Preface

Recently, the non-associative algebraic analytic structures of the spaces of bounded complex harmonic functions and harmonic functionals, which are eigenfunctions of convolution operators on locally compact groups and their Fourier algebras, have been studied in detail in [13, 14]. It was proposed in [13] to further the investigation in the non-abelian matrix setting which should have wider applications. This research monograph presents some new results and developments in this connection. Indeed, we develop a general theory of matrix convolution operators on L^p spaces of matrix functions on a locally compact group G, for $1 \leq p \leq \infty$, focusing on the spectral properties of these operators and their eigenfunctions, as well as convolution semigroups, and thereby the results in [9, 13, 14] can be subsumed and viewed in perspective in this matrix context. In particular, we describe the L^p-spectrum of these operators and study the algebraic structures of eigenspaces, of which the one corresponding to the largest possible positive eigenvalue is the space of L^p matrix harmonic functions. Of particular interest are the L^∞ matrix harmonic functions which carry the structure of a Jordan triple system. We study contractivity properties of a convolution semigroup of matrix measures and its eigenspaces. Connections with harmonic functions on Riemannian manifolds are discussed.

Some results of this work have been presented in seminars and colloquia in London, Cergy-Pontoise, Hong Kong, Taiwan, Tübingen and York. We thank warmly the audience at these institutions for their inspiration and hospitality, and hope this monograph will also serve as a useful reference for the interested audience.

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Chapter 2
Lebesgue Spaces of Matrix Functions

In this Chapter, we introduce the notations and define the spaces $L^p(G, M_n)$ of matrix L^p functions on locally compact groups G as a setting for later developments. We recall some basic definitions and derive some results for convolution operators in the scalar case. We discuss differentiability of the norm in $L^p(G, M_n)$ which is needed later, and compute the Gateaux derivative of the norm when the matrix space M_n is equipped with the Hilbert-Schmidt norm.

2.1 Preliminaries

We denote by G throughout a locally compact group with identity e and a *right* invariant Haar measure λ. To avoid the inconvenience of additional measure-theoretic technicalities, we assume throughout that λ is σ-finite. If G is compact, λ is normalized to $\lambda(G) = 1$.

Let $1 \leq p < \infty$. Given a complex Banach space E, we denote by $L^p(G, E)$ the Banach space of (equivalence classes of) E-valued Bochner integrable functions f on G satisfying

$$\|f\|_p = \left(\int_G \|f(x)\|^p d\lambda(x) \right)^{\frac{1}{p}} < \infty$$

(cf. [22, p.97]). We write $L^p(G)$ for $L^p(G, E)$ if $\dim E = 1$. In the sequel, E is usually the C^*-algebra M_n of $n \times n$ complex matrices in which case, a function $f : G \longrightarrow M_n$ is an $n \times n$ matrix (f_{ij}) of complex functions f_{ij} on G.

We denote by $\mathcal{B}(E)$ the Banach algebra of bonded linear self-maps on a Banach space E.

Let $\mathrm{Tr} : M_n \to \mathbb{C}$ be the canonical trace of M_n. Every continuous linear functional $\varphi : M_n \to \mathbb{C}$ is of the form $\varphi(\cdot) = \mathrm{Tr}(\cdot A_\varphi)$ where the matrix $A_\varphi \in M_n$ is unique and $\|\varphi\| = \mathrm{Tr}(|A_\varphi|) = \mathrm{Tr}((A_\varphi^* A_\varphi)^{1/2})$ which is the trace-norm $\|A_\varphi\|_{tr}$ of A_φ. We will identify the dual M_n^*, via the map $\varphi \in M_n^* \mapsto A_\varphi \in M_n$, with the vector space M_n equipped with the trace-norm $\|\cdot\|_{tr}$. If we equip M_n with the Hilbert-Schmidt norm

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$\|A\|_{hs} = \mathrm{Tr}(A^*A)^{1/2}$, then M_n is a Hilbert space with inner product $\langle A, B\rangle = \mathrm{Tr}(B^*A)$. We note that the C*-norm, the trace-norm and the Hilbert-Schmidt norm on M_n are related by

$$\|\cdot\| \le \|\cdot\|_{tr} \le \sqrt{n}\|\cdot\|_{hs} \le n\|\cdot\|$$

and norm convergence is equivalent to entry-wise convergence in M_n.

If M_n is equipped with the Hilbert-Schmidt norm, then $L^2(G, (M_n, \|\cdot\|_{hs}))$ is a Hilbert space, with inner product

$$\langle f, g\rangle_2 = \int_G \mathrm{Tr}(f(x)g(x)^*)d\lambda(x).$$

Since $\|f(x)g(x)^*\|_{hs} \le \|f(x)\|_{hs}\|g(x)\|_{hs}$ for $f, g \in L^2(G, (M_n, \|\cdot\|_{hs}))$, the Bochner integral

$$\langle\langle f, g\rangle\rangle = \int_G f(x)g(x)^* d\lambda(x)$$

exists in M_n and defines an M_n-valued inner product, turning $L^2(G, (M_n, \|\cdot\|_{hs}))$ into an inner product (left) M_n-module.

We denote by $L^\infty(G, M_n)$ the complex Banach space of M_n-valued essentially bounded (locally) λ-measurable functions on G, where M_n is equipped with the C*-norm. It is a von Neumann algebra, with predual $L^1(G, M_n^*)$, under the pointwise product and involution:

$$(fg)(x) = f(x)g(x), \quad f^*(x) = f(x)^* \qquad (f, g \in L^\infty(G, M_n),\ x \in G).$$

We will study convolution operators on $L^p(G, M_n)$ defined by matrix-valued measures. In this section, we first recall some basic definitions and derive some results for convolution operators on $L^p(G)$, for later reference. One important difference in the matrix setting is the presence of non-commutative and non-associative algebraic structures.

We equip the vector space $C(G)$ of complex continuous functions on G with the topology of uniform convergence on compact sets in G, and denote by $C_c(G)$ the subspace of functions with compact support. The Banach space of bounded complex continuous functions on G is denoted by $C_b(G)$. Let $C_0(G)$ be the Banach space of complex continuous functions on G vanishing at infinity. The dual $C_0(G)^*$ identifies with the space $M(G)$ of complex regular Borel measures on G. Each $\mu \in M(G)$ has finite total variation $|\mu|$ and $M(G)$ is a unital Banach algebra in the total variation norm and the convolution product:

$$\|\mu\| = |\mu|(G), \quad \langle f, \mu * \nu\rangle = \int_G\int_G f(xy)d\mu(x)d\nu(y) \qquad (f \in C_0(G), \mu, \nu \in M(G))$$

where we always denote the duality of a dual pair of Banach spaces E and F by

$$\langle\cdot,\cdot\rangle : E \times F \longrightarrow \mathbb{C}.$$

We also write $\mu(f)$ for $\langle f, \mu \rangle = \int_G f d\mu$. The unit mass at a point $a \in G$ is denoted by δ_a where δ_e is the identity in $M(G)$. A measure $\mu \in M(G)$ is called *absolutely continuous* if its total variation $|\mu|$ is absolutely continuous with respect to the Haar measure λ.

Given $\sigma \in M(G)$, the *support* of σ is defined to be the support of its total variation $|\sigma|$ and is denoted by $\operatorname{supp}\sigma$. We denote by G_σ the closed subgroup of G generated by the support of $|\sigma|$. A measure $\sigma \in M(G)$ is called *adapted* if $G_\sigma = G$. A measure $\sigma \in M(G)$ is said to be *non-degenerate* if $\operatorname{supp}|\sigma|$ generates a dense semigroup in G. Evidently, every non-degenerate measure is adapted. An absolutely continuous (non-zero) measure on a *connected* group must be adapted.

By a (*complex*) *measure* μ on G, we will mean a measure $\mu \in M(G)\backslash\{0\}$.

The convolutions for Borel functions f and g on G, when exit, are defined by

$$\begin{aligned}(f*g)(x) &= \int_G f(xy^{-1})g(y)d\lambda(y);\\ (f*\mu)(x) &= \int_G f(xy^{-1})d\mu(y);\\ (\mu*f)(x) &= \int_G f(y^{-1}x)\triangle_G(y^{-1})d\mu(y)\end{aligned}$$

where $\triangle_G$ is the modular function satisfying $d\lambda(xy) = \triangle_G(x)d\lambda(y)$ and $d\lambda(x^{-1}) = \triangle_G(x^{-1})d\lambda(x)$.

We denote by ℓ_x and r_x, respectively, the left and right translations by an element $x \in G$:

$$\ell_x f(y) = f(x^{-1}y), \qquad r_x f(y) = f(yx) \qquad (y \in G)$$

for any function f on G. A complex function f on G is left uniformly continuous if $\|r_x f - f\|_\infty \longrightarrow 0$ as $x \to e$. It is right uniformly continuous if $\|\ell_x f - f\|_\infty \longrightarrow 0$ as $x \to e$. We also write ${}_x f = \ell_{x^{-1}} f$ and f_x for $r_x f$.

We note that each $f \in C_c(G)$ is both left and right uniformly continuous, and for any $\mu \in M(G)$, we have $f*\mu \in C_b(G)$ since $|f*\mu(x) - f*\mu(y)| \le \|\ell_{xy^{-1}} f - f\|\|\mu\|$. We also have

$$\langle f, \mu*\nu \rangle = \langle \widetilde{f}, \widetilde{\nu}*\widetilde{\mu} \rangle \tag{2.1}$$

where $\nu \in M(G)$ and we define $\widetilde{f}(x) = f(x^{-1})$ and $d\widetilde{\mu}(x) = d\mu(x^{-1})$. Note that

$$\widetilde{\mu}(f) = \mu(\widetilde{f}) = (f*\mu)(e) \quad \text{and} \quad \widetilde{\mu*\nu} = \widetilde{\nu}*\widetilde{\mu}$$

for $f \in C_c(G)$.

Let $\sigma \in M(G)$. For $1 \le p \le \infty$, we define the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ by

$$T_\sigma(f) = f*\sigma \qquad (f \in L^p(G)).$$

To avoid triviality, σ is always non-zero for T_σ. The definition of T_σ depends on its domain $L^p(G)$ although we often omit referring to it if there is no ambiguity. When regarded as an operator on $L^p(G)$, the operator T_σ is easily seen to be bounded and we denote its norm by $\|T_\sigma\|_p$, or simply $\|T_\sigma\|$ in obvious context. We have $\|T_\sigma\|_p \le \|\sigma\|$.

A convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ commutes with left translations:

$$\ell_x T_\sigma = T_\sigma \ell_x \qquad (x \in G).$$

Conversely, for abelian groups G, every translation invariant operator $T : L^1(G) \longrightarrow L^1(G)$ is a convolution operator T_σ for some $\sigma \in M(G)$ [55, 3.8.4]. However, this result does not hold for $1 < p \leq \infty$, even if G is compact and abelian [44, p.85]. We will characterise the more general matrix convolution operators in Chapter 3. In particular, the above L^1 result is generalized to the matrix-valued case, for all locally compact groups.

For $1 \leq p \leq \infty$, we denote by q its conjugate exponent throughout, that is, $\frac{1}{p} + \frac{1}{q} = 1$, and for the dual pairing $\langle \cdot, \cdot \rangle$ between $L^p(G)$ and $L^q(G)$, we have

$$\langle f * \sigma, h \rangle = \langle f, h * \widetilde{\sigma} \rangle \tag{2.2}$$

for $f \in L^p(G)$ and $h \in L^q(G)$. This implies that T_σ is weakly continuous on $L^p(G)$ for $1 \leq p < \infty$, and is weak* continuous on $L^\infty(G)$. In particular, T_σ is a weakly compact operator on $L^p(G)$ for $1 < p < \infty$. For $p = 1, \infty$, we will discuss presently weak compactness of $T_\sigma : L^p(G) \longrightarrow L^p(G)$, but we note the following two lemmas first.

Lemma 2.1.1. *Let $\sigma \in M(G)$ and $p < \infty$. Let $T_\sigma^* : L^q(G) \longrightarrow L^q(G)$ be the dual map of the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$. Then $T_\sigma^* = T_{\widetilde{\sigma}}$. The operator $T_\sigma : L^2(G) \longrightarrow L^2(G)$ is self-adjoint if $\widetilde{\sigma} = \sigma$ is a real measure. The weak* continuous operator $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ has predual $T_{\widetilde{\sigma}} : L^1(G) \longrightarrow L^1(G)$.*

Proof. By (2.2), we have $\langle f, T_\sigma^* h \rangle = \langle f, T_{\widetilde{\sigma}} h \rangle$ for $f \in L^p(G)$ and $h \in L^q(G)$. The adjoint of T_σ in $\mathcal{B}(L^2(G))$ is $T_{\widetilde{\overline{\sigma}}}$ where $\overline{\sigma}$ is the complex conjugate of σ. □

Lemma 2.1.2. *Let $\sigma \in M(G)$ and let T_σ be the convolution operator on $L^p(G)$ for $p = 1, \infty$. We have $\|T_\sigma\|_1 = \|T_\sigma\|_\infty = \|\sigma\|$.*

Proof. We have $\|\sigma\| = \sup\{|\int_G f d\sigma| : f \in C_c(G) \text{ and } \|f\| \leq 1\}$ in which

$$\left| \int_G f d\sigma \right| = |\widetilde{f} * \sigma(e)| \leq \|\widetilde{f} * \sigma\|_\infty \leq \|T_\sigma\|_\infty$$

where $\widetilde{f} * \sigma \in C_b(G)$. Next, we have $\|T_\sigma\|_1 = \|T_\sigma^*\|_\infty = \|T_{\widetilde{\sigma}}\|_\infty = \|\widetilde{\sigma}\| = \|\sigma\|$. □

Remark 2.1.3. We note that $\|T_\sigma\|_p$ need not equal $\|\sigma\|$ if $1 < p < \infty$. Indeed, if σ is an adapted probability measure whose support contains the identity e and if $\|T_\sigma\|_p = 1$ for some $1 < p < \infty$, then G is amenable (see, for example, [4, Theorem 1]). On the other hand, if G is amenable and σ is a probability measure, then $\|T_\sigma\|_p = 1$ for all p (cf. [33, p.48]).

By Lemma 2.1.2, the spectral radius of $T_\sigma \in \mathcal{B}(L^p(G))$, for $p = 1, \infty$, is $\lim_n \|T_\sigma^n\|^{\frac{1}{n}} = \lim_n \|T_{\sigma^n}\|^{\frac{1}{n}} = \lim_n \|\sigma^n\|^{\frac{1}{n}}$ where σ^n is the n-fold convolution of σ with itself.

Lemma 2.1.4. *Let G be a compact group and let $\sigma \in M(G)$ be absolutely continuous. Then the convolution operator $T_\sigma : L^p(G) \longrightarrow L^p(G)$ is compact for every $p \in [1,\infty]$.*

Proof. Let $\sigma = h \cdot \lambda$ for some $h \in L^1(G)$. Consider first $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$. By absolute continuity of σ, we have $T_\sigma(L^\infty(G)) \subset C(G)$. Hence, by Arzela-Ascoli theorem, we need only show that the set

$$\{T_\sigma(f) : \|f\|_\infty \leq 1\}$$

is equicontinuous in $C(G)$. Let $\varepsilon > 0$. Pick $\varphi \in C_c(G)$ with support K and $\|\varphi - h\|_1 < \frac{\varepsilon}{4}$. Let W be a compact neighbourhood of the identity $e \in G$. By uniform continuity, we can choose a compact neighbourhood $V \subset W$ of e such that

$$|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2\lambda(KW)}$$

whenever $x^{-1}y \in V$. Then

$$\begin{aligned}\|\varphi_x - \varphi_y\|_1 &= \int_G |\varphi(zx) - \varphi(zy)| d\lambda(z) \\ &= \int_{KW} |\varphi(z) - \varphi(zx^{-1}y)| d\lambda(z) < \frac{\varepsilon}{2}.\end{aligned}$$

It follows that, for $x^{-1}y \in V$ and $\|f\|_\infty \leq 1$, we have

$$\begin{aligned}|T_\sigma(f)(x) - T_\sigma(f)(y)| &= \left| \int_G f(xz^{-1})h(z)d\lambda(z) - \int_G f(yz^{-1})h(z)d\lambda(z) \right| \\ &\leq \int_G |f(z^{-1})h(zx) - f(z^{-1})h(zy)| d\lambda(z) \\ &\leq \|f\|_\infty \|h_x - h_y\|_1 \\ &\leq \|f\|_\infty (\|h_x - \varphi_x\|_1 + \|\varphi_x - \varphi_y\|_1 + \|h_y - \varphi_y\|_1) < \varepsilon\end{aligned}$$

which proves equicontinuity and hence, compactness of $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$.

Likewise $T_{\tilde{\sigma}} : L^\infty(G) \longrightarrow L^\infty(G)$ is compact and therefore $T_\sigma : L^1(G) \longrightarrow L^1(G)$ is compact.

Let $1 < p < \infty$. Let (h_n) be a sequence in $C(G)$ such that $\|h_n - h\|_1 \longrightarrow 0$. Then $T_\sigma = \lim_{n\to\infty} T_{\sigma_n}$ in $\mathcal{B}(L^p(G))$, where $\sigma_n = h_n \cdot \lambda$. Hence it suffices to show compactness of T_σ on $L^p(G)$ for the case $h \in C(G)$.

Let (f_n) be a sequence in the unit ball of $L^p(G)$. Then $\|f_n\|_1 \leq 1$ for all n and compactness of $T_\sigma : L^1(G) \longrightarrow L^1(G)$ implies that the sequence $(f_n * \sigma)$ contains a subsequence L^1-converging to some $f \in L^1(G)$, and hence a subsequence $(f_k * \sigma)$ converging pointwise to f λ-almost everywhere. Since $h \in C(G)$, we have $\|f_k * \sigma\|_\infty \leq \|f_k\|_p \|h\|_q \leq \|h\|_q$ for all k, and $f \in L^\infty(G)$. It follows that

$$\|f_k * \sigma - f\|_p^p \leq \|f_k * \sigma - f\|_1 \|f_k * \sigma - f\|_\infty^{p-1} \longrightarrow 0 \qquad \text{as } k \to \infty.$$

This proves compactness of $T_\sigma : L^p(G) \longrightarrow L^p(G)$. □

Remark 2.1.5. The above result is clearly false if σ is not absolute continuous, for instance, T_σ is the identity operator if $\sigma = \delta_e$.

A compactness criterion has been given in [48] for a class of convolution operators of the form $f \in L^1(G) \mapsto f * F \in C(G)$ where $F \in L^\infty(G)$ and G is compact abelian. Compactness of the composition of a convolution operator with a multiplier has also been considered in [59, 60]. Fredholmness of convolution operators on locally compact groups has been studied in [54, 59, 61].

Proposition 2.1.6. *Let σ be a positive measure on a group G such that $\sigma^2 * \widetilde{\sigma}^2$ is adapted. Let T_σ be the associated convolution operator. The following conditions are equivalent.*

(i) $T_\sigma : L^1(G) \longrightarrow L^1(G)$ *is weakly compact.*
(ii) $T_\sigma : L^1(G) \longrightarrow L^1(G)$ *is compact.*
(iii) $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ *is weakly compact.*
(iv) $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ *is compact.*
(v) $T_\sigma : L^p(G) \longrightarrow L^p(G)$ *is compact for all* $p \in [1, \infty]$.
(vi) *G is compact and σ is absolutely continuous.*

Proof. (i) $\Longrightarrow$ (vi). We first prove compactness of G. Note that $L^1(G)$ has the Dunford-Pettis property and in particular, every weakly compact operator on $L^1(G)$ sends weakly compact subsets to norm compact sets [22, p.154]. Hence weak compactness of T_σ implies that the operator $T_\sigma^2 : L^1(G) \longrightarrow L^1(G)$ is compact, and so is the operator $T_{\sigma*\sigma*\widetilde{\sigma}*\widetilde{\sigma}} = T_{\widetilde{\sigma}}^2 T_\sigma^2$. Since $\sigma^2 * \widetilde{\sigma}^2$ is a positive measure, the spectral radius of $T_{\sigma^2*\widetilde{\sigma}^2}$ is $\sigma(G)^4$, by a remark before Lemma 2.1.4. On the Hilbert space $L^2(G)$, the operator $T_{\sigma^2*\widetilde{\sigma}^2} = T_{\sigma^2}^* T_{\sigma^2}$ is a positive operator and therefore has only non-negative eigenvalues. The eigenvalues of $T_{\sigma^2*\widetilde{\sigma}^2} \in \mathcal{B}(L^1(G))$ are also eigenvalues of $T_{\sigma^2*\widetilde{\sigma}^2} \in \mathcal{B}(L^2(G))$ and therefore non-negative. It follows that $\sigma(G)^4$ is an eigenvalue of the compact operator $T_{\sigma^2*\widetilde{\sigma}^2} \in \mathcal{B}(L^1(G))$, that is, there is a non-zero function $f \in L^1(G)$ satisfying $f * \sigma^2 * \widetilde{\sigma}^2 = \sigma(G)^4 f$. Note that the measure $\sigma(G)^{-4}\sigma^2 * \widetilde{\sigma}^2$ is an adapted probability measure on G. Now, by [10, Theorem 3.12], f is constant which implies that G must be compact.

Next, we show that σ is absolutely continuous. By the Dunford-Pettis-Phillips Theorem [22, p.75], there is an essentially bounded function $g : G \longrightarrow L^1(G)$ such that

$$T_\sigma(f) = \int_G f g d\lambda \qquad (f \in L^1(G)).$$

Now the arguments in [22, p.91] can still be applied without commutativity of G. Let $a \in G$. For each $f \in L^1(G)$, we have, for λ-a.e. y,

$$\begin{aligned}\int_G f(x)g(x)(y)d\lambda(x) &= T_\sigma f(y) = \ell_{a^{-1}} T_\sigma(\ell_a f)(y)\\ &= \int_G (\ell_a f)(x)g(x)(ay)d\lambda(x)\\ &= \int_G f(a^{-1}x)g(x)(ay)d\lambda(x)\\ &= \int_G f(x)g(ax)(ay)d\lambda(x).\end{aligned}$$

It follows that

$$g(ax)(ay) = g(x)(y)$$

for λ-a.e. x and y. This implies that, for each $f \in C(G)$, the function

$$F(y) = \int_G f(x)g(yx^{-1})(y)d\lambda(x) \qquad (y \in G)$$

is invariant under the left translations ℓ_a for all $a \in G$. Using compactness of G, one can show that F is constant λ-almost every on G, as in [22, p.91], and hence we have,

$$F(y) = \int_G F(z)d\lambda(z) = \int_G \int_G f(x)g(zx^{-1})(x)d\lambda(x)d\lambda(z) \tag{2.3}$$

for λ-a.e. y. Let $h \in L^1(G)$ be defined by

$$h(x) = \int_G g(yx^{-1})(y)d\lambda(y).$$

We show that $f * \sigma = f * h$ for each $f \in C(G) \subset L^1(G)$ which then yields absolutely continuity of σ. Indeed, for each $k \in L^\infty(G)$, we have

$$\begin{aligned}\langle k, f*h\rangle &= \int_G k(y)\int_G f(yx^{-1})h(x)d\lambda(x)d\lambda(y)\\ &= \int_G k(y)\int_G\int_G f(yx^{-1})g(zx^{-1})(z)d\lambda(x)d\lambda(z)d\lambda(y)\\ &= \int_G k(y)\int_G f(yx^{-1})g(yx^{-1})(y)d\lambda(x)d\lambda(y) \qquad \text{(by (2.3))}\\ &= \int_G k(y) f*\sigma(y)d\lambda(y) = \langle k, f*\sigma\rangle\end{aligned}$$

which concludes the proof.

(vi) $\Longrightarrow$ (v). By Lemma 2.1.4.

(v) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii). Trivial.

(iii) $\Longrightarrow$ (ii). The given condition implies that $T_{\widetilde{\sigma}} : L^1G) \longrightarrow L^1(G)$ is weakly compact. Repeating (i) $\Longrightarrow$ (v) $\Longrightarrow$ (iv) for $\widetilde{\sigma}$, we see that $T_{\widetilde{\sigma}} : L^\infty(G) \longrightarrow L^\infty(G)$ is compact, and hence $T_\sigma : L^1(G) \longrightarrow L^1(G)$ is compact.

(ii) $\Longrightarrow$ (i). Trivial. □

Remark 2.1.7. In (i) $\Longrightarrow$ (vi) above, the proof of absolute continuity of σ from weak compactness of $T_\sigma \in B(L^1(G))$ is valid for *any* measure σ on a compact group G, without adaptedness of $\sigma^2 * \widetilde{\sigma}^2$.

Corollary 2.1.8. *Given a positive absolutely continuous measure σ on a connected group G, the following conditions are equivalent.*

(i) $T_\sigma : L^1(G) \longrightarrow L^1(G)$ *is weakly compact.*
(ii) $T_\sigma : L^\infty(G) \longrightarrow L^\infty(G)$ *is weakly compact.*
(iii) $T_\sigma : L^p(G) \longrightarrow L^p(G)$ *is compact for all* $p \in [1,\infty]$.
(iv) G *is compact.*

Proof. This is because absolutely continuous measures on a connected group are adapted. □

Definition 2.1.9. The spectrum of an element a in a unital Banach algebra $\mathcal{A}$ is denoted by $\operatorname{Spec}_{\mathcal{A}} a$ which is often shortened to $\operatorname{Spec} a$ if the Banach algebra $\mathcal{A}$ is understood. For $1 \le p \le \infty$, we write $\operatorname{Spec}(T_\sigma, L^p(G))$, or simply, $\operatorname{Spec}(T_\sigma, L^p)$, for the spectrum $\operatorname{Spec} T_\sigma$, when regarding $T_\sigma \in \mathcal{B}(L^p(G))$. We denote by $\Lambda(T_\sigma, L^p(G))$, or simply, $\Lambda(T_\sigma, L^p)$, the set of eigenvalues of $T_\sigma : L^p(G) \longrightarrow L^p(G)$.

Given any Banach algebra $\mathcal{A}$ and an element $a \in \mathcal{A}$, we define, as usual, the *quasi-spectrum* of a, denoted by $\operatorname{Spec}'_{\mathcal{A}} a$, to be the spectrum $\operatorname{Spec}_{\mathcal{A}_1} a$ of a in the unit extension $\mathcal{A}_1$ of $\mathcal{A}$. We always have $0 \in \operatorname{Spec}'_{\mathcal{A}} a$. If $\mathcal{A}$ has an identity, then we have

$$\operatorname{Spec}'_{\mathcal{A}} a = \operatorname{Spec}_{\mathcal{A}} a \cup \{0\}.$$

We recall that

$$\operatorname{Spec}(T_\sigma, L^p) = \Lambda(T_\sigma, L^p) \cup \operatorname{Spec}^r(T_\sigma, L^p) \cup \operatorname{Spec}^c(T_\sigma, L^p)$$

where $\operatorname{Spec}^r(T_\sigma, L^p)$ denotes the *residue spectrum* of T_σ, consisting of $\alpha \in \operatorname{Spec}(T_\sigma, L^p) \backslash \Lambda(T_\sigma, L^p)$ satisfying

$$\overline{(T_\sigma - \alpha I)(L^p(G))} \neq L^p(G)$$

and $\operatorname{Spec}^c(T_\sigma, L^p)$ denotes the *continuous spectrum* of T_σ, consisting of $\alpha \in \operatorname{Spec}(T_\sigma, L^p) \backslash \Lambda(T_\sigma, L^p)$ such that

$$\overline{(T_\sigma - \alpha I)(L^p(G))} = L^p(G).$$

Since $T_\sigma^* = T_{\widetilde{\sigma}}$ for $p < \infty$, we have

$$\operatorname{Spec}(T_\sigma, L^p) = \operatorname{Spec}(T_{\widetilde{\sigma}}, L^q)$$

for $1 \le p < \infty$, and also $\operatorname{Spec}(T_\sigma, L^\infty) = \operatorname{Spec}(T_{\widetilde{\sigma}}, L^1)$.

We denote by $\operatorname{Spec}\sigma$ the spectrum of σ in the measure algebra $M(G)$. Note that $\operatorname{Spec}\sigma = \operatorname{Spec}\widetilde{\sigma}$ since $\widetilde{\sigma} * \widetilde{\mu} = \widetilde{\mu * \sigma}$ for each $\mu \in M(G)$.

Given a locally compact group G, we let $\widehat{G}$ be the dual space consisting of (the equivalence classes of) continuous unitary irreducible representations $\pi : G \longrightarrow \mathcal{B}(H_\pi)$, where H_π is a Hilbert space. Let $\iota \in \widehat{G}$ be the one-dimensional identity representation. For $\pi \in \widehat{G}$, $\sigma \in M(G)$ and $f \in L^1(G)$, we define the *Fourier transforms*:

$$\widehat{\sigma}(\pi) = \int_G \pi(x^{-1}) d\sigma(x) \in \mathcal{B}(H_\pi),$$
$$\widehat{f}(\pi) = \int_G f(x)\pi(x^{-1}) d\lambda(x) \in \mathcal{B}(H_\pi).$$

We have $\widehat{f * \sigma}(\pi) = \widehat{\sigma}(\pi)\widehat{f}(\pi)$ and $\widehat{\mu * \sigma}(\pi) = \widehat{\sigma}(\pi)\widehat{\mu}(\pi)$ for $\mu \in M(G)$.

The spectrum $\mathrm{Spec}_{\mathcal{B}(H_\pi)} \widehat{\sigma}(\pi)$ of $\widehat{\sigma}(\pi) \in \mathcal{B}(H_\pi)$ will be written as $\mathrm{Spec}\, \widehat{\sigma}(\pi)$ if no confusion is likely.

If G is abelian, $\widehat{G}$ is the group of characters and we often use the letter χ to denote an element in $\widehat{G}$. For $1 < p < 2$ and $f \in L^p(G)$, we define the Fourier transform $\widehat{f} \in L^q(\widehat{G})$ via Riesz-Thorin interpolation.

A continuous homomorphism χ from an abelian group G to the multiplicative group $\mathbb{C}\backslash\{0\}$ is called a *generalized character*. For such a character χ with $|\chi(\cdot)| \le 1$, one can still define $\widehat{\sigma}(\chi)$ as above. The spectrum $\Omega(G)$ of the Banach algebra $M(G)$, i.e., the non-zero multiplicative functionals on $M(G)$, identifies with the generalized characters χ of G with $|\chi(\cdot)| \le 1$, and by Gelfand theory, we have $\mathrm{Spec}\, \sigma = \widehat{\sigma}(\Omega(G))$ which contains $\widehat{\sigma}(\widehat{G})$. The spectrum of $L^1(G)$ identifies with the dual group $\widehat{G}$ and if G is discrete, then $M(G) = \ell^1(G)$ and $\mathrm{Spec}\, \sigma = \mathrm{Spec}_{\ell^1(\mathrm{G})}\, \sigma = \widehat{\sigma}(\widehat{\mathrm{G}})$. For arbitrary groups, we have the following result.

Lemma 2.1.10. *Let σ be a complex measure on a group G. Then*

$$\Lambda(T_\sigma, L^1) \subset \bigcup_{\pi \in \widehat{G}} \mathrm{Spec}\, \widehat{\sigma}(\pi) \subset \mathrm{Spec}\, \sigma.$$

The inclusions are strict in general.

Proof. Similar inclusions hold in the more general matrix setting for which a simple proof will be given in Proposition 3.3.8. If σ is an adapted probability measure and G is non-compact, then by [10, Theorem 3.12], $1 \notin \Lambda(T_\sigma, L^1)$ while $1 \in \mathrm{Spec}\, \iota(\sigma)$ where $\iota \in \widehat{G}$ is the identity representation.

If G is abelian, then $\widehat{\sigma}(\widehat{G}) = \bigcup_{\pi \in \widehat{G}} \mathrm{Spec}\, \widehat{\sigma}(\pi)$ and Example 3.3.4 shows that the last inclusion can be strict. In fact, even the closure $\overline{\widehat{\sigma}(\widehat{G})}$ may not equal $\mathrm{Spec}\, \sigma$ by Remark 3.3.24. □

It has been shown in [10, Lemma 3.11] that $1 \notin \bigcup_{\pi \in \widehat{G} \backslash \{\iota\}} \mathrm{Spec}\, \widehat{\sigma}(\pi)$ if σ is an adapted probability measure on a locally compact group G. In general, there seem to be few definitive results concerning the spectrum of T_σ for non-abelian groups. We will consider this case in Chapter 3 and prove various results there.

We will make use of a version of the Wiener-Levy theorem, stated below, which has been proved in [55, Theorem 6.2.4] and will be generalized to the matrix setting in Chapter 3.

Lemma 2.1.11. *Let Ω be an open set in $\mathbb{C}$ and let $F : \Omega \longrightarrow \mathbb{C}$ be a real analytic function satisfying $F(0) = 0$ if $0 \in \Omega$. Given an abelian group G and a function $f \in L^1(G)$ such that $\overline{\widehat{f}(\widehat{G})} \subset \Omega$, then $F(\widehat{f})$ is the Fourier transform of an $L^1(G)$-function.*

Example 2.1.12. For the Cauchy distribution

$$d\sigma_t(x) = \frac{t}{\pi(t^2 + x^2)} dx \qquad (t > 0)$$

on $\mathbb{R}$, we have $\widehat{\sigma}_t(\widehat{\mathbb{R}}) = \{\exp(-t|x|) : x \in \mathbb{R}\} = (0,1] = \mathrm{Spec}(T_\sigma, L^p) \backslash \{0\} = \Lambda(T_{\sigma_t}, L^\infty)$.

Example 2.1.13. Let G be any locally compact group and let $\sigma = \delta_a$ be the unit mass at $a \in G$. Then T_σ is a translation on $L^p(G)$ and we have

$$\mathrm{Spec}(T_\sigma, L^\infty) \subset \{\alpha : |\alpha| = 1\}.$$

If $G = \mathbb{T}$ and $a = i$, then $L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$ and $\mathrm{Spec}(T_\sigma, L^\infty) = \mathrm{Spec}(T_\sigma, L^2) = \widehat{\sigma}(\mathbb{Z}) = \{\exp(-in\pi/2) : n \in \mathbb{Z}\} = \{\pm 1, \pm i\} \neq \{\alpha : |\alpha| = 1\}$.

If $G = \mathbb{Z}$ and $a = 1$, then $\mathrm{Spec}(T_\sigma, \ell^2) = \{\alpha : |\alpha| = 1\} = \mathrm{Spec}(T_\sigma, \ell^\infty)$.

If $G = \mathbb{R}$ and $a \neq 0$, then $\mathrm{Spec}(T_\sigma, L^p) = \{\alpha : |\alpha| = 1\} = \Lambda(T_\sigma, L^\infty)$ as $\widehat{\delta}_a(\widehat{\mathbb{R}}) = \{\exp(-ia\theta) : \theta \in \mathbb{R}\}$.

Next consider the measure $\mu = \frac{1}{2}(\delta_0 + \delta_1)$ on $\mathbb{R}$. Its n-fold convolution

$$\mu^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \delta_k$$

is a convex sum of discrete measures and we have

$$\mathrm{Spec}(T_{\mu^n}, L^\infty) = \Lambda(T_{\mu^n}, L^\infty) = \left\{ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp(-ik\theta) : \theta \in \mathbb{R} \right\}$$

where, for example, $\mathrm{Spec}(T_\mu, L^\infty)$ is the circle containing 0 and internally tangent to the unit circle at 1, with $\sin \pi x$ as a 0-eigenfunction.

2.2 Differentiability of Norm in $L^p(G, M_n)$

We will be working with the complex Lebesgue spaces $L^p(G, M_n)$ where, for convenience and consistency with previous and related works elsewhere, we equip M_n with the C*-norm unless otherwise stated. Some remarks are in order here. First, there is no essential difference if one chooses to equip M_n with the trace norm since it amounts to considering the space $L^p(G,M_n^*)$ which is, for $p>1$, the dual of $L^q(G,M_n)$. Also, the Lebesgue spaces $L^p(G,M_n)$ defined in terms of the C*, trace and Hilbert-Schmidt norms on M_n are all isomorphic and most results for these three cases are identical. There is, however, a difference among the three cases if one considers the differentiability of the norm of $L^p(G,M_n)$ which will be needed later.

Let us first consider the differentiability of the C*-norm $\|\cdot\|$, the trace norm $\|\cdot\|_{tr}$ and the Hilbert-Schmidt norm $\|\cdot\|_{hs}$ on M_n, regarded as a real Banach space.

We recall that the norm $\|\cdot\|$ of a real Banach space E is said to be *Gateaux differentiable* at a point $u \in E$ if the following limit exists

$$\partial\|u\|(x) = \lim_{t\to 0}\frac{\|u+tx\|-\|u\|}{t}$$

for each $x \in E$, in which case, the limit is called the *Gateaux derivative* of the norm at u, *in the direction of* x. We note that the *right directional derivative*

$$\partial^+\|u\|(x) = \lim_{t\downarrow 0}\frac{\|u+tx\|-\|u\|}{t}$$

always exists. In fact, it is equal to

$$\sup\{\psi(x) : \psi \text{ is a subdifferential at } u\}$$

where a linear functional ψ in the dual E^* is called a *subdifferential* at u if

$$\psi(x-u) \le \|x\| - \|u\|$$

for each $x \in E$. The norm is Gateaux differentiable at u if, and only if, there is a unique subdifferential at u, in which case, the subdifferential is the Gateaux derivative (cf. [53, Proposition 1.8]).

The Hilbert-Schmidt norm $\|\cdot\|_{hs}$ on M_n is Gateaux differentiable at every $A \in M_n\backslash\{0\}$. Indeed, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\|A+tX\|_{hs}-\|A\|_{hs}}{t} &= \lim_{t\to 0}\frac{\mathrm{Tr}((A+tX)^*(A+tX))-\mathrm{Tr}(A^*A)}{t(\|A+tX\|_{hs}+\|A\|_{hs})}\\ &= \frac{\mathrm{Tr}(A^*X+X^*A)}{2\|A\|_{hs}}\\ &= \frac{1}{\|A\|_{hs}}\mathrm{Re}\,\mathrm{Tr}(A^*X).\end{aligned}$$

Although the norm of a separable Banach space is Gateaux differentiable on a dense G_δ set, it is easy to see that the C*-norm and the trace norm need not be Gateaux differentiable at every non-zero $A \in M_n$.

Lemma 2.2.1. *Let $A \in M_n\backslash\{0\}$. The C*-norm on M_n is Gateaux differentiable at A if, and only if, given any unit vectors $\xi,\eta \in \mathbb{C}^n$ with $\|A\xi\| = \|A\eta\| = \|A\|$, we have*

$$\langle A\xi, X\xi\rangle = \langle A\eta, X\eta\rangle \qquad (X \in M_n).$$

In the above case, the Gateaux derivative at A is given by

$$\partial\|A\|(X) = \frac{1}{\|A\|}\operatorname{Re}\langle A\xi, X\xi\rangle \qquad (X \in M_n)$$

where $\xi \in \mathbb{C}^n$ is a unit vector satisfying $\|A\xi\| = \|A\|$.

Proof. Suppose the norm is Gateaux differentiable at A. Let $\xi \in \mathbb{C}^n$ be a unit vector such that $\|A\| = \|A\xi\|$. Define a real continuous linear functional $\psi_\xi : M_n \longrightarrow \mathbb{R}$ by

$$\psi_\xi(X) = \frac{1}{\|A\|}\operatorname{Re}\langle A\xi, X\xi\rangle \qquad (X \in M_n).$$

Then for each $X \in M_n$, we have

$$\begin{aligned}\psi_\xi(X-A) &= \frac{1}{\|A\|}\operatorname{Re}\langle A\xi, X-A\xi\rangle\\ &= \frac{1}{\|A\|}\operatorname{Re}(\langle A\xi, X\xi\rangle - \langle A\xi, A\xi\rangle) \le \|X\| - \|A\|.\end{aligned}$$

Hence ψ_ξ is a subdifferential at A. If η is a unit vector in $\mathbb{C}^n$ such that $\|A\eta\| = \|A\|$, then we must have $\psi_\eta = \psi_\xi$, by uniqueness of the subdifferential, which gives $\langle A\xi, X\xi\rangle = \langle A\eta, X\eta\rangle$ for every $X \in M_n$.

To show the converse, we note that (cf. [5, Proposition 4.12]),

$$\lim_{t\downarrow 0}\frac{\|A+tX\| - \|A\|}{t} = \sup\left\{\lim_{t\downarrow 0}\frac{\|(A+tX)\xi\| - \|A\xi\|}{t} : \|\xi\| = 1, \|A\xi\| = \|A\|\right\}$$

where

$$\begin{aligned}\lim_{t\downarrow 0}\frac{\|(A+tX)\xi\| - \|A\xi\|}{t} &= \lim_{t\downarrow 0}\frac{\langle (A+tX)\xi, (A+tX)\xi\rangle - \langle A\xi, A\xi\rangle}{t(\|(A+tX)\xi\| + \|A\xi\|)}\\ &= \frac{\langle A\xi, X\xi\rangle + \langle X\xi, A\xi\rangle}{2\|A\|}.\end{aligned}$$

Hence the necessary condition implies that the above set on the right reduces to a singleton which gives the right directional derivative. We also have

$$\begin{aligned}\lim_{t\uparrow 0}\frac{\|(A+tX)\xi\| - \|A\xi\|}{t} &= -\lim_{t\downarrow 0}\frac{\|(A-tX)\xi\| - \|A\xi\|}{t}\\ &= -\frac{\langle A\xi, -X\xi\rangle + \langle -X\xi, A\xi\rangle}{2\|A\|}\\ &= \lim_{t\downarrow 0}\frac{\|(A+tX)\xi\| - \|A\xi\|}{t}.\end{aligned}$$

This proves Gateaux differentiability at A. The last assertion is clear from the above computation. $\square$

Example 2.2.2. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2$. Then the unit vectors in $\mathbb{C}^2$ where A achieves its norm are of the form $(\alpha, 0)$ with $|\alpha| = 1$. For any matrix $X = (x_{ij})$ in M_2, we have $\langle A(\alpha,0)^T, X(\alpha,0)^T\rangle = \overline{x_{12}} + \overline{x_{21}}$ which is independent of α, and the C*-norm is Gateaux differentiable at A with derivative

$$\partial\|A\|(X) = \mathrm{Re}\,\langle A(1,0)^T, X(1,0)^T\rangle = \mathrm{Re}\,x_{11}.$$

The matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ achieves its norm at $(\sqrt{2},0)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$; but

$$\left\langle B(\sqrt{2},0)^T, X(\sqrt{2},0)^T \right\rangle \neq \left\langle B\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T, X\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)^T \right\rangle$$

if X is the identity matrix, say. Hence the C*-norm is not Gateaux differentiable at B, however, we have the *right* I-directional derivative

$$\begin{aligned}\partial^+\|B\|(I) &= \lim_{t\downarrow 0} \frac{\|B+tI\| - \|B\|}{t} \\ &= \lim_{t\downarrow 0} \frac{\sqrt{1+t+t^2+\sqrt{1+2t+2t^2}} - \sqrt{2}}{t} = \frac{\sqrt{2}}{2}.\end{aligned}$$

On the other hand, the trace norm $\|\cdot\|_{tr}$ is not Gateaux differentiable at A since

$$\frac{\|A+tX\|_{tr} - \|A\|_{tr}}{t} = \frac{|t|}{t}$$

for $X = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, say.

Lemma 2.2.3. *Let $A \in M_n \backslash \{0\}$ with polar decomposition $A = u|A|$. If the trace norm $\|\cdot\|_{tr}$ on M_n is Gateaux differentiable at A, then the Gateaux derivative is given by*

$$\partial\|A\|_{tr}(X) = \mathrm{Re}\,\mathrm{Tr}(u^*X) \qquad (X \in M_n).$$

Proof. We only need to show that $\psi(X) = \mathrm{Re}\,Tr(u^*X)$ is a subdifferential. Indeed, we have $|A| = u^*A$ and

$$\begin{aligned}\psi(X-A) &= \mathrm{Re}\,\mathrm{Tr}(u^*X) - \mathrm{Re}\,\mathrm{Tr}(u^*A) \\ &\leq \|u^*\|\|X\|_{tr} - \|A\|_{tr} \\ &= \|X\|_{tr} - \|A\|_{tr}.\end{aligned}$$

$\square$

Example 2.2.4. In Example 2.2.2 above, we have $u = A$ in the polar decomposition of A and $\operatorname{Re} Tr(u^*X) = 0$ for $X = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, while the *right* X-directional derivative is given by

$$\lim_{t \downarrow 0} \frac{\|A+tX\|_{tr} - \|A\|_{tr}}{t} = \lim_{t \downarrow 0} \frac{|t|}{t} = 1.$$

Due to the non-smoothness of the C*-norm and trace norm on M_n, we will consider the Lebesgue spaces $L^p(G,(M_n,\|\cdot\|_{hs}))$ with M_n equipped with the Hilbert-Schmidt norm when we need to make use of norm differentiability later. We compute below the Gateaux derivatives for $L^p(G,(M_n,\|\cdot\|_{hs}))$.

Since the function $u \in E \mapsto \|u\|^p$ is convex on any Banach space E, we have, for $0 < t < 1$ and $u, v \in E$,

$$\|u+tv\|^p \leq (1-t)\|u\|^p + t\|u+v\|^p$$

and

$$\|u\|^p \leq \frac{t}{1+t}\|u-v\|^p + \frac{1}{1+t}\|u+tv\|^p$$

which gives

$$\|u\|^p - \|u-v\|^p \leq \frac{1}{t}(\|u+tv\|^p - \|u\|^p) \leq \|u+v\|^p - \|u\|^p. \tag{2.4}$$

Proposition 2.2.5. *Let $1 < p < \infty$. The norm of $L^p(G,(M_n,\|\cdot\|_{hs}))$ is Gateaux differentiable at each non-zero f with Gateaux derivative*

$$\partial\|f\|_p(g) = \operatorname{Re}\|f\|_p^{1-p}\int_{\{x:f(x)\neq 0\}} \|f(x)\|_{hs}^{p-2}\operatorname{Tr}(f(x)^*g(x))d\lambda(x)$$

for $g \in L^p(G,(M_n,\|\cdot\|_{hs}))$.

Proof. Given $A \in M_n \backslash \{0\}$, we have, by the chain rule,

$$\frac{d}{dt}\bigg|_{t=0} \|A+tX\|_{hs}^p = p\|A\|_{hs}^{p-1} \frac{d}{dt}\bigg|_{t=0} \|A+tX\|_{hs} = p\|A\|_{hs}^{p-1}\operatorname{Re}\operatorname{Tr}(A^*X)$$

for $X \in M_n$.

Fix a non-zero f in $L^p(G,(M_n,\|\cdot\|_{hs}))$. Given $p > 1$ and $g \in L^p(G,(M_n,\|\cdot\|_{hs}))$, we have

$$\frac{d}{dt}\bigg|_{t=0} \|tg(x)\|_{hs}^p = 0.$$

By (2.4) and the dominated convergence theorem, we have

$$
\begin{aligned}
p\|f\|_p^{p-1} \left.\frac{d}{dt}\right|_{t=0} \|f+tg\|_p &= \left.\frac{d}{dt}\right|_{t=0} \|f+tg\|_p^p \\
&= \int_G \left.\frac{d}{dt}\right|_{t=0} \|f(x)+tg(x)\|_{hs}^p d\lambda(x) \\
&= \int_{\{x:f(x)\neq 0\}} \left.\frac{d}{dt}\right|_{t=0} \|f(x)+tg(x)\|_{hs}^p d\lambda(x) \\
&= \int_{\{x:f(x)\neq 0\}} p\|f(x)\|_{hs}^{p-2} \mathrm{Re}\,\mathrm{Tr}(f(x)^* g(x)) d\lambda(x)
\end{aligned}
$$

which gives the formula for the Gateaux derivative at f. □

Corollary 2.2.6. *For $1 < p < \infty$, the Lebesgue space $L^p(G,(M_n,\|\cdot\|_{hs}))$ is strictly convex, that is, the extreme points of its closed unit ball are exactly the functions of unit norm.*

Proof. This follows from the fact that a Banach space E is strictly convex if, and only if, the norm of its dual E^* is Gateaux differentiable on the unit sphere. □