ be the log PD-envelope of the admissible immersions $(X,D) \xrightarrow{\subset} (\mathcal{X}_i,\mathcal{D}_i)$ and $(X,D) \xrightarrow{\subset} (\mathcal{X}_{12},\mathcal{D}_{12})$, respectively.

Then the induced morphisms $(\mathcal{X}_{12},\mathcal{D}_{12}) \longrightarrow (\mathcal{X}_i,\mathcal{D}_i)$ $(i=1,2)$ induce morphisms of filtered complexes

(2.5.2.2)
$$(Q^*_{X/S}L_{X/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{D}_i)), Q^*_{X/S}P) \longrightarrow (Q^*_{X/S}L_{X/S}(\Omega^\bullet_{\mathcal{X}_{12}/S}(\log \mathcal{D}_{12})), Q^*_{X/S}P),$$

and

(2.5.2.3)
$$(\mathcal{O}_{\mathfrak{D}_i}\otimes_{\mathcal{O}_{\mathcal{X}_i}}\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{D}_i), P) \longrightarrow (\mathcal{O}_{\mathfrak{D}_{12}}\otimes_{\mathcal{O}_{\mathcal{X}_{12}}}\Omega^\bullet_{\mathcal{X}_{12}/S}(\log \mathcal{D}_{12}), P),$$

which are filtered quasi-isomorphisms by (2.5.1). Thus the proof for (2.5.2.3) gives a simpler proof of a filtered version of the last lemma in [47] (cf. [48, (1.7)], [64, 3.4]). Because we allow not only local lifts of (X,D) but also local admissible immersions in the constructions of $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$ and $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$, we can use the Poincaré lemma implicitly for the proof of the quasi-isomorphism (2.5.2.3). We can also use a complicated version of [64, 3.4] to prove that (2.5.2.3) is a filtered quasi-isomorphism; however we omit this proof because this proof is lengthy.

Next we prove that $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ and $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ are independent of the data (2.4.0.1) and (2.4.0.2) for $D \cup Z$ and Δ.

Let the notations be as in §2.4. Let $\{X_{i_0}\}_{i_0\in I_0}$ and $\{X_{j_0}\}_{j_0\in J_0}$ be two open coverings of X, where I_0 and J_0 are two sets. Let I and J be two sets in §1.5. By §1.6 we have a diagram of ringed topoi $(((\widetilde{X_{\bullet\bullet}, Z_{\bullet\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S})$ and $(\widetilde{X_{\bullet\bullet}}_{\mathrm{zar}}, f^{-1}_{\bullet\bullet}(\mathcal{O}_S))$.

Let i and j be arbitrary elements of I and J, respectively. For simplicity of notation, set $E := D\cup Z$. Let $\{D_\lambda\}_\lambda$ and $\{Z_\mu\}_\mu$ be decompositions of D and Z by smooth components of D and Z, respectively. Set $\Delta := \{E_\nu\}_\nu := \{D_\lambda, Z_\mu\}_{\lambda,\mu}$. Then Δ is a decomposition of E by smooth components of E. Assume that there exist two diagrams of admissible immersions $(X_i, E_i; \Delta|_{X_i})_{i\in I} \xrightarrow{\subset} (\mathcal{X}_i, \mathcal{E}_i; \widetilde{\Delta}_i)_{i\in I}$ and $(X_j, E_j; \Delta|_{X_j})_{j\in J} \xrightarrow{\subset} (\mathcal{X}_j, \mathcal{E}_j; \widetilde{\Delta}_j)_{j\in J}$ over S. Set $X_{ij} := X_i \cap X_j$ and $E_{ij} := E_i \cap E_j$. Let $\mathcal{X}_{(i,ij)} :=$

$\mathcal{X}_i \setminus (\overline{X}_i \setminus X_{ij})$ (resp. $\mathcal{X}_{(j,ij)} := \mathcal{X}_j \setminus (\overline{X}_j \setminus X_{ij})$) and set $\mathcal{X}'_{ij} := \mathcal{X}_{(i,ij)} \times_S \mathcal{X}_{(j,ij)}$. Then we have a locally closed immersion $X_{ij} \overset{\subset}{\longrightarrow} \mathcal{X}'_{ij}$. Set $\{\mathcal{E}_{(\nu;i)}\}_\nu := \widetilde{\Delta}_i$ and $\{\mathcal{E}_{(\nu;j)}\}_\nu := \widetilde{\Delta}_j$. Set also $\mathcal{E}_{(i,ij)} := \mathcal{E}_i \cap \mathcal{X}_{(i,ij)}$, $\mathcal{E}_{(j,ij)} := \mathcal{E}_j \cap \mathcal{X}_{(j,ij)}$, $\mathcal{E}_{(\nu;i,ij)} := \mathcal{E}_{(\nu;i)} \cap \mathcal{X}_{(i,ij)}$ and $\mathcal{E}_{(\nu;j,ij)} := \mathcal{E}_{(\nu;j)} \cap \mathcal{X}_{(j,ij)}$. Blow up $\mathcal{X}'_{ij}$ along $\bigcup_\nu (\mathcal{E}_{(\nu;i,ij)} \times_S \mathcal{E}_{(\nu;j,ij)})$. Let $\mathcal{X}''_{ij}$ be the resulting scheme. Let $\mathcal{X}_{ij}$ be the complement of the strict transform of

$$[\bigcup_\nu \mathcal{E}_{(\nu;i,ij)} \times_S \mathcal{X}_{(j,ij)})] \cup [\bigcup_\nu \mathcal{X}_{(i,ij)} \times_S \mathcal{E}_{(\nu;j,ij)}]$$

in $\mathcal{X}''_{ij}$. Let $\mathcal{E}_{ij}$ be the exceptional divisor on $\mathcal{X}_{ij}$. Then $\mathcal{E}_{ij}$ is a relative SNCD on $\mathcal{X}_{ij}$ by (2.4.2). Considering the strict transform of the image of X_{ij} in $\mathcal{X}_{ij}$, we have a locally closed immersion $X_{ij} \overset{\subset}{\longrightarrow} \mathcal{X}_{ij}$, in fact, an admissible immersion $(X_{ij}, E_{ij}) \overset{\subset}{\longrightarrow} (\mathcal{X}_{ij}, \mathcal{E}_{ij})$ by (2.4.2). Let $\{\mathcal{E}_{(\nu;ij)}\}_\nu$ be the resulting decomposition of $\mathcal{E}_{ij}$ by smooth components of $\mathcal{E}_{ij}$. We also have a relative SNCD $\mathcal{Z}_{ij}$ on $\mathcal{X}_{ij}/S$ by using Z instead of E. Let $\mathfrak{D}_{ij}$ be the log PD-envelope of the locally closed immersion $(X_{ij}, Z_{ij}) \overset{\subset}{\longrightarrow} (\mathcal{X}_{ij}, \mathcal{Z}_{ij})$.

Let

$$R\eta^{\log}_{\mathrm{Rcrys}*} \colon \mathrm{D}^+\mathrm{F}(Q^*_{(X_{\bullet\bullet}, Z_{\bullet\bullet})/S}(\mathcal{O}_{(X_{\bullet\bullet}, Z_{\bullet\bullet})/S})) \longrightarrow \mathrm{D}^+\mathrm{F}((Q^*_{(X_\bullet, Z_\bullet)/S}(\mathcal{O}_{(X_\bullet, Z_\bullet)/S}))_{\bullet \in I})$$

and

$$R\eta^{\log}_{i,\mathrm{Rcrys}*} \colon \mathrm{D}^+\mathrm{F}(Q^*_{(X_{i\bullet}, Z_{i\bullet})/S}(\mathcal{O}_{(X_{i\bullet}, Z_{i\bullet})/S})) \longrightarrow \mathrm{D}^+\mathrm{F}(Q^*_{(X_i, Z_i)/S}(\mathcal{O}_{(X_i, Z_i)/S}))$$

be the natural morphisms defined in (1.6.0.2) and (1.6.0.3), respectively.

Let

$$R\eta_{\mathrm{zar}*} \colon \mathrm{D}^+\mathrm{F}(f^{-1}_{\bullet\bullet}(\mathcal{O}_S)) \longrightarrow \mathrm{D}^+\mathrm{F}((f^{-1}_\bullet(\mathcal{O}_S))_{\bullet \in I}),$$
$$R\eta_{i,\mathrm{zar}*} \colon \mathrm{D}^+\mathrm{F}(f^{-1}_{i\bullet}(\mathcal{O}_S)) \longrightarrow \mathrm{D}^+\mathrm{F}(f^{-1}_i(\mathcal{O}_S))$$

be the natural morphisms defined in (1.6.0.6) and (1.6.0.7), respectively. Then we have the following:

Theorem 2.5.3.

(2.5.3.1)
$$\begin{aligned} &R\eta^{\log}_{\mathrm{Rcrys}*}(Q^*_{(X_{\bullet\bullet}, Z_{\bullet\bullet})/S} L_{(X_{\bullet\bullet}, Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{E}_{\bullet\bullet})), Q^*_{(X_{\bullet\bullet}, Z_{\bullet\bullet})/S} P^{D_{\bullet\bullet}}) \\ &\quad = (Q^*_{(X_\bullet, Z_\bullet)/S} L_{(X_\bullet, Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{E}_\bullet)), Q^*_{(X_\bullet, Z_\bullet)/S} P^{D_\bullet})_{\bullet \in I}. \end{aligned}$$

(2.5.3.2)
$$\begin{aligned} &R\eta_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_{\bullet\bullet}} \otimes_{\mathcal{O}_{\mathcal{X}_{\bullet\bullet}}} \Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{E}_{\bullet\bullet}), P^{D_{\bullet\bullet}}) \\ &\quad = (\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}_\bullet}} \Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{E}_\bullet), P^{D_\bullet})_{\bullet \in I}. \end{aligned}$$

Proof. Because (2.5.3.2) follows from (2.5.3.1) by (2.2.22) and by the commutative diagram (1.6.4.7), we have only to prove (2.5.3.1).

Let $\gamma_{ij}\colon X_{ij} \longrightarrow X_i$ $(i \in I, j \in J)$ be the natural morphism. Then

$$\eta^{\log}_{\mathrm{Rcrys}*}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{E}_{\bullet\bullet})), Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}P^{D_{\bullet\bullet}}) =$$

$$\mathrm{Ker}\{\prod_{j_0} \gamma_{\bullet j_0 \mathrm{Rcrys}*}(Q^*_{(X_{\bullet j_0},Z_{\bullet j_0})/S}L_{(X_{\bullet j_0},Z_{\bullet j_0})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet j_0}/S}(\log \mathcal{E}_{\bullet j_0})), Q^*_{(X_{\bullet j_0},Z_{\bullet j_0})/S}P^{D_{\bullet j_0}})$$

$$\longrightarrow \prod_{j_0<j_1} \gamma_{\bullet j_0 j_1 \mathrm{Rcrys}*}(Q^*_{(X_{\bullet j_0 j_1},Z_{\bullet j_0 j_1})/S}L_{(X_{\bullet j_0 j_1},Z_{\bullet j_0 j_1})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet j_0 j_1}/S}(\log \mathcal{E}_{\bullet j_0 j_1})),$$

$$Q^*_{(X_{\bullet j_0 j_1},Z_{\bullet j_0 j_1})/S}P^{D_{\bullet j_0 j_1}})\} \quad (j_0, j_1 \in J_0).$$

Thus there exists a natural composite morphism

$$\begin{aligned}&(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{E}_\bullet)), Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet})\\ &\longrightarrow \eta^{\log}_{\mathrm{Rcrys}*}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{E}_{\bullet\bullet})), Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}P^{D_{\bullet\bullet}})\\ &\longrightarrow R\eta^{\log}_{\mathrm{Rcrys}*}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{E}_{\bullet\bullet})), Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}P^{D_{\bullet\bullet}}).\end{aligned}$$

For $i \in I$, let

$$e_i\colon (((\widetilde{X_{i\bullet}, Z_{i\bullet}})/S)^{\log}_{\mathrm{Rcrys}}, \mathcal{O}_{(X_{i\bullet},Z_{i\bullet})/S}) \longrightarrow (((\widetilde{X_{\bullet\bullet}, Z_{\bullet\bullet}})/S)^{\log}_{\mathrm{Rcrys}}, \mathcal{O}_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S})$$

be a morphism defined in §1.5. Let $(I^\bullet_{\bullet\bullet}, \{(I^\bullet_{\bullet\bullet})_k\})$ be a filtered flasque resolution of $(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{E}_{\bullet\bullet})), Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}P^{D_{\bullet\bullet}})$ such that, for each i, $(I^\bullet_{i\bullet}, \{(I^\bullet_{i\bullet})_k\})$ is a filtered flasque resolution of

$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}).$$

Obviously we have $e_i^{-1}(\eta^{\log}_{\mathrm{Rcrys}*}(I^\bullet_{\bullet\bullet}, \{(I^\bullet_{\bullet\bullet})_k\}) = \eta^{\log}_{i,\mathrm{Rcrys}*}(I^\bullet_{i\bullet}, \{(I^\bullet_{i\bullet})_k\})$. Hence it suffices to prove that the morphism

$$(2.5.3.3) \quad \begin{aligned}&(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), Q^*_{(X_i,Z_i)/S}P^{D_i})\\ &\qquad\longrightarrow \eta^{\log}_{i,\mathrm{Rcrys}*}(I^\bullet_{i\bullet}, \{(I^\bullet_{i\bullet})_k\})\end{aligned}$$

is a filtered quasi-isomorphism. Henceforth we fix $i \in I$ in this proof.

If there exists a morphism $j' \longrightarrow j$ in J, then there exists the natural open immersion $(\mathcal{X}_{(i,ij)}, \mathcal{E}_{(i,ij)}) \overset{\subset}{\longrightarrow} (\mathcal{X}_{(i,ij')}, \mathcal{E}_{(i,ij')})$. By the definition of $\eta^{\log}_{i,\mathrm{Rcrys}}$, we obtain an equality

$$(2.5.3.4) \quad \eta^{\log,-1}_{i,\mathrm{Rcrys}}(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), Q^*_{(X_i,Z_i)/S}P^{D_i}) =$$

$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{(i,i\bullet)}/S}(\log \mathcal{E}_{(i,i\bullet)})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}).$$

Next, we construct two morphisms (2.5.3.5) and (2.5.3.6) below (cf. [47], [48, (1.7)], [64, 3.4]). Blow up $\mathcal{X}_{i\bullet} \times_S \mathcal{X}_{(i,i\bullet)}$ along $\bigcup_\nu (\mathcal{E}_{(\nu;i\bullet)} \times_S \mathcal{E}_{(\nu;i,i\bullet)})$. Let $\mathcal{W}_{i\bullet}$ be the complement of the strict transform of

$$\bigcup_{\nu}((\mathcal{E}_{(\nu;i\bullet)} \times_S \mathcal{X}_{(i,i\bullet)}) \cup (\mathcal{X}_{i\bullet} \times_S \mathcal{E}_{(\nu;i,i\bullet)}))$$

in this blowing up. Let $\mathcal{F}_{i\bullet}$ be the exceptional divisor on $\mathcal{W}_{i\bullet}$. By considering the strict transform of the image of $X_{i\bullet}$ in $\mathcal{W}_{i\bullet}$, we have a locally closed immersion $X_{i\bullet} \overset{\subset}{\longrightarrow} \mathcal{W}_{i\bullet}$.

The two projections $\mathcal{W}_{i\bullet} \longrightarrow \mathcal{X}_{i\bullet}$ and $\mathcal{W}_{i\bullet} \longrightarrow \mathcal{X}_{(i,i\bullet)}$ induce two morphisms

(2.5.3.5)
$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \longrightarrow$$
$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{W}_{i\bullet}/S}(\log \mathcal{F}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}})$$

and

(2.5.3.6)
$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{(i,i\bullet)}/S}(\log \mathcal{E}_{(i,i\bullet)})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \longrightarrow$$
$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{W}_{i\bullet}/S}(\log \mathcal{F}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}).$$

Because there exists the following commutative diagram

$$\begin{CD}
(X_{ij}, E_{ij}) @>{\subset}>> (\mathcal{W}_{ij}, \mathcal{F}_{ij}) \\
@| @VVV \\
(X_{ij}, E_{ij}) @>{\subset}>> (\mathcal{X}_{ij}, \mathcal{E}_{ij})
\end{CD}$$

such that the horizontal arrows are admissible immersions, we see that (2.5.3.5) is a filtered quasi-isomorphism by (2.5.1). By the same proof, we see that (2.5.3.6) is a filtered quasi-isomorphism.

Now we can prove that (2.5.3.3) is a filtered quasi-isomorphism. Indeed, let $(J^{\bullet}_{i\bullet}, \{(J^{\bullet}_{i\bullet})_k\})$ be a filtered flasque resolution of

$$(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{W}_{i\bullet}/S}(\log \mathcal{F}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}).$$

Because (2.5.3.6) is a filtered quasi-isomorphism, so is the following composite morphism

$$\begin{aligned}
&\eta^{\log,-1}_{i,\mathrm{Rcrys}}(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^{\bullet}_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), Q^*_{(X_i,Z_i)/S}P^{D_{i\bullet}}) \\
&= (Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{(i,i\bullet)}/S}(\log \mathcal{E}_{(i,i\bullet)})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \\
&\longrightarrow (Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^{\bullet}_{\mathcal{W}_{i\bullet}/S}(\log \mathcal{F}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \\
&\longrightarrow (J^{\bullet}_{i\bullet}, \{(J^{\bullet}_{i\bullet})_k\}).
\end{aligned}$$

Hence, by the filtered cohomological descent (1.5.1) (2), the following composite morphism

(2.5.3.7)

$$
\begin{aligned}
&(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), Q^*_{(X_i,Z_i)/S}P^{D_i}) \\
&\longrightarrow \eta^{\log}_{i,\mathrm{Rcrys}*}\eta^{\log,-1}_{i,\mathrm{Rcrys}}(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), Q^*_{(X_i,Z_i)/S}P^{D_i}) \\
&\longrightarrow \eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\})
\end{aligned}
$$

is a filtered quasi-isomorphism. Because (2.5.3.5) is a filtered quasi-isomorphism, so is the following composite morphism

(2.5.3.8)

$$
\begin{aligned}
&(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \\
&\longrightarrow (Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{W}_{i\bullet}/S}(\log \mathcal{F}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \\
&\longrightarrow (J_{i\bullet}, \{(J_{i\bullet})_k\}).
\end{aligned}
$$

The filtered quasi-isomorphism (2.5.3.8) induces a morphism

$$
\eta^{\log}_{i,\mathrm{Rcrys}*}(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}}) \\
\longrightarrow \eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\}).
$$

By the definition of the composite morphisms (2.5.3.7) and (2.5.3.8), the following diagram is commutative:

$$
\begin{array}{ccc}
\begin{array}{r}(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), \\ Q^*_{(X_i,Z_i)/S}P^{D_i})\end{array} & \xrightarrow{\simeq} & \eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\}) \\
\downarrow & & \| \\
\begin{array}{r}\eta^{\log}_{i,\mathrm{Rcrys}*}(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), \\ Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}})\end{array} & \longrightarrow & \eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\}).
\end{array}
$$

We also have the following diagram

$$
\begin{array}{ccc}
\begin{array}{r}\eta^{\log}_{i,\mathrm{Rcrys}*}(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), \\ Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}})\end{array} & \longrightarrow & \eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\}) \\
\downarrow & & \\
\eta^{\log}_{i,\mathrm{Rcrys}*}(I^\bullet_{i\bullet}, \{(I^\bullet_{i\bullet})_k\}) & & .
\end{array}
$$

Since $(I^\bullet_{i\bullet}, \{(I^\bullet_{i\bullet})_k\})$ and $(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\})$ are filtered flasque resolutions of the same complex $(Q^*_{(X_{i\bullet},Z_{i\bullet})/S}L_{(X_{i\bullet},Z_{i\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{i\bullet}/S}(\log \mathcal{E}_{i\bullet})), Q^*_{(X_{i\bullet},Z_{i\bullet})/S}P^{D_{i\bullet}})$, we have an isomorphism $\eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet}, \{(J^\bullet_{i\bullet})_k\}) \xrightarrow{\sim} \eta^{\log}_{i,\mathrm{Rcrys}*}(I^\bullet_{i\bullet}, \{(I^\bullet_{i\bullet})_k\})$ in $\mathrm{D}^+\mathrm{F}(Q^*_{(X_i,Z_i)/S}(\mathcal{O}_{(X_i,Z_i)/S}))$ which makes the diagram of the triangle above commutative. Hence the composite morphism

$$
\begin{aligned}
&(Q^*_{(X_i,Z_i)/S}L_{(X_i,Z_i)/S}(\Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{E}_i)), Q^*_{(X_i,Z_i)/S}P^{D_i})\\
\longrightarrow&\eta^{\log}_{i,\mathrm{Rcrys}*}(J^\bullet_{i\bullet},\{(J^\bullet_{i\bullet})_k\}) \longrightarrow \eta^{\log}_{i,\mathrm{Rcrys}*}(I^\bullet_{i\bullet},\{(I^\bullet_{i\bullet})_k\})
\end{aligned}
$$

is an isomorphism in $\mathrm{D}^+\mathrm{F}(Q^*_{(X_i,Z_i)/S}(\mathcal{O}_{(X_i,Z_i)/S}))$. Therefore we have proved that the morphism (2.5.3.3) is a filtered quasi-isomorphism. We finish the proof of (2.5.3). □

Corollary 2.5.4. *Fix decompositions of D and Z by their smooth components. Then the following hold*:

(1) $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ *is independent of the data* (2.4.0.1) *and* (2.4.0.2).

(2) *The following formula holds in* $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$:

(2.5.4.1)
$$R\overline{u}_{(X,Z)/S*}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) = (C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D).$$

As a result, $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ *is independent of the data* (2.4.0.1) *and* (2.4.0.2).

Proof. (1): By (2.5.3), $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ is equal to

$$
\begin{aligned}
&R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}(C^{\log,Z_\bullet}_{\mathrm{Rcrys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}), P^{D_\bullet})_{\bullet\in I}\\
=&R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}R\eta^{\log}_{\mathrm{Rcrys}*}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log\mathcal{E}_{\bullet\bullet})),\\
&\qquad\qquad Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}P^{D_{\bullet\bullet}})\\
=&R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}(C^{\log,Z_\bullet}_{\mathrm{Rcrys}}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}), P^{D_\bullet})_{\bullet\in J}.
\end{aligned}
$$

(2): We have

(2.5.4.2)
$$
\begin{aligned}
&R\overline{u}_{(X,Z)/S*}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)\\
=&R\pi_{\mathrm{zar}*}R\overline{u}_{(X_\bullet,Z_\bullet)/S*}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log\mathcal{E}_\bullet)),\\
&\qquad\qquad Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet})\\
=&R\pi_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log\mathcal{E}_\bullet), P^{D_\bullet})\\
=&(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D).
\end{aligned}
$$

Here the first (resp. second) equality follows from (1.6.4.6) (resp. (2.2.22.2)). The fact that the isomorphism (2.5.4.1) is independent of the data (2.4.0.1) and (2.4.0.2) immediately follows from (2.5.3.1) and (2.5.3.2). □

Remark 2.5.5. In §2.7 we shall prove that $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ is independent of the choice of the decompositions of D and Z by their smooth components. As a result, $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ is also independent of the choice above.

Corollary 2.5.6. *Let $\iota\colon (X, D\cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D}\cup\mathcal{Z})$ be an admissible immersion over S with respect to the union of decompositions Δ_D and Δ_Z of D and Z by smooth components of D and Z, respectively. Let $\mathfrak{D}$ be the log PD-envelope of the locally closed immersion $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{Z})$ over $(S, \mathcal{I}, \gamma)$. Then the following hold*:

(1)

$$(2.5.6.1) \qquad \begin{aligned}&(C^{\log, Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X, D\cup Z)/S}), P^D)\\ =&(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))), Q^*_{(X,Z)/S}P^D).\end{aligned}$$

In particular, the filtered complex $(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))),$ $Q^*_{(X,Z)/S}P^D)$ is independent of the choice of the admissible immersion of $(X, D\cup Z)$ over S if one fixes Δ_D and Δ_Z.

(2)

$$(2.5.6.2) \quad (C^{\log, Z}_{\mathrm{zar}}(\mathcal{O}_{(X, D\cup Z)/S}), P^D) = (\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})), P^D)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$. In particular, the filtered complex $(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup \mathcal{Z})), P^D)$ is independent of the choice of the admissible immersion of $(X, D\cup Z)$ over S if one fixes Δ_D and Δ_Z.

Proof. By (2.5.4) (1), we have

$$\begin{aligned}&(C^{\log, Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X, D\cup Z)/S}), P^D)\\ =&(Q^*_{(X,Z)/S}L_{(X,Z)/S}(\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))), Q^*_{(X,Z)/S}P^D).\end{aligned}$$

Hence (1) follows. The proof of (2) is the same. □

Proposition 2.5.7. *Let $(S, \mathcal{I}, \gamma)$ and $f\colon (X, E) := (X, D\cup Z) \longrightarrow S_0$ be as in §2.4. Let Δ be a decomposition of E by smooth components of D and Z. Let $X = \bigcup_{i_0\in I_0} X_{i_0}$ be an affine open covering of X, where I_0 is a set. Set $(X_0, E_0) := (\coprod_{i_0} X_{i_0}, \coprod_{i_0}(E\cap X_{i_0}))$ and $(X_n, E_n) := (\mathrm{cosk}^X_0(X_0)_n, \mathrm{cosk}^E_0(E_0)_n)$ $(n\in\mathbb{N})$. Let (X_n, Z_n) and (X_n, D_n) be the analogues of (X_n, E_n) for Z and D, respectively. Set $\Delta_0 := \coprod_{i_0}\Delta|_{X_{i_0}}$ ((2.1.12)) and let Δ_n $(n\in\mathbb{Z}_{>0})$ be the induced decomposition of E_n of smooth components of E_n. Let*

$$\begin{aligned}{\pi'}^{\log}_{\mathrm{Rcrys}}\colon &(((\widetilde{X_n, Z_n})/S)^{\log}_{\mathrm{Rcrys}}, Q^*_{(X_n,Z_n)/S}(\mathcal{O}_{(X_n,Z_n)/S}))_{n\in\mathbb{N}}\\ &\longrightarrow (((\widetilde{X, Z})/S)^{\log}_{\mathrm{Rcrys}}, Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))\end{aligned}$$

be a natural morphism of ringed topoi. Then there exists an admissible immersion $(X_n, E_n)_{n\in\mathbb{N}} \overset{\subset}{\longrightarrow} (\mathcal{X}_n, \mathcal{E}_n)_{n\in\mathbb{N}}$ of simplicial smooth schemes with simplicial relative SNCD's over S with respect to $(\Delta_n)_{n\in\mathbb{N}}$. Moreover,

(2.5.7.1)
$(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,E)/S}), P^D) =$

$R\pi'^{\log}_{(X,Z)\mathrm{Rcrys}*}((Q^*_{(X_n,Z_n)/S}L_{(X_n,Z_n)/S}(\Omega^\bullet_{\mathcal{X}_n/S}(\log \mathcal{E}_n)), Q^*_{(X_n,Z_n)/S}P^{D_n})_{n\in\mathbb{N}}).$

Proof. Let I' be a category whose objects are $(i_0,\ldots,i_r)$'s ($r\in\mathbb{N}, i_0,\ldots,i_r\in I_0$) and the morphism from $i := (i_0,\ldots,i_r) \longrightarrow j := (j_0,\ldots,j_s)$ is one point if $\{i_0,\ldots,i_r\}\subset\{j_0,\ldots,j_s\}$ and empty otherwise. For an object $i = (i_0,\ldots,i_r)$, set $X_i := \bigcap_{s=0}^r X_{i_s}$, $E_i := \bigcap_{s=0}^r(E\cap X_{i_s})$. Then we have the following contravariant functor:

$$(X_\bullet, E_\bullet)\colon I'^o \longrightarrow \{\text{smooth schemes with relative SNCD's over } S_0\}.$$

The construction in §2.4 shows the existence of a diagram of admissible immersions into a diagram of smooth schemes with relative SNCD's over S: $(X_\bullet,E_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}_\bullet,\mathcal{E}_\bullet)$ ($\bullet\in I'$) with respect to $\Delta_\bullet$, where $\Delta_\bullet$ is the induced decomposition of E by Δ ((2.1.12)). For an element $j_1,j_2\in I_0$, there exists two natural morphisms $\delta'_k\colon (\mathcal{X}_{(j_1,j_2)},\mathcal{E}_{(j_1,j_2)}) \longrightarrow (\mathcal{X}_{j_k},\mathcal{E}_{j_k})$ ($k=1,2$). Using these morphisms, we have natural face morphisms $\delta_m\colon (\mathcal{X}_n,\mathcal{E}_n)\longrightarrow (\mathcal{X}_{n-1},\mathcal{E}_{n-1})$ ($m=0,\ldots,n$). Moreover, note that $\mathcal{X}_{(i,i)}$ ($i\in I_0$) is an open scheme of the blow up of $\mathcal{X}_i\times_S\mathcal{X}_i$ by a closed subscheme of it. By considering the strict transform of the diagonal immersion $\mathcal{X}_i \overset{\subset}{\longrightarrow} \mathcal{X}_i\times_S\mathcal{X}_i$, we have a natural morphism $s'\colon \mathcal{X}_i\longrightarrow\mathcal{X}_{(i,i)}$. Using this morphism, we have natural degeneracy morphisms $s_m\colon (\mathcal{X}_{n-1},\mathcal{E}_{n-1})\longrightarrow(\mathcal{X}_n,\mathcal{E}_n)$ ($m=0,\ldots,n-1$). The morphisms s_m and δ_m ($m\in\mathbb{N}$) satisfy the standard relations in [90, (8.1.3)]. Hence we have a desired simplicial log scheme $(\mathcal{X}_n,\mathcal{E}_n)_{n\in\mathbb{N}}$.

Fix a total order $<$ on I_0. Let I be a subcategory of I' whose objects are $(i_0,\ldots,i_r)$'s ($r\in\mathbb{N}, i_0<\cdots<i_r, i_j\in I_0$). Let

$$\begin{aligned}\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}}\colon &(((\widetilde{X_\bullet,Z_\bullet})/S)^{\log}_{\mathrm{Rcrys}}, Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S}))_{\bullet\in I}\\ &\longrightarrow (((\widetilde{X,Z})/S)^{\log}_{\mathrm{Rcrys}}, Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))\end{aligned}$$

be a natural morphism of ringed topoi. Then we have

$$\begin{aligned}&(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,E)/S}), P^D)\\ =&R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}((Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log\mathcal{E}_\bullet)), Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet})_{\bullet\in I})\end{aligned}$$

by the definition of $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,E)/S}), P^D)$. Because Čech complexes are calculated by alternating cochains as in [80, §3], the right hand side is canonically isomorphic to

$$R\pi'^{\log}_{(X,Z)/S\mathrm{Rcrys}*}((Q^*_{(X_n,Z_n)/S}L_{(X_n,Z_n)/S}(\Omega^\bullet_{\mathcal{X}_n/S}(\log\mathcal{E}_n)), Q^*_{(X_n,Z_n)/S}P^{D_n})_{n\in\mathbb{N}}).$$

□

Corollary 2.5.8. *With the notation of* (2.5.7), *let* $\mathfrak{D}_n$ *be the log PD-envelope of the locally closed immersion* $(X_n, Z_n) \overset{\subset}{\longrightarrow} (\mathcal{X}_n, \mathcal{Z}_n)$ *over* $(S, \mathcal{I}, \gamma)$. *Let* $\pi'_{\mathrm{zar}} \colon (\widetilde{X}_n)_{n\in\mathbb{N}} \longrightarrow \widetilde{X}$ *be a natural morphism of topoi. Then the following holds*:

(2.5.8.1)
$$(C_{\mathrm{zar}}^{\log,Z}(\mathcal{O}_{(X,E)/S}), P^D) = R\pi'_{\mathrm{zar}*}((\mathcal{O}_{\mathfrak{D}_n} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{\bullet}_{\mathcal{X}_n/S}(\log \mathcal{E}_n), P^{\mathcal{D}_n})_{n\in\mathbb{N}}).$$

Proof. We immediately have (2.5.8) since we have the analogue of (2.5.4.1) for

$$R\pi'^{\log}_{(X,Z)/S\mathrm{Rcrys}*}((Q^*_{(X_n,Z_n)/S}L_{(X_n,Z_n)/S}(\Omega^{\bullet}_{\mathcal{X}_n/S}(\log \mathcal{E}_n)), Q^*_{(X_n,Z_n)/S}P^{\mathcal{D}_n})_{n\in\mathbb{N}}).$$

□

2.6 The Preweight Spectral Sequence

Let the notations be as in §2.4 and §2.5. Recall the projections $u_{(X,Z)/S}$ and $u_{(X,D\cup Z)/S}$ ((2.2.22.1), (2.4.6.4)). Set $f_{(X,Z)/S} := f \circ u_{(X,Z)/S}$ and $f_{(X,D\cup Z)/S} := f \circ u_{(X,D\cup Z)/S}$. Then we have the log crystalline cohomology sheaf $R^h f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})$ ($h \in \mathbb{Z}$). We also have the log crystalline cohomology sheaf $R^h f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S})$ of $(D^{(k)}, Z|_{D^{(k)}})/(S,\mathcal{I},\gamma)$. In this section we construct the following spectral sequence of $\mathcal{O}_S$-modules:

(2.6.0.1)
$$\begin{aligned} E_1^{-k,h+k} &= R^{h-k} f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)) \\ &\Longrightarrow R^h f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}). \end{aligned}$$

Theorem 2.6.1. *Let* $a^{(k)} \colon (D^{(k)}, Z|_{D^{(k)}}) \longrightarrow (X, Z)$ $(k \in \mathbb{N})$ *be the natural morphism. Let*

$$a^{(k)\log}_{\mathrm{crys}*} \colon \mathrm{D}^+(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}) \longrightarrow \mathrm{D}^+(\mathcal{O}_{(X,Z)/S})$$

and

$$a^{(k)}_{\mathrm{zar}*} \colon \mathrm{D}^+((f \circ a^{(k)})^{-1}(\mathcal{O}_S)) \longrightarrow \mathrm{D}^+(f^{-1}(\mathcal{O}_S))$$

be the induced morphisms by $a^{(k)}$. *Fix decompositions of* D *and* Z *by their smooth components. Then there exist the following canonical isomorphisms*

(2.6.1.1)
$$\begin{aligned} &\mathrm{gr}_k^{P^D}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})) \\ &= Q^*_{(X,Z)/S} a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\} \end{aligned}$$

and

$$\begin{aligned}(2.6.1.2)\quad &\mathrm{gr}_k^{P^D}(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}))\\ &=a^{(k)}_{\mathrm{zar}*}Ru_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(D/S_0))\{-k\}.\end{aligned}$$

Proof. Let the notations be as in §2.4. By applying $R\overline{u}_{(X,Z)/S*}$ to both hands of (2.6.1.1), we immediately have (2.6.1.2) by (1.3.4.1) and (2.5.4.1); hence we have only to prove (2.6.1.1).

Let

$$\begin{aligned}(2.6.1.3)\quad \pi^{\log}_{(D^{(k)},Z|_{D^{(k)}})/S\mathrm{crys}}\colon &(((D^{(k)}_\bullet,\widetilde{Z_\bullet|_{D^{(k)}_\bullet}})/S)^{\log}_{\mathrm{crys}},\mathcal{O}_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S})\\ &\longrightarrow(((D^{(k)},\widetilde{Z|_{D^{(k)}}})/S)^{\log}_{\mathrm{crys}},\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S})\end{aligned}$$

be the natural morphism of ringed topoi (§1.6). Then we have the following equalities:

$$\begin{aligned}(2.6.1.4)\quad&\\ &\mathrm{gr}_k^{P^D}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}))\\ &=\mathrm{gr}_k^{P^D}R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))))\\ &=R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}\mathrm{gr}_k^{Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}\\ &\qquad(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))))\\ &=R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}a^{(k)\log}_{\bullet\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}\\ &\qquad\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet))\{-k\}\\ &=Q^*_{(X,Z)/S}R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}a^{(k)\log}_{\bullet\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}\\ &\qquad\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet))\{-k\}\\ &=Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}R\pi^{\log}_{(D^{(k)},Z|_{D^{(k)}})/S\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}\\ &\qquad\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet))\{-k\}\\ &=Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\}.\end{aligned}$$

Here the second, the third, the fourth and the fifth equalities follow from (1.3.4.1), (2.2.21.2), (1.6.4.1) and (1.6.0.13), respectively. The last equality follows from the cohomological descent.

Next we prove that the isomorphism (2.6.1.4) is independent of the choice of the data (2.4.0.1) and (2.4.0.2). Assume that we are given the other data (2.4.0.1) and (2.4.0.2) as in §2.5. By the trivially filtered version of (2.5.3), we have

$$R\eta^{\log}_{\mathrm{Rcrys}*}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}a^{(k)\log}_{\bullet\bullet\mathrm{crys}*}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_{\bullet\bullet}/S}(\log \mathcal{Z}_{\bullet\bullet}|_{\mathcal{D}^{(k)}_{\bullet\bullet}}) \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D_{\bullet\bullet}/S;Z_{\bullet\bullet})))$$

$$= Q^*_{(X_\bullet,Z_\bullet)/S}a^{(k)\log}_{\bullet\mathrm{crys}*}(L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(k)}_\bullet})) \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet)).$$

Since $R\eta^{\log}_{\mathrm{Rcrys}*}\mathrm{gr}_k^{P^{D_{\bullet\bullet}}} = \mathrm{gr}_k^{P^{D_\bullet}}R\eta^{\log}_{\mathrm{Rcrys}*}$ by (1.3.4.1), we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{gr}_k^{Q^*_{(X_\bullet,Z_\bullet)/S}P^{D_\bullet}}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)))) & \xrightarrow{\sim} & Q^*_{(X_\bullet,Z_\bullet)/S}a^{(k)\log}_{\bullet\mathrm{crys}*}(L_{(D^{(k)}_\bullet,Z_\bullet|_{D^{(k)}_\bullet})/S}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(k)}_\bullet}))\{-k\} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet)) \\
\Vert & & \Vert \\
R\eta^{\log}_{\mathrm{Rcrys}*}\mathrm{gr}_k^{Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}P^{D_{\bullet\bullet}}}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{\bullet\bullet}/S}(\log(\mathcal{D}_{\bullet\bullet} \cup \mathcal{Z}_{\bullet\bullet})))) & \xrightarrow{\sim} & Q^*_{(X_\bullet,Z_\bullet)/S}R\eta^{\log}_{\mathrm{crys}*}a^{(k)\log}_{\bullet\bullet\mathrm{crys}*}(L_{(D^{(k)}_{\bullet\bullet},Z_{\bullet\bullet}|_{D^{(k)}_{\bullet\bullet}})/S}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_{\bullet\bullet}/S}(\log \mathcal{Z}_{\bullet\bullet}|_{\mathcal{D}^{(k)}_{\bullet\bullet}}))\{-k\} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D_{\bullet\bullet}/S;Z_{\bullet\bullet})).
\end{array}$$

Hence we see that the isomorphism (2.6.1.1) (and hence (2.6.1.2)) is independent of the choice of the data (2.4.0.1) and (2.4.0.2). $\square$

Corollary 2.6.2. *Let k' be a nonnegative integer. For integers k and h, set*

$$E_1^{-k,h+k}((X,D\cup Z)/S;k') := \begin{cases} R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)) & (k \leq k'), \\ 0 & (k > k'). \end{cases}$$

Set $\overline{f}_{(X,Z)/S} := f \circ \overline{u}_{(X,Z)/S}$. Then there exists the following spectral sequence

$$E_1^{-k,h+k} = E_1^{-k,h+k}((X,D\cup Z)/S;k') \Longrightarrow R^h\overline{f}_{(X,Z)/S*}(P^D_{k'}C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})). \tag{2.6.2.1}$$

In particular, there exists the following spectral sequence

(2.6.2.2)
$$\begin{aligned}E_1^{-k,h+k} &= E_1^{-k,h+k}((X,D\cup Z)/S)\\ &= R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\\ &\Longrightarrow R^hf_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}).\end{aligned}$$

Proof. Let $(I^\bullet_{k'},\{I^\bullet_l\}_{l\leq k'})\in \mathrm{K}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$ be a filtered flasque resolution of a representative of $(P^D_{k'}C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),\{P^D_lC^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})\}_{l\leq k'})\in \mathrm{D}^+\mathrm{F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$. Consider the following spectral sequence

$$E_1^{-k,h+k}=\mathcal{H}^h(\overline{f}_{(X,Z)/S*}\mathrm{gr}_k(I^\bullet_{k'}))\Longrightarrow \mathcal{H}^h(\overline{f}_{(X,Z)/S*}I^\bullet_{k'}).$$

Obviously we have $\mathcal{H}^h(\overline{f}_{(X,Z)/S*}I_{k'})=R^h\overline{f}_{(X,Z)/S*}(P^D_{k'}C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}))$. By the proof of (1.3.4.1), $\mathrm{gr}_k(I^\bullet_{k'})$ is a flasque resolution of $\mathrm{gr}_k^{P^D}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}))$ for $k\leq k'$. Hence, for $k\leq k'$, we have

$$\begin{aligned}E_1^{-k,h+k} &=R^h\overline{f}_{(X,Z)/S*}(\mathrm{gr}_k^{P^D}(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})))\\ &=R^h\overline{f}_{(X,Z)/S*}(Q^*_{(X,Z)/S}(a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\{-k\}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))))\\ &=R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)).\end{aligned}$$

Here, in the last equality, we have used the commutativity of the diagram (1.6.3.1) for the trivially filtered case. Therefore we obtain (2.6.2.1). By using (2.4.7.2), we obtain (2.6.2.2) similarly. □

Corollary 2.6.3. *Fix decompositions Δ_D and Δ_Z of D and Z by their smooth components, respectively. Let $\iota\colon (X,D\cup Z)\overset{\subset}{\longrightarrow}(\mathcal{X},\mathcal{D}\cup\mathcal{Z})$ be an admissible immersion over S with respect to Δ_D and Δ_Z. Let $f\colon (X,D\cup Z)\longrightarrow S_0$ and $f_S\colon (\mathcal{X},\mathcal{D}\cup\mathcal{Z})\longrightarrow S$ be the structural morphisms. Let $\mathfrak{D}$ be the log PD-envelope of the locally closed immersion $(X,Z)\overset{\subset}{\longrightarrow}(\mathcal{X},\mathcal{Z})$ over $(S,\mathcal{I},\gamma)$. Let $f^{(k)}_S\colon \mathfrak{D}^{(k)}\longrightarrow S$ be the PD-envelope of the locally closed immersion $D^{(k)}\overset{\subset}{\longrightarrow}\mathcal{D}^{(k)}$ over $(S,\mathcal{I},\gamma)$. Let k' be a nonnegative integer. For integers k and h, set*

$$\begin{aligned}&E_1^{-k,h+k}((\mathcal{X},\mathcal{D}\cup\mathcal{Z})/S;k')\\ &:=\begin{cases}R^{h-k}f^{(k)}_{S*}(\mathcal{O}_{\mathfrak{D}^{(k)}}\otimes_{\mathcal{O}_{\mathcal{D}^{(k)}}}\Omega^\bullet_{\mathcal{D}^{(k)}/S}(\log\mathcal{Z}|_{\mathcal{D}^{(k)}})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S)) & (k\leq k'),\\ 0 & (k>k').\end{cases}\end{aligned}$$

Then the following spectral sequence

(2.6.3.1)
$$\begin{aligned}E_1^{-k,h+k} &:= E_1^{-k,h+k}((\mathcal{X},\mathcal{D}\cup\mathcal{Z})/S;k')\\ &\Longrightarrow R^hf_{S*}(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}P^{\mathcal{D}}_{k'}\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))\end{aligned}$$

is isomorphic to (2.6.2.1)*, and hence it is independent of the choice of the admissible immersion* f_S*. In particular, if* $f_S\colon(\mathcal{X},\mathcal{D}\cup\mathcal{Z})\longrightarrow S$ *is a lift of* $f\colon(X,D\cup Z)\longrightarrow S_0$*, then the following spectral sequence*

(2.6.3.2)

$$E_1^{-k,h+k}=E_1^{-k,h+k}((\mathcal{X},\mathcal{D}\cup\mathcal{Z})/S;k')\Longrightarrow R^hf_{S*}(P^{\mathcal{D}}_{k'}\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))$$

is independent of the choice of the lift. Here

$$E_1^{-k,h+k}((\mathcal{X},\mathcal{D}\cup\mathcal{Z})/S;k')$$
$$=\begin{cases}R^{h-k}f^{(k)}_{S*}(\Omega^{\bullet}_{\mathcal{D}^{(k)}/S}(\log\mathcal{Z}|_{\mathcal{D}^{(k)}})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\rm zar}(\mathcal{D}/S)) & (k\leq k'),\\ 0 & (k>k').\end{cases}$$

Proof. (2.6.3) immediately follows from (2.5.4.1) and (2.6.2.1). □

Remark 2.6.4. In §2.9 below, we consider the functoriality of (2.6.2.2); in particular, in the case where S_0 is of characteristic p, we shall consider the compatibility of (2.6.2.2) with the relative Frobenius $F\colon(X,D)\longrightarrow(X',D')$ over S_0.

2.7 The Vanishing Cycle Sheaf and the Preweight Filtration

Let S, S_0 and $f\colon(X,D\cup Z)\longrightarrow S_0$ be as in §2.4. Let $a^{(k)}\colon(D^{(k)},Z|_{D^{(k)}})\longrightarrow(X,Z)$ be as in §2.2 **(2)**. In §2.4 and §2.5, we have constructed the preweight-filtered restricted crystalline complex

$$(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\in{\rm D}^+{\rm F}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$$

such that

$$C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S})=Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})$$

in ${\rm D}^+(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$. Here

$$\epsilon_{(X,D\cup Z,Z)/S}\colon((X,\widetilde{D\cup Z})/S)^{\log}_{\rm crys}\longrightarrow((\widetilde{X,Z})/S)^{\log}_{\rm crys}$$

is the forgetting log morphism along D ((2.3.2)). Let $j\colon U:=X\setminus D\overset{\subset}{\longrightarrow}X$ be the natural open immersion. Let n be a positive integer. Let $(X,D\cup Z)$ be as above or an analogous log scheme over $\mathbb{C}$ or an algebraically closed field of characteristic $p>0$. Then we have the following translation if $Z=\emptyset$:

(2.7.0.1)

$/\mathbb{C}$	l-adic	crystal
U_{an} $(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}$, $\widetilde{(X_{\mathrm{an}}, D_{\mathrm{an}})}{}^{\log}_{\mathrm{et}}$ X_{an}, $\widetilde{X_{\mathrm{an}}}$ $j_{\mathrm{an}} \colon U_{\mathrm{an}} \overset{\subset}{\longrightarrow} X_{\mathrm{an}}$	$\widetilde{U}_{\mathrm{et}}$ $\widetilde{(X, D)}{}^{\log}_{\mathrm{et}}$ $\widetilde{X}_{\mathrm{et}}$ $j_{\mathrm{et}} \colon \widetilde{U}_{\mathrm{et}} \longrightarrow \widetilde{X}_{\mathrm{et}}$	? $(\widetilde{(X, D)}/S)^{\log}_{\mathrm{crys}}$ $\widetilde{(X/S)}_{\mathrm{crys}}$?
$\epsilon_{\mathrm{top}} \colon (X_{\mathrm{an}}, D_{\mathrm{an}})^{\log} \longrightarrow X_{\mathrm{an}}$ $\epsilon_{\mathrm{an}} \colon \widetilde{(X_{\mathrm{an}}, D_{\mathrm{an}})}{}^{\log}_{\mathrm{et}} \longrightarrow \widetilde{X_{\mathrm{an}}}$	$\epsilon_{\mathrm{et}} \colon \widetilde{(X, D)}{}^{\log}_{\mathrm{et}}$ $\longrightarrow \widetilde{X}_{\mathrm{et}}$	$\epsilon_{(X,D)/S} \colon (\widetilde{(X, D)}/S)^{\log}_{\mathrm{crys}}$ $\longrightarrow \widetilde{(X/S)}_{\mathrm{crys}}$
$Rj_{\mathrm{an}*}(\mathbb{Z}) = R\epsilon_{\mathrm{top}*}(\mathbb{Z})$ $R\epsilon_{\mathrm{top}*}(\mathbb{Z}/n) = R\epsilon_{\mathrm{an}*}(\mathbb{Z}/n)$	$Rj_{\mathrm{et}*}(\mathbb{Z}/l^n) =$ $R\epsilon_{\mathrm{et}*}(\mathbb{Z}/l^n)$	? $R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$
$X_{\mathrm{an}} \longrightarrow X$	$\widetilde{X}_{\mathrm{et}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$	$u_{X/S} \colon \widetilde{(X/S)}_{\mathrm{crys}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$
$\mathbb{Z}_{(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}}$ $(\mathbb{Z}/n)_{(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}}$ $(\mathbb{Z}/n)_{\widetilde{(X_{\mathrm{an}}, D_{\mathrm{an}})}{}^{\log}_{\mathrm{et}}}$	$(\mathbb{Z}/n)_{\widetilde{(X,D)}{}^{\log}_{\mathrm{et}}}$ $(p \nmid n)$	$\mathcal{O}_{(X,D)/S}$
$\mathbb{Z}_{X_{\mathrm{an}}}$ $(\mathbb{Z}/n)_{X_{\mathrm{an}}}$ $(n \in \mathbb{Z})$	$(\mathbb{Z}/n)_{\widetilde{X}_{\mathrm{et}}}$ $(p \nmid n)$	$\mathcal{O}_{X/S}$
$(\Omega^{\bullet}_{X/\mathbb{C}}(\log D), P)$ $(\Omega^{\bullet}_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), P)$ $(\Omega^{\bullet}_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), \tau)$	? ? ?	$(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$ $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), \tau)$

Here $(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}$ is the real blow up of $(X_{\mathrm{an}}, D_{\mathrm{an}})$ ([58, (1.2)]) and ϵ_{top} is the natural morphism of topological spaces, $\widetilde{(X_{\mathrm{an}}, D_{\mathrm{an}})}{}^{\log}_{\mathrm{et}}$ is the analytic log etale topos of $(X_{\mathrm{an}}, D_{\mathrm{an}})$ ([51]) and ϵ_{an} is the forgetting log morphism to the topos $\widetilde{X}_{\mathrm{an}}$ defined by the local isomorphisms to X_{an}; the morphism ϵ_{et} in the middle column is the forgetting log morphism ([30], cf. [67, (1.1.2)]); the upper (resp. lower) equality in the left column has been obtained in [58, (1.5.1)] (resp. [72]), and the equality in the middle column ([30, (3.6)]) follows from the following composite equality

(2.7.0.2)

$$R^h\epsilon_{\mathrm{et}*}(\mathbb{Z}/l^n) = \bigwedge^{h}(M_D^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Z}/l^n(-h) = R^h j_{\mathrm{et}*}(\mathbb{Z}/l^n) \ (h \in \mathbb{Z}, n \in \mathbb{Z}_{>0}).$$

Here the first equality follows from [58, (2.4)] and the second equality is Gabber's purity ([33]) which has solved Grothendieck's purity conjecture. Recall that, in the crystalline case, $Rj_{\mathrm{crys}*}(\mathcal{O}_{U/S})$ is not a good object ([3, VI Lemme 1.2.2]).

The purpose of this section is to give another intrinsic description of the preweight-filtered restricted crystalline complex $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ and, as a corollary, to obtain the spectral sequence (2.6.2.2) in a different way.

We start with the following, which includes a crystalline analogue of Gabber's purity.

Theorem 2.7.1 (p-adic purity). *Let k be a nonnegative integer. Then*

(2.7.1.1)
$$Q^*_{(X,Z)/S}R^k\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) = Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)).$$

Proof. The "increasing filtration" $\{P^D_kC^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})\}_{k\in\mathbb{Z}}$ on $C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})$ gives us the following spectral sequence

(2.7.1.2)
$$E_1^{-k,h+k} = \mathcal{H}^h(\mathrm{gr}^{P^D}_kC^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}))\Longrightarrow\mathcal{H}^h(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})).$$

Let $I^\bullet$ be a flasque resolution of $\mathcal{O}_{(X,D\cup Z)/S}$. By (2.4.7.1) and by the exactness of $Q^*_{(X,Z)/S}$, we have

$$\begin{aligned}\mathcal{H}^h(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})) &= \mathcal{H}^h(Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}))\\ &= \mathcal{H}^h(Q^*_{(X,Z)/S}\epsilon_{(X,D\cup Z,Z)/S*}(I^\bullet))\\ &= Q^*_{(X,Z)/S}\mathcal{H}^h(\epsilon_{(X,D\cup Z,Z)/S*}(I^\bullet))\\ &= Q^*_{(X,Z)/S}R^h\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}),\end{aligned}$$

and by (2.6.1.1) we have

$$\begin{aligned}&\mathcal{H}^h(\mathrm{gr}^{P^D}_kC^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}))\\ =&\mathcal{H}^{h-k}(Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)));\end{aligned}$$

this is equal to $Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))$, 0 for $k=h$ and $k\neq h$, respectively. Hence (2.7.1.2) degenerates at E_1; thus we have a canonical isomorphism

$$\begin{aligned}&Q^*_{(X,Z)/S}R^k\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})\\ =&Q^*_{(X,Z)/S}a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)).\end{aligned}$$

□

By the Leray spectral sequence for the functor $\epsilon_{(X,D\cup Z,Z)/S*}\colon((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}}\longrightarrow((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}}$ and $f_{(X,Z)/S*}\colon((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}}\longrightarrow\widetilde{X}_{\mathrm{zar}}$, we obtain the following spectral sequence

(2.7.1.3)
$$E_2^{st}:=R^sf_{(X,Z)/S*}R^t\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})\Longrightarrow$$

$$R^{s+t}f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}).$$

Set $\overline{f}_{(X,Z)/S} := f_{(X,Z)/S} \circ Q_{(X,Z)/S} \colon ((\widetilde{X,Z)/S})^{\log}_{\mathrm{Rcrys}} \longrightarrow ((\widetilde{X,Z)/S})^{\log}_{\mathrm{crys}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$. Because $R\overline{f}_{(X,Z)/S*} \circ Q^*_{(X,Z)/S} = Rf_{(X,Z)/S*}$, (2.7.1.3) is equal to the following spectral sequence

(2.7.1.4) $$E_2^{st} = R^s f_{(D^{(t)},Z|_{D^{(t)}})/S*}(\mathcal{O}_{(D^{(t)},Z|_{D^{(t)}})/S} \otimes_{\mathbb{Z}} \varpi^{(t)\log}_{\mathrm{crys}}(D/S;Z)) \Longrightarrow$$
$$R^{s+t}f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})$$

by (2.7.1).

Using (2.7.1), we can give another simpler expression of $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$. To do this, let us recall the canonical filtration of a complex.

Let $(\mathcal{T},\mathcal{A})$ be a ringed topos and let $E^\bullet$ be an object in $\mathrm{C}(\mathcal{A})$. Then the canonical filtration $\tau := \{\tau_k E^\bullet\}_{k\in\mathbb{Z}}$ of $E^\bullet$ is defined as follows: $\tau_k E^i := E^i$ $(i<k)$, $:= \mathrm{Ker}(E^k \longrightarrow E^{k+1})$ $(i=k)$, $:= 0$ $(i>k)$. Let $E^\bullet$ and $F^\bullet$ be objects in $\mathrm{C}^+(\mathcal{A})$. Then a homotopy h between two morphisms $f,g\colon E^\bullet \longrightarrow F^\bullet$ also gives a filtered homotopy between two morphisms $f,g\colon (E^\bullet,\tau) \longrightarrow (F^\bullet,\tau)$ of filtered complexes. Furthermore, a quasi-isomorphism $f\colon E^\bullet \longrightarrow F^\bullet$ induces a filtered quasi-isomorphism $f\colon (E^\bullet,\tau) \longrightarrow (F^\bullet,\tau)$; thus a functor $\mathrm{C}^+(\mathcal{A}) \ni E^\bullet \longmapsto (E^\bullet,\tau) \in \mathrm{C}^+\mathrm{F}(\mathcal{A})$ induces a functor $\mathrm{D}^+(\mathcal{A}) \longrightarrow \mathrm{D}^+\mathrm{F}(\mathcal{A})$, which is also denoted by $E^\bullet \mapsto (E^\bullet,\tau)$.

We prove the following lemma for a main result (2.7.3) below in this section:

Lemma 2.7.2. *Let $f\colon (\mathcal{T},\mathcal{A}) \longrightarrow (\mathcal{T}',\mathcal{A}')$ be a morphism of ringed topoi. Then, for an object $E^\bullet$ in $\mathrm{D}^+(\mathcal{A})$, there exists a canonical morphism*

(2.7.2.1) $$(Rf_*(E^\bullet),\tau) \longrightarrow Rf_*((E^\bullet,\tau))$$

in $\mathrm{D}^+\mathrm{F}(\mathcal{A}')$.

Proof. Let $E^\bullet \longrightarrow I^\bullet$ be a quasi-isomorphism into a complex of flasque $\mathcal{A}$-modules. Let $(I^\bullet,\tau) \longrightarrow (J^\bullet,\{J^\bullet_k\})$ be a filtered flasque resolution of $(I^\bullet,\tau)$. Then, by applying the functor f_* to the morphism of this resolution, we obtain a morphism

(2.7.2.2) $$(f_*(I^\bullet),\{f_*(\tau_k I^\bullet)\}) \longrightarrow (f_*(J^\bullet),\{f_*(J^\bullet_k)\}).$$

By (1.1.12) (2), the right hand side of (2.7.2.2) is equal to $Rf_*((E^\bullet,\tau))$. On the other hand, there exists a natural morphism $f_*(\tau_k I^\bullet) \longrightarrow \tau_k f_*(I^\bullet)$; in fact, by the left exactness of f_*, we have $f_*(\tau_k I^\bullet) \xrightarrow{\sim} \tau_k f_*(I^\bullet)$. Hence the left hand side of (2.7.2.2) is equal to $(f_*(I^\bullet),\{\tau_k f_*(I^\bullet)\}) = (Rf_*(E^\bullet),\tau)$. It is easy to check that the induced morphism in $\mathrm{D}^+\mathrm{F}(\mathcal{A}')$ by the morphism (2.7.2.2) is independent of the choice of $I^\bullet$ and $(J^\bullet,\{J^\bullet_k\})$. Therefore we have a canonical morphism (2.7.2.1). $\square$

Now we give another description of $(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$.

Theorem 2.7.3 (Comparison theorem). *Let S_0, S, X, D and Z be as in §2.4. Set*

$$(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D) := (R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}),\tau)\in \mathrm{D}^+\mathrm{F}(\mathcal{O}_{(X,Z)/S}). \tag{2.7.3.1}$$

Then there exists a canonical isomorphism

$$Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D) \overset{\sim}{\longrightarrow} (C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D). \tag{2.7.3.2}$$

In particular,

$$(C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),\tau) = (C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D). \tag{2.7.3.3}$$

Proof. Fix the data (2.4.0.1) and (2.4.0.2) for $D\cup Z$. Then, as usual, there exists a natural morphism of filtered $\mathcal{O}_{(X_\bullet,Z_\bullet)/S}$-modules:

$$Q^*_{(X_\bullet,Z_\bullet)/S}(L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),\tau)\longrightarrow (Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),Q^*_{(X_\bullet,Z_\bullet)/S}P^{\mathcal{D}_\bullet}). \tag{2.7.3.4}$$

By (2.7.2) there exists a canonical morphism

$$(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),\tau)\longrightarrow R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),\tau). \tag{2.7.3.5}$$

By composing (2.7.3.5) with the morphism $R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}((2.7.3.4))$, we obtain a morphism

$$(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),\tau)\longrightarrow R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}(Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))),Q^*_{(X_\bullet,Z_\bullet)/S}P^{\mathcal{D}_\bullet}) \tag{2.7.3.6}$$

which is nothing but a morphism

$$(Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}),\tau)\longrightarrow (C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D) \tag{2.7.3.7}$$

by (1.6.4.1). (We have not yet claimed that the morphism (2.7.3.7) is independent of the data (2.4.0.1) and (2.4.0.2).) To prove that the morphism (2.7.3.7) is a filtered quasi-isomorphism, it suffices to prove that the induced morphism

$$\mathrm{gr}^\tau_k Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})\longrightarrow \mathrm{gr}^{P^D}_k C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}) \tag{2.7.3.8}$$

is a quasi-isomorphism for each $k \in \mathbb{Z}$. By the definition of the canonical filtration τ and by the proof of (2.7.1), we have

(2.7.3.9)

$$\begin{aligned}
&\mathcal{H}^i(\mathrm{gr}_k^{\tau} Q^*_{(X,Z)/S} R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})) \\
&= \begin{cases} Q^*_{(X,Z)/S} R^k\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) & (i=k) \\ 0 & (i \neq k) \end{cases} \\
&= \begin{cases} Q^*_{(X,Z)/S} a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)) & (i=k), \\ 0 & (i \neq k). \end{cases}
\end{aligned}$$

By the proof of (2.7.1) again, $\mathcal{H}^i(\mathrm{gr}_k^{P^D} C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S}))$ is also equal to the last formulas in (2.7.3.9). Hence the morphism (2.7.3.7) is a quasi-isomorphism.

Finally we show that the morphism (2.7.3.7) is independent of the data (2.4.0.1) and (2.4.0.2). Indeed, let the notations be as in §2.5. Using (2.5.3.1), we have the following commutative diagram:

$$\begin{array}{ccc}
\begin{array}{r}(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*} Q^*_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S} \\ (\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))), \tau)\end{array} & \xrightarrow{\sim} & \begin{array}{r}(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*} Q^*_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))), \\ Q^*_{(X_\bullet,Z_\bullet)/S} P^{D_\bullet})\end{array} \\
\| & & \| \\
\begin{array}{r}(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*} R\eta^{\log}_{\mathrm{Rcrys}*} Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S} L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S} \\ (\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log(\mathcal{D}_{\bullet\bullet} \cup \mathcal{Z}_{\bullet\bullet}))), \tau)\end{array} & \longrightarrow & \begin{array}{r}(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*} R\eta^{\log}_{\mathrm{Rcrys}*} Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S} L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log(\mathcal{D}_{\bullet\bullet} \cup \mathcal{Z}_{\bullet\bullet}))), \\ Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S} P^{D_{\bullet\bullet}}).\end{array}
\end{array}$$

Thus the independence in question follows. □

Definition 2.7.4. We call $(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \in \mathrm{D}^+\mathrm{F}(\mathcal{O}_{(X,Z)/S})$ the *preweight-filtered vanishing cycle crystalline complex* of $(X, D\cup Z)/S$ with respect to D. Set

$$(E^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) := Ru_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$$

and we call it the *preweight-filtered vanishing cycle zariskian complex* of $(X, D \cup Z)/S$ with respect to D.

Corollary 2.7.5. *There exists a canonical isomorphism*

$$(2.7.5.1) \qquad (E^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \xrightarrow{\sim} (C^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D).$$

Proof. The left hand side of (2.7.5.1) is equal to

$$\begin{aligned}&R\overline{u}_{(X,Z)/S*}Q^*_{(X,Z)/S}(E^{\log,Z}_{\rm crys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)\\ =&R\overline{u}_{(X,Z)/S*}(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) = (C^{\log,Z}_{\rm zar}(\mathcal{O}_{(X,D\cup Z)/S}), P^D).\end{aligned}$$

Here we have used (2.5.4.1) for the last equality. □

Corollary 2.7.6. *The spectral sequence* (2.7.1.4) *is equal to* (2.6.2.2) *if we make the renumbering* $E_r^{-k,h+k} = E_{r+1}^{h-k,k}$ $(r \geq 1)$.

Proof. By [23, (1.4.8)], the spectral sequence (2.7.1.4) is obtained from the increasing filtration $\{\tau_k C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S})\}_{k\in\mathbb{Z}}$; this filtration is equal to $\{P^D_k C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S})\}_{k\in\mathbb{Z}}$ by (2.7.3). Hence (2.7.6) follows. □

Corollary 2.7.7. (1) *The filtered complex* $(C^{\log,Z}_{\rm Rcrys}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ *is independent of the choice of the decompositions of D and Z by their smooth components. The spectral sequence* (2.6.2.2) *is also independent of the choice of them.*

(2) *Let the assumptions be as in* (2.5.6). *Then the right hand sides of* (2.5.6.1) *and* (2.5.6.2) *are independent of the choice of the decompositions of D and Z by their smooth components.*

Proof. The proof is obvious. □

Corollary 2.7.8. *The isomorphism* (2.6.1.1) *is independent of the choice of the decompositions of D and Z by their smooth components. Consequently the isomorphism* (2.6.1.2) *and the spectral sequences* (2.6.2.1), (2.6.2.2), (2.6.3.1) *and* (2.6.3.2) *are also independent of the choice.*

Proof. Since both hands of (2.6.1.1) is independent of the choice by (2.7.3) and (2.2.15), the problem is local. By (A.0.1) below, we may assume that two choices of the decompositions of D and Z by their smooth components are the same. Now the independence follows from the proof of (2.5.1) and the argument in (2.5.3). □

The following is another proof of (2.5.7):

Corollary 2.7.9. (2.5.7) *and* (2.5.8) *hold.*

Proof. By (1.6.4.1), (2.3.10.1) and the cohomological descent, we have

$$\begin{aligned} &R\pi'^{\log}_{(X,Z)/S\mathrm{crys}*}(Q^*_{(X_n,Z_n)/S}L_{(X_n,Z_n)/S}(\Omega^\bullet_{\mathcal{X}_n/S}(\log \mathcal{E}_n))_{n\in\mathbb{N}}) \\ =&Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}). \end{aligned}$$

By the same proof as that for the formula (2.7.3.2), we also have

$$\begin{aligned} &R\pi'^{\log}_{(X,Z)/S\mathrm{crys}*}((Q^*_{(X_n,Z_n)/S}L_{(X_n,Z_n)/S}(\Omega^\bullet_{\mathcal{X}_n/S}(\log \mathcal{E}_n)), \\ &\qquad\qquad\qquad\qquad\qquad\qquad Q^*_{(X_n,Z_n)/S}P^{D_n}_k)_{n\in\mathbb{N}}) \\ =&(R\pi'^{\log}_{(X,Z)/S\mathrm{crys}*}(L_{(X_n,Z_n)/S}(\Omega^\bullet_{\mathcal{X}_n/S}(\log \mathcal{E}_n))_{n\in\mathbb{N}}),\tau). \end{aligned}$$

Hence we have (2.5.7) and (2.5.8). □

We shall use the following for the preweight-filtered Künneth formula:

Proposition 2.7.10. *Assume that X is quasi-compact. Then the filtered complex $(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$ is bounded.*

Proof. By (2.3.11), $R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})$ is bounded. Hence (2.7.10) immediately follows. □

Remark 2.7.11. In this remark we show an unexpected nonequality

(2.7.11.1)
$$R^k\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) \neq a^{(k)}_{\mathrm{crys}*}(\mathcal{O}_{D^{(k)}/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{crys}}(D/S)) \quad (k\in\mathbb{N})$$

in general. More specially, in this remark, we prove that the natural morphism

(2.7.11.2) $$\mathcal{O}_{X/S}\longrightarrow \epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) = R^0\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$$

is not surjective in general if $p=0$ on S_0.

Let $(X,D)\overset{\subset}{\longrightarrow}(\mathcal{X},\mathcal{D})$ be an exact closed immersion into a smooth scheme with a relative SNCD over S. Let $\iota\colon L_{X/S}(\Omega^1_{\mathcal{X}/S})\longrightarrow L_{X/S}(\Omega^1_{\mathcal{X}/S}(\log\mathcal{D}))$ be a natural morphism of $\mathcal{O}_{X/S}$-modules, and let $d\colon L_{X/S}(\mathcal{O}_{\mathcal{X}})\longrightarrow L_{X/S}(\Omega^1_{\mathcal{X}/S})$ be the natural boundary morphism. By the crystalline Poincaré lemma and the Poincaré lemma of a vanishing cycle sheaf ((2.3.10)), we have the following:

(2.7.11.3)
$$\begin{array}{ccc} \mathcal{O}_{X/S} & \xrightarrow{=} & \mathrm{Ker}(d\colon L_{X/S}(\mathcal{O}_{\mathcal{X}})\longrightarrow L_{X/S}(\Omega^1_{\mathcal{X}/S})) \\ \big\downarrow & & \big\downarrow \\ \epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) & \xrightarrow{=} & \mathrm{Ker}(\iota\circ d\colon L_{X/S}(\mathcal{O}_{\mathcal{X}})\longrightarrow L_{X/S}(\Omega^1_{\mathcal{X}/S}(\log\mathcal{D}))) \end{array}$$

Consider the following commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & L_{X/S}(\mathcal{O}_{\mathcal{X}}) \arrow[r, equal] \arrow[d, "d"'] & L_{X/S}(\mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d, "\iota\circ d"'] & 0 \\
0 \arrow[r] & \operatorname{Ker}\iota \arrow[r] & L_{X/S}(\Omega^1_{\mathcal{X}/S}) \arrow[r] & \operatorname{Im}\iota \arrow[r] & 0.
\end{tikzcd}$$

Hence, by the snake lemma and (2.7.11.3), we obtain the following exact sequence

$$(2.7.11.4)\quad 0 \longrightarrow \mathcal{O}_{X/S} \longrightarrow \epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) \longrightarrow \operatorname{Ker}(\iota) \longrightarrow \operatorname{Coker}(d).$$

Now set $\mathcal{X} := \underline{\operatorname{Spec}}_S(\mathcal{O}_S[x])$ and let $\mathcal{D}$ be a relative smooth divisor on $\mathcal{X}$ defined by an equation $x = 0$. Set $(X, D) := (\mathcal{X}, \mathcal{D}) \times_S S_0$. In this case, $\operatorname{Coker}(d) = 0$ by the crystalline Poincaré lemma. Hence, to prove that (2.7.11.2) is not an isomorphism in this case, it suffices to prove that $\operatorname{Ker}(\iota) \neq 0$. Set $\mathcal{A}_0 := \mathcal{O}_{S_0}[x, y]/(xy)$. Let $f : \mathcal{A}_0 \longrightarrow \mathcal{O}_{S_0}[x]$ be a morphism of sheaves of rings over $\mathcal{O}_{S_0}$ defined by equations $f(x) = x$ and $f(y) = 0$. Let $\mathcal{A}_0^{\mathrm{PD}}$ be the PD-envelope of $\mathcal{A}_0$ with respect to $\operatorname{Ker}(f)$. Let δ be the PD-structure on $\overline{\operatorname{Ker}(f)}$ and let $f^{\mathrm{PD}} \colon \mathcal{A}_0^{\mathrm{PD}} \longrightarrow \mathcal{O}_{S_0}[x]$ be the induced morphism of sheaves of rings over $\mathcal{O}_{S_0}$ by f. Set $T := \underline{\operatorname{Spec}}_{S_0}(\mathcal{A}_0^{\mathrm{PD}})$. Then f induces a PD closed immersion $X \overset{\subset}{\longrightarrow} T$; the triple (X, T, δ) is an object of $(X/S)_{\mathrm{crys}}$.

Let $g \colon \mathcal{A}_0^{\mathrm{PD}} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_{S_0}[x] \longrightarrow \mathcal{O}_{S_0}[x]$ be a morphism of sheaves of rings over $\mathcal{O}_{S_0}$ defined by $g(s \otimes t) := f^{\mathrm{PD}}(s)t$ ($s \in \mathcal{A}_0^{\mathrm{PD}}$, $t \in \mathcal{O}_{S_0}[x]$) and let $\mathcal{B}$ be the PD-envelope of $\mathcal{A}_0^{\mathrm{PD}} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_{S_0}[x]$ with respect to $\operatorname{Ker}(g)$. Then, by the proof of [11, (6.10)], the value $L_{X/S}(\Omega^1_{\mathcal{X}/S})_T$ of $L_{X/S}(\Omega^1_{\mathcal{X}/S})$ at T is given by the following formula

$$L_{X/S}(\Omega^1_{\mathcal{X}/S})_T = \mathcal{B} \otimes_{\mathcal{O}_S[x]} \mathcal{O}_S[x]dx = \mathcal{B}dx,$$

while the value $L_{X/S}(\Omega^1_{\mathcal{X}}(\log \mathcal{D}))_T$ is given by the following formula

$$L_{X/S}(\Omega^1_{\mathcal{X}/S}(\log \mathcal{D}))_T = \mathcal{B}d\log x.$$

Let $\iota_T \colon L_{X/S}(\Omega^1_{\mathcal{X}/S})_T \longrightarrow L_{X/S}(\Omega^1_{\mathcal{X}/S}(\log \mathcal{D}))_T$ be the value of ι at T. Then $\iota_T(dx) = (1 \otimes x)d\log x$.

To prove that ι_T is not injective, it suffices to prove that a morphism $\mathcal{B} \longrightarrow \mathcal{B}$ given by multiplication by $1 \otimes x$ is not injective. Here we denote the image of a local section s of $\mathcal{A}_0^{\mathrm{PD}} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_{S_0}[x]$ in $\mathcal{B}$ by the same symbol s by abuse of notation. We check

(A) $y \otimes 1 \neq 0$ in $\mathcal{B}$

and

(B) $(1 \otimes x)^p(y \otimes 1) = 0$ in $\mathcal{B}$.

First we check (A). Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, "f"] \arrow[d] & \mathcal{O}_{S_0}[x] \arrow[d] \\ \mathcal{O}_{S_0}[y] \arrow[r] & \mathcal{O}_{S_0}, \end{tikzcd}$$

where the vertical morphisms are defined by sending x to 0 and the lower horizontal morphism is defined by sending y to 0. By taking the PD-envelopes with respect to the kernels of the horizontal morphisms, we obtain the following commutative diagram:

(2.7.11.5)
$$\begin{tikzcd} \mathcal{A}_0^{\mathrm{PD}} \arrow[r, "f^{\mathrm{PD}}"] \arrow[d] & \mathcal{O}_{S_0}[x] \arrow[d] \\ \mathcal{O}_{S_0}\langle y\rangle \arrow[r] & \mathcal{O}_{S_0}. \end{tikzcd}$$

Denote by φ the left vertical morphism in (2.7.11.5) and let $\psi\colon \mathcal{A}_0^{\mathrm{PD}} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_{S_0}[x] \longrightarrow \mathcal{O}_{S_0}\langle y\rangle$ be a morphism defined by $\psi(s\otimes t) := \varphi(s)\cdot(t \bmod x\mathcal{O}_{S_0}[x])$ $(s \in \mathcal{A}_0^{\mathrm{PD}}, t \in \mathcal{O}_{S_0}[x])$. Then the diagram (2.7.11.5) gives the following commutative diagram

$$\begin{tikzcd} \mathcal{A}_0^{\mathrm{PD}} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_{S_0}[x] \arrow[r, "g"] \arrow[d, "\psi"'] & \mathcal{O}_{S_0}[x] \arrow[d] \\ \mathcal{O}_{S_0}\langle y\rangle \arrow[r] & \mathcal{O}_{S_0} \end{tikzcd}$$

and then the following commutative diagram:

(2.7.11.6)
$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{O}_{S_0}[x] \arrow[d] \\ \mathcal{O}_{S_0}\langle y\rangle \arrow[r] & \mathcal{O}_{S_0}. \end{tikzcd}$$

Since the image of $y \otimes 1 \in \mathcal{B}$ by the left vertical morphism in (2.7.11.6) is equal to $y \in \mathcal{O}_{S_0}\langle y\rangle$, $y \otimes 1 \neq 0$ in $\mathcal{B}$.

Next we check (B). It is clear that $1 \otimes x - x \otimes 1 \in \mathcal{B}$ is a local section of the PD-ideal sheaf of $\mathcal{B}$. Hence we have the following equalities in $\mathcal{B}$

$$\begin{aligned}(1 \otimes x)^p(y \otimes 1) &= x^p y \otimes 1 + (1 \otimes x^p - x^p \otimes 1)(y \otimes 1) \\ &= 0 + (1 \otimes x - x \otimes 1)^p(y \otimes 1) \\ &= p!(1 \otimes x - x \otimes 1)^{[p]}(y \otimes 1) = 0\end{aligned}$$

because $p = 0$ in $\mathcal{B}$.

Now we have proved that the morphism (2.7.11.2) is not an isomorphism in general.

We also remark the following.

By the p-adic purity in $(\widetilde{X/S})_{\mathrm{Rcrys}}$ ((2.7.1)), $(\mathcal{O}_{X/S})_{\mathcal{X}} = (\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}))_{\mathcal{X}}$. Hence the exact sequence (2.7.11.4) tells us that $\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$ is not a crystal of $\mathcal{O}_{X/S}$-modules in general.

Remark 2.7.12. (1) Let (X, D) be a smooth analytic variety with (not necessarily simple) NCD over the complex number field. Let U be the complement of D in X and let j be the natural inclusion $U \overset{\subset}{\longrightarrow} X$. Let $\widetilde{D}^{(0)}$ be the normalization of D and for a positive integer k, define $\widetilde{D}^{(k)}$ in the way described in (2.2.15) from $\widetilde{D}^{(0)}$. Let $\widetilde{a}^{(k)} \colon \widetilde{D}^{(k)} \longrightarrow X$ be the natural morphism. Then, in [23, (3.1.8)], Deligne has proved that

$$(2.7.12.1) \qquad (\Omega^\bullet_{X/\mathbb{C}}(\log D), \tau) \longrightarrow (\Omega^\bullet_{X/\mathbb{C}}(\log D), P)$$

is a quasi-isomorphism by using the Poincaré residue isomorphism and the Poincaré lemma

$$\begin{aligned}\mathrm{gr}_k^P \Omega^\bullet_{X/\mathbb{C}}(\log D) &\underset{\sim}{\overset{\mathrm{Res}}{\longrightarrow}} \widetilde{a}^{(k)}_*(\Omega^\bullet_{\widetilde{D}^{(k)}/\mathbb{C}}\{-k\} \otimes_{\mathbb{Z}} \widetilde{\varpi}^{(k)}(D/\mathbb{C})(-k)) \\ &= \widetilde{a}^{(k)}_*(\mathbb{C}_{\widetilde{D}^{(k)}}\{-k\} \otimes_{\mathbb{Z}} \widetilde{\varpi}^{(k)}(D/\mathbb{C})(-k)),\end{aligned}$$

where $\widetilde{\varpi}^{(k)}(D/\mathbb{C})$ is the orientation sheaf of $\widetilde{D}^{(k)}$ (Since we have used the notation ϵ as a forgetting log morphism, we cannot use the notation ϵ in [23]). Note that, in (2.7.1), (2.7.3) and (2.7.12.1), the graded pieces gr_k^P is isomorphic to the complex which consists of one component; this property is a key point for (2.7.1) and the quasi-isomorphism (2.7.12.1). It is reasonable to expect that, if D is an SNCD, if we use the log infinitesimal topos and if we develop analogous theory for this topos by the same method as that in this book, we will be able to prove that

$$(2.7.12.2) \qquad R^k\epsilon_*(\mathcal{O}_{X/\mathbb{C}}) = a^{(k)}_*(\mathcal{O}_{D^{(k)}/\mathbb{C}} \otimes_{\mathbb{Z}} \varpi^{(k)}(D/\mathbb{C})(-k)),$$

where $\epsilon \colon (\widetilde{X/\mathbb{C}})^{\log}_{\mathrm{inf}} \longrightarrow (\widetilde{X/\mathbb{C}})_{\mathrm{inf}}$ is the forgetting log morphism of infinitesimal topoi, $\mathcal{O}_{X/\mathbb{C}}$ (resp. $\mathcal{O}_{D^{(k)}/\mathbb{C}}$) is the structure sheaf in $(\widetilde{X/\mathbb{C}})^{\log}_{\mathrm{inf}}$ (resp. $(\widetilde{D^{(k)}/\mathbb{C}})_{\mathrm{inf}}$), $a^{(k)} := \widetilde{a}^{(k)} \colon D^{(k)} = \widetilde{D}^{(k)} \longrightarrow X$ and $\varpi^{(k)}(D/\mathbb{C}) := \widetilde{\varpi}^{(k)}(D/\mathbb{C})$.

(2) The morphism (2.7.2.1) is not a filtered isomorphism in general. Indeed, if it were so, we would have the following contradiction.

Assume that $Z = \emptyset$ and that it were an isomorphism. Then, by applying $R\overline{u}_{X/S*}$ to (2.7.3.2), we would have

$$(2.7.12.3) \qquad \begin{aligned}(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P) &= Ru_{X/S*}(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P)\\ &= Ru_{X/S*}(R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}),\tau)\\ &= (Ru_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}),\tau)\\ &= (C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),\tau).\end{aligned}$$

Here the first equality follows from (2.7.5.1). The third equality follows from our assumption. The fourth equality follows from (2.4.7.3). However it is practically well-known that the equality (2.7.12.3) does not hold in general. Indeed, let κ be a field of characteristic $p > 0$ and let (X, D) be a smooth scheme with an SNCD over κ. Assume that $S = S_0 = \mathrm{Spec}(\kappa)$. Then (2.7.12.3) is an isomorphism

$$(\Omega^\bullet_{X/\kappa}(\log D),\tau) = (\Omega^\bullet_{X/\kappa}(\log D),P).$$

If we take $X := \mathbb{A}^1_\kappa$, D: the origin of X and $k = 0$, we have a contradiction. Hence (2.7.2.1) is not a filtered isomorphism in general.

2.8 Boundary Morphisms

In this section we define the log cycle class of a smooth divisor which intersects the log locus transversally (cf. [29, §2]).

As an application, we give the description of the boundary morphism between the E_1-terms of the spectral sequence (2.6.2.2).

Let $f\colon (X,Z) \longrightarrow S_0$ be a smooth scheme with a relative SNCD over a scheme S_0. Let D be a smooth divisor on X which intersects Z transversally over S_0; for a decomposition $\Delta = \{Z_\mu\}_\mu$ of Z by smooth components of Z, $\Delta(D) := \{D, Z_\mu\}_\mu$ is a decomposition of $D \cup Z$ by smooth components of $D \cup Z$. The closed subscheme $Z|_D := Z \cap D$ in D is a relative SNCD on D/S_0; $\Delta|_D := \{Z_\mu|_D\}_\mu$ be a decomposition of $Z|_D$ by smooth components of $Z|_D$. Let $a\colon (D,Z|_D) \xrightarrow{\subset} (X,Z)$ be the natural closed immersion over S_0. Let $a_{\mathrm{zar}}\colon (\widetilde{D}_{\mathrm{zar}},\mathcal{O}_D) \longrightarrow (\widetilde{X}_{\mathrm{zar}},\mathcal{O}_X)$ be the induced morphism of Zariski ringed topoi. Let $a^{\log}_{\mathrm{crys}}\colon (((\widetilde{D,Z|_D})/S)^{\log}_{\mathrm{crys}},\mathcal{O}_{(D,Z|_D)/S}) \longrightarrow (((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}},\mathcal{O}_{(X,Z)/S})$ be also the induced morphism of log crystalline ringed topoi. Let

(2.8.0.1)
$$\mathrm{Res}^D\colon \Omega^\bullet_{X/S_0}(\log(D\cup Z)) \longrightarrow a_{\mathrm{zar}*}(\Omega^\bullet_{D/S_0}(\log(Z|_D)) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(D/S_0))\{-1\}$$

be the Poincaré residue morphism with respect to D/S_0. Then we have the following exact sequence:

$$(2.8.0.2) \qquad 0 \longrightarrow \Omega^\bullet_{X/S_0}(\log Z) \longrightarrow \Omega^\bullet_{X/S_0}(\log(D\cup Z))$$

$$\xrightarrow{\mathrm{Res}^D} a_{\mathrm{zar}*}(\Omega^{\bullet}_{D/S}(\log(Z|_D)) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(D/S_0))\{-1\} \longrightarrow 0.$$

Let $(S, \mathcal{I}, \gamma)$ and S_0 be as in §2.4. As in §2.4, by abuse of notation, we also denote by f the composite morphism $(X, Z) \longrightarrow S_0 \overset{\subset}{\longrightarrow} S$.

As in §2.4, we have the following data:

(2.8.0.3): An open covering $X = \bigcup_{i_0 \in I_0} X_{i_0}$ with $X_i = \bigcap_{s=0}^{r} X_{i_s}$ $(i = (i_0, \ldots, i_r))$. The family $\{(X_i, D_i \cup Z_i)\}_{i \in I}$ $(D_i := D \cap X_i,\ Z_i := Z \cap X_i)$ of log schemes form a diagram of log schemes over $(X, D \cup Z)$, which we denote by $(X_\bullet, D_\bullet \cup Z_\bullet)$. That is, $(X_\bullet, D_\bullet \cup Z_\bullet)$ is a contravariant functor

$$\begin{aligned} I^o \longrightarrow \{&\text{smooth schemes with relative SNCD's over } S_0 \\ &\text{which are augmented to } (X, D \cup Z)\}. \end{aligned}$$

We have a diagram $\Delta_\bullet(D_\bullet)$ of a decomposition of $D_\bullet \cup Z_\bullet$ by a diagram of smooth components of $D_\bullet \cup Z_\bullet$.

(2.8.0.4): A family $(X_\bullet, D_\bullet \cup Z_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)$ $(\bullet \in I)$ of admissible immersions into a diagram of smooth schemes with relative SNCD's over S with respect to $\Delta_\bullet(D_\bullet)$.

Let $b_\bullet \colon \mathcal{D}_\bullet \longrightarrow \mathcal{X}_\bullet$ be a diagram of the natural closed immersions. By using the Poincaré residue isomorphism with respect to $\mathcal{D}_\bullet$, we have the following exact sequence ([29, §2]):

(2.8.0.5)
$$0 \longrightarrow \Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \longrightarrow \Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)) \xrightarrow{\mathrm{Res}}$$
$$b_{\bullet\mathrm{zar}*}(\Omega^{\bullet}_{\mathcal{D}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}_\bullet})) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S))\{-1\} \longrightarrow 0.$$

Let $L_{(X_\bullet, Z_\bullet)/S}$ (resp. $L_{(D_\bullet, Z_\bullet|_{D_\bullet})/S}$) be the log linearization functor with respect to the diagram of the locally closed immersions $(X_\bullet, Z_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}_\bullet, \mathcal{Z}_\bullet)$ (resp. $(D_\bullet, Z_\bullet|_{D_\bullet}) \overset{\subset}{\longrightarrow} (\mathcal{D}_\bullet, \mathcal{Z}_\bullet|_{\mathcal{D}_\bullet})$). By (2.2.12) and (2.2.16), $L_{(X_\bullet, Z_\bullet)/S} b_{\bullet\mathrm{zar}*} = a^{\log}_{\mathrm{crys}\bullet *} L_{(D_\bullet, Z_\bullet|_{D_\bullet})/S}$. Hence we have the following exact sequence by (2.2.17) (2) and (2.2.21.2):

(2.8.0.6)
$$\begin{aligned} 0 &\longrightarrow Q^*_{(X_\bullet, Z_\bullet)/S} L_{(X_\bullet, Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet)) \\ &\longrightarrow Q^*_{(X_\bullet, Z_\bullet)/S} L_{(X_\bullet, Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))) \\ &\longrightarrow Q^*_{(X_\bullet, Z_\bullet)/S} a^{\log}_{\mathrm{crys}\bullet *}(L_{(D_\bullet, Z_\bullet|_{D_\bullet})/S}(\Omega^{\bullet}_{\mathcal{D}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}_\bullet}))) \\ &\qquad\qquad \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D_\bullet/S; Z_\bullet))\{-1\}) \longrightarrow 0. \end{aligned}$$

Recall the morphisms $\pi^{\log}_{(X,Z)/S\mathrm{crys}}$ and $\pi^{\log}_{(D,Z|_D)/S\mathrm{crys}}$ of ringed topoi in (2.4.7.4) for the case $D = \phi$ and (2.6.1.3). By (1.6.0.23) we have the following triangle

(2.8.0.7)

$$\longrightarrow Q^*_{(X,Z)/S}R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet)) \longrightarrow$$
$$Q^*_{(X,Z)/S}R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))) \longrightarrow$$
$$Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}R\pi^{\log}_{(D,Z|_D)/S\mathrm{crys}*}L_{(D_\bullet,Z_\bullet|_{D_\bullet})/S}(\Omega^\bullet_{\mathcal{D}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}_\bullet}))$$
$$\otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D_\bullet/S;Z_\bullet))\{-1\} \xrightarrow{+1} \cdots .$$

By (2.2.7), (2.3.10.1) and by the cohomological descent, we have the following triangle:

(2.8.0.8)

$$\longrightarrow Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}) \longrightarrow Q^*_{(X,Z)/S}R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) \longrightarrow$$
$$Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S} \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z))\{-1\} \xrightarrow{+1} \cdots .$$

Using the Convention (4), we have the boundary morphism

(2.8.0.9)

$$d\colon Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S} \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)))\{-1\}$$
$$\longrightarrow Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})[1]$$

in $\mathrm{D}^+(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$. Equivalently, we have the following morphism

(2.8.0.10)

$$d\colon Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S} \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)))$$
$$\longrightarrow Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})[1]\{1\}$$

in $\mathrm{D}^+(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$. Set

(2.8.0.11) $$G_{D/(X,Z)} := -d.$$

and call $G_{D/(X,Z)}$ the *Gysin morphism* of D. Then we have a cohomology class

(2.8.0.12)

$$c_{(X,Z)/S}(D) := G_{D/(X,Z)} \in \mathcal{E}xt^0_{Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})}(Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S}$$
$$\otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)), Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})[1]\{1\}).$$

Since $\varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)$ is canonically isomorphic to $\mathbb{Z}$ and since there exists a natural morphism $Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}) \longrightarrow Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S})$, we have a cohomology class

$$c_{(X,Z)/S}(D) \in \mathcal{E}xt^0_{Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})}(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}), Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})[1]\{1\})$$
$$=: Q^*_{(X,Z)/S}\underline{\mathcal{H}}^2_{\text{log-crys}}((X,Z)/S).$$

As usual, if $Z = \emptyset$, we denote $G_{D/(X,Z)}$ and $c_{(X,Z)/S}(D)$ simply by $G_{D/X}$ and $c_{X/S}(D)$, respectively.

Remark 2.8.1. (cf. [35, (1.6)]) Let $t_\bullet = 0$ be a local equation of $\mathcal{D}_\bullet$ in $\mathcal{X}_\bullet$. If we use a Poincaré residue morphism

$$\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)) \ni d\log t_\bullet \wedge \omega_\bullet \longmapsto \omega_\bullet|_{\mathcal{D}_\bullet} \in b_{\bullet\mathrm{zar}*}(\Omega^\bullet_{\mathcal{D}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}_\bullet}))$$
$$\otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S; Z_\bullet))[-1]$$

instead of the Poincaré residue morphism in (2.8.0.5), then we have a Gysin morphism

$$G_{D/(X,Z)} \colon Q^*_{(X,Z)/S} a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S} \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S; Z))[-1]$$
$$\longrightarrow Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S})[1].$$

Here we have used the Convention (4). Hence, by the Convention (2), we have a Gysin morphism

$$G_{D/(X,Z)} \colon Q^*_{(X,Z)/S} a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S} \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S; Z))[-2] \longrightarrow Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}). \tag{2.8.1.1}$$

However we do not use this Gysin morphism in this book.

Proposition 2.8.2. *The morphism* $G_{D/(X,Z)}$ *and the class* $c_{(X,Z)/S}(D)$ *are independent of the data* (2.8.0.3) *and* (2.8.0.4).

Proof. Use notations in §2.5. Assume that we are given two data in (2.8.0.3) and two data in (2.8.0.4). Because the question is local, we may assume that the two admissible immersions are admissible immersions with respect to the same decompositions of D and Z by their smooth components. As in §2.5 we have two morphisms

$$\eta_{(X,Z)/S} \colon (((\widetilde{X_{\bullet\bullet}, Z_{\bullet\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S})$$
$$\longrightarrow (((\widetilde{X_\bullet, Z_\bullet})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_\bullet,Z_\bullet)/S}),$$

and

$$\eta_{(D,Z|_D)/S} \colon (((\widetilde{D_{\bullet\bullet}, Z_{\bullet\bullet}|_{D_{\bullet\bullet}}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(D_{\bullet\bullet},Z_{\bullet\bullet}|_{D_{\bullet\bullet}})/S})$$
$$\longrightarrow (((\widetilde{D_\bullet, Z_\bullet|_{D_\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(D_\bullet,Z_\bullet|_{D_\bullet})/S})$$

of ringed topoi. Then we have the following commutative diagram of triangles:

(2.8.2.1)

$$\begin{array}{ccc}
\longrightarrow & Q^*_{(X_\bullet,Z_\bullet)/S}R\eta_{(X,Z)/S*}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{Z}_{\bullet\bullet})) & \longrightarrow \\
& \big\uparrow & \\
\longrightarrow & Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet)) & \longrightarrow
\end{array}$$

$$\begin{array}{cc}
Q^*_{(X_\bullet,Z_\bullet)/S}R\eta_{(X,Z)/S*}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^\bullet_{\mathcal{X}_{\bullet\bullet}/S}(\log(\mathcal{D}_{\bullet\bullet}\cup \mathcal{Z}_{\bullet\bullet}))) & \longrightarrow \\
\big\uparrow & \\
Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup \mathcal{Z}_\bullet))) & \longrightarrow \\
Q^*_{(X_\bullet,Z_\bullet)/S}a^{\log}_{\mathrm{crys}\bullet*}R\eta_{(D,Z|_D)/S*}L_{(D_{\bullet\bullet},Z_{\bullet\bullet}|_{D_{\bullet\bullet}})/S} & \\
(\Omega^\bullet_{\mathcal{D}_{\bullet\bullet}/S}(\log(\mathcal{Z}_{\bullet\bullet}|_{\mathcal{D}_{\bullet\bullet}}))\{-1\}) & \xrightarrow{+1} \\
\big\uparrow & \\
Q^*_{(X_\bullet,Z_\bullet)/S}a^{\log}_{\mathrm{crys}\bullet*}L_{(D_\bullet,Z_\bullet|_{D_\bullet})/S}(\Omega^\bullet_{\mathcal{D}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}_\bullet}))\{-1\}) & \xrightarrow{+1}\ .
\end{array}$$

By the proof of (2.5.3), the three vertical morphisms above are isomorphisms. Hence (2.8.2) follows. □

Remark 2.8.3. We can also construct $c_{(X,Z)/S}(D)$ by using the vanishing cycle sheaf as follows.

Let $\epsilon_{(X,D\cup Z,Z)/S}\colon ((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}} \longrightarrow ((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}}$ be the forgetting log morphism along D ((2.3.2)). By (2.3.2.9), there exists a natural morphism

$$\mathcal{O}_{(X,Z)/S} \longrightarrow R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) \tag{2.8.3.1}$$

in $\mathrm{D}^+(\mathcal{O}_{(X,Z)/S})$. Let $\underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S})$ be the mapping fiber of (2.8.3.1). Then we have a triangle

(2.8.3.2)
$$\longrightarrow \underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S}) \longrightarrow \mathcal{O}_{(X,Z)/S} \longrightarrow R\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) \xrightarrow{+1}.$$

Set $\mathcal{H}^i_D(\mathcal{O}_{(X,Z)/S}) := \mathcal{H}^i(\underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S}))$ $(i\in\mathbb{Z})$. Then we have the following exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathcal{H}^i_D(\mathcal{O}_{(X,Z)/S}) &\longrightarrow \mathcal{H}^i(\mathcal{O}_{(X,Z)/S}) \\ &\longrightarrow R^i\epsilon_{(X,D\cup Z,Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}) \longrightarrow \cdots .\end{aligned} \tag{2.8.3.3}$$

Here we have used the Convention (4) and (5). By (2.7.1), we have

$$\begin{aligned}&Q^*_{(X,Z)/S}\mathcal{H}^i_D(\mathcal{O}_{(X,Z)/S}) \\ &= \begin{cases} Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)) & (i=2),\\ 0 & (i\neq 2).\end{cases}\end{aligned} \tag{2.8.3.4}$$

Let $E^\bullet$ be a representative of $Q^*_{(X,Z)/S}\underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S})$. Then we have an isomorphism

$$\tau_2 E^\bullet \xrightarrow{\sim} E^\bullet$$

and we can take an isomorphism

$$\tau_2 E^\bullet \xrightarrow{\sim} Q^*_{(X,Z)/S}\mathcal{H}^2_D(\mathcal{O}_{(X,Z)/S})\{-1\}[-1]. \tag{2.8.3.5}$$

Therefore we have a canonical isomorphism

$$\begin{aligned}&Q^*_{(X,Z)/S}\underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S})\\ =&Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z))\{-1\}[-1].\end{aligned}$$

Since there exists a natural morphism $\underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S}) \longrightarrow \mathcal{O}_{(X,Z)/S}$ by the definition of $\underline{R\Gamma}_D(\mathcal{O}_{(X,Z)/S})$, we have a canonical morphism

$$\begin{aligned}Q^*_{(X,Z)/S}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z))\{-1\}[-1]&\\ \longrightarrow Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}).&\end{aligned} \tag{2.8.3.6}$$

By (2.8.0.8), we see that the morphism (2.8.3.6) is equal to $-G_{D/(X,Z)}$.

If we take the canonical isomorphism

$$\tau_2 E^\bullet \xrightarrow{\sim} Q^*_{(X,Z)/S}\mathcal{H}^2_D(\mathcal{O}_{(X,Z)/S})[-2]. \tag{2.8.3.7}$$

instead of (2.8.3.5), we obtain the Gysin morphism (2.8.1.1) again.

Proposition 2.8.4. *Let $u\colon (S',\mathcal{I}',\gamma') \longrightarrow (S,\mathcal{I},\gamma)$ be a morphism of PD-schemes. Set $S'_0 := \underline{\mathrm{Spec}}_{S'}(\mathcal{O}_{S'}/\mathcal{I}')$. Let $h\colon Y \longrightarrow S'_0$ be a smooth morphism of schemes fitting into the following commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S'_0 & \longrightarrow & S_0. \end{array}$$

Set $E := D\times_X Y$ and $W := Z\times_X Y$. Assume that $E\cup W$ is a relative SNCD on Y over S_0. Let $b\colon (E,W|_E) \xrightarrow{\subset} (Y,W)$ be a natural closed immersion of log schemes. Then the image of $g^{-1}_{\mathrm{zar}}Ru_{(X,Z)/S*}(c_{(X,Z)/S}(D))$ in (2.8.0.12) *by the natural morphism*

$$\begin{aligned}&g^{-1}_{\mathrm{zar}}\mathcal{E}xt^0_{f^{-1}(\mathcal{O}_S)}(Ru_{(X,Z)/S*}a^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(D,Z|_D)/S}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)),\\ &\qquad\qquad Ru_{(X,Z)/S*}(\mathcal{O}_{(X,Z)/S})[1]\{1\})\\ \longrightarrow\ &\mathcal{E}xt^0_{h^{-1}(\mathcal{O}_{S'})}(Ru_{(Y,W)/S*}b^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(E,W|_E)/S'}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\mathrm{crys}}(E/S';W)),\\ &\qquad\qquad Ru_{(Y,W)/S*}(\mathcal{O}_{(Y,W)/S})[1]\{1\})\end{aligned}$$

is equal to $Ru_{(Y,W)/S*}(c_{(Y,W)/S'}(E))$.

Proof. (2.8.4) immediately follows from the functoriality of the construction given in (2.8.3). □

Finally we prove that the boundary morphism $d_1^{\bullet\bullet}$ of (2.6.2.2) is expressed by summation of Gysin morphisms with signs.

Henceforth D denotes a (not necessarily smooth) relative SNCD on X over S_0 which meets Z transversally. First, fix a decomposition $\{D_\lambda\}_{\lambda\in\Lambda}$ of D by smooth components of D over S_0. Assume that $D_{\{\lambda_0,\ldots,\lambda_{k-1}\}}\neq\emptyset$. Set $\underline{\lambda}:=\{\lambda_0,\ldots,\lambda_{k-1}\}$, $\underline{\lambda}_j:=\{\lambda_0,\ldots,\widehat{\lambda}_j,\ldots,\lambda_{k-1}\}$, $D_{\underline{\lambda}}:=D_{\{\lambda_0,\ldots,\lambda_{k-1}\}}$, and $D_{\underline{\lambda}_j}:=D_{\{\lambda_0,\ldots,\widehat{\lambda}_j,\ldots,\lambda_{k-1}\}}$ for $k\geq 2$ and $D_{\underline{\lambda}_0}:=X$. Here $\widehat{\ }$ means the elimination. Then $D_{\underline{\lambda}}$ is a smooth divisor on $D_{\underline{\lambda}_j}$ over S_0. Let $\iota_{\underline{\lambda}}^{\lambda_j}\colon(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})\overset{\subset}{\longrightarrow}(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})$ be the closed immersion. Set

$$\varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S;Z):=\varpi^{\log}_{\lambda_0\cdots\lambda_{k-1}\mathrm{crys}}(D/S;Z)$$

and

$$\varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S;Z):=\varpi^{\log}_{\lambda_0\cdots\widehat{\lambda}_j\cdots\lambda_{k-1}\mathrm{crys}}(D/S;Z).$$

By (2.8.0.11) we have a morphism

$$\begin{aligned}(2.8.4.1)\qquad & G_{\underline{\lambda}}^{\lambda_j}:=G_{D_{\underline{\lambda}}/(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})}\colon\\ & Q^*_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S}\iota^{\lambda_j\log}_{\underline{\lambda}\mathrm{crys}*}(\mathcal{O}_{(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})/S}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\lambda_j\mathrm{crys}}(D/S;Z))\\ & \longrightarrow Q^*_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S}(\mathcal{O}_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S})[1]\{1\}.\end{aligned}$$

We fix an isomorphism

$$(2.8.4.2)\qquad \varpi^{\log}_{\lambda_j\mathrm{crys}}(D/S;Z)\otimes_{\mathbb{Z}}\varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S;Z)\overset{\sim}{\longrightarrow}\varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S;Z)$$

by the following morphism

$$(\lambda_j)\otimes(\lambda_0\cdots\widehat{\lambda}_j\cdots\lambda_{k-1})\longmapsto(-1)^j(\lambda_0\cdots\lambda_{k-1}).$$

We identify $\varpi^{\log}_{\lambda_j\mathrm{crys}}(D/S;Z)\otimes_{\mathbb{Z}}\varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S;Z)$ with $\varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S;Z)$ by this isomorphism. We also have the following composite morphism

(2.8.4.3)

$$\begin{aligned}&(-1)^jG_{\underline{\lambda}}^{\lambda_j}\colon Q^*_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S}\iota^{\lambda_j\log}_{\underline{\lambda}\mathrm{crys}*}(\mathcal{O}_{(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})/S}\otimes_{\mathbb{Z}}\varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S;Z))\overset{\sim}{\longrightarrow}\\ &Q^*_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S}\iota^{\lambda_j\log}_{\underline{\lambda}\mathrm{crys}*}(\mathcal{O}_{(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})/S}\otimes_{\mathbb{Z}}\varpi^{\log}_{\lambda_j\mathrm{crys}}(D/S;Z)\otimes_{\mathbb{Z}}\varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S;Z))\\ &\overset{G_{\underline{\lambda}}^{\lambda_j}\otimes 1}{\longrightarrow}Q^*_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S}(\mathcal{O}_{(D_{\underline{\lambda}_j},Z|_{D_{\underline{\lambda}_j}})/S}\otimes_{\mathbb{Z}}\varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S;Z))[1]\{1\}\end{aligned}$$

defined by

$$\text{(2.8.4.4)} \qquad \text{“}x \otimes (\lambda_0 \cdots \lambda_{k-1}) \longmapsto (-1)^j G^{\lambda_j}_{\underline{\lambda}}(x) \otimes (\lambda_0 \cdots \widehat{\lambda}_j \cdots \lambda_{k-1})\text{”}.$$

The morphism (2.8.4.3) induces a morphism of log crystalline cohomologies:

(2.8.4.5)
$$(-1)^j G^{\lambda_j}_{\underline{\lambda}} \colon R^{h-k} f_{(D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}})/S*}(\mathcal{O}_{(D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}})/S} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S; Z)) \longrightarrow$$
$$R^{h-k+2} f_{(D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}})/S*}(\mathcal{O}_{(D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}})/S} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S; Z)).$$

Here we have used the Convention (6). If $D_{\{\lambda_0,\ldots,\lambda_{k-1}\}} = \emptyset$, set $(-1)^j G^{\lambda_j}_{\underline{\lambda}} := 0$.

Denote by $a_{\underline{\lambda}}$ (resp. $a_{\underline{\lambda}_j}$) the natural exact closed immersion $(D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}}) \overset{\subset}{\longrightarrow} (X, Z)$ (resp. $(D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}}) \overset{\subset}{\longrightarrow} (X, Z)$).

Proposition 2.8.5. *Let* $d_1^{-k,h+k} \colon E_1^{-k,h+k} \longrightarrow E_1^{-k+1,h+k}$ *be the boundary morphism of* (2.6.2.2). *Set* $G := \sum_{\{\lambda_0,\ldots,\lambda_{k-1} \mid \lambda_i \neq \lambda_j \ (i\neq j)\}} \sum_{j=0}^{k-1} (-1)^j G^{\lambda_j}_{\underline{\lambda}}$. *Then* $d_1^{-k,h+k} = -G$.

Proof. (cf. [64, 4.3]) Assume that we are given the data (2.4.0.1) and (2.4.0.2) for $D \cup Z$. Consider the following exact sequence

$$0 \longrightarrow \mathrm{gr}_{k-1}^{Q^*_{(X_\bullet,D_\bullet)/S} P^{D_\bullet}} (Q^*_{(X_\bullet,D_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)))) \longrightarrow$$
$$(Q^*_{(X_\bullet,D_\bullet)/S} P^{D_\bullet}_k / Q^*_{(X_\bullet,D_\bullet)/S} P^{D_\bullet}_{k-2})(Q^*_{(X_\bullet,D_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))))$$
$$\longrightarrow \mathrm{gr}_k^{Q^*_{(X_\bullet,D_\bullet)S} P^{D_\bullet}} (Q^*_{(X_\bullet,D_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)))) \longrightarrow 0.$$

Then the boundary morphism $d_1^{-k,h+k}$ is induced by the boundary morphism of the following triangle

$$\longrightarrow R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*} \mathrm{gr}_{k-1}^{Q^*_{(X_\bullet,D_\bullet)/S} P^{D_\bullet}} (Q^*_{(X_\bullet,D_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))))$$
$$\longrightarrow R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}((Q^*_{(X_\bullet,D_\bullet)/S} P^{D_\bullet}_k / Q^*_{(X_\bullet,D_\bullet)/S} P^{D_\bullet}_{k-2})$$
$$(Q^*_{(X_\bullet,D_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))))) \longrightarrow$$
$$R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*} \mathrm{gr}_k^{Q^*_{(X_\bullet,D_\bullet)S} P^{D_\bullet}} (Q^*_{(X_\bullet,D_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)))) \xrightarrow{+1} .$$

Here we have used the Convention (4).

Assume that $\mathcal{D}_{(\underline{\lambda};\bullet)} := \mathcal{D}_{(\lambda_0;\bullet)} \cap \cdots \cap \mathcal{D}_{(\lambda_{k-1};\bullet)} \neq \emptyset$. Set $\mathcal{D}_{(\underline{\lambda}_j;\bullet)} := \mathcal{D}_{(\lambda_0;\bullet)} \cap \cdots \cap \mathcal{D}_{(\lambda_{j-1};\bullet)} \cap \mathcal{D}_{(\lambda_{j+1};\bullet)} \cap \cdots \cap \mathcal{D}_{(\lambda_{k-1};\bullet)}$. We use a shorter notation $\varpi_{\underline{\lambda}\mathrm{zar}}(\mathcal{D}_\bullet/S; \mathcal{Z}_\bullet)$ for a zariskian orientation sheaf $\varpi_{\lambda_0 \cdots \lambda_{k-1}\mathrm{zar}}(\mathcal{D}_\bullet/S; \mathcal{Z}_\bullet)$ and so on as for crystalline orientation sheaves.

The Poincaré residue morphisms with respect to $\mathcal{D}_{\underline{\lambda}_j}$ $(0 \leq j \leq k-1)$ and $\mathcal{D}_{\underline{\lambda}}$ induce the following morphisms

$$\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j} : \mathrm{gr}^{P^{\mathcal{D}_\bullet}}_{k-1} \Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)) \longrightarrow$$

$$\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}})\{-(k-1)\} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)$$

and

$$\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}} : \mathrm{gr}^{P^{\mathcal{D}_\bullet}}_{k} \Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)) \longrightarrow$$

$$\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}})\{-k\} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda} \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet).$$

As in (2.8.4.2), we fix an isomorphism

(2.8.5.1) $$\varpi_{\lambda_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet) \xrightarrow{\sim} \varpi_{\underline{\lambda} \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)$$

by the following morphism

$$(\lambda_j) \otimes (\lambda_0 \cdots \widehat{\lambda}_j \cdots \lambda_{k-1}) \longmapsto (-1)^j (\lambda_0 \cdots \lambda_{k-1}).$$

We identify $\varpi_{\lambda_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)$ with $\varpi_{\underline{\lambda} \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)$ by this isomorphism. Let Res_j be the Poincaré residue morphism

(2.8.5.2) $$\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}/S}(\log(\mathcal{D}_{(\underline{\lambda};\bullet)} \cup \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}})) \longrightarrow$$

$$\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}})\{-1\} \otimes_{\mathbb{Z}} \varpi_{\lambda_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)$$

with respect to the divisor $\mathcal{D}_{(\underline{\lambda};\bullet)}$ on $\mathcal{D}_{(\underline{\lambda}_j;\bullet)}$. Then we have a composite morphism

$$\begin{aligned}
&(-1)^j\mathrm{Res}_j : \Omega^\bullet_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}/S}(\log(\mathcal{D}_{(\underline{\lambda};\bullet)} \cup \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}})) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)\\
&\longrightarrow \Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}})\{-1\} \otimes_{\mathbb{Z}} \varpi_{\lambda_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_j \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet)\\
&\xrightarrow{\sim} \Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}})\{-1\} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda} \mathrm{zar}}(\mathcal{D}_\bullet/S;\mathcal{Z}_\bullet).
\end{aligned}$$

defined by

(2.8.5.3) $$x \otimes (\lambda_0 \cdots \widehat{\lambda}_j \cdots \lambda_{k-1}) \longmapsto (-1)^j \mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j}(x) \otimes (\lambda_0 \cdots \lambda_{k-1}).$$

It is easy to check that $(-1)^j\mathrm{Res}_j$ is well-defined. The morphism $(-1)^j\mathrm{Res}_j$ induces a morphism

(2.8.5.4)

$$\begin{aligned}
&L_{(X_\bullet,Z_\bullet)/S}((-1)^j\mathrm{Res}_j) :\\
&L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}/S}(\log(\mathcal{D}_{(\underline{\lambda};\bullet)} \cup \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}})) \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_j \mathrm{crys}}(D/S;Z_\bullet))\\
&\longrightarrow L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}})\{-1\} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda} \mathrm{crys}}(D_\bullet/S;Z_\bullet)).
\end{aligned}$$

As in [64, 4.3], the morphism $Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j})$ uniquely extends to a morphism $Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j,\underline{\lambda}})$ fitting into the following commutative diagram:

(2.8.5.5)

$$
\begin{array}{ccc}
0 \longrightarrow & \mathrm{gr}_{k-1}^{Q^*_{(X_\bullet,D_\bullet)/S}P^{D_\bullet}} Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))) & \\
 & \Big\downarrow {\scriptstyle Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j})} & \\
0 \longrightarrow & Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}}))\{-(k-1)\} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D_\bullet/S;Z_\bullet) &
\end{array}
$$

$$
\begin{array}{c}
\longrightarrow \\
(Q^*_{(X_\bullet,D_\bullet)/S}P^{D_\bullet}_k/Q^*_{(X_\bullet,D_\bullet)/S}P^{D_\bullet}_{k-2})Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))) \\
\Big\downarrow {\scriptstyle Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j,\underline{\lambda}})} \\
Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}/S}(\log(\mathcal{D}_{(\underline{\lambda};\bullet)} \cup \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda}_j;\bullet)}})))\{-(k-1)\} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D_\bullet/S;Z_\bullet) \\
\longrightarrow
\end{array}
$$

$$
\begin{array}{cc}
\xrightarrow{Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}((-1)^j\mathrm{Res}_j)} & \\
\mathrm{gr}_k^{Q^*_{(X_\bullet,D_\bullet)/S}P^{D_\bullet}} Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{D}_\bullet)) & \longrightarrow 0 \\
\Big\downarrow {\scriptstyle Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}})} & \\
Q^*_{(X_\bullet,D_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}}))\{-k\} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D_\bullet/S;Z_\bullet) & \longrightarrow 0.
\end{array}
$$

Here the morphism $\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j,\underline{\lambda}}$ is defined by a formula

$$\mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}_j,\underline{\lambda}}(yd\log x_{\lambda_0}\cdots d\log x_{\lambda_{k-1}}) = (-1)^j yd\log x_{\lambda_j} \otimes (\lambda_0 \cdots \widehat{\lambda}_j \cdots \lambda_{k-1}),$$

where $x_{\lambda_i} = 0$ ($x_{\lambda_i} \in \mathcal{O}_{\mathcal{X}_\bullet}$) is a local equation of $\mathcal{D}_{(\lambda_i;\bullet)}$ in $\mathcal{X}_\bullet$ and y is a local section of $\Omega^\bullet_{\mathcal{X}_\bullet}(\log \mathcal{Z}_\bullet)$ (the formula $\mathrm{Rés}^I_{I_q}(\omega) = \alpha \wedge dx_{i_q}/x_{i_q}|_{D_{I_q}}$ in [64, p. 323, l. -9] have to be replaced by $\mathrm{Rés}^I_{I_q}(\omega) = (-1)^{q-1}\alpha \wedge dx_{i_q}/x_{i_q}|_{D_{I_q}}$). By the formulas (2.8.4.4) and (2.8.5.3), by the definition of the Gysin morphism for smooth divisors ((2.8.0.11)) and by the Convention (4) and (5), we see that $(-1)^j(-G^{\lambda_j}_{\underline{\lambda}})$ is the boundary morphism of the lower exact sequence. Hence we obtain (2.8.5). $\square$

2.9 The Functoriality of the Preweight-Filtered Zariskian Complex

Let S_0, S and $(X, D \cup Z)$ be as in §2.4. In this section we prove the functoriality of $(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$; (2.7.3) is indispensable for the proof of the functoriality.

Let $(S', \mathcal{I}', \gamma')$ be another PD-scheme satisfying the same conditions in the beginning of §2.4. Set $S'_0 := \underline{\mathrm{Spec}}_{S'}(\mathcal{O}_{S'}/\mathcal{I}')$. Let $u\colon (S, \mathcal{I}, \gamma) \longrightarrow (S', \mathcal{I}', \gamma')$ be a morphism of PD-schemes. Let $u_0\colon S_0 \longrightarrow S'_0$ be the induced morphism by u. Let $(X', D' \cup Z')$ be a smooth scheme with a relative SNCD over S'_0. Let

$$
\begin{array}{ccc}
(X, D \cup Z) & \xrightarrow{\ g\ } & (X', D' \cup Z') \\
\big\downarrow & & \big\downarrow \\
S_0 & \xrightarrow{\ u_0\ } & S'_0
\end{array} \tag{2.9.0.1}
$$

be a commutative diagram of log schemes. Assume that the morphism g induces $g_{(X,D)}\colon (X, D) \longrightarrow (X', D')$ and $g_{(X,Z)}\colon (X, Z) \longrightarrow (X', Z')$ over $u_0\colon S_0 \longrightarrow S'_0$. Let

$$\epsilon\colon ((X, \widetilde{D \cup Z})/S)^{\log}_{\mathrm{crys}} \longrightarrow ((\widetilde{X, Z})/S)^{\log}_{\mathrm{crys}}$$

and

$$\epsilon'\colon ((X', \widetilde{D' \cup Z'})/S')^{\log}_{\mathrm{crys}} \longrightarrow ((\widetilde{X', Z'})/S')^{\log}_{\mathrm{crys}}$$

be the forgetting log morphisms along D and D', respectively.

Theorem 2.9.1 (Functoriality). *Let the notations be as above. Then the following hold*:

(1) *There exists a canonical morphism*

$$
\begin{aligned}
g^{\log *}_{(X,Z)\mathrm{crys}}\colon (E^{\log, Z'}_{\mathrm{crys}}(\mathcal{O}_{(X', D' \cup Z')/S'}), P^{D'}) & \\
\longrightarrow Rg^{\log}_{(X,Z)\mathrm{crys}*}(E^{\log, Z}_{\mathrm{crys}}(\mathcal{O}_{(X, D \cup Z)/S}), P^{D}). &
\end{aligned} \tag{2.9.1.1}
$$

(2) *There exists a canonical morphism*

(2.9.1.2)
$$g^*_{\mathrm{zar}}\colon (C^{\log, Z'}_{\mathrm{zar}}(\mathcal{O}_{(X', D' \cup Z')/S'}), P^{D'}) \longrightarrow Rg_{\mathrm{zar}*}(C^{\log, Z}_{\mathrm{zar}}(\mathcal{O}_{(X, D \cup Z)/S}), P^{D}).$$

Proof. (1): (1) is clear.

(2): Let

$$
\begin{aligned}
g^{\log}_{\mathrm{crys}}\colon (((X, \widetilde{D \cup Z})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X, D \cup Z)/S}) & \\
\longrightarrow (((X', \widetilde{D' \cup Z'})/S')^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X', D' \cup Z')/S'}) &
\end{aligned}
$$

be the morphism of log crystalline ringed topoi induced by g. Then we construct a desired morphism in the following way:

$$
\begin{aligned}
& (C^{\log, Z'}_{\mathrm{zar}}(\mathcal{O}_{(X', D' \cup Z')/S'}), P^{D'}) \\
& = Ru_{(X', Z')/S'*}(E^{\log, Z'}_{\mathrm{crys}}(\mathcal{O}_{(X', D' \cup Z')/S'}), P^{D'})
\end{aligned}
$$

$$
\begin{aligned}
&= Ru_{(X',Z')/S'*}(R\epsilon'_*(\mathcal{O}_{(X',D'\cup Z')/S'}),\tau)\\
&\longrightarrow Ru_{(X',Z')/S'*}(R\epsilon'_*Rg^{\log}_{\mathrm{crys}*}(\mathcal{O}_{(X,D\cup Z)/S}),\tau)\\
&= Ru_{(X',Z')/S'*}(Rg^{\log}_{(X,Z)\mathrm{crys}*}R\epsilon_*(\mathcal{O}_{(X,D\cup Z)/S}),\tau)\\
&\longrightarrow Ru_{(X',Z')/S'*}Rg^{\log}_{(X,Z)\mathrm{crys}*}(R\epsilon_*(\mathcal{O}_{(X,D\cup Z)/S}),\tau)\\
&= Ru_{(X',Z')/S'*}Rg^{\log}_{(X,Z)\mathrm{crys}*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\\
&= Rg_{\mathrm{zar}*}Ru_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)\\
&= Rg_{\mathrm{zar}*}(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D).
\end{aligned}
$$

Here the first and the last equalities follow from (2.7.5.1); the first arrow is induced by $g^{\log *}_{\mathrm{crys}}$ and the second arrow is obtained from (2.7.2). □

Corollary 2.9.2. *Let* $E_{\mathrm{ss}}((X,D\cup Z)/S)$ *(resp.* $E_{\mathrm{ss}}((X',D'\cup Z')/S'))$ *be the spectral sequence* (2.6.2.2) *(resp.* (2.6.2.2) *for* $(X',D'\cup Z')/S')$*. Then the morphism* $g^{\log*}_{\mathrm{crys}}$ *induces a morphism*

$$
(2.9.2.1) \qquad g^{\log*}_{\mathrm{crys}}\colon E_{\mathrm{ss}}((X',D'\cup Z')/S') \longrightarrow E_{\mathrm{ss}}((X,D\cup Z)/S)
$$

of spectral sequences.

Proof. The proof is straightforward. □

Let $a'^{(k)}\colon (D'^{(k)},Z'|_{D'^{(k)}}) \longrightarrow (X',Z')$ be a natural morphism. Assume that g induces a morphism $g_{D^{(k)}}\colon (D^{(k)},Z|_{D^{(k)}}) \longrightarrow (D'^{(k)},Z'|_{D'^{(k)}})$ for any $k\in\mathbb{N}$. By (2.6.1.1), (2.9.1) and (1.3.4.1), the morphism $g^{\log*}_{(X,Z)\mathrm{crys}}$ induces the following morphism

(2.9.2.2)

$$
\begin{aligned}
&\mathrm{gr}^P_k(g^{\log*}_{(X,Z)\mathrm{crys}})\colon\\
&Ru_{(X',Z')/S'*}a'^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D'^{(k)},Z'|_{D'^{(k)}})/S'}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D'/S';Z'))\{-k\}\longrightarrow\\
&Ru_{(X',Z')/S'*}a'^{(k)\log}_{\mathrm{crys}*}Rg^{\log}_{D^{(k)}\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z))\{-k\}.
\end{aligned}
$$

In the following, we make the morphism $\mathrm{gr}^P_k(g^{\log*}_{(X,Z)\mathrm{crys}})$ in (2.9.2.2) explicit in certain cases by using a notion which is analogous to the *D-twist* in [71].

Assume that the following two conditions hold:

(2.9.2.3): there exists the same cardinality of smooth components of D and D' over S_0 and S'_0, respectively: $D=\bigcup_{\lambda\in\Lambda}D_\lambda$, $D'=\bigcup_{\lambda\in\Lambda}D'_\lambda$, where D_λ and D'_λ are smooth divisors over S_0 and S'_0, respectively.

(2.9.2.4): there exist positive integers e_λ $(\lambda\in\Lambda)$ such that $e_\lambda D_\lambda = g^*(D'_\lambda)$.

As in the previous section, set $\underline{\lambda} := \{\lambda_1, \ldots, \lambda_k\}$ $(\lambda_j \in \Lambda,\ (\lambda_i \neq \lambda_j\ (i \neq j)))$. Let $a_{\underline{\lambda}} \colon (D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}}) \longrightarrow (X, Z)$ and $a'_{\underline{\lambda}} \colon (D'_{\underline{\lambda}}, Z'|_{D'_{\underline{\lambda}}}) \longrightarrow (X', Z')$ be natural morphisms. Consider the following direct factor of the morphism (2.9.2.2):

(2.9.2.5)

$$\begin{aligned}
&Ru_{(X',Z')/S'*}{a'}^{\log}_{\underline{\lambda}\mathrm{crys}*}(g^{\log*}_{\underline{\lambda}\mathrm{crys}}) \colon \\
&Ru_{(X',Z')/S'*}{a'}^{\log}_{\underline{\lambda}\mathrm{crys}*}(\mathcal{O}_{(D'_{\underline{\lambda}},Z|_{D'_{\underline{\lambda}}})/S'} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D'/S'; Z'))\{-k\} \\
&\longrightarrow Ru_{(X',Z')/S'*}{a'}^{\log}_{\underline{\lambda}\mathrm{crys}*}Rg^{\log}_{\underline{\lambda}\mathrm{crys}*}(\mathcal{O}_{(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})/S} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S; Z))\{-k\}.
\end{aligned}$$

Proposition 2.9.3. *Let the notations and the assumptions be as above. Let*

$$g_{(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})} \colon (D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}}) \longrightarrow (D'_{\underline{\lambda}}, Z'|_{D'_{\underline{\lambda}}})$$

be the induced morphism by g. Then the morphism $Ru_{(X',Z')/S'*}(g^{\log*}_{\underline{\lambda}\mathrm{crys}})$ in (2.9.2.5) *is equal to $(\prod_{j=1}^{k} e_{\lambda_j})Ru_{(X',Z')/S'*}{a'}^{\log}_{\underline{\lambda}\mathrm{crys}*}(g^{\log*}_{(D_{\underline{\lambda}},Z|_{D_{\underline{\lambda}}})\mathrm{crys}})$ for $k \geq 0$. Here we define $\prod_{j=1}^{k} e_{\lambda_j}$ as 1 for $k = 0$.*

Proof. We may assume that $k \geq 1$. Let us take affine open coverings $X = \bigcup_{i_0 \in I_0} X_{i_0}, X' = \bigcup_{i_0 \in I_0} X'_{i_0}$ of X, X' by the same index set I_0 satisfying $g(X_{i_0}) \subseteq X'_{i_0}$ $(i_0 \in I_0)$ and let us form diagrams of log schemes $(X_\bullet, D_\bullet \cup Z_\bullet)$ and $(X'_\bullet, D'_\bullet \cup Z'_\bullet)$ indexed by I as in (2.4.0.1). Then we have a morphism $g_\bullet : (X_\bullet, D_\bullet \cup Z_\bullet) \longrightarrow (X'_\bullet, D'_\bullet \cup Z'_\bullet)$ of diagrams of log schemes over g. Next let us take log smooth lifts

$$(X_{i_0}, D_{i_0} \cup Z_{i_0}) \overset{\subset}{\longrightarrow} (\mathcal{X}_{i_0}, \mathcal{D}_{i_0} \cup \mathcal{Z}_{i_0}), \quad (X'_{i_0}, D'_{i_0} \cup Z'_{i_0}) \overset{\subset}{\longrightarrow} (\mathcal{X}'_{i_0}, \mathcal{D}'_{i_0} \cup \mathcal{Z}'_{i_0})$$

for each $i_0 \in I_0$ and from these data, let us construct the diagrams of admissible immersions

$$(X_\bullet, D_\bullet \cup Z_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet), \quad (X'_\bullet, D'_\bullet \cup Z'_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}'_\bullet, \mathcal{D}'_\bullet \cup \mathcal{Z}'_\bullet)$$

by the method explained in §2.4 before (2.4.1). Let $g_{(X_\bullet,Z_\bullet)} : (X_\bullet, Z_\bullet) \longrightarrow (X'_\bullet, Z'_\bullet)$ be the morphism induced by $g_\bullet$, which exists by assumption on g and let π_{zar} be the morphism defined in (2.4.5.2). Then we have

(2.9.3.1)

$$\mathrm{gr}_k^{P^D} C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/S}) = R\pi_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}_\bullet}} \mathrm{gr}_k^{P^{\mathcal{D}_\bullet}} \Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet \cup \mathcal{Z}_\bullet))),$$

(2.9.3.2)

$$\begin{aligned}
&\mathrm{gr}_k^{P^{D'}} C^{\log,Z'}_{\mathrm{zar}}(\mathcal{O}_{(X',D'\cup Z')/S}) \\
&= R\pi_{\mathrm{zar}*}Rg_{(X_\bullet,Z_\bullet)\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}'_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}'_\bullet}} \mathrm{gr}_k^{P^{\mathcal{D}'_\bullet}} \Omega^\bullet_{\mathcal{X}'_\bullet/S}(\log(\mathcal{D}'_\bullet \cup \mathcal{Z}'_\bullet))),
\end{aligned}$$

where $\mathfrak{D}_\bullet$ (resp. $\mathfrak{D}'_\bullet$) denotes the log PD-envelope of $(X_\bullet, Z_\bullet) \xrightarrow{\subset} (\mathcal{X}_\bullet, \mathcal{Z}_\bullet)$ (resp. $(X'_\bullet, Z'_\bullet) \xrightarrow{\subset} (\mathcal{X}'_\bullet, \mathcal{Z}'_\bullet)$). Because $(\mathcal{X}'_{i_0}, \mathcal{D}'_{i_0} \cup \mathcal{Z}'_{i_0})$ is log smooth over S' and the exact closed immersion $(X_{i_0}, D_{i_0} \cup Z_{i_0}) \xrightarrow{\subset} (\mathcal{X}_{i_0}, \mathcal{D}_{i_0} \cup \mathcal{Z}_{i_0})$ is defined by the nil-ideal sheaf $\mathcal{I}\mathcal{O}_{\mathcal{X}_{i_0}}$, there exists a morphism $\widetilde{g}_{i_0} \colon (\mathcal{X}_{i_0}, \mathcal{D}_{i_0} \cup \mathcal{Z}_{i_0}) \longrightarrow (\mathcal{X}'_{i_0}, \mathcal{D}'_{i_0} \cup \mathcal{Z}'_{i_0})$ which is a lift of $g|_{(X_{i_0}, D_{i_0} \cup Z_{i_0})}$ (cf. [11, N.B. in 5.27]). The family $\{\widetilde{g}_{i_0}\}_{i_0 \in I_0}$ induces a morphism

$$\widetilde{g}_\bullet \colon (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet) \longrightarrow (\mathcal{X}'_\bullet, \mathcal{D}'_\bullet \cup \mathcal{Z}'_\bullet) \tag{2.9.3.3}$$

of diagrams of log schemes by the universality of blow-up. Let

$$\widetilde{h}_{(\underline{\lambda};\bullet)} \colon (\mathcal{D}_{(\underline{\lambda};\bullet)}, \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}}) \longrightarrow (\mathcal{D}'_{(\underline{\lambda};\bullet)}, \mathcal{Z}'|_{\mathcal{D}'_{(\underline{\lambda};\bullet)}})$$

be the induced morphism. (Here we put $\mathcal{D}_{(\underline{\lambda};\bullet)} := \bigcap_{i=1}^k \mathcal{D}_{(\lambda_i;\bullet)}, \mathcal{D}'_{(\underline{\lambda};\bullet)} := \bigcap_{i=1}^k \mathcal{D}'_{(\lambda_i;\bullet)}$, where $\mathcal{D}_{(\lambda_i;\bullet)}, \mathcal{D}'_{(\lambda_i;\bullet)}$ are as in §2.4 before (2.4.1).)

For $i_0 \in I_0$, Let $x_{(j;i_0)} = 0$ (resp. $x'_{(j;i_0)} = 0$) be a local equation of $\mathcal{D}_{(\lambda_j;i_0)}$ in $\mathcal{X}_{i_0}$ (resp. $\mathcal{D}'_{(\lambda_j;i_0)}$ in $\mathcal{X}'_{i_0}$) $(1 \leq j \leq k)$. Then we have $\widetilde{g}^*_{i_0}(x'_{(j;i_0)}) = u_{(j;i_0)} x^{e_{\lambda_j}}_{(j;i_0)}$ for some unit $u_{(j;i_0)}$. For $i = (i_0, ..., i_r) \in I$, let us put $x_{(j;i)} := x_{(j;i_0)}, x'_{(j;i)} := x'_{(j;i_0)}, u_{(j;i)} := u_{(j;i_0)}$. Then, by definition of $\mathcal{D}_{(\lambda_j;i)}, \mathcal{D}'_{(\lambda_j;i)}$ (via the blow-up construction), $x_{(j;i)} = 0$ (resp. $x'_{(j;i)} = 0$) is a local equation of $\mathcal{D}_{(\lambda_j;i)}$ in $\mathcal{X}_i$ (resp. $\mathcal{D}'_{(\lambda_j;i)}$ in $\mathcal{X}_i$) $(1 \leq j \leq k)$ and we have the equality $\widetilde{g}^*_i(x'_{(j;i)}) = u_{(j;i)} x^{e_{\lambda_j}}_{(j;i)}$. So, for a local section $\omega = a d\log x'_{(1;i)} \cdots d\log x'_{(k;i)}$ of $P^{\mathcal{D}'}_k \Omega^\bullet_{\mathcal{X}'/S'}(\log(\mathcal{D}' \cup \mathcal{Z}'))$ $(a \in \Omega^{\bullet-k}_{\mathcal{X}'/S'}(\log \mathcal{Z}'))$, we have $\widetilde{g}^*_i(\omega) = (\prod_{j=1}^k e_{\lambda_j}) \widetilde{g}^*_i(a) d\log x_{(1;i)} \cdots d\log x_{(k;i)} + \omega'$, where $\omega' \in P^{\mathcal{D}_i}_{k-1} \Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{D}_i \cup \mathcal{Z}_i))$. So, if we put

$$\Omega^\bullet_{(\underline{\lambda};\bullet)} := \Omega^\bullet_{\mathcal{D}_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}_{(\underline{\lambda};\bullet)}}) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}\mathrm{zar}}(D_\bullet/S),$$

$$\Omega^\bullet_{(\underline{\lambda};\bullet)'} := \Omega^\bullet_{\mathcal{D}'_{(\underline{\lambda};\bullet)}/S}(\log \mathcal{Z}'_\bullet|_{\mathcal{D}'_{(\underline{\lambda};\bullet)}}) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}\mathrm{zar}}(D'_\bullet/S),$$

we have the following commutative diagram (the vertical arrows are Poincaré residue morphisms with respect to $\mathcal{D}'_{\underline{\lambda}}$ and $\mathcal{D}_{\underline{\lambda}}$):

$$\begin{array}{ccc}
\mathrm{gr}_k^{P^{D'}}(C_{\mathrm{zar}}^{\log, Z'_\bullet}(\mathcal{O}_{(X'_\bullet, D'_\bullet \cup Z'_\bullet)/S})) & \xrightarrow{\mathrm{gr}_k^{P^{D_\bullet}}(\widetilde{g}^*_\bullet)} & \\
\Vert & & \\
(\mathcal{O}_{\mathfrak{D}_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}_\bullet}} \mathrm{gr}_k^{P^{D'_\bullet}} \Omega^\bullet_{\mathcal{X}'_\bullet/S}(\log(\mathcal{D}'_\bullet \cup \mathcal{Z}'_\bullet))) & & \\
\Big\downarrow {\scriptstyle \mathrm{Res}^{\mathcal{D}'_\bullet}_{\underline{\lambda}}} & & \\
(\mathcal{O}_{\mathfrak{D}'_\bullet} \otimes_{\mathcal{O}_{\mathcal{X}'_\bullet}} (a'_{\underline{\lambda}}|_{(D'_\bullet, Z'|_{D'_\bullet})})_{\mathrm{zar}*} \Omega^\bullet_{(\underline{\lambda};\bullet)'})\{-k\} & \xrightarrow{(\prod_{j=1}^k e_{\lambda_j}) \widetilde{h}^*_{(\underline{\lambda};\bullet)}} &
\end{array} \tag{2.9.3.4}$$

$$
\begin{array}{c}
g_{(X_\bullet,Z_\bullet)\mathrm{zar}*}\mathrm{gr}_k^{P^D}(C_{\mathrm{zar}}^{\log,Z_\bullet}(\mathcal{O}_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}))\\
\Big\|\\
g_{(X_\bullet,Z_\bullet)\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\mathrm{gr}_k^{P^{\mathcal{D}_\bullet}}\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)))\\
\Big\downarrow{\scriptstyle \mathrm{Res}^{\mathcal{D}_\bullet}_{\underline{\lambda}}}\\
g_{(X_\bullet,Z_\bullet)\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}(a_{\underline{\lambda}}|_{(D_\bullet,Z|_{D_\bullet})})_{\mathrm{zar}*}\Omega^\bullet_{(\underline{\lambda};\bullet)})\{-k\}.
\end{array}
$$

Now, by (2.9.3.1), (2.9.3.2), (2.9.3.4) and log crystalline Poincaré lemma for $(D_\bullet, Z|_{D_\bullet})$, $(D'_\bullet, Z'|_{D'_\bullet})$, (2.9.3) is reduced to the following obvious lemma. □

Lemma 2.9.4. *Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of abelian categories. Let $M^\bullet$ and $M'^\bullet$ (resp. $N^\bullet$ and $N'^\bullet$) be objects of* $\mathrm{K}^+(\mathcal{B})$ *(resp.* $\mathrm{K}^+(\mathcal{A})$*). Let*

$$
\begin{array}{ccc}
M^\bullet & \xrightarrow{f} & F(N^\bullet)\\
{\scriptstyle\simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle\simeq}\\
M'^\bullet & \xrightarrow{f'} & F(N'^\bullet)
\end{array}
$$

be the commutative diagram in $\mathrm{K}^+(\mathcal{B})$*. Assume that $\mathcal{A}$ has enough injectives. Then the following diagram is commutative*:

$$
\begin{array}{ccc}
M^\bullet & \xrightarrow{f} & RF(N^\bullet)\\
{\scriptstyle\simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle\simeq}\\
M'^\bullet & \xrightarrow{f'} & RF(N'^\bullet).
\end{array}
$$

Proof. The proof is obvious. □

Definition 2.9.5. (1) We call $\{e_\lambda\}_{\lambda\in\Lambda} \in \mathbb{Z}^\Lambda_{>0}$ the *multi-degree* of g with respect to a decomposition $\Delta := \{D_\lambda\}_\lambda$ and $\Delta' := \{D'_\lambda\}_\lambda$ of D and D', respectively. We denote it by $\deg_{\Delta,\Delta'}(g) \in \mathbb{Z}^\Lambda_{>0}$. If e_λ's for all λ's are equal, we also denote $e_\lambda \in \mathbb{Z}_{>0}$ by $\deg_{\Delta,\Delta'}(g) \in \mathbb{Z}_{>0}$.

(2) Assume that e_λ's for all λ's are equal. Let $u\colon \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of $\mathcal{O}_S$-modules. Let k be a nonnegative integer. The *k-twist*

$$
u(-k)\colon \mathcal{E}(-k; g; \Delta, \Delta') \longrightarrow \mathcal{F}(-k; g; \Delta, \Delta')
$$

of u with respect to g, Δ and Δ' is, by definition, the morphism $\deg_{\Delta,\Delta'}(g)^k u\colon \mathcal{E} \longrightarrow \mathcal{F}$.

Corollary 2.9.6. *Assume that e_λ's for all λ's are equal. Let $E_{\mathrm{ss}}((X, D \cup Z)/S)$ be the following spectral sequence*

$$\begin{aligned}&E_1^{-k,h+k}((X,D\cup Z)/S)\\&=R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S})(-k;g;\Delta,\Delta')\\&\Longrightarrow R^hf_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S})\end{aligned}$$

and let $E_{\rm ss}((X',D'\cup Z')/S')$ be the obvious analogue of the above for $(X',D'\cup Z')/S'$. Then there exists a morphism

(2.9.6.1) $$g^{\log *}_{\rm crys}\colon E_{\rm ss}((X',D'\cup Z')/S')\longrightarrow E_{\rm ss}((X,D\cup Z)/S)$$

of spectral sequences.

Proof. (2.9.6) immediately follows from (2.9.3). □

Assume that S_0 is a scheme of characteristic $p>0$. Let $F_{S_0}\colon S_0\longrightarrow S_0$ be the p-th power endomorphism. Let $(X',D'\cup Z')$ be the base change of $(X,D\cup Z)$ by F_{S_0}. The relative Frobenius morphism

$$F\colon (X,D\cup Z)\longrightarrow (X',D'\cup Z')$$

over S_0 induces the relative Frobenius morphisms

$$F_{(X,Z)}\colon (X,Z)\longrightarrow (X',Z')$$

and

$$F^{(k)}\colon (D^{(k)},Z|_{D^{(k)}})\longrightarrow (D^{(k)\prime},Z'|_{D^{(k)\prime}}).$$

Let

$$a^{(k)}\colon (D^{(k)},Z|_{D^{(k)}})\longrightarrow (X,D\cup Z)$$

and

$$a^{(k)\prime}\colon (D^{(k)\prime},Z'|_{D^{(k)\prime}})\longrightarrow (X',D'\cup Z')$$

be the natural morphisms. We define the relative Frobenius action

$$\Phi_{(D^{(k)},Z|_{D^{(k)}})/S}\colon a^{(k)\prime\log}_{\rm crys*}\varpi^{(k)\log}_{\rm crys}(D'/S;Z')\longrightarrow F^{\log}_{\rm crys*}a^{(k)\log}_{\rm crys*}\varpi^{(k)\log}_{\rm crys}(D/S;Z)$$

as the identity under the natural identification

$$\varpi^{(k)\log}_{\rm crys}(D'/S;Z')\overset{\sim}{\longrightarrow}F^{(k)\log}_{\rm crys*}\varpi^{(k)\log}_{\rm crys}(D/S;Z).$$

When g is the relative Frobenius $F\colon (X,D\cup Z)\longrightarrow (X',D'\cup Z')$, we denote (2.9.6.1) by

(2.9.6.2)
$$\begin{aligned}E_1^{-k,h+k}((X,D\cup Z)/S)=R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(&\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\\&\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\rm crys}(D/S;Z))(-k)\\\Longrightarrow R^hf_{(X,D\cup Z)/S*}(&\mathcal{O}_{(X,D\cup Z)/S}).\end{aligned}$$

((2.9.6.2) is equal to (2.6.2.2)+(the compatibility with Frobenius).) (2.9.6.2) is generalized to the following spectral sequence

(2.9.6.3)
$$\begin{aligned} E_1^{-k,h+k} &= E_1^{-k,h+k}((X,D\cup Z)/S;k')(-k) \\ &\Longrightarrow R^h\overline{f}_{(X,D\cup Z)/S*}(P^D_{k'}C^{\log,Z}_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D\cup Z)/S})) \\ &= R^h f_{(X,D\cup Z)/S*}(P^D_{k'}E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})) \end{aligned}$$

by (2.6.2.1) and (2.7.3.2).

Definition 2.9.7. We call the sequence (2.9.6.2) the *preweight spectral sequence of* $(X, D\cup Z)/(S,\mathcal{I},\gamma)$ *with respect to* D. If $Z=\emptyset$, then we call it the *preweight spectral sequence of* $(X,D)/(S,\mathcal{I},\gamma)$.

By the proof of (2.8.5) and (2.9.3), the morphism G in (2.8.5) is a morphism

(2.9.7.1)
$$\begin{aligned} &G\colon R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)))(-k) \\ &\longrightarrow R^{h-k+2}f_{(D^{(k-1)},Z|_{D^{(k-1)}})/S*}(\mathcal{O}_{(D^{(k-1)},Z|_{D^{(k-1)}})/S} \\ &\qquad\qquad\otimes_{\mathbb{Z}}\varpi^{(k-1)\log}_{\mathrm{crys}}(D/S;Z))(-(k-1)). \end{aligned}$$

By (2.7.6) we also have the following Leray spectral sequence

(2.9.7.2)
$$\begin{aligned} E_2^{st} &:= R^s f_{(D^{(t)},Z|_{D^{(t)}})/S*}(\mathcal{O}_{(D^{(t)},Z|_{D^{(t)}})/S}\otimes_{\mathbb{Z}}\varpi^{(t)\log}_{\mathrm{crys}}(D/S;Z))(-t) \\ &\Longrightarrow R^{s+t}f_{(X,D\cup Z)/S*}(\mathcal{O}_{(X,D\cup Z)/S}). \end{aligned}$$

2.10 The Base Change Theorem and the Künneth Formula

In this section we prove the base change theorem of a preweight-filtered vanishing cycle crystalline complex and the Künneth formula of it. (2.7.5) plays an important role in this section.

We keep the notations in §2.4. In this section we assume that X is quasi-compact. Hence we can assume that the cardinality of the family $\{X_{i_0}\}_{i_0\in I_0}$ of an open covering of X is finite.

(1) Base change theorem.

Proposition 2.10.1. *Let*

(2.10.1.1)
$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (T', \mathcal{J}', \gamma') & \xrightarrow{u} & (T, \mathcal{J}, \gamma) \end{array}$$

be a commutative diagram of fine log schemes, where a PD-structure γ (resp. γ') on a PD-ideal sheaf $\mathcal{J}$ (resp. $\mathcal{J}'$) of $\mathcal{O}_T$ (resp. $\mathcal{O}_{T'}$) extends to Y (resp. Y') and u is a PD-morphism of PD-log schemes. Let $(E^\bullet, \{E_k^\bullet\})$ be a bounded below filtered complex of $\mathcal{O}_{Y/T}$-modules. Assume that $Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\})$ is bounded above. Then there exists a canonical morphism

(2.10.1.2)
$$Lu^* Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\}) \longrightarrow Rf'_{Y'/T'*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})$$

in $\mathrm{DF}(\mathcal{O}_{T'})$.

Proof. By (1.2.3.2) we have only to find an element in

$$\mathcal{H}^0[\{\mathrm{RHom}_{\mathcal{O}_{T'}}(Lu^* Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\}), Rf'_{Y'/T'*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})\}_0].$$

Using (1.2.2), we have the following formula

(2.10.1.3)
$$\begin{aligned} &\mathrm{RHom}_{\mathcal{O}_{T'}}(Lu^* Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\}), Rf'_{Y'/T'*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})) \\ &= \mathrm{RHom}_{\mathcal{O}_T}(Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\}), Ru_* Rf'_{Y'/T'*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})) \\ &= \mathrm{RHom}_{\mathcal{O}_T}(Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\}), Rf_{Y/T*} Rg^{\log}_{\mathrm{crys}*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})). \end{aligned}$$

The adjunction morphism $(E^\bullet, \{E_k^\bullet\}) \longrightarrow g^{\log}_{\mathrm{crys}*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})$ induces a morphism $(E^\bullet, \{E_k^\bullet\}) \longrightarrow Rg^{\log}_{\mathrm{crys}*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})$. This morphism induces a morphism

$$Rf_{Y/T*}(E^\bullet, \{E_k^\bullet\}) \longrightarrow Rf_{Y/T*} Rg^{\log}_{\mathrm{crys}*} g_{\mathrm{crys}}^{\log -1}(E^\bullet, \{E_k^\bullet\})$$

in $\mathrm{DF}(\mathcal{O}_T)$. □

Proposition 2.10.2. (1) *Let $f\colon (X, D \cup Z) \longrightarrow S_0 (\overset{\subset}{\longrightarrow} S)$ and $(S, \mathcal{I}, \gamma)$ be as in §2.4. Assume moreover that S is quasi-compact and that $\overset{\circ}{f}\colon X \longrightarrow S_0$ is quasi-separated and quasi-compact. Let $f_{(X,Z)}\colon (X, Z) \longrightarrow S_0 (\overset{\subset}{\longrightarrow} S)$ be the induced morphism by f. Then $R^h f_{(X,Z)/S*} P_k^D(E_{\mathrm{crys}}^{\log, Z}(\mathcal{O}_{(X, D\cup Z)/S})$ $(h, k \in \mathbb{Z})$ are quasi-coherent $\mathcal{O}_S$-modules and $Rf_{(X,Z)/S*}(E_{\mathrm{crys}}^{\log, Z}(\mathcal{O}_{(X, D\cup Z)/S}), P^D)$ is isomorphic to a bounded filtered complex of $\mathcal{O}_S$-modules.*

(2) *Let $(S, \mathcal{I}, \gamma)$ and S_0 be as in §2.4. Let Y be a quasi-compact smooth scheme over S_0 (with trivial log structure). Let $f\colon (X, D \cup Z) \longrightarrow Y$ be a morphism of log schemes such that $\overset{\circ}{f}\colon X \longrightarrow Y$ is smooth, quasi-compact and quasi-separated and such that $D \cup Z$ is a relative SNCD over Y. (In particular, $D \cup Z$ is also a relative SNCD on X over S_0.) Let $f_{(X,Z)}\colon (X, Z) \longrightarrow Y$ be*

the induced morphism by f. Then $Rf^{\log}_{(X,Z)\mathrm{crys}*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ is isomorphic to a bounded filtered complex of $\mathcal{O}_{Y/S}$-modules.

Proof. (1): Let $(I^\bullet, \{I^\bullet_k\})$ be a filtered flasque resolution of $(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$. Then $Rf_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) = (f\circ u_{(X,Z)/S})_*(I^\bullet, \{I^\bullet_k\})$.

Now, fix a decomposition $\{D_\lambda\}_\lambda$ of D by its smooth components and give a total order on λ's. Then there exists an isomorphism $\mathbb{Z} \xrightarrow{\sim} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)$. Furthermore, for each k, fix a decomposition $\{(Z|_{D^{(k)}})_\mu\}$ of $Z|_{D^{(k)}}$ by its smooth components and give a total order on μ's. Because X is quasi-compact, the sets λ's and μ's are finite. By (2.6.2.2) we have the following spectral sequence

$$E_1^{-l,h+l} = R^{h-l}f_{Z^{(l)}|_{D^{(k)}}/S*}(\mathcal{O}_{Z^{(l)}|_{D^{(k)}}/S}) \Longrightarrow R^h f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}). \tag{2.10.2.1}$$

By [11, 7.6 Theorem] and by the spectral sequences (2.6.2.2) and (2.10.2.1), $\mathcal{H}^h((f_{(X,Z)}\circ u_{(X,Z)/S})_*(I^\bullet_k))$ $(h,k\in\mathbb{Z})$ are quasi-coherent $\mathcal{O}_S$-modules and there exists an integer h_0 such that, for all $h\geq h_0$ and for all $k\in\mathbb{Z}$, $\mathcal{H}^h((f_{(X,Z)}\circ u_{(X,Z)/S})_*(I^\bullet_k))=0$. Hence $R^hf_{(X,Z)/S*}P^D_k(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})$ $(h,k\in\mathbb{Z})$ are quasi-coherent $\mathcal{O}_S$-modules and $Rf_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) = ((f_{(X,Z)}\circ u_{(X,Z)/S})_*(I^\bullet), (f\circ u_{(X,Z)/S})_*(I^\bullet_k))$ is isomorphic to a bounded filtered complex of $\mathcal{O}_S$-modules.

(2): (2) immediately follows from (1) and from the proof of [3, V Corollaire 3.2.3] (cf. the proof of [11, 7.11 Corollary]). $\square$

Theorem 2.10.3 (Base change theorem). *Let $f\colon (X, D\cup Z)\longrightarrow S_0(\overset{\subset}{\longrightarrow} S)$ and $(S,\mathcal{I},\gamma)$ be as in (2.10.2). Let $u\colon (S',\mathcal{I}',\gamma')\longrightarrow(S,\mathcal{I},\gamma)$ be a morphism of PD-schemes. Assume that $\mathcal{I}'$ is a quasi-coherent ideal sheaf of $\mathcal{O}_{S'}$. Set $S'_0 := \underline{\mathrm{Spec}}_{S'}(\mathcal{O}_{S'}/\mathcal{I}')$. Let $f'\colon (X', D'\cup Z') := (X\times_{S_0}S'_0, (D\cup Z)\times_{S_0}S'_0)\longrightarrow S'_0$ be the base change morphism of f with respect to $u|_{S'_0}$. Then there exists a canonical isomorphism*

$$Lu^*Rf_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \xrightarrow{\sim} Rf'_{(X',Z')/S'*}(E^{\log,Z'}_{\mathrm{crys}}(\mathcal{O}_{(X',D'\cup Z')/S'}), P^{D'}) \tag{2.10.3.1}$$

in the filtered derived category $\mathrm{DF}(f'^{-1}(\mathcal{O}_{S'}))$.

Proof. Let $g_{(X,Z)}\colon (X',Z')\longrightarrow(X,Z)$ and $g_{(X,D\cup Z)}\colon (X',D'\cup Z')\longrightarrow(X,D\cup Z)$ be the natural morphisms of log schemes. First we use the general theory in §1.5 as follows.

Consider a small category $I:=\{i,i'\}$ consisting of two elements. The morphisms in I, by definition, consist of three elements id_i, $\mathrm{id}_{i'}$ and a morphism $i\longrightarrow i'$. By corresponding the natural morphism

$$g^{\log}_{(X,D\cup Z)\mathrm{crys}}\colon (((X',\widetilde{D'\cup Z'})/S')^{\log}_{\mathrm{crys}},\mathcal{O}_{(X',D'\cup Z')/S'})$$
$$\longrightarrow (((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}},\mathcal{O}_{(X,D\cup Z)/S})$$

to the morphism $i\longrightarrow i'$, we have a ringed topos $(((X_j,\widetilde{D_j\cup Z_j})/S_j)^{\log}_{\mathrm{crys}},\mathcal{O}_{(X_j, D_j\cup Z_j)/S_j})_{j\in I}$. Let $(I^\bullet_j)_{j\in I}$ be a flasque resolution of $(\mathcal{O}_{(X_j,D_j\cup Z_j)/S_j})_{j\in I}$ ((1.5.0.2)). Let $\epsilon\colon ((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}}\longrightarrow ((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}}$ and $\epsilon'\colon ((X',\widetilde{D'\cup Z'})/S')^{\log}_{\mathrm{crys}}\longrightarrow ((\widetilde{X',Z'})/S')^{\log}_{\mathrm{crys}}$ be the forgetting log morphisms along D and D', respectively. Then $(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)$ and $(E^{\log,Z'}_{\mathrm{crys}}(\mathcal{O}_{(X',D'\cup Z')/S}),P^{D'})$ are represented by $(\epsilon_*(I^\bullet_i),\tau)$ and $(\epsilon'_*(I^\bullet_{i'}),\tau)$, respectively. Since $g^{\log\,-1}_{(X,Z)\mathrm{crys}}$ is exact, $g^{\log\,-1}_{(X,Z)\mathrm{crys}}(\epsilon_*(I^\bullet_i),\tau)=(g^{\log\,-1}_{(X,Z)\mathrm{crys}}\epsilon_*(I^\bullet_i),\tau)$. By the following commutative diagram

$$\begin{array}{ccc}(X',D'\cup Z') & \xrightarrow{g_{(X,D\cup Z)}} & (X,D\cup Z)\\ {\scriptstyle\epsilon'}\downarrow & & \downarrow{\scriptstyle\epsilon}\\ (X',Z') & \xrightarrow{g_{(X,Z)}} & (X,Z),\end{array}$$

we have a natural morphism $(g^{\log\,-1}_{(X,Z)\mathrm{crys}}\epsilon_*(I^\bullet_i),\tau)\longrightarrow(\epsilon'_*g^{\log\,-1}_{(X,D\cup Z)\mathrm{crys}}(I^\bullet_i),\tau)$. By the definition of $(I^\bullet_j)_{j\in I}$, we have the morphism $g^{\log\,-1}_{(X,D\cup Z)\mathrm{crys}}(I^\bullet_i)\longrightarrow I^\bullet_{i'}$. Hence we have a composite morphism

$$(g^{\log\,-1}_{(X,Z)\mathrm{crys}}\epsilon_*(I^\bullet_i),\tau)\longrightarrow(\epsilon'_*(I^\bullet_{i'}),\tau).$$

Therefore we have a canonical morphism (2.10.3.1) by (2.10.1) and (2.10.2) (1).

We prove that (2.10.3.1) is an isomorphism. By the filtered cohomological descent (1.5.1) (2) and by the same argument as that in the proof of [3, V Proposition 3.5.2] ([11, 7.8 Theorem]), we may assume that S is affine and that X is an affine scheme over S_0. Then $(X,D\cup Z)$ has a lift $(\mathcal{X},\mathcal{D}\cup\mathcal{Z})/S$ $(D=\mathcal{D}\times_{\mathcal{X}}X,\ Z=\mathcal{Z}\times_{\mathcal{X}}X)$ by (2.3.14). In this case, we may assume that the morphism (2.4.5.1) is the identity of $(((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}},\mathcal{O}_{(X,Z)/S})$. Let $\mathfrak{f}\colon(\mathcal{X},\mathcal{D}\cup\mathcal{Z})\longrightarrow S$ be the lift of f. Set $\mathfrak{f}_*(P^{\mathcal{D}}_k):=\mathfrak{f}_*(P^{\mathcal{D}}_k\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))$ $(k\in\mathbb{Z})$ and $\mathfrak{f}_*(P^{\mathcal{D}}):=\{\mathfrak{f}_*(P^{\mathcal{D}}_k)\}_{k\in\mathbb{Z}}$ for simplicity of notation. Then, by (2.7.5), we have

$$Rf_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}),P^D)=(\mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))),\mathfrak{f}_*(P^{\mathcal{D}})).$$

and we have the same formula for $(X',D'\cup Z')/S'$. We claim that $\mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))/\mathfrak{f}_*(P^{\mathcal{D}}_k)$ is a flat $\mathcal{O}_S$-module for any k. Indeed, the filtration $P^{\mathcal{D}}_k$ on $\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$ is finite and $\mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))$ is a flat $\mathcal{O}_S$-module. Because $\mathcal{X}$ is affine over S, we have the following exact sequence

(2.10.3.2)

$$0 \longrightarrow \mathfrak{f}_*(\mathrm{gr}_k^{P^{\mathcal{D}}}\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))) \longrightarrow \mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))/\mathfrak{f}_*(P^{\mathcal{D}}_{k-1}) \longrightarrow \mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})))/\mathfrak{f}_*(P^{\mathcal{D}}_{k}) \longrightarrow 0.$$

By the Poincaré residue isomorphism, the left term of (2.10.3.2) is isomorphic to $\mathfrak{f}_*(b^{(k)}_*\Omega^\bullet_{\mathcal{D}^{(k)}/S}(\log\mathcal{Z}|_{\mathcal{D}^{(k)}})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}/S))\{-k\}$, where $b^{(k)}\colon \mathcal{D}^{(k)} \longrightarrow \mathcal{X}$ is the natural morphism. Hence, the descending induction on k shows the claim. Therefore the left hand side of (2.10.3.1) is equal to $u^*\mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z})), P^{\mathcal{D}})$. Since $\mathfrak{f}\colon \mathcal{X} \longrightarrow S$ is an affine morphism, we obtain (2.10.3) by the affine base change theorem ([39, (1.5.2)]) as in the classical case ([11, 7.8 Theorem]). □

As in [3, V] and [11, §7], we have some important consequences of (2.10.3).

Corollary 2.10.4. *Let $f\colon (X, D\cup Z) \longrightarrow Y$ be as in* (2.10.2) (2). *Then*

$$Rf^{\log}_{(X,Z)\mathrm{crys}*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$$

is a filtered crystal in $\mathrm{DF}(\mathcal{O}_{Y/S})$. *That is, for a morphism $v\colon (U',T',\delta') \longrightarrow (U,T,\delta)$ of the crystalline site $(Y/S)_{\mathrm{crys}}$, the canonical morphism*

$$Lv^*((Rf^{\log}_{(X,Z)\mathrm{crys}*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D))_T) \longrightarrow$$

$$Rf^{\log}_{(X',Z')\mathrm{crys}*}(E^{\log,Z'}_{\mathrm{crys}}(\mathcal{O}_{(X',D'\cup Z')/S}), P^{D'})_{T'}$$

is an isomorphism, where $(X', D'\cup Z') := (X', D'\cup Z')\times_U U'$.

Corollary 2.10.5. *Let $f\colon (X, D\cup Z) \longrightarrow Y$ be as in* (2.10.2) (2). *Assume that Y has a smooth lift $\mathcal{Y}$ over S. Let h be an integer. Then the following holds*:

(1) *There exists a quasi-nilpotent integrable connection*

(2.10.5.1)
$$R^hf_{(X,Z)/\mathcal{Y}*}(P^D_kE^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})) \xrightarrow{\nabla_k} R^hf_{(X,Z)/\mathcal{Y}*}(P^D_kE^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}))\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^1_{\mathcal{Y}/S} \quad (k\in\mathbb{Z})$$

making the following diagram commutative for any two nonnegative integers $k\le l$:

(2.10.5.2)

$$\begin{array}{ccc} R^hf_{(X,Z)/\mathcal{Y}*}(P^D_kE^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})) & \xrightarrow{\nabla_k} & R^hf_{(X,Z)/\mathcal{Y}*}(P^D_kE^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}))\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^1_{\mathcal{Y}/S} \\ \downarrow & & \downarrow \\ R^hf_{(X,Z)/\mathcal{Y}*}(P^D_lE^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S})) & \xrightarrow{\nabla_l} & R^hf_{(X,Z)/\mathcal{Y}*}(P^D_lE^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}))\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^1_{\mathcal{Y}/S}. \end{array}$$

(2) *For $k \in \mathbb{Z}$, set*

$$
\begin{aligned}
&P_k^D R^h f_{(X,D\cup Z)/\mathcal{Y}*}(\mathcal{O}_{(X,D\cup Z)/S}) := \\
&\mathrm{Im}(R^h f_{(X,Z)/\mathcal{Y}*}(P_k^D E_{\mathrm{crys}}^{\log,Z}(\mathcal{O}_{(X,D\cup Z)/S})) \longrightarrow R^h f_{(X,D\cup Z)/\mathcal{Y}*}(\mathcal{O}_{(X,D\cup Z)/S})).
\end{aligned}
$$

Then there exists a quasi-nilpotent connection

$$
\begin{aligned}
P_k^D R^h f_{(X,D\cup Z)/\mathcal{Y}*}&(\mathcal{O}_{(X,D\cup Z)/S}) \\
&\longrightarrow P_k^D R^h f_{(X,D\cup Z)/\mathcal{Y}*}(\mathcal{O}_{(X,D\cup Z)/S})\otimes_{\mathcal{O}_{\mathcal{Y}}}\Omega^1_{\mathcal{Y}/S}.
\end{aligned}
$$

Corollary 2.10.6. *Let $f\colon (X, D\cup Z) \longrightarrow Y$ be as in* (2.10.2) (2)*. Let*

$$
\begin{array}{ccc}
(X', D'\cup Z') & \xrightarrow{g} & (X, D\cup Z) \\
{\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow{h} & Y \\
\big\downarrow & & \big\downarrow \\
(S', \mathcal{I}', \gamma') & \longrightarrow & (S, \mathcal{I}, \gamma)
\end{array}
$$

be a commutative diagram such that the upper rectangle is cartesian. Assume that Y' is a quasi-compact smooth scheme over S'. Then the natural morphism

$$
\begin{aligned}
&Lh^*_{\mathrm{crys}} Rf^{\log}_{(X,Z)\mathrm{crys}*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D) \longrightarrow \\
&Rf'^{\log}_{(X',Z')\mathrm{crys}*}(E^{\log,Z'}_{\mathrm{crys}}(\mathcal{O}_{(X',D'\cup Z')/S'}), P^{D'})
\end{aligned}
$$

is an isomorphism.

Corollary 2.10.7. *Let the notations and the assumptions be as in* (2.10.2) (1)*. Then $Rf^{\log}_{(X,Z)/S\mathrm{crys}*}(P_k^D E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}))$ $(k \in \mathbb{N})$ has finite tor-dimension. Moreover, if S is noetherian and if f is proper, then $Rf_{(X,Z)/S*}$ $(P_k^D E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}))$ is a perfect complex of $\mathcal{O}_S$-module.*

Definition 2.10.8. Let A be a noetherian commutative ring. Let $(E^\bullet, \{E^\bullet_k\}) \in \mathrm{CF}(A)$ be a filtered complex of A-modules. We say that $(E^\bullet, \{E^\bullet_k\})$ is *filteredly strictly perfect* if it is bounded, if the filtration $\{E^q_k\}$ is finite for any q and if all E^q_k's are finitely generated projective A-modules.

Definition 2.10.9. Let A be a commutative ring with unit element. For a filtered A-module $(E, \{E_k\})$ whose filtration is finite and for a family $\{T_l\}_{l\in\mathbb{Z}}$ of A-modules, we say that $(E, \{E_k\})$ is the *direct sum* of $\{T_l\}_{l\in\mathbb{Z}}$ if $E_k = \bigoplus_{l\leq k} T_l$ $(\forall k \in \mathbb{Z})$.

The following is a nontrivial filtered version of [11, 7.15 Lemma]:

Theorem 2.10.10. *Let A be a noetherian commutative ring. Let $(E^\bullet, \{E_k^\bullet\})$ be a filtered complex of A-modules. Assume that there exist integers $k_0 \leq k_1$ such that $E_{k_1}^q = E^q$ and $E_{k_0}^q = 0$ for all $q \in \mathbb{Z}$. Then $(E^\bullet, \{E_k^\bullet\})$ is quasi-isomorphic to a filteredly strictly perfect complex if and only if $E_k^\bullet$ $(\forall k)$ has finite tor-dimension and finitely generated cohomologies.*

Proof. Roughly speaking, the proof is dual to that of (1.1.7) with some additional calculations.

We have only to prove the "if" part. Let k be an integer such that $k_0 < k \leq k_1$. By the assumption, we may assume that $E^q = 0$ $(q > 0)$. Since $H^0(E_k^\bullet)$ is finitely generated, there exists a free A-module T_k^0 of finite rank with a morphism $T_k^0 \longrightarrow E_k^0$ such that the induced morphism $T_k^0 \longrightarrow H^0(E_k^\bullet)$ is surjective. Set $T_k^0 := 0$ for $k \leq k_0$ or $k > k_1$. Let $(Q^0, \{Q_k^0\})$ be the direct sum of $\{T_k^0\}$. Then we have a natural filtered morphism $(Q^0, \{Q_k^0\}) \longrightarrow (E^0, \{E_k^0\})$.

Assume that, for a nonpositive integer q, we are given a morphism

$$(Q^{\bullet \geq q}, \{Q_k^{\bullet \geq q}\}) \longrightarrow (E^{\bullet \geq q}, \{E_k^{\bullet \geq q}\})$$

of $(\geq q)$-truncated filtered complexes such that the induced morphism $H^*(Q_k^\bullet) \longrightarrow H^*(E_k^\bullet)$ is an isomorphism for $* > q$, $\operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1}) \longrightarrow H^q(E_k^\bullet)$ is surjective, $Q^\bullet = 0$ for $\bullet \geq 0$, $Q^\bullet = Q_{k_1}^\bullet$, $Q_{k_0}^\bullet = 0$ $(q \leq \bullet \leq 0)$ and that $(Q^r, \{Q_k^r\})$ $(\forall r \geq q)$ is the direct sum of some family $\{T_k^r\}_{k \in \mathbb{Z}}$ of free A-modules of finite rank.

For an integer $k_0 < k \leq k_1$, consider the fiber product $E_k^{q-1} \times_{E_k^q} \operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1})$. Let I_k^q be the image of the following composite morphism

$$E_k^{q-1} \times_{E_k^q} \operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1}) \longrightarrow \operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1}) \xrightarrow{\subset} Q_k^q.$$

Since A is noetherian, I_k^q is finitely generated. Let $\{y_i\}_{i \in I}$ be a system of finite generators of I_k^q. Take an element $(x_i, y_i) \in E_k^{q-1} \times_{E_k^q} \operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1})$. Because $H^{q-1}(E_k^\bullet)$ is finitely generated, we can take a family $\{z_j\}_{j \in J}$ of finite elements of $\operatorname{Ker}(E_k^{q-1} \longrightarrow E_k^q)$ whose images in $H^{q-1}(E_k^\bullet)$ form a system of generators of $H^{q-1}(E_k^\bullet)$.

Now consider a finitely generated A-module S_k^{q-1} generated by $\{(x_i, y_i)\}_{i \in I}$ and $\{(z_j, 0)\}_{j \in J}$ in $E_k^{q-1} \times_{E_k^q} \operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1})$. Let T_k^{q-1} be a free A-module of finite rank such that there exists a surjection $T_k^{q-1} \longrightarrow S_k^{q-1}$. Set $T_k^{q-1} := 0$ for $k \leq k_0$ or $k > k_1$. Let $(Q^{q-1}, \{Q_k^{q-1}\})$ be the direct sum of $\{T_k^{q-1}\}_{k \in \mathbb{Z}}$. Then we have a natural filtered morphism $(Q^{q-1}, \{Q_k^{q-1}\}) \longrightarrow (E^{q-1}, \{E_k^{q-1}\})$.

By assumption, $\operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1}) \longrightarrow H^q(E_k^\bullet)$ is a surjection. Moreover, if the image of an element of $\operatorname{Ker}(Q_k^q \longrightarrow Q_k^{q+1})$ vanishes in $H^q(E_k^\bullet)$, then this element belongs to $\operatorname{Im}(T_k^{q-1} \longrightarrow Q_k^q)$ by the definition of T_k^{q-1}. In particular, this element belongs to $\operatorname{Im}(Q_k^{q-1} \longrightarrow Q_k^q)$. Hence the natural morphism

$\mathrm{Ker}(Q^q_k \longrightarrow Q^{q+1}_k) \longrightarrow H^q(E^\bullet_k)$ induces an isomorphism $H^q(Q^\bullet_k) \xrightarrow{\sim} H^q(E^\bullet_k)$. Moreover, it is easy to see that $\mathrm{Ker}(Q^{q-1}_k \longrightarrow Q^q_k) \longrightarrow H^{q-1}(E^\bullet_k)$ is surjective. Hence the induction works well and so we have constructed a filtered complex $(Q^\bullet, \{Q^\bullet_k\})$ such that $Q^q = 0$ $(q > 0)$, such that $Q^\bullet_{k_0} = 0$ and $Q^\bullet_{k_1} = Q^\bullet$, such that $(Q^q, \{Q^q_k\})$ $(q \in \mathbb{Z})$ is the direct sum of a family $\{T^q_k\}_{k\in\mathbb{Z}}$ of free A-modules of finite rank and such that there exists a filtered quasi-isomorphism $(Q^\bullet, \{Q^\bullet_k\}) \longrightarrow (E^\bullet, \{E^\bullet_k\})$. Because $E^\bullet_k$ $(\forall k)$ has finite tor-dimension, $\mathrm{gr}_k E^\bullet$ $(\forall k)$ also has it. Since $(Q^\bullet, \{Q^\bullet_k\})$ is filteredly quasi-isomorphic to $(E^\bullet, \{E^\bullet_k\})$, $\mathrm{gr}_k Q^\bullet$ $(\forall k)$ also has it. Since the filtration on $Q^\bullet$ is finite, there exists a nonpositive integer r and a complex $F^\bullet_k$ of flat A-modules for each $k \in \mathbb{Z}$ satisfying the following properties:

(a) $F^\bullet_k$ is quasi-isomorphic to $\mathrm{gr}_k Q^\bullet$,

(b) $F^\bullet_k = 0$ for $\bullet > 0$ or $\bullet \leq r$.

Set $B^q_k := \mathrm{Im}(Q^{q-1}_k \longrightarrow Q^q_k)$. Let $l \leq k_1 - k_0$ be a positive integer. Set

$$R^q_{k_0+l} = \begin{cases} 0 & (q < r-l+1 \text{ or } q > 0), \\ Q^q_{k_0+l}/(Q^q_{k_0+r-q} + B^q_{k_0+r-q+1}) & (r-l+1 \leq q \leq r), \\ Q^q_{k_0+l} & (r < q \leq 0). \end{cases}$$

Then we claim that $R^q_{k_0+l}$ is a flat A-module. We proceed on induction on l. Unusually we assume that the initial case $l = 1$ holds and that $l \geq 2$. Consider the following exact sequence

$$0 \longrightarrow R^q_{k_0+l-1} \longrightarrow R^q_{k_0+l} \longrightarrow \mathrm{gr}_{k_0+l} Q^q \longrightarrow 0 \quad (r-l+1 < q \leq r).$$

By the induction hypothesis, we may assume that $R^q_{k_0+l-1}$ $(r-l+1 < q \leq r)$ is a flat A-module. Since $\mathrm{gr}_{k_0+l} Q^q$ is a flat A-module, so is $R^q_{k_0+l}$ $(r-l+1 < q \leq r)$. Now we show that $R^{r-l+1}_{k_0+l}$ is a flat A-module. By the properties (a) and (b), we have the following exact sequence

$$\cdots \longrightarrow \mathrm{gr}_{k_0+l} Q^{r-l} \longrightarrow \mathrm{gr}_{k_0+l} Q^{r-l+1} \longrightarrow R^{r-l+1}_{k_0+l} \longrightarrow 0.$$

For a positive integer i and for any A-module M,

$$\begin{aligned} &\mathrm{Tor}^A_i(R^{r-l+1}_{k_0+l}, M) \\ =&H^{-i}(\cdots \longrightarrow \mathrm{gr}_{k_0+l} Q^{r-l} \otimes_A M \longrightarrow \mathrm{gr}_{k_0+l} Q^{r-l+1} \otimes_A M \longrightarrow 0) \\ =&H^{r-l+1-i}(\mathrm{gr}_{k_0+l} Q^\bullet \otimes_A M) = H^{r-l+1-i}(F^\bullet_{k_0+l} \otimes_A M) = 0. \end{aligned}$$

Hence $R^{r-l+1}_{k_0+l}$ is a flat A-module. The rest for showing the claim is to prove that $R^r_{k_0+1}$ is a flat A-module. As above, we can prove this using the following resolution

$$\cdots \longrightarrow Q^{r-1}_{k_0+1} \longrightarrow Q^r_{k_0+1} \longrightarrow R^r_{k_0+1} \longrightarrow 0.$$

Set $R^\bullet := R^\bullet_k := R^\bullet_{k_1}$ for $k \geq k_1$ and $R^\bullet_k := 0$ for $k \leq k_0$. Then $\{R^\bullet_k\}_{k\in\mathbb{Z}}$ is an increasing filtration on $R^\bullet$ since the natural morphism $R^\bullet_{k_0+l-1} \longrightarrow R^\bullet_{k_0+l}$ is injective. Note that $R^\bullet$ is a bounded complex of projective A-modules.

Finally we claim that the natural morphism $(Q^\bullet, \{Q^\bullet_k\}) \longrightarrow (R^\bullet, \{R^\bullet_k\})$ is a filtered quasi-isomorphism. Indeed, for a positive integer $l \leq k_1 - k_0$, $\mathrm{gr}_{k_0+l}R^\bullet$ is the following complex

$$
\begin{aligned}
0 \longrightarrow \mathrm{gr}_{k_0+l}Q^{r-l+1}/\mathrm{Im}(&\underset{r-l+1}{\mathrm{gr}_{k_0+l}Q^{r-l}} \longrightarrow \mathrm{gr}_{k_0+l}Q^{r-l+1}) \\
&\longrightarrow \underset{r-l+2}{\mathrm{gr}_{k_0+l}Q^{r-l+2}} \longrightarrow \cdots .
\end{aligned}
$$

This complex is isomorphic to $\mathrm{gr}_{k_0+l}Q^\bullet$ by the properties (a) and (b).

Hence we have finished the proof of (2.10.10). $\square$

Corollary 2.10.11. *Let the notations and the assumptions be as in* (2.10.7). *Then the filtered complex* $Rf_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys}}(\mathcal{O}_{(X,D\cup Z)/S}), P^D)$ *is a filtered perfect complex of* $\mathcal{O}_S$*-modules, that is, locally on* S_{zar}*, filteredly quasi-isomorphic to a filtered strictly perfect complex.*

Proof. (2.10.11) immediately follows from (2.10.7) and (2.10.10). $\square$

(2) Künneth formula.

Next, we give the Künneth formula of preweight-filtered vanishing cycle crystalline complexes.

Let X_j/S_0 $(j = 1, 2)$ be a smooth scheme with transversal relative SNCD's D_j and Z_j over S_0. Set $X_3 := X_1 \times_{S_0} X_2$, $D_3 = (D_1 \times_{S_0} X_2) \cup (X_1 \times_{S_0} D_2)$ and $Z_3 = (Z_1 \times_{S_0} X_2) \cup (X_1 \times_{S_0} Z_2)$. Let $f_j\colon (X_j, D_j \cup Z_j) \longrightarrow S_0$ $(j = 1, 2, 3)$ be the structural morphism. Assume that S is quasi-compact and that $\overset{\circ}{f}_j$ $(j = 1, 2)$ is quasi-compact and quasi-separated. We denote $Rf_{j(X_j,Z_j)/S*}(E^{\log,Z_j}_{\mathrm{crys}}(\mathcal{O}_{(X_j,D_j\cup Z_j)/S}), P^{D_j})$ simply by $Rf_{(X_j,Z_j)/S*}(E^{\log,Z_j}_{\mathrm{crys}}(\mathcal{O}_{(X_j,D_j\cup Z_j)/S}), P^{D_j})$. We have the following commutative diagram of ringed topoi for $j = 1, 2$:

(2.10.11.1)

$$
\begin{CD}
(((X_j, \widetilde{D_j \cup Z_j})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_j,D_j\cup Z_j)/S}) @<{q^{\log}_{j\mathrm{crys}}}<< (((X_3, \widetilde{D_3 \cup Z_3})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_3,D_3\cup Z_3)/S}) \\
@V{\epsilon_{(X_j,D_j\cup Z_j,Z_j)/S}}VV @VV{\epsilon_{(X_3,D_3\cup Z_3,Z_3)/S}}V \\
(((\widetilde{X_j, Z_j})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_j,Z_j)/S}) @<{p^{\log}_{j\mathrm{crys}}}<< (((\widetilde{X_3, Z_3})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_3,Z_3)/S}) \\
@V{f_{(X_j,Z_j)/S}}VV @VV{f_{(X_3,Z_3)/S}}V \\
(\widetilde{S}_{\mathrm{zar}}, \mathcal{O}_S) @= (\widetilde{S}_{\mathrm{zar}}, \mathcal{O}_S),
\end{CD}
$$

where $q_j\colon (X_3, D_3 \cup Z_3) \longrightarrow (X_j, D_j \cup Z_j)$ and $p_j\colon (X_3, Z_3) \longrightarrow (X_j, Z_j)$ are the projections. We shall construct a canonical morphism

$$Rf_{(X_1,Z_1)/S*}(E^{\log,Z_1}_{\rm crys}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}), P^{D_1})\otimes^L_{\mathcal{O}_S} Rf_{(X_2,Z_2)/S*}(E^{\log,Z_2}_{\rm crys}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}), P^{D_2}) \longrightarrow Rf_{(X_3,Z_3)/S*}(E^{\log,Z_3}_{\rm crys}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}), P^{D_3}). \tag{2.10.11.2}$$

For simplicity of notation, set $\epsilon_j := \epsilon_{(X_j,D_j\cup Z_j,Z_j)/S}$ $(j = 1, 2, 3)$. We have to construct a morphism

(2.10.11.3)

$$Rf_{(X_1,Z_1)/S*}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}), \tau)\otimes^L_{\mathcal{O}_S} Rf_{(X_2,Z_2)/S*}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}), \tau) \longrightarrow Rf_{(X_3,Z_3)/S*}(R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}), \tau).$$

To construct it, we need the following two lemmas:

Lemma 2.10.12 (cf. (2.7.2)). *Let $f\colon (\mathcal{T}, \mathcal{A}) \longrightarrow (\mathcal{T}', \mathcal{A}')$ be a morphism of ringed topoi. Then, for an object $E^\bullet$ in $\mathrm{D}^-(\mathcal{A}')$, there exists a canonical morphism*

$$Lf^*((E^\bullet, \tau)) \longrightarrow (Lf^*(E^\bullet), \tau) \tag{2.10.12.1}$$

in $\mathrm{D}^-\mathrm{F}(\mathcal{A})$.

Proof. Let $Q^\bullet \longrightarrow E^\bullet$ be a quasi-isomorphism from a complex of flat $\mathcal{A}'$-modules. Let $(R^\bullet, \{R^\bullet_k\}) \longrightarrow (Q^\bullet, \tau)$ be a filtered flat resolution of $(Q^\bullet, \tau)$. Then, by applying the functor f^* to the morphism of this resolution, we obtain a diagram

$$\begin{CD} f^*(R^\bullet_k) @>>> f^*(\tau_k Q^\bullet) \\ @VVV @VVV \\ f^*(R^\bullet) @>>> f^*(Q^\bullet). \end{CD} \tag{2.10.12.2}$$

By (1.1.19) (2), the left hand side of (2.10.12.2) is equal to $Lf^*((E^\bullet, \tau))$. On the other hand, there exists a natural morphism $f^*(\tau_k Q^\bullet) \longrightarrow \tau_k f^*(Q^\bullet)$. Hence there exists a natural diagram

$$\begin{CD} f^*(\tau_k Q^\bullet) @>>> \tau_k f^*(Q^\bullet) \\ @VVV @VVV \\ f^*(Q^\bullet) @= f^*(Q^\bullet). \end{CD} \tag{2.10.12.3}$$

Composing (2.10.12.2) with (2.10.12.3), we have a morphism (2.10.12.1). □

Lemma 2.10.13. *Let $(\mathcal{T}, \mathcal{A})$ be a ringed topos. Let $E^\bullet$ and $F^\bullet$ be two complexes of $\mathcal{A}$-modules. Assume that $E^\bullet$ is bounded above. Then there exists a canonical morphism*

(2.10.13.1) $$(E^\bullet,\tau)\otimes^L_{\mathcal{A}}(F^\bullet,\tau)\longrightarrow(E^\bullet\otimes^L_{\mathcal{A}}F^\bullet,\tau).$$

Proof. Let $P^\bullet\longrightarrow E^\bullet$ be a flat resolution of $E^\bullet$. Let $(Q^\bullet,\{Q^\bullet_k\})\longrightarrow(P^\bullet,\tau)$ be a filtered flat resolution of $(P^\bullet,\tau)$. Then we have the following:

$$\begin{aligned}
&(E^\bullet,\tau)\otimes^L_{\mathcal{A}}(F^\bullet,\tau)\\
&=(Q^\bullet,\{Q^\bullet_k\})\otimes_{\mathcal{A}}(F^\bullet,\tau)\\
&=(Q^\bullet\otimes_{\mathcal{A}}F^\bullet,\{\mathrm{Im}(\sum_{l+m=k}Q^\bullet_l\otimes_{\mathcal{A}}\tau_mF^\bullet\longrightarrow Q^\bullet\otimes_{\mathcal{A}}F^\bullet)\}_{k\in\mathbb{Z}})\\
&\longrightarrow(P^\bullet\otimes_{\mathcal{A}}F^\bullet,\{\mathrm{Im}(\sum_{l+m=k}\tau_lP^\bullet\otimes_{\mathcal{A}}\tau_mF^\bullet\longrightarrow\sum_{l+m=k}P^\bullet\otimes_{\mathcal{A}}F^\bullet)\}_{k\in\mathbb{Z}})\\
&\longrightarrow(E^\bullet\otimes^L_{\mathcal{A}}F^\bullet,\tau).
\end{aligned}$$

□

Now we construct the canonical morphism (2.10). We need a canonical element in

(2.10.13.2)
$$\begin{aligned}
&\mathcal{H}^0[\mathrm{RHom}_{\mathcal{O}_{(X_1,Z_1)/S}}(Lf^*_{(X_3,Z_3)/S}\{Rf_{(X_1,Z_1)/S*}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),\tau)\otimes^L_{\mathcal{O}_S}\\
&\qquad Rf_{(X_2,Z_2)/S*}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),\tau)\},(R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau))].
\end{aligned}$$

First we have the following morphism

$$\begin{aligned}
&Lf^*_{(X_3,Z_3)/S}\{Rf_{(X_1,Z_1)/S*}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),\tau)\otimes^L_{\mathcal{O}_S}\\
&\qquad\qquad Rf_{(X_2,Z_2)/S*}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),\tau)\}\\
&=Lf^*_{(X_3,Z_3)/S}Rf_{(X_1,Z_1)/S*}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),\tau)\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}\\
&\qquad\qquad Lf^*_{(X_3,Z_3)/S}Rf_{(X_2,Z_2)/S*}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),\tau)\\
&=Lp^{\log*}_{1\mathrm{crys}}Lf^*_{(X_1,Z_1)/S}Rf_{(X_1,Z_1)/S*}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),\tau)\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}\\
&\qquad\qquad Lp^{\log*}_{2\mathrm{crys}}Lf^*_{(X_2,Z_2)/S}Rf_{(X_2,Z_2)/S*}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),\tau)\\
&\longrightarrow Lp^{\log*}_{1\mathrm{crys}}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),\tau)\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}\\
&\qquad\qquad Lp^{\log*}_{2\mathrm{crys}}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),\tau).
\end{aligned}$$

Note that $R\epsilon_{j*}(\mathcal{O}_{(X_j,D_j\cup Z_j)/S})$ $(j=1,2,3)$ is bounded above by (2.7.10). Therefore it suffices to construct a canonical morphism

(2.10.13.3)
$$\begin{aligned}
&Lp^{\log*}_{1\mathrm{crys}}(R\epsilon_{1*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),\tau)\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}Lp^{\log*}_{2\mathrm{crys}}(R\epsilon_{2*}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),\tau)\\
&\qquad\longrightarrow(R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau).
\end{aligned}$$

We also have the following composite morphism

$$\begin{aligned}Lp^{\log *}_{j\mathrm{crys}}(R\epsilon_{j*}(\mathcal{O}_{(X_j,D_j\cup Z_j)/S}),\tau) &\longrightarrow (Lp^{\log *}_{j\mathrm{crys}}R\epsilon_{j*}(\mathcal{O}_{(X_j,D_j\cup Z_j)/S}),\tau)\\ &\longrightarrow (R\epsilon_{3*}Lq^{\log *}_{j\mathrm{crys}}(\mathcal{O}_{(X_j,D_j\cup Z_j)/S}),\tau)\\ &= (R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau)\end{aligned}$$

Here we have obtained the first morphism by (2.10.12), and the second morphism by the commutative diagram (2.10.11.1) and the adjunction morphism. Thus we have only to construct a canonical morphism

$$\begin{aligned}(R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau)\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}(R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau)&\\ \longrightarrow (R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau).&\end{aligned}$$

By (2.10.13), it suffices to construct a canonical morphism

$$\begin{aligned}(R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau)&\\ \longrightarrow (R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),\tau)&\end{aligned}$$

and, furthermore, to construct a canonical morphism

$$\begin{aligned}R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}}R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})&\\ \longrightarrow R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}).&\end{aligned}$$

Hence we have only to have a canonical element of

(2.10.13.4) $$\mathcal{H}^0[\mathrm{RHom}_{\mathcal{O}_{(X_3,D_3\cup Z_3)/S}}(L\epsilon_3^*\{R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\otimes^L_{\mathcal{O}_{(X_3,Z_3)/S}} R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\},\mathcal{O}_{(X_3,D_3\cup Z_3)/S})].$$

The source of [ ] in (2.10.13.4) is

$$L\epsilon_3^*R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\otimes^L_{\mathcal{O}_{(X_3,D_3\cup Z_3)/S}}L\epsilon_3^*R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}).$$

Using the adjunction, we have a composite morphism

$$\begin{aligned}L\epsilon_3^*R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\otimes^L_{\mathcal{O}_{(X_3,D_3\cup Z_3)/S}}L\epsilon_3^*R\epsilon_{3*}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S})\longrightarrow&\\ \mathcal{O}_{(X_3,D_3\cup Z_3)/S}\otimes^L_{\mathcal{O}_{(X_3,D_3\cup Z_3)/S}}\mathcal{O}_{(X_3,D_3\cup Z_3)/S}=\mathcal{O}_{(X_3,D_3\cup Z_3)/S}.&\end{aligned}$$

Thus we have a morphism (2.10.11.2).

Theorem 2.10.14 (Künneth formula). (1) *Let the notation be as above. Then there exists a canonical isomorphism*

(2.10.14.1) $$Rf_{(X_1,Z_1)/S*}(E^{\log,Z_1}_{\mathrm{crys}}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),P^{D_1})\otimes^L_{\mathcal{O}_S} Rf_{(X_2,Z_2)/S*}(E^{\log,Z_2}_{\mathrm{crys}}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),P^{D_2})$$

$$\xrightarrow{\sim} Rf_{(X_3,Z_3)/S*}(E^{\log,Z_3}_{\text{crys}}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),P^{D_3}).$$

(2) *Let* Y *and* $f_j\colon (X_j,D_j\cup Z_j)\longrightarrow Y$ $(j=1,2)$ *be as in* (2.10.2) (2). *Set* $f_3:=f_1\times_Y f_2$. *Then there exists a canonical isomorphism*

$$\begin{aligned}(2.10.14.2)\qquad & Rf^{\log}_{(X_1,Z_1)\text{crys}*}(E^{\log,Z_1}_{\text{crys}}(\mathcal{O}_{(X_1,D_1\cup Z_1)/S}),P^{D_1})\otimes^L_{\mathcal{O}_{Y/S}}\\ & Rf^{\log}_{(X_2,Z_2)\text{crys}*}(E^{\log,Z_2}_{\text{crys}}(\mathcal{O}_{(X_2,D_2\cup Z_2)/S}),P^{D_2})\\ & \xrightarrow{\sim} Rf^{\log}_{(X_3,Z_3)\text{crys}*}(E^{\log,Z_3}_{\text{crys}}(\mathcal{O}_{(X_3,D_3\cup Z_3)/S}),P^{D_3}).\end{aligned}$$

Proof. (1): By virtue of the filtered cohomological descent (1.5.1) (2), we may assume that X_j $(j=1,2)$ and S are affine as in the proof of [3, V Corollary 4.2.2], and hence that $(X_j,D_j\cup Z_j)$ $(j=1,2)$ has a log smooth lift $(\mathcal{X}_j,\mathcal{D}_j\cup\mathcal{Z}_j)$ over S. Let $(\mathcal{X}_3,\mathcal{D}_3\cup\mathcal{Z}_3)$ be the fiber product of $(\mathcal{X}_1,\mathcal{D}_1\cup\mathcal{Z}_1)$ and $(\mathcal{X}_2,\mathcal{D}_2\cup\mathcal{Z}_2)$ over S. Let $g_j\colon(\mathcal{X}_j,\mathcal{D}_j\cup\mathcal{Z}_j)\longrightarrow S$ $(j=1,2,3)$ be the structural morphism. In this case, by (2.7.5), the proof of (1) is reduced to showing an isomorphism

$$\begin{aligned}&(g_{1*}\Omega^{\bullet}_{\mathcal{X}_1/S}(\log(\mathcal{D}_1\cup\mathcal{Z}_1))\otimes_{\mathcal{O}_S}g_{2*}\Omega^{\bullet}_{\mathcal{X}_2/S}(\log(\mathcal{D}_2\cup\mathcal{Z}_2)),\\ &\{\sum_{l+m=k}g_{1*}P^{\mathcal{D}_1}_l\Omega^{\bullet}_{\mathcal{X}_1/S}(\log(\mathcal{D}_1\cup\mathcal{Z}_1))\otimes_{\mathcal{O}_S}g_{2*}P^{\mathcal{D}_2}_m\Omega^{\bullet}_{\mathcal{X}_2/S}(\log(\mathcal{D}_2\cup\mathcal{Z}_2))\}_{k\in\mathbb{Z}})\\ &\longrightarrow g_{3*}(\Omega^{\bullet}_{\mathcal{X}_3/S}(\log(\mathcal{D}_3\cup\mathcal{Z}_3)),P^{\mathcal{D}_3}),\end{aligned}$$

which is easily verified.

(2): (2) follows from (1) as in [3, V Theorem 4.2.1]. □

The following is the compatibility of the preweight-filtered Künneth formula with the base change formula.

Proposition 2.10.15. *Let* u *be the morphism in* (2.10.3). *Let* $'$ *mean the base change of an object over* S *by* $u|_{S_0}$. *Let* K *be the preweight-filtered Künneth isomorphism* (2.10.14.1) *and* K' *the preweight-filtered Künneth isomorphism for* $(X'_i,D'_i\cup Z'_i)$ $(i=1,2,3)$. *Set*

$$H_i:=Rf_{(X_i,Z_i)/S*}(E^{\log,Z_i}_{\text{crys}}(\mathcal{O}_{(X_i,D_i\cup Z_i)/S}),P^{D_i})$$

and

$$H'_i:=Rf_{(X'_i,Z'_i)/S'*}(E^{\log,Z'_i}_{\text{crys}}(\mathcal{O}_{(X'_i,D'_i\cup Z'_i)/S'}),P^{D'_i})$$

$(i=1,2,3)$. *Then the following diagram is commutative*:

$$(2.10.15.1)\qquad \begin{array}{ccc} Lu^*H_1\otimes^L_{\mathcal{O}_{S'}}Lu^*H_2 & \xrightarrow{Lu^*(K)} & Lu^*H_3\\ \simeq\big\downarrow & & \big\downarrow\simeq\\ H'_1\otimes^L_{\mathcal{O}_{S'}}H'_2 & \xrightarrow{K'} & H'_3.\end{array}$$

Proof. We leave the proof of (2.10.15) to the reader because the proof is a straightforward (but long) exercise by recalling the constructions of the base change isomorphism and the Künneth isomorphism (cf. [3, V Proposition 4.1.3]). □

2.11 Log Crystalline Cohomology with Compact Support

Let the notations be as in §2.4. Let us define a variant of a special case of the definition of the log crystalline cohomology sheaf with compact support in [85, §5] briefly (cf. [29, §2]). Let $(U, T, \iota, M_T, \delta)$ be an object of the log crystalline site $((X, D\cup Z)/S)^{\log}_{\rm crys} = ((X, M(D\cup Z))/S)^{\log}_{\rm crys}$. Set $M_U := M(D\cup Z)|_U$. Because $\iota\colon (U, M_U) \longrightarrow (T, M_T)$ is an exact closed immersion, $M_T/\mathcal{O}_T^* = M_U/\mathcal{O}_U^*$ on $U_{\rm zar} = T_{\rm zar}$. Hence the defining local equation of the relative SNCD $D\cap U$ on U lifts to a local section t of M_T. We define an ideal sheaf $\mathcal{I}^D_{(X,D\cup Z)/S} \subset \mathcal{O}_{(X,D\cup Z)/S}$ by the following: $\mathcal{I}^D_{(X,D\cup Z)/S}(T)$= the ideal generated by the image of t by the structural morphism $M_T \longrightarrow \mathcal{O}_T$. One can prove that $Q^*_{(X,D\cup Z)/S}(\mathcal{I}^D_{(X,D\cup Z)/S})$ is a crystal on the restricted log crystalline site $((X, D\cup Z)/S)^{\log}_{\rm Rcrys}$ in the same way as [85, (5.3)].

Definition 2.11.1. We call the higher direct image sheaf $R^h f_{(X,D\cup Z)/S*}(\mathcal{I}^D_{(X,D\cup Z)/S})$ in $\widetilde{S}_{\rm zar}$ the *log crystalline cohomology sheaf with compact support with respect to* D and denote it by $R^h f_{(X,D\cup Z)/S*,{\rm c}}(\mathcal{O}_{(X,D\cup Z;Z)/S})$.

The local description of $R^h f_{(X,D\cup Z)/S*,{\rm c}}(\mathcal{O}_{(X,D\cup Z;Z)/S})$ is as follows; assume that there exists an exact closed immersion $\iota\colon (X, D\cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D}\cup\mathcal{Z})$ into a smooth scheme with a relative SNCD over S such that ι induces exact closed immersions $(X, D) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D})$ and $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{Z})$. Let $\mathfrak{D}$ be the log PD-envelope of the exact closed immersion $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{Z})$ over $(S, \mathcal{I}, \gamma)$ with structural morphism $f_S\colon \mathfrak{D} \longrightarrow S$. Let $u_{(X,D\cup Z)/S}\colon ((X, \widetilde{D\cup Z})/S)^{\log}_{\rm crys} \longrightarrow \widetilde{X}_{\rm zar}$ be the canonical projection. Let $\mathcal{F}$ be the crystal on $((X, D\cup Z)/S)^{\log}_{\rm crys}$ corresponding to the integrable log connection $\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{D}) \longrightarrow \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}(-\mathcal{D})\Omega^1_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$. Then there exists a natural morphism $\mathcal{F} \longrightarrow \mathcal{I}^D_{(X,D\cup Z)/S}$ and it induces an isomorphism $Q^*_{(X,D\cup Z)/S}(\mathcal{F}) \overset{=}{\longrightarrow} Q^*_{(X,D\cup Z)/S}(\mathcal{I}^D_{(X,D\cup Z)/S})$ by [85, (5.3)]. Hence we have the following formula:

(2.11.1.1)
$$
\begin{aligned}
&Ru_{(X,D\cup Z)/S*}(\mathcal{I}^D_{(X,D\cup Z)/S})\\
&= R\overline{u}_{(X,D\cup Z)/S*}Q^*_{(X,D\cup Z)/S}(\mathcal{I}^D_{(X,D\cup Z)/S})\\
&= R\overline{u}_{(X,D\cup Z)/S*}Q^*_{(X,D\cup Z)/S}(\mathcal{F})\\
&= Ru_{(X,D\cup Z)/S*}(\mathcal{F}) = \mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^\bullet_{\mathcal{X}/S}(\log(\mathcal{Z}-\mathcal{D})),
\end{aligned}
$$

where $\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{Z}-\mathcal{D})) := \mathcal{O}_{\mathcal{X}}(-\mathcal{D})\Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{D}\cup\mathcal{Z}))$. As a result, we have

$$R^h f_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S}) = R^h f_{S*}(\mathcal{O}_{\mathfrak{D}} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^{\bullet}_{\mathcal{X}/S}(\log(\mathcal{Z}-\mathcal{D}))).$$

Let $\{D_\lambda\}_\lambda$ be a decomposition of D by smooth components of D. Let the notations be as in §2.8. The exact closed immersion $\iota^{\underline{\lambda}_j}_{\underline{\lambda}} : (D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}}) \overset{\subset}{\longrightarrow} (D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}})$ induces the morphism

(2.11.1.2)
$$(-1)^j \iota^{\underline{\lambda}_j \log *}_{\underline{\lambda}\mathrm{crys}} : \mathcal{O}_{(D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}})/S} \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_j\mathrm{crys}}(D/S;Z) \longrightarrow$$
$$\iota^{\underline{\lambda}_j \log}_{\underline{\lambda}\mathrm{crys}*}(\mathcal{O}_{(D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}})/S}) \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}\mathrm{crys}}(D/S;Z)$$

defined by $x \otimes (\lambda_0 \cdots \widehat{\lambda}_j \cdots \lambda_{k-1}) \longmapsto (-1)^j \iota^{\underline{\lambda}_j \log *}_{\underline{\lambda}\mathrm{crys}}(x) \otimes (\lambda_0 \cdots \lambda_{k-1})$. It is easy to check that the morphism $(-1)^j \iota^{\underline{\lambda}_j \log *}_{\underline{\lambda}\mathrm{crys}}$ is well-defined. Set

(2.11.1.3)
$$\iota^{(k-1)\log *}_{\mathrm{crys}} := \sum_{\{\lambda_0,\lambda_1,\cdots,\lambda_{k-1} \ | \ \lambda_i \neq \lambda_l \ (i\neq l)\}} \sum_{j=0}^{k-1} a^{\log}_{\underline{\lambda}_j\mathrm{crys}*} \circ ((-1)^j \iota^{\underline{\lambda}_j \log *}_{\underline{\lambda}\mathrm{crys}}) :$$
$$a^{(k-1)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k-1)}, Z|_{D^{(k-1)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k-1)\log}_{\mathrm{crys}}(D/S;Z)) \longrightarrow$$
$$a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}, Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)).$$

The composite morphism $\iota^{(k)\log *}_{\mathrm{crys}} \circ \iota^{(k-1)\log *}_{\mathrm{crys}}$ is the zero. Indeed, the question is local. By taking trivializations of orientation sheaves, we can reduce this vanishing to the usual well-known case.

In this section we start with the following:

Lemma 2.11.2. *The morphism $\iota^{(k-1)\log *}_{\mathrm{crys}}$ is independent of the choice of the decomposition of $\{D_\lambda\}_\lambda$ by smooth components of D/S_0.*

Proof. The question is local. Let Δ and Δ' be two decompositions of D by smooth components of D. Let x be a point of X. By (A.0.1) below, there exists an open neighborhood U of x such that $\Delta'|_U = \Delta|_U$. Thus we have (2.11.2). □

Theorem 2.11.3. *Let ϵ: $((X, \widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}} \longrightarrow ((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}}$ be the forgetting log morphism along D ((2.3.2)). Set*

(2.11.3.1)
$$E^{\log, Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S})$$
$$:= (\mathcal{O}_{(X,Z)/S} \otimes_{\mathbb{Z}} \varpi^{(0)\log}_{\mathrm{crys}}(D/S;Z) \overset{\iota^{(0)\log *}_{\mathrm{crys}}}{\longrightarrow}$$
$$a^{(1)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(1)}, Z|_{D^{(1)}})/S} \otimes_{\mathbb{Z}} \varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)) \overset{\iota^{(1)\log *}_{\mathrm{crys}}}{\longrightarrow}$$

$$a^{(2)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(2)},Z|_{D^{(2)}})/S}\otimes_{\mathbb{Z}}\varpi^{(2)\log}_{\mathrm{crys}}(D/S;Z))\overset{\iota^{(2)\log*}_{\mathrm{crys}}}{\longrightarrow}\cdots).$$

Then there exists the following canonical isomorphism in $\mathrm{D}^+(Q^*_{(X,Z)/S}(\mathcal{O}_{(X,Z)/S}))$:

$$Q^*_{(X,Z)/S}R\epsilon_*(\mathcal{I}^D_{(X,D\cup Z)/S})\overset{\sim}{\longrightarrow}Q^*_{(X,Z)/S}E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}). \tag{2.11.3.2}$$

Before the proof of (2.11.3), we prove two lemmas.

Lemma 2.11.4. *There exists a morphism of topoi*

$$\epsilon_{\mathrm{Rcrys}}:((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{Rcrys}}\longrightarrow((\widetilde{X,Z})/S)^{\log}_{\mathrm{Rcrys}}$$

fitting into the following commutative diagram of topoi:

$$\begin{array}{ccc}((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{Rcrys}} & \xrightarrow{\epsilon_{\mathrm{Rcrys}}} & ((\widetilde{X,Z})/S)^{\log}_{\mathrm{Rcrys}}\\ {\scriptstyle Q_{(X,D\cup Z)/S}}\downarrow & & \downarrow{\scriptstyle Q_{(X,Z)/S}}\\ ((X,\widetilde{D\cup Z})/S)^{\log}_{\mathrm{crys}} & \xrightarrow{\epsilon} & ((\widetilde{X,Z})/S)^{\log}_{\mathrm{crys}}.\end{array} \tag{2.11.4.1}$$

Proof. First we show the existence of $\epsilon_{\mathrm{Rcrys}}$. To show this, it suffices to see that, for an object $T:=(U,T,M_T,\iota,\delta)\in((X,D\cup Z)/S)^{\log}_{\mathrm{Rcrys}}$, the object $(U,T,N^{\mathrm{inv}}_T,\iota,\delta)$ constructed in §2.3 belongs to $((X,Z)/S)^{\log}_{\mathrm{Rcrys}}$ Zariski locally on T. (Then we can define the exact functor $\epsilon^*_{\mathrm{Rcrys}}$ in the same way as ϵ^* in §2.3.) Let us assume that T is the log PD-envelope of the closed immersion $i:(U,(D\cup Z)|_U)\overset{\subset}{\longrightarrow}(\mathcal{U},M_{\mathcal{U}})$, where $(\mathcal{U},M_{\mathcal{U}})$ is log smooth over S. Since the log structure $M_{\mathcal{U}}$ is defined on the Zariski site of $\mathcal{U}$, we have a factorization

$$(U,(D\cup Z)|_U)\overset{\subset}{\longrightarrow}(\mathcal{U}',M_{\mathcal{U}'})\longrightarrow(\mathcal{U},M_{\mathcal{U}})$$

of i Zariski locally on $\mathcal{U}$ such that the first morphism is an exact closed immersion and that the second morphism is log etale. Then T is the log PD-envelope of the first morphism. Hence we may suppose that i is an exact closed immersion. Then, by (2.1.5), we may assume that i is an admissible closed immersion $(U,(D\cup Z)|_U)\overset{\subset}{\longrightarrow}(\mathcal{U},\mathcal{D}\cup\mathcal{Z})$. In this case, the log structure N^{inv}_T on T is nothing but the pull-back of the log structure on $\mathcal{U}$ defined by $\mathcal{Z}$. Hence (T,N^{inv}_T) is the log PD-envelope of the exact closed immersion $(U,Z|_U)\overset{\subset}{\longrightarrow}(\mathcal{U},\mathcal{Z})$. Hence $(U,T,N^{\mathrm{inv}}_T,\iota,\delta)$ belongs to $((X,Z)/S)^{\log}_{\mathrm{Rcrys}}$ Zariski locally on T. Now it is clear that we have the morphism $\epsilon_{\mathrm{Rcrys}}$ of topoi. It is easy to see that we have the commutative diagram (2.11.4.1). □

Lemma 2.11.5. *Let the notations be as in* (2.11.4)*. Then the following natural morphism of functors*

$$Q^*_{(X,Z)/S}R\epsilon_*\longrightarrow R\epsilon_{\mathrm{Rcrys}*}Q^*_{(X,D\cup Z)/S}$$

for $\mathcal{O}_{(X,D\cup Z)/S}$-modules is an isomorphism.

Proof. By the same argument as that in the proof of (1.6.4), we are reduced to showing that, for any parasitic $\mathcal{O}_{(X,D\cup Z)/S}$-module F of $((X,D\cup Z)/S)^{\log}_{\text{crys}}$, $R^q\epsilon_*(F)$ is also parasitic for any $q\geq 0$. To see this, it suffices to prove that, for any object $T:=(U,T,M_T,\iota,\delta)\in((X,Z)/S)^{\log}_{\text{Rcrys}}$ with T sufficiently small, the sheaf $(R^q\epsilon_*(F))_T$ on T_{zar} induced by $R^q\epsilon_*(F)$ is equal to zero. Hence we may assume that there exists a closed immersion $i:(U,Z|_U)\overset{\subset}{\longrightarrow}\mathcal{X}$ into an affine log smooth scheme over S such that (T,M_T) is the log PD-envelope of i. On the other hand, let us take a closed immersion $i':(U,(D\cup Z)|_U)\overset{\subset}{\longrightarrow}\mathcal{Y}$ into an affine log scheme which is log smooth over S. Then, for any $n\in\mathbb{Z}_{\geq 1}$, we have the closed immersion $i_n:(U,(D\cup Z)|_U)\overset{\subset}{\longrightarrow}\mathcal{X}\times_S\mathcal{Y}^n$ induced by $i\circ(\epsilon|_U)$ and i'. Let $\mathfrak{D}(n)$ be the log PD-envelope of the closed immersion i_n over $(S,\mathcal{I},\gamma)$. Then it is isomorphic to the log PD-envelope of the closed immersion $(U,(D\cup Z)|_U)\overset{\subset}{\longrightarrow}(T,M_T)\times_S\mathcal{Y}^n$ (induced by the composite $\iota\circ(\epsilon|_U)$ and i') compatible with $\overline{\delta}$, where $\overline{\delta}$ is the PD-structure on $\text{Ker}(\mathcal{O}_T\longrightarrow\mathcal{O}_U)+\mathcal{I}\mathcal{O}_T$ extending γ and δ. By the log version of [3, V 1.2.5], we have

$$(R^q\epsilon_*(F))_T=R^qf_{(U,(D\cup Z)|_U)/T*}F=R^q(\iota\circ(\epsilon|_U))_*\check{\text{C}}\text{A}(F),$$

where $\check{\text{C}}\text{A}(F)=F_{\mathfrak{D}(\bullet)}$ is the log version of the Čech-Alexander complex of F ([3, V 1.2.3]). Since F is parasitic, we have $F_{\mathfrak{D}(n)}=0$ for any n. Now we have $(R^q\epsilon_*(F))_T=0$. □

Proof (of Theorem 2.11.3). Assume that we are given the data (2.4.0.1) and (2.4.0.2) for $(X,D\cup Z)$. Let $b^{(k)}_\bullet:\mathcal{D}^{(k)}_\bullet\longrightarrow\mathcal{X}_\bullet$ be the natural morphism. Let $\pi^{\log}_{(X,D\cup Z)/S\text{crys}}$ be the morphism of topoi defined in (2.4.7.4). Let $\pi^{\log}_{(X,Z)/S\text{crys}}$ be the morphism of topoi defined in (2.4.7.4) for the case $D=\phi$. Let $\mathcal{F}_\bullet$ be the crystal on $(X_\bullet,D_\bullet\cup Z_\bullet)/S$ corresponding to the integrable log connection $\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\mathcal{O}_{\mathcal{X}_\bullet}(-\mathcal{D}_\bullet)\longrightarrow\mathcal{O}_{\mathfrak{D}_\bullet}\otimes_{\mathcal{O}_{\mathcal{X}_\bullet}}\mathcal{O}_{\mathcal{X}_\bullet}(-\mathcal{D}_\bullet)\Omega^1_{\mathcal{X}_\bullet/S}(\log(\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet))$, where $\mathfrak{D}_\bullet$ denotes the log PD-envelope of $(X_\bullet,D_\bullet\cup Z_\bullet)$ in $(\mathcal{X}_\bullet,\mathcal{D}_\bullet\cup\mathcal{Z}_\bullet)$. Then we have

$$\begin{aligned}
&Q^*_{(X,Z)/S}R\epsilon_*(\mathcal{I}^D_{(X,D\cup Z)/S})\\
&\overset{=}{\longrightarrow}Q^*_{(X,Z)/S}R\epsilon_*R\pi^{\log}_{(X,D\cup Z)/S\text{crys}*}\pi^{\log,-1}_{(X,D\cup Z)/S\text{crys}}(\mathcal{I}^D_{(X,D\cup Z)/S})\\
&\overset{=}{\longrightarrow}R\epsilon_{\text{Rcrys}*}R\pi^{\log}_{(X,D\cup Z)/S\text{Rcrys}*}Q^*_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}\pi^{\log,-1}_{(X,D\cup Z)/S\text{crys}}(\mathcal{I}^D_{(X,D\cup Z)/S})\\
&\overset{=}{\longleftarrow}R\epsilon_{\text{Rcrys}*}R\pi^{\log}_{(X,D\cup Z)/S\text{Rcrys}*}Q^*_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}(\mathcal{F}_\bullet)\\
&\overset{=}{\longleftarrow}R\pi^{\log}_{(X,Z)/S\text{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}R\epsilon_{\bullet*}(\mathcal{F}_\bullet)\\
&\overset{=}{\longrightarrow}R\pi^{\log}_{(X,Z)/S\text{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}R\epsilon_{\bullet*}L_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}(\Omega_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet-\mathcal{D}_\bullet)))\\
&\overset{=}{\longleftarrow}R\pi^{\log}_{(X,Z)/S\text{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet-\mathcal{D}_\bullet))).
\end{aligned}$$

By the same argument as that in [27, (4.2.2) (a), (c)], the following sequence

(2.11.5.1)
$$0 \longrightarrow \Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet - \mathcal{D}_\bullet)) \longrightarrow \Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)$$
$$\xrightarrow{\iota^{(0)*}_{\bullet,\mathrm{zar}}} b^{(1)}_{\bullet *}(\Omega^{\bullet}_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) \xrightarrow{\iota^{(1)*}_{\bullet,\mathrm{zar}}} \cdots$$

is exact. Here we define $\iota^{(k)*}_{\bullet,\mathrm{zar}}$ similarly as for $\iota^{(k)\log *}_{\mathrm{crys}}$. Hence $\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet - \mathcal{D}_\bullet))$ is quasi-isomorphic to the single complex of the following double complex

(2.11.5.2)
$$\begin{array}{ccc}
\cdots & \longrightarrow & \cdots \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow \\
\Omega^{2}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S) & \xrightarrow{\iota^{(0)*}_{\bullet,\mathrm{zar}}} & b^{(1)}_{\bullet *}(\Omega^{2}_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow \\
\Omega^{1}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S) & \xrightarrow{\iota^{(0)*}_{\bullet,\mathrm{zar}}} & b^{(1)}_{\bullet *}(\Omega^{1}_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) \\
{\scriptstyle d}\uparrow & & {\scriptstyle -d}\uparrow \\
\mathcal{O}_{\mathcal{X}_\bullet} \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S) & \xrightarrow{\iota^{(0)*}_{\bullet,\mathrm{zar}}} & b^{(1)}_{\bullet *}(\mathcal{O}_{\mathcal{D}^{(1)}_\bullet} \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S))
\end{array}$$
$$\begin{array}{ccccc}
\longrightarrow & & \cdots & & \longrightarrow \cdots \\
& & {\scriptstyle d}\uparrow & & \\
\xrightarrow{\iota^{(1)*}_{\bullet,\mathrm{zar}}} & & b^{(2)}_{\bullet *}(\Omega^{2}_{\mathcal{D}^{(2)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(2)}_\bullet}) \otimes_{\mathbb{Z}} \varpi^{(2)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) & & \xrightarrow{\iota^{(2)*}_{\bullet,\mathrm{zar}}} \cdots \\
& & {\scriptstyle d}\uparrow & & \\
\xrightarrow{\iota^{(1)*}_{\bullet,\mathrm{zar}}} & & b^{(2)}_{\bullet *}(\Omega^{1}_{\mathcal{D}^{(2)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(2)}_\bullet}) \otimes_{\mathbb{Z}} \varpi^{(2)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) & & \xrightarrow{\iota^{(2)*}_{\bullet,\mathrm{zar}}} \cdots \\
& & {\scriptstyle d}\uparrow & & \\
\xrightarrow{\iota^{(1)*}_{\bullet,\mathrm{zar}}} & & b^{(2)}_{\bullet *}(\mathcal{O}_{\mathcal{D}^{(2)}_\bullet} \otimes_{\mathbb{Z}} \varpi^{(2)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) & & \xrightarrow{\iota^{(2)*}_{\bullet,\mathrm{zar}}} \cdots .
\end{array}$$

We claim that the following sequence

(2.11.5.3)
$$0 \longrightarrow Q^{*}_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet - \mathcal{D}_\bullet))) \longrightarrow$$
$$Q^{*}_{(X_\bullet,Z_\bullet)/S} L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet)$$
$$\otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) \xrightarrow{Q^{*}_{(X_\bullet,Z_\bullet)/S}(\iota^{(0)*}_{\bullet,\mathrm{zar}})}$$

$$Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(b^{(1)}_{\bullet *}(\Omega^\bullet_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S))) \xrightarrow{Q^*_{(X_\bullet,Z_\bullet)/S}(\iota^{(1)*}_{\bullet,\mathrm{zar}})} \cdots$$

is exact. Indeed, the question is local and we have only to prove that the sequence (2.11.5.3) for $\bullet = i$ is exact for a fixed $i \in I$. As in (2.2.17), we have only to prove that the following sequence

(2.11.5.4)

$$\begin{aligned}0 &\longrightarrow \mathcal{O}_{\mathfrak{D}_i} \otimes_{\mathcal{O}_{\mathcal{X}_i}} \Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{Z}_i - \mathcal{D}_i)) \\ &\longrightarrow \mathcal{O}_{\mathfrak{D}_i} \otimes_{\mathcal{O}_{\mathcal{X}_i}} \Omega^\bullet_{\mathcal{X}_i/S}(\log \mathcal{Z}_i) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_i/S) \\ &\xrightarrow{\iota^{(0)*}_{i,\mathrm{zar}}} \mathcal{O}_{\mathfrak{D}_i} \otimes_{\mathcal{O}_{\mathcal{X}_i}} b^{(1)}_{i*}(\Omega^\bullet_{\mathcal{D}^{(1)}_i/S}(\log \mathcal{Z}_i|_{\mathcal{D}^{(1)}_i}) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_i/S)) \xrightarrow{\iota^{(1)*}_{i,\mathrm{zar}}} \cdots\end{aligned}$$

is exact. The following argument is the same as that in the proof of (2.2.17) (1). We may have cartesian diagrams (2.1.13.1) and (2.1.13.2) for SNCD $\mathcal{D}_i \cup \mathcal{Z}_i$ on $\mathcal{X}_i$; we assume that $\mathcal{D}_i$ (resp. $\mathcal{Z}_i$) is defined by an equation $x_1 \cdots x_t = 0$ (resp. $x_{t+1} \cdots x_s = 0$). Set $\mathcal{J}_i := (x_{d+1}, \ldots, x_{d'})\mathcal{O}_{\mathcal{X}_i}$. We may assume that there exists a positive integer N such that $\mathcal{J}_i^N \mathcal{O}_{\mathfrak{D}_i} = 0$. Set $\mathcal{X}'_i := \underline{\mathrm{Spec}}_{\mathcal{X}_i}(\mathcal{O}_{\mathcal{X}_i}/\mathcal{J}_i)$ and $\mathcal{X}'' := \underline{\mathrm{Spec}}_S(\mathcal{O}_S[x_{d+1}, \ldots, x_{d'}])$. Let $\mathcal{D}'_i$ (resp. $\mathcal{Z}'_i$) be the closed subscheme of $\mathcal{X}'_i$ defined by an equation $x_1 \cdots x_t = 0$ (resp. $x_{t+1} \cdots x_s = 0$). As in [11, 3.32 Proposition], we may assume that there exists a morphism

$$\mathcal{O}_{\mathcal{X}'_i}[x_{d+1}, \ldots, x_{d'}] \longrightarrow \mathcal{O}_{\mathcal{X}_i}/\mathcal{J}_i^N$$

such that the induced morphism $\mathcal{O}_{\mathcal{X}'_i}[x_{d+1}, \ldots, x_{d'}]/\mathcal{J}_{0i}^N \longrightarrow \mathcal{O}_{\mathcal{X}_i}/\mathcal{J}_i^N$ is an isomorphism, where $\mathcal{J}_{0i} := (x_{d+1}, \ldots, x_{d'})$. By [11, 3.32 Proposition], $\mathcal{O}_{\mathfrak{D}_i}$ is locally isomorphic to the PD-polynomial algebra $\mathcal{O}_{\mathcal{X}'_i}\langle x_{d+1}, \ldots, x_{d'}\rangle$. Let $b'^{(k)}_i$ ($k \in \mathbb{Z}_{>0}$) and $\iota'^{(k)*}_{i,\mathrm{zar}}$ ($k \in \mathbb{Z}_{\geq 0}$) be analogous morphisms to $b^{(k)}_i$ and $\iota^{(k)*}_{i,\mathrm{zar}}$, respectively, for $\mathcal{X}'_i$, $\mathcal{D}'_i$ and $\mathcal{Z}'_i$. Then we have an exact sequence

$$(2.11.5.5) \quad 0 \longrightarrow \Omega^\bullet_{\mathcal{X}'_i/S}(\log(\mathcal{Z}'_i - \mathcal{D}'_i)) \longrightarrow \Omega^\bullet_{\mathcal{X}'_i/S}(\log \mathcal{Z}'_i) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}'_i/S)$$

$$\xrightarrow{\iota'^{(0)*}_{i,\mathrm{zar}}} b'^{(1)}_{i*}(\Omega^\bullet_{\mathcal{D}'^{(1)}_i/S}(\log \mathcal{Z}'_i|_{\mathcal{D}'^{(1)}_i}) \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}'_i/S)) \xrightarrow{\iota'^{(1)*}_{i,\mathrm{zar}}} \cdots$$

Since $\mathcal{O}_S\langle x_{d+1}, \ldots, x_{d'}\rangle \otimes_{\mathcal{O}_{\mathcal{X}''}} \Omega^q_{\mathcal{X}''/S}$ ($q \in \mathbb{N}$) is a free $\mathcal{O}_S$-module, applying the tensor product $\otimes_{\mathcal{O}_S}\mathcal{O}_S\langle x_{d+1}, \ldots, x_{d'}\rangle \otimes_{\mathcal{O}_{\mathcal{X}''}} \Omega^q_{\mathcal{X}''/S}$ ($q \in \mathbb{N}$) to the exact sequence (2.11.5.5) preserves the exactness. Because

$$\mathcal{O}_{\mathfrak{D}_i} \otimes_{\mathcal{O}_{\mathcal{X}_i}} \Omega^\bullet_{\mathcal{X}_i/S}(\log(\mathcal{Z}_i - \mathcal{D}_i)) \simeq \Omega^\bullet_{\mathcal{X}'_i/S}(\log(\mathcal{Z}'_i - \mathcal{D}'_i)) \otimes_{\mathcal{O}_S} \mathcal{O}_S\langle x_{d+1}, \ldots, x_{d'}\rangle \otimes_{\mathcal{O}_{\mathcal{X}''}} \Omega^\bullet_{\mathcal{X}''/S}$$

and because the similar formulas for $\mathcal{O}_{\mathfrak{D}_i} \otimes_{\mathcal{X}_i} b^{(k)}_{i*}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_i/S}(\log \mathcal{Z}_i|_{\mathcal{D}^{(k)}_i}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}_i/S))$ ($k \in \mathbb{N}$) hold, we have the exactness of (2.11.5.4).

By (2.2.12) and (2.11.5.3), we have the following quasi-isomorphism

$$
\begin{aligned}
(2.11.5.6)\quad & Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet - \mathcal{D}_\bullet)))\\
&\overset{\sim}{\longrightarrow}\{Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S))\\
&\longrightarrow (Q^*_{(X_\bullet,Z_\bullet)/S}a^{(1)\log}_{\bullet\mathrm{crys}*}L_{(D^{(1)}_\bullet,Z_\bullet|_{D^{(1)}_\bullet})/S}(\Omega^{\bullet}_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}))\\
&\qquad\otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)), -d) \longrightarrow \cdots\}.
\end{aligned}
$$

Applying the direct image $R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}$ to (2.11.5.6), we have

$$
\begin{aligned}
(2.11.5.7)\quad & R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet - \mathcal{D}_\bullet))) \overset{\sim}{\longrightarrow}\\
&\{R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) \longrightarrow\\
&(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}a^{(1)\log}_{\bullet\mathrm{crys}*}L_{(D^{(1)}_\bullet,Z_\bullet|_{D^{(1)}})/S}(\Omega^{\bullet}_{\mathcal{D}^{(1)}_\bullet/S}(\log(\mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}))\\
&\qquad\otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S), -d) \longrightarrow \cdots\}.
\end{aligned}
$$

(See (2.11.8) below.) By (1.6.4.1) and (2.2.20.1), the isomorphism (2.11.5.7) is nothing but an isomorphism (2.11.3.2).

Now we show that the isomorphism (2.11.3.2) is independent of the data (2.4.0.1) and (2.4.0.2).

Let the notations be as in the proof of (2.5.3). Let

$$
\begin{aligned}
R\eta^{\log}_{\mathrm{Rcrys}*}\colon & \mathrm{D}^+\mathrm{F}(Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\mathcal{O}_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}))\\
&\longrightarrow \mathrm{D}^+\mathrm{F}((Q^*_{(X_\bullet,Z_\bullet)/S}(\mathcal{O}_{(X_\bullet,Z_\bullet)/S}))_{\bullet\in I})
\end{aligned}
$$

be a morphism of filtered derived categories in §2.5. Then we have the following commutative diagram by the cohomological descent:

$$
\begin{array}{cc}
R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet - \mathcal{D}_\bullet))) & \overset{\sim}{\longrightarrow}\\
\downarrow & \\
R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}R\eta^{\log}_{\mathrm{Rcrys}*}Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{\bullet\bullet}/S}(\log(\mathcal{Z}_{\bullet\bullet} - \mathcal{D}_{\bullet\bullet}))) & \overset{\sim}{\longrightarrow}
\end{array}
$$

$$
\begin{array}{c}
\{R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}Q^*_{(X_\bullet,Z_\bullet)/S}L_{(X_\bullet,Z_\bullet)/S}(\Omega^{\bullet}_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)) \longrightarrow\\
\downarrow\\
\{R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}R\eta^{\log}_{\mathrm{Rcrys}*}Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}L_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{\bullet\bullet}/S}(\log \mathcal{Z}_{\bullet\bullet}) \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_{\bullet\bullet}/S)) \longrightarrow
\end{array}
$$

$$\begin{array}{c}
(R(\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}}Q^*_{(X_\bullet,Z_\bullet)/S}a^{(1)\log}_{\bullet\mathrm{crys}})_*L_{(D^{(1)}_\bullet,Z_\bullet|_{D^{(1)}})/S}(\Omega^\bullet_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}) \\
\otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)), -d) \longrightarrow \cdots\} \\
\downarrow \\
(R\pi^{\log}_{(X,Z)/S\mathrm{Rcrys}*}R\eta^{\log}_{\mathrm{Rcrys}*}Q^*_{(X_{\bullet\bullet},Z_{\bullet\bullet})/S}a^{(1)\log}_{\bullet\bullet\mathrm{crys}*}L_{(D^{(1)}_{\bullet\bullet},Z_{\bullet\bullet}|_{D^{(1)}})/S}(\Omega^\bullet_{\mathcal{D}^{(1)}_{\bullet\bullet}/S}(\log \mathcal{Z}_{\bullet\bullet}|_{\mathcal{D}^{(1)}_{\bullet\bullet}}) \\
\otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_{\bullet\bullet}/S)), -d) \longrightarrow \cdots\}.
\end{array}$$

Hence the isomorphism (2.11.3.2) is independent of the data (2.4.0.1) and (2.4.0.2). □

Remark 2.11.6. Let the notation be as in the proof of (2.11.3) and let $L^\bullet_{(X_\bullet,Z_\bullet)/S}$ be the complex

$$\{L_{(X_\bullet,Z_\bullet)/S}(\Omega^\bullet_{\mathcal{X}_\bullet/S}(\log \mathcal{Z}_\bullet)\otimes_{\mathbb{Z}}\varpi^{(0)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S))$$

$$(a^{(1)\log}_{\bullet\mathrm{crys}*}L_{(D^{(1)}_\bullet,Z_\bullet|_{D^{(1)}_\bullet})/S}(\Omega^\bullet_{\mathcal{D}^{(1)}_\bullet/S}(\log \mathcal{Z}_\bullet|_{\mathcal{D}^{(1)}_\bullet}))\otimes_{\mathbb{Z}}\varpi^{(1)}_{\mathrm{zar}}(\mathcal{D}_\bullet/S)), -d) \longrightarrow \cdots\}.$$

Then, by the proof of (2.11.3), we see that the isomorphism (2.11.3.2) is obtained by applying $Q^*_{(X,Z)/S}$ to the following diagram:

$$(2.11.6.1) \qquad \begin{aligned}
& R\epsilon_*(\mathcal{I}^D_{(X,D\cup Z)/S}) \\
\xrightarrow{=}\ & R\epsilon_*R\pi^{\log}_{(X,D\cup Z)/S\mathrm{crys}*}\pi^{\log,-1}_{(X,D\cup Z)/S\mathrm{crys}}(\mathcal{I}^D_{(X,D\cup Z)/S}) \\
\longleftarrow\ & R\epsilon_*R\pi^{\log}_{(X,D\cup Z)/S\mathrm{crys}*}(\mathcal{F}_\bullet) \\
\xleftarrow{=}\ & R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}R\epsilon_{\bullet*}(\mathcal{F}_\bullet) \\
\xrightarrow{=}\ & R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}R\epsilon_{\bullet*}L_{(X_\bullet,D_\bullet\cup Z_\bullet)/S}(\Omega_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet-\mathcal{D}_\bullet))) \\
\xleftarrow{=}\ & R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}L_{(X_\bullet,Z_\bullet)/S}(\Omega_{\mathcal{X}_\bullet/S}(\log(\mathcal{Z}_\bullet-\mathcal{D}_\bullet))) \\
\longrightarrow\ & R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}L^\bullet_{(X_\bullet,Z_\bullet)/S} \\
\xleftarrow{=}\ & R\pi^{\log}_{(X,Z)/S\mathrm{crys}*}\pi^{\log,-1}_{(X,Z)/S\mathrm{crys}}E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}) \\
\xleftarrow{=}\ & E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}).
\end{aligned}$$

Note that the arrows in the above diagram without = are not necessarily isomorphisms: they become isomorphic only after we apply $Q^*_{(X,Z)/S}$. Note also that they become isomorphic if we apply $Ru_{(X,Z)/S*}$ or $Rf_{(X,Z)/S*}$ because $Ru_{(X,Z)/S*} = R\overline{u}_{(X,Z)/S}\circ Q^*_{(X,Z)/S}$ and $Rf_{(X,Z)/S*} = R\overline{f}_{(X,Z)/S}\circ Q^*_{(X,Z)/S}$.

Let $P^D_{\mathrm{c}} := \{P^{D,k}_{\mathrm{c}}\}_{k\in\mathbb{Z}}$ be the stupid filtration on $E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S})$. Then, by (2.11.3), we have a filtered complex $(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}), P^D_{\mathrm{c}}) \in \mathrm{D}^+\mathrm{F}(\mathcal{O}_{(X,Z)/S})$.

Definition 2.11.7. We call $(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}})$ the *preweight-filtered vanishing cycle crystalline complex with compact support* of $\mathcal{O}_{(X,D\cup Z)/S}$ (or $(X,D\cup Z)/S$) with respect to D. Set

$$(C^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}}) := Q^*_{(X,Z)/S}(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}}).$$

We call $(C^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}})$ the *preweight-filtered crystalline complex with compact support* of $\mathcal{O}_{(X,D\cup Z)/S}$ (or $(X,D\cup Z)/S$) with respect to D. Set

$$(E^{\log,Z}_{\mathrm{zar},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}}) := Ru_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}}).$$

We call $(E^{\log,Z}_{\mathrm{zar},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}})$ the *preweight-filtered vanishing cycle zariskian complex with compact support* of $\mathcal{O}_{(X,D\cup Z)/S}$ (or $(X,D\cup Z)/S$) with respect to D.

By the definition of $(E^{\log,Z}_{\mathrm{zar},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}})$, there exists the following canonical isomorphism in $\mathrm{D}^+(f^{-1}(\mathcal{O}_S))$:

(2.11.7.1)

$$\begin{aligned}
&E^{\log,Z}_{\mathrm{zar},c}(\mathcal{O}_{(X,D\cup Z)/S})\\
&\overset{\sim}{\longrightarrow}\{Ru_{(X,Z)/S*}(\mathcal{O}_{(X,Z)/S}\otimes_{\mathbb{Z}}\varpi^{(0)\log}_{\mathrm{crys}}(D/S;Z))\longrightarrow\\
&\qquad a^{(1)}_{\mathrm{zar}*}(Ru_{(D^{(1)},Z|_{D^{(1)}})/S*}(\mathcal{O}_{(D^{(1)},Z|_{D^{(1)}})/S}\otimes_{\mathbb{Z}}\varpi^{(1)\log}_{\mathrm{crys}}(D/S;Z)),-d)\\
&\qquad\qquad\longrightarrow\cdots\}.
\end{aligned}$$

Remark-Definition 2.11.8. Because the notation for the right hand side of (2.11.7.1) is only suggestive, we have to give the strict definition of it. Let $I^{\bullet\bullet}$ be a double complex of $\mathcal{O}_{(X,Z)/S}$-modules such that, for each nonnegative integer k, $I^{k\bullet}$ is a $u_{(X,Z)/S*}$-acyclic resolution of $(a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)),(-1)^k d)$. Then the right hand side of (2.11.7.1) is, by definition, an object in $\mathrm{D}^+(f^{-1}(\mathcal{O}_S))$ which is given by the single complex of $u_{(X,Z)/S*}(I^{\bullet\bullet})$. Let $P^D_{\mathrm{c}} := \{P^{D,k}_{\mathrm{c}}\}_{k\in\mathbb{Z}}$ be the stupid filtration with respect to the first degree of $u_{(X,Z)/S*}(I^{\bullet\bullet})$. Then $(E^{\log,Z}_{\mathrm{zar},c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\mathrm{c}}) = (u_{(X,Z)/S*}(I^{\bullet\bullet}),P^D_{\mathrm{c}})$ in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$.

Corollary 2.11.9. $E^{\log,Z}_{\mathrm{zar},c}(\mathcal{O}_{(X,D\cup Z)/S}) = Ru_{(X,D\cup Z)/S*}(\mathcal{I}^D_{(X,D\cup Z)/S})$.

Proof. We have only to apply the direct image $Ru_{(X,Z)/S*}$ to (2.11.3.2) and to use the commutative diagram (1.6.3.1) for the case of the trivial filtration. □

By applying Rf_* to both hands of (2.11.7.1) (cf. (2.11.8)), we have a canonical isomorphism

(2.11.9.1)

$$Rf_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S}) \xrightarrow{\sim} \{Rf_{(X,Z)/S*}(\mathcal{O}_{(X,Z)/S} \otimes_{\mathbb{Z}} \varpi^{(0)}_{\mathrm{crys}}(D/S;Z))$$

$$\longrightarrow (Rf_{(D^{(1)},Z|_{D^{(1)}})/S*}(\mathcal{O}_{(D^{(1)},Z|_{D^{(1)}})/S} \otimes_{\mathbb{Z}} \varpi^{(1)}_{\mathrm{crys}}(D/S;Z)),-d) \longrightarrow \cdots\}.$$

Next we prove the base change theorem of $(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}), P^D_{\mathrm{c}})$.

Proposition 2.11.10. *Let the notations and the assumptions be as in* (2.10.2) (1). *Then* $R^h f_{(X,Z)/S*}(P^{D,k}_{\mathrm{c}} E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}))$ $(h,k \in \mathbb{Z})$ *is a quasi-coherent* $\mathcal{O}_S$*-module and* $Rf_{(X,Z)/S*}(P^{D,k}_{\mathrm{c}} E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}))$ $(k \in \mathbb{Z})$ *has finite tor-dimension.*

Proof. This immediately follows from the spectral sequence (2.10.2.1) and [11, 7.6 Theorem], [11, 7.13 Corollary]. □

Theorem 2.11.11 (Base change theorem). *Let the notations and the assumptions be as in* (2.10.3). *Then there exists the following canonical isomorphism*

(2.11.11.1)
$$Lu^* Rf_{(X,Z)/S*}(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}), P^D_{\mathrm{c}}) \xrightarrow{\sim} Rf_{(X',Z')/S'*}(E^{\log,Z'}_{\mathrm{crys},c}(\mathcal{O}_{(X',D'\cup Z')/S'}), P^{D'}_{\mathrm{c}}).$$

Proof. Let $I^{\bullet\bullet}$ be a double complex of $\mathcal{O}_{(X,Z)/S}$-modules such that, for each $k \in \mathbb{N}$, $I^{k\bullet}$ is an injective resolution of $(a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)), (-1)^k d)$. Then we have a double complex $((fu_{(X,Z)/S})_*(I^{0\bullet}) \longrightarrow (fu_{(X,Z)/S})_*(I^{1\bullet}) \longrightarrow \cdots)$. This double complex is a representative of $Rf_{(X,D\cup Z)/S*}(E^{\log,Z}_{\mathrm{crys},c}(\mathcal{O}_{(X,D\cup Z)/S}))$. For a nonnegative integer r, let $\tau_r(fu_{(X,Z)/S})_*(I^{k\bullet})$ be the canonical filtration of the complex $(fu_{(X,Z)/S})_*(I^{k\bullet})$. Because $Rf_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S})$ is "bounded" by (2.10.2.1) and [11, 7.6 Theorem], and because $D^{(k)} = \emptyset$ if $k \gg 0$ (since X is quasi-compact), if r is large enough, the natural inclusions $\tau_r(fu_{(X,Z)/S})_*(I^{k\bullet}) \xrightarrow{\subset} (fu_{(X,Z)/S})_*(I^{k\bullet})$ are quasi-isomorphisms for all k. Hence the natural morphism

$$s(\tau_r(fu_{(X,Z)/S})_*(I^{\bullet\bullet})) \longrightarrow s((fu_{(X,Z)/S})_*(I^{\bullet\bullet}))$$

is a quasi-isomorphism. Let

$$d' \colon (fu_{(X,Z)/S})_*(I^{\bullet l}) \longrightarrow (fu_{(X,Z)/S})_*(I^{\bullet+1,l})$$

and

$$d'' \colon (fu_{(X,Z)/S})_*(I^{k\bullet}) \longrightarrow (fu_{(X,Z)/S})_*(I^{k,\bullet+1})$$

be the boundary morphisms. Using the functor L^0 in [11, §7], we have a flat resolution $Q^{\bullet k*}$ of $\tau_r(fu_{(X,Z)/S})_*(I^{k*})$ for a fixed $r \gg 0$. The morphisms d' and d'' induce morphisms $d'_Q \colon Q^{j\bullet l} \longrightarrow Q^{j,\bullet+1,l}$ and $d''_Q \colon Q^{jk\bullet} \longrightarrow Q^{jk,\bullet+1}$, respectively. We fix the boundary morphisms as follows: $(-1)^j d'_Q \colon Q^{j\bullet l} \longrightarrow$

$Q^{j,\bullet+1,l}$ and $(-1)^j d''_Q \colon Q^{jk\bullet} \longrightarrow Q^{jk,\bullet+1}$. We also have a natural boundary morphism $Q^{\bullet kl} \longrightarrow Q^{\bullet+1,k,l}$. By these three boundary morphisms, we have a triple complex $Q^{\bullet\bullet\bullet}$. Then $Lu^*P^{D,k}_{\rm c}Rf_{(X,D\cup Z)/S*}(E^{\log,Z}_{{\rm crys,c}}(\mathcal{O}_{(X,D\cup Z)/S})) = (u^*Q^{\bullet k\bullet} \longrightarrow u^*Q^{\bullet,k+1,\bullet} \longrightarrow \cdots)\{-k\}$. By the base change theorem of Kato ([54, (6.10)]), this complex is isomorphic to

$$\{(Rf_{(D'^{(k)},Z'|_{D'^{(k)}})/S'*}(\mathcal{O}_{(D'^{(k)},Z'|_{D'^{(k)}})/S'} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\rm crys}(D'/S';Z')))\{-k\},(-1)^k d) \longrightarrow$$

$$\cdots\} = Rf_{(X',Z')/S'*}P^{D',k}_{\rm c}(E^{\log,Z'}_{\rm crys,c}(\mathcal{O}_{(X',D'\cup Z')/S'})).$$

□

Proposition 2.11.12. *Let the notations and the assumptions be as those in* (2.10.2) (1). *Assume moreover that* $f \colon X \longrightarrow S_0$ *is proper. Then*

$$Rf_{(X,Z)/S*}(E^{\log,Z}_{\rm crys,c}(\mathcal{O}_{(X,D\cup Z)/S}), P^D_{\rm c})$$

is a filteredly strictly perfect complex.

Proof. We use the criterion (2.10.10); we have checked the condition as to the tor-dimension in (2.11.10) and we obtain the finiteness from the spectral sequence (2.10.2.1) and [11, 7.16 Theorem]. □

We prove the boundedness property of the log crystalline cohomology for the coefficient $\mathcal{I}_{(X,D\cup Z)/S}$:

Proposition 2.11.13. *Let* $f : (X, D\cup Z) \longrightarrow Y$ *be as in* (2.10.2) (2) *and let* $\epsilon : (X, D\cup Z) \longrightarrow (X,Z)$ *be the forgetting log morphism along* D. *Then* $Rf^{\log}_{{\rm crys}*}(\mathcal{I}^D_{(X,D\cup Z)/S})$ *and* $R\epsilon_*(\mathcal{I}^D_{(X,D\cup Z)/S})$ *are bounded.*

Proof. Let us first prove that $Rf^{\log}_{{\rm crys}*}(\mathcal{I}^D_{(X,D\cup Z)/S})$ is bounded. Let the notations be as that in the proof of (2.3.11). By the same argument as [3, V Théorème 3.2.4], we are reduced to proving the following claim: there exists a positive integer r such that, for any $(U,T,\delta) \in (Y/S)^{\log}_{\rm crys}$, we have $R^i f_{X_U/T*}(\mathcal{I}^D_{(X,D\cup Z)/S}) = 0$ for $i > r$. Again by the same argument as that in the proof of [3, V Théorème 3.2.4, Proposition 3.2.5], we are reduced to showing the above claim in the case where X and Y are sufficiently small affine schemes. Hence we may assume that the log structure $M(D\cup Z)$ associated to $D\cup Z$ admits a chart of the form $\mathbb{N}^b \longrightarrow M(D\cup Z)$. If we take a surjection $\varphi_1 : \mathcal{O}_Y[\mathbb{N}^a] \longrightarrow \mathcal{O}_X$ and if we set $\varphi_2 := {\rm id} \colon \mathbb{N}^b \xrightarrow{=} \mathbb{N}^b$, we can construct a commutative diagram

(2.11.13.1)
$$\begin{array}{ccc} X_U & \xrightarrow{\psi} & \widetilde{T} \\ \downarrow & & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\iota} & T \end{array}$$

in the same way as in the proof of (2.3.11) such that ψ is an exact closed immersion and that g is log smooth. Then we can form a crystal $\mathcal{F}$ on $(X_U/T)^{\log}_{\rm crys}$ satisfying the equality $Q^*_{X_U/S}(\mathcal{I}^D_{(X,D\cup Z)/S}) = Q^*_{X_U/S}(\mathcal{F})$. Then we have $R^if_{X_U/T*}(\mathcal{I}^D_{(X,D\cup Z)/S}) = R^if_{X_U/T*}(\mathcal{F})$ and it vanishes for $i > a+b$ by (2.3.11). Now we have proved that $Rf^{\log}_{{\rm crys}*}(\mathcal{I}^D_{(X,D\cup Z)/S})$ is bounded.

Let us prove that $R\epsilon_*(\mathcal{I}^D_{(X,D\cup Z)/S})$ is bounded. It suffices to prove that there exists a positive integer r such that, for any $(U,T,\delta) \in ((X,Z)/S)^{\log}_{\rm crys}$, $R^if_{X_U/T*}(\mathcal{I}^D_{(X,D\cup Z)/S}) = 0$ (where $X_U := U \times_{(X,Z)} (X, D\cup Z) = (U, (D\cup Z)|_U))$ for $i > r$. We may also assume that X is sufficiently small. Hence we may assume that the log structure $M(D)$ associated to D admits a chart of the form $\alpha : \mathbb{N}^b \longrightarrow M(D)$. Let us denote the log structure on X associated to $D\cup Z$ by M_X. If we put $\varphi_1 := {\rm id}_{\mathcal{O}_X}$ and $\varphi_2 := {\rm id}_{\mathbb{N}^b}$, we can construct the commutative diagram (2.11.13.1) in the same way as (2.3.11) and then we can form a crystal $\mathcal{F}$ on $(X_U/T)^{\log}_{\rm crys}$ which satisfies the equality $Q^*_{X_U/S}(\mathcal{I}^D_{(X,D\cup Z)/S}) = Q^*_{X_U/S}(\mathcal{F})$. Then we have $R^if_{X_U/T*}(\mathcal{I}^D_{(X,D\cup Z)/S}) = R^if_{X_U/T*}(\mathcal{F})$ and it vanishes for $i > b$ by (2.3.11). Hence we have also proved that $R\epsilon_*(\mathcal{I}^D_{(X,D\cup Z)/S})$ is bounded. □

Using (2.11.11) and (2.11.13), we can prove the following:

Proposition 2.11.14. *Let the notations and the assumptions as in* (2.10.6).
(1) *The natural morphism*

(2.11.14.1)
$$Lh^*_{\rm crys}Rf^{\log}_{(X,Z){\rm crys}*}(E^{\log,Z}_{{\rm crys},c}(\mathcal{O}_{(X,D\cup Z)/S}), P^D_{\rm c}) \longrightarrow$$
$$Rf'^{\log}_{(X',Z'){\rm crys}*}(E^{\log,Z'}_{{\rm crys},c}(\mathcal{O}_{(X',D'\cup Z')/S}), P^{D'}_{\rm c})$$

is an isomorphism.
(2) *There exists a natural isomorphism*

(2.11.14.2)
$$Lh^*_{\rm crys}Rf^{\log}_{{\rm crys}*}(\mathcal{I}^D_{(X,D\cup Z)/S}) \longrightarrow Rf'^{\log}_{{\rm crys}*}(\mathcal{I}^{D'}_{(X',D'\cup Z')/S'})$$

which is compatible with the isomorphism (2.11.14.1).

Proof. (1) follows from (2.11.12) in the same way as [3, V], [11, §7] (see also §17).

Let us prove (2). One can construct the morphism (2.11.14.2) in usual way ([3, V Théorème 3.5.1]), using the boundedness of $Rf^{\log}_{{\rm crys}*}(\mathcal{I}^D_{(X,D\cup Z)/S})$ which has been proved in (2.11.13). We can take data $(X_\bullet, D_\bullet \cup Z_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)$ as (2.4.0.1), (2.4.0.2) for $(X, D\cup Z)$. If we put $(X'_\bullet, D'_\bullet \cup Z'_\bullet) := (X_\bullet, D_\bullet \cup Z_\bullet)\times_S S'$ and $(\mathcal{X}'_\bullet, \mathcal{D}'_\bullet \cup \mathcal{Z}'_\bullet) := (\mathcal{X}_\bullet, \mathcal{D}_\bullet \cup \mathcal{Z}_\bullet)\times_S S'$, we obtain the data $(X'_\bullet, D'_\bullet \cup Z'_\bullet) \overset{\subset}{\longrightarrow} (\mathcal{X}'_\bullet, \mathcal{D}'_\bullet \cup \mathcal{Z}'_\bullet)$ as (2.4.0.1), (2.4.0.2) for $(X', D'\cup Z')$. Then we see from the diagram (2.11.6.1) that there exists a diagram of base change morphisms

$$\begin{array}{ccc}
Lh^*_{\rm crys}Rf^{\log}_{{\rm crys}*}(\mathcal{I}^D_{(X,D\cup Z)/S}) & \longleftarrow & Lh^*_{\rm crys}Rf^{\log}_{(X_\bullet,D_\bullet\cup Z_\bullet){\rm crys}*}(\mathcal{F}_\bullet) \\
\Big\downarrow & & \Big\downarrow \\
Rf'^{\log}_{{\rm crys}*}(\mathcal{I}^{D'}_{(X',D'\cup Z')/S'}) & \longleftarrow & Rf'^{\log}_{(X'_\bullet,D'_\bullet\cup Z'_\bullet){\rm crys}*}(\mathcal{F}'_\bullet)
\end{array}$$

$$\begin{array}{c}
\longrightarrow Lh^*_{\rm crys}Rf^{\log}_{(X,Z){\rm crys}*}(E^{\log,Z}_{{\rm crys},c}(\mathcal{O}_{(X,D\cup Z)/S})) \\
\Big\downarrow \\
\longrightarrow Rf'^{\log}_{(X',Z'){\rm crys}*}(E^{\log,Z'}_{{\rm crys},c}(\mathcal{O}_{(X',D'\cup Z')/S})),
\end{array}$$

where $f_{(X_\bullet,D_\bullet\cup Z_\bullet)}$ (resp. $f'_{(X'_\bullet,D'_\bullet\cup Z'_\bullet)}$) denotes the composite morphism of $(X_\bullet,D_\bullet\cup Z_\bullet)\longrightarrow(X,D\cup Z)$ with f (resp. $(X'_\bullet,D'_\bullet\cup Z'_\bullet)\longrightarrow(X',D'\cup Z')$ with f') and $\mathcal{F}'_\bullet$ is the crystal on $(X'_\bullet,D'_\bullet\cup Z'_\bullet)/S'$ defined in the same way as $\mathcal{F}$. To prove (2), it suffices to prove that the horizontal arrows are isomorphisms. We are reduced to showing (as in [3, V 3.5.5]) that, in the situation in (2.10.3), the horizontal arrows in the following diagram of base change morphisms

$$\begin{array}{ccc}
Lu^*Rf_{(X,D\cup Z)/S*}(\mathcal{I}^D_{(X,D\cup Z)/S}) & \longleftarrow & Lu^*Rf_{(X_\bullet,D_\bullet\cup Z_\bullet)/S*}(\mathcal{F}_\bullet) \\
\Big\downarrow & & \Big\downarrow \\
Rf_{(X',D'\cup Z')/S'*}(\mathcal{I}^{D'}_{(X',D'\cup Z')/S'}) & \longleftarrow & Rf_{(X'_\bullet,D'_\bullet\cup Z'_\bullet)/S'*}(\mathcal{F}'_\bullet)
\end{array}$$

$$\begin{array}{c}
\longrightarrow Lu^*Rf_{(X,Z)/S*}(E^{\log,Z}_{{\rm crys},c}(\mathcal{O}_{(X,D\cup Z)/S})) \\
\Big\downarrow \\
\longrightarrow Rf_{(X',Z')/S'*}(E^{\log,Z'}_{{\rm crys},c}(\mathcal{O}_{(X',D'\cup Z')/S'}))
\end{array}$$

are isomorphisms. This follows from (2.11.6) because the arrows in (2.11.6.1) become isomorphic if we apply $Rf_{(X,Z)/S*}$. Hence we have proved (2). □

By using the filtered complex $(E^{\log,Z}_{\rm crys,c}(\mathcal{O}_{(X,D\cup Z)/S}),P^D_{\rm c})$, by (2.11.9) and by the Convention (6), we have the following spectral sequence

$$\begin{aligned}
(2.11.14.3)\quad & E_{1,{\rm c}}^{k,h-k}((X,D\cup Z)/S) \\
&= R^{h-k}f_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\rm crys}(D/S;Z))) \\
&\Longrightarrow R^hf_{(X,D\cup Z)/S*,{\rm c}}(\mathcal{O}_{(X,D\cup Z;Z)/S}).
\end{aligned}$$

Let k be a fixed integer. Set

$$E_{1,{\rm c}}^{k',h-k'}((X,D\cup Z)/S)=$$

$$\begin{cases}R^{h-k'}f_{(D^{(k')},Z|_{D^{(k')}})/S*}(\mathcal{O}_{(D^{(k')},Z|_{D^{(k')}})/S}\otimes_{\mathbb{Z}}\varpi^{(k')\log}_{\rm crys}(D/S;Z)) & (k'\geq k),\\ 0 & (k'<k).\end{cases}$$

We shall also need the following spectral sequence later

$$E_1^{k',h-k'} = E_{1,\mathrm{c}}^{k',h-k'}((X, D \cup Z)/S) \Longrightarrow \tag{2.11.14.4}$$

$$R^{h-k} f_{(X,D\cup Z)/S*}((a^{(k)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)), (-1)^k d) \longrightarrow$$
$$a^{(k+1)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(k+1)},Z|_{D^{(k+1)}})/S} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\mathrm{crys}}(D/S;Z)), (-1)^{k+1} d) \longrightarrow \cdots).$$

Definition 2.11.15. We call the spectral sequence (2.11.14.3) the *preweight spectral sequence of* $(X, D \cup Z)/(S, \mathcal{I}, \gamma)$ *with respect to* D *for the log crystalline cohomology with compact support.* If $Z = \emptyset$, then we call (2.11.14.3) the *preweight spectral sequence of* $(X, D)/(S, \mathcal{I}, \gamma)$ *for the log crystalline cohomology with compact support.*

Let $P_{\mathrm{c}}^{D,\bullet}$ be the filtration on $R^h f_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S})$ induced from the spectral sequence (2.11.14.3). Since $P_{\mathrm{c}}^{D,\bullet}$ is the decreasing filtration, we also consider the following increasing filtration $P^D_{\bullet,\mathrm{c}}$:

(2.11.15.1)
$$P^D_{h-\bullet,\mathrm{c}} R^h f_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S}) = P_{\mathrm{c}}^{D,\bullet} R^h f_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S}).$$

Proposition 2.11.16. *Let the notations be as in* (2.10.3)*. There exists a canonical morphism of spectral sequences*

(2.11.16.1)
$$\begin{aligned}&\{E_{1,\mathrm{c}}^{-k,h+k}((X, D\cup Z)/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \\ &\qquad\Longrightarrow R^h f_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}\} \\ \longrightarrow &\{(E_{1,\mathrm{c}}^{-k,h+k}((X', D'\cup Z')/S') \\ &\qquad\Longrightarrow R^h f_{(X',D'\cup Z')/S'*,\mathrm{c}}(\mathcal{O}_{(X',D'\cup Z';Z')/S'})\}\end{aligned}$$

of $\mathcal{O}_{S'}$*-modules.*

Proof. (2.11.16) immediately follows from the construction of (2.11.14.3). □

Proposition 2.11.17. *The boundary morphism* $d_1^{k,h-k} \colon E_{1,c}^{k,h-k}((X, D\cup Z)$ $/S) \longrightarrow E_{1,c}^{k+1,h-k}((X, D\cup Z)/S)$ *is equal to the morphism induced by* $\iota^{(k)\log *}_{\mathrm{crys}}$.

Proof. Though the proof is the same as that of [68, (5.1)], we give the proof here.

We have the following triangle

$$\longrightarrow Rf_{(D^{(k+1)},Z|_{D^{(k+1)}})/S*}(\mathcal{O}_{(D^{(k+1)},Z|_{D^{(k+1)}})/S})[-(k+1)] \longrightarrow \tag{2.11.17.1}$$
$$P_{\mathrm{c}}^{D,k}/P_{\mathrm{c}}^{D,k+2}((2.11.9.1)) \longrightarrow Rf_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S})[-k] \xrightarrow{+1} .$$

Hence the boundary morphism $d_1^{k,h-k}$ is equal to the boundary morphism

$$Rf_{(D^{(k)},Z|_{D^{(k)}})/S*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/S})[-k] \longrightarrow$$
$$Rf_{(D^{(k+1)},Z|_{D^{(k+1)}})/S*}(\mathcal{O}_{(D^{(k+1)},Z|_{D^{(k+1)}})/S})[-k]$$

by the Convention (4) and (5). By the Convention (3), (4), (6) and by taking the Godement resolution of the complex $a^{(l)\log}_{\mathrm{crys}*}(\mathcal{O}_{(D^{(l)},Z|_{D^{(l)}})/S*}\otimes_{\mathbb{Z}} \varpi^{(l)\log}_{\mathrm{crys}}(D/S;Z))$ $(l=k,k+1)$, we can check that $d_1^{k,h-k}$ is equal to the morphism induced by $\iota^{(k)\log*}_{\mathrm{crys}}$. □

Proposition 2.11.18. *Let u be as in* (2.10.3)*. Let $u_0\colon S'_0 \longrightarrow S_0$ be the induced morphism by u. Let $(Y,E\cup W)$ and $(X,D\cup Z)$ be smooth schemes with relative SNCD's over S'_0 and S_0, respectively. Let*

(2.11.18.1)
$$\begin{array}{ccc} (Y,E\cup W) & \xrightarrow{g} & (X,D\cup Z) \\ \big\downarrow & & \big\downarrow \\ S'_0 & \xrightarrow{u_0} & S_0 \end{array}$$

be a commutative diagram of log schemes such that the morphism g induces morphisms $g^{(k)}\colon (E^{(k)},W|_{E^{(k)}}) \longrightarrow (D^{(k)},Z|_{D^{(k)}})$ of log schemes over $u_0\colon S'_0 \longrightarrow S_0$ for all $k\in\mathbb{N}$. Then the isomorphism in (2.11.11.1) *and the spectral sequence* (2.11.14.3) *are functorial with respect to $g^{\log*}_{\mathrm{crys}}$.*

Proof. The proof is obvious. □

The following is the Künneth formula for the log crystalline cohomology sheaf with compact support $R^hf_{(X,D\cup Z)/S*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/S})$.

Theorem 2.11.19 (Künneth formula). *Let the notations be as in those in* (2.10.14) (2)*. Then the following hold*:

(1) *Set $H_{i.\mathrm{c}} := Rf^{\log}_{(X_i,Z_i)\mathrm{crys}*}(E^{\log,Z_i}_{\mathrm{crys,c}}(\mathcal{O}_{(X_i,D_i\cup Z_i)/S}),P^{D_i}_{\mathrm{c}})$ $(i=1,2,3)$. Then there exists a canonical isomorphism*

(2.11.19.1)
$$H_{1,\mathrm{c}}\otimes^L_{\mathcal{O}_{Y/S}} H_{2,\mathrm{c}} \xrightarrow{\sim} H_{3,\mathrm{c}}.$$

(2) *There exists a canonical isomorphism*

(2.11.19.2)
$$Rf^{\log}_{(X_1,D_1\cup Z_1)\mathrm{crys}*}(\mathcal{I}^{D_1}_{(X_1,D_1\cup Z_1)/S})\otimes^L_{\mathcal{O}_{Y/S}} Rf^{\log}_{(X_2,D_2\cup Z_2)\mathrm{crys}*}(\mathcal{I}^{D_2}_{(X_2,D_2\cup Z_2)/S})$$
$$\xrightarrow{\sim} Rf^{\log}_{(X_3,D_3\cup Z_3)\mathrm{crys}*}(\mathcal{I}^{D_3}_{(X_3,D_3\cup Z_3)/S})$$

which is compatible with the isomorphism (2.11.19.1)*.*

(3) *The isomorphisms* (2.11.19.1), (2.11.19.2) *are compatible with the base change isomorphism* (cf. (2.10.15))*.*

Proof. (1): Let k be a nonnegative integer. Then $D_3^{(k)} = \coprod_{i+j=k} D_1^{(i)} \times_{S_0} D_2^{(j)}$. Hence (1) follows from the usual Künneth formula.

(2): We have to check that the diagram (2.11.6.1) is compatible with log Künneth morphisms.

Let $(X_{j\bullet}, D_{j\bullet} \cup Z_{j\bullet})_{\bullet \in I} \overset{\subset}{\longrightarrow} (\mathcal{X}_{j\bullet}, \mathcal{D}_{j\bullet} \cup \mathcal{Z}_{j\bullet})_{\bullet \in I}$ $(j = 1, 2)$ be the data (2.4.0.1) and (2.4.0.2) with $\Delta_{j\bullet}$ for $(X_j, D_j \cup Z_j)/S_0/S$. Here note that we may assume that the index set I is independent of $j = 1, 2$ since I_0 in §2.4 can be assumed to be independent of $j = 1, 2$ by considering the product of two index sets. Set $\mathcal{X}_{3\bullet} := \mathcal{X}_{1\bullet} \times_S \mathcal{X}_{2\bullet}$, $\mathcal{D}_{3\bullet} := (\mathcal{D}_{1\bullet} \times_S \mathcal{X}_{2\bullet}) \cup (\mathcal{X}_{1\bullet} \times_S \mathcal{D}_{2\bullet})$ and $\mathcal{Z}_{3\bullet} := (\mathcal{Z}_{1\bullet} \times_S \mathcal{X}_{2\bullet}) \cup (\mathcal{X}_{1\bullet} \times_S \mathcal{Z}_{2\bullet})$. Then we have a natural datum $(X_{3\bullet}, D_{3\bullet} \cup Z_{3\bullet})_{\bullet \in I} \overset{\subset}{\longrightarrow} (\mathcal{X}_{3\bullet}, \mathcal{D}_{3\bullet} \cup \mathcal{Z}_{3\bullet})_{\bullet \in I}$ with $\Delta_{3\bullet}$. Set $\epsilon_{j\bullet} := \epsilon_{(X_{j\bullet}, D_{j\bullet} \cup Z_{j\bullet}, Z_{j\bullet})/S}$ $(j = 1, 2, 3)$. Then we have the following diagram

(2.11.19.3)
$$\begin{array}{ccc}
(((X_{j\bullet}, \widetilde{D_{j\bullet} \cup Z_{j\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{j\bullet}, D_{j\bullet} \cup Z_{j\bullet})/S}) & \xleftarrow{q^{\log}_{j\bullet\mathrm{crys}}} & (((X_{3\bullet}, \widetilde{D_{3\bullet} \cup Z_{3\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{3\bullet}, D_{3\bullet} \cup Z_{3\bullet})/S}) \\
{\scriptstyle \epsilon_{j\bullet}} \downarrow & & \downarrow {\scriptstyle \epsilon_{3\bullet}} \\
(((X_{j\bullet}, \widetilde{Z_{j\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{j\bullet}, Z_{j\bullet})/S}) & \xleftarrow{p^{\log}_{j\bullet\mathrm{crys}}} & (((X_{3\bullet}, \widetilde{Z_{3\bullet}})/S)^{\log}_{\mathrm{crys}}, \mathcal{O}_{(X_{3\bullet}, Z_{3\bullet})/S}) \\
{\scriptstyle f_{(X_{j\bullet}, Z_{j\bullet})/S}} \downarrow & & \downarrow {\scriptstyle f_{(X_{3\bullet}, Z_{3\bullet})/S}} \\
(\widetilde{S}_{\mathrm{zar}}, \mathcal{O}_S) & = & (\widetilde{S}_{\mathrm{zar}}, \mathcal{O}_S)
\end{array}$$

as (2.10.11.1) $(j = 1, 2)$. Let $\mathcal{F}_{j\bullet}$ $(j = 1, 2, 3)$ be the crystal $\mathcal{F}_\bullet$ in the proof of (2.11.3) for the admissible immersion $(X_{j\bullet}, D_{j\bullet} \cup Z_{j\bullet}) \overset{\subset}{\longrightarrow} (\mathcal{X}_{j\bullet}, \mathcal{D}_{j\bullet} \cup \mathcal{Z}_{j\bullet})$. Then we have a natural morphism

$$q^{\log *}_{1\bullet\mathrm{crys}}(\mathcal{F}_{1\bullet}) \otimes_{\mathcal{O}_{(X_{3\bullet}, D_{3\bullet} \cup Z_{3\bullet})/S}} q^{\log *}_{2\bullet\mathrm{crys}}(\mathcal{F}_{2\bullet}) \longrightarrow \mathcal{F}_{3\bullet}$$

and hence a natural morphism

(2.11.19.4)
$$Lq^{\log *}_{1\bullet\mathrm{crys}}(\mathcal{F}_{1\bullet}) \otimes^L_{\mathcal{O}_{(X_{3\bullet}, D_{3\bullet} \cup Z_{3\bullet})/S}} Lq^{\log *}_{2\bullet\mathrm{crys}}(\mathcal{F}_{2\bullet}) \longrightarrow \mathcal{F}_{3\bullet}.$$

Using the adjunction formula, we have a natural morphism

(2.11.19.5)
$$\begin{aligned} & Lp^{\log *}_{1\bullet\mathrm{crys}} R\epsilon_{1\bullet *}(\mathcal{F}_{1\bullet}) \otimes^L_{\mathcal{O}_{(X_{3\bullet}, D_{3\bullet} \cup Z_{3\bullet})/S}} Lp^{\log *}_{2\bullet\mathrm{crys}} R\epsilon_{2\bullet *}(\mathcal{F}_{2\bullet}) \\ & \longrightarrow R\epsilon_{3\bullet *}(Lq^{\log *}_{1\bullet\mathrm{crys}}(\mathcal{F}_{1\bullet}) \otimes^L_{\mathcal{O}_{(X_{3\bullet}, D_{3\bullet} \cup Z_{3\bullet})/S}} Lq^{\log *}_{2\bullet\mathrm{crys}}(\mathcal{F}_{2\bullet})). \end{aligned}$$

Here, note that $R\epsilon_{j\bullet *}(\mathcal{F}_{j\bullet})$ $(j = 1, 2)$ is bounded above by (2.3.12). Composing (2.11.19.5) with (2.11.19.4), we have a morphism

(2.11.19.6)
$$Lp^{\log *}_{1\bullet\mathrm{crys}} R\epsilon_{1\bullet *}(\mathcal{F}_{1\bullet}) \otimes^L_{\mathcal{O}_{(X_{3\bullet}, Z_{3\bullet})/S}} Lp^{\log *}_{2\bullet\mathrm{crys}} R\epsilon_{2\bullet *}(\mathcal{F}_{2\bullet}) \longrightarrow R\epsilon_{3\bullet *}(\mathcal{F}_{3\bullet}).$$

Now let us set

$$L_{j\bullet}^{\log} := L_{(X_{j\bullet},D_{j\bullet}\cup Z_{j\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{j\bullet}/S}(\log(\mathcal{Z}_{j\bullet}-\mathcal{D}_{j\bullet}))),$$

$$L_{j\bullet} := L_{(X_{j\bullet},Z_{j\bullet})/S}(\Omega^{\bullet}_{\mathcal{X}_{j\bullet}/S}(\log(\mathcal{Z}_{j\bullet}-\mathcal{D}_{j\bullet})))$$

and

$$L_{j\bullet}^{k} := a^{(k)\log}_{j\bullet\mathrm{crys}*}L_{(D^{(k)}_{j\bullet},Z_{j\bullet}|_{D^{(k)}_{j\bullet}})/S}(\Omega^{\bullet}_{\mathcal{D}^{(k)}_{j\bullet}/S}(\log \mathcal{Z}_{j\bullet}|_{\mathcal{D}^{(k)}_{j\bullet}})\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_{j\bullet}/S;Z_{j\bullet}))$$

for $k\in\mathbb{N}$, where

$$a^{(k)\log}_{j\bullet}\colon (D^{(k)}_{j\bullet},Z_{j\bullet}|_{D^{(k)}_{j\bullet}})\longrightarrow (X_{j\bullet},Z_{j\bullet})$$

is a natural morphism. Then we have a morphism

(2.11.19.7)
$$Lp^{\log *}_{1\bullet\mathrm{crys}}R\epsilon_{1\bullet*}(L^{\log}_{1\bullet})\otimes^{L}_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}}Lp^{\log *}_{2\bullet\mathrm{crys}}R\epsilon_{2\bullet*}(L^{\log}_{2\bullet})\longrightarrow R\epsilon_{3\bullet*}(L^{\log}_{3\bullet})$$

which is constructed in the same way as (2.11.19.6) and we also have natural morphisms

(2.11.19.8)
$$Lp^{\log *}_{1\bullet\mathrm{crys}}(L_{1\bullet})\otimes^{L}_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}}Lp^{\log *}_{2\bullet\mathrm{crys}}(L_{2\bullet})\longrightarrow L_{3\bullet},$$

(2.11.19.9)
$$Lp^{\log *}_{1\bullet\mathrm{crys}}(L^{\bullet}_{1\bullet})\otimes_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}}Lp^{\log *}_{2\bullet\mathrm{crys}}(L^{\bullet}_{2\bullet})\longrightarrow L^{\bullet}_{3\bullet}.$$

We can check that the canonical morphism $\mathcal{F}_{j\bullet}\longrightarrow L^{\log}_{j\bullet}$ induces the morphism

$$(2.11.19.6)\longrightarrow(2.11.19.7),$$

the isomorphism $R\epsilon_{j\bullet*}(L^{\log}_{j\bullet})\overset{=}{\longrightarrow}L_{j\bullet}$ induces the isomorphism

$$(2.11.19.7)\overset{=}{\longrightarrow}(2.11.19.8)$$

and the morphism $L_{j\bullet}\longrightarrow L^{\bullet}_{j\bullet}$ induces the morphism

$$(2.11.19.8)\longrightarrow(2.11.19.9).$$

Hence we have the commutative diagram

(2.11.19.10)
$$\begin{CD}
Lp^{\log *}_{1\bullet\mathrm{crys}}R\epsilon_{1\bullet*}(\mathcal{F}_{1\bullet})\otimes^{L}_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}}Lp^{\log *}_{2\bullet\mathrm{crys}}R\epsilon_{2\bullet*}(\mathcal{F}_{2\bullet}) @>>> Lp^{\log *}_{1\bullet\mathrm{crys}}(L^{\bullet}_{1\bullet})\otimes_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}}Lp^{\log *}_{2\bullet\mathrm{crys}}(L^{\bullet}_{2\bullet})\\
@VVV @VVV\\
R\epsilon_{3\bullet*}(\mathcal{F}_{3\bullet}) @>>> L^{\bullet}_{3\bullet}.
\end{CD}$$

By applying $R\pi^{\log}_{(X_3,Z_3)/S\mathrm{crys}*}$ to the diagram (2.11.19.10) and by using the adjunction formula, we obtain the commutative diagram

(2.11.19.11)

$$
\begin{array}{c}
Lp^{\log *}_{1\mathrm{crys}}R\epsilon_{1*}R\pi^{\log}_{(X_1,D_1\cup Z_1)/S\mathrm{crys}*}(\mathcal{F}_{1\bullet})\otimes^{L}_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}} Lp^{\log *}_{2\mathrm{crys}}R\epsilon_{2*}R\pi^{\log}_{(X_2,D_2\cup Z_2)/S\mathrm{crys}*}(\mathcal{F}_{2\bullet}) \\
\downarrow \\
R\epsilon_{3*}R\pi^{\log}_{(X_3,D_3\cup Z_3)/S\mathrm{crys}*}(\mathcal{F}_{3\bullet})
\end{array}
$$

$$
\begin{array}{cc}
\longrightarrow & Lp^{\log *}_{1\mathrm{crys}}R\pi^{\log}_{(X_1,Z_1)/S\mathrm{crys}*}(L^{\bullet}_{1\bullet})\otimes_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}} Lp^{\log *}_{2\mathrm{crys}}R\pi^{\log}_{(X_2,Z_2)/S\mathrm{crys}*}(L^{\bullet}_{2\bullet}) \\
& \downarrow \\
\longrightarrow & R\pi^{\log}_{(X_3,Z_3)/S\mathrm{crys}*}(L^{\bullet}_{3\bullet}).
\end{array}
$$

(Note that, by (2.3.11), $R\epsilon_{j*}R\pi^{\log}_{(X_j,D_j\cup Z_j)/S\mathrm{crys}*}(\mathcal{F}_{j\bullet})$ and $R\pi^{\log}_{(X_j,Z_j)/S\mathrm{crys}*}$ $(L^{\bullet}_{j\bullet})$ are bounded.) Let us put

$$\mathcal{O}^{k}_{j} := a^{(k)\log}_{j\mathrm{crys}*}(\mathcal{O}_{(D^{(k)}_j,Z_j|_{D^{(k)}_j})/S}\otimes_{\mathbb{Z}}\varpi^{(k)\log}_{\mathrm{crys}}(D_j/S;Z_j)),$$

where

$$a^{(k)\log}_{j}\colon (D^{(k)}_j,Z_j|_{D^{(k)}_j})\longrightarrow (X_j,Z_j)$$

is a natural morphism. Then we have a natural morphism

$$Lp^{\log *}_{1\mathrm{crys}}(\mathcal{O}^{\bullet}_1)\otimes^{L}_{\mathcal{O}_{(X_3,Z_3)/S}} Lp^{\log *}_{2\mathrm{crys}}(\mathcal{O}^{\bullet}_2)\longrightarrow \mathcal{O}^{\bullet}_3, \tag{2.11.19.12}$$

and the isomorphism $R\pi^{\log}_{(X_j,Z_j)/S\mathrm{crys}*}(L^{\bullet}_{j\bullet})\overset{=}{\longleftarrow}\mathcal{O}^{\bullet}_j$ induces the isomorphism

$$(\text{the right column of (2.11.19.11)})\overset{=}{\longleftarrow}(2.11.19.12).$$

On the other hand, we have a natural morphism

(2.11.19.13)

$$
\begin{aligned}
Lp^{\log *}_{1\mathrm{crys}}R\epsilon_{1*}(\mathcal{I}^{D_1}_{(X_1,D_1\cup Z_1)/S})\otimes^{L}_{\mathcal{O}_{(X_3,Z_3)/S}} & Lp^{\log *}_{2\mathrm{crys}}R\epsilon_{2*}(\mathcal{I}^{D_2}_{(X_2,D_2\cup Z_2)/S}) \\
& \longrightarrow R\epsilon_{3*}(\mathcal{I}^{D_3}_{(X_3,D_3\cup Z_3)/S}),
\end{aligned}
$$

(note that $R\epsilon_{j*}(\mathcal{I}^{D_j}_{(X_j,D_j\cup Z_j)/S})$ is bounded by (2.11.13)) and the morphism $\mathcal{I}^{D_j}_{(X_j,D_j\cup Z_j)}\longleftarrow R\pi^{\log}_{(X_j,D_j\cup Z_j)/S\mathrm{crys}*}(\mathcal{F}_{j\bullet})$ induces the morphism

$$(2.11.19.13)\longleftarrow(\text{the left column of (2.11.19.11)}).$$

Hence we obtain the diagram

(2.11.19.14)

$$
\begin{tikzcd}
Lp^{\log *}_{1\mathrm{crys}} R\epsilon_{1*}\mathcal{I}^{D_1}_{(X_1,D_1\cup Z_1)/S} \otimes^{L}_{\mathcal{O}_{(X_3,Z_3)/S}} Lp^{\log *}_{2\mathrm{crys}} R\epsilon_{2*}\mathcal{I}^{D_2}_{(X_2,D_2\cup Z_2)/S} \arrow[d] & Lp^{\log *}_{1\mathrm{crys}} R\epsilon_{1*} R\pi^{\log}_{(X_1,D_1\cup Z_1)/S\mathrm{crys}*}(\mathcal{F}_{1\bullet}) \otimes^{L}_{\mathcal{O}_{(X_{3\bullet},Z_{3\bullet})/S}} Lp^{\log *}_{2\mathrm{crys}} R\epsilon_{2*} R\pi^{\log}_{(X_2,D_2\cup Z_2)/S\mathrm{crys}*}(\mathcal{F}_{2\bullet}) \arrow[l] \arrow[r] \arrow[d] & Lp^{\log *}_{1\mathrm{crys}}(\mathcal{O}^{\bullet}_{1}) \otimes^{L}_{\mathcal{O}_{(X_3,Z_3)/S}} Lp^{\log *}_{2\mathrm{crys}}(\mathcal{O}^{\bullet}_{2}) \arrow[d] \\
R\epsilon_{3*}(\mathcal{I}^{D_3}_{(X_3,D_3\cup Z_3)/S}) & R\epsilon_{3*} R\pi^{\log}_{(X_3,D_3\cup Z_3)/S\mathrm{crys}*}(\mathcal{F}_{3\bullet}) \arrow[l] \arrow[r] & \mathcal{O}^{\bullet}_{3}.
\end{tikzcd}
$$

By applying $Rf^{\log}_{(X_3,Z_3)}$ to the diagram (2.11.19.14) and by using the adjunction formula, we obtain the diagram

(2.11.19.15)

$$
\begin{tikzcd}
Rf^{\log}_{(X_1,D_1\cup Z_1)\mathrm{crys}*}(\mathcal{I}^{D_1}_{(X_1,D_1\cup Z_1)/S}) \otimes^{L}_{\mathcal{O}_{Y/S}} Rf^{\log}_{(X_2,D_2\cup Z_2)\mathrm{crys}*}(\mathcal{I}^{D_2}_{(X_2,D_2\cup Z_2)/S}) \arrow[d] & Rf^{\log}_{(X_1,D_1\cup Z_1)\mathrm{crys}*} R\pi^{\log}_{(X_1,D_1\cup Z_1)/S\mathrm{crys}*}(\mathcal{F}_{1\bullet}) \otimes^{L}_{\mathcal{O}_{Y/S}} Rf^{\log}_{(X_2,D_2\cup Z_2)\mathrm{crys}*} R\pi^{\log}_{(X_2,D_2\cup Z_2)/S\mathrm{crys}*}(\mathcal{F}_{2\bullet}) \arrow[l] \arrow[r] \arrow[d] & Rf^{\log}_{(X_1,Z_1)\mathrm{crys}*}(\mathcal{O}^{\bullet}_{1}) \otimes^{L}_{\mathcal{O}_{Y/S}} Rf^{\log}_{(X_2,Z_2)\mathrm{crys}*}(\mathcal{O}^{\bullet}_{2}) \arrow[d] \\
Rf^{\log}_{(X_3,D_3\cup Z_3)\mathrm{crys}*}(\mathcal{I}^{D_3}_{(X_3,D_3\cup Z_3)/S}) & Rf^{\log}_{(X_3,D_3\cup Z_3)\mathrm{crys}*} R\pi^{\log}_{(X_3,D_3\cup Z_3)/S\mathrm{crys}*}(\mathcal{F}_{3\bullet}) \arrow[l] \arrow[r] & Rf^{\log}_{(X_3,Z_3)\mathrm{crys}*}(\mathcal{O}^{\bullet}_{3}).
\end{tikzcd}
$$

The left vertical morphism in (2.11.19.15) is the morphism in the statement of (2) and the right vertical morphism is (the non-filtered version of) the morphism (2.11.19.1). Therefore, to prove (2), it suffices to prove that the horizontal morphisms in (2.11.19.15) are isomorphisms. We can check this in the same way as (2.11.14).

(3): (3) immediately follows from [3, V Corollary 4.1.4], (2.11.14) and (2). □

2.12 Filtered Log de Rham-Witt Complex

Let κ be a perfect field of characteristic $p > 0$. Let W (resp. W_n) be the Witt ring of κ (resp. the Witt ring of length $n \in \mathbb{Z}_{>0}$). Let K_0 be the fraction field of W. Let (X, D) be a smooth scheme with an SNCD over κ. In this section, as a special case, we prove that $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ in the case $S = \mathrm{Spec}(W_n)$ is canonically isomorphic to the filtered log de Rham-Witt complex $(W_n\Omega_X^\bullet(\log D), P) := (W_n\Omega_X^\bullet(\log D), \{P_k W_n\Omega_X^\bullet(\log D)\}_{k\in\mathbb{Z}})$ constructed by Mokrane ([64, 1.4]).

Before proceeding on our way, we have to give the following remarks. Let $s = (\mathrm{Spec}(\kappa), L)$ be a fine log scheme. Let $g\colon Y := (\overset{\circ}{Y}, M) \longrightarrow s$ be a log smooth morphism of Cartier type. Let $W_n\Lambda_Y^\bullet$ be the "reverse" log de Rham-Witt complex defined in [46, (4.1)] and denoted by $W_n\omega_Y^\bullet$ in [loc. cit.]. Then, in [46, (4.19)], we find the following statements:

(1) There exists a canonical isomorphism

$$Ru_{Y/W_n*}(\mathcal{O}_{Y/W_n}) \xrightarrow{\sim} W_n\Lambda_Y^\bullet \quad (n \in \mathbb{Z}_{>0}).$$

(2) These isomorphisms for various $n \in \mathbb{Z}_{>0}$ are compatible with transition morphisms with respect to n.

However, as pointed out in [68, §7], the proofs of these two claims have gaps: especially we cannot find a proof of (2) in the proof of [46, (4.19)]; in [68, (7.19)], we have completed the proof of [46, (4.19)]. Hence we can use [46, (4.19)]. In addition, we have to note one more point as in [68, (7.20)] for the completeness of this book; in the definition of the embedding system in [46, p. 237], we allow the (not necessarily closed) immersion as in [82, Definition 2.2.10].

Now we come back to our situation. We keep the notations in §2.4. For example, the morphism $f\colon X \longrightarrow \mathrm{Spec}(\kappa)$ is smooth and $D \cup Z$ is a transversal SNCD on X; by abuse of notation, we also denote by f the composite morphism $X \longrightarrow \mathrm{Spec}(\kappa) \overset{\subset}{\longrightarrow} \mathrm{Spec}(W_n)$ $(n \in \mathbb{Z}_{>0})$. Because the morphism $(X, D \cup Z) \longrightarrow (\mathrm{Spec}(\kappa), \kappa^*)$ of log schemes is of Cartier type, we can apply the general theory of the log de Rham-Witt complexes in [46, §4] and [68, §6, §7] (cf. [48]) to our situation above. In particular, we have a canonical isomorphism

$$Ru_{(X,D\cup Z)/W_n*}(\mathcal{O}_{(X,D\cup Z)/W_n}) \xrightarrow{\sim} W_n\Omega_X^\bullet(\log(D \cup Z)) \tag{2.12.0.1}$$

by the Zariski analogue of [46, (4.19)]=[68, (7.19)]. In other words, we have a canonical isomorphism

$$C_{\mathrm{zar}}^{\log, Z}(\mathcal{O}_{(X,D\cup Z)/W_n}) \xrightarrow{\sim} W_n\Omega_X^\bullet(\log(D \cup Z)) \tag{2.12.0.2}$$

Let $(\mathcal{Y}_n, \mathcal{E}_n \cup \mathcal{W}_n)$ be a lift of $(X, D \cup Z)$ over W_n. Then we have $W_n\Omega^i_X(\log(D \cup Z)) = \mathcal{H}^i(\Omega^\bullet_{\mathcal{Y}_n/W_n}(\log(\mathcal{E}_n \cup \mathcal{W}_n)))$. Set

$$P^D_k W_n\Omega^i_X(\log(D \cup Z)) = \mathcal{H}^i(P^{\mathcal{E}_n}_k \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log(\mathcal{E}_n \cup \mathcal{W}_n))). \tag{2.12.0.3}$$

Definition 2.12.1. We call the filtration $P^D := \{P^D_k W_n\Omega^i_X(\log(D \cup Z))\}_{k\in\mathbb{Z}}$ the *preweight filtration* on $W_n\Omega^i_X(\log(D \cup Z))$ with respect to D.

We shall prove, in (2.12.4) below, that $P^D_k W_n\Omega^i_X(\log(D\cup Z))$ is independent of the choice of the lift $(\mathcal{Y}_n, \mathcal{E}_n \cup \mathcal{W}_n)$. If $Z = \emptyset$, $P^D_k W_n\Omega^i_X(\log(D\cup Z))$ is the preweight filtration defined in [64, (1.4.1)]. Here, as noted in [68, (4.3)], we use the terminology "preweight filtration" instead of the terminology "weight filtration" since $W_n\Omega^i_X(\log(D\cup Z))$ is a sheaf of torsion W-modules in $\widetilde{X}_{\rm zar}$.

To prove a filtered version of (2.12.0.2), we need some lemmas (cf. [64, 1.2, 1.4.3]).

Let $\Delta_D := \{D_\lambda\}_\lambda$ (resp. $\Delta_Z := \{Z_\mu\}_\mu$) be a decomposition of D (resp. Z) by smooth components of D (resp. Z). Set $\Delta := \{D_\lambda, Z_\mu\}_{\lambda,\mu}$. Let $\iota\colon (X, D \cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X}_n, \mathcal{D}_n \cup \mathcal{Z}_n)$ be an admissible immersion over W_n with respect to Δ which induces an admissible immersion $(X, D) \overset{\subset}{\longrightarrow} (\mathcal{X}_n, \mathcal{D}_n)$ (resp. $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{X}_n, \mathcal{Z}_n)$) with respect to Δ_D (resp. Δ_Z). Let $\iota'\colon (X, D \cup Z) \overset{\subset}{\longrightarrow} (\mathcal{Y}_n, \mathcal{E}_n \cup \mathcal{W}_n)$ be a lift of $(X, D \cup Z)$ over W_n such that ι' induces a lift $(X, D) \overset{\subset}{\longrightarrow} (\mathcal{Y}_n, \mathcal{E}_n)$ (resp. $(X, Z) \overset{\subset}{\longrightarrow} (\mathcal{Y}_n, \mathcal{W}_n)$). Assume that $(\mathcal{Y}_n, \mathcal{E}_n \cup \mathcal{W}_n)$ and $(\mathcal{X}_n, \mathcal{D}_n \cup \mathcal{Z}_n)$ are affine log schemes. Because $(\mathcal{X}_n, \mathcal{D}_n \cup \mathcal{Z}_n)$ is log smooth over W_n, there exists a morphism of log schemes $\mathfrak{f}\colon (\mathcal{Y}_n, \mathcal{E}_n \cup \mathcal{W}_n) \longrightarrow (\mathcal{X}_n, \mathcal{D}_n \cup \mathcal{Z}_n)$ over W_n such that $\mathfrak{f}$ induces morphisms $(\mathcal{Y}_n, \mathcal{E}_n) \longrightarrow (\mathcal{X}_n, \mathcal{D}_n)$ and $(\mathcal{Y}_n, \mathcal{W}_n) \longrightarrow (\mathcal{X}_n, \mathcal{Z}_n)$ and such that $\mathfrak{f} \circ \iota' = \iota$. Let $\mathfrak{D}_X(\mathcal{X}_n)$ be the PD-envelope of the closed immersion $\iota\colon X \longrightarrow \mathcal{X}_n$ over $(\mathrm{Spec}(W_n), pW_n, [\ ])$. The morphism $\mathfrak{f}$ also induces a morphism $\mathfrak{f}\colon (\mathcal{Y}_n, p\mathcal{O}_{\mathcal{Y}_n}) \longrightarrow (\mathfrak{D}_X(\mathcal{X}_n), \overline{\mathrm{Ker}(\mathcal{O}_{\mathcal{X}_n} \longrightarrow \mathcal{O}_X)\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}})$ of PD-schemes. Hence $\mathfrak{f}$ induces a morphism $\mathfrak{f}^*\colon \mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)) \longrightarrow \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log(\mathcal{E}_n \cup \mathcal{W}_n))$ of complexes. By (2.2.17) (1), we have the following exact sequence

$$\begin{aligned}(2.12.1.1)\qquad & 0 \longrightarrow \mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} P^{\mathcal{D}_n}_{k-1}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)) \\ & \longrightarrow \mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} P^{\mathcal{D}_n}_k \Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)) \longrightarrow \\ & \mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{\bullet - k}_{\mathcal{D}_n{}^{(k)}/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}^{(k)}_n}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm zar}(\mathcal{D}_n/W_n)(-k) \longrightarrow 0\end{aligned}$$

by using the Poincaré residue isomorphism with respect to $\mathcal{D}_n$ ((2.2.21.3)). (The compatibility of the Poincaré residue isomorphism with the Frobenius can be checked as in [68, (9.3) (1)].) Note that the derivative

$$d\colon P^{\mathcal{D}_n}_k \Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)) \longrightarrow P^{\mathcal{D}_n}_k \Omega^{\bullet+1}_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n))$$

extends to a derivative of $\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} P_k^{\mathcal{D}_n}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n))$ (cf. [50, 0 (3.1.4)], [54, (6.7)]).

Lemma 2.12.2. *The long exact sequence associated to* (2.12.1.1) *is decomposed into the following short exact sequences*:

$$\begin{aligned}(2.12.2.1)\qquad 0 &\longrightarrow \mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} P_{k-1}^{\mathcal{D}_n}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)))\\ &\longrightarrow \mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} P_{k}^{\mathcal{D}_n}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)))\\ &\longrightarrow \mathcal{H}^{q-k}(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})\\ &\qquad\qquad \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}_n/W_n)(-k)) \longrightarrow 0 \quad (q\in\mathbb{Z}).\end{aligned}$$

Proof. (cf. [64, 1.2]) The problem is Zariski local. In the following, we fix an isomorphism $\varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}_n/W_n) \xrightarrow{\sim} \mathbb{Z}$.

Let $u : \widetilde{X}_{\mathrm{et}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$ be a canonical morphism of topoi. For a coherent $\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}$-module (resp. a coherent $\mathcal{O}_{\mathcal{Y}_n}$-module) $\mathcal{F}$ on $\mathfrak{D}_X(\mathcal{X}_n)_{\mathrm{zar}} \simeq X_{\mathrm{zar}}$ (resp. $\mathcal{Y}_{n\mathrm{zar}} \simeq X_{\mathrm{zar}}$), let $\mathcal{F}_{\mathrm{et}}$ be the corresponding coherent $\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}$-module (resp. a coherent $\mathcal{O}_{\mathcal{Y}_n}$-module) on $\mathfrak{D}_X(\mathcal{X}_n)_{\mathrm{et}} \simeq X_{\mathrm{et}}$ (resp. $\mathcal{Y}_{n\mathrm{et}} \simeq X_{\mathrm{et}}$). Let us consider the following diagram

$$\begin{aligned}(2.12.2.2)\qquad 0 &\longrightarrow \mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} P_{k-1}^{\mathcal{D}_n}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)))_{\mathrm{et}})\\ &\longrightarrow \mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} P_{k}^{\mathcal{D}_n}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)))_{\mathrm{et}})\\ &\longrightarrow \mathcal{H}^{q-k}((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}_n^{(k)}}))_{\mathrm{et}}) \longrightarrow 0\\ &\qquad\qquad (q\in\mathbb{Z}),\end{aligned}$$

which is the etale analogue of the diagram (2.12.2.1). We prove that the diagram (2.12.2.1) is exact for any $k, q \in \mathbb{Z}$ if and only if the diagram (2.12.2.2) is exact for any $k, q \in \mathbb{Z}$.

By the Zariski analogue of [54, (6.4)], both

$$\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))$$

and

$$\mathcal{H}^q(\Omega^\bullet_{\mathcal{E}_n^{(k)}/W_n}(\log(\mathcal{W}_n^{(k)}|_{\mathcal{E}_n^{(k)}}))) = W_n\Omega^q_{D^{(k)}}(\log(Z|_{D^{(k)}}))$$

calculate $R^q u_{(D^{(k)},Z|_{D^{(k)}})/W_n*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/W_n})$. Hence we have

$$\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}}))) = W_n\Omega^q_{D^{(k)}}(\log(Z|_{D^{(k)}}))$$

and it is a quasi-coherent $W_n(\mathcal{O}_X)$-module on X_{zar}. On the other hand, let $((D^{(k)}, Z|_{D^{(k)}})/W_n)^{\log}_{\mathrm{crys,et}}$ be the log crystalline site of $(D^{(k)}, Z|_{D^{(k)}})$ over W_n with respect to the etale topology and let

$$u_{(D^{(k)},Z|_{D^{(k)}})/W_n,\mathrm{et}} : \widetilde{((D^{(k)}, Z|_{D^{(k)}})/W_n)}^{\log}_{\mathrm{crys,et}} \longrightarrow \widetilde{X}_{\mathrm{et}}$$

be the morphism of topoi which is defined in the same way as the morphism $u_{(D^{(k)},Z|_{D^{(k)}})/W_n}$. Then, by [54, (6.4)], both

$$\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))_{\mathrm{et}})$$

and

$$\mathcal{H}^q((\Omega^\bullet_{\mathcal{E}_n^{(k)}/W_n}(\log(\mathcal{W}_n^{(k)}|_{\mathcal{E}_n^{(k)}})))_{\mathrm{et}})=W_n\Omega^q_{D^{(k)}}(\log(Z|_{D^{(k)}}))$$

calculate $R^qu_{(D^{(k)},Z|_{D^{(k)}})/W_n,\mathrm{et}*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/W_n})$. Hence we have $\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))_{\mathrm{et}})=W_n\Omega^q_{D^{(k)}}(\log(Z|_{D^{(k)}}))$ on X_{et} and it is the quasi-coherent $W_n(\mathcal{O}_X)$-module on X_{et} corresponding to $\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))$. Hence there exists the canonical isomorphism

(2.12.2.3)
$$\begin{aligned}&\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))\\ \xrightarrow{=}\ &Ru_*\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))_{\mathrm{et}})\end{aligned}$$

and for any etale morphism $\varphi\colon X'\longrightarrow X$, there exists the following canonical isomorphism

(2.12.2.4)
$$\begin{aligned}&W_n(\mathcal{O}_{X'})\otimes_{\varphi^{-1}(W_n(\mathcal{O}_X))}\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))\\ \xrightarrow{=}\ &\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log(\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}})))_{\mathrm{et}})|_{X'_{\mathrm{zar}}}.\end{aligned}$$

Now let us assume that the diagram (2.12.2.2) is exact for any $k,q\in\mathbb{Z}$. Then, by (2.12.2.3) and the induction on k, we see that each term of (2.12.2.2) is u_*-acyclic and that $u_*((2.12.2.2))$ gives the exact sequence (2.12.2.1). On the other hand, assume that the diagram (2.12.2.1) is exact for any $k,q\in\mathbb{Z}$. In this case, note that the morphisms in the diagram (2.12.2.1) and the long exact sequence

$$\begin{aligned}\cdots\longrightarrow\ &\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}P^{\mathcal{D}_n}_{k-1}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n\cup\mathcal{Z}_n)))_{\mathrm{et}})\\ \longrightarrow\ &\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}P^{\mathcal{D}_n}_{k}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n\cup\mathcal{Z}_n)))_{\mathrm{et}})\\ \longrightarrow\ &\mathcal{H}^{q-k}((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log\mathcal{Z}_n|_{\mathcal{D}_n^{(k)}}))_{\mathrm{et}})\longrightarrow\cdots\end{aligned}$$

are $W_n(\mathcal{O}_X)$-linear with respect to the natural action of $W_n(\mathcal{O}_X)=\mathcal{H}^0(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}\Omega^\bullet_{\mathcal{X}_n/W_n})$. Then, by (2.12.2.4) and the induction on k, we see that there exists the canonical isomorphism

$$\begin{aligned}&W_n(\mathcal{O}_{X'})\otimes_{\varphi^{-1}(W_n(\mathcal{O}_X))}\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}P^{\mathcal{D}_n}_{k}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n\cup\mathcal{Z}_n)))\\ \xrightarrow{=}\ &\mathcal{H}^q((\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}}P^{\mathcal{D}_n}_{k}\Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n\cup\mathcal{Z}_n)))_{\mathrm{et}})|_{X'_{\mathrm{zar}}}\end{aligned}$$

for any etale morphism $\varphi : X' \longrightarrow X$. Hence the diagram $(2.12.2.2)|_{X'_{\rm zar}}$ is exact for any $\varphi : X' \longrightarrow X$ as above, and this implies the exactness of (2.12.2.2) for any $k, q \in \mathbb{Z}$. Hence the exactness of (2.12.2.1) for any $k, q \in \mathbb{Z}$ is equivalent to the exactness of (2.12.2.2) for any $k, q \in \mathbb{Z}$.

By the claim we have shown in the previous paragraph, we may work etale locally to prove the lemma. Hence we may assume that $\mathcal{X}_n$ is the scheme $\mathrm{Spec}(W_n[x_1, \ldots, x_d])$ and that $\mathcal{D}_n \overset{\subset}{\longrightarrow} \mathcal{X}_n$ is the closed immersion defined by the ideal $(x_1 \cdots x_s)$ for some $0 \leq s \leq d$. In this case, by the proof of [64, 1.2], the morphism

$$Z(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} P_k^{\mathcal{D}_n} \Omega^q_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n))) \\ \longrightarrow Z(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{q-k}_{\mathcal{D}_n^{(k)}/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}_n^{(k)}}))$$

is surjective on $X_{\rm zar}$. Hence we obtain the exactness of (2.12.2.1). □

By the Zariski analogue of [54, (6.4)] we have the following commutative diagram:

(2.12.2.5)

$$\begin{CD}
R^q u_{(X,D\cup Z)/W_n *}(\mathcal{O}_{(X,D\cup Z)/W_n}) @>{\sim}>> \mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n))) \\
@| @VV{\mathcal{H}^q(\mathfrak{f}^*)}V \\
R^q u_{(X,D\cup Z)/W_n *}(\mathcal{O}_{(X,D\cup Z)/W_n}) @>{\sim}>> \mathcal{H}^q(\Omega^\bullet_{\mathcal{Y}_n/W_n}(\log(\mathcal{E}_n \cup \mathcal{W}_n))).
\end{CD}$$

Lemma 2.12.3. *Let k be a nonnegative integer. Then $\mathcal{H}^q(\mathfrak{f}^*)$ induces an isomorphism*

$$\mathcal{H}^q(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} P_k^{\mathcal{D}_n} \Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n))) \\ \overset{\sim}{\longrightarrow} \mathcal{H}^q(P_k^{\mathcal{E}_n} \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log(\mathcal{E}_n \cup \mathcal{W}_n))).$$

Proof. We have two proofs.

First proof: The morphism $\mathfrak{f}$ induces a morphism

$$\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} P_k^{\mathcal{D}_n} \Omega^\bullet_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)) \longrightarrow P_k^{\mathcal{E}_n} \Omega^\bullet_{\mathcal{Y}_n/W_n}(\log(\mathcal{E}_n \cup \mathcal{W}_n)).$$

By using the Poincaré residue isomorphisms with respect $\mathcal{D}_n$, by (2.12.2) and by induction on k, it suffices to prove that $\mathfrak{f}^*$ induces an isomorphism

$$\mathcal{H}^{q-k}(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^\bullet_{\mathcal{D}_n^{(k)}/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}_n^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm zar}(\mathcal{D}_n/W_n)(-k)) \overset{\sim}{\longrightarrow}$$

$$\mathcal{H}^{q-k}(\Omega^\bullet_{\mathcal{E}_n^{(k)}/W_n}(\log \mathcal{W}_n|_{\mathcal{E}_n^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm zar}(\mathcal{E}_n/W_n)(-k)).$$

By noting that $\mathfrak{D}_X(\mathcal{X}_n) \times_{\mathcal{X}_n} \mathcal{D}_n^{(k)}$ is the PD-envelope of the closed immersion $D^{(k)} \longrightarrow \mathcal{D}_n^{(k)}$ ((2.2.16) (2)), we see that the morphism above is an isomorphism by [11, 7.1 Theorem].

Second proof: (2.12.3) immediately follows from (2.5.4) (2). $\square$

The following is a generalization of the preweight filtration on $W_n\Omega^i_X(\log D)$ ([64, (1.4.1)]) for an admissible closed immersion $(X, D\cup Z) \xrightarrow{\subset} (\mathcal{X}_n, \mathcal{D}_n\cup \mathcal{Z}_n)$ over $(\mathrm{Spec}(W_n), pW_n, [\ ])$ even if $Z = \emptyset$:

Corollary 2.12.4. (1) *The preweight filtration on* $W_n\Omega^\bullet_X(\log(D\cup Z))$ *with respect to* D *is well-defined. More generally,* $\{\mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)}\otimes_{\mathcal{O}_{\mathcal{X}_n}} P^{\mathcal{D}_n}_k\Omega^*_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n\cup\mathcal{Z}_n)))\}_{k\in\mathbb{N}}$ *induces the preweight filtration on* $W_n\Omega^\bullet_X(\log(D\cup Z))$.

(2) *Let* i *be a nonnegative integer. Then*

$$P^D_kW_n\Omega^i_X(\log(D\cup Z)) = \mathcal{H}^i(P^D_kC^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/W_n})) \tag{2.12.4.1}$$

(3) *There exists the following canonical isomorphism*

(2.12.4.2)
$$\mathrm{Res}^D\colon \mathrm{gr}^{P^D}_k W_n\Omega^\bullet_X(\log(D\cup Z)) \xrightarrow{\sim} W_n\Omega^\bullet_{D^{(k)}}(\log Z|_{D^{(k)}})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(D/\kappa)(-k)$$

which is compatible with the Frobenius endomorphisms.

Proof. (1): We can show the well-definedness by the standard method in, e.g., [64, 3.4] and by (2.12.3). The latter statement is obvious by (2.12.3).

(2): (2) is obvious by the definition (2.12.0.3).

(3): (2.12.4.2) is an isomorphism of complexes of W_n-modules by (2.6.1.1) and the definition of the boundary morphism of the two log de Rham Witt complexes in (2.12.4.2). The compatibility with the Frobenius endomorphisms is obtained by the same argument as that in [68, (9.3) (1)]. $\square$

Let $g\colon Y := (\overset{\circ}{Y}, M) \longrightarrow s$ be as in the beginning of this section. By abuse of notation, we denote $\overset{\circ}{Y}$ by Y. Let $\iota\colon (Y,M) \xrightarrow{\subset} (\mathcal{Y},\mathcal{M})$ be a closed immersion into a fine formally log smooth scheme over $(\mathrm{Spf}(W), W(L))$, where $W(L)$ is the canonical lift of L over $\mathrm{Spf}(W)$ (cf. [46, (3.1)]). Let $\widetilde{g}\colon (\mathcal{Y},\mathcal{M}) \longrightarrow (\mathrm{Spf}(W), W(L))$ be the structural morphism. Let $(\mathfrak{D}_Y(\mathcal{Y}), \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})})$ be the log PD-envelope of the closed immersion $(Y,M) \xrightarrow{\subset} (\mathcal{Y},\mathcal{M})$. Set $(\mathcal{Y}_n,\mathcal{M}_n) := (\mathcal{Y},\mathcal{M})\otimes_W W_n$, $(\mathfrak{D}_Y(\mathcal{Y}_n), \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y}_n)}) := (\mathfrak{D}_Y(\mathcal{Y}), \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})})\otimes_W W_n$ and $g_n := \widetilde{g}\otimes_W W_n$ ($n\in\mathbb{Z}_{>0}$). Let $\iota_n\colon (Y,M) \xrightarrow{\subset} (\mathcal{Y}_n,\mathcal{M}_n)$ be the induced natural closed immersion. Let $W_n(M)$ be the canonical lift of M over $W_n(Y)$. Assume that there exists an endomorphism Φ of $(\mathcal{Y},\mathcal{M})$ which is a lift of the Frobenius morphism of $(\mathcal{Y}_1,\mathcal{M}_1)$. Then there exists a morphism

$$W_n(\iota)\colon (W_n(Y), W_n(M)) \longrightarrow (\mathcal{Y}_n,\mathcal{M}_n) \tag{2.12.4.3}$$

of log schemes which has been constructed in ([68, (7.17)]) by using a log version of a lemma of Dieudonné-Cartier ([68, (7.10)]). In this book we only review the definition of the morphism $W_n(\iota)$. As a morphism of schemes,

$W_n(\iota)$ is well-known (e.g., [50, 0 (1.3.21), II (1.1.4)]). Let $\widetilde{m}$ be a local section of $\mathcal{M}$ with image $m \in \mathcal{M}_n$. Let z_j $(1 \leq j \leq n-1)$ be a unique local section of $1 + p\mathcal{O}_{\mathcal{Y}}$ satisfying an equality $\Phi^{*j}(\widetilde{m}) = \widetilde{m}^{p^j} z_j$. Let $\{\widetilde{s}_j\}_{j=1}^{n-1}$ be a family of local sections of $\mathcal{O}_{\mathcal{Y}}$ satisfying the following equalities

$$1 + p\widetilde{s}_1^{p^{j-1}} + \cdots + p^j \widetilde{s}_j = z_j.$$

(The existence of $\{\widetilde{s}_j\}_{j=1}^{n-1}$ has been proved in the proof of the log version of a lemma of Dieudonné-Cartier ([68, (7.10)]) by using the argument in [61, VII 4].) Set $s_j := \iota^*(\widetilde{s}_j)$ $(1 \leq j \leq n-1)$ and $s_0 := 1$. Then $W_n(\iota)$ as a morphism of log structures is, by definition, the following morphism:

$$(2.12.4.4) \qquad W_n(\iota)^*(\mathcal{M}_n) \ni m \longmapsto (\iota_n^*(m), (s_0, \ldots, s_{n-1})) \in M \oplus (1 + VW_{n-1}(\mathcal{O}_Y)) = W_n(M).$$

Here we denote $W_n(\iota)^*(m)$ simply by m.

By the universality of the log PD-envelope, Φ induces a natural morphism

$$\Phi_{\mathfrak{D}_Y(\mathcal{Y})} \colon (\mathfrak{D}_Y(\mathcal{Y}), \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})}) \longrightarrow (\mathfrak{D}_Y(\mathcal{Y}), \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})}).$$

Following [31], let us denote by $\Lambda^i_{\mathcal{Y}_n/W_n}$ the sheaf of log differential forms of degree i on $(\mathcal{Y}_n, \mathcal{M}_n)/(\mathrm{Spec}(W_n), W_n(L))$, and by $W_n\Lambda^i_Y$ the Hodge-Witt sheaf of log differential forms of degree i on $(Y, M)/s$. The morphism $W_n(\iota)$ induces a morphism

$$(2.12.4.5) \qquad \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^\bullet_{\mathcal{Y}_n/W_n} \longrightarrow \Lambda^\bullet_{W_n(Y)/(W_n, W_n(L)), [\ ]}$$

of complexes of $g_n^{-1}(W_n)$-modules, where $\Lambda^\bullet_{W_n(Y)/(W_n, W_n(L)), [\ ]}$ is defined in the proof of [46, (4.19)] and denoted by $\omega^\bullet_{W_n(Y)/(W_n, W_n(L)), [\ ]}$ in [loc. cit.]. By [46, (4.9)] there exists a canonical morphism

$$(2.12.4.6) \quad \Lambda^\bullet_{W_n(Y)/(W_n, W_n(L)), [\ ]} \longrightarrow \mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^*_{\mathcal{Y}_n/W_n}) (= W_n\Lambda^\bullet_Y).$$

Composing (2.12.4.5) with (2.12.4.6), we have a morphism

$$(2.12.4.7) \qquad \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^\bullet_{\mathcal{Y}_n/W_n} \longrightarrow \mathcal{H}^\bullet(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^*_{\mathcal{Y}_n/W_n}).$$

As usual ([50, II (1.1)]), the induced morphism by (2.12.4.7) in the derived category is independent of the choice of $\mathcal{Y}$ and Φ.

Lemma 2.12.5. *Set* $\varphi := \Phi^*_{\mathfrak{D}_Y(\mathcal{Y})}$. *Then the morphism* (2.12.4.7) *is equal to the morphism* $(\varphi/p^\bullet)^n \bmod p^n$.

Proof. First consider the case $\bullet = 0$. Because $\mathcal{O}_{\mathcal{Y}}$ is p-torsion-free, the following morphism s_φ is well-defined:

$$(2.12.5.1) \qquad s_\varphi \colon \mathcal{O}_{\mathcal{Y}} \ni x \longmapsto (s_0, s_1, \ldots, s_{n-1}, \ldots) \in W(\mathcal{O}_{\mathcal{Y}}),$$

where s_i's satisfy the following equations $s_0^{p^{m-1}} + ps_1^{p^{m-2}} + \cdots + p^{m-1}s_{m-1} = \varphi^{m-1}(x)$ $(m \in \mathbb{Z}_{>0})$ (e.g., [50, 0 (1.3.16)]). The morphism (2.12.4.5) for $\bullet = 0$ is induced by the following composite morphism

$$\mathcal{O}_{\mathcal{Y}_n} \overset{s_\varphi \bmod V^n W(\mathcal{O}_{\mathcal{Y}_1})}{\longrightarrow} W_n(\mathcal{O}_{\mathcal{Y}_1}) \longrightarrow W_n(\mathcal{O}_Y).$$

The morphism (2.12.4.6) for $\bullet = 0$ is defined by

$$(2.12.5.2) \qquad W_n(\mathcal{O}_Y) \ni (t_0, t_1, \ldots, t_{n-1}) \longmapsto$$

$$\widetilde{t}_0^{p^n} + p\widetilde{t}_1^{p^{n-1}} + \cdots + p^{n-1}\widetilde{t}_{n-1}^{p} \in \mathcal{H}^0(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^\bullet_{\mathcal{Y}_n/W_n}),$$

where $\widetilde{t}_j \in \mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)}$ $(1 \leq j \leq n-1)$ is a lift of $t_j \in \mathcal{O}_Y$ ([46, (4.9)]). Since $\varphi(s^{p^{n-i}}) \equiv s^{p^{n-i+1}} \bmod p^{n-i+1}\mathcal{O}_{\mathcal{Y}_n}$ $(s \in \mathcal{O}_{\mathcal{Y}_n}, i \in \{0, 1, \ldots, n\})$, (2.12.4.7) for $\bullet = 0$ is induced by the morphism $x \longmapsto \varphi^n(x)$ $(x \in \mathcal{O}_{\mathcal{Y}_n})$.

Next, consider the case $\bullet = 1$. Because the image of (2.12.4.5) is contained in the image of $W_n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} W_n(M)^{\mathrm{gp}}$ in $\Lambda^1_{W_n(Y)/(W_n, W_n(L)), [\ ]}$, consider the following composite morphism

$$(2.12.5.3) \qquad W_n(\mathcal{O}_Y) \otimes_{\mathbb{Z}} W_n(M)^{\mathrm{gp}} \longrightarrow \Lambda^1_{W_n(Y)/(W_n, W_n(L)), [\ ]}$$
$$\overset{(2.12.4.6)}{\longrightarrow} \mathcal{H}^1(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^\bullet_{\mathcal{Y}_n/W_n}).$$

The morphism (2.12.5.3) is defined by the morphisms (2.12.5.2) and $d\log m \longmapsto d\log \widetilde{m} \bmod p^n$ $(m \in M)$, where $\widetilde{m} \in \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})}$ is a lift of m. Since $\varphi \colon \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})} \longrightarrow \mathcal{M}_{\mathfrak{D}_Y(\mathcal{Y})}$ is a lift of the Frobenius endomorphism, there exists a section a of $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y})}$ such that $\varphi(\widetilde{m}) = \widetilde{m}^p(1 + pa)$. Then

$$\begin{aligned} p^{-1} d\log \varphi(\widetilde{m}) &= p^{-1} d\log(\widetilde{m}^p(1 + pa)) = d\log \widetilde{m} + p^{-1} d\log(1 + pa) \\ &= d\log \widetilde{m} + d\Big(\sum_{i=1}^{\infty} (-1)^{i-1}(p^{i-1}/i)a^i\Big). \end{aligned}$$

Hence

$$\begin{aligned} d\log \widetilde{m} \bmod p^n &= p^{-1} d\log \varphi(\widetilde{m}) \bmod p^n \\ &= \cdots \\ &= p^{-n} d\log \varphi^n(\widetilde{m}) \bmod p^n. \end{aligned}$$

in $\mathcal{H}^1(\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)} \otimes_{\mathcal{O}_{\mathcal{Y}_n}} \Lambda^\bullet_{\mathcal{Y}_n})$. Furthermore the image of $1 \otimes (1 + VW_{n-1}(\mathcal{O}_Y))$ by the morphism (2.12.5.3) is the zero.

Let m and $\{s_j\}_{j=1}^{n-1}$ be local sections in (2.12.4.4). Then the image of $d\log \widetilde{m}$ by the morphism (2.12.4.7) is the class of $d\log \widetilde{m} + d\log(1 + \sum_{j=1}^{n-1} p^j \widetilde{s}_j^{p^{n-j}})$, where $\widetilde{s}_j$ is a lift of s_j in $\mathcal{O}_{\mathfrak{D}_Y(\mathcal{Y}_n)}$. As in the argument above, the second form is exact. Hence the morphism (2.12.4.7) for $\bullet = 1$ is induced from $(\varphi/p)^n \bmod p^n$.

When $\bullet \geq 2$, (2.12.5) follows from the definition of (2.12.4.6), from [46, (4.9)] and from the calculation above. $\square$

Corollary 2.12.6. *Let* $\iota\colon (X, D\cup Z) \overset{\subset}{\longrightarrow} (\mathcal{X}, \mathcal{D}\cup\mathcal{Z})$ *be an admissible immersion into a formally smooth scheme over* $\mathrm{Spf}(W)$ *with a relative transversal SNCD over* $\mathrm{Spf}(W)$. *Set* $(\mathcal{X}_n, \mathcal{D}_n \cup \mathcal{Z}_n) := (\mathcal{X}, \mathcal{D} \cup \mathcal{Z}) \otimes_W W_n$. *Assume that there exists a lift* $\Phi\colon (\mathcal{X}, \mathcal{D} \cup \mathcal{Z}) \longrightarrow (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$ *of the Frobenius endomorphism of* $(\mathcal{X}_1, \mathcal{D}_1 \cup \mathcal{Z}_1)$. *Let* $\mathfrak{D}_X(\mathcal{X})$ *be the PD-envelope of the closed immersion* $X \overset{\subset}{\longrightarrow} \mathcal{X}$ *over* $(\mathrm{Spf}(W), pW, [\ ])$. *Set* $\mathfrak{D}_X(\mathcal{X}_n) := \mathfrak{D}_X(\mathcal{X}) \otimes_W W_n$. *Then the morphism*

$$\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{\bullet}_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)) \longrightarrow W_n\Omega^{\bullet}_X(\log(D \cup Z)) \tag{2.12.6.1}$$

defined in (2.12.4.7) *induces an isomorphism*

(2.12.6.2)
$$(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{\bullet}_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)), P^{\mathcal{D}_n}) \longrightarrow (W_n\Omega^{\bullet}_X(\log(D \cup Z)), P^D)$$

in $\mathrm{D}^+\mathrm{F}(f^{-1}(W_n))$.

Proof. The endomorphism Φ induces an endomorphism $\Phi_{\mathfrak{D}_X(\mathcal{X})}\colon \mathfrak{D}_X(\mathcal{X}) \longrightarrow \mathfrak{D}_X(\mathcal{X})$. Set $\varphi := \Phi^*_{\mathfrak{D}_X(\mathcal{X})}$. By the definition of $W_n\Omega^{\bullet}_X(\log(D \cup Z))$ ([46, (4.1)]), we have $W_n\Omega^{\bullet}_X(\log(D \cup Z)) = \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^*_{\mathcal{X}_n/W_n}(\log(\mathcal{D}_n \cup \mathcal{Z}_n)))$. By (2.12.5), the morphism (2.12.6.1) is induced by $\varphi_n := (\varphi/p^{\bullet}) \mathrm{mod}\, p^n$. By a calculation in [68, (8.1), (8.4)], φ_n preserves the preweight filtrations with respect to $\mathcal{D}_n$:

$$\begin{aligned}\varphi_n(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X})} \otimes_{\mathcal{O}_{\mathcal{X}}} P^{\mathcal{D}}_k \Omega^{\bullet}_{\mathcal{X}/W}(\log(\mathcal{D} \cup \mathcal{Z}))/p^n)& \\ \subset \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X})} \otimes_{\mathcal{O}_{\mathcal{X}}} P^{\mathcal{D}}_k \Omega^{*}_{\mathcal{X}/W}(\log(\mathcal{D} \cup \mathcal{Z}))/p^n).&\end{aligned}$$

Hence, by using the Poincaré residue isomorphism and by (2.12.2), it suffices to prove that φ_n induces an isomorphism

$$\begin{aligned}\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{\bullet}_{\mathcal{D}^{(k)}_n/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}^{(k)}_n}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}_n/W_n)(-k) \overset{\sim}{\longrightarrow}& \\ \mathcal{H}^{\bullet}(\mathcal{O}_{\mathfrak{D}_X(\mathcal{X}_n)} \otimes_{\mathcal{O}_{\mathcal{X}_n}} \Omega^{*}_{\mathcal{D}^{(k)}_n/W_n}(\log \mathcal{Z}_n|_{\mathcal{D}^{(k)}_n}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(\mathcal{D}_n/W_n)(-k)).&\end{aligned} \tag{2.12.6.3}$$

This immediately follows from (2.2.16) (2) and [46, (4.19)]=[68, (7.19)]. $\square$

Lemma 2.12.7. *Let* Y *be a scheme over* $\mathbb{F}_p$ *with a closed subscheme* E. *Let* U *be the complement of* E *in* Y *and* $j\colon U \overset{\subset}{\longrightarrow} Y$ *the open immersion. Denote by* (Y, E) *a log scheme* $(Y, j_*(\mathcal{O}^*_U) \cap \mathcal{O}_Y)$. *Let* $(W_n(Y), W_n(E))$ *be a similar log scheme over* $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n : (W_n(Y), W_n(E)) := (W_n(Y), W_n(j)_*(W_n(\mathcal{O}_U)^*) \cap W_n(\mathcal{O}_Y))$. *Assume that the natural morphism* $\mathcal{O}_Y \longrightarrow j_*(\mathcal{O}_U)$ *is injective. Then* $(W_n(Y), W_n(E))$ *is the canonical lift of* (Y, E) *in the sense of* [46, (3.1)].

Proof. Let $[\,]\colon \mathcal{O}_Y \ni a \longmapsto (a, 0, \ldots, 0) \in W_n(\mathcal{O}_Y)$ be the Teichmüller lift. By noting that $VW_n(\mathcal{O}_U)$ is a nilpotent ideal sheaf of $W_n(\mathcal{O}_U)$ ([50, 0 (1.3.13)]), we have a formula $W_n(\mathcal{O}_U)^* = [\mathcal{O}_U^*] \oplus \mathrm{Ker}(W_n(\mathcal{O}_U)^* \longrightarrow \mathcal{O}_U^*)$. We claim that

$$\mathrm{Ker}\{W_n(j)_*(W_n(\mathcal{O}_U)^*) \cap W_n(\mathcal{O}_Y) \longrightarrow j_*(\mathcal{O}_U^*)\} = \mathrm{Ker}(W_n(\mathcal{O}_Y)^* \longrightarrow \mathcal{O}_Y^*).$$

The inclusion $\supset$ is obvious. Let a be a local section on the left hand side. Then the image of a in $\mathcal{O}_Y$ is 1 since $\mathcal{O}_Y \longrightarrow j_*(\mathcal{O}_U)$ is injective. Hence we have $a \in W_n(\mathcal{O}_Y)^*$ since $VW_n(\mathcal{O}_Y)$ is a nilpotent ideal sheaf of $W_n(\mathcal{O}_Y)$. Therefore

$$W_n(j)_*(W_n(\mathcal{O}_U)^*) \cap W_n(\mathcal{O}_Y) = [j_*(\mathcal{O}_U^*) \cap \mathcal{O}_Y] \oplus \mathrm{Ker}(W_n(\mathcal{O}_Y)^* \longrightarrow \mathcal{O}_Y^*).$$

This equality shows (2.12.7). $\square$

Let us also consider the case of the log crystalline cohomology with compact support.

Assume that $Z = \emptyset$ for the time being. Fix a total order on λ's only in (2.12.7.1) below. In [64, Lemma 3.15.1], it is claimed that the following sequence

$$0 \longrightarrow W_n\Omega_X^\bullet(-\log D) \longrightarrow W_n\Omega_X^\bullet \longrightarrow W_n\Omega_{D^{(1)}}^\bullet \longrightarrow \cdots \tag{2.12.7.1}$$

is exact. Let R be the Cartier-Dieudonné-Raynaud algebra over κ ([52, I (1.1)]). Set $R_n := R/(V^nR + dV^nR)$. The second isomorphism

$$R_n \otimes_R^L W\Omega_X^\bullet(-\log D) \xrightarrow{\sim} W_n\Omega_X^\bullet(-\log D)$$

in [64, 1.3.3] (we have to say that the turn of the tensor product in [64, 1.3.3] is not desirable) is necessary for the proof of [64, Lemma 3.15.1]. However the proof of the second isomorphism in [64, 1.3.3] is too sketchy. In [68, §6] we have given a precise proof of the second isomorphism in [64, 1.3.3]. Hence we can use [64, Lemma 3.15.1] without anxiety, and we identify $W_n\Omega_X^\bullet(-\log D)$ with the following complex

$$\begin{aligned} W_n\Omega_X^\bullet &\longrightarrow (W_n\Omega_{D^{(1)}}^\bullet \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(1)}(D/\kappa), -d) \\ &\longrightarrow W_n\Omega_{D^{(2)}}^\bullet \otimes_{\mathbb{Z}} \varpi_{\mathrm{zar}}^{(2)}(D/\kappa) \longrightarrow \cdots \end{aligned} \tag{2.12.7.2}$$

in $\mathrm{D}^+(f^{-1}(W_n))$.

We generalize the exact sequence (2.12.7.2) to the case $Z \neq \emptyset$ as follows.

First assume that X is affine. Let $(\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$ be a formal lift of $(X, D \cup Z)$ over $\mathrm{Spf}(W)$ with a lift $\Phi\colon (\mathcal{X}, \mathcal{D} \cup \mathcal{Z}) \longrightarrow (\mathcal{X}, \mathcal{D} \cup \mathcal{Z})$ of the Frobenius of $(X, D \cup Z)$. Let $\widetilde{f}\colon \mathcal{X} \longrightarrow \mathrm{Spf}(W)$ be the structural morphism. Set $\Omega^\bullet := \Omega^\bullet_{\mathcal{X}/W}(\log(\mathcal{Z} - \mathcal{D}))$, $\Omega_1^\bullet := \Omega^\bullet/p\Omega^\bullet$ and $\phi = \Phi^*\colon \Omega^\bullet \longrightarrow \Omega^\bullet$. Then $(\Omega^\bullet, \phi)$ satisfies the axioms of (6.0.1) $\sim$ (6.0.5) in [68], that is, (2.12.7.3): $\Omega^i = 0$ for $i < 0$.

(2.12.7.4): Ω^i ($i \in \mathbb{N}$) are p-torsion-free, p-adically complete $\mathbb{Z}_p$-modules in $\mathrm{C}^+(\widetilde{f}^{-1}(\mathbb{Z}_p))$.

(2.12.7.5): $\phi(\Omega^i) \subset \{\omega \in p^i\Omega^i \mid d\omega \in p^{i+1}\Omega^{i+1}\} \quad (i \in \mathbb{N})$.

(2.12.7.6): There exists an $\mathbb{F}_p$-linear isomorphism

$$C^{-1} \colon \Omega_1^i \overset{\sim}{\longrightarrow} \mathcal{H}^i(\Omega_1^\bullet) \quad (i \in \mathbb{N}).$$

((19.7.6) is an isomorphism in [27, (4.2.1.3)].)

(2.12.7.7): A composite morphism $(\mathrm{mod}\ p) \circ p^{-i}\phi \colon \Omega^i \longrightarrow \Omega^i \longrightarrow \Omega_1^i$ factors through $\mathrm{Ker}(d \colon \Omega_1^i \longrightarrow \Omega_1^{i+1})$, and the following diagram is commutative:

$$\begin{tikzcd} \Omega^i \arrow[r, "\mathrm{mod}\ p"] \arrow[d, "p^{-i}\phi"'] & \Omega_1^i \arrow[d, "C^{-1}"] \\ \Omega^i \arrow[r, "\mathrm{mod}\ p"] & \mathcal{H}^i(\Omega_1^\bullet). \end{tikzcd}$$

Theorem 2.12.8 ([68, (6.2), (6.3), (6.4)]). (1) *For a gauge* $\epsilon \colon \mathbb{Z} \longrightarrow \mathbb{N}$ ([11, 8.7 Definition])*, let* $\eta \colon \mathbb{Z} \longrightarrow \mathbb{N}$ *be the associated cogauge to* ϵ *defined by*

$$\eta(i) := \begin{cases} \epsilon(i) + i & (i \geq 0), \\ \epsilon(0) & (i \leq 0). \end{cases}$$

Let $\Omega_\epsilon^\bullet$ (*resp.* $\Omega_\eta^\bullet$) *be the largest complex of* $\Omega^\bullet$ *whose i-th degree is contained in* $p^{\epsilon(i)}\Omega^i$ (*resp.* $p^{\eta(i)}\Omega^i$). *Then the morphism* $\phi \colon \Omega^\bullet \longrightarrow \Omega^\bullet$ *induces a quasi-isomorphism* $\phi_\epsilon \colon \Omega_\epsilon^\bullet \longrightarrow \Omega_\eta^\bullet$.

(2) *Assume that* $\Omega_\epsilon^\bullet$ *and* $\Omega_\eta^\bullet$ *are bounded above and that they consist of flat* $\mathbb{Z}_p$*-modules. Let* $\mathcal{M}$ *be an* $\widetilde{f}^{-1}(\mathbb{Z}_p) = \mathbb{Z}_p$*-module. Then the morphism*

$$\phi_\epsilon \otimes_{\mathbb{Z}_p} \mathrm{id}_{\mathcal{M}} \colon \Omega_\epsilon^\bullet \otimes_{\mathbb{Z}_p} \mathcal{M} \longrightarrow \Omega_\eta^\bullet \otimes_{\mathbb{Z}_p} \mathcal{M} \tag{2.12.8.1}$$

is a quasi-isomorphism.

(3) (cf. [52, III (1.5)]) *Let* i (*resp.* n) *be a nonnegative* (*resp. positive*) *integer. Then*

(2.12.8.2)
$$\frac{p^i\{\omega \in \Omega^i \mid d\omega \in p^{n+1}\Omega^{i+1}\}}{p^{i+n}\{\omega \in \Omega^i \mid d\omega \in p\Omega^{i+1}\} + p^i d\Omega^{i-1}} \overset{\phi}{\underset{\sim}{\longleftarrow}} \frac{\{\omega \in \Omega^i \mid d\omega \in p^n\Omega^{i+1}\}}{p^n\Omega^i + pd\Omega^{i-1}}.$$

Proof. (1): We only remark that the proof is the same as that in [11, 8.8 Theorem].

(2): By the assumption, the complex $\mathrm{MC}(\phi_\epsilon) \otimes_{\mathbb{Z}_p} \mathcal{M}$ is acyclic.

(3): Set $\mathcal{M} := \mathbb{Z}/p^n$ in (2). Let ϵ be any gauge such that $\epsilon(i-1) = 1$ and $\epsilon(i) = 0$. Then (2.12.8.1) at the degree i is equal to (2.12.8.2). $\square$

Set

(2.12.8.3)
$$Z_n^i := \{\omega \in \Omega^i \mid d\omega \in p^n\Omega^{i+1}\}, \quad B_n^i := p^n\Omega^i + d\Omega^{i-1}, \quad \mathfrak{W}_n\Omega^i := Z_n^i/B_n^i.$$

As usual (e.g., [68, §6]), we can define the following operators:

$$F\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i, \quad V\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i, \quad d\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_n\Omega^{i+1},$$

$$\mathbf{p}\colon \mathfrak{W}_n\Omega^i \longrightarrow \mathfrak{W}_{n+1}\Omega^i \quad \text{and} \quad \pi\colon \mathfrak{W}_{n+1}\Omega^i \longrightarrow \mathfrak{W}_n\Omega^i.$$

We only remark that $\mathbf{p}$ is an injective morphism induced by $p^{-(i-1)}\phi\colon \Omega^i \longrightarrow \Omega^i$ (note that $-(i-1)$ is positive if $i=0$) and that π is the following composite surjective morphism ([68, (6.5)]):

(2.12.8.4)

$$\mathfrak{W}_{n+1}\Omega^i = Z_{n+1}^i/B_{n+1}^i \xrightarrow{\text{proj.}} Z_{n+1}^i/(p^nZ_1^i + d\Omega^{i-1}) \underset{\sim}{\xrightarrow{(p^{-i}\phi)^{-1}}} \frac{Z_n^i}{p^n\Omega^i + pd\Omega^{i-1}} \xrightarrow{\text{proj.}} Z_n^i/B_n^i = \mathfrak{W}_n\Omega^i.$$

Here the isomorphism $p^{-i}\phi$ in (2.12.8.4) is given by (2.12.8.2). As usual, one can endow $\mathfrak{W}_n\Omega^i$ with a natural $W_n(\mathcal{O}_X)$-module structure, and the following formulas hold:

$$d^2 = 0,\ FdV = d,\ FV = VF = p\ ,$$
$$F\mathbf{p} = \mathbf{p}F,\ V\mathbf{p} = \mathbf{p}V,\ d\mathbf{p} = \mathbf{p}d,\ \mathbf{p}\pi = \pi\mathbf{p} = p.$$

Set $\mathfrak{W}\Omega^\bullet = \varprojlim_{\pi} \mathfrak{W}_n\Omega^\bullet$. Then $\mathfrak{W}\Omega^\bullet$ is a complex of sheaves of $W(\mathcal{O}_X)$-modules and torsion-free W-modules in $\mathrm{C}^+(\widetilde{f}^{-1}(W))$. In fact, $\mathfrak{W}\Omega^\bullet$ (resp. $\mathfrak{W}_n\Omega^\bullet$) is naturally an R-module (resp. R_n-module). Set

$$\mathrm{Fil}^r\mathfrak{W}\Omega^i := \begin{cases} \mathrm{Ker}(\mathfrak{W}\Omega^i \longrightarrow \mathfrak{W}_r\Omega^i) & (r>0), \\ \mathfrak{W}\Omega^i & (r \leq 0). \end{cases}$$

We recall the following (cf. [50, I (3.31)], [50, I (3.21.1.5)], [62, (1.20)], [52, II (1.2)], [62, (2.16)]):

Proposition 2.12.9 ([68, §6, (A), (B), (C)]). *The following formulas hold*:

(1) $\mathrm{Fil}^r\mathfrak{W}\Omega^i = V^r\mathfrak{W}\Omega^i + dV^r\mathfrak{W}\Omega^{i-1}$ $(i \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0})$.
(2) $d^{-1}(p^n\mathfrak{W}\Omega^\bullet) = F^n\mathfrak{W}\Omega^\bullet$.
(3) $R_n \otimes_R^L \mathfrak{W}\Omega^\bullet = \mathfrak{W}_n\Omega^\bullet$ $(n \in \mathbb{Z}_{>0})$.

Proof. Here we only remark that (3) is a formal consequence of (1) and (2) (see [52, II (1.2)]). □

Let us come back to the general case.

Recall the ideal sheaf $\mathcal{I}^D_{(X,D\cup Z)/S}$ of $\mathcal{O}_{(X,D\cup Z)/S}$ in §2.11. Set

$$W_n\Omega^i_X(\log(Z-D)) := R^iu_{(X,D\cup Z)/W_n*}(\mathcal{I}^D_{(X,D\cup Z)/W_n}) \quad (i\in\mathbb{N}). \tag{2.12.9.1}$$

Zariski locally on X, we have an isomorphism $W_n\Omega^i_X(\log(Z-D)) \xrightarrow{\sim} \mathfrak{W}_n\Omega^i$. It is a routine work to check that the family $\{W_n\Omega^\bullet_X(\log(Z-D))\}_{n\in\mathbb{Z}_{>0}}$ of complexes has the operators F, V, d, $\mathbf{p}$ and π (cf. [46, (4.1), (4.2)]) (especially one can check that $\mathbf{p}$ and π are well-defined by considering embedding systems of $(X, D\cup Z)$ over W); in fact, $W\Omega^\bullet_X(\log(Z-D))$ is naturally an R-module. Then the following holds:

Proposition 2.12.10. *The complex $W_n\Omega^\bullet_X(\log(Z-D))$ $(n\in\mathbb{Z}_{>0})$ is quasi-isomorphic to the single complex of the following double complex*:

$$\begin{array}{ccc}
\cdots & \longrightarrow & \cdots \\
{\scriptstyle d}\big\uparrow & & {\scriptstyle -d}\big\uparrow \\
W_n\Omega^2_X(\log Z) & \xrightarrow{\iota^{(0)*}} & W_n\Omega^2_{D^{(1)}}(\log Z|_{D^{(1)}})\otimes_{\mathbb{Z}}\varpi^{(1)}_{\mathrm{zar}}(D/\kappa) \\
{\scriptstyle d}\big\uparrow & & {\scriptstyle -d}\big\uparrow \\
W_n\Omega^1_X(\log Z) & \xrightarrow{\iota^{(0)*}} & W_n\Omega^1_{D^{(1)}}(\log Z|_{D^{(1)}})\otimes_{\mathbb{Z}}\varpi^{(1)}_{\mathrm{zar}}(D/\kappa) \\
{\scriptstyle d}\big\uparrow & & {\scriptstyle -d}\big\uparrow \\
W_n\Omega^0_X(\log Z) & \xrightarrow{\iota^{(0)*}} & W_n\Omega^0_{D^{(1)}}(\log Z|_{D^{(1)}})\otimes_{\mathbb{Z}}\varpi^{(1)}_{\mathrm{zar}}(D/\kappa)
\end{array} \tag{2.12.10.1}$$

$$\begin{array}{cccc}
\longrightarrow & \cdots & \longrightarrow & \cdots \\
& {\scriptstyle d}\big\uparrow & & \\
\xrightarrow{\iota^{(1)*}} & W_n\Omega^2_{D^{(2)}}(\log Z|_{D^{(2)}})\otimes_{\mathbb{Z}}\varpi^{(2)}_{\mathrm{zar}}(D/\kappa) & \xrightarrow{\iota^{(2)*}} & \cdots \\
& {\scriptstyle d}\big\uparrow & & \\
\xrightarrow{\iota^{(1)*}} & W_n\Omega^1_{D^{(2)}}(\log Z|_{D^{(2)}})\otimes_{\mathbb{Z}}\varpi^{(2)}_{\mathrm{zar}}(D/\kappa) & \xrightarrow{\iota^{(2)*}} & \cdots \\
& {\scriptstyle d}\big\uparrow & & \\
\xrightarrow{\iota^{(1)*}} & W_n\Omega^0_{D^{(2)}}(\log Z|_{D^{(2)}})\otimes_{\mathbb{Z}}\varpi^{(2)}_{\mathrm{zar}}(D/\kappa) & \xrightarrow{\iota^{(2)*}} & \cdots .
\end{array}$$

Proof. The proof is the same as that of [64, Lemma 3.15.1]: by using (2.12.9) (3) and Ekedahl's Nakayama duality, we can reduce the exactness to that for the case $n = 1$, and in this case, we obtain the exactness by the argument of [27, (4.2.2) (a), (c)] (cf. (2.11.5.1)). □

The complex (2.12.10.1) has a stupid filtration σ^k $(k\in\mathbb{Z})$ with respect to the columns and we set $P^{D,k}_{\mathrm{c}} := \sigma^k$. Hence we obtain a filtered complex in

$\mathrm{C}^+\mathrm{F}(f^{-1}(W_n))$, $\mathrm{K}^+\mathrm{F}(f^{-1}(W_n))$ and $\mathrm{D}^+\mathrm{F}(f^{-1}(W_n))$, and we denote it by

(2.12.10.2) $$(W_n\Omega^{\bullet}_X(\log(Z-D)), P^D_{\mathrm{c}}).$$

The following is the main result in this section:

Theorem 2.12.11 (Comparison theorem). (1) *In* $\mathrm{D}^+\mathrm{F}(f^{-1}(W_n))$*, there exists the following canonical isomorphism*:

(2.12.11.1) $$(C^{\log,Z}_{\mathrm{zar}}(\mathcal{O}_{(X,D\cup Z)/W_n}), P^D) \overset{\sim}{\longrightarrow} (W_n\Omega^{\bullet}_X(\log(D\cup Z)), P^D).$$

The isomorphisms (2.12.11.1) *for* n*'s are compatible with two projections of both hands of* (2.12.11.1).

(2) *In* $\mathrm{D}^+\mathrm{F}(f^{-1}(W_n))$*, there exists the following canonical isomorphism*:

(2.12.11.2) $$(E^{\log}_{\mathrm{zar,c}}(\mathcal{O}_{(X,D\cup Z)/W_n}), P^D_{\mathrm{c}}) \overset{\sim}{\longrightarrow} (W_n\Omega^{\bullet}_X(\log(Z-D)), P^D_{\mathrm{c}}).$$

If one forgets the filtrations of both hands of (2.12.11.2)*, one can identify the isomorphism* (2.12.11.2) *with the isomorphism*

(2.12.11.3) $$Ru_{(X,D\cup Z)/W_n*}(\mathcal{I}^D_{(X,D\cup Z)/W_n}) \overset{\sim}{\longrightarrow} W_n\Omega^{\bullet}_X(\log(Z-D))$$

induced by the isomorphism (2.12.0.2)*. The isomorphisms* (2.12.11.2) *for* n*'s are compatible with two projections of both hands of* (2.12.11.2)*. The isomorphism* (2.12.11.2) *is functorial for the commutative diagram* (2.11.18.1) *for the case* $S_0 = \mathrm{Spec}(\kappa)$*,* $S = \mathrm{Spec}(W_n)$*,* $S'_0 = \mathrm{Spec}(\kappa')$ *and* $S' = \mathrm{Spec}(W_n(\kappa'))$*, where* κ' *is a perfect field of characteristic* p*.*

Proof. (1): Let $\{X_{i_0}\}_{i_0\in I_0}$ be an affine open covering of X. Set $D_{i_0} := D\cap X_{i_0}$ and $Z_{i_0} := Z\cap X_{i_0}$. Then there exists an affine formal log scheme $(\mathcal{X}_{i_0}, \mathcal{D}_{i_0}\cup \mathcal{Z}_{i_0})_{i_0\in I_0}$ over $\mathrm{Spf}(W)$ such that each $(\mathcal{X}_{i_0}, \mathcal{D}_{i_0}\cup\mathcal{Z}_{i_0})$ is a lift of the log scheme $(X_{i_0}, D_{i_0}\cup Z_{i_0})$. The Frobenius morphism $(X_{i_0}, D_{i_0}\cup Z_{i_0}) \longrightarrow (X_{i_0}, D_{i_0}\cup Z_{i_0})$ lifts to a morphism $\Phi_{i_0}\colon (\mathcal{X}_{i_0}, \mathcal{D}_{i_0}\cup\mathcal{Z}_{i_0}) \longrightarrow (\mathcal{X}_{i_0}, \mathcal{D}_{i_0}\cup\mathcal{Z}_{i_0})$. Then, using $\{(\mathcal{X}_{i_0}, \mathcal{D}_{i_0}\cup\mathcal{Z}_{i_0})\}_{i_0\in I_0}$, we have a diagram of log schemes $(\mathcal{X}_{\bullet}, \mathcal{D}_{\bullet}\cup\mathcal{Z}_{\bullet})_{\bullet\in I}$ over $\mathrm{Spf}(W)$ as in §2.4. Using $\{\Phi_{i_0}\}_{i_0\in I_0}$, we have an endomorphism $\Phi_{\bullet}\colon (\mathcal{X}_{\bullet}, \mathcal{D}_{\bullet}\cup \mathcal{Z}_{\bullet}) \longrightarrow (\mathcal{X}_{\bullet}, \mathcal{D}_{\bullet}\cup\mathcal{Z}_{\bullet})$ of a diagram of log schemes; Φ_i is a lift of the Frobenius of $(\mathcal{X}_i, \mathcal{D}_i\cup\mathcal{Z}_i)\otimes_W \kappa$ $(i\in I)$. Let $\mathfrak{D}_{X_{\bullet}}(\mathcal{X}_{\bullet})$ be the PD-envelope of the locally closed immersion $X_{\bullet} \overset{\subset}{\longrightarrow} \mathcal{X}_{\bullet}$ over $(\mathrm{Spec}(W_n), pW_n, [\;])$. Then the morphism $\Phi_{\bullet}$ induces a natural morphism $\mathfrak{D}_{X_{\bullet}}(\mathcal{X}_{\bullet}) \longrightarrow \mathfrak{D}_{X_{\bullet}}(\mathcal{X}_{\bullet})$.

Set $(\mathcal{X}_{\bullet,n}, \mathcal{D}_{\bullet,n}\cup\mathcal{Z}_{\bullet,n})_{\bullet\in I} := (\mathcal{X}_{\bullet}\otimes_W W_n, (\mathcal{D}_{\bullet}\otimes_W W_n)\cup(\mathcal{Z}_{\bullet}\otimes_W W_n))_{\bullet\in I}$ and set $\Phi_{\bullet,n} := \Phi_{\bullet} \bmod p^n$. Then there exists a morphism $(W_n(X_{\bullet}), W_n(D_{\bullet})\cup W_n(Z_{\bullet})) \longrightarrow (\mathcal{X}_{\bullet,n}, \mathcal{D}_{\bullet,n}\cup\mathcal{Z}_{\bullet,n})$ of diagrams of log schemes, where $(W_n(X_{\bullet}), W_n(D_{\bullet})\cup W_n(Z_{\bullet}))$ is a log scheme defined in (2.12.7). By (2.12.4.7), this morphism induces a morphism

(2.12.11.4)
$$\mathcal{O}_{\mathfrak{D}_{X_{\bullet}}(\mathcal{X}_{\bullet,n})}\otimes_{\mathcal{O}_{\mathcal{X}_{\bullet,n}}}\Omega^{\bullet}_{\mathcal{X}_{\bullet,n}/W_n}(\log(\mathcal{D}_{\bullet,n}\cup\mathcal{Z}_{\bullet,n})) \longrightarrow W_n\Omega^{\bullet}_{X_{\bullet}}(\log(D_{\bullet}\cup Z_{\bullet})).$$

(Note that $(W_n(X_i), W_n(D_i) \cup W_n(Z_i))$ is the canonical lift of $(X_i, D_i \cup Z_i)$ over W_n by (2.12.7); thus, by applying the filtered higher direct image of the natural morphism $\pi_{\mathrm{zar}} \colon (\widetilde{X}_{\bullet\mathrm{zar}}, f_\bullet^{-1}(W_n)) \longrightarrow (\widetilde{X}_{\mathrm{zar}}, f^{-1}(W_n))$ to the morphism in (2.12.11.4), we obtain a morphism which is equal to a special case of a morphism defined in [46, (4.19)].)

The morphism (2.12.11.4) induces a filtered quasi-isomorphism with respect to preweight filtrations. Indeed, the problem is local; in this case, it follows from (2.12.6). By applying the filtered higher direct image of π_{zar} to (2.12.11.4), we have an isomorphism (2.12.11.1). As in the proof of (2.6.1), we can check that the morphism (2.12.11.1) is independent of the choice of the open covering of X and the lift of each open scheme.

Let $g \colon (X_1, D_1 \cup Z_1) \longrightarrow (X_2, D_2 \cup Z_2)$ be a morphism of smooth schemes with SNCD's over κ which induces morphisms $(X_1, D_1) \longrightarrow (X_2, D_2)$ and $(X_1, Z_1) \longrightarrow (X_2, Z_2)$. Then, by the proof of [68, (9.3) (2)], g induces a morphism

$$g^* \colon (W_n\Omega^i_{X_2}(\log(D_2 \cup Z_2)), P^{D_2}) \longrightarrow (W_n\Omega^i_{X_1}(\log(D_1 \cup Z_1)), P^{D_1}).$$

Using the diagram of log schemes, we see that the proof of the functoriality of (2.12.11.1) is reduced to the local question on $(X_i, D_i \cup Z_i)$ $(i = 1, 2)$. In this case, by the functoriality of the morphisms (2.12.4.3) and (2.12.4.6) and by (2.12.6), we obtain the functoriality of (2.12.11.1).

In [68, (7.18)] we have proved that the morphism (2.12.4.6) is compatible with two projections; as a result, the morphism (2.12.4.7) is also compatible with them. In particular, we have the following commutative diagram

$$\begin{array}{ccc}
(\mathcal{O}_{\mathfrak{D}_{X_\bullet}(\mathcal{X}_{\bullet,n+1})} \otimes_{\mathcal{O}_{\mathcal{X}_{\bullet,n+1}}} \Omega^\bullet_{\mathcal{X}_{\bullet,n+1}/W_{n+1}}(\log(\mathcal{D}_{\bullet,n+1} \cup \mathcal{Z}_{\bullet,n+1})), P^D) & \xrightarrow{(2.12.11.4)} & (W_{n+1}\Omega^\bullet_{X_\bullet}(\log(D_\bullet \cup Z_\bullet)), P^D) \\
\Big\downarrow{\scriptstyle \mathrm{proj.}} & & \Big\downarrow{\scriptstyle \pi} \\
(\mathcal{O}_{\mathfrak{D}_{X_\bullet}(\mathcal{X}_{\bullet,n})} \otimes_{\mathcal{O}_{\mathcal{X}_{\bullet,n}}} \Omega^\bullet_{\mathcal{X}_{\bullet,n}/W_n}(\log(\mathcal{D}_{\bullet,n} \cup \mathcal{Z}_{\bullet,n})), P^D) & \xrightarrow{(2.12.11.4)} & (W_n\Omega^\bullet_{X_\bullet}(\log(D_\bullet \cup Z_\bullet)), P^D).
\end{array}$$

Applying the direct image $R\pi_{\mathrm{zar}*}$, we obtain the compatibility with two projections.

(2): The morphism (2.12.11.4) induces a morphism

(2.12.11.5)
$$\mathcal{O}_{\mathfrak{D}_{X_\bullet}(\mathcal{X}_{\bullet,n})} \otimes_{\mathcal{O}_{\mathcal{X}_{\bullet,n}}} \Omega^\bullet_{\mathcal{X}_{\bullet,n}/W_n}(\log(\mathcal{Z}_{\bullet,n} - \mathcal{D}_{\bullet,n})) \longrightarrow W_n\Omega^\bullet_{X_\bullet}(\log(Z_\bullet - D_\bullet)).$$

By (2.2.16) (2), $\mathfrak{D}_{X_\bullet}(\mathcal{X}_\bullet) \times_{\mathcal{X}_\bullet} \mathcal{D}^{(k)}$ is the PD-envelope of the locally closed immersion $D^{(k)}_\bullet \xrightarrow{\subset} \mathcal{D}^{(k)}_\bullet$. Set $\mathfrak{D}_{D^{(k)}_\bullet}(\mathcal{D}^{(k)}_{\bullet,n}) := (\mathfrak{D}_{X_\bullet}(\mathcal{X}_\bullet) \times_{\mathcal{X}_\bullet} \mathcal{D}^{(k)}_\bullet) \otimes_W W_n$. By (2.11.9) and (2.12.10), we have the following commutative diagram:

$$
(2.12.11.6)\quad
\begin{array}{c}
R\pi_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_{X_\bullet}(\mathcal{X}_{\bullet,n})}\otimes_{\mathcal{O}_{\mathcal{X}_{\bullet,n}}}\Omega^\bullet_{\mathcal{X}_{\bullet,n}/W_n}(\log(\mathcal{Z}_{\bullet,n}-\mathcal{D}_{\bullet,n}))) \xrightarrow{\sim} \\
\big\downarrow \\
R\pi_{\mathrm{zar}*}(W_n\Omega^\bullet_{X_\bullet}(\log(Z_\bullet - D_\bullet))) \xrightarrow{\sim} \\
(R\pi_{\mathrm{zar}*}(\mathcal{O}_{\mathfrak{D}_{D^{(0)}_\bullet}(\mathcal{D}^{(0)}_{\bullet,n})}\otimes_{\mathcal{O}_{\mathcal{D}^{(0)}_{\bullet,n}}}\Omega^\bullet_{\mathcal{D}^{(0)}_{\bullet,n}/W_n}(\log \mathcal{Z}_{\bullet,n}|_{\mathcal{D}^{(0)}_{\bullet,n}}))\otimes_{\mathbb{Z}}\varpi^{(0)}_{\mathrm{zar}}(D/\kappa)\longrightarrow\cdots) \\
\big\downarrow \\
(R\pi_{\mathrm{zar}*}(W_n\Omega^\bullet_{X_\bullet}(\log Z_\bullet))\otimes_{\mathbb{Z}}\varpi^{(0)}_{\mathrm{zar}}(D/\kappa)\longrightarrow\cdots).
\end{array}
$$

By the cohomological descent, the lower vertical morphism in (2.12.11.6) is equal to

$$
\{Ru_{(X,Z)/W_n*}(\mathcal{O}_{(X,Z)/W_n})\otimes_{\mathbb{Z}}\varpi^{(0)}_{\mathrm{zar}}(D/\kappa)\longrightarrow\cdots\}\longrightarrow
$$
$$
\{W_n\Omega^\bullet_X(\log Z)\otimes_{\mathbb{Z}}\varpi^{(0)}_{\mathrm{zar}}(D/\kappa)\longrightarrow\cdots\}.
$$

By [46, (4.19)]=[68, (7.19)], this is an isomorphism. The claim as to the compatibility of the filtrations is obvious by the definitions. As usual (cf. [50, II (1.1)], §2.5), we see that the lower vertical morphism in (2.12.11.6) is independent of the choice of the open covering of X, that of the lift of each open subscheme and that of the lift of the Frobenius.

The compatibility with respect to two projections follows from the following commutative diagram:

$$
\begin{array}{ccc}
Ru_{(D^{(\bullet)},Z|_{D^{(\bullet)}})/W_{n+1}}(\mathcal{O}_{(D^{(\bullet)},Z|_{D^{(\bullet)}})/W_{n+1}}) & \xrightarrow{\sim} & W_{n+1}\Omega^\bullet_{D^{(\bullet)}}(\log Z|_{D^{(\bullet)}}) \\
{\scriptstyle\mathrm{proj.}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\
Ru_{(D^{(\bullet)},Z|_{D^{(\bullet)}})/W_{n}}(\mathcal{O}_{(D^{(\bullet)},Z|_{D^{(\bullet)}})/W_{n}}) & \xrightarrow{\sim} & W_{n}\Omega^\bullet_{D^{(\bullet)}}(\log Z|_{D^{(\bullet)}}),
\end{array}
$$

which we can prove in the same way as [46, (4.19)]=[68, (7.19)].

The functoriality claimed in (2) is obvious by the proof above. □

Let i be a nonnegative integer. We conclude this section by constructing the preweight spectral sequences of $W_n\Omega^i_X(\log(D\cup Z))$ and $W_n\Omega^i_X(\log(Z\text{–}D))$ with respect to D and describing the boundary morphisms between the E_1-terms of the spectral sequences.

The following is a generalization of [68, (5.7.1;n)]:

Proposition 2.12.12. *Let i be a nonnegative integer. Then there exists the following spectral sequence*

$$
(2.12.12.1)\quad E_1^{-k,h+k} = H^{h-i}(D^{(k)}, W_n\Omega^{i-k}_{D^{(k)}}(\log Z|_{D^{(k)}})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(D/\kappa))(-k)
$$
$$
\Longrightarrow H^{h-i}(X, W_n\Omega^i_X(\log(D\cup Z))).
$$

The spectral sequences (2.12.12.1) *for n's are compatible with the projections.*

Proof. (2.12.12.1) immediately follows from (2.12.4.2). The compatibility with the projection immediately follows from the same proof as that of [68, (8.4) (2)]. $\square$

Next we describe the boundary morphism between the E_1-terms of the spectral sequence (2.12.12.1).

Let the notations be before (2.8.5). Consider the following exact sequence

(2.12.12.2)
$$0 \longrightarrow W_n\Omega^i_{D_{\underline{\lambda}_j}}(\log Z|_{D_{\underline{\lambda}_j}}) \longrightarrow W_n\Omega^i_{D_{\underline{\lambda}_j}}(\log(Z \cup D_{\underline{\lambda}}))$$
$$\xrightarrow{\mathrm{Res}^{D_{\underline{\lambda}}}} \iota^{\lambda_j}_{\underline{\lambda}*}(W_n\Omega^{i-1}_{D_{\underline{\lambda}}}(\log Z|_{D_{\underline{\lambda}}}))(-1) \longrightarrow 0.$$

We have the boundary morphism

(2.12.12.3)
$$-G^{\lambda_j}_{\underline{\lambda}} : \iota^{\lambda_j \log}_{\underline{\lambda}*} W_n\Omega^{i-1}_{D_{\underline{\lambda}}}(\log Z_{D_{\underline{\lambda}}})(-1) \longrightarrow W_n\Omega^i_{D_{\underline{\lambda}_j}}(\log Z|_{D_{\underline{\lambda}_j}})[1].$$

of (2.12.12.2). Here we have used the Convention (4). As in (2.8.4.5), the morphism (2.12.12.3) induces the following morphism

(2.12.12.4)
$$(-1)^j G^{\lambda_j}_{\underline{\lambda}} : H^{h-i}(D_{\underline{\lambda}}, W_n\Omega^{i-k}(\log Z|_{D_{\underline{\lambda}}}) \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda},\mathrm{zar}}(D/\kappa))(-k) \longrightarrow$$
$$H^{h-i+1}(D_{\underline{\lambda}_j}, W_n\Omega^{i+1-k}(\log Z|_{D_{\underline{\lambda}_j}}) \otimes_{\mathbb{Z}} \varpi^{\log}_{\underline{\lambda}_j,\mathrm{zar}}(D/\kappa))(-(k-1)).$$

Definition 2.12.13. We call the morphism (2.12.12.4) the *Gysin morphism in log Hodge-Witt cohomologies* associated to the closed immersion $(D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}}) \xrightarrow{\subset} (D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}})$.

Proposition 2.12.14. *Set* $G := \sum_{\{\lambda_0,\ldots,\lambda_{k-1} \mid \lambda_i \neq \lambda_j \ (i \neq j)\}} \sum_{j=0}^{k-1} (-1)^j G^{\lambda_j}_{\underline{\lambda}}$. *Then the boundary morphism* $d_1^{-k,h+k} \colon E_1^{-k,h+k} \longrightarrow E_1^{-k+1,h+k}$ *of* (2.12.12.1) *is equal to* $-G$.

Proof. The proof is the same as that of (2.8.5). $\square$

Proposition 2.12.15. *Let i be a nonnegative integer. Then there exists the following spectral sequence*

(2.12.15.1)
$$E_1^{k,h-k} = H^{h-i-k}(D^{(k)}, W_n\Omega^i_{D^{(k)}}(\log Z|_{D^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{zar}}(D/\kappa))$$
$$\Longrightarrow H^{h-i}(X, W_n\Omega^i_X(\log(Z-D))).$$

The spectral sequences (2.12.15.1) *for n's are compatible with the projections. The boundary morphism*

(2.12.15.2) $d_1^{k,h-k}\colon H^{h-i-k}(D^{(k)}, W_n\Omega^i_{D^{(k)}}(\log Z|_{D^{(k)}}) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm zar}(D/\kappa)) \longrightarrow$

$$H^{h-i-k}(D^{(k+1)}, W_n\Omega^i_{D^{(k+1)}}(\log Z|_{D^{(k+1)}}) \otimes_{\mathbb{Z}} \varpi^{(k+1)}_{\rm zar}(D/\kappa))$$

is equal to $\iota^{(k)*}$.

Proof. (2.12.15) immediately follows from (2.12.10.1). (The compatibility with the projection is easy to check.) □

Remark 2.12.16. If X is proper over κ and if $Z = \emptyset$, the first-named author has proved the E_2-degeneration of the following spectral sequences modulo torsion ([68, (5.9)]):

$$\begin{aligned} E_1^{-k,h+k} &= H^{h-i}(D^{(k)}, W\Omega^{i-k}_{D^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm zar}(D/\kappa))(-k) \\ &\Longrightarrow H^{h-i}(X, W\Omega^i_X(\log D)), \end{aligned}$$

$$\begin{aligned} E_1^{k,h-k} &= H^{h-i-k}(D^{(k)}, W\Omega^{i}_{D^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm zar}(D/\kappa)) \\ &\Longrightarrow H^{h-i}(X, W\Omega^i_X(-\log D)). \end{aligned}$$

2.13 Filtered Convergent F-isocrystal

So far we have worked over a base scheme whose structure sheaf is killed by a power of p. We can also work over a (not necessarily affine) P-adic base in the sense of [11, 7.17 Definition], and the analogues of results in previous sections hold in this case.

Let V be a complete discrete valuation ring of mixed characteristics with perfect residue field κ of characteristics $p > 0$. Let W be the Witt ring of κ with fraction field K_0. Let K be the fraction field of V. For a V-module M, M_K denotes the tensor product $M \otimes_V K$. Unless otherwise stated, from this section to §2.19, S denotes a p-adic formal V-scheme in the sense of [74, §1], i.e., S is a noetherian formal scheme over V with the p-adic topology such that, for any affine open formal subscheme U, there exists a surjective morphism $V\{x_1, \ldots, x_n\} \longrightarrow \Gamma(U, \mathcal{O}_U)$ of topological rings for some n. Let $f\colon (X, D \cup Z) \longrightarrow S$ denote a proper smooth morphism of p-adic formal V-schemes (e.g., V/p-schemes) of finite type with relative transversal SNCD. Following [74], for a p-adic formal scheme $T/\mathrm{Spf}(V)$, set $T_1 := \underline{\mathrm{Spec}}_T(\mathcal{O}_T/p\mathcal{O}_T)$.

By virtue of results in previous sections, we can give the compatibility of the weight filtrations on log crystalline cohomologies as convergent F-isocrystals with some canonical operations, e.g., the base change, the Künneth formula, the functoriality. Later, in §2.19, we shall give the compatibility of them with the Poincaré duality.

(1) Base change theorem

Theorem 2.13.1. *Let k, h be two nonnegative integers. Then there exists a convergent F-isocrystal E_k^h on S/V such that*

$$(E_k^h)_T = R^h f_{(X_{T_1},Z_{T_1})/T*}(P_k^{D_{T_1}} E_{\rm crys}^{\log, Z_{T_1}}(\mathcal{O}_{(X_{T_1},D_{T_1}\cup Z_{T_1})/T}))_K$$

for any p-adic enlargement T of S/V. In particular, there exists a convergent F-isocrystal $R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$ on S/V such that

$$R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})_T = R^h f_{(X_{T_1},D_{T_1}\cup Z_{T_1})/T*}(\mathcal{O}_{(X_{T_1},D_{T_1}\cup Z_{T_1})/T})_K$$

for any p-adic enlargement T of S/V.

Proof. The base change theorem (2.10.3) and the argument in [74, (3.1)] show the existence of a p-adically convergent isocrystal E_k^h.

As in the proof of [74, (3.7)], we may assume that $V = W$; furthermore, by the log version of [74, (3.4)], we may assume that $p\mathcal{O}_S = 0$. The spectral sequence in (2.9.6.3) for

$$R^h f_{(X_{T_1},Z_{T_1})/T*}(P_k^{D_{T_1}} E_{\rm crys}^{\log, Z_{T_1}}(\mathcal{O}_{(X_{T_1},D_{T_1}\cup Z_{T_1})/T}))$$

shows that the Frobenius action $F_S^*(E_k^h) \longrightarrow E_k^h$ is an isomorphism. Thus E_k^h prolongs to a convergent F-isocrystal as in [74, (3.7)]. $\square$

Remark 2.13.2. The existence of the convergent F-isocrystal $R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$ is a special case of [76, Theorem 4] and [29, §2 (e), (f)]. This existence also follows from the log base change theorem ([54, (6.10)]), the bijectivity of the Frobenius [46, (2.24)], and the same proof of [74, (3.1), (3.7)].

Corollary 2.13.3. *The weight filtration on $R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$ with respect to D is a convergent F-isocrystal on S/V. That is, the image $P_k^D R^h f_*(\mathcal{O}_{(X,D\cup Z)/K}) := \mathrm{Im}(E_k^h \longrightarrow R^h f_*(\mathcal{O}_{(X,D\cup Z)/K}))$ $(k \in \mathbb{N})$ is a convergent F-isocrystal.*

Proof. The category of the convergent isocrystals on S/V is abelian ([74, (2.10)]); hence the image $\mathrm{Im}(E_k^h \longrightarrow R^h f_*(\mathcal{O}_{(X,D\cup Z)/K}))$ is a convergent isocrystal.

Now, by [74, (2.18), (2.21)], we have only to prove that $P_k^D R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$ gives a p-adically convergent F-isocrystal for the case $V = W$. The existence of the Frobenius on $P_k^D R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$ is clear by the functoriality which will be stated in (2.13.9) below soon. Because the Frobenius F's on the E_1-terms of (2.9.6.3)$\otimes_V K$ for a p-adic formal V-scheme T are isomorphisms, the Frobenius on $P_k^D R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})$ is also an isomorphism. This completes the proof of (2.13.3). $\square$

Remark 2.13.4. We can also develop theory of weight filtrations by virtue of theory of log convergent topoi ([82]). See [73] for details.

Corollary 2.13.5. *Let k, h be two nonnegative integers. For any p-adic enlargement T of S/V,*

$$P_k^{D_{T_1}} R^h f_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T*}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T})_K := \tag{2.13.5.1}$$

$$\mathrm{Im}(R^h f_{(X_{T_1}, Z_{T_1})/T*}(P_k^{D_{T_1}} E_{\mathrm{crys}}^{\log, Z_{T_1}}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T}))_K \longrightarrow$$
$$R^h f_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T*}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T})_K)$$

is a flat $\mathcal{O}_T \otimes_V K$-module.

Proof. (2.13.5) follows from [74, (2.9)] and (2.13.3). □

Remark 2.13.6. The flatness of $R^h f_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T*}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T})_K$ is a special case of [76, Lemma 36] and [29, §2 (e), (f)].

(2) Künneth formula

Theorem 2.13.7. *Let $(X_j, D_j \cup Z_j)$ $(j = 1, 2)$ be a log scheme stated in the beginning of this section. Let $(X_3, D_3 \cup Z_3)$ be the product $(X_1, D_1 \cup Z_1) \times_S (X_2, D_2 \cup Z_2)$ in the category of fine log schemes. Then the there exists the following canonical isomorphism*

$$\bigoplus_{i+j=h} R^i f_*(\mathcal{O}_{(X_1, D_1 \cup Z_1)/K}) \otimes_{\mathcal{O}_{S/K}} R^j f_*(\mathcal{O}_{(X_2, D_2 \cup Z_2)/K}) \tag{2.13.7.1}$$
$$\longrightarrow R^h f_*(\mathcal{O}_{(X_3, D_3 \cup Z_3)/K})$$

of convergent F-isocrystals on S/V which is compatible with the weight filtrations with respect to D_1, D_2 and D_3.

Proof. The existence of the canonical isomorphism in (2.13.7.1) as weight-filtered convergent F-isocrystals immediately follows from (2.10.15). □

(3) Log crystalline cohomology sheaf with compact support

Using (2.11.11) and (2.11.19), we obtain the following as in **(1)** and **(2)**.

Theorem 2.13.8. *Let k, h be two nonnegative integers.*

(1) *There exists a convergent F-isocrystal $E_{k,\mathrm{c}}^h$ on S/V such that*

$$(E_{k,\mathrm{c}}^h)_T = P_k^{D_{T_1}} R^h f_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T*,\mathrm{c}}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1}; Z_{T_1})/T})_K$$

for any p-adic enlargement T of S/V. In particular, there exists a convergent F-isocrystal $R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X, D \cup Z; Z)/K})$ on S/V such that

$$R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X, D \cup Z; Z)/K})_T = R^h f_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T*,\mathrm{c}}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1}; Z_{T_1})/T})_K$$

for any p-adic enlargement T of S/V.

(2) *The* $\mathcal{O}_T \otimes_V K$*-module*

$$P_k^{D_{T_1}} R^h f_{(X_{T_1}, D_{T_1} \cup Z_{T_1})/T*, \mathrm{c}}(\mathcal{O}_{(X_{T_1}, D_{T_1} \cup Z_{T_1}; Z_{T_1})/T})_K$$

is flat for any p*-adic enlargement* T *of* S/V.

(3) *Let* $(X_j, D_j \cup Z_j)$ $(j = 1, 2)$ *be as in* (2.13.7). *Then there exists the following canonical isomorphism*

$$\bigoplus_{i+j=h} R^i f_{*,\mathrm{c}}(\mathcal{O}_{(X_1, D_1 \cup Z_1; Z_1)/K}) \otimes_{\mathcal{O}_{S/K}} R^j f_{*,\mathrm{c}}(\mathcal{O}_{(X_2, D_2 \cup Z_2; Z_2)/K})$$

$$\overset{\sim}{\longrightarrow} R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X_3, D_3 \cup Z_3; Z_3)/K})$$

of convergent F*-isocrystals on* S/V *which is compatible with the weight filtrations with respect to* D_1, D_2 *and* D_3.

(4) Functoriality

Theorem 2.13.9. *Let* $f\colon (X, D \cup Z) \longrightarrow S$ *be as in the beginning of this section. Let* k, h *be nonnegative integers. Then the following hold*:

(1) *The convergent* F*-isocrystal* $P_k^D R^h f_*(\mathcal{O}_{(X, D\cup Z)/K})$ $(k \in \mathbb{Z})$ *is functorial.*

(2) *The convergent* F*-isocrystal* $P_k^D R^h f_{\mathrm{c}*}(\mathcal{O}_{(X, D\cup Z; Z)/K})$ $(k \in \mathbb{Z})$ *is functorial with respect to the obvious analogue of the morphism in* (2.11.18).

Proof. (1) and (2) immediately follow from (2.9.1) and (2.11.18), respectively. □

(5) Gysin morphisms

Proposition 2.13.10. *The Gysin morphism* (2.8.4.5) *induces the following morphism*

(2.13.10.1) $$(-1)^j G_{\underline{\lambda}}^{\lambda_j} : R^{h-k} f_*(\mathcal{O}_{(D_{\underline{\lambda}}, Z|_{D_{\underline{\lambda}}})/K} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}}^{\log}(D/K; Z))(-k) \longrightarrow$$

$$R^{h-k+2} f_*(\mathcal{O}_{(D_{\underline{\lambda}_j}, Z|_{D_{\underline{\lambda}_j}})/K} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}_j}^{\log}(D/K; Z))(-(k-1)).$$

of convergent F*-isocrystals on* S/V. *Here* $R^h f_*(\mathcal{O}_{(D_{\underline{\lambda}}, Z_{\underline{\lambda}}|_{D_{\underline{\lambda}}})/K} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}}^{\log}(D/K; Z))$ *is a convergent* F*-isocrystal on* S/V *such that* $R^h f_*(\mathcal{O}_{(D_{\underline{\lambda}}, Z_{\underline{\lambda}}|_{D_{\underline{\lambda}}})/K} \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}}^{\log}(D/K; Z))_T = R^h f_{X_{T_1}/T*}(\mathcal{O}_{((D_{\underline{\lambda}})_{T_1}, (Z_{\underline{\lambda}})_{T_1}|_{(D_{\underline{\lambda}})_{T_1}}/T}) \otimes_{\mathbb{Z}} \varpi_{\underline{\lambda}\mathrm{crys}}^{\log}(D_{T_1}/T; Z_{T_1}))$ *for a* p*-adic enlargement* T *of* S/V.

Proof. (2.13.10) immediately follows from (2.8.4). □

Using **(1), (3), (4)** and **(5)**, we obtain the following:

Theorem 2.13.11. *Let* $R^h f_*(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/K} \otimes_{\mathbb{Z}} \varpi^{(k)\log}(D/K;Z))$ *be a convergent* F*-isocrystal on* S/V *such that*

$$R^h f_*(\mathcal{O}_{D^{(k)}/K} \otimes_{\mathbb{Z}} \varpi^{(k)\log}(D/K;Z))_T =$$

$$R^h f_{X_{T_1}/T*}(\mathcal{O}_{(D^{(k)}_{T_1},Z|_{D^{(k)}_{T_1}})/T} \otimes_{\mathbb{Z}} \varpi^{(k)\log}_{\rm crys}(D_{T_1}/T;Z_{T_1}))$$

for any p*-adic enlargement* T *of* S/V*. Then the following hold*:

(1) *There exist the following weight spectral sequences of convergent* F*-isocrystals*

(2.13.11.1)
$$\begin{aligned} &E_1^{-k,h+k}((X,D\cup Z)/K) \\ &= R^{h-k} f_*(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/K} \otimes_{\mathbb{Z}} \varpi^{(k)\log}(D/K;Z))(-k) \\ &\Longrightarrow R^h f_*(\mathcal{O}_{(X,D\cup Z)/K}), \end{aligned}$$

(2.13.11.2)
$$\begin{aligned} &E_{1,{\rm c}}^{k,h-k}((X,D\cup Z)/K) \\ &= R^{h-k} f_*(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})/K} \otimes_{\mathbb{Z}} \varpi^{(k)\log}(D/K;Z)) \\ &\Longrightarrow R^h f_{*,{\rm c}}(\mathcal{O}_{(X,D\cup Z;Z)/K}). \end{aligned}$$

The boundary morphism of (2.13.11.1) (*resp.* (2.13.11.2)) *is given by* $-G$ (*resp.* $\iota^{(k)*}$) *induced by the morphism in* (2.8.5) (*resp.* (2.11.1.3)).

(2) *The spectral sequences* (2.13.11.1) *and* (2.13.11.2) *are functorial with respect to the obvious analogue of the morphism in* (2.9.0.1) *and* (2.11.18), *respectively.*

Proof. (1): (1) follows from (2.9.6.2) and (2.11.14.3).
(2): Obvious. □

Definition 2.13.12. In the case $Z=\emptyset$, we call (2.13.11.1) (resp. (2.13.11.2)) the *p-adic weight spectral sequence* of $R^h f_*(\mathcal{O}_{(X,D)/K})$ (resp. $R^h f_{*,{\rm c}}(\mathcal{O}_{(X,D)/K})$).

2.14 Specialization Argument in Log Crystalline Cohomology

Let us recall a specialization argument of Deligne-Illusie in log crystalline cohomologies (cf. [49, (3.10)], [68, §3]) for later sections §2.15 and §2.18.

Let p be a prime number. Let $\overset{\circ}{T}$ be a noetherian formal scheme with an ideal sheaf of definition $a\mathcal{O}_T$, where a is a global section of $\Gamma(\overset{\circ}{T}, \mathcal{O}_T)$. Assume that there exists a positive integer n such that $p\mathcal{O}_T = a^n\mathcal{O}_T$. Let T be a fine formal log scheme with underlying formal scheme $\overset{\circ}{T}$. Assume that $\mathcal{O}_T$ is a-torsion-free, that is, the endomorphism $a \times \mathrm{id}_{\mathcal{O}_T} \in \mathrm{End}_{\mathcal{O}_T}(\mathcal{O}_T)$ is injective, and that the ideal sheaf $a\mathcal{O}_T$ has a PD-structure γ. We call $T = (T, a\mathcal{O}_T, \gamma)$ above an *adic fine formal log PD-scheme*. We define the notion of a morphism $g' \colon T' \longrightarrow T$ of adic fine formal log PD-schemes in the following way: the morphism g' is nothing but a morphism of formal fine log PD-schemes, and T' is a'-adically complete and separated and a'-torsion-free, where $a' := g'^*(a)$. In this section we assume that, for each affine open set $\mathrm{Spf}(R)$ of T, aR is a prime ideal and that the localization ring R_a at the ideal aR is a discrete valuation ring.

Let $\mathcal{H}$ be an $\mathcal{O}_T$-module of finite type. Since R_a is a PID, there exists a non-empty open log formal subscheme T' of T such that there exists an isomorphism $\mathcal{H}|_{T'} \simeq \mathcal{O}_{T'}^r \oplus \mathcal{H}_{\mathrm{tor}}$, where $\mathcal{H}_{\mathrm{tor}}$ is a direct sum of $\mathcal{O}_{T'}$-modules $\mathcal{O}_{T'}/a^e$ ($e \in \mathbb{Z}_{>0}$) (Deligne's remark ([49, (3.10)])). Let $\mathcal{E}$ be an a-torsion-free $\mathcal{O}_T$-module. Then, as in [68, (3.1)], it is easy to see that

$$\mathcal{T}or_r^{\mathcal{O}_{T'}}(\mathcal{H}|_{T'}, \mathcal{E}|_{T'}) = 0 \quad (\forall r \in \mathbb{Z}_{>0}) \tag{2.14.0.1}$$

and

$$\mathcal{T}or_r^{g^{-1}(\mathcal{O}_{T'})}(g^{-1}(\mathcal{H}|_{T'}), \mathcal{O}_{T''}) = 0 \quad (\forall r \in \mathbb{Z}_{>0}) \tag{2.14.0.2}$$

for any morphism $g \colon T'' \longrightarrow T'$ of adic fine formal log PD-schemes.

Set $T_1 := \underline{\mathrm{Spec}}_T(\mathcal{O}_T/a)$, and set $T'_1 := \underline{\mathrm{Spec}}_{T'}(\mathcal{O}_{T'}/a)$ for an open log formal subscheme T' of T. Let $f \colon X \longrightarrow T_1$ be a proper log smooth integral morphism. By the finiteness of log crystalline cohomologies (cf. [11, 7.24 Theorem]), there exists a non-empty open log formal subscheme T' of T such that

$$\mathcal{T}or_r^{\mathcal{O}_{T'}}(R^h f_{X_{T'_1}/T'*}(\mathcal{O}_{X_{T'_1}/T'}), \mathcal{E}|_{T'}) = 0 \quad (\forall r \in \mathbb{Z}_{>0}) \tag{2.14.0.3}$$

for any a-torsion-free $\mathcal{O}_T$-module $\mathcal{E}$ and for any $h \in \mathbb{Z}$. Assume furthermore that the log structures on X, T are fs. Let $\mathcal{I}_{X/T}$ be the ideal sheaf on $\mathcal{O}_{X/T}$ defined in [85, §5]. (In [85, §5], $\mathcal{I}_{X/T}$ is defined under the condition that $\overset{\circ}{T}$ is equal to $\mathrm{Spec}\, W_m(\kappa)$ (κ is a perfect field of characteristic $p > 0$), the log structure on T is associated to the morphism $\mathbb{N} \ni 1 \mapsto b \in W_m(\kappa)$ for some b and that the morphism f is universally saturated. However, for the definition of $\mathcal{I}_{X/T}$, we do not need these assumptions.) Set $R^h f_{X/T*,\mathrm{c}}(\mathcal{O}_{X/T}) := R^h f_{X/T*}(\mathcal{I}_{X/T})$. One can see that $\mathcal{I}_{X/S}$ is a crystal on the restricted log crystalline site $(X/T)^{\log}_{\mathrm{Rcrys}}$ as in [85, (5.3)] and that, for any log smooth integral lift $\mathcal{X} \longrightarrow T$ of f, the sheaf $(\mathcal{I}_{X/T})_{\mathcal{X}}$ is flat over $\mathcal{O}_T$ by [85, (2.22)]. By using these facts, we see that the log version of the proofs of [11, (7.8), (7.13), (7.16), (7.24)] and [74, (3.3)] work for the coefficient $\mathcal{I}_{X/S}$. Hence $R^h f_{X/T*,\mathrm{c}}(\mathcal{O}_{X/T})$ is a perfect complex of $\mathcal{O}_T$-modules and it satisfies the base change property. Therefore, if T' is sufficiently small, we have

$$(2.14.0.4) \quad \mathcal{T}or_r^{\mathcal{O}_{T'}}(R^h f_{X_{T'_1}/T'*,\mathrm{c}}(\mathcal{O}_{X_{T'_1}/T'}), \mathcal{E}|_{T'}) = 0 \quad (\forall r \in \mathbb{Z}_{>0}, \forall h \in \mathbb{Z}).$$

Proposition 2.14.1. *Let $T = (T, a\mathcal{O}_T, \gamma)$ be as above. Let $g\colon T'' \longrightarrow T'$ be a morphism from an adic fine formal log scheme into an open log formal subscheme of T. If T' is small enough, then the following hold*:

(1) *The canonical morphism*

$$g^* R^h f_{X_{T'_1}/T'*}(\mathcal{O}_{X_{T'_1}/T'}) \longrightarrow R^h f_{X_{T''_1}/T''*}(\mathcal{O}_{X_{T''_1}/T''})$$

is an isomorphism of $\mathcal{O}_{T''}$-modules.

(2) *The canonical morphism*

$$g^* R^h f_{X_{T'_1}/T'*,\mathrm{c}}(\mathcal{O}_{X_{T'_1}/T'}) \longrightarrow R^h f_{X_{T''_1}/T''*,\mathrm{c}}(\mathcal{O}_{X_{T''_1}/T''})$$

is an isomorphism of $\mathcal{O}_{T''}$-modules.

Proof. We may assume that (2.14.0.2), (2.14.0.3) and (2.14.0.4) hold.

(1): As in [68, (3.2)], we immediately obtain (1) using the existence of a strictly perfect complex of $\mathcal{O}_{T'}$-modules representing $R\Gamma(X_{T'_1}/T', \mathcal{O}_{X_{T'_1}/T'})$ (cf. [11, 7.14 Definition, 7.24.3 Theorem]), using (2.14.0.2) and (2.14.0.3), and using the log base change theorem ([54, (6.10)], cf. [74, (3.3)]).

(2): By the facts described before (2.14.0.4), the same proof as that of (1) works.

□

We will use the following proposition in §2.18 below.

Proposition 2.14.2. *Let T be an adic formal scheme. Let $g\colon T'' \longrightarrow T'$ be a morphism from an adic scheme into an open formal subscheme of T. Let $f\colon (X, D \cup Z) \longrightarrow T_1$ be a proper smooth scheme with a relative SNCD over T_1. If T' is small enough, then the following hold*:

(1) *The canonical morphism*

$$g^* P_k^{D_{T'_1}} R^h f_{(X,D\cup Z)_{T'_1}/T'*}(\mathcal{O}_{(X,D\cup Z)_{T'_1}/T'}) \longrightarrow$$

$$P_k^{D_{T''_1}} R^h f_{(X,D\cup Z)_{T''_1}/T''*}(\mathcal{O}_{(X,D\cup Z)_{T''_1}/T''})$$

is an isomorphism.

(2) *The canonical morphism*

$$g^* P_k^{D_{T'_1}} R^h f_{(X,D\cup Z)_{T'_1}/T'*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)_{T'_1}/T'}) \longrightarrow$$

$$P_k^{D_{T_1''}} R^h f_{(X,D\cup Z)_{T_1''}/T''*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)_{T_1''}/T''})$$

is an isomorphism

Proof. By (2.9.6.2), there exist the following two spectral sequences

(2.14.2.1)

$$\begin{aligned} E_1^{-k,h+k} = R^{h-k} f_{(D^{(k)},Z|_{D^{(k)}})_{T_1'}/T'*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})_{T_1'}/T'} \\ \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(k)\log}(D_{T_1'}/T';Z_{T_1'}))(-k) \\ \Longrightarrow R^h f_{(X,D\cup Z)_{T_1'}/T'*}(\mathcal{O}_{(X,D\cup Z)_{T_1'}/T'}), \end{aligned}$$

(2.14.2.2)

$$\begin{aligned} E_1^{-k,h+k} = R^{h-k} f_{(D^{(k)},Z|_{D^{(k)}})_{T_1''}/T''*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})_{T_1''}/T''} \\ \otimes_{\mathbb{Z}} \varpi_{\mathrm{crys}}^{(k)\log}(D_{T_1''}/T'';Z_{T''}))(-k) \\ \Longrightarrow R^h f_{(X,D\cup Z)_{T_1''}/T''*}(\mathcal{O}_{(X,D\cup Z)_{T_1''}/T''}). \end{aligned}$$

By (2.9.1) (2), there exists a canonical morphism

$$g^{-1}((2.14.2.1)) \otimes_{g^{-1}(\mathcal{O}_{T'})} \mathcal{O}_{T''} \longrightarrow (2.14.2.2).$$

Then, by (2.14.0.3), there exists a non-empty open formal subscheme T' such that

$$\mathcal{T}or_r^{\mathcal{O}_{T'}}(R^h f_{(D^{(k)},Z|_{D^{(k)}})_{T_1'}/T'*}(\mathcal{O}_{(D^{(k)},Z|_{D^{(k)}})_{T_1'}/T'}),\mathcal{E}|_{T'}) = 0$$

for any $\mathcal{O}_T$-module $\mathcal{E}$ without a-torsion and for all $r \in \mathbb{Z}_{>0}$. Hence we have an isomorphism

$$g^{-1}E_1^{-k,h+k}((X,D\cup Z)_{T'_1}/T')\otimes_{g^{-1}(\mathcal{O}_{T'})}\mathcal{O}_{T''} \xrightarrow{\sim} E_1^{-k,h+k}((X,D\cup Z)_{T_1''}/T'')$$

as in the proof of (2.14.1) (1), and therefore the morphism in (1) is an isomorphism.

The proof of (2) is the same as that of (1). □

2.15 The E_2-degeneration of the p-adic Weight Spectral Sequence of an Open Smooth Variety

Let κ be a perfect field of characteristic $p > 0$. Let W be the Witt ring of κ. Let K_0 be the fraction field of W. In [68, (5.2)] we have proved the E_2-degenerations modulo torsion of the weight spectral sequences (2.9.6.2) and (2.11.14.3) when $Z = \emptyset$ and $S = \mathrm{Spf}(W)$. To prove the degenerations, we have used a somewhat tricky argument in [68, (5.2)] (cf. [68, (3.2), (3.4), (3.5), (3.6)]) based on Deligne's remark ([49, 3.10]). Though we also use Deligne's remark in this book, the proof in this section is not tricky by virtue of the existence of the weight spectral sequences (2.9.6.2) and (2.11.14.3) over a general base (cf. [68, (3.7)]).

Let (X,D) be a proper smooth scheme with an SNCD over κ. By [40, 3, (8.9.1) (iii), (8.10.5)] and [40, 4, (17.7.8)] , there exist a smooth affine scheme S_1 over a finite field $\mathbb{F}_q$ and a model $(\mathcal{X},\mathcal{D})$ of (X,D) over S_1. By a standard deformation theory ([41, III (6.10)]), there exists a formally smooth scheme S such that $S\otimes_{W(\mathbb{F}_q)}\mathbb{F}_q=S_1$. Let T be an affine open subscheme of S, and set $T_1:=T\otimes_{W(\mathbb{F}_q)}\mathbb{F}_q$. Take a closed point t of T_1. The point t is the spectrum of a finite field κ_t. We fix a lift $F_T\colon T\longrightarrow T$ of the Frobenius(=p-th power morphism) F_{T_1} of T_1. Then we have the Teichmüller lift $\Gamma(T,\mathcal{O}_T)\longrightarrow W(\kappa_t)$ (resp. $\Gamma(T,\mathcal{O}_T)\longrightarrow W$) of the morphism $\Gamma(T_1,\mathcal{O}_{T_1})\longrightarrow \kappa_t$ (resp. $\Gamma(T_1,\mathcal{O}_{T_1})\longrightarrow \kappa$) (e.g., [50, 0 1.3]). The rings $W(\kappa_t)$ and W become $\Gamma(T,\mathcal{O}_T)$-algebras by these lifts.

To prove the E_2-degenerations, we prove some elementary lemmas.

Let A be a p-adically complete and separated p-torsion-free ring with a lift f of the Frobenius endomorphism of $A_1:=A/p$. Then there exists a unique section $\widetilde{\tau}\colon A\longrightarrow W(A)$ of the projection $W(A)\longrightarrow A$ such that $\widetilde{\tau}\circ f=F\circ\widetilde{\tau}$, where F is the Frobenius of $W(A)$ (e.g., [50, 0 (1.3.16)]). This morphism induces morphisms $\tau\colon A\longrightarrow W(A_1)$ and $\tau_n\colon A/p^n\longrightarrow W_n(A_1)$. Then the following holds:

Lemma 2.15.1. *If A_1 is reduced, then the morphism $\tau\colon A\longrightarrow W(A_1)$ is injective.*

Proof. Let $F^n_*(A_1)$ be the restriction of scalars of A_1 by the n-th power of the Frobenius endomorphism of A_1. By the assumption, the morphism $F^n\colon A_1\longrightarrow F^n_*(A_1)$ is injective. (2.15.1) follows from the following commutative diagram in [50, 0 (1.3.22)]:

$$\begin{array}{ccc} A_1 & \xrightarrow{F^n} & F^n_*(A_1) \\ {\scriptstyle p^n}\big\downarrow{\scriptstyle\simeq} & & {\scriptstyle V^n}\big\downarrow{\scriptstyle\simeq} \\ p^nA/p^{n+1}A & \xrightarrow{\mathrm{gr}^n\tau_{n+1}} & V^nW(A_1)/V^{n+1}W(A_1). \end{array} \qquad (\forall n\in\mathbb{N})$$

□

Lemma 2.15.2. (1) *Let B be a commutative ring whose Jacobson radical* $\mathrm{rad}(B)$ *is the zero. Let $\mathcal{M}(B)$ be the set of the maximal ideals of B. Then the morphism $W(B)\longrightarrow \prod_{\mathfrak{m}\in\mathcal{M}(B)} W(B/\mathfrak{m})$ is injective.*

(2) *Let C be a commutative ring with unit element and let D be a smooth C-algebra. If* $\mathrm{rad}(C)=0$, *then* $\mathrm{rad}(D)=0$.

Proof. (1): By the assumption, the natural morphism $B \longrightarrow \prod_{\mathfrak{m}\in\mathcal{M}(B)} B/\mathfrak{m}$ is injective. Thus $W(B) \longrightarrow W(\prod_{\mathfrak{m}\in\mathcal{M}(B)} B/\mathfrak{m}) = \prod_{\mathfrak{m}\in\mathcal{M}(B)} W(B/\mathfrak{m})$ is injective.

(2): Let $\{f_i\}_i$ be a family of elements of D such that $\mathrm{Spec}(D) = \bigcup_i \mathrm{Spec}(D_{f_i})$. Then the natural morphism $D \longrightarrow \prod_i D_{f_i}$ is injective since $D \longrightarrow \prod_{\mathfrak{m}\in\mathcal{M}(D)} D_{\mathfrak{m}}$ is injective. Thus the problem is local; we may assume that there exists a finite etale morphism $C[X_1, \ldots, X_m] \longrightarrow D$. Let $(\sqrt{0})_C$ and $(\sqrt{0})_D$ be the nilpotent radicals of C and D, respectively. Since $(\sqrt{0})_C \subset \mathrm{rad}(C) = 0$, $(\sqrt{0})_C = 0$. Hence C is a Jacobson ring and D is also by [13, V §3, n°4, Theorem 3]. Therefore $(\sqrt{0})_D = \mathrm{rad}(D)$. Since $C[X_1, \ldots, X_m]$ is reduced, D is also by [41, I Proposition 9.2]. Hence $(\sqrt{0})_D = 0$. □

Corollary 2.15.3. *Let κ' be a perfect field of characteristic $p > 0$. Let A be a p-adically complete and separated formally smooth algebra over $W(\kappa')$ with a lift of the Frobenius morphism of A_1. Then the morphism $A \longrightarrow \prod_{\mathfrak{m}\in\mathcal{M}(A_1)} W(A_1/\mathfrak{m})$ is injective.*

Proof. (2.15.3) follows from (2.15.1) and (2.15.2). □

Theorem 2.15.4 ([68, (5.2)]). *If $Z = \emptyset$ and $S = \mathrm{Spf}(W)$, then* (2.9.6.2) *and* (2.11.14.3) *degenerate at E_2 modulo torsion.*

Proof. For a $W(\mathbb{F}_q)$-module M, $M_{K_0(\mathbb{F}_q)}$ denotes $M \otimes_{W(\mathbb{F}_q)} K_0(\mathbb{F}_q)$. First we prove (2.15.4) for (2.9.6.2). Replace T by a sufficiently small affine open sub log formal scheme in order that, for any $h, k \in \mathbb{Z}, r \in \mathbb{Z}_{>0}$, $E_r^{-k,h+k}((\mathcal{X}_{T_1}, \mathcal{D}_{T_1})/T)$ has the form $\mathcal{O}_T^{\oplus n} \oplus \mathcal{N}$ $(n \in \mathbb{N})$, where $\mathcal{N}$ is a direct sum of modules of type $\mathcal{O}_T/p^e$ $(e \in \mathbb{Z}_{>0})$. Then we have

$$\mathcal{T}or_s^{g^{-1}(\mathcal{O}_T)}(g^{-1}E_r^{-k,h+k}((\mathcal{X}_{T_1}, \mathcal{D}_{T_1})/T), \mathcal{O}_{T'}) = 0 \quad (\forall s \in \mathbb{Z}_{>0})$$

for any morphism $g : T' \longrightarrow T$ of p-adic fine log PD-schemes and for any $h, k \in \mathbb{Z}, r \in \mathbb{Z}_{>0}$. Then we have

$$g^* E_r^{-k,h+k}((\mathcal{X}_{T_1}, \mathcal{D}_{T_1})/T) = E_r^{-k,h+k}((\mathcal{X}_{T'_1}, \mathcal{D}_{T'_1})/T')$$

for any morphism $g : T' \longrightarrow T$ of p-adic fine log PD-schemes and for any $h, k \in \mathbb{Z}, r \in \mathbb{Z}_{>0}$. Indeed, for $r = 1$, it is nothing but (2.14.1) (1); for general r, it follows from the functoriality of the spectral sequence (2.9.6.2) and induction. Hence, to prove the theorem for the spectral sequence (2.9.6.2), we have to only to prove that the morphism

$$\begin{aligned} d_r^{-k,h+k}((\mathcal{X}_{T_1}, \mathcal{D}_{T_1})/T)_{K_0(\mathbb{F}_q)} \colon E_r^{-k,h+k}&((\mathcal{X}_{T_1}, \mathcal{D}_{T_1})/T)_{K_0(\mathbb{F}_q)} \longrightarrow \\ &E_r^{-k+r,h+k-r+1}((\mathcal{X}_{T_1}, \mathcal{D}_{T_1})/T)_{K_0(\mathbb{F}_q)} \end{aligned}$$

is zero for any $r \geq 2$. Let us express

$$E_r^{-k,h+k}((\mathcal{X}_{T_1},\mathcal{D}_{T_1})/T)=\mathcal{O}_T^{\oplus n}\oplus\mathcal{N},$$

$$E_r^{-k+r,h+k-r+1}((\mathcal{X}_{T_1},\mathcal{D}_{T_1})/T)=\mathcal{O}_T^{\oplus n'}\oplus\mathcal{N}',$$

where $\mathcal{N},\mathcal{N}'$ are direct sums of modules of type $\mathcal{O}_T/p^e$ $(e\in\mathbb{Z}_{>0})$. Then we have

$$\begin{aligned}d_r^{-k,h+k}((\mathcal{X}_{T_1},\mathcal{D}_{T_1})/T)&\in\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T^{\oplus n}\oplus\mathcal{N},\mathcal{O}_T^{\oplus n'}\oplus\mathcal{N}')\\&=\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_T^{\oplus n},\mathcal{O}_T^{\oplus n'})\oplus N,\end{aligned}$$

where N is a direct sum of modules of type $\Gamma(T,\mathcal{O}_T)/p^e$ $(e\in\mathbb{Z}_{>0})$. Then, for any closed point t of T_1, we have

$$\begin{aligned}&d_r^{-k,h+k}((\mathcal{X}_t,\mathcal{D}_t)/W(\kappa_t))\\=&d_r^{-k,h+k}((\mathcal{X}_{T_1},\mathcal{D}_{T_1})/T)\otimes_{\mathcal{O}_T}W(\kappa_t)\\\in&\mathrm{Hom}_{W(\kappa_t)}(W(\kappa_t)^{\oplus n},W(\kappa_t)^{\oplus n'})\oplus(N\otimes_{\Gamma(T,\mathcal{O}_T)}W(\kappa_t)).\end{aligned}$$

By the purity of the weight [15, (1.2)] or [68, (2.2) (4)], we have $d_r^{-k,h+k}((\mathcal{X}_t,\mathcal{D}_t)/W(\kappa_t))_{K_0(\mathbb{F}_q)}=0$ for any closed point t of T_1, that is, $d_r^{-k,h+k}((\mathcal{X}_t,\mathcal{D}_t)/W(\kappa_t))$ is contained in $N\otimes_{\Gamma(T,\mathcal{O}_T)}W(\kappa_t)$. From this and (2.15.3), we see that $d_r^{-k,h+k}((\mathcal{X}_{T_1},\mathcal{D}_{T_1})/T)$ is contained in N. Hence $d_r^{-k,h+k}((\mathcal{X}_{T_1},\mathcal{D}_{T_1})/T)_{K_0(\mathbb{F}_q)}=0$.

The proof of the degeneration of (2.11.14.3) is the same as the above. (One may use the duality between (2.9.6.2) $\otimes_W K_0$ and (2.11.14.3) $\otimes_W K_0$ for the case $Z=\emptyset$ and $S=\mathrm{Spf}(W)$.) □

2.16 The Filtered Log Berthelot-Ogus Isomorphism

In this section we prove a filtered version of Berthelot-Ogus isomorphism. Because the proof of this isomorphism is almost the same as that in [12] and [74], we give only the sketch of the proof.

Proposition 2.16.1. *Let S be a scheme of characteristic $p>0$ and let $S_0\overset{\subset}{\longrightarrow}S$ be a nilpotent immersion. Let $S\overset{\subset}{\longrightarrow}T$ be a PD-closed immersion into a formal scheme with p-adic topology such that $\mathcal{O}_T$ is p-torsion-free. Let $f\colon(X,D\cup Z)\longrightarrow S$ and $f'\colon(X',D'\cup Z')\longrightarrow S$ be smooth schemes with relative transversal SNCD's. Assume that X, X', S and T are noetherian. Set $(X_0,D_0\cup Z_0):=(X,D\cup Z)\times_S S_0$ and $(X'_0,D'_0\cup Z'_0):=(X',D'\cup Z')\times_S S_0$. Let $g\colon(X_0,D_0\cup Z_0)\longrightarrow(X'_0,D'_0\cup Z'_0)$ be a morphism of log schemes over S_0 which induces morphisms $(X_0,D_0)\longrightarrow(X'_0,D'_0)$ and $(X_0,Z_0)\longrightarrow(X'_0,Z'_0)$. Then the following hold*:

(1) *There exists a canonical filtered morphism*

(2.16.1.1)
$$g^* : (Rf'_{(X',D'\cup Z')/T*}(\mathcal{O}_{(X',D'\cup Z')/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^{D'}) \longrightarrow$$
$$(Rf_{(X,D\cup Z)/T*}(\mathcal{O}_{(X,D\cup Z)/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^D),$$

which is compatible with compositions. If g has a lift $\widetilde{g}\colon (X, D\cup Z) \longrightarrow (X', D'\cup Z')$, then $g^* = \widetilde{g}^{\log *}_{\mathrm{crys}}$.

(2) *Assume that g induces a morphism $g^{(k)}\colon (D^{(k)}_0, Z_0|_{D^{(k)}_0}) \longrightarrow (D'^{(k)}_0, Z'_0|_{D'^{(k)}_0})$ for all $k \in \mathbb{N}$. Then there exists a canonical filtered morphism*

(2.16.1.2)
$$g^*_c : (Rf'_{(X',D'\cup Z')/T*,\mathrm{c}}(\mathcal{O}_{(X',D'\cup Z';Z')/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^{D'}_{\mathrm{c}}) \longrightarrow$$
$$(Rf_{(X,D\cup Z)/T*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^D_{\mathrm{c}}),$$

which is compatible with compositions. If g has a lift $\widetilde{g}\colon (X, D\cup Z) \longrightarrow (X', D'\cup Z')$, then $g^*_c = \widetilde{g}^{\log *}_{\mathrm{crys}}$.

Proof. (1): The relative Frobenius $F_{(X,D\cup Z)/S}\colon (X, D\cup Z) \longrightarrow (X^{(p)}, D^{(p)} \cup Z^{(p)})$ over S induces an isomorphism

$$P^{D^{(p)}}_k Rf_{(X^{(p)},D^{(p)}\cup Z^{(p)})/T*}(\mathcal{O}_{(X^{(p)},D^{(p)}\cup Z^{(p)})/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}$$
$$\xrightarrow{\sim} P^D_k Rf_{(X,D\cup Z)/T*}(\mathcal{O}_{(X,D\cup Z)/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q} \quad (k \in \mathbb{Z})$$

by (2.9.6.3) and (2.10.2.1) because the relative Frobenius induces an isomorphism of the classical iso-crystalline cohomology of a smooth scheme over S ([12, (1.3)]). Hence the same proof as that in [12, (2.1)] shows that we have the morphism (2.16.1.1).

(2): The proof for (2.16.1.2) is the same as that for (2.16.1.1) by using (2.11.14.4) instead of (2.9.6.3) and using (2.11.18). □

Corollary 2.16.2. *If $(X_0, D_0 \cup Z_0) = (X'_0, D'_0 \cup Z'_0)$, then*

$$(Rf_{(X,D\cup Z)/T*}(\mathcal{O}_{(X,D\cup Z)/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^D) =$$
$$(Rf_{(X',D'\cup Z')/T*}(\mathcal{O}_{(X',D'\cup Z')/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^{D'})$$

and

$$(Rf_{(X,D\cup Z)/T*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^D_{\mathrm{c}}) =$$
$$(Rf_{(X',D'\cup Z')/T*,\mathrm{c}}(\mathcal{O}_{(X',D'\cup Z';Z')/T}) \otimes^L_{\mathbb{Z}} \mathbb{Q}, P^{D'}_{\mathrm{c}}).$$

Proof. Obvious (cf. [12, (2.2)]). □

Theorem 2.16.3 (Filtered log Berthelot-Ogus isomorphism). *Let V be a complete discrete valuation ring of mixed characteristics with perfect residue field κ. Let p be the characteristic of κ. Set $K := \mathrm{Frac}(V)$. Let S be a p-adic formal V-scheme in the sense of* [74, §1]. *Let $(X, D\cup Z) \longrightarrow S$ be a proper formally smooth scheme with a relative transversal SNCD over S. Let*

T be an enlargement of S with morphism $z\colon T_0 := (\underline{\mathrm{Spec}}_T(\mathcal{O}_T/p))_{\mathrm{red}} \longrightarrow S$. Set $T_1 := \underline{\mathrm{Spec}}_T(\mathcal{O}_T/p)$. Let $f_0\colon (X_0, D_0 \cup Z_0) := (X, D \cup Z) \times_{S,z} T_0 \longrightarrow T_0$ be the base change of $f\colon (X, D \cup Z) \longrightarrow S$. Then the following hold:

(1) *If there exists a log smooth lift $f_1\colon (X_1, D_1 \cup Z_1) \longrightarrow T_1$ of f_0, then there exist the following canonical filtered isomorphisms*

$$\sigma_T\colon (R^h f_{(X_1,D_1\cup Z_1)/T*}(\mathcal{O}_{(X_1,D_1\cup Z_1)/T})_K, P^{D_1}) \xrightarrow{\sim} (R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})_T, P^D),$$

$$\sigma_{T,\mathrm{c}}\colon (R^h f_{(X_1,D_1\cup Z_1)/T*,\mathrm{c}}(\mathcal{O}_{(X_1,D_1\cup Z_1;Z_1)/T})_K, P_{\mathrm{c}}^{D_1}) \xrightarrow{\sim} (R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/K})_T, P_{\mathrm{c}}^{D}).$$

(2) *If there exists a log smooth lift $\mathfrak{f}\colon (\mathcal{X}, \mathcal{D} \cup \mathcal{Z}) \longrightarrow T$ of f_0, then there exist the following canonical filtered isomorphisms*

$$\sigma^{\log}_{\mathrm{crys},T}\colon (R^h \mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/T}(\log(\mathcal{D} \cup \mathcal{Z})))_K, P^{\mathcal{D}}) \xrightarrow{\sim} (R^h f_*(\mathcal{O}_{(X,D\cup Z)/K})_T, P^D),$$

$$\sigma^{\log}_{\mathrm{crys},T,\mathrm{c}}\colon (R^h \mathfrak{f}_*(\Omega^\bullet_{\mathcal{X}/T}(\log(\mathcal{Z}-\mathcal{D})))_K, P_{\mathrm{c}}^{\mathcal{D}}) \xrightarrow{\sim} (R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D\cup Z;Z)/K})_T, P_{\mathrm{c}}^{D}).$$

Proof. The proof is the same as that of [74, (3.8)]. □

Remark 2.16.4. Let V, κ and p be as in (2.16.3). Then V/p is a κ-algebra by [79, II Proposition 8].

(1) Let $(\mathcal{X}, \mathcal{D}\cup\mathcal{Z})$ a proper smooth scheme over $\mathrm{Spec}(V)$ with an (S)NCD. Set $\mathcal{U}_K := \mathcal{X}_K \setminus (\mathcal{D}_K \cup \mathcal{Z}_K)$. Then, by (2.16.3) and (2.16.2) and the base change theorem of the log crystalline cohomology ([54, (6.10)]), there are canonical isomorphisms:

(2.16.4.1)

$$H^h_{\text{log-crys}}((\mathcal{X}_\kappa, \mathcal{D}_\kappa \cup \mathcal{Z}_\kappa)/W(\kappa))_K \xrightarrow{\sim} H^h(\mathcal{X}_K, \Omega^\bullet_{\mathcal{X}_K/K}(\log(\mathcal{D}_K \cup \mathcal{Z}_K))) = H^h_{\mathrm{dR}}(\mathcal{U}_K/K),$$

(2.16.4.2)

$$H^h_{\text{log-crys},\mathrm{c}}((\mathcal{X}_\kappa, \mathcal{D}_\kappa \cup \mathcal{Z}_\kappa; \mathcal{Z}_\kappa)/W(\kappa))_K \xrightarrow{\sim} H^h(\mathcal{X}_K, \Omega^\bullet_{\mathcal{X}_K/K}(\log(\mathcal{Z}_K - \mathcal{D}_K)))$$

which are compatible with the weight filtrations with respect to $\mathcal{D}_\kappa$ and $\mathcal{D}_K$. See also [17] for analogous statements by the rigid analytic method in the case $\mathcal{Z} = \emptyset$.

(2) Let (X, D) be a proper smooth scheme with a relative SNCD over κ. Set $U := X \setminus D$. By the finite base change theorem ([5, (1.8)]) and by Shiho's comparison theorems [82, Theorem 2.4.4, Corollary 2.3.9, Theorem 3.1.1]), there exists a canonical isomorphism $H^h_{\mathrm{rig}}(U/K) \xrightarrow{\sim} H^h_{\text{log-crys}}((X, D)/W) \otimes_W K$. As a result, $H^h_{\mathrm{rig}}(U/K)$ has a weight filtration.

By [85], [82, Theorem 2.4.4, Corollary 2.3.9, Theorem 3.1.1] and [6, (2.4)], we obtain $H^h_{\text{log-crys,c}}((X,D)/W) \otimes_W K = H^h_{\text{rig,c}}(U/K)$. In particular, $H^h_{\text{rig,c}}(U/K)$ has a weight filtration.

If (X,D) is the special fiber of $(\mathcal{X},\mathcal{D})$ in (1), there exists a weight-filtered isomorphism $H^h_{\text{rig}}(U/K) \xrightarrow{\sim} H^h_{\text{dR}}(\mathcal{U}_K/K)$. An analogous statement can be found in [17].

(3) Let U be a separated scheme of finite type over κ. Let Z/κ be a closed subscheme of U. In [70] the first-named author has defined a finite increasing filtration on $H^h_{\text{rig},Z}(U/K)$ which deserves the name "weight filtration". In particular, the weight filtration on $H^h_{\text{rig}}(U/K)$ defined in (2) is independent of the choice of (X,D). See §3.4 below for more details. In [loc. cit.] he has also defined a finite increasing filtration on $H^h_{\text{rig,c}}(U/K)$ which deserves the name "weight filtration" in the case where U is embeddable into a smooth scheme over κ as a closed subscheme. See also §3.6 below for more details.

2.17 The E_2-degeneration of the p-adic Weight Spectral Sequence of a Family of Open Smooth Varieties

Let V be a complete discrete valuation ring of mixed characteristics with perfect residue field κ of characteristic $p>0$. Let B be a topologically finitely generated ring over V. For a V-module M, M_K denotes the tensor product $M \otimes_V K$. In particular, $B_K = B \otimes_V K$. Let $\mathfrak{m}$ be a maximal ideal of B_K. By the proof of [84, (4.5)], $B_K/\mathfrak{m}$ is a finite extension of K. Set $K' := B_K/\mathfrak{m}$. Let C be the image of B in $B_K/\mathfrak{m} = K'$. Let V' be the integer ring of K'. Then the following is well-known (cf. [74, the proof of (4.2)]):

Lemma 2.17.1. *$V \subset C \subset V'$.*

Proof. The inclusion $V \subset C$ is obvious. Let π be a uniformizer of V. Let v be a normalized valuation of V'. Let e be the ramification index of V'/V. By the definition of B, there exists a surjection $V\{x_1,\ldots,x_r\} \longrightarrow B$. It suffices to show that the image y_i $(1 \leq i \leq r)$ of x_i in K' belongs to V'. If not, $v(y_i) < 0$ for some i. Set

$$a_n = \begin{cases} \pi^{n/(e+1)} & (n \in \mathbb{N},\ e+1|n), \\ 0 & (n \in \mathbb{N},\ e+1 \nmid n). \end{cases}$$

Then the image of an element $\sum_{n=0}^{\infty} a_n x_i^n \in V\{x_1,\ldots,x_r\}$ in K' does not converge in K'. This is a contradiction. $\square$

We keep the notations in §2.4 except that S is a p-adic formal V-scheme in the sense of [74, §1] and that X is a proper smooth scheme with a relative SNCD D over $S_1 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$. The main result in this section is the following:

Theorem 2.17.2 (E_2-degeneration). *Assume that S is a p-adic formal V-scheme and that X is a proper smooth scheme over S_1. Then* (2.9.6.2)$\otimes_V K$ *and* (2.11.14.3) $\otimes_V K$ *degenerate at E_2 in the case $Z = \emptyset$ and $S_0 = S_1$.*

Proof. (Compare the following proof with [20, (5.5)].)

We first prove the theorem for (2.9.6.2)$\otimes_V K$ for the case $Z = \emptyset$ and $S_0 = S_1$. We may assume that S is a p-adic affine flat formal scheme $\mathrm{Spf}(B)$ over $\mathrm{Spf}(V)$. Consider the following boundary morphism:

(2.17.2.1)
$$d_r^{-k,h+k} \colon E_r^{-k,h+k}((X,D)/S)_K \longrightarrow E_r^{-k+r,h+k-r+1}((X,D)/S)_K \quad (r \geq 2).$$

We prove that $d_r^{-k,h+k} = 0$ $(r \geq 2)$.

Case I: First we consider a case where B is a topologically finitely generated ring over V such that B_K is an artinian local ring. Let $\mathfrak{m}$ be the maximal ideal of B_K. Then $\mathfrak{m}$ is nilpotent. Set $K' := B_K/\mathfrak{m}$. Consider the following ideal of B: $I := \mathrm{Ker}(B \longrightarrow B_K/\mathfrak{m})$. Then $C = B/I$, $C_K = K'$ and $V \subset C \subset V'$ ((2.17.1)). Let $\iota \colon \mathrm{Spf}(C) \overset{\subset}{\longrightarrow} \mathrm{Spf}(B)$ be the nilpotent closed immersion. Since the characteristic of K is 0, the morphism $\mathrm{Spec}(C_K) \longrightarrow \mathrm{Spec}(K)$ is smooth and hence there exists a section $s_K \colon \mathrm{Spec}(B_K) \longrightarrow \mathrm{Spec}(C_K)$ of the nilpotent closed immersion $\mathrm{Spec}(C_K) \longrightarrow \mathrm{Spec}(B_K)$. By [74, (1.17)], there exists a finite modification $\pi \colon \mathrm{Spf}(B') \longrightarrow \mathrm{Spf}(B)$, a nilpotent closed immersion $\iota' \colon \mathrm{Spf}(C) \overset{\subset}{\longrightarrow} \mathrm{Spf}(B')$ with $\pi \circ \iota' = \iota$ and a morphism $s \colon \mathrm{Spf}(B') \longrightarrow \mathrm{Spf}(C)$ such that s induces s_K and that $s \circ \iota' = \mathrm{id}$. Set $S' := \mathrm{Spf}(B')$. Because the boundary morphisms $\{d_1^{-k,h+k}\}$ are summations of Gysin morphisms (with signs) ((2.8.5)), the E_2-terms of (2.9.6.2) $\otimes_V K$ are convergent F-isocrystals by [74, (3.7), (3.13), (2.10)]. Hence we have $E_2^{-k,h+k}((X,D)/S)_K = E_2^{-k,h+k}((X_{S'_1}, D_{S'_1})/S')_K$ since $B'_K = B_K$. Let $\{d_r'^{\bullet\bullet}\}$ $(r \geq 1)$ be the boundary morphism of (2.9.6.2)$\otimes_V K$ for $(X_{S'_1}, D_{S'_1})/S'$. Because $\{d_r^{\bullet\bullet}\}$ $(r \geq 2)$ are functorial with respect to a morphism of p-adic enlargements, we have the following commutative diagram for $r \geq 2$:

$$\begin{array}{ccc}
E_r^{-k,h+k}((X,D)/S)_K & \longrightarrow & E_r^{-k,h+k}((X_{S'_1}, D_{S'_1})/S')_K \\
{\scriptstyle d_r^{-k,h+k}}\big\downarrow & & \big\downarrow{\scriptstyle d_r'^{-k,h+k}} \\
E_r^{-k+r,h+k-r+1}((X,D)/S)_K & \longrightarrow & E_r^{-k+r,h+k-r+1}((X_{S'_1}, D_{S'_1})/S')_K.
\end{array}$$

Here, if $r = 2$, then the two horizontal morphisms above are isomorphisms. By induction on $r \geq 2$, we see that $d_r^{\bullet\bullet}$ vanishes if $d_r'^{\bullet\bullet}$ does. Hence it suffices to prove that the boundary morphism

(2.17.2.2)
$$\begin{aligned} d_r'^{-k,h+k} \colon & E_r^{-k,h+k}((X_{S'_1}, D_{S'_1})/S')_K \\ & \longrightarrow E_r^{-k+r,h+k-r+1}((X_{S'_1}, D_{S'_1})/S')_K \quad (r \geq 2) \end{aligned}$$

is the zero. Let $l(M)$ be the length of a finitely generated $B'_K = B_K$-module M. Furthermore, to prove the vanishing of $d_r'^{\bullet\bullet}$, it suffices to prove that

(2.17.2.3)
$$l(R^h f_{(X_{S'_1},D_{S'_1})/S'*}(\mathcal{O}_{(X_{S'_1},D_{S'_1})/S'})_K) = l(\bigoplus_k E_2^{-k,h+k}((X_{S'_1},D_{S'_1})/S')_K).$$

Set $S'' := \mathrm{Spf}(C)$. Then we have the morphism $(X_{S''_1},D_{S''_1}) \longrightarrow S''$. Let us denote the pull-back of the morphism $(X_{S''_1},D_{S''_1}) \longrightarrow S''$ by $s : S' \longrightarrow S''$ by $(X'_{S'_1},D'_{S'_1}) \longrightarrow S'$. Then, since we have $\pi \circ \iota' = \iota$ and $s \circ \iota' = \mathrm{id}$, both $(X_{S'_1},D_{S'_1})$ and $(X'_{S'_1},D'_{S'_1})$ are deformations of $(X_{S''_1},D_{S''_1})$ to S'_1. Hence, by (2.16.2), the spectral sequence (16.6.2)$\otimes_V K$ for $(X_{S'_1},D_{S'_1})/S'$ and that for $(X'_{S'_1},D'_{S'_1})/S'$ are isomorphic. Therefore we have

$$\begin{aligned} E_2^{-k,h+k}((X_{S'_1},D_{S'_1})/S')_K &= E_2^{-k,h+k}((X'_{S'_1},D'_{S'_1})/S')_K \\ &= B' \otimes_C E_2^{-k,h+k}((X_{S''_1},D_{S''_1})/S'')_K. \end{aligned}$$

Hence, to prove (2.17.2.3), it suffices to prove that

(2.17.2.4)
$$\begin{aligned} &\dim_{K'}(R^h f_{(X_{S''_1},D_{S''_1})/S''*}(\mathcal{O}_{(X_{S''_1},D_{S''_1})/S''})_K) \\ &\qquad = \dim_{K'}(\bigoplus_k E_2^{-k,h+k}((X_{S''_1},D_{S''_1})/S'')_K). \end{aligned}$$

Set $V'_1 := V'/p$. Because there exists a morphism $\mathrm{Spf}(V') \longrightarrow \mathrm{Spf}(C)$ of p-adic enlargements of S, it suffices to prove that

(2.17.2.5)
$$\begin{aligned} &\dim_{K'}(R^h f_{(X_{V'_1},D_{V'_1})/V'*}(\mathcal{O}_{(X_{V'_1},D_{V'_1})/V'})_K) \\ &\qquad = \dim_{K'}(\bigoplus_k E_2^{-k,h+k}((X_{V'_1},D_{V'_1})/V')_K). \end{aligned}$$

We reduce (2.17.2.5) to a result of [68, (5.2) (1)](=(2.15.4) for (2.9.6.2) in this book) by using (a log version of) a result of Berthelot-Ogus ([12, §2]) as follows.

Let κ' be the residue field of V'. Since κ is perfect and since κ' is a finite extension of κ, κ' is also perfect. Let W' be the Witt ring of κ'. The ring V'_1 is an artinian local κ'-algebra with residue field κ' ([79, II Proposition 8]). Set $X' := X_{V'_1} \otimes_{V'_1} \kappa'$ and $D' := D_{V'_1} \otimes_{V'_1} \kappa'$. Then $(X' \otimes_{\kappa'} V'_1, D' \otimes_{\kappa'} V'_1)$ and $(X_{V'_1},D_{V'_1})$ are two log deformations of (X',D'). Therefore, by (2.16.2), the spectral sequence (2.9.6.2)$\otimes_V K$ for $(X' \otimes_{\kappa'} V'_1, D' \otimes_{\kappa'} V'_1)/V'$ and that for $(X_{V'_1},D_{V'_1})/V'$ are isomorphic. From this fact, the log base change theorem ([54, (6.10)]) and the compatibility of Gysin morphisms with base change ([3, VI Théorème 4.3.12]), we have

(2.17.2.6)
$$\begin{aligned} &R^h f_{(X_{V'_1},D_{V'_1})/V'*}(\mathcal{O}_{(X_{V'_1},D_{V'_1})/V'}) \otimes_{V'} K' \\ &\qquad \xrightarrow{\sim} R^h f_{(X',D')/W'*}(\mathcal{O}_{(X',D')/W'}) \otimes_{W'} K', \end{aligned}$$

(2.17.2.7)
$$E_2^{-k,h+k}((X_{V_1'}, D_{V_1'})/V') \otimes_{V'} K' \overset{\sim}{\longrightarrow} E_2^{-k,h+k}((X', D')/W') \otimes_{W'} K'.$$

Hence it suffices to prove that

$$E_2^{-k,h+k}((X', D')/W') \otimes_{W'} K' = E_\infty^{-k,h+k}((X', D')/W') \otimes_{W'} K'.$$

We have already proved this in [68, (5.2) (1)](=(2.15.4)).

Case II: Next, we consider the general case. Let $\mathfrak{m}$ be a maximal ideal of B_K. Consider the following ideal $I^{(n)}$ and the following ring $B_{(n)}$ in [74, p. 780]:

$$I^{(n)} := \mathrm{Ker}(B \longrightarrow B_K/\mathfrak{m}^n), \quad B_{(n)} := B/I^{(n)} \quad (n \in \mathbb{N}).$$

The ring $B_{(n)}$ defines a p-adic enlargement $S_{(n)}$ of S. Let

$$d_{r,(n)}^{-k,h+k} \colon E_r^{-k,h+k}((X_{(S_{(n)})_1}, D_{(S_{(n)})_1})/S_{(n)})_K \longrightarrow E_r^{-k+r,h+k-r+1}((X_{(S_{(n)})_1}, D_{(S_{(n)})_1})/S_{(n)})_K$$

be the boundary morphism. Because $\{d_r^{\bullet\bullet}\}$ is functorial, we have the following commutative diagram:

$$\begin{tikzcd}
E_r^{-k,h+k}((X,D)/S)\otimes_B (B_{(n)})_K \arrow[r] \arrow[d, "d_r^{-k,h+k}\otimes_{B_K}(B_{(n)})_K"'] & E_r^{-k,h+k}((X_{(S_{(n)})_1}, D_{(S_{(n)})_1})/S_{(n)})_K \arrow[d, "d_{r,(n)}^{-k,h+k}"] \\
E_r^{-k+r,h+k-r+1}((X,D)/S)\otimes_B (B_{(n)})_K \arrow[r] & E_r^{-k+r,h+k-r+1}((X_{(S_{(n)})_1}, D_{(S_{(n)})_1})/S_{(n)})_K.
\end{tikzcd}$$

Because $E_2^{-k,h+k}((X,D)/S)_K$ is a convergent F-isocrystal, the two horizontal morphisms are isomorphisms if $r = 2$. By induction on r and by the proof for the Case I, the boundary morphism $d_r^{\bullet\bullet}\otimes_{B_K}(B_{(n)})_K$ $(r \geq 2)$ vanishes. Thus $\varprojlim_n (d_r^{\bullet\bullet}\otimes_{B_K} B_K/\mathfrak{m}^n) = 0$. Because B_K is a noetherian ring and $E_2^{-k,h+k}((X,D)/S)_K$ is a finitely generated B_K-module, we have

$$d_r^{\bullet\bullet}\otimes_{B_K}(\varprojlim_n B_K/\mathfrak{m}^n) = \varprojlim_n (d_r^{\bullet\bullet}\otimes_{B_K} B_K/\mathfrak{m}^n) = 0.$$

Since $(B_K)_\mathfrak{m}$ is a Zariski ring, $\varprojlim_n (B_K)_\mathfrak{m}/\mathfrak{m}^n(B_K)_\mathfrak{m}$ is faithfully flat over $(B_K)_\mathfrak{m}$ ([13, III §3 Proposition 9]). Therefore $d_r^{\bullet\bullet}\otimes_{B_K}(B_K)_\mathfrak{m} = 0$. Since $\mathfrak{m}$ is an arbitrary maximal ideal of B_K, $d_r^{\bullet\bullet} = 0$ $(r \geq 2)$. Hence we have proved (2.17.2) for (2.9.6.2) $\otimes_V K$.

Next we prove (2.17.2) for (2.11.14.3)$\otimes_V K$ for the case $Z = \emptyset$ and $S_0 = S_1$.

As we remarked before (2.14.0.4), we have the base change property for $R^q f_{(X,D)/S*,\mathrm{c}}(\mathcal{O}_{(X,D)/S})_K = (R^q f_{(X,D)/S*}\mathcal{I}_{(X,D)/S}) \otimes_V K$. Hence the proof is analogous to the proof of (2.17.2) for (2.9.6.2)$\otimes_V K$ for the case $Z = \emptyset$ and $S_0 = S_1$: we have only to use (2.16.2) for $Rf_{(X,D)/S*,\mathrm{c}}(\mathcal{O}_{(X,D)/S})_K$, (2.11.17) and use [68, (5.2) (2)] (=(2.15.4) for (2.11.14.3)). □

We can reprove (2.13.3) in the case $Z = \emptyset$ and more:

Corollary 2.17.3. *Let k be a nonnegative integer. Then the following hold*:

(1) *There exists a convergent F-isocrystal $E_2^{-k,h+k}((X,D)/K)$ such that*

$$E_2^{-k,h+k}((X,D)/K)_T = \mathrm{gr}^P_{h+k} R^h f_{(X_{T_1},D_{T_1})/T*}(\mathcal{O}_{(X_{T_1},D_{T_1})/T})_K$$

for any p-adic enlargement T of S over $\mathrm{Spf}(V)$.

(2) *There exists a convergent F-isocrystal $P_kR^h f_*(\mathcal{O}_{(X,D)/K})$ such that*

$$P_kR^h f_*(\mathcal{O}_{(X,D)/K})_T = P_kR^h f_{(X_{T_1},D_{T_1})/T*}(\mathcal{O}_{(X_{T_1},D_{T_1})/T})_K$$

for any p-adic enlargement T of S over $\mathrm{Spf}(V)$.

(3) *There exists a spectral sequence of convergent F-isocrystals on (X,D) $/S$ over* $\mathrm{Spf}(V)$:

$$(2.17.3.1)\qquad E_1^{-k,h+k}((X,D)/K) = R^{h-k} f_*(\mathcal{O}_{D^{(k)}/K} \otimes_{\mathbb{Z}} \varpi^{(k)}(D/K))(-k) \Longrightarrow R^h f_*(\mathcal{O}_{(X,D)/K}).$$

This spectral sequence degenerates at E_2.

Proof. (1): By (2.8.5), the boundary morphism $d_1^{\bullet\bullet}$ of (2.9.6.2) $\otimes_V K$ is a summation (with signs) of Gysin morphisms, and thus $d_1^{\bullet\bullet}$ is a morphism of convergent F-isocrystals by [74, (3.13)]. By [74, (3.1)] and by (2.17.2), we obtain (1).

(2): By (1), for a morphism $g\colon T' \longrightarrow T$ of p-adic affine enlargements of S over $\mathrm{Spf}(V)$, $P_kR^h f_{(X_{T'_1},D_{T'_1})/T'*}(\mathcal{O}_{(X_{T'_1},D_{T'_1})/T'})_K = g^*P_kR^h f_{(X_{T_1},D_{T_1})/T*}(\mathcal{O}_{(X_T,D_T)/T})_K$. The claim on the F-isocrystal follows as in [74, (3.7)].

(3): (3) immediately follows from (2.17.2). $\square$

We can reprove (2.13.8) (1) and (2) in the case $Z = \emptyset$ and more:

Corollary 2.17.4. *Let k be a nonnegative integer. Then the following hold*:

(1) *There exists a convergent F-isocrystal $E_{2,\mathrm{c}}^{k,h-k}((X,D)/K)$ such that*

$$E_{2,\mathrm{c}}^{k,h-k}((X,D)/K)_T = \mathrm{gr}^P_{h-k} R^h f_{(X_{T_1},D_{T_1})/T*,\mathrm{c}}(\mathcal{O}_{(X_{T_1},D_{T_1})/T})_K$$

for any p-adic enlargement T of S over $\mathrm{Spf}(V)$.

(2) *There exists a convergent F-isocrystal $P_kR^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K})$ on $S/$* $\mathrm{Spf}(V)$ *such that*

$$(P_kR^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}))_T = P_kR^h f_{(X_{T_1},D_{T_1})/T*,\mathrm{c}}(\mathcal{O}_{(X_{T_1},D_{T_1})/T})_K$$

for any p-adic enlargement T of $S/$ $\mathrm{Spf}(V)$.

(3) *There exists a spectral sequence of convergent F-isocrystals on X/S over* $\mathrm{Spf}(V)$:

$$E_{1,c}^{k,h-k}((X,D)/K) = R^{h-k}f_*(\mathcal{O}_{D^{(k)}/K} \otimes_{\mathbb{Z}} \varpi^{(k)}(D/K)) \Longrightarrow R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}). \tag{2.17.4.1}$$

This spectral sequence degenerates at E_2.

Proof. (1), (2), (3): We obtain (1), (2) and (3) as in (2.17.3). □

As in [11, §7], for a p-adic formal V-scheme S, we have a log crystalline topos $((\widetilde{X,D})/S)^{\log}_{\mathrm{crys}}$ and the forgetting log morphism $\epsilon_{(X,D)/S} \colon ((\widetilde{X,D})/S)^{\log}_{\mathrm{crys}} \longrightarrow (\widetilde{\overset{\circ}{X}}/S)_{\mathrm{crys}}$. The following is nothing but a restatement of a part of (2.17.2) by the p-adic version of (2.7.6):

Corollary 2.17.5. *The following Leray spectral sequence*

$$E_2^{k,h-k} = R^{h-k}\overset{\circ}{f}_{(X,D)/S*}R^k\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})_K \Longrightarrow R^h f_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})_K \tag{2.17.5.1}$$

degenerates at E_3.

2.18 Strict Compatibility

In this section, using a specialization argument of Deligne-Illusie (§2.14) and by using the convergence of the weight filtration (§2.13, §2.17), we prove the strictness of the induced morphism of log crystalline cohomologies by a morphism of log schemes with respect to the weight filtration.

Let V be a complete discrete valuation ring of mixed characteristics with perfect residue field κ of characteristic $p > 0$ and with fraction field K. Let $g \colon (X', D') \longrightarrow (X, D)$ be a morphism of two proper smooth schemes with SNCD's over κ. Let W be the Witt ring of κ and K_0 the fraction field of W. Then the following holds:

Theorem 2.18.1. *Let h be an integer. Then the following hold*:

(1) *The induced morphism*

$$g^{\log*}_{\mathrm{crys}} \colon H^h_{\text{log-crys}}(X/W)_K \longrightarrow H^h_{\text{log-crys}}(X'/W)_K \tag{2.18.1.1}$$

is strictly compatible with the weight filtration.

(2) *Assume that g induces morphisms $g^{(k)} \colon D'^{(k)} \longrightarrow D^{(k)}$ for all $k \in \mathbb{N}$. Then the induced morphism*

$$g^{\log*}_{\mathrm{crys,c}} \colon H^h_{\text{log-crys,c}}(X/W)_K \longrightarrow H^h_{\text{log-crys,c}}(X'/W)_K \tag{2.18.1.2}$$

is strictly compatible with the weight filtration.

Proof. (1): In this proof, for the sake of clarity, denote by P and P' the weight filtrations on $H^h_{\text{log-crys}}(X/W)_{K_0}$ and $H^h_{\text{log-crys}}(X'/W)_{K_0}$, respectively.

Since $P_kH^h_{\text{log-crys}}(X/W)_{K_0}\otimes_{K_0}K=(P_kH^h_{\text{log-crys}}(X/W))_K$ $(k\in\mathbb{Z}\cup\{\infty\})$, we may assume that $V=W$. By (2.9.1) the morphism g induces a morphism

(2.18.1.3)
$$g^{\log *}_{\text{crys}}\colon P_kH^h_{\text{log-crys}}(X/W)_{K_0}\longrightarrow P'_kH^h_{\text{log-crys}}(X'/W)_{K_0}\quad (k\in\mathbb{Z}\cup\{\infty\}).$$

Let $P''_kH^h_{\text{log-crys}}(X'/W)_{K_0}$ be the image of $P_kH^h_{\text{log-crys}}(X/W)_{K_0}$ by $g^{\log *}_{\text{crys}}$. Then we prove that

(2.18.1.4)
$$P'_k\cap P''_\infty=P''_k.$$

By [40, 3, (8.9.1) (iii), (8.10.5)] and [40, 4, (17.7.8)], there exists a model of g, that is, there exists a morphism $\mathfrak{g}\colon(\mathcal{X}',\mathcal{D}')\longrightarrow(\mathcal{X},\mathcal{D})$ of proper smooth schemes with relative SNCD's over the spectrum $S_1:=\text{Spec}(A_1)$ of a smooth algebra $A_1(\subset\kappa)$ over a finite field $\mathbb{F}_q$ such that $\mathfrak{g}\otimes_{A_1}\kappa=g$. By a standard deformation theory ([41, III (6.10)]), there exists a formally smooth scheme $S=\text{Spf}(A)$ over $\text{Spf}(W(\mathbb{F}_q))$ such that $S\otimes_{W(\mathbb{F}_q)}\mathbb{F}_q=S_1$. We fix a lift $F\colon S\longrightarrow S$ of the Frobenius of S_1. Then, as in §2.15, W is an A-algebra. Let $\mathcal{P}'$ and $\mathcal{P}''$ be the analogous filtrations on $R^hf_{\mathcal{X}'/S*}(\mathcal{O}_{\mathcal{X}'/S})\otimes_{W(\mathbb{F}_q)}K_0(\mathbb{F}_q)$, where $K_0(\mathbb{F}_q)$ is the fraction field of $W(\mathbb{F}_q)$. By (2.14.2), in order to prove (2.18.1.4), it suffices to prove that

(2.18.1.5)
$$\mathcal{P}'_k\cap\mathcal{P}''_\infty=\mathcal{P}''_k$$

by shrinking S. Here, note that the extension $\kappa/\text{Frac}(A_1)$ of fields may be infinite and transcendental. Because $\mathcal{P}'_k$ and $\mathcal{P}''_\infty$ are convergent isocrystals ((2.13.3) or (2.17.3)), so is $\mathcal{P}'_k\cap\mathcal{P}''_\infty$ by [74, (2.10)]. Since two inclusions $(\mathcal{P}'_k\cap\mathcal{P}''_\infty)\cap\mathcal{P}''_k\longrightarrow\mathcal{P}''_k$ and $(\mathcal{P}'_k\cap\mathcal{P}''_\infty)\cap\mathcal{P}''_k\longrightarrow\mathcal{P}'_k\cap\mathcal{P}''_\infty$ are morphisms of convergent isocrystals, it suffice to prove that

(2.18.1.6)
$$(\mathcal{P}'_k\cap\mathcal{P}''_\infty)_s=(\mathcal{P}''_k)_s$$

for any closed point $s\in S$ by [74, (3.17)]. In this case, (2.18.1.6) immediately follows from the purity of the weight of the crystalline cohomologies ([15, (1.2)] or [68, (2.2) (4))]) and by the spectral sequence (2.9.6.2). Thus we have proved (1).

(2): By the assumption of g, the analogue of (2.18.1.3) for the log crystalline cohomology with compact support holds. Using (2.13.8) instead of (2.13.3), we obtain (2) in a similar way. □

Theorem 2.18.2 (Strict compatibility). *Let S be a p-adic formal V-scheme. Let $f\colon(X,D)\longrightarrow S_1$ and $f'\colon(X',D')\longrightarrow S_1$ be proper smooth schemes with relative SNCD's over S_1. Let $g\colon(X',D')\longrightarrow(X,D)$ be a morphism of log schemes over S_1. Let h be an integer. Then the following hold*:

(1) *The induced morphism*

(2.18.2.1)
$$g^*\colon R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})_K \longrightarrow R^hf'_{(X',D')/S*}(\mathcal{O}_{(X',D')/S})_K \quad (h\in\mathbb{Z})$$

is strictly compatible with the weight filtration.
(2) *Assume that g induces morphisms $g^{(k)}\colon D'^{(k)} \longrightarrow D^{(k)}$ for all $k\in\mathbb{N}$. Then the induced morphism*

(2.18.2.2)
$$g^*_{\rm c}\colon R^hf_{(X,D)/S*,{\rm c}}(\mathcal{O}_{(X,D)/S})_K \longrightarrow R^hf'_{(X',D')/S*,{\rm c}}(\mathcal{O}_{(X',D')/S})_K \quad (h\in\mathbb{Z})$$

is strictly compatible with the weight filtration.

Proof. Since the proofs of (1) and (2) are similar, we give only the proof of (1).

By (2.13.3) (or (2.17.3)) and by the proof of [74, (3.17)], we may assume that S is the formal spectrum of a finite extension V' of V. Let κ' be the residue field of V'. As mentioned in the proof of (2.17.2), V'/p is an κ'-algebra; the two pairs (X,D) and $((X,D)\otimes_{V'}\kappa')\otimes_{\kappa'}V'/p$ are two deformations of $(X,D)\otimes_{V'}\kappa'$; the obvious analogue for (X',D') also holds. Hence, by the deformation invariance of log crystalline cohomologies with weight filtrations ((2.16.2)), we may assume that $S=\mathrm{Spf}(W(\kappa'))$ and that (X,D) and (X',D') are smooth schemes with SNCD's over a perfect field κ' of characteristic $p>0$. Hence (1) follows from (2.18.1) (1). □

Corollary 2.18.3. *Let the notations be as in* (2.18.2). *Let $g\colon(X',D')\longrightarrow(X,D)$ be a log etale morphism such that $Rg_*(\mathcal{O}_{X'})=\mathcal{O}_X$ (e.g., the blowing up along center a smooth component of $D^{(k)}$). Then g^* in* (2.18.2.1) *is a filtered isomorphism.*

Proof. We may assume that S is flat over $\mathrm{Spf}(V)$. By the second proof of [65, (2.2)] and by [loc. cit., (2.4)], the induced morphism

$$Rf_*(\Omega^\bullet_{X/S_1}(\log D)) \longrightarrow Rf'_*(\Omega^\bullet_{X'/S_1}(\log D'))$$

is an isomorphism (cf. [43, VII (3.5)], (2.18.7) below). By the log version of a triangle in the proof of [11, 7.16 Theorem] and by the log version of [11, 7.22.2], the induced morphism

$$g^*\colon Rf_{X/S*}(\mathcal{O}_{(X,D)/S}) \longrightarrow Rf'_{(X',D')/S*}(\mathcal{O}_{(X',D')/S})$$

is an isomorphism; in particular, $g^*\colon R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})_K \longrightarrow R^hf_{(X',D')/S*}(\mathcal{O}_{(X',D')/S})_K$ is an isomorphism. (2.18.3) follows from (2.18.2) (1). □

Remark 2.18.4. Let the notations be as in (2.18.2). We do not know an example such that the induced morphism $g^*\colon (R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}),P) \longrightarrow (R^hf'_{(X',D')/S*}(\mathcal{O}_{(X',D')/S}),P)$ is not strictly compatible with the weight filtration.

Theorem 2.18.5. *Let the notations be as in* (2.18.2)*. Assume that g induces morphisms $g^{(k)}\colon D'^{(k)} \longrightarrow D^{(k)}$ for all $k \in \mathbb{N}$. Assume, moreover, that g is log etale, that $Rg_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and that $g^*(\mathcal{O}_X(-D)) = \mathcal{O}_{X'}(-D')$. Then g_{c}^* in* (2.18.2.2) *is a filtered isomorphism.*

Proof. We may assume that S is flat over $\mathrm{Spf}(V)$. Because g is log etale, we have $g^*(\Omega^i_{X/S_1}(\log D)) = \Omega^i_{X'/S_1}(\log D')$ $(i \in \mathbb{N})$. Hence, by the assumption, we have $g^*(\Omega^i_{X/S_1}(-\log D)) = \Omega^i_{X'/S_1}(-\log D')$. By using the projection formula as in [65, p. 168], we have $\Omega^i_{X/S_1}(-\log D) = Rg_*(\Omega^i_{X'/S_1}(-\log D'))$. Consequently, as in [65, (2.4)], we have $\Omega^\bullet_{X/S_1}(-\log D) = Rg_*(\Omega^\bullet_{X'/S_1}(-\log D'))$ by using the spectral sequence

$$E_1^{ij} = R^j g_*(\Omega^i_{X'/S_1}(-\log D')) \Longrightarrow R^{i+j} g_*(\Omega^\bullet_{X'/S_1}(-\log D')).$$

Let n be a positive integer, and set $S_n := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/p^n)$. Then we have an exact sequence

$$0 \longrightarrow p^n\mathcal{O}_S/p^{n+1}\mathcal{O}_S \longrightarrow \mathcal{O}_{S_{n+1}} \longrightarrow \mathcal{O}_{S_n} \longrightarrow 0.$$

By using the base change theorem of the log crystalline cohomology sheaf with compact support ((2.11.11.1)), we have the following triangle as in [11, 7.16 Theorem]:

$$\begin{aligned}
&\longrightarrow Rf_{(X,D)/S_1*,\mathrm{c}}(\mathcal{O}_{(X,D)/S}) \otimes^L_{\mathcal{O}_{S_1}} p^n\mathcal{O}_S/p^{n+1}\mathcal{O}_S \\
&\longrightarrow Rf_{(X,D)/S_{n+1}*,\mathrm{c}}(\mathcal{O}_{(X,D)/S_{n+1}}) \\
&\longrightarrow Rf_{(X,D)/S_n*,\mathrm{c}}(\mathcal{O}_{(X,D)/S_n}) \xrightarrow{+1} \cdots .
\end{aligned} \tag{2.18.5.1}$$

Hence, by induction on n and by (2.11.7.1) and [11, 7.22.2], we have

$$R^h f_{(X',D')/S*,\mathrm{c}}(\mathcal{O}_{(X',D')/S}) = R^h f_{(X,D)/S*,\mathrm{c}}(\mathcal{O}_{(X,D)/S}).$$

In particular, g_{c}^* is an isomorphism of $\mathcal{O}_S \otimes_V K$-modules. Moreover, by (2.18.2) (2), g_{c}^* is a filtered isomorphism. $\square$

Remark 2.18.6. It is straightforward to generalize (2.18.2), (2.18.3), (2.18.5) into the framework of convergent F-isocrystals.

Remark 2.18.7. The following example (=a very special case of [65, (2.3)]) shows that the strictness of the induced morphism on sheaves of log differential forms by a morphism of smooth schemes with relative SNCD's does not hold.

Let S be a scheme and let X be an affine plane $\mathbb{A}^2_S = \underline{\mathrm{Spec}}_S(\mathcal{O}_S[x,y])$. Let D be a relative SNCD on X/S defined by $xy = 0$. Let $g\colon X' \longrightarrow X$ be the blow up of X along the center $(0,0)$. Let D' be the union of the strict transform of D and the exceptional divisor of g; then D' is a relative SNCD on X'/S. Let i be an integer. Then Mokrane has proved that

$R^jg_*(\Omega^i_{X'/S}(\log D'))=0$ ($j\in \mathbb{Z}_{>0}$) and $g_*(\Omega^i_{X'/S}(\log D'))=\Omega^i_{X/S}(\log D)$ (a very special case of [65, (2.2)]; however, note that in the notations in [loc. cit.], the condition that the closed immersion $Y \overset{\subset}{\longrightarrow} X$ is a regular embedding is necessary for [loc. cit.] because the fact $Rf_*(\mathcal{O}_{X'})=\mathcal{O}_X$ in [43, VII (3.5)] has been shown under this assumption.). The pull-back morphism

$$g^* \colon (\Omega^2_{X/S}(\log D), P) \longrightarrow g_*(\Omega^2_{X'/S}(\log D'), P)$$

is a morphism of filtered sheaves; however, as remarked in [loc. cit.], g^* is not strict. (Consequently g^* does not induce an isomorphism of filtered sheaves of log differential forms.)

Note that the number of smooth components of D' is more than those of D; the log structure of (X', D') is "bigger" than that of (X, D).

Remark 2.18.8. The following remark is the crystalline analogue of a part of results in [24, (9.2)].

Let $(S,\mathcal{I},\gamma)$ and S_0 be as in §2.4. Let $f\colon (X,D)\longrightarrow S_0$ be a smooth scheme with a smooth relative divisor over S_0. Let $a\colon D \overset{\subset}{\longrightarrow} X$ be the natural closed immersion. Then, by (2.6.1.1), we have the following exact sequence

$$\begin{aligned}(2.18.8.1)\qquad 0 \longrightarrow Q^*_{X/S}(\mathcal{O}_{X/S}) &\longrightarrow Q^*_{X/S}C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}) \\ &\longrightarrow Q^*_{X/S}a_{\mathrm{crys}*}(\mathcal{O}_{D/S})(-1)\{-1\} \longrightarrow 0.\end{aligned}$$

Applying the higher direct image functor $R^\bullet\overline{f}_{X/S*}$ to (2.18.8.1), we have the following exact sequence

(2.18.8.2)

$$\begin{aligned}\cdots \longrightarrow R^{h-2}f_{D/S*}(\mathcal{O}_{D/S})(-1) &\longrightarrow R^hf_{X/S*}(\mathcal{O}_{X/S}) \\ &\longrightarrow R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) \longrightarrow \cdots .\end{aligned}$$

The spectral sequence (2.9.6.2) degenerates at E_2 in this case since $E_2^{ij}=0$ if $i\neq 0$ or $i\neq -1$. It is easy to check that the exact sequence (2.18.8.2) is strictly compatible with the preweight filtration.

Using (2.11.7.1), we also have the following exact sequence which is strictly compatible with the preweight filtration

(2.18.8.3)

$$\begin{aligned}\cdots \longrightarrow R^hf_{X/S*}(\mathcal{O}_{X/S}) &\longrightarrow R^hf_{D/S*}(\mathcal{O}_{D/S}) \\ &\longrightarrow R^{h+1}f_{(X,D)/S*,\mathrm{c}}(\mathcal{O}_{(X,D)/S}) \longrightarrow \cdots .\end{aligned}$$

Now assume that S is a p-adic formal V-scheme (in the sense of [74, §1]) over a complete discrete valuation ring V of mixed characteristics with perfect residue field. Assume also that $S_0=\underline{\mathrm{Spec}}_S(\mathcal{O}_S/p)$, that X is projective over S_0 of pure relative dimension d and that D is a smooth hypersurface section. Let K be the fraction field of V. Then the induced morphism

(2.18.8.4) $$R^h f_{X/S*}(\mathcal{O}_{X/S})_K \longrightarrow R^h f_{D/S*}(\mathcal{O}_{D/S})_K$$

by the closed immersion $D \overset{\subset}{\longrightarrow} X$ is an isomorphism for $h \leq d-2$ and an injection for $h = d-1$ (cf. [2, Théorème]). Indeed, first consider the case $h \leq d-2$. Then we can assume that S is the formal spectrum of a finite extension of V by [74, (3.17)]. In this case, the argument in the proof of (2.18.2) and the specialization argument of Deligne-Illusie ([49, 3.10], cf. the argument in (2.18.1)) show that the hard Lefschetz theorem holds for $R^h f_{X/S*}(\mathcal{O}_{X/S})_K$ (cf. [49, 3.8]). Hence the proof of [57, p. 76 Corollary] shows that (2.18.8.4) is an isomorphism for $h \leq d-2$. As to the case $h = d-1$, the same proof works by considering the image of $R^{d-1} f_{X/S*}(\mathcal{O}_{X/S})_K$ in $R^{d-1} f_{D/S*}(\mathcal{O}_{D/S})_K$. By the Poincaré duality ([74, (3.12)]), the Gysin morphism

$$G_h \colon R^{h-2} f_{D/S*}(\mathcal{O}_{D/S})_K(-1) \longrightarrow R^h f_{X/S*}(\mathcal{O}_{X/S})_K$$

is an isomorphism for $h \geq d+2$ and a surjection for $h \geq d+1$. Set

$$R^{d-1} f_{D/S*,\mathrm{ev}}(\mathcal{O}_{D/S})_K(-1) := \mathrm{Ker}\ G_{d+1}.$$

Then $R^{d-1} f_{D/S*,\mathrm{ev}}(\mathcal{O}_{D/S})_K(-1)$ is the orthogonal part of the image of the injective morphism $R^{d-1} f_{X/S*}(\mathcal{O}_{X/S})_K \longrightarrow R^{d-1} f_{D/S*}(\mathcal{O}_{D/S})_K$. Therefore we have the following direct decomposition:

(2.18.8.5)
$$R^{d-1} f_{D/S*}(\mathcal{O}_{D/S})_K = R^{d-1} f_{D/S*,\mathrm{ev}}(\mathcal{O}_{D/S})_K \oplus R^{d-1} f_{X/S*}(\mathcal{O}_{X/S})_K.$$

2.19 The Weight-Filtered Poincaré Duality

The following is the Poincaré duality:

Theorem 2.19.1 (Weight-filtered Poincaré duality). *Let V be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic $p > 0$. Let S be a p-adic formal V-scheme. Let (X, D) be a formally smooth scheme with a relative SNCD over S. Assume that X/S is projective and that the relative dimension of X/S is of pure dimension d. Then there exists a perfect pairing of convergent F-isocrystal on $S/\mathrm{Spf}(V)$*

(2.19.1.1) $$R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}) \otimes R^{2d-h} f_*(\mathcal{O}_{(X,D)/K}) \longrightarrow \mathcal{O}_{S/K}(-d),$$

which is strictly compatible with the weight filtration. That is, the natural morphism

(2.19.1.2)
$$R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{S/K}}(R^{2d-h} f_*(\mathcal{O}_{(X,D)/K}), \mathcal{O}_{S/K}(-d))$$

is an isomorphism of weight-filtered convergent F-isocrystals on S/V.

Proof. By (2.11.3), there exists a canonical morphism $R^{2d}f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}) \longrightarrow R^{2d}f_*(\mathcal{O}_{X/K})$ of convergent isocrystals on $S/\mathrm{Spf}(V)$, which is constructed from natural morphisms $R^{2d}f_{*,\mathrm{c}}(\mathcal{O}_{(X_{T_1},D_{T_1})/T}) \longrightarrow R^{2d}f_*(\mathcal{O}_{X_{T_1}/T})$ for p-adic enlargements T of $S/\mathrm{Spf}(V)$. Using the cup product, we have the following composite morphism

(2.19.1.3)

$$R^h f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}) \otimes R^{2d-h} f_*(\mathcal{O}_{(X,D)/K}) \xrightarrow{\cup} R^{2d} f_{*,\mathrm{c}}(\mathcal{O}_{(X,D)/K}) \longrightarrow R^{2d} f_*(\mathcal{O}_{X/K}) \xrightarrow{\mathrm{Tr}_{\overset{\circ}{f}}} \mathcal{O}_{S/K}(-d).$$

by [74, (3.12.1)]. The morphism (2.19.1.2) is an isomorphism. Indeed, by [74, (3.17)], we may assume that S is the spectrum of a perfect field κ of finite characteristic. In this case $\mathrm{Tr}_{\overset{\circ}{f}}$ is the classical trace map ([74, pp. 809–810]), Therefore (2.19.1.2) for $S = \mathrm{Spec}(\kappa)$ is an isomorphism by [85, (5.6)], and hence we have an isomorphism (2.19.1.2).

By using the arguments in (2.18.1) and (2.18.2), we obtain the strict compatibility of the isomorphism (2.19.1.2) with the weight filtration. $\square$

2.20 *l*-adic Weight Spectral Sequence

Let S be a scheme. Let $(X,D)/S$ be a proper smooth scheme with a relative SNCD. Set $U := X \setminus D$ and let $f\colon U \longrightarrow S$ be the structural morphism. Let $f^{(k)}\colon D^{(k)} \longrightarrow S$ ($k \in \mathbb{Z}_{\geq 0}$) be the structural morphism and $a^{(k)}\colon D^{(k)} \longrightarrow X$ also the natural morphism. Let l be a prime number which is invertible on S. Let $\varpi^{(k)}_{\mathrm{et}}(D/S)(-k)$ ($k \in \mathbb{N}$) be the etale orientation sheaf of $D^{(k)}$: $\varpi^{(k)}_{\mathrm{et}}(D/S)(-k) := \{u^{-1}(\bigwedge^{k}(M(D)/\mathcal{O}^*_X))\}|_{D^{(k)}_{\mathrm{et}}}$, where u is the canonical morphism $\widetilde{X}_{\mathrm{et}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$ of topoi. Here note that we do not define "$\varpi^{(k)}_{\mathrm{et}}(D/S)$". If S is of characteristic $p > 0$, then the Frobenius of (X,D) acts on $\varpi^{(k)}_{\mathrm{et}}(D/S)(-k)$ by the multiplication by p^k. Almost all the results in the previous sections have l-adic analogues. For example, the excision spectral sequence

(2.20.0.1) $$E_1^{k,h-k} = R^{h-k} f^{(k)}_*(\mathbb{Q}_l(k) \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{et}}(D/S)(-k)) \Longrightarrow R^h f_{*,\mathrm{c}}(\mathbb{Q}_l).$$

calculates $R^h f_{*,\mathrm{c}}(\mathbb{Q}_l)$.

Let $j\colon U \overset{\subset}{\longrightarrow} X$ be the open immersion. By Grothendieck's absolute purity, which has been solved by O. Gabber ([33]), we obtain $R^k j_*(\mathbb{Q}_l) \xrightarrow{\sim} a^{(k)}_*(\mathbb{Q}_{l,D^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{et}}(D/S)(-k))$. As in the Introduction, we use the following isomorphism

$$R^k j_*(\mathbb{Q}_l) \xrightarrow{\sim} a_*^{(k)}(\mathbb{Q}_{l,D^{(k)}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{et}}^{(k)}(D/S)(-k)) \xrightarrow[\sim]{(-1)^k} a_*^{(k)}(\mathbb{Q}_{l,D^{(k)}} \otimes_{\mathbb{Z}} \varpi_{\mathrm{et}}^{(k)}(D/S)(-k)). \tag{2.20.0.2}$$

Then we have the following spectral sequence:

$$E_2^{k,h-k} = R^{h-k} f_*^{(k)}(\mathbb{Q}_l \otimes_{\mathbb{Z}} \varpi_{\mathrm{et}}^{(k)}(D/S)(-k)) \Longrightarrow R^k f_*(\mathbb{Q}_l). \tag{2.20.0.3}$$

The spectral sequence (2.20.0.1) (resp. (2.20.0.3)) degenerates at E_2 (resp. E_3) by the standard specialization argument (e.g., [34]) and the Weil conjecture ([26, (3.3.9)]).