Preface

Computational Fluid Dynamics: An Introduction grew out of a von Karman Institute (VKI) Lecture Series by the same title first presented in 1985 and repeated with modifications every year since that time.

The objective, then and now, was to present the subject of computational fluid dynamics (CFD) to an audience unfamiliar with all but the most basic numerical techniques and to do so in such a way that the practical application of CFD would become clear to everyone.

A second edition appeared in 1995 with updates to all the chapters and when that printing came to an end, the publisher requested that the editor and authors consider the preparation of a third edition. Happily, the authors received the request with enthusiasm.

The third edition has the goal of presenting additional updates and clarifications while preserving the introductory nature of the material.

The book is divided into three parts. John Anderson lays out the subject in Part I by first describing the governing equations of fluid dynamics, concentrating on their mathematical properties which contain the keys to the choice of the numerical approach. Methods of discretizing the equations are discussed and transformation techniques and grids are presented. Two examples of numerical methods close out this part of the book: source and vortex panel methods and the explicit method.

Part II is devoted to four self-contained chapters on more advanced material. Roger Grundmann treats the boundary layer equations and methods of solution. Gerard Degrez treats implicit time-marching methods for inviscid and viscous compressible flows; relative to the second edition, figures in the section on stability properties have been added and the section on numerical dissipation has been expanded with examples. Eric Dick, in two separate articles, treats both finite volume and finite element methods; the sections on current developments have been updated and references to a number of essential recent publications have been added.

Part III brings a new contribution by Jan Vierendeels and Joris Degroote which provides insight into the steps that are needed to obtain a CFD solution of a flow field using commercial CFD software packages. The wide availability of such codes

provides advantages for the non-specialist in numerical techniques, but requires an appreciation of their limitations and knowledge of an application methodology.

The editor and authors will consider this book to have been successful if the readers conclude they have been well prepared to examine the literature in the field and to begin the application of CFD methods to the resolution of problems in their area of interest.

The editor takes this opportunity to thank the authors for their contributions to this book and for their enthusiasm to continue the tradition of continually improving the VKI Lecture Series on which it is based.

Eagle River, WI, USA *John F. Wendt*

Chapter 2
Governing Equations of Fluid Dynamics

J.D. Anderson, Jr.

2.1 Introduction

The cornerstone of computational fluid dynamics is the fundamental governing equations of fluid dynamics—the continuity, momentum and energy equations. *These equations speak physics*. They are the mathematical statements of three fundamental physical principles upon which all of fluid dynamics is based:

(1) mass is conserved;
(2) F = ma (Newton's second law);
(3) energy is conserved.

The purpose of this chapter is to derive and discuss these equations.

The purpose of taking the time and space to derive the governing equations of fluid dynamics in this course are three-fold:

(1) Because all of CFD is based on these equations, it is important for each student to feel very comfortable with these equations before continuing further with his or her studies, and certainly before embarking on any application of CFD to a particular problem.
(2) This author assumes that the attendees of the present VKI short course come from varied background and experience. Some of you may not be totally familiar with these equations, whereas others may use them every day. For the former, this chapter will hopefully be some enlightenment; for the latter, hopefully this chapter will be an interesting review.
(3) The governing equations can be obtained in various different forms. For most aerodynamic theory, the particular form of the equations makes little difference. However, for CFD, the use of the equations in one form may lead to success, whereas the use of an alternate form may result in oscillations (wiggles) in the numerical results, or even instability. Therefore, in the world of CFD, the various forms of the equations are of vital interest. In turn, it is important to *derive* these equations in order to point out their differences and similarities, and to reflect on possible implications in their application to CFD.

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J.F. Wendt (ed.), *Computational Fluid Dynamics*, 3rd ed.,

2.2 Modelling of the Flow

In obtaining the basic equations of fluid motion, the following philosophy is always followed:

(1) Choose the appropriate fundamental physical principles from the laws of physics, such as

   (a) Mass is conserved.
   (b) F = ma (Newton's 2nd Law).
   (c) Energy is conserved.

(2) Apply these physical principles to a suitable model of the flow.
(3) From this application, extract the mathematical equations which embody such physical principles.

This section deals with item (2) above, namely the definition of a suitable model of the flow. This is not a trivial consideration. A solid body is rather easy to see and define; on the other hand, a fluid is a 'squishy' substance that is hard to grab hold of. If a solid body is in translational motion, the velocity of each part of the body is the same; on the other hand, if a fluid is in motion the velocity may be different at each location in the fluid. How then do we visualize a moving fluid so as to apply to it the fundamental physical principles?

For a continuum fluid, the answer is to construct one of the two following models.

2.2.1 Finite Control Volume

Consider a general flow field as represented by the streamlines in Fig. 2.1(a). Let us imagine a closed volume drawn within a *finite* region of the flow. This volume defines a *control volume, V*, and a *control surface, S*, is defined as the closed surface which bounds the volume. The control volume may be *fixed* in space with the fluid moving through it, as shown at the left of Fig. 2.1(a). Alternatively, the control volume may be moving with the fluid such that the same fluid particles are always inside it, as shown at the right of Fig. 2.1(a). In either case, the control volume is a reasonably large, finite region of the flow. The fundamental physical principles are applied to the fluid inside the control volume, and to the fluid crossing the control surface (if the control volume is fixed in space). Therefore, instead of looking at the whole flow field at once, with the control volume model we limit our attention to just the fluid in the finite region of the volume itself. The fluid flow equations that we *directly* obtain by applying the fundamental physical principles to a finite control volume are in *integral form*. These integral forms of the governing equations can be manipulated to *indirectly* obtain partial differential equations. The equations so obtained from the finite control volume fixed in space (left side of Fig. 2.1a), in either integral or partial differential form, are called the *conservation* form of the governing equations. The equations obtained from the finite control volume moving

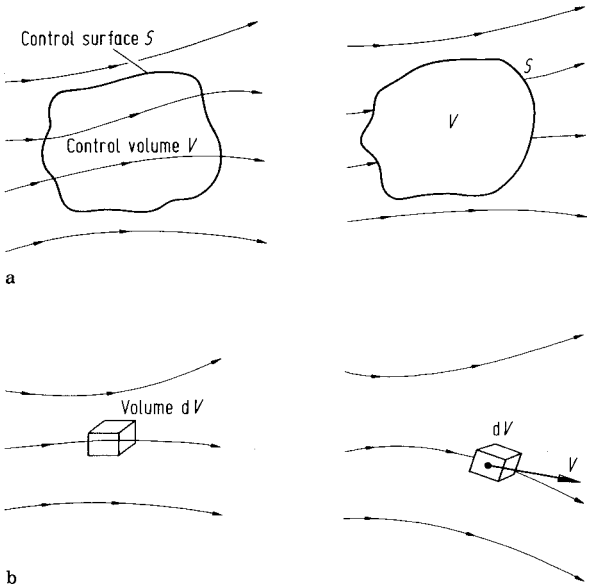

Fig. 2.1 (**a**) Finite control volume approach. (**b**) Infinitesimal fluid element approach

with the fluid (right side of Fig. 2.1a), in either integral or partial differential form, are called the *non-conservation* form of the governing equations.

2.2.2 Infinitesimal Fluid Element

Consider a general flow field as represented by the streamlines in Fig. 2.1b. Let us imagine an infinitesimally small fluid element in the flow, with a differential volume, dV. The fluid element is infinitesimal in the same sense as differential calculus; however, it is large enough to contain a huge number of molecules so that it can be viewed as a continuous medium. The fluid element may be fixed in space with the fluid moving through it, as shown at the left of Fig. 2.1(b). Alternatively, it may be moving along a streamline with a vector velocity V equal to the flow velocity at each point. Again, instead of looking at the whole flow field at once, the fundamental physical principles are applied to just the fluid element itself. This application leads *directly* to the fundamental equations in *partial differential equation form*. Moreover, the particular partial differential equations obtained directly from the fluid element fixed in space (left side of Fig. 2.1b) are again the *conservation form* of the equations. The partial differential equations obtained *directly* from the moving fluid element (right side of Fig. 2.1b) are again called the *non-conservation form* of the equations.

In general aerodynamic theory, whether we deal with the conservation or nonconservation forms of the equations is irrelevant. Indeed, through simple manipulation, one form can be obtained from the other. However, there are cases in CFD where it is important which form we use. In fact, the nomenclature which is used to distinguish these two forms (conservation versus nonconservation) has arisen primarily in the CFD literature.

The comments made in this section become more clear after we have actually derived the governing equations. Therefore, when you finish this chapter, it would be worthwhile to re-read this section.

As a final comment, in actuality, the motion of a fluid is a ramification of the mean motion of its atoms and molecules. Therefore, a third model of the flow can be a microscopic approach wherein the fundamental laws of nature are applied directly to the atoms and molecules, using suitable statistical averaging to define the resulting fluid properties. This approach is in the purview of *kinetic theory*, which is a very elegant method with many advantages in the long run. However, it is beyond the scope of the present notes.

2.3 The Substantial Derivative

Before deriving the governing equations, we need to establish a notation which is common in aerodynamics—that of the substantial derivative. In addition, the substantial derivative has an important physical meaning which is sometimes not fully appreciated by students of aerodynamics. A major purpose of this section is to emphasize this physical meaning.

As the model of the flow, we will adopt the picture shown at the right of Fig. 2.1(b), namely that of an infinitesimally small fluid element moving with the flow. The motion of this fluid element is shown in more detail in Fig. 2.2. Here, the fluid element is moving through cartesian space. The unit vectors along the x, y, and z axes are $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively. The vector velocity field in this cartesian space is given by

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

where the x, y, and z components of velocity are given respectively by

$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$

Note that we are considering in general an *unsteady flow*, where u, v, and w are functions of both space and time, t. In addition, the scalar density field is given by

$$\rho = \rho(x, y, z, t)$$

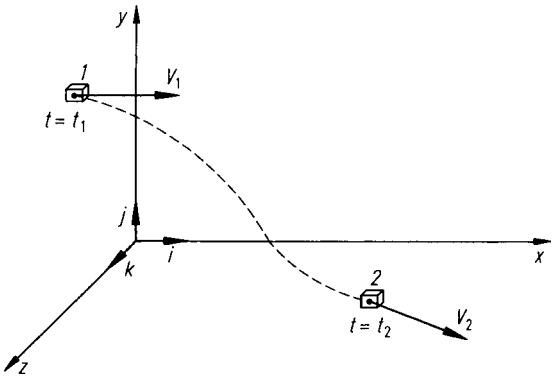

Fig. 2.2 Fluid element moving in the flow field—illustration for the substantial derivative

At time t_1, the fluid element is located at point 1 in Fig. 2.2. At this point and time, the density of the fluid element is

$$\rho_1 = \rho(x_1, y_1, z_1, t_1)$$

At a later time, t_2, the same fluid element has moved to point 2 in Fig. 2.2. Hence, at time t_2, the density of this *same* fluid element is

$$\rho_2 = \rho(x_2, y_2, z_2, t_2)$$

Since $\rho = \rho(x,y,z,t)$, we can expand this function in a Taylor's series about point 1 as follows:

$$\rho_2 = \rho_1 + \left(\frac{\partial \rho}{\partial x}\right)_1 (x_2 - x_1) + \left(\frac{\partial \rho}{\partial y}\right)_1 (y_2 - y_1) + \left(\frac{\partial \rho}{\partial z}\right)_1 (z_2 - z_1) + \left(\frac{\partial \rho}{\partial t}\right)_1 (t_2 - t_1) + (\text{higher order terms})$$

Dividing by $(t_2 - t_1)$, and ignoring higher order terms, we obtain

$$\frac{\rho_2 - \rho_1}{t_2 - t_1} = \left(\frac{\partial \rho}{\partial x}\right)_1 \left(\frac{x_2 - x_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial y}\right)_1 \left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial z}\right)_1 \left(\frac{z_2 - z_1}{t_2 - t_1}\right) + \left(\frac{\partial \rho}{\partial t}\right)_1 \tag{2.1}$$

Examine the left side of Eq. (2.1). This is physically the *average* time-rate-of-change in density of the fluid element as it moves from point 1 to point 2. In the limit, as t_2 approaches t_1, this term becomes

$$\lim_{t_2 \to t_1} \left(\frac{\rho_2 - \rho_1}{t_2 - t_1}\right) \equiv \frac{D\rho}{Dt}$$

Here, $D\rho/Dt$ is a symbol for the *instantaneous* time rate of change of density of the fluid element as it moves through point 1. By definition, this symbol is called the substantial derivative, D/Dt. Note that $D\rho/Dt$ is the time rate of change of density of the *given fluid element* as it moves through space. Here, our eyes are locked on the fluid element as it is moving, and we are watching the density of the element change as it moves through point 1. This is different from $(\partial\rho/\partial t)_1$, which is physically the time rate of change of density at the fixed point 1. For $(\partial\rho/\partial t)_1$, we fix our eyes on the stationary point 1, and watch the density change due to transient fluctuations in the flow field. Thus, $D\rho/Dt$ and $\partial\rho/\rho t$ are physically and numerically different quantities.

Returning to Eq. (2.1), note that

$$\lim_{t_2 \to t_1} \left(\frac{x_2 - x_1}{t_2 - t_1} \right) \equiv u$$

$$\lim_{t_2 \to t_1} \left(\frac{y_2 - y_1}{t_2 - t_1} \right) \equiv v$$

$$\lim_{t_2 \to t_1} \left(\frac{z_2 - z_1}{t_2 - t_1} \right) \equiv w$$

Thus, taking the limit of Eq. (2.1) as $t_2 \to t_1$, we obtain

$$\frac{D\rho}{Dt} = u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \frac{\partial\rho}{\partial t} \tag{2.2}$$

Examine Eq. (2.2) closely. From it, we can obtain an expression for the substantial derivative in cartesian coordinates:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{2.3}$$

Furthermore, in cartesian coordinates, the vector operator ∇ is defined as

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{2.4}$$

Hence, Eq. (2.3) can be written as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left(\vec{V} \cdot \nabla \right) \tag{2.5}$$

Equation (2.5) represents a definition of the substantial derivative operator in vector notation; thus, it is valid for any coordinate system.

Focusing on Eq. (2.5), we once again emphasize that D/Dt is the substantial derivative, which is physically the time rate of change following a moving fluid element; $\partial/\partial t$ is called the *local derivative*, which is physically the time rate of change at a fixed point; $\vec{V} \cdot \nabla$ is called the *convective derivative*, which is physically the time rate of change due to the movement of the fluid element from one

location to another in the flow field where the flow properties are spatially different. The substantial derivative applies to any flow-field variable, for example, Dp/Dt, DT/Dt, Du/Dt, etc., where p and T are the static pressure and temperature respectively. For example:

$$\frac{DT}{Dt} \equiv \underbrace{\frac{\partial T}{\partial t}}_{\substack{\text{local}\\ \text{derivative}}} + \underbrace{(\vec{V}\cdot\nabla)}_{\substack{\text{convective}\\ \text{derivative}}} T \equiv \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{2.6}$$

Again, Eq. (2.6) states physically that the temperature of the fluid element is changing as the element sweeps past a point in the flow because at that point the flow field temperature itself may be fluctuating with time (the local derivative) and because the fluid element is simply on its way to another point in the flow field where the temperature is different (the convective derivative).

Consider an example which will help to reinforce the physical meaning of the substantial derivative. Imagine that you are hiking in the mountains, and you are about to enter a cave. The temperature inside the cave is cooler than outside. Thus, as you walk through the mouth of the cave, you feel a temperature decrease—this is analagous to the convective derivative in Eq. (2.6). However, imagine that, at the same time, a friend throws a snowball at you such that the snowball hits you just at the same instant you pass through the mouth of the cave. You will feel an additional, but momentary, temperature drop when the snowball hits you—this is analagous to the local derivative in Eq. (2.6). The net temperature drop you feel as you walk through the mouth of the cave is therefore a combination of both the act of moving into the cave, where it is cooler, and being struck by the snowball at the same instant—this net temperature drop is analagous to the substantial derivative in Eq. (2.6).

The above derivation of the substantial derivative is essentially taken from this author's basic aerodynamics text book given as Ref. [1]. It is used there to introduce new aerodynamics students to the full physical meaning of the substantial derivative. The description is repeated here for the same reason—to give you a physical feel for the substantial derivative. We could have circumvented most of the above discussion by recognizing that the substantial derivative is essentially the same as the total differential from calculus. That is, if

$$\rho = \rho(x,y,z,t)$$

then the chain rule from differential calculus gives

$$\mathrm{d}\rho = \frac{\partial \rho}{\partial x}\mathrm{d}x + \frac{\partial \rho}{\partial y}\mathrm{d}y + \frac{\partial \rho}{\partial z}\mathrm{d}z + \frac{\partial \rho}{\partial t}\mathrm{d}t \tag{2.7}$$

From Eq. (2.7), we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \rho}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \rho}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{2.8}$$

Since $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, and $\frac{dz}{dt} = w$, Eq. (2.8) becomes

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \tag{2.9}$$

Comparing Eqs. (2.2) and (2.9), we see that $d\rho/dt$ and $D\rho/Dt$ are one-in-the-same.

Therefore, the substantial derivative is nothing more than a total derivative with respect to time. However, the derivation of Eq. (2.2) highlights more of the physical significance of the substantial derivative, whereas the derivation of Eq. (2.9) is more formal mathematically.

2.4 Physical Meaning of $\nabla \cdot \vec{V}$

As one last item before deriving the governing equations, let us consider the divergence of the velocity, $\nabla \cdot \vec{V}$. This term appears frequently in the equations of fluid dynamics, and it is well to consider its physical meaning.

Consider a control volume moving with the fluid as sketched on the right of Fig. 2.1(a). This control volume is always made up of the same fluid particles as it moves with the flow; hence, its mass is fixed, invariant with time. However, its volume $\mathcal{V}$ and control surface S are changing with time as it moves to different regions of the flow where different values of ρ exist. That is, this moving control volume of fixed mass is constantly increasing or decreasing its volume and is changing its shape, depending on the characteristics of the flow. This control volume is shown in Fig. 2.3 at some instant in time. Consider an infinitesimal element of the surface dS moving at the local velocity $\vec{V}$, as shown in Fig. 2.3. The change in the volume of the control volume $\Delta\mathcal{V}$, due to just the movement of dS over a time increment Δt, is, from Fig. 2.3, equal to the volume of the long, thin cylinder with base area dS and altitude $(\vec{V}\Delta t)\cdot\vec{n}$, where $\vec{n}$ is a unit vector perpendicular to the surface at dS. That is,

$$\Delta\mathcal{V} = \left[(\vec{V}\Delta t)\cdot\vec{n}\right]dS = (\vec{V}\Delta t)\cdot d\vec{S} \tag{2.10}$$

where the vector $d\vec{S}$ is defined simply as $d\vec{S} \equiv \vec{n}\, dS$. Over the time increment Δt, the total change in volume of the whole control volume is equal to the summation of Eq. (2.10) over the total control surface. In the limit as $dS \to 0$, the sum becomes the surface integral

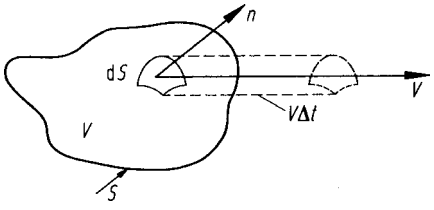

Fig. 2.3 Moving control volume used for the physical interpretation of the divergence of velocity

$$\oiint_S (\vec{V}\Delta t)\cdot \mathrm{d}S$$

If this integral is divided by Δt, the result is physically the time rate of change of the control volume, denoted by DV/Dt, i.e.

$$\frac{D\mathcal{V}}{Dt} = \frac{1}{\Delta t}\oiint_S (\vec{V}\cdot\Delta t)\cdot \mathrm{d}\vec{S} = \oiint_S \vec{V}\cdot \mathrm{d}\vec{S} \tag{2.11}$$

Note that we have written the left side of Eq. (2.11) as the substantial derivative of $\mathcal{V}$, because we are dealing with the time rate of change of the control volume *as the volume moves with the flow* (we are using the picture shown at the right of Fig. 2.1a), and this is physically what is meant by the substantial derivative. Applying the divergence theorem from vector calculus to the right side of Eq. (2.11), we obtain

$$\frac{D\mathcal{V}}{Dt} = \oiiint_V (\nabla\cdot\vec{V})\mathrm{d}\mathcal{V} \tag{2.12}$$

Now, let us image that the moving control volume in Fig. 2.3 is shrunk to a very small volume, $\delta\mathcal{V}$, essentially becoming an infinitesimal moving fluid element as sketched on the right of Fig. 2.1(a). Then Eq. (2.12) can be written as

$$\frac{D(\delta\mathcal{V})}{Dt} = \oiiint_{\delta\mathcal{V}} (\nabla\cdot\vec{V})\mathrm{d}\mathcal{V} \tag{2.13}$$

Assume that $\delta\mathcal{V}$ is small enough such that $\nabla\cdot\vec{V}$ is essentially the same value throughout $\delta\mathcal{V}$. Then the integral in Eq. (2.13) can be approximated as $(\nabla\cdot\vec{V})\delta\mathcal{V}$. From Eq. (2.13), we have

$$\frac{D(\delta\mathcal{V})}{Dt} = (\nabla\cdot\vec{V})\delta\mathcal{V}$$

or

$$\boxed{\nabla\cdot\vec{V} = \frac{1}{\delta\mathcal{V}}\frac{D(\delta\mathcal{V})}{Dt}} \tag{2.14}$$

Examine Eq. (2.14) closely. On the left side we have the divergence of the velocity; on the right side we have its physical meaning. That is,

$\nabla\cdot\vec{V}$ is physically the time rate of change of the volume of a moving fluid element, per unit volume.

2.5 The Continuity Equation

Let us now apply the philosophy discussed in Sect. 2.2; that is, (a) write down a fundamental physical principle, (b) apply it to a suitable model of the flow, and (c) obtain an equation which represents the fundamental physical principle. In this section we will treat the following case:

2.5.1 Physical Principle: Mass is Conserved

We will carry out the application of this principle to *both* the finite control volume and infinitesimal fluid element models of the flow. This is done here specifically to illustrate the physical nature of both models. Moreover, we will choose the finite control volume to be *fixed in space* (left side of Fig. 2.1a), whereas the infinitesimal fluid element will be *moving with the flow* (right side of Fig. 2.1b). In this way we will be able to contrast the differences between the conservation and non-conservation forms of the equations, as described in Sect. 2.2.

First, consider the model of a moving fluid element. The mass of this element is fixed, and is given by δm. Denote the volume of this element by $\delta \mathcal{V}$, as in Sect. 2.4. Then

$$\delta m = \rho \delta \mathcal{V} \tag{2.15}$$

Since mass is conserved, we can state that the time-rate-of-change of the mass of the fluid element is zero as the element moves along with the flow. Invoking the physical meaning of the substantial derivative discussed in Sect. 2.3, we have

$$\frac{D(\delta m)}{Dt} = 0 \tag{2.16}$$

Combining Eqs. (2.15) and (2.16), we have

$$\frac{D(\rho \delta \mathcal{V})}{Dt} = \delta \mathcal{V} \frac{D\rho}{Dt} + \rho \frac{D(\delta \mathcal{V})}{Dt} = 0$$

or,

$$\frac{D\rho}{Dt} + \rho \left[\frac{1}{\delta \mathcal{V}} \frac{D(\delta \mathcal{V})}{Dt} \right] = 0 \tag{2.17}$$

We recognize the term in brackets in Eq. (2.17) as the physical meaning of $\nabla \cdot \vec{V}$, discussed in Sect. 2.4. Hence, combining Eqs. (2.14) and (2.17), we obtain

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla . \vec{V} = 0} \tag{2.18}$$

Equation (2.18) is the *continuity equation* in *non-conservation* form. In light of our philosophical discussion in Sect. 2.2, note that:

(1) By applying the model of an *infinitesimal fluid element*, we have obtained Eq. (2.18) *directly* in partial differential form.
(2) By choosing the model to be *moving with the flow*, we have obtained the *non-conservation form* of the continuity equation, namely Eq. (2.18).

Now, consider the model of a finite control volume fixed in space, as sketched in Fig. 2.4. At a point on the control surface, the flow velocity is $\vec{V}$ and the vector elemental surface area (as defined in Sect. 2.4) is $d\vec{S}$. Also let $d\mathcal{V}$ be an elemental volume inside the finite control volume. Applied to this control volume, our fundamental physical principle that mass is conserved means

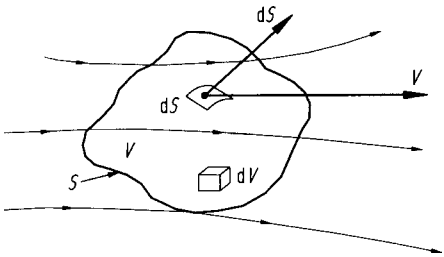

Fig. 2.4 Finite control volume fixed in space

$$\left\{\begin{array}{l}\text{Net mass flow } \textit{out}\\ \text{of control volume}\\ \text{through surface } S\end{array}\right\} = \left\{\begin{array}{l}\text{time rate of decrease}\\ \text{of mass inside control}\\ \text{volume}\end{array}\right\} \tag{2.19a}$$

or,

$$B = C \tag{2.19b}$$

where B and C are just convenient symbols for the left and right sides, respectively, of Eq. (2.19a). First, let us obtain an expression for B in terms of the quantities shown in Fig. 2.4. The mass flow of a moving fluid across any fixed surface (say, in kg/s, or slug/s) is equal to the product of (density) × (area of surface) × (component of velocity perpendicular to the surface). Hence the elemental mass flow across the area dS is

$$\rho V_{\mathrm{n}}\mathrm{d}S = \rho\vec{V}\cdot\vec{\mathrm{d}}S \tag{2.20}$$

Examining Fig. 2.4, note that by convention, $\vec{\mathrm{d}}S$ always points in a direction *out* of the control volume. Hence, when $\vec{V}$ also points out of the control volume (as shown in Fig. 2.4), the product $\rho\vec{V}\cdot\mathrm{d}\vec{S}$ is *positive*. Moreover, when $\vec{V}$ points out of the control volume, the mass flow is physically leaving the control volume, i.e. it is an *outflow*. Hence, a positive $\rho\vec{V}\cdot\vec{\mathrm{d}}S$ denotes an outflow. In turn, when $\vec{V}$ points into the control volume, $\rho\vec{V}\cdot\vec{\mathrm{d}}S$ is *negative*. Moreover, when $\vec{V}$ points inward, the mass flow is physically entering the control volume, i.e. it is an *inflow*. Hence, a negative $\rho\vec{V}\cdot\vec{\mathrm{d}}S$ denotes an inflow. The net mass flow *out* of the entire control volume through the control surface S is the summation over S of the elemental mass flows shown in Eq. (2.20). In the limit, this becomes a surface integral, which is physically the left side of Eqs. (2.19a and b), i.e.

$$B = \oiint_S \rho\vec{V}\cdot\vec{\mathrm{d}}S \tag{2.21}$$

Now consider the right side of Eqs. (2.19a and b). The mass contained within the elemental volume d$\mathscr{V}$ is $\rho\,\mathrm{d}\mathscr{V}$. The total mass inside the control volume is therefore

$$\iiint_{\mathscr{V}} \rho\,\mathrm{d}\mathscr{V}$$

The time rate of *increase* of mass inside $\mathscr{V}$ is then

$$-\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho \, d\mathscr{V}$$

In turn, the time rate of decrease of mass inside $\mathscr{V}$ is the negative of the above, i.e.

$$-\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho \, d\mathscr{V} = C \tag{2.22}$$

Thus, substituting Eqs. (2.21) and (2.22) into (2.19b), we have

$$\iint_S \rho \vec{V} \cdot \vec{dS} = -\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho \, d\mathscr{V}$$

or,

$$\boxed{\frac{\partial}{\partial t}\iiint_{\mathscr{V}} \rho \, d\mathscr{V} + \iint_S \rho \vec{V} \cdot \vec{dS} = 0} \tag{2.23}$$

Equation (2.23) is the *integral form of the continuity equation*; it is also in *conservation* form.

Let us cast Eq. (2.23) in the form of a differential equation. Since the control volume in Fig. 2.4 is fixed in space, the limits of integration for the integrals in Eq. (2.23) are constant, and hence the time derivative $\partial/\partial t$ can be placed inside the integral.

$$\iiint_{\mathscr{V}} \frac{\partial \rho}{\partial t} d\mathscr{V} + \iint_S \rho \vec{V} \cdot \vec{dS} = 0 \tag{2.24}$$

Applying the divergence theorem from vector calculus, the surface integral in Eq. (2.24) can be expressed as a volume integral

$$\iint_S (\rho \vec{V}) \cdot \vec{dS} = \iiint_{\mathscr{V}} \nabla \cdot (\rho \vec{V}) d\mathscr{V} \tag{2.25}$$

Substituting Eq. (2.25) into Eq. (2.24), we have

$$\iiint_{\mathscr{V}} \frac{\partial \rho}{\partial t} d\mathscr{V} + \iiint_{\mathscr{V}} \nabla \cdot (\rho \vec{V}) d\mathscr{V} = 0$$

or

$$\iiint_{\mathscr{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right] d\mathscr{V} = 0 \tag{2.26}$$

Since the finite control volume is *arbitrarily* drawn in space, the only way for the integral in Eq. (2.26) to equal zero is for the integrand to be zero at every point within the control volume. Hence, from Eq. (2.26)

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0} \tag{2.27}$$

Equation (2.27) is the *continuity equation* in *conservation* form.

Examining the above derivation in light of our discussion in Sect. 2.2, we note that:

(1) By applying the model of a finite control volume, we have obtained Eq. (2.23) *directly* in integral form.
(2) Only after some manipulation of the integral form did we *indirectly* obtain a partial differential equation, Eq. (2.27).
(3) By choosing the model to be *fixed in space*, we have obtained the *conservation* form of the continuity equation, Eqs. (2.23) and (2.27).

Emphasis is made that Eqs. (2.18) and (2.27) are both statements of the conservation of mass expressed in the form of partial differential equations. Eq. (2.18) is in non-conservation form, and Eq. (2.27) is in conservation form; both forms are equally valid. Indeed, one can easily be obtained from the other, as follows. Consider the vector identity involving the divergence of the product of a scalar times a vector, such as

$$\nabla \cdot (\rho \vec{V}) \equiv \rho \nabla \cdot \vec{V} + \vec{V} \cdot \nabla \rho \tag{2.28}$$

Substitute Eq. (2.28) in the conservation form, Eq. (2.27):

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0 \tag{2.29}$$

The first two terms on the left side of Eq. (2.29) are simply the substantial derivative of density. Hence, Eq. (2.29) becomes

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

which is the non-conservation form given by Eq. (2.18).

Once again we note that the use of conservation or non-conservation forms of the governing equations makes little difference in most of theoretical aerodynamics. In contrast, which form is used can make a difference in some CFD applications, and this is why we are making a distinction between these two different forms in the present notes.

2.6 The Momentum Equation

In this section, we apply another fundamental physical principle to a model of the flow, namely:

$$\textit{Physical Principle}: \quad \vec{F} = m\vec{a} \text{ (Newton's 2nd law)}$$

We choose for our flow model the moving fluid element as shown at the right of Fig. 2.1(b). This model is sketched in more detail in Fig. 2.5.

Newton's 2nd law, expressed above, when applied to the moving fluid element in Fig. 2.5, says that the net force on the fluid element equals its mass times the

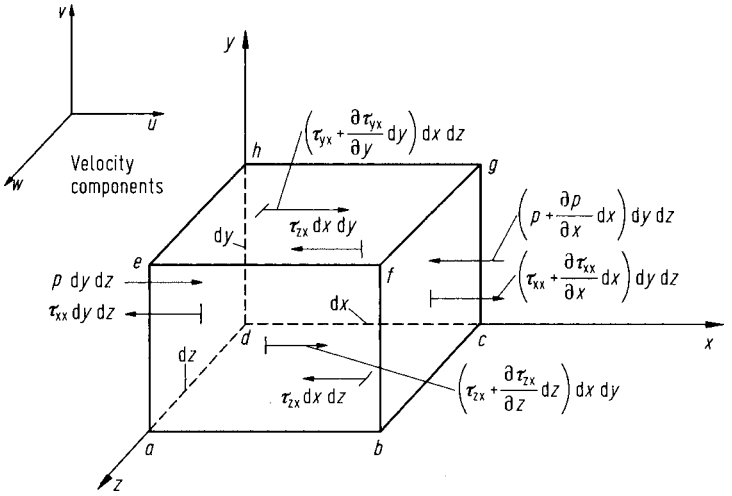

Fig. 2.5 Infinitesimally small, moving fluid element. Only the forces in the *x* direction are shown

acceleration of the element. This is a vector relation, and hence can be split into three scalar relations along the *x*, *y*, and *z*-axes. Let us consider only the *x*-component of Newton's 2nd law,

$$F_x = ma_x \tag{2.30}$$

where F_x and a_x are the scalar *x*-components of the force and acceleration respectively.

First, consider the left side of Eq. (2.30). We say that the moving fluid element experiences a force in the *x*-direction. What is the source of this force? There are two sources:

(1) *Body forces*, which act directly on the volumetric mass of the fluid element. These forces 'act at a distance'; examples are gravitational, electric and magnetic forces.
(2) *Surface forces*, which act directly on the surface of the fluid element. They are due to only two sources: (a) the pressure distribution acting on the surface, imposed by the outside fluid surrounding the fluid element, and (b) the shear and normal stress distributions acting on the surface, also imposed by the outside fluid 'tugging' or 'pushing' on the surface by means of friction.

Let us denote the body force per unit mass acting on the fluid element by $\vec{f}$, with f_x as its *x*-component. The volume of the fluid element is (d*x* d*y* d*z*); hence,

$$\left\{\begin{array}{l}\text{Body force on the}\\ \text{fluid element acting}\\ \text{in the } x\text{-direction}\end{array}\right\} = \rho f_x(\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z) \tag{2.31}$$

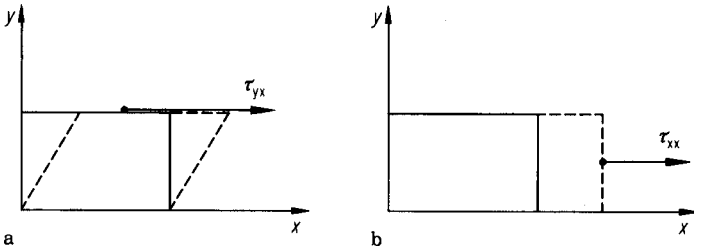

Fig. 2.6 Illustration of shear and normal stresses

The shear and normal stresses in a fluid are related to the time-rate-of-change of the deformation of the fluid element, as sketched in Fig. 2.6 for just the *xy* plane. The shear stress, denoted by τ_{xy} in this figure, is related to the time rate-of-change of the shearing deformation of the fluid element, whereas the normal stress, denoted by τ_{xx} in Fig. 2.6, is related to the time-rate-of-change of volume of the fluid element. As a result, both shear and normal stresses depend on velocity gradients in the flow, to be designated later. In most viscous flows, normal stresses (such as τ_{xx}) are much smaller than shear stresses, and many times are neglected. Normal stresses (say τ_{xx} in the *x*-direction) become important when the normal velocity gradients (say $\partial u/\partial x$) are very large, such as *inside* a shock wave.

The surface forces in the *x*-direction exerted on the fluid element are sketched in Fig. 2.5. The convention will be used here that τ_{ij} denotes a stress in the *j*-direction exerted on a plane perpendicular to the *i*-axis. On face *abcd*, the only force in the *x*-direction is that due to shear stress, $\tau_{yx}\,dx\,dz$. Face *efgh* is a distance d*y* above face *abcd*; hence the shear force in the *x*-direction on face *efgh* is $[\tau_{yx} + (\partial\tau_{yx}/\partial y)\,dy]\,dx\,dz$. Note the directions of the shear force on faces *abcd* and *efgh*; on the bottom face, τ_{yx} is to the left (the negative *x*-direction), whereas on the top face, $[\tau_{yx} + (\partial\tau_{yx}/\partial y)\,dy]$ is to the right (the positive *x*-direction). These directions are consistent with the convention that positive increases in all three components of velocity. *u*, *v* and *w*, occur in the positive directions of the axes. For example, in Fig. 2.5, *u* increases in the positive *y*-direction. Therefore, concentrating on face *efgh*, *u* is higher just above the face than on the face; this causes a 'tugging' action which tries to pull the fluid element in the positive *x*-direction (to the right) as shown in Fig. 2.5. In turn, concentrating on face *abcd*, *u* is lower just beneath the face than on the face; this causes a retarding or dragging action on the fluid element, which acts in the negative *x*-direction (to the left) as shown in Fig. 2.5. The directions of all the other viscous stresses shown in Fig. 2.5, including τ_{xx}, can be justified in a like fashion. Specifically on face *dcgh*, τ_{zx} acts in the negative *x*-direction, whereas on face *abfe*, $[\tau_{zx} + (\partial\tau_{zx}/\partial z)\,dz]$ acts in the positive *x*-direction. On face *adhe*, which is perpendicular to the *x*-axis, the only forces in the *x*-direction are the pressure force $p\,dx\,dz$, which always acts in the direction *into* the fluid element, and $\tau_{xx}\,dy\,dz$, which is in the negative *x*-direction. In Fig. 2.5, the reason why τ_{xx} on face *adhe* is to the left hinges on the convention mentioned earlier for the direction of increasing

velocity. Here, by convention, a positive increase in u takes place in the positive x-direction. Hence, the value of u just to the left of face *adhe* is smaller than the value of u on the face itself. As a result, the viscous action of the normal stress acts as a 'suction' on face *adhe*, i.e. there is a dragging action toward the left that wants to retard the motion of the fluid element. In contrast, on face *bcgf*, the pressure force $[p+(\partial p/\partial x)\,\mathrm{d}x]\,\mathrm{d}y\,\mathrm{d}z$ presses inward on the fluid element (in the negative x-direction), and because the value of u just to the right of face *bcgf* is larger than the value of u on the face, there is a 'suction' due to the viscous normal stress which tries to pull the element to the right (in the positive x-direction) with a force equal to $[\tau_{xx}+(\partial\tau_{xx}/\partial x)]\,\mathrm{d}y\,\mathrm{d}z$.

With the above in mind, for the moving fluid element we can write

$$\begin{aligned}\left\{\begin{array}{l}\text{Net surface force}\\ \text{in the } x\text{-direction}\end{array}\right\} &= \left[p-\left(p+\frac{\partial p}{\partial x}\mathrm{d}x\right)\right]\mathrm{d}y\,\mathrm{d}z\\ &+\left[\left(\tau_{xx}+\frac{\partial\tau_{xx}}{\partial x}\mathrm{d}x\right)-\tau_{xx}\right]\mathrm{d}y\,\mathrm{d}z\\ &+\left[\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\mathrm{d}y\right)-\tau_{yx}\right]\mathrm{d}x\,\mathrm{d}z\\ &+\left[\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\mathrm{d}z\right)-\tau_{zx}\right]\mathrm{d}x\,\mathrm{d}y \qquad (2.32)\end{aligned}$$

The total force in the x-direction, F_x, is given by the sum of Eqs. (2.31) and (2.32). Adding, and cancelling terms, we obtain

$$F_x=\left(-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z+\rho f_x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (2.33)$$

Equation (2.33) represents the left-hand side of Eq. (2.30).

Considering the right-hand side of Eq. (2.30), recall that the mass of the fluid element is fixed and is equal to

$$m=\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (2.34)$$

Also, recall that the acceleration of the fluid element is the time-rate-of-change of its velocity. Hence, the component of acceleration in the x-direction, denoted by a_x, is simply the time-rate-of-change of u; since we are following a moving fluid element, this time-rate-of-change is given by the substantial derivative. Thus,

$$a_x=\frac{Du}{Dt} \qquad (2.35)$$

Combining Eqs. (2.30), (2.33), (2.34) and (2.35), we obtain

$$\boxed{\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\rho f_x} \qquad (2.36a)$$

which is the *x*-component of the momentum equation for a viscous flow. In a similar fashion, the *y* and *z* components can be obtained as

$$\boxed{\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y} \tag{2.36b}$$

and

$$\boxed{\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z} \tag{2.36c}$$

Equations (2.36a, b and c) are the *x*-, *y*- and *z*-components respectively of the momentum equation. Note that they are partial differential equations obtained *directly* from an application of the fundamental physical principle to an infinitesimal fluid element. Moreover, since this fluid element is moving with the flow, Eqs. (2.36a, b and c) are in *non-conservation* form. They are scalar equations, and are called the *Navier–Stokes equations* in honour of two men—the Frenchman M. Navier and the Englishmen G. Stokes—who independently obtained the equations in the first half of the nineteenth century.

The Navier–Stokes equations can be obtained in conservation form as follows. Writing the left-hand side of Eq. (2.36a) in terms of the definition of the substantial derivative,

$$\rho\frac{Du}{Dt} = \rho\frac{\partial u}{\partial t} + \rho\vec{V}\cdot\nabla u \tag{2.37}$$

Also, expanding the following derivative,

$$\frac{\partial(\rho u)}{\partial t} = \rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t}$$

or,

$$\rho\frac{\partial u}{\partial t} = \frac{\partial(\rho u)}{\partial t} - u\frac{\partial \rho}{\partial t} \tag{2.38}$$

Recalling the vector identity for the divergence of the product of a scalar times a vector, we have

$$\nabla\cdot(\rho u\vec{V}) = u\nabla\cdot(\rho\vec{V}) + (\rho\vec{V})\cdot\nabla u$$

or

$$\rho\vec{V}\cdot\nabla u = \nabla\cdot(\rho u\vec{V}) - u\nabla\cdot(\rho\vec{V}) \tag{2.39}$$

Substitute Eqs. (2.38) and (2.39) into Eq. (2.37).

$$\begin{aligned}\rho\frac{Du}{Dt} &= \frac{\partial(\rho u)}{\partial t} - u\frac{\partial \rho}{\partial t} - u\nabla\cdot(\rho\vec{V}) + \nabla\cdot(\rho u\vec{V})\\ \rho\frac{Du}{Dt} &= \frac{\partial(\rho u)}{\partial t} - u\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] + \nabla\cdot(\rho u\vec{V})\end{aligned} \tag{2.40}$$

The term in brackets in Eq. (2.40) is simply the left-hand side of the continuity equation given as Eq. (2.27); hence the term in brackets is zero. Thus Eq. (2.40) reduces to

$$\rho\frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\vec{V}) \tag{2.41}$$

Substitute Eq. (2.41) into Eq. (2.36a).

$$\boxed{\frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x} \tag{2.42a}$$

Similarly, Eqs. (2.36b and c) can be expressed as

$$\boxed{\frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\vec{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y} \tag{2.42b}$$

and

$$\boxed{\frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w\vec{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z} \tag{2.42c}$$

Equations (2.42a–c) are the Navier-Stokes equations in *conservation* form.

In the late seventeenth century Isaac Newton stated that shear stress in a fluid is proportional to the time-rate-of-strain, i.e. velocity gradients. Such fluids are called *Newtonian* fluids. (Fluids in which τ is *not* proportional to the velocity gradients are non-Newtonian fluids; blood flow is one example.) In virtually all practical aerodynamic problems, the fluid can be assumed to be Newtonian. For such fluids, Stokes, in 1845, obtained:

$$\tau_{xx} = \lambda\nabla\cdot\vec{V} + 2\mu\frac{\partial u}{\partial x} \tag{2.43a}$$

$$\tau_{yy} = \lambda\nabla\cdot\vec{V} + 2\mu\frac{\partial v}{\partial y} \tag{2.43b}$$

$$\tau_{zz} = \lambda\nabla\cdot\vec{V} + 2\mu\frac{\partial w}{\partial z} \tag{2.43c}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{2.43d}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{2.43e}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{2.43f}$$

where μ is the molecular viscosity coefficient and λ is the bulk viscosity coefficient. Stokes made the hypothesis that

$$\lambda = -\frac{2}{3}\mu$$

which is frequently used but which has still not been definitely confirmed to the present day.

Substituting Eq. (2.43) into Eq. (2.42), we obtain the complete Navier–Stokes equations in conservation form:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}$$
$$=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$
$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x \tag{2.44a}$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho vw)}{\partial z}$$
$$=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial v}{\partial y}\right)$$
$$+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\rho f_y \tag{2.44b}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}+\frac{\partial(\rho w^2)}{\partial z}$$
$$=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]$$
$$+\frac{\partial}{\partial z}\left(\lambda\nabla\cdot\vec{V}+2\mu\frac{\partial w}{\partial z}\right)+\rho f_z \tag{2.44c}$$

2.7 The Energy Equation

In the present section, we derive the energy equation using as our model an infinitesimal moving fluid element. This will be in keeping with our derivation of the Navier–Stokes equations in Sect. 2.6, where the infinitesimal element was shown in Fig. 2.5.

We now invoke the following fundamental physical principle:

2.7.1 Physical Principle: Energy is Conserved

A statement of this principle is the first law of thermodynamics, which, when applied to the moving fluid element in Fig. 2.5, becomes

$$\left\{\begin{array}{l}\text{Rate of change of}\\ \text{energy inside the}\\ \text{fluid element}\end{array}\right\} = \left\{\begin{array}{l}\text{Net flux of}\\ \text{heat into}\\ \text{the element}\end{array}\right\} + \left\{\begin{array}{l}\text{Rate of working done on}\\ \text{the element due to body}\\ \text{and surface forces}\end{array}\right\}$$

or,

$$A = B + C \tag{2.45}$$

where A, B and C denote the respective terms above.

Let us first evaluate C, i.e. obtain an expression for the rate of work done on the moving fluid element due to body and surface forces. It can be shown that the rate of doing work by a force exerted on a moving body is equal to the product of the force and the component of velocity in the direction of the force (see References 3 and 14 for such a derivation). Hence the rate of work done by the body force acting on the fluid element moving at a velocity $\vec{V}$ is

$$\rho \vec{f} \cdot \vec{V}(\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z)$$

With regard to the surface forces (pressure plus shear and normal stresses), consider just the forces in the x-direction, shown in Fig. 2.5. The rate of work done on the moving fluid element by the pressure and shear forces in the x-direction shown in Fig. 2.5 is simply the x-component of velocity, u, multiplied by the forces, e.g. on face $abcd$ the rate of work done by $\tau_{yx}\mathrm{d}x\ \mathrm{d}z$ is $u\tau_{yx}\mathrm{d}x\ \mathrm{d}z$, with similar expressions for the other faces. To emphasize these energy considerations, the moving fluid element is redrawn in Fig. 2.7, where the rate of work done on each face by surface forces in the x-direction is shown explicitly. To obtain the *net* rate of work done on the fluid element by the surface forces, note that forces in the positive x-direction do positive work and that forces in the negative x-direction do negative work. Hence,

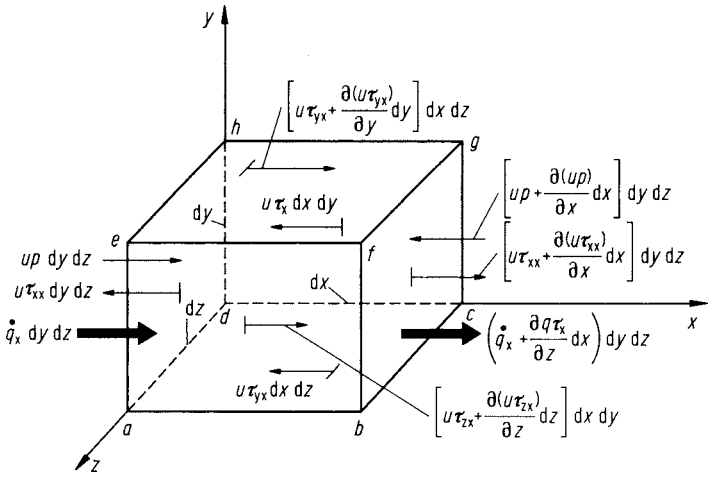

Fig. 2.7 Energy fluxes associated with an infinitesimally small, moving fluid element. For simplicity, only the fluxes in the x direction are shown

comparing the pressure forces on face *adhe* and *bcgf* in Fig. 2.7, the net rate of work done by pressure in the *x*-direction is

$$\left[up - \left(up + \frac{\partial(up)}{\partial x}\mathrm{d}x\right)\right]\mathrm{d}y\,\mathrm{d}z = -\frac{\partial(up)}{\partial x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Similarly, the net rate of work done by the shear stresses in the *x*-direction on faces *abcd* and *efgh* is

$$\left[\left(u\tau_{\mathrm{yx}} + \frac{\partial(u\tau_{\mathrm{yx}})}{\partial y}\mathrm{d}y\right) - u\tau_{\mathrm{yx}}\right]\mathrm{d}x\,\mathrm{d}z = \frac{\partial(u\tau_{\mathrm{yx}})}{\partial y}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Considering all the surface forces shown in Fig. 2.7, the net rate of work done on the moving fluid element due to these forces is simply

$$\left[-\frac{\partial(up)}{\partial x} + \frac{\partial(u\tau_{\mathrm{xx}})}{\partial x} + \frac{\partial(u\tau_{\mathrm{yx}})}{\partial y} + \frac{\partial(u\tau_{\mathrm{zx}})}{\partial z}\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

The above expression considers only surface forces in the *x*-direction. When the surface forces in the *y*- and *z*-directions are also included, similar expressions are obtained. In total, the net rate of work done on the moving fluid element is the sum of the surface force contributions in the *x*-, *y*- and *z*-directions, as well as the body force contribution. This is denoted by *C* in Eq. (2.45), and is given by

$$\begin{aligned} C = \Bigg[&-\left(\frac{\partial(up)}{\partial x} + \frac{\partial(vp)}{\partial y} + \frac{\partial(wp)}{\partial z}\right) + \frac{\partial(u\tau_{\mathrm{xx}})}{\partial x} + \frac{\partial(u\tau_{\mathrm{yx}})}{\partial y} \\ &+ \frac{\partial(u\tau_{\mathrm{zx}})}{\partial z} + \frac{\partial(v\tau_{\mathrm{xy}})}{\partial x} + \frac{\partial(v\tau_{\mathrm{yy}})}{\partial y} + \frac{\partial(v\tau_{\mathrm{zy}})}{\partial z} + \frac{\partial(w\tau_{\mathrm{xz}})}{\partial x} \\ &+ \frac{\partial(w\tau_{\mathrm{yz}})}{\partial y} + \frac{\partial(w\tau_{\mathrm{zz}})}{\partial z}\Bigg]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \rho\vec{f}\cdot\vec{V}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \end{aligned} \tag{2.46}$$

Note in Eq. (2.46) that the first three terms on the right-hand side are simply $\nabla\cdot(p\vec{V})$.

Let us turn our attention to *B* in Eq. (2.45), i.e. the net flux of heat into the element. This heat flux is due to: (1) volumetric heating such as absorption or emission of radiation, and (2) heat transfer across the surface due to temperature gradients, i.e. thermal conduction. Define $\dot{q}$ as the rate of volumetric heat addition per unit mass. Noting that the mass of the moving fluid element in Fig. 2.7 is $\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, we obtain

$$\left\{\begin{array}{l}\text{Volumetric heating}\\ \text{of the element}\end{array}\right\} = \rho\dot{q}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.47}$$

In Fig. 2.7, the heat transferred by thermal conduction into the moving fluid element across face *adhe* is $\dot{q}_{\mathrm{x}}\,\mathrm{d}y\,\mathrm{d}z$ where $\dot{q}_{\mathrm{x}}$ is the heat transferred in the *x*-direction per unit time per unit area by thermal conduction. The heat transferred out of the element across face *bcgf* is $[\dot{q}_{\mathrm{x}} + (\partial\dot{q}_{\mathrm{x}}/\partial x)\,\mathrm{d}x]\,\mathrm{d}y\,\mathrm{d}z$. Thus, the net heat transferred in the *x*-direction into the fluid element by thermal conduction is

$$\left[\dot{q}_x - \left(\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x}\mathrm{d}x\right)\right]\mathrm{d}y\,\mathrm{d}z = -\frac{\partial \dot{q}_x}{\partial x}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Taking into account heat transfer in the *y*- and *z*-directions across the other faces in Fig. 2.7, we obtain

$$\left\{\begin{array}{l}\text{Heating of the}\\ \text{fluid element by}\\ \text{thermal conduction}\end{array}\right\} = -\left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.48}$$

The term *B* in Eq. (2.45) is the sum of Eqs. (2.47) and (2.48).

$$B = \left[\rho\dot{q} - \left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.49}$$

Heat transfer by thermal conduction is proportional to the local temperature gradient:

$$\dot{q}_x = -k\frac{\partial T}{\partial x}; \qquad \dot{q}_y = -k\frac{\partial T}{\partial y}; \qquad \dot{q}_z = -k\frac{\partial T}{\partial z}$$

where *k* is the thermal conductivity. Hence, Eq. (2.49) can be written

$$B = \left[\rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.50}$$

Finally, the term *A* in Eq. (2.45) denotes the time-rate-of-change of energy of the fluid element. The total energy of a moving fluid per unit mass is the sum of its internal energy per unit mass, *e*, and its kinetic energy per unit mass, $V^2/2$. Hence, the total energy is $(e + V^2/2)$. Since we are following a moving fluid element, the time-rate-of-change of energy per unit mass is given by the substantial derivative. Since the mass of the fluid element is $\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, we have

$$A = \rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.51}$$

The final form of the energy equation is obtained by substituting Eqs. (2.46), (2.50) and (2.51) into Eq. (2.45), obtaining:

$$\begin{aligned}\rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = {} & \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\ & - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y}\\ & + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z}\\ & + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho\vec{f}\cdot\vec{V}\end{aligned} \tag{2.52}$$

This is the *non-conservation* form of the energy equation; also note that it is in terms of the *total* energy, $(e + V^2/2)$. Once again, the non-conservation form results from the application of the fundamental physical principle to a *moving* fluid element.

The left-hand side of Eq. (2.52) involves the total energy, $(e + V^2/2)$. Frequently, the energy equation is written in a form that involves just the internal energy, e. The derivation is as follows. Multiply Eqs. (2.36a, b, and c) by u, v, and w respectively.

$$\rho \frac{D\left(\frac{u^2}{2}\right)}{Dt} = -u\frac{\partial p}{\partial x} + u\frac{\partial \tau_{xx}}{\partial x} + u\frac{\partial \tau_{yx}}{\partial y} + u\frac{\partial \tau_{zx}}{\partial z} + \rho u f_x \tag{2.53a}$$

$$\rho \frac{D\left(\frac{v^2}{2}\right)}{Dt} = -v\frac{\partial p}{\partial y} + v\frac{\partial \tau_{xy}}{\partial x} + v\frac{\partial \tau_{yy}}{\partial y} + v\frac{\partial \tau_{zy}}{\partial z} + \partial v f_y \tag{2.53b}$$

$$\rho \frac{D\left(\frac{w^2}{2}\right)}{Dt} = -w\frac{\partial p}{\partial z} + w\frac{\partial \tau_{xz}}{\partial x} + w\frac{\partial \tau_{yz}}{\partial y} + w\frac{\partial \tau_{zz}}{\partial z} + \rho w f_z \tag{2.53c}$$

Add Eqs. (2.53a, b and c), and note that $u^2 + v^2 + w^2 = V^2$. We obtain

$$\begin{aligned}\rho \frac{DV^2/2}{Dt} = &-u\frac{\partial p}{\partial x} - v\frac{\partial p}{\partial y} - w\frac{\partial p}{\partial z} + u\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \\ &+ v\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) \\ &+ \rho(u f_x + v f_y + w f_z)\end{aligned} \tag{2.54}$$

Subtracting Eq. (2.54) from Eq. (2.52), noting that $\rho \vec{f} \cdot \vec{V} = \rho(u f_x + v f_y + w f_z)$, we have

$$\boxed{\begin{aligned}\rho \frac{De}{Dt} = &\rho \dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\ &- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} \\ &+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} \\ &+ \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}\end{aligned}} \tag{2.55}$$

Equation (2.55) is the energy equation in terms of internal energy, e. Note that the body force terms have cancelled; the energy equation when written in terms of e does not explicitly contain the body force. Eq. (2.55) is still in *non-conservation* form.

Equations (2.52) and (2.55) can be expressed totally in terms of flow field variables by replacing the viscous stress terms τ_{xy}, τ_{xz}, etc. with their equivalent expressions from Eqs (2.43a, b, c, d, e and f). For example, from Eq. (2.55), noting that $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$,

$$\rho\frac{De}{Dt} = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$
$$- p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$
$$+ \tau_{yx}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Substituting Eqs. (2.43a, b, c, d, e and f) into the above equation, we have

$$\boxed{\begin{aligned}\rho\frac{De}{Dt} &= \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\ &\quad - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\\ &\quad + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right.\\ &\quad \left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right]\end{aligned}} \tag{2.56}$$

Equation (2.56) is a form of the energy equation completely in terms of the flow-field variables. A similar substitution of Eqs. (2.43a, b, c, d, e and f) can be made into Eq. (2.52); the resulting form of the energy equation in terms of the flow-field variables is lengthy, and to save time and space it will not be given here.

The energy equation in *conservation* form can be obtained as follows. Consider the left-hand side of Eq. (2.56). From the definition of the substantial derivative:

$$\rho\frac{De}{Dt} = \rho\frac{\partial e}{\partial t} + \rho\vec{V}\cdot\nabla e \tag{2.57}$$

However,

$$\frac{\partial(\rho e)}{\partial t} = \rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t}$$

or,

$$\rho\frac{\partial e}{\partial t} = \frac{\partial(\rho e)}{\partial t} - e\frac{\partial \rho}{\partial t} \tag{2.58}$$

From the vector identity concerning the divergence of the product of a scalar times a vector,

$$\nabla\cdot(\rho e\vec{V}) = e\nabla\cdot(\rho\vec{V}) + \rho\vec{V}\cdot\nabla e$$

or

$$\rho\vec{V}\cdot\nabla e = \nabla\cdot(\rho e\vec{V}) - e\nabla\cdot(\rho\vec{V}) \tag{2.59}$$

Substitute Eqs. (2.58) and (2.59) into Eq. (2.57)

$$\rho\frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} - e\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] + \nabla\cdot(\rho e\vec{V}) \tag{2.60}$$

The term in square brackets in Eq. (2.60) is zero, from the continuity equation, Eq. (2.27). Thus, Eq. (2.60) becomes

$$\rho \frac{De}{Dt} = \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \vec{V}) \tag{2.61}$$

Substitute Eq. (2.61) into Eq. (2.56):

$$\begin{aligned} \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \vec{V}) = {} & \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \\ & + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ & + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 + \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2\right. \\ & + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ & \left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right] \end{aligned} \tag{2.62}$$

Equation (2.62) is the *conservation* form of the energy equation, written in terms of the internal energy.

Repeating the steps from Eq. (2.57) to Eq. (2.61), except operating on the *total* energy, $(e + V^2/2)$, instead of just the internal energy, e, we obtain

$$\rho \frac{D\left(e + \frac{V^2}{2}\right)}{Dt} = \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla\left[\rho\left(e + \frac{V^2}{2}\right)\vec{V}\right] \tag{2.63}$$

Substituting Eq. (2.63) into the left-hand side of Eq. (2.52), we obtain

$$\begin{aligned} & \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{V^2}{2}\vec{V}\right)\right] \\ & \quad = \rho \dot{q} + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) \\ & \qquad + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} \\ & \qquad + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} \\ & \qquad + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho \vec{f} \cdot \vec{V} \end{aligned} \tag{2.64}$$

Equation (2.64) is the *conservation* form of the energy equation, written in terms of the *total* energy, $(e + V^2/2)$.

As a final note in this section, there are many other possible forms of the energy equation; for example, the equation can be written in terms of enthalpy, h, or total enthalpy, $(h + V^2/2)$. We will not take the time to derive these forms here; see Refs. [1–3] for more details.

2.8 Summary of the Governing Equations for Fluid Dynamics: With Comments

By this point in our discussions, you have seen a large number of equations, and they may seem to you at this stage to be 'all looking alike'. Equations by themselves can be tiring, and this chapter would seem to be 'wall-to-wall' equations. However, *all* of theoretical and computational fluid dynamics is based on these equations, and therefore it is absolutely *essential* that you are familiar with them, and that you understand their physical significance. That is why we have spent so much time and effort in deriving the governing equations.

Considering this time and effort, it is important to now summarize the important forms of these equations, and to sit back and digest them.

2.8.1 Equations for Viscous Flow

The equations that have been derived in the preceding sections apply to a *viscous* flow, i.e. a flow which includes the dissipative, transport phenomena of viscosity and thermal conduction. The additional transport phenomenon of mass diffusion has not been included because we are limiting our considerations to a homogenous, non-chemically reacting gas. If diffusion were to be included, there would be additional continuity equations—the species continuity equations involving mass transport of chemical species i due to a concentration gradient in the species. Moreover, the energy equation would have an additional term to account for energy transport due to the diffusion of species. See, for example, Ref. [4] for a discussion of such matters.

With the above restrictions in mind, the governing equations for an unsteady, three-dimensional, compressible, viscous flow are:

Continuity equations
(Non-conservation form—Eq. (2.18))

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

(Conservation form—Eq. (2.27))

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Momentum equations
(Non-conservation form—Eqs. (2.36a–c))

$$x\text{-component}: \quad \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$y\text{-component}: \quad \rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$z\text{-component}: \quad \rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

(Conservation form—Eqs. (2.42a–c))

$$x\text{-component}: \quad \frac{\partial(\rho u)}{\partial t} + \nabla\cdot(\rho u\vec{V}) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$y\text{-component}: \quad \frac{\partial(\rho v)}{\partial t} + \nabla\cdot(\rho v\vec{V}) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$

$$z\text{-component}: \quad \frac{\partial(\rho w)}{\partial t} + \nabla\cdot(\rho w\vec{V}) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z$$

Energy equation
(Non-conservation form—Eq. (2.52))

$$\begin{aligned}\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right) = {} & \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\ & - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} \\ & + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} \\ & + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho\vec{f}\cdot\vec{V}\end{aligned}$$

(Conservation form—Eq. (2.64))

$$\begin{aligned}& \frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right] \\ & = \rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\ & \quad + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) - \frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \frac{\partial(u\tau_{xx})}{\partial x} \\ & \quad + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} \\ & \quad + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z} + \rho\vec{f}\cdot\vec{V}\end{aligned}$$

2.8.2 Equations for Inviscid Flow

Inviscid flow is, by definition, a flow where the dissipative, transport phenomena of viscosity, mass diffusion and thermal conductivity are *neglected*. The governing equations for an unsteady, three-dimensional, compressible inviscid flow are obtained by dropping the viscous terms in the above equations.

Continuity equation
(Non-conservation form)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$

(Conservation form)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Momentum equations
(Non-conservation form)

$$x\text{-component}: \quad \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x$$
$$y\text{-component}: \quad \rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y$$
$$z\text{-component}: \quad \rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z$$

(Conservation form)

$$x\text{-component}: \quad \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{V}) = -\frac{\partial p}{\partial x} + \rho f_x$$
$$y\text{-component}: \quad \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v \vec{V}) = -\frac{\partial p}{\partial y} + \rho f_y$$
$$z\text{-component}: \quad \frac{\partial (\rho w)}{\partial t} + \nabla \cdot (\rho w \vec{V}) = -\frac{\partial p}{\partial z} + \rho f_z$$

Energy equation
(Non-conservation form)

$$\rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = p\dot{q} - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \rho \vec{f} \cdot \vec{V}$$

(Conservation form)

$$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla \cdot \left[\rho\left(e + \frac{V^2}{2}\right)\vec{V}\right] = \rho\dot{q} - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \rho \vec{f} \cdot \vec{V}$$

2.8.3 Comments on the Governing Equations

Surveying the above governing equations, several comments and observations can be made.

(1) They are a coupled system of non-linear partial differential equations, and hence are very difficult to solve analytically. To date, there is no general closed-form solution to these equations.
(2) For the momentum and energy equations, the difference between the non-conservation and conservation forms of the equations is just the left-hand side. The right-hand side of the equations in the two different forms is the same.
(3) Note that the conservation form of the equations contain terms on the left-hand side which include the divergence of some quantity, such as $\nabla\cdot(\rho\vec{V})$, $\nabla\cdot(\rho u\vec{V})$, etc. For this reason, the conservation form of the governing equations is sometimes called the *divergence form*.
(4) The normal and shear stress terms in these equations are functions of the velocity gradients, as given by Eqs. (2.43a, b, c, d, e and f).
(5) The system contains five equations in terms of six unknown flow-field variables, ρ, p, u, v, w, e. In aerodynamics, it is generally reasonable to assume the gas is a perfect gas (which assumes that intermolecular forces are negligible—see Refs. [1, 3]. For a perfect gas, the equation of state is

$$p = \rho RT$$

where R is the specific gas constant. This provides a sixth equation, but it also introduces a seventh unknown, namely temperature, T. A seventh equation to close the entire system must be a thermodynamic relation between state variables. For example,

$$e = e(T, p)$$

For a calorically perfect gas (constant specific heats), this relation would be

$$e = c_v T$$

where c_v is the specific heat at constant volume.
(6) In Sect. 2.6, the momentum equations for a viscous flow were identified as the *Navier–Stokes equations*, which is historically accurate. However, in the modern CFD literature, this terminology has been expanded to include the *entire system* of flow equations for the solution of a viscous flow—continuity and energy as well as momentum. Therefore, when the computational fluid dynamic literature discusses a numerical solution to the 'complete Navier–Stokes equations', it is usually referring to a numerical solution of the *complete system of equations*, say for example Eqs. (2.27), (2.42a, b, c, d, e and c) and (2.64). In this sense, in the CFD literature, a 'Navier–Stokes solution' simply means a solution of a *viscous flow* problem using the *full governing equations*.

2.8.4 Boundary Conditions

The equations given above govern the flow of a fluid. They are the same equations whether the flow is, for example, over a Boeing 747, through a subsonic wind tunnel or past a windmill. However, the flow fields are quite *different* for these cases, although the governing equations are the *same*. Why? Where does the difference enter? The answer is through the *boundary conditions*, which are quite different for each of the above examples. The boundary conditions, and sometimes the initial conditions, dictate the particular solutions to be obtained from the governing equations. For a viscous fluid, the boundary condition on a surface assumes no relative velocity between the surface and the gas immediately at the surface. This is called the *no-slip* condition. If the surface is stationary, with the flow moving past it, then

$$u = v = w = 0 \quad \text{at the surface (for a viscous flow)}$$

For an inviscid fluid, the flow slips over the surface (there is no friction to promote its 'sticking' to the surface); hence, at the surface, the flow must be *tangent* to the surface.

$$\vec{V} \cdot \vec{n} = 0 \quad \text{at the surface (for an inviscid flow)}$$

where $\vec{n}$ is a unit vector perpendicular to the surface. The boundary conditions elsewhere in the flow depend on the type of problem being considered, and usually pertain to inflow and outflow boundaries at a finite distance from the surfaces, or an 'infinity' boundary condition infinitely far from the surfaces.

The boundary conditions discussed above are *physical boundary conditions* imposed by nature. In computational fluid dynamics we have an additional concern, namely, the *proper numerical implementation of the boundary conditions*. In the same sense as the real flow field is dictated by the physical boundary conditions, the computed flow field is driven by the numerical boundary conditions. The subject of proper and accurate boundary conditions in CFD is very important, and is the subject of much current CFD research. We will return to this matter at appropriate stages in these chapters.

2.9 Forms of the Governing Equations Particularly Suited for CFD: Comments on the Conservation Form

We have already noted that all the previous equations in *conservation form* have a divergence term on the left-hand side. These terms involve the divergence of the *flux* of some physical quantity, such as:

(From Eq. (2.27)): $\rho\vec{V}$ — mass flux
(From Eq. (2.42b)): $\rho u\vec{V}$ —flux of x-component of momentum
(From Eq. (2.42b)): $\rho v\vec{V}$ —flux of y-component of momentum

(From Eq. (2.42c)): $\rho w\vec{V}$ —flux of z-component of momentum
(From Eq. (2.62)): $\rho e\vec{V}$ — flux of internal energy
(From Eq. (2.64)): $\rho\left(e+V^2/2\right)\vec{V}$ — flux of total energy

Recall that the conservation form of the equations was obtained *directly* from a control volume that was *fixed in space*, rather than moving with the fluid. When the volume is fixed in space, we are concerned with the *flux* of mass, momentum and energy into and out of the volume. In this case, the *fluxes* themselves become important dependent variables in the equations, rather than just the primitive variables such as p, ρ, $\vec{V}$, etc.

Let us pursue this idea further. Examine the *conservation* form of *all* the governing equations—continuity, momentum and energy. Note that they all have the same generic form, given by

$$\boxed{\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=J} \tag{2.65}$$

Equation (2.65) can represent the *entire system* of governing equations in conservation form if U, F, G, H and J are interpreted as column vectors, given by

$$U=\begin{Bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho(e+V^2/2)\end{Bmatrix}$$

$$F=\begin{Bmatrix}\rho u\\ \rho u^2+p-\tau_{xx}\\ \rho vu-\tau_{xy}\\ \rho wu-\tau_{xz}\\ \rho(e+V^2/2)u+pu-k\dfrac{\partial T}{\partial x}-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}\end{Bmatrix}$$

$$G=\begin{Bmatrix}\rho v\\ \rho uv-\tau_{yx}\\ \rho v^2+p-\tau_{yy}\\ \rho wv-\tau_{yz}\\ \rho(e+V^2/2)v+pv-k\dfrac{\partial T}{\partial y}-u\tau_{yx}-v\tau_{yy}-w\tau_{yz}\end{Bmatrix}$$

$$H=\begin{Bmatrix}\rho w\\ \rho uw-\tau_{zx}\\ \rho vw-\tau_{zy}\\ \rho w^2+p-\tau_{zz}\\ \rho(e+V^2/2)w+pw-k\dfrac{\partial T}{\partial z}-u\tau_{zx}-v\tau_{zy}-w\tau_{zz}\end{Bmatrix}$$

$$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(u f_x + v f_y + w f_z) + \rho\dot{q} \end{Bmatrix}$$

In Eq. (2.65), the column vectors F, G, and H are called the flux terms (or flux vectors), and J represents a 'source term' (which is zero if body forces are negligible). For an unsteady problem, U is called the solution vector because the elements in U (ρ, ρu, ρv, etc.) are the dependent variables which are usually solved numerically in steps of time. Please note that, in this formalism, it is the elements of U that are obtained computationally, i.e. numbers are obtained for the products ρu, ρv, ρw and $\rho(e + V^2/2)$ rather than for the primitive variables u, v, w and e by themselves. Hence, in a computational solution of an unsteady flow problem using Eq. (2.65), the dependent variables are treated as ρ, ρu, ρv, ρw and $\rho(e + V^2/2)$. Of course, once numbers are known for these dependent variables (which includes ρ by itself), obtaining the primitive variables is simple:

$$\begin{aligned} \rho &= \rho \\ u &= \frac{\rho u}{\rho} \\ v &= \frac{\rho v}{\rho} \\ w &= \frac{\rho w}{\rho} \\ e &= \frac{\rho(e + V^2/2)}{\rho} - \frac{u^2 + v^2 + w^2}{2} \end{aligned}$$

For an *inviscid* flow, Eq. (2.65) remains the same, except that the elements of the column vectors are simplified. Examining the conservation form of the inviscid equations summarized in Sect. 2.8.2, we find that

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho(e + V^2/2) \end{Bmatrix}; \qquad F = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ \rho u(e + V^2/2) + pu \end{Bmatrix}$$

$$G = \begin{Bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho wv \\ \rho v(e + V^2/2) + pv \end{Bmatrix}; \qquad H = \begin{Bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ \rho w(e + V^2/2) + pw \end{Bmatrix}$$

$$J = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho(u f_x + v f_y + w f_z) + \rho\dot{q} \end{Bmatrix}$$

For the numerical solution of an unsteady inviscid flow, once again the solution vector is U, and the dependent variables for which numbers are directly obtained are ρ, ρu, ρv, ρw, and $\rho(e + V^2/2)$. For a steady inviscid flow, $\partial U/\partial t = 0$. Frequently, the numerical solution to such problems takes the form of 'marching' techniques; for example, if the solution is being obtained by marching in the x-direction, then Eq. (2.65) can be written as

$$\frac{\partial F}{\partial x} = J - \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \tag{2.66}$$

Here, F becomes the 'solutions' vector, and the dependent variables for which numbers are obtained are $\rho u, (\rho u^2 + p)$, ρuv, ρuw and $[\rho u(e + V^2/2) + pu]$. From these dependent variables, it is still possible to obtain the primitive variables, although the algebra is more complex than in our previously discussed case (see Ref. [5] for more details).

Notice that the governing equations, when written in the form of Eq. (2.65), have no flow variables outside the single x, y, z and t derivatives. Indeed, the terms in Eq. (2.65) have everything buried inside these derivatives. The flow equations in the form of Eq. (2.65) are said to be in *strong* conservation form. In contrast, examine the form of Eqs. (2.42a, b and c) and (2.64). These equations have a number of x, y and z derivatives explicitly appearing on the right-hand side. These are the *weak* conservation form of the equations.

The form of the governing equations given by Eq. (2.65) is popular in CFD; let us explain why. In flow fields involving shock waves, there are sharp, discontinuous changes in the primitive flow-field variables p, ρ, u, T, etc., across the shocks. Many computations of flows with shocks are designed to have the shock waves appear naturally within the computational space as a direct result of the overall flow-field solution, i.e. as a direct result of the general algorithm, without any special treatment to take care of the shocks themselves. Such approaches are called *shock-capturing methods*. This is in contrast to the alternate approach, where shock waves are *explicitly* introduced into the flow-field solution, the exact Rankine–Hugoniot relations for changes across a shock are used to relate the flow immediately ahead of and behind the shock, and the governing flow equations are used to calculate the remainder of the flow field. This approach is called the *shock-fitting method*. These two different approaches are illustrated in Figs. 2.8 and 2.9. In Fig. 2.8, the computational domain for calculating the supersonic flow over the body extends both upstream and downstream of the nose. The shock wave is allowed to form within the computational domain as a consequence of the general flow-field algorithm,

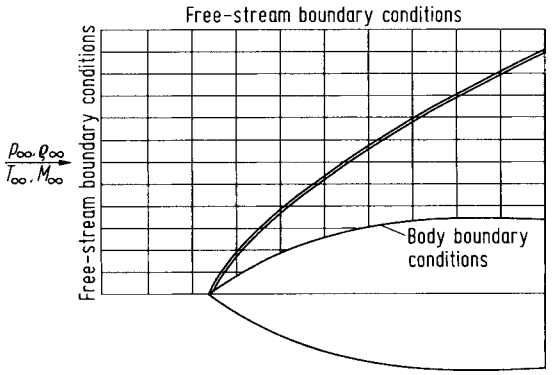

Fig. 2.8 Mesh for the shock-capturing approach

without any special shock relations being introduced. In this manner, the shock wave is 'captured' within the domain by means of the computational solution of the governing partial differential equations. Therefore, Fig. 2.8 is an example of the *shock-capturing method*. In contrast, Fig. 2.9 illustrates the same flow problem, except that now the computational domain is the flow between the shock and the body. The shock wave is introduced directly into the solution as an explicit discontinuity, and the standard oblique shock relations (the Rankine–Hugoniot relations) are used to fit the freestream supersonic flow ahead of the shock to the flow computed by the partial differential equations downstream of the shock. Therefore, Fig. 2.9 is an example of the *shock-fitting method*. There are advantages and disadvantages of both methods. For example, the shock-capturing method is ideal for complex flow problems involving shock waves for which we do not know either the location or number of shocks. Here, the shocks simply form within the computational domain as nature would have it. Moreover, this takes place without requiring any special treatment of the shock within the algorithm, and hence simplifies the computer programming. However, a disadvantage of this approach is that the shocks are generally smeared over a number of grid points in the computational mesh, and hence the numerically obtained shock thickness bears no relation what-so-ever to the actual physical shock thickness, and the *precise* location of the shock discontinuity is uncertain within a few mesh sizes. In contrast, the advantage of the shock-fitting method is

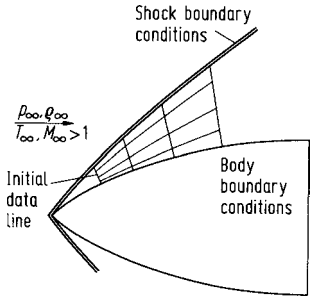

Fig. 2.9 Mesh for the shock-fitting approach

that the shock is always treated as a discontinuity, and its location is well-defined numerically. However, for a given problem you have to know in advance approximately where to put the shock waves, and how many there are. For complex flows, this can be a distinct disadvantage. Therefore, there are pros and cons associated with both shock-capturing and shock-fitting methods, and both have been employed extensively in CFD. In fact, a combination of these two methods is possible, wherein a shock-capturing approach during the course of the solution is used to predict the formation and approximate location of shocks, and then these shocks are fit with explicit discontinuities midway through the solution. Another combination is to fit shocks explicitly in those parts of a flow field where you know in advance they occur, and to employ a shock-capturing method for the remainder of the flow field in order to generate shocks that you cannot predict in advance.

Again, what does all of this discussion have to do with the conservation form of the governing equations as given by Eq. (2.65)? Simply this. For the *shock-capturing* method, experience has shown that the *conservation form* of the governing equations should be used. When the conservation form is used, the computed flow-field results are generally smooth and stable. However, when the non-conservation form is used for a shock-capturing solution, the computed flow-field results usually exhibit unsatisfactory spatial oscillations (wiggles) upstream and downstream of the shock wave, the shocks may appear in the wrong location and the solution may even become unstable. In contrast, for the shock-fitting method, satisfactory results are usually obtained for either form of the equations—conservation or non-conservation.

Why is the use of the conservation form of the equations so important for the shock-capturing method? The answer can be seen by considering the flow across a normal shock wave, as illustrated in Fig. 2.10. Consider the density distribution across the shock, as sketched in Fig. 2.10(a). Clearly, there is a discontinuous increase in ρ across the shock. If the non-conservation form of the governing equations were used to calculate this flow, where the primary dependent variables are the primitive variables such as ρ and p, then the equations would see a large discontinuity in the dependent variable ρ. This in turn would compound the numerical errors associated with the calculation of ρ. On the other hand, recall the continuity equation for a normal shock wave (see Refs. [1, 3]):

$$\rho_1 u_1 = \rho_2 u_2 \tag{2.67}$$

From Eq. (2.67), the *mass flux*, ρu, is *constant* across the shock wave, as illustrated in Fig. 2.10(b). The conservation form of the governing equations uses the product ρu as a dependent variable, and hence the conservation form of the equations see *no* discontinuity in this dependent variable across the shock wave. In turn, the numerical accuracy and stability of the solution should be greatly enhanced. To reinforce this discussion, consider the momentum equation across a normal shock wave [1, 3]:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{2.68}$$

As shown in Fig. 2.10(c), the pressure itself is discontinuous across the shock; however, from Eq. (2.68) the flux variable $(p + \rho u^2)$ is constant across the shock.

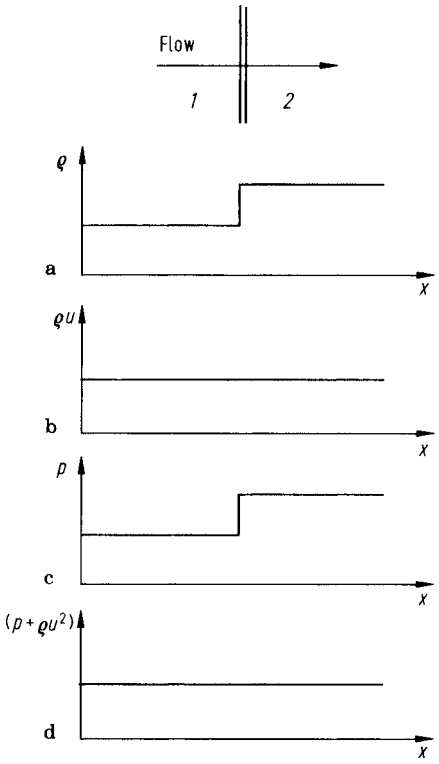

Fig. 2.10 Variation of flow properties through a normal shock wave

This is illustrated in Fig. 2.10(d). Examining the inviscid flow equations in the conservation form given by Eq. (2.65), we clearly see that the quantity $(p+\rho u^2)$ is one of the dependent variables. Therefore, the conservation form of the equations would see no discontinuity in this dependent variable across the shock. Although this example of the flow across a normal shock wave is somewhat simplistic, it serves to explain why the use of the conservation form of the governing equations are so important for calculations using the shock-capturing method. Because the conservation form uses flux variables as the dependent variables, and because the changes in these flux variables are either zero or small across a shock wave, the numerical quality of a shock-capturing method will be enhanced by the use of the conservation form in contrast to the non-conservation form, which uses the primitive variables as dependent variables.

In summary, the previous discussion is one of the primary reasons why CFD makes a distinction between the two forms of the governing equations—conservation and non-conservation. And this is why we have gone to great lengths in this chapter to derive these different forms, to explain what basic physical models lead to the different forms, and why we should be aware of the differences between the two forms.

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