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General Introduction

"As far as the laws of mathematics refer to reality, they are not certain; as far as they are certain, they do not refer to reality."

Albert Einstein

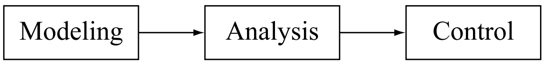
In the analysis of contemporary dynamical systems, scientists and engineers are often confronted with increasingly complex models that can simultaneously include terms taking into account non-linear dynamics, time delays and hysteresis effects, and uncertainties in parameters. Classical optimal control techniques have allowed them to optimize the control systems they build for cost and performance, but these techniques are not always tolerant of fluctuations in the dynamical system or in the real world. The goal of robust control theory is to estimate the performance changes of a dynamical system with changing system parameters and functions, and to develop alternatives that are insensitive to changes in the system in order to maintain the stability and the performance. In a broad sense, the goal of the robust control is to maintain the transformation from the desired state to the output state as close to unity as possible, despite these fluctuations.

In this chapter, we explain the motivations and our general ideas for using the robust control theory to study non-linear dynamical systems. Then, we present the general process for our robust control approach and finally, in order to explain our theoretical proposals on practical cases, we briefly give various applications, that exhibit graceful degradation subject to many disturbances, as well as more fluctuations which affect considerably the model of the dynamical system, where the use of robust control theory is extremely important.

1.1 Motivations and Objectives

The differential systems that we want to study are spatially and temporally distributed and they are governed by partial differential equations (PDEs) which are mostly non-linear. They may represent many fields of physics or bio-technological processes, which are modeled by systems with parameters distributed and governed by time-dependent PDEs (with or without time delays), in which it is of interest to prescribe a suitable dynamical behavior.

The mathematical robust control theory is a part of applied mathematics serving perhaps the most important link between modeling, mathematics (either from a theoretical or computational point of view), industrial processes and technology. We note that the three main steps in the area of research in robust control of dynamical systems are inextricably linked, as shown below:



The investigated problems are of various nature and deal both with the analysis of the structural properties of parametrically dependent differential equations and with their regulation according to some task or cost. Three types of problems arise then naturally:

(*i*) *Identification*: Certain parameters or functions intervening in these models are unknown, or rather badly known (for example, coefficients of diffusion, non-linear source, initial conditions or boundary conditions, *etc.*). We propose to identify these parameters or functions starting from experimental observations: these problems are called "inverse problems" (in opposition to the resolution of equations themselves which constitutes the direct problem). Indeed, certain parameters or functions can influence considerably the material behavior or modify phenomena in environmental, bio-economic, biological or medical matter; then their knowledge is an invaluable help for the physicists, biologists or chemists who, in general, use a mathematical model for their problem, but with a great uncertainty on its parameters. The resolution of the inverse problems thus provides them essential informations which are necessary to the comprehension of the various processes which can intervene in these models.

(*ii*) *Regulation*: The most real physical, biological or chemical systems can only be described by means of an uncertain model may induce instability. Moreover, the systems are destabilizing by unmeasured noises and disturbances. Consequently, even if some "well-posedness" property is verified, the systems become often unstable. The idea is to regulate the response of systems by modifying the dynamical nature of the system.

(*iii*) *Optimization*: The physicists, biologists and chemists control, in general, their experimental devices by using a certain number of functions of control which enable them to optimize and/or to stabilize the system. The work of the mathematician consists in determining these functions in an optimal way. The methods consist in designing a trajectory for the control inputs and are normally based on optimization of the performance of the system relative to some performance functional.

The optimal control methods are used to determine the unknown parameters or control certain functions for problems where uncertainties (disturbances, noises, fluctuations, *etc.*) are neglected. But it is well known that many uncertainties occur in more realistic studies of physical, biological or chemical problems. The presence of these uncertainties may induce complex behaviors, *e.g.*, oscillations, instability, bad performances, *etc.* Problems with uncertainties are the most challenging and difficult in control theory but their analysis are necessary and important for applications.

The fundament of robust control theory, which is a generalization of the optimal control theory, is to take into account these uncertain behaviours and to analyze how the control system can deal with this problem. From Chandrasekharan [70], "*Robust control refers to the control of unknown plants with unknown dynamics subject to unknown disturbances.*"

The uncertainty can be of two types: first, the errors (or imperfections) coming from the model (difference between the reality and the mathematical model, in particular if some parameters are badly known) and, second, the unmeasured noises and fluctuations that act on the physical, biological or chemical systems. These uncertainty terms can have additive and/or multiplicative components. They often lead to great instability: for example, the *El Niño* phenomenon, tropical instability waves, which are essentially the consequence of a small perturbation of ocean-surface temperature (these equatorial waves are the precursors for hurricanes and typhoons). The goal of robust control theory is to control these instabilities, either by acting on some parameters to maintain the system in a desired state (target), or by calculating the limit of these parameters before the system becomes unstable ("predict to act"). In other words, the robust control allows engineers to analyze instabilities and their consequences and helps them to determine the most acceptable conditions for which a system remains stable. The goal is then to define the maximum of noises and fluctuations that can be accepted if we want to keep the system stable. Therefore, we can predict that if the disturbances exceed this threshold, the system becomes unstable (for example, we can predict the fluctuation of ocean-surface temperature from which the *El Niño* phenomenon can occur). It also allows us, in a system where we can control the perturbations, to provide the threshold at which the system becomes unstable. The robust control theory for dynamical systems can be described by the block diagram Figure 1.1, where the function ψ corresponds to the disturbance, ϕ is

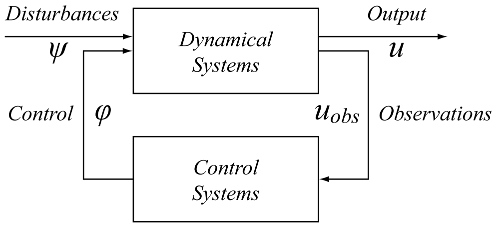


Figure 1.1. Process of robust control

the control function, u_{obs} is the observation (*e.g.*, measurement), u represents the perturbation of the desired target (that it is desired to keep small).

The fundamental idea of our approach is the connection between the game theory approach and the problem of stabilizing uncertain non-linear distributed parameter systems[1] where the fluctuation dynamics and noises are *deterministic*. This is motivated, by the fact that the robust control theory can be represented as a *differential game*[2] between an engineer seeking the best control which stabilizes the system perturbations with limited control efforts, and simultaneously plant (during physical, biological or chemical experiences) or unexpected events (during the physical, biological or chemical dynamics) seeking the maximally malevolent disturbance which destabilizes the system perturbations with limited disturbance magnitude.

In other words, the idea of our approach is to transform robust stability and performance problems into constrained game-type minimax optimization ones (of infinite-dimensional dynamical systems). The objective of a robust control is then to compensate the undesirable effects of system disturbances through control actions such that a performance functional achieves its minimum for the worst disturbances, *i.e.*, to find the best control which takes into account the worst-case disturbance. This area concerns investigation of the control, stability and adjoint control optimization of infinite-dimensional dynamical systems.

Since the late 1970s, a great variety of techniques have been developed for this new area of research. Many different models describing physical, biological or chemical systems (from chemical to mechanical engineering, from biological systems to population dynamics, *etc.*) with taking into account some uncertainties are giving raise to different research directions. In the framework of many robust control problems studied in the literature, the considered problems mainly correspond to the construction of controllers for linear (or linearized) plants with additive disturbances (because by assuming these considerations, the analysis is greatly simplified). But it is well-known that in the real world, the systems have non-linear dynamical behaviors and dis-

[1] The goal of the game is to find a controller in the presence of an adversary that changes the process.

[2] In the sense of two-person zero-sum game, see Section 5.3.2.

turbances may act additively and/or multiplicatively. Moreover, the systems may present hard control, disturbance or state constraints (*e.g.*, pointwise constraints).

Therefore, we are interested in the robust regulation of the deviation of the systems from the desired target, by analyzing the *full non-linear and time varying systems* (with or without time delays), which models large perturbations to the desired target, and by considering different actions of disturbances and controls with various constraints.

In our approach it is not assumed that the system is stabilizable or detectable, as opposed to other books in the treatment of robust control problems to some classes of finite (or infinite)-dimensional systems which are considered in terms of a general Riccati operator (see, *e.g.*, Başar and Bernhard, [26], Chen [76], Foias *et al.* [125], Dullerud and Paganini [232], Petersen *et al.* [241], Sanchez-Pena and Sznaier [257], Van Keulen [288], Whittle [300], Zhou *et al.* [311, 312], and references therein), a hypothesis that is difficult to verify in practice. Moreover, in the more general case, numerical realizations based on the adjoint control optimization are preferred over techniques based on Riccati approaches. To the best of our knowledge, the approaches and results developed in the previous and below references are not applicable to the applications analyzed in this book.

The previous references give an interesting background though most of these references consider linear systems (of finite or infinite dimensions) and optimizations over the infinite time horizon, and are referred to differential games and $\mathcal{H}_\infty$-control theory[3] (that is well described in Green and Limebeer [140] and Zhou *et al.* [311]) or its stochastic counterpart, *i.e.*, risk-sensitive control theory. For the robust design where the $\mathcal{H}_\infty$-control problem is regarded as loop-shaping problem, the reader can refer to Vinnicombe [290], in which the author introduces a new metric for systems. Other methods based on Lyapunov design method were proposed for robust stabilization of non-linear uncertain systems see, *e.g.*, Qu [246], in which the author considers the robust stabilization of systems described by ordinary differential equations. For practical examples, we can cite Ackermann *et al.* [3], in which the authors present stability analysis for problems described by linear time-invariant, based on Kharitonov-type criteria. Finally, for robust control of time-delay systems, the reader may refer to Zhong [310], in which used tools are chain-scattering approach and J-spectral factorizations and Niculescu [230], in which the author treats the stability of finite-dimensional delay differential equations (we can also refer to Mahmoud [213]). The reader can find other references which complete this survey in the introduction of each chapter.

The approach described in this book, which is based on minimax theorems in relation with non-linear PDEs in finite time horizon and motivated by practical application, has been highlighted from the beginning of the 2000s. For

[3] $\mathcal{H}_\infty$-control problem is worked and posed in Hardy spaces (spaces of all stable transfer functions).

these developments, we can mention for the robust control of a class of nonlinear parabolic systems with time-varying delays, Belmiloudi [38, 39]; for robust control of the incompressible Navier–Stokes equations, Bewley *et al.* [52]; for the Kuramoto–Sivashinsky model, Hu and Temam [162]; for the stability of solidification processes, Belmiloudi *et al.* [40, 41]; and for the Ginzburg–Landau system and superconductivity, Belmiloudi [45].

We shall now present the process of our control robust approach.

1.2 General Process of the Robust Control Theory

In contrast to the optimal control problems,[4] the relation between the problems of identification, regulation and optimization, lies in the fact that it acts, in these cases, to find a saddle point of a functional calculus depending on the control, the disturbance and the solution of the perturbed PDEs. Indeed, the problems of control can be formulated as the robust regulation of the deviation of the systems from the desired target; the considered control and disturbance variables, in this case, can be in the parameters or in the functions to be identified. This optimization problem (a minimax problem), depending on the solution of PDEs, with respect to control and disturbance variables (intervening either in the initial conditions, or boundary conditions or equation itself), is the base of the robust control theory of PDEs.

The essential data used in our robust control problem are the following:

1. A "control" variable φ in a set U_{ad} (known as set of "admissible controls") and a "disturbance" variable ψ in a set V_{ad} (known as set of "admissible disturbances").
2. The state $u(\varphi, \psi)$ of the system to be controlled, which is given, for a chosen control-disturbance (φ, ψ), by the resolution of a perturbed equation
$$\tilde{\mathcal{F}}(t)(u(\varphi,\psi)) = \text{"given function of } (\varphi,\psi)\text{"}$$
where $\tilde{\mathcal{F}}(.)$ is an operator (supposed to be known) which represents the system to be controlled and u is the perturbation of the desired target U. The operator $\tilde{\mathcal{F}}(.)$, which depends on U, is the perturbation of the model $\mathcal{F}(.)$ of the studied system.
3. An "observation" u_{obs} which is supposed to be known exactly (for example, the desired tolerance for the perturbation or the offset given by measurements).
4. A "cost" functional (or "objective" functional) $J(\varphi, \psi)$ which is defined from a real-valued and positive function $G(X, Y)$ by
$$J(\varphi,\psi) = G((\varphi,\psi), u(\varphi,\psi)).$$

[4] The optimal control problem corresponds to minimize or maximize a calculus function depending on the control and the solution of PDEs.

The goal is to find a saddle point of J, *i.e.*, a solution $(\varphi^*, \psi^*) \in U_{\mathrm{ad}} \times V_{\mathrm{ad}}$ of

$$J(\varphi^*, \psi) \leq J(\varphi^*, \psi^*) \leq J(\varphi, \psi^*) \quad \forall \varphi \in U_{\mathrm{ad}}, \psi \in V_{\mathrm{ad}}.$$

It should be noted that there is no general method to analyze the problems of robust control (it is necessary to adapt it in each situation). Moreover, in non-linear systems or bilinear systems, the analysis is more complicated than in the case of optimal control problems, because we are interested in the robust regulation of the deviation of the systems from the desired target, by analyzing the *full non-linear* systems which model large perturbations to the desired target. Consequently the perturbations of the initial models, governed by PDEs, which show additional operators (and then difficulties) generate new primal problem and then new dual problem which, often, seem of a new type.

On the other hand, we can define the process to be followed for each situation:

(*i*) Solve the initial problem (analysis of PDEs, existence of solutions, stability according to the data, regularity, *etc.*).

(*ii*) Define the function or the parameter to be identified and the type of disturbance to be controlled.

(*iii*) Introduce and solve the perturbed problem which plays the role of the primal problem (analysis of PDEs, existence of solutions, stability according to the data, regularity, differentiability of the operator solution, *etc.*).

(*iv*) Define the cost (or objective) functional, which depends on control and disturbance functions.

(*v*) Obtain the existence of an optimal solution (as a saddle point of the cost functional) and analyze the necessary (and if possible the sufficient) conditions of optimality (which require to obtain before a very fine regularity on the state functions).

(*vi*) Characterize the optimal solutions.

(*vii*) Define an algorithm allowing to solve numerically the robust control problem (which requires sometimes the development of new methods of numerical resolution).

We now present some applications to biological and physical sciences (which are studied and analyzed in the third part of this book).

1.3 Applications to Biological and Physical Sciences

The mathematical, physical and biological systems studied in the third part of the book include the following:

- reaction-diffusion equations from population growth (*e.g.*, Lotka–Volterra model)
- bioheat transfer and Pennes-type model
- Ginzburg–Landau system and superconductivity
- Warren–Boettinger-type model and solidification
- incompressible Navier–Stokes equations coupled with transport-diffusion equations and other equations of fluid mechanics
- micropolar fluid model.

For all these problems, the following questions will be addressed:

(a) modeling and governing system
(b) existence, uniqueness, regularity and continuous dependence on the data of the model
(c) perturbation model and same points as in (b)
(d) motivation and formulation of the robust control problem in different situations
(e) existence of an optimal solution (a saddle point)
(f) optimality conditions and identification of gradients
(g) uniqueness of the optimal solution.

Applications are of three main types and are described next.

1.3.1 Material Sciences

Material sciences concern with the synthesis and manufacture of new materials, the modification of materials, the understanding and prediction of material properties, and the evolution and control of these properties over a time period. Today it is a vast growing body of knowledge based on physical sciences, engineering, and mathematics.

The goal is to present some mathematical treatments for the non-linear evolution systems which arise in material sciences. During the manufacture of the material, small perturbations caused by the introduction of noises terms in the data (which are regarded as impurities) give rise to surface and convective instabilities. To manufacture the materials free from impurities, it is essential to control both surface and convective instabilities.

• *Vortex dynamics in superconducting films with Ginzburg–Landau systems*: The phase transitions taking place in superconductor films with variable thickness is modeled by a two-dimensional, time-dependent Ginzburg–Landau type model with Robin boundary conditions on a phase-field parameter. To take into account thermal fluctuations and material impurities (which affect the motion of vortices in superconductors), we use a variant of Ginzburg–Landau type model containing additive noise (this work is a generalization of the recent research developed by Belmiloudi in [45]); we can note that the unknown phase-field parameters are complex valued. The objective is to control and stabilize the motion of vortices in superconductors.

• *Multi-scale modeling solidification of binary alloys and phase-field model*: The isothermal solidification of a binary alloy (*i.e.*, a mixture of two elements) is modeled by a two-dimensional, time-dependent and solutal phase field model of Warren–Boettinger type. To take into account thermal fluctuations and material impurities, we use a variant of the Warren–Boettinger model containing additive noise due to thermal fluctuations (this work is a generalization of the recent research developed by Belmiloudi in [40]). This studied model involves dentrite growth of highly supersatured binary melts. The objective is to predict and stabilize the microstructure dynamics by taking into account thermal fluctuations and material impureties. The developed technique can be used to study different general physical models concerning the solidification process, for example the problems presented recently by Granasy *et al.* [139], and Warren *et al.* [295].

1.3.2 Fluid Mechanics

Our aim is to describe some non-linear time-dependent PDEs which arise in fluid mechanics. In each case, we present briefly the physical model and the governing equations. Then we present the mathematical setting of the obtained equation and we study the robust control problem in order to control the fluctuations of the system. We also discuss the mechanisms of control of these instabilities. Different techniques and methods, used to investigate these instabilities, are developed.

• *Large-scale ocean in the climate system*: The phenomenon of long waves in tropical ocean is modeled by equations of non-linear Navier–Stokes type for the velocity and pressure, and of transport-diffusion type for the temperature and the salinity. The oceanic currents, which play a key role in the regulation of the climate, are characterized, in the tropical zone, by steady zonal currents and by long waves propagating westward along the equator and superimposed to the mean currents. The equatorial waves can be connected with strong vertical currents, which are very sensitive to small changes in temperature (for example the *El Niño* phenomenon begins with a temperature elevation of 2 or 3 °C of surface waters). The goal is to predict the deviation of circulation from the mean circulation caused by these small variations of the surface temperature. Our work therefore complements and generalizes research works developed for several years by Belmiloudi (by using the optimal control techniques) on the analysis of fluctuations and perturbations in the equatorial zone. We study this problem with two types of hypothesis: the Boussinesq approximation and the Hydrostatic approximation with vertical viscosity.

1.3.3 Biological Models

The goal is to present some mathematical treatments for the non-linear evolution systems which arise in life sciences. The questions that we address are the same as developed in the previous applications. We provide a brief review of various models in mathematical biology, and describes how these models arise. Then we study the properties of solutions of non-linear time-dependent partial differential systems (with and without time delays). In particular, the existence and the uniqueness of these solutions are discussed. Various mechanisms of control of the perturbations of the system by using different techniques under different boundary conditions, are also presented.

• *Impact of heat transfer laws on temperature distribution in biological tissues*: The temperature distribution in living tissues is modeled by some generalized transient bioheat transfer type models with directional blood flow and Robin boundary conditions. The model equation depends on the blood perfusion rate, the heat transfer parameter, the distributed energy source terms and the heat flux due to the evaporation, which affect the effects of thermal and physical properties on the transient temperature of biological tissues. The knowledge about the behavior of the temperature in tissues can be very beneficial for thermal diagnostics and treatments in medical practices, for example thermotherapy for regional hyperthermia, often used in treatment of cancer (the aim of the thermal therapy is to destroy the pathological tissues by rising the temperature with minimal damage to the surrounding tissues). The goal of our study is to control and stabilize the desired online temperature.

• *Parabolic Lotka–Volterra type systems with logistic time-varying delays*: The studied systems are governed by parabolic equations governing diffusive biological species with logistic growth terms and multiple time-varying delays. A very important ecological and economical problem is resource management, *i.e.*, the stabilization of uncertain biological species taking into account the influence of the population at earlier times on the regulatory effect. In population dynamics, this includes the multiple time-varying delay model, for example the birth rate, which does not act instantaneously (time to reach maturity), the finished period of gestation, *etc.* The applications are varied and include: forest or agriculture (trapping animals, damage cost to environment), fisheries (resource stock to prevent overfishing), *etc.*

1.3.4 Other Systems

We also develop two very interesting systems: first, we analyze the motion of animal blood which is described by micropolar fluid models and, second, we present the semiconductor melts in zone-melting and Czochralski growth configuration.

Most of the topics developed here are new or have recently appeared. Moreover, the methods developed in this book can be extended to the well-posedness non-linear hyperbolic systems. Relevant questions which are not developed here include those of hybrid systems involving a mixture of continuous and discrete dynamics. These aspects are currently being investigated and will be the object of publications in the near future.