Preface

This volume contains the courses delivered at the CIME meeting "Pseudo-differential Operators, Quantization and Signals" held in Cetraro, Italy, from June 19, 2006 to June 24, 2006 and includes the courses by H.-G. Feichtinger presenting new results for Gabor multipliers on modulation and Wiener amalgam spaces, by B. Helffer analyzing non-self-adjoint operators using microlocal techniques, by M. Lamoureux addressing applications of pseudo-differential operators in geophysics, and by N. Lerner applying the techniques of Wick quantization to problems on subellipticity and lower bounds. The lectures by J. Toft on Schatten–von Neumann classes of Weyl pseudo-differential operators are also included.

This introduction is written for non-specialists. We first recall the basic notions and give an account of some developments of pseudo-differential operators. Our starting point is the class of pseudo-differential operators studied in the 1965 seminal paper of Kohn and Nirenberg published in "Communications on Pure and Applied Mathematics." Then we give a brief overview of several pre-eminent ancestors and successors in the study of pseudo-differential operators before and after the Kohn–Nirenberg milestone. The connections with quantization envisaged by Hermann Weyl in his classic "Group Theory and Quantum Mechanics," first observed by Grossmann, Loupias and Stein in the 1968 paper "Annales de l'Institute Fourier (Grenoble)," will then be described in the context of Wigner transforms. These connections give new insights into the role of pseudo-differential operators in the analysis of signals and images in the perspectives of Gabor transforms and wavelet transforms. From these come the Stockwell transform that has numerous applications in geophysics and medical imaging. The recently developed mathematical underpinnings of the Stockwell transform will be highlighted.

1. Pseudo-differential Operators

The starting point is the class of classical pseudo-differential operators introduced by Kohn and Nirenberg [19] and modified almost immediately by Hörmander [16] about 40 years ago. To wit, let $m \in \mathbb{R}$. Then we let $S^m_{1,0}$ or simply S^m be the set of all C^∞ functions σ on $\mathbb{R}^n \times \mathbb{R}^n$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

for all x and ξ in $\mathbb{R}^n$. A function σ in S^m is called a symbol of order m.
Let $\sigma \in S^m$. Then we define the pseudo-differential operator T_σ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi)\, d\xi$$

for all φ in $\mathcal{S}(\mathbb{R}^n)$ and all x in $\mathbb{R}^n$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx$$

for all ξ in $\mathbb{R}^n$. It is easy to prove that T_σ maps $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$ continuously. The most fundamental properties of pseudo-differential operators which are useful in the study of partial differential equations are listed as Theorems 1.1–1.3.

Theorem 1.1. *Let $\sigma \in S^0$. Then T_σ, initially defined on $\mathcal{S}(\mathbb{R}^n)$, can be uniquely extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.*

Theorem 1.2. *If $\sigma \in S^m$, then $T_\sigma^* = T_\tau$, where $\tau \in S^m$ and*

$$\tau \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma}.$$

Here, T_σ^* is the formal adjoint of T_σ.
To recall, the formal adjoint T_σ^* of T_σ is defined by

$$(T_\sigma \varphi, \psi)_{L^2(\mathbb{R}^n)} = (\varphi, T_\sigma^* \psi)_{L^2(\mathbb{R}^n)}$$

for all φ and ψ in $L^2(\mathbb{R}^n)$, where $(\,,\,)_{L^2(\mathbb{R}^n)}$ is the inner product in $L^2(\mathbb{R}^n)$. The asymptotic expansion $\tau \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma}$ means that

$$\tau - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!} \partial_x^\mu \partial_\xi^\mu \overline{\sigma} \in S^{m-N}$$

for all positive integers N.

Theorem 1.3. *If $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$, then $T_\sigma T_\tau = T_\lambda$, where $\lambda \in S^{m_1+m_2}$ and*

$$\lambda \sim \sum_\mu \frac{(-i)^{|\mu|}}{\mu!} (\partial_\xi^\mu \sigma)(\partial_x^\mu \tau).$$

The asymptotic expansion

$$\lambda \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu \sigma)(\partial_x^\mu \tau)$$

means that

$$\lambda - \sum_{|\mu|<N} \frac{(-i)^{|\mu|}}{\mu!}(\partial_\xi^\mu \sigma)(\partial_x^\mu \tau) \in S^{m_1+m_2-N}$$

for all positive integers N.
All these results are very well known and can be found in the books [17] by Hörmander [20] by Kumano-go, [23] by Rodino, [29] by Wong and many others. We can see variants of these results in other settings in this presentation.

2. Ancestors and Successors
Earliest sources of pseudo-differential operators can be traced to problems for n-dimensional singular integral equations. The first contributions to the theory of multi-dimensional singular integrals appear to be those of Tricomi [27] in 1928. To recall, let (r, θ) be the polar coordinates of a generic point $y = (y_1, y_2)$ in $\mathbb{R}^2$ and define for suitable functions φ on $\mathbb{R}^2$,

$$(P\varphi)(x) = \lim_{\varepsilon \to 0} \int_{r>\varepsilon} \frac{h(\theta)}{r^2} \varphi(x-y)\, dy, \quad x \in \mathbb{R}^2.$$

In general, the integral $\int_{\mathbb{R}^2} \frac{h(\theta)}{r^2}\varphi(x-y)\,dy$ is not absolutely convergent, but under the so-called Tricomi condition stipulating that

$$\int_0^{2\pi} h(\theta)\, d\theta = 0$$

and appropriate assumptions on h and φ, the limit exists and $(P\varphi)(x)$ is well defined for almost all x in $\mathbb{R}^2$. If we assume for simplicity that h is C^∞ on the unit circle S^1 with center at the origin, then P is a bounded linear operator from $L^2(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$. Despite unsuccessful attempts by Tricomi in solving the equation

$$P\varphi = \psi$$

by finding another singular integral operator P^{-1} for which

$$P^{-1}P = I$$

and

$$PP^{-1} = I,$$

where I is the identity operator, we all know nowadays that this can be done using the Fourier transform. Indeed, P can be regarded as the convolution operator given by

$$P\varphi = K * \varphi,$$

where the singular kernel K given by

$$K(y) = \frac{h(\theta)}{r^2}, \quad y = (r, \theta) \in \mathbb{R}^2,$$

has to be suitably seen as a tempered distribution on $\mathbb{R}^2$. Applying the Fourier transform, we get

$$(P\varphi)^\wedge(\xi) = \sigma(\xi)\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^2,$$

where

$$\sigma(\xi) = 2\pi \hat{K}(\xi), \quad \xi \in \mathbb{R}^2.$$

In view of the Tricomi condition on $h \in C^\infty(S^1)$, σ turns out to be C^∞ and homogeneous of degree 0 on $\mathbb{R}^2 \setminus \{0\}$. Hence, apart from the singularity at the origin, σ is a symbol in S^0 depending on ξ only, and with the notation of the preceding section,

$$P = T_\sigma.$$

Furthermore, if σ is elliptic in the sense that there exists a positive constant C such that

$$|\sigma(\xi)| \geq C, \quad \xi \in \mathbb{R}^2,$$

then $\sigma^{-1} \in S^0$ and we can define P^{-1} to be $T_{\sigma^{-1}}$. Such applications of the Fourier transform were not known to Tricomi and it took almost 30 years for mathematicians to come to these simple conclusions. Milestones of the developments in this direction are the works of Giraud [13] in 1934, Calderón and Zygmund [4] in 1952 and Mihlin [21] in 1965. Additional references can be found in the introduction of [21] and the survey paper [24] of Seeley. In fact, the analysis has been extended to the case when h also depends on x, i.e., the kernel K is a function of x and y given by

$$K(x, y) = \frac{h(x, \theta)}{r^2},$$

where $y = (r, \theta)$. In the final formulation of these results in the setting of $\mathbb{R}^n$, the symbol $\sigma \in S^0$ is the Fourier transform with respect to y of the kernel $K(x, y)$ in terms of the singular integral given by

$$\sigma(x, \xi) = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} e^{-iy\cdot\xi} K(x, y)\, dy, \quad x, \xi \in \mathbb{R}^n.$$

If σ is elliptic in the sense that there exists a positive constant C such that

$$|\sigma(x, \xi)| \geq C, \quad x, \xi \in \mathbb{R}^n,$$

then we still have

$$\sigma^{-1} \in S^0.$$

However, it is important to note that $T_{\sigma^{-1}}$ is no longer the inverse of P in this case. But, as in Theorem 1.3, we obtain

$$T_{\sigma^{-1}} P = I + K_1$$

and

$$P T_{\sigma^{-1}} = I + K_2,$$

where K_1 and K_2 are pseudo-differential operators of order -1. When we transfer the definition of P to a compact manifold M, the operators K_1 and K_2 are compact and P is then a Fredholm operator on $L^2(M)$. It is remarkable to note that this very rudimentary symbolic calculus with remainders of order -1 plays an important role in the proof of the Atiyah–Singer index formula in [1].

In addition to the obvious extension to an arbitrary order $m \in \mathbb{R}$, the most novel ideas of the Kohn–Nirenberg paper [19] in the context of the theory of singular integral operators are the precise asymptotic formulas articulated in Theorems 1.2 and 1.3. Almost immediately after the appearance of the work of Kohn and Nirenberg is the far-reaching calculus of Hörmander [16] concerning symbols σ of type (ρ, δ), $0 \leq \delta < \rho \leq 1$. Let us recall that a function σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$ and type (ρ, δ) if for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|}$$

for all x and ξ in $\mathbb{R}^n$. Since then, other generalizations and variants of pseudo-differential operators have appeared. Among many interesting classes is the very general class of pseudo-differential operators developed by Beals [2] in 1975 in which the Hörmander estimates are replaced by

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta} \lambda(x, \xi) \Psi(x, \xi)^{-|\beta|} \Phi(x, \xi)^{|\alpha|}$$

for all x and ξ in $\mathbb{R}^n$, where

$$\Psi(x, \xi) = (\Psi_1(x, \xi), \Psi_2(x, \xi), \ldots, \Psi_n(x, \xi))$$

and

$$\Phi(x, \xi) = (\Phi_1(x, \xi), \Phi_2(x, \xi), \ldots, \Phi_n(x, \xi))$$

are n-tuples of suitable weight functions, and $\lambda(x, \xi)$ is now the "order" of the corresponding pseudo-differential operator. Recasting the calculus of Beals, another achievement is due to Hörmander [16] using the Weyl expression for pseudo-differential operators. We refer the readers to [16] for a wide range of applications to linear partial differential equations. Weyl quantization is described in the next section, and for the sake of simplicity, we begin with a motivation based on symbols in S^m, i.e., Hörmander symbols with $\rho = 1$ and $\delta = 0$.

3. Weyl Transforms

Let $\sigma \in S^m$. Then we can associate to it the pseudo-differential operator T_σ, but T_σ is not the only operator that can be assigned to σ. To see what else can be done, let us note that for all φ in $\mathcal{S}(\mathbb{R}^n)$ and all x in $\mathbb{R}^n$,

$$\begin{aligned}(T_\sigma\varphi)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)\,d\xi \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)\varphi(y)\,dy\,d\xi,\end{aligned}$$

where the last integral is to be understood as an oscillatory integral in which the integral with respect to y has to be performed first. With this formula in hand, it requires a huge amount of ingenuity (certainly not logic) to see that we can associate to σ another useful linear operator W_σ on $\mathcal{S}$ defined by the same formula with $\sigma(x,\xi)$ replaced by $\sigma\left(\frac{x+y}{2},\xi\right)$. The linear operator W_σ can be traced back to the work [28] by Hermann Weyl and hence we call W_σ the Weyl transform associated to the symbol σ. In fact, we have the following connection between Weyl transforms and pseudo-differential operators.

Theorem 3.1. *Let $\sigma \in S^m$. Then there exists a symbol τ in S^m such that*

$$T_\sigma = W_\tau$$

and there exists a symbol κ in S^m such that

$$W_\sigma = T_\kappa.$$

Thus, there is a one-to-one correspondence between pseudo-differential operators and Weyl transforms. We have the following result, which can be thought of as the fundamental Theorem of pseudo-differential operators.

Theorem 3.2. *Let $\sigma \in S^m$, $m \in \mathbb{R}$. Then for all φ and ψ in $\mathcal{S}(\mathbb{R}^n)$,*

$$(W_\sigma\varphi,\psi)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sigma(x,\xi)W(\varphi,\psi)(x,\xi)\,dx\,d\xi,$$

where $W(\varphi,\psi)$ is the Wigner transform of φ and ψ defined by

$$W(\varphi,\psi)(x,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\xi\cdot p}\varphi\left(x+\frac{p}{2}\right)\overline{\psi\left(x-\frac{p}{2}\right)}\,dp$$

for all x and ξ in $\mathbb{R}^n$.

The Wigner transform is a very well-behaved bilinear form on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and it satisfies the so-called Moyal identity or the Plancherel formula to the effect that

$$\|W(\varphi, \psi)\|_{L^2(\mathbb{R}^{2n})} = \|\varphi\|_{L^2(\mathbb{R}^n)} \|\psi\|_{L^2(\mathbb{R}^n)}$$

for all φ and ψ in $L^2(\mathbb{R}^n)$.
A *tour de force* from Theorems 3.1 and 3.2 shows that we can now define pseudo-differential operators with nonsmooth symbols not in the Hörmander class S^m. To be specific, we look at symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ only.
Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we define the Weyl transform W_σ on $L^2(\mathbb{R}^n)$ by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi)\, W(f, g)(x, \xi)\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$. Then we have the following analogs of Theorems 1.1–1.3.

Theorem 3.3. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert–Schmidt operator.*

Theorem 3.4. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the adjoint W_σ^* of W_σ is given by*

$$W_\sigma^* = W_{\overline{\sigma}}.$$

Theorem 3.5. *Let σ and τ be symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$W_\sigma W_\tau = W_\lambda,$$

where $\lambda \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and is given by

$$\hat{\lambda} = (2\pi)^{-n}(\hat{\sigma} *_{1/4} \hat{\tau}).$$

Theorem 3.5, which is attributed to Grossmann, Loupias and Stein [15], tells us that the product of two Weyl transforms with symbols in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is again a Weyl transform with symbol in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and is given by a twisted convolution. Let us recall that the twisted convolution $f *_{1/4} g$ of two measurable functions f and g on $\mathbb{C}^n (= \mathbb{R}^n \times \mathbb{R}^n)$ is defined by

$$(f *_{1/4} g)(z) = \int_{\mathbb{C}^n} f(z - w) g(w) e^{i[z,w]/4} dw$$

for all z in $\mathbb{C}^n$, where $[z, w]$ is the symplectic form of z and w given by

$$[z, w] = 2\,\mathrm{Im}(z \cdot \overline{w}).$$

See the books [3] by Boggiatto, Buzano and Rodino, [12] by Folland, [25] by Stein and [30] by Wong for details and related topics.

4. Gabor Transforms

If we make a change of variables in the definition of the Wigner transform, then we get for all f and g in $L^2(\mathbb{R}^n)$, and all x and ξ in $\mathbb{R}^n$,

$$W(f,g)(x,\xi) = 2^n e^{2ix\cdot\xi}(G_{\tilde{g}}f)(2x,2\xi),$$

where

$$\tilde{g}(x) = g(-x)$$

for all x in $\mathbb{R}^n$ and $G_{\tilde{g}}f$ is the well-known Gabor transform or the short-time Fourier transform of f with window $\tilde{g}$ given by

$$(G_{\tilde{g}}f)(x,\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-it\cdot\xi}f(t)\overline{\tilde{g}(t-x)}\,dt$$

for all x and ξ in $\mathbb{R}^n$. In image analysis, we can think of $(G_{\tilde{g}}f)(x,\xi)$ as the spectral content of the image f with frequency ξ at the point x.

Let us now fix a window φ in $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)\,dx = 1$. Then the Gabor transform $G_\varphi f$ of f is given by

$$(G_\varphi f)(x,\xi) = (2\pi)^{-n/2}(f, M_\xi T_{-x}\varphi)_{L^2(\mathbb{R}^n)}$$

for all x and ξ in $\mathbb{R}^n$, where M_ξ and T_{-x} are the modulation operator and the translation operator given by

$$(M_\xi h)(t) = e^{it\cdot\xi}h(t)$$

and

$$(T_{-x}h)(t) = h(t-x)$$

for all measurable functions h on $\mathbb{R}^n$ and all t in $\mathbb{R}^n$. Now, for all x and ξ in $\mathbb{R}^n$, we define the function $\varphi_{x,\xi}$ on $\mathbb{R}^n$ by

$$\varphi_{x,\xi} = M_\xi T_{-x}\varphi.$$

We call the functions $\varphi_{x,\xi}$, $x,\xi\in\mathbb{R}^n$, the Gabor wavelets generated from the Gabor mother wavelet φ by translations and modulations.

The usefulness of the Gabor wavelets in signal and image analysis is enhanced by the following resolution of the identity formula, which allows the reconstruction of a signal or an image from its Gabor spectrum.

Theorem 4.1. For all f in $L^2(\mathbb{R}^n)$,

$$f = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(f,\varphi_{x,\xi})_{L^2(\mathbb{R}^n)}\varphi_{x,\xi}dx\,d\xi.$$

Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we define the Gabor multiplier $G_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$(G_{\sigma,\varphi} f, g)_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi)(G_\varphi f)(x,\xi)\overline{(G_\varphi g)(x,\xi)}\, dx\, d\xi$$

for all f and g in $L^2(\mathbb{R}^n)$. Using the Gabor wavelets, we see that $G_{\sigma,\varphi} f$ is equal to

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x,\xi)(f, \varphi_{x,\xi})_{L^2(\mathbb{R}^n)} \varphi_{x,\xi}\, dx\, d\xi$$

for all f in $L^2(\mathbb{R}^n)$.
Gabor multipliers are also known as localization operators, Daubechies operators, anti-Wick quantization and Wick quantization. The following results are the analogs of Theorems 1.1–1.3 for Gabor multipliers.

Theorem 4.2. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Gabor multiplier $G_{\sigma,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert–Schmidt operator.*

Theorem 4.3. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the adjoint $G^*_{\sigma,\varphi}$ of $G_{\sigma,\varphi}$ is given by*

$$G^*_{\sigma,\varphi} = G_{\overline{\sigma},\varphi}.$$

Theorem 4.4. *Let σ and τ be functions in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$G_{\sigma,\varphi} G_{\tau,\varphi} = G_{\lambda,\varphi},$$

where

$$\hat{\lambda} = (2\pi)^{-n}(\hat{\sigma} *^{1/2} \hat{\tau}).$$

In Theorem 4.4, we have a new twisted convolution. To wit, the new twisted convolution $f *^{1/2} g$ of two measurable functions f and g on $\mathbb{C}^n$, is defined by

$$(f *^{1/2} g)(z) = \int_{\mathbb{C}^n} f(z-w)\, g(w) e^{(z\cdot\overline{w} - |w|^2)/2} dw$$

for all z in $\mathbb{C}^n$ provided that the integral exists. Theorem 4.4 can be found in the 2000 paper [10] by Du and Wong.
The interesting feature with Theorem 4.4 is that the new twisted convolution $f *^{1/2} g$ of two functions f and g in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ need not be in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. This phenomenon is the motivation for many interesting research papers on the product of Gabor multipliers. It suffices to mention the works [5] by Coburn, [7] by Cordero and Gröchenig and [8] by Cordero and Rodino.
What is a Gabor multiplier? Is it something already well known to us? The answer is yes.

Theorem 4.5. *Let $\sigma \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then*

$$G_{\sigma,\varphi} = W_{\sigma * V(\varphi,\varphi)},$$

where

$$V(\varphi,\varphi)^\wedge = W(\varphi,\varphi).$$

References for the materials in this section are the books [9] by Daubechies, [14] by Gröchenig, [31] by Wong and many others.

5. Wavelet Transforms

Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$ and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Then we call φ a mother wavelet and φ is said to satisfy the admissibility condition.

Let φ be a mother wavelet. Then for all b in $\mathbb{R}$ and a in $\mathbb{R}\backslash\{0\}$, we can define the wavelet $\varphi_{b,a}$ by

$$\varphi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \varphi\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}.$$

We call $\varphi_{b,a}$ the affine wavelet generated from the mother wavelet φ by translation and dilation. To put things in perspective, let $b \in \mathbb{R}$ and let $a \in \mathbb{R}\backslash\{0\}$. Then we let T_b be the translation operator as before and D_a be the dilation operator defined by

$$(D_a f)(x) = \sqrt{|a|} f(ax)$$

for all x in $\mathbb{R}$ and all measurable functions f on $\mathbb{R}$. So, the wavelet $\varphi_{b,a}$ can be expressed as

$$\varphi_{b,a} = T_{-b} D_{1/a} \varphi.$$

Let φ be a mother wavelet. Then the wavelet transform $\Omega_\varphi f$ of a function f in $L^2(\mathbb{R})$ is defined to be the function on $\mathbb{R} \times \mathbb{R}\backslash\{0\}$ by

$$(\Omega_\varphi f)(b,a) = (f, \varphi_{b,a})_{L^2(\mathbb{R})}$$

for all b in $\mathbb{R}$ and a in $\mathbb{R}\backslash\{0\}$. At the heart of the analysis of the wavelet transform is the following resolution of the identity formula.

Theorem 5.1. *Let φ be a mother wavelet. Then for all functions f and g in $L^2(\mathbb{R})$,*

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\Omega_\varphi f)(b,a) \overline{(\Omega_\varphi g)(b,a)} \, \frac{db\, da}{a^2},$$

where

$$c_\varphi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi.$$

The resolution of the identity formula leads to the reconstruction formula which says that

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} \varphi_{b,a} \frac{db\, da}{a^2}$$

for all f in $L^2(\mathbb{R})$. In other words, we have a reconstruction formula for the signal f from a knowledge of its time-scale spectrum.
Let φ be a mother wavelet and let $\sigma \in L^2(\mathbb{R} \times \mathbb{R})$. Then we define the wavelet multiplier $\Omega_{\sigma,\varphi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\Omega_{\sigma,\varphi} f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b,a)(f, \varphi_{b,a})_{L^2(\mathbb{R})} \varphi_{b,a} \frac{db\, da}{a^2}$$

for all f in $L^2(\mathbb{R})$.
As in the case of the Gabor multipliers, we have the following results.

Theorem 5.2. *The wavelet multiplier*

$$\Omega_{\sigma,\varphi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

is a Hilbert–Schmidt operator.

Theorem 5.3. *The adjoint* $\Omega^*_{\sigma,\varphi}$ *of the wavelet multiplier* $\Omega_{\sigma,\varphi}$ *is given by*

$$\Omega^*_{\sigma,\varphi} = \Omega_{\overline{\sigma},\varphi}.$$

What is the product of two wavelet multipliers? The answer is not so simple and seems to depend on the availability of a useful formula for a wavelet multiplier. Some technical information in this direction can be found in the paper [32] by Wong. If

$$\sigma(b,a) = \alpha(a)\beta(b)$$

for all b in $\mathbb{R}$ and all a in $\mathbb{R}\setminus\{0\}$, then $\Omega_{\sigma,\varphi}$ is a paracommutator in the sense of Janson and Peetre [18], and Peng and Wong [22]. If σ is a function of a only, then $\Omega_{\sigma,\varphi}$ is a paraproduct in the sense of Coifman and Meyer [6]. If σ is a function of b only, then $\Omega_{\sigma,\varphi}$ is a Fourier multiplier.

6. Stockwell Transforms

Let us recall that for a signal f in $L^2(\mathbb{R})$, the Gabor transform $(G_\varphi f)(x,\xi)$ with respect to the window φ gives the time–frequency content of f at time x and frequency ξ by using the window φ at time x. The drawback here is that a window of fixed width is used for all time x. It is more accurate if we can have an adaptive window that gives a wide window for low frequency and a narrow window for high frequency. That this can be done comes from our experiences with the wavelet transform. Indeed, we see that the window $\varphi_{b,a}$ is narrow if the scale a is small and the window is wide when the scale is big.

Now, the Stockwell transform $S_\varphi f$ with window φ of a signal f is defined by

$$(S_\varphi f)(x,\xi) = (2\pi)^{-1/2}|\xi| \int_{-\infty}^{\infty} e^{-it\xi} f(t)\overline{\varphi(\xi(t-x))}\, dt$$

for all x and ξ in $\mathbb{R}$. Formally, we note that for all f in $L^2(\mathbb{R})$, all x in $\mathbb{R}$ and all ξ in $\mathbb{R}\backslash\{0\}$,

$$(S_\varphi f)(x,\xi) = (f, \varphi^{x,\xi})_{L^2(\mathbb{R})},$$

where

$$\varphi^{x,\xi} = (2\pi)^{-1/2} M_\xi T_{-x} \tilde{D}_\xi \varphi.$$

Here, the dilation operator $\tilde{D}_\xi$ is defined by

$$(\tilde{D}_\xi f)(t) = |\xi| f(\xi t)$$

for all t in $\mathbb{R}$ and all measurable functions f on $\mathbb{R}$. Besides the modulation, a notable feature in the Stockwell transform is the normalizing factor in the dilation operator, which is $|\cdot|$ and not $|\cdot|^{1/2}$ as in the case of the wavelet transforms. These features distinguish the Stockwell transform from the wavelet transforms.

The Stockwell transform has recently been successfully used in seismic waves [26] by Stockwell, Mansinha and Lowe and in medical imaging [34] by Zhu and others. An attempt in understanding the mathematical underpinnings of the Stockwell transform is underway by Wong and Zhu. See [33] in this direction and we describe some of the results therein.

Theorem 6.1. *Let φ be a window with*

$$\int_{-\infty}^{\infty} \varphi(x)dx = 1.$$

Then for all f in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} (S_\varphi f)(x,\xi)\, dx = \hat{f}(\xi)$$

for all ξ in $\mathbb{R}$.

See Fig. 1 for an illustration of Theorem 6.1. In view of Theorem 6.1, we have a reconstruction formula for a signal f in terms of its Stockwell spectrum, which says that

$$f = \mathcal{F}^{-1} A S_\varphi f,$$

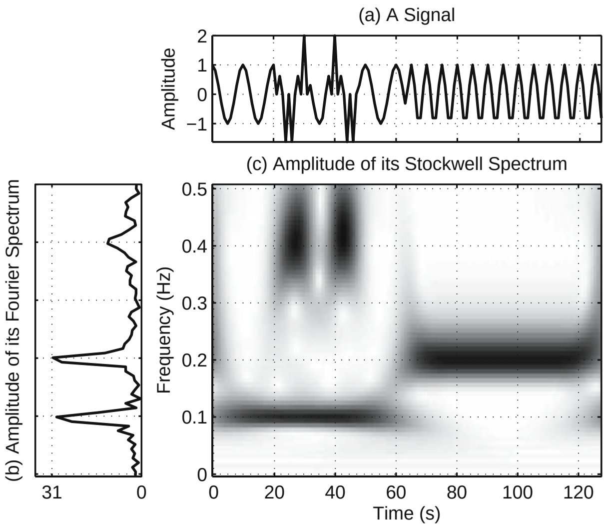

Fig. 1 Time–frequency representation of the Stockwell transform: (**a**) a signal consisting of multiple frequency components (**b**) the amplitude of the corresponding Fourier spectrum, i.e., $|(\mathcal{F}f)(k)|$ (**c**) the contour plotting the amplitude of the corresponding Stockwell transform, i.e., $|(Sf)(\tau, k)|$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform and A is the time average operator given by

$$(AF)(\xi) = \int_{-\infty}^{\infty} F(x,\xi)\, dx$$

for all ξ in $\mathbb{R}$ and all measurable functions F on $\mathbb{R} \times \mathbb{R}$.
For the second result, we let M be the set of all measurable functions F on $\mathbb{R} \times \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} F(x,\xi)\, dx \right|^2 d\xi < \infty.$$

Then M is an indefinite Hilbert space in which the indefinite inner product $(\ ,\ )_M$ is given by

$$(F, G)_M = (AF, AG)_{L^2(\mathbb{R})}$$

for all F and G in M.
Then we have a characterization of the Stockwell spectra given by the following theorem.

Theorem 6.2. $\{S_\varphi f : f \in L^2(\mathbb{R})\} = M/Z$, where

$$Z = \{F : \mathbb{R} \times \mathbb{R} \to \mathbb{C} : AF = 0\}.$$

Can we reconstruct a signal from its Stockwell spectrum? The answer is yes provided that we choose the right window. To do this, we say that a function φ in $L^2(\mathbb{R})$ satisfies the admissibility condition if and only if

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi < \infty.$$

For a function in $L^2(\mathbb{R})$ satisfying the admissibility condition, we define the constant c_φ by

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi.$$

Theorem 6.3. Let φ be a function in $L^2(\mathbb{R})$ with $\|\varphi\|_2 = 1$ satisfying the admissibility condition. Then for all f in $L^2(\mathbb{R})$,

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{x,\xi})_{L^2(\mathbb{R})} \varphi^{x,\xi} \frac{dx\, d\xi}{|\xi|}.$$

Remark: It is important to note that an admissible wavelet φ for the Stockwell transform has to satisfy the condition

$$\hat{\varphi}(-1) = 0.$$

So, the Gaussian window that has been used exclusively for the Stockwell transform in the literature is not admissible.
This formula and its discretization can be found in the paper [11] by Du, Wong and Zhu.

L. Rodino,
M.W. Wong

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Four Lectures in Semiclassical Analysis for Non Self-Adjoint Problems with Applications to Hydrodynamic Instability

B. Helffer

Abstract Our aim is to show how semi-classical analysis can be useful in questions of stability appearing in hydrodynamics. We will emphasize on the motivating examples and see how these problems can be solved or by harmonic approximation techniques used in the semi-classical analysis of the Schrödinger operator or by recently obtained semi-classical versions of estimates for operators of principal type (mainly subelliptic estimates). These notes correspond to an extended version of the course given at the school in Cetraro. We have in particularly kept the structure of these lectures with an alternance between the motivating examples and the presentation of the theory. Many of the results which are presented have been obtained in collaboration with Olivier Lafitte.

1 General Introduction

In Hydrodynamics an important question is to analyze the stability or the instability of the solutions. This question appears at least at the first stage (analysis of the linearized problem) to be a question of spectral analysis. This question appears to depend strongly on the various physical parameters. In some asymptotics regime, this question can be analyzed by techniques coming from semi-classical analysis: this means that there is a small parameter h which plays in the analysis the role of the Planck constant in an analogous way to the Quantum Mechanics.

We will emphasize on the motivating examples and see how these problems can be solved or by harmonic approximation techniques used in the semi-classical analysis of the Schrödinger operator or by recently obtained

Bernard Helffer
Laboratoire de Mathématiques, Université Paris-Sud, 91405 Orsay Cedex, France
e-mail: bernard.helffer@math.u-psud.fr

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semi-classical versions of estimates for operators of principal type (mainly subelliptic estimates). In this way, we hope to show that these recent results are much more than academic transpositions of former theorems developed more than thirty years ago when analyzing the main properties of Partial Differential Equations: local solvability, hypoellipticity, propagation of singularities... (see Egorov [Eg], Trèves [Trev], the treatise by Hörmander [Ho3] and references therein).

Actually, we will not need at the moment the most sophisticated theorems of this theory (see the lectures by N. Lerner [Le]) but the most generic. We will give explicit proofs for the simple examples we have. They are based mainly on two tools: the semiclassical elliptic theory for h-pseudodifferential operators and the construction of WKB solutions.

We consider four different models coming from different modelizations appearing in hydrodynamics. The first one is the *Rayleigh–Taylor model*. Although the subject has a long story starting with [St] (see also [Cha]), the semi-classical analysis appears in [La1, La2, HelLaf1]. The problem we meet in this case is self-adjoint and related to the analysis of the bottom of the spectrum for a Schrödinger operator. The three other examples are not selfadjoint. We will see that we meet problems related to the notion of pseudospectrum. The second one extends the previous one by introducing some velocity at the surface between the two fluids. This is an extension of the *Kelvin–Helmholtz* classical model which is analyzed in [CCLa]. The third one, the *Rayleigh with convection model* was studied in [CCLaRa] and is a natural generalization with a convective velocity of the classical Rayleigh problem for a transition region. The fourth one is called the *Kull–Anisimov ablation front model*. It has been analyzed by many physicists and more recently in the PhD theses of L. Masse [Mas] and V. Goncharov [Go].

Finally as other relevant references, we quote [Ag], [BeSh], [Bo], [BudkoL], [ChLa], [Col], [Da1], [Da3], [GH], [He1], [HeRo1], [HeRo2], [HelSj2], [KeSu], [Kull], [Si2], [Tay].

Organization of the Course

This course is divided in four (unequal) lectures.

Lecture 1 is devoted to the analysis of the Rayleigh–Taylor model. We show how the initial problem of analyzing the possible instability of the model leads to a spectral problem for a compact selfadjoint operator which appears to be an h-pseudodifferential operator.

When needed, we will recall various basic things on the h-pseudodifferential operators.

We are let to the analysis of the largest eigenvalue of a compact operator. We show that either harmonic analysis or WKB solutions permit to have a good asymptotic of this eigenvalue.

Lecture 2 is devoted to the presentation of some mathematical tools adapted to the analysis of non-selfadjoint problems. We first start by presenting a new example (Kelvin–Helmholtz) as a motivation. We then give the main definitions related to the pseudo-spectrum. Here we will emphasize on the "elliptic" h-pseudodifferential theory and on what can be done by WKB constructions. We then apply the techniques for analyzing our Kelvin–Helmholtz model.

Lecture 3 is devoted to the presentation of the results on subellipticity in the semi-classical context. We will see how the question of the subellipticity of h-pseudodifferential operators can appear naturally. In comparison with what was done in the course of N. Lerner [Le], this will illustrate the most simple examples which were presented!

Lecture 4 explains the origin of two other models. We will show that they lead to similar questions for some suitable regimes of parameters. Again, we arrive to the analysis of a system, which can be reduced to a high order non symmetric differential operator. We then sketch the mathematical treatment of these two models. This gives us also a good opportunity for presenting other results in subellipticity mainly obtained by Dencker–Sjöstrand–Zworski.

2 Lecture 1: The Rayleigh–Taylor Model

2.1 The Rayleigh–Taylor Model: Physical Origin

The starting point for this model is the analysis of the following differential system in $\mathbb{R}^4 = \mathbb{R}^3_x \times \mathbb{R}_t$. With $x = (x_1, x_2, x_3)$ this system reads:

$$\begin{aligned} &\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \\ &\partial_t(\varrho \mathbf{u}) + \nabla \cdot (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \varrho \mathbf{g} \,. \end{aligned} \tag{1}$$

The unknowns are $\mathbf{u} = (u_1, u_2, u_3)$, the density ϱ and the pressure p. We assume that $\mathbf{g} = (0, 0, 1)g$. The second line in (1) corresponds to three equations and reads more explicitly:

$$\begin{aligned} &\partial_t(\varrho u_1) + \operatorname{div}(\varrho\, u_1\, \mathbf{u}) + \partial_1 p = 0 \;; \\ &\partial_t(\varrho u_2) + \operatorname{div}(\varrho\, u_2\, \mathbf{u}) + \partial_2 p = 0 \;; \\ &\partial_t(\varrho u_3) + \operatorname{div}(\varrho\, u_3\, \mathbf{u}) + \partial_3 p = \varrho g \,. \end{aligned} \tag{2}$$

Here we have used the short notations:

$$\partial_t = \frac{\partial}{\partial t} \,,\; \partial_i = \frac{\partial}{\partial x_i} \text{ for } i = 1, 2, 3 \,.$$

The reader can look in the first pages of the book by P.L. Lions [Li] for the way to get these equations from the principles of conservation of mass (for the first line of (1)) and of momentum (for the second line of (1)).

This system models the so-called Rayleigh–Taylor instability, which occurs when a heavy fluid is above a light fluid in a gravity field directed from the heavy to the light fluid. We refer to Chap. X in Chandrasekhar's book [Cha] for a presentation of the theory. Here we intend to study the linear growth rate of this instability in a situation where there is a mixing region. This linear growth rate will corresponds to γ in (16) below.

We would like to analyze the linearized problem around a stationary solution (i.e. t-independent):

$$\varrho = \rho^0 \ , \ \mathbf{u} = \mathbf{u^0} = 0 \ , \ p = p^0 \ , \tag{3}$$

where ρ^0 is assumed to depend only on x_3 and p^0 and ρ^0 are related, as imposed by the second line in (1), by:

$$\nabla p^0 = \rho^0 \mathbf{g} \ . \tag{4}$$

We assume that the perturbation $(\hat{\mathbf{u}}, \hat{p}, \hat{\rho})$ is incompressible that is satisfying:

$$\operatorname{div} \hat{\mathbf{u}} = 0 \ . \tag{5}$$

The linearized system takes the form:

$$\partial_t \hat{\rho} + (\rho^0)' \hat{u}_3 = 0 \ ; \tag{6}$$

$$\rho^0 \partial_t \hat{u}_1 + \partial_1 \hat{p} = 0 \ ; \tag{7}$$

$$\rho^0 \partial_t \hat{u}_2 + \partial_2 \hat{p} = 0 \ ; \tag{8}$$

$$\rho^0 \partial_t \hat{u}_3 + \partial_3 \hat{p} = g \hat{\rho} \ . \tag{9}$$

In order to analyze (at least formally this system) we extract from the system an equation involving only $\hat{u}_3$ (by eliminating the other unknowns). This is done along the following lines.
We first differentiate with respect to t (9). This leads to:

$$\rho^0 \partial_t^2 \hat{u}_3 + \partial_t \partial_3 \hat{p} = g \frac{\partial \hat{\rho}}{\partial t} \ . \tag{10}$$

We now use (6) in order to eliminate $\frac{\partial \hat{\rho}}{\partial t}$ and get:

$$\rho^0 \partial_t^2 \hat{u}_3 + \partial_t \partial_3 \hat{p} + g (\rho^0)'(x_3) \hat{u}_3 = 0 \ . \tag{11}$$

We now differentiate (7) and (8) respectively with respect to x_1 and x_2. This gives:

$$\rho^0 \partial_t \partial_1 \hat{u}_1 + \partial_1^2 \hat{p} = 0 \ , \tag{12}$$

and

$$\rho^0 \partial_t \partial_2 \hat{u}_2 + \partial_2^2 \hat{p} = 0 \ . \tag{13}$$

Differentiating (5) with respect to t and using (12) and (13), we get:

$$\Delta_{12}\,\hat{p} = \rho^0 \partial_t \partial_3 \hat{u}_3 \ , \tag{14}$$

where Δ_{12} is the Laplacian with respect to the two first variables (x_1, x_2):

$$\Delta_{12} = \partial_1^2 + \partial_2^2 \ .$$

It remains to eliminate $\hat{p}$ between (11) and (14):

$$\Delta_{12}\ \left(\rho^0 \partial_t^2 \hat{u}_3 + (\rho^0)' g \hat{u}_3\right) + \partial_3 \rho^0 \partial_3 \partial_t^2 \hat{u}_3 = 0 \ . \tag{15}$$

We now look for a solution $\hat{u}_3$ in the form:

$$\mathbb{R}^3 \times \mathbb{R} \ni (x,t) \mapsto \hat{u}_3(x_1, x_2, x_3, t) = v(x_3) \exp(\gamma t + i k_1 x_1 + i k_2 x_2) \ , \tag{16}$$

where:

- v is an unknown real function in $L^2(\mathbb{R})$.
- γ is a real parameter.
- (k_1, k_2) is in $\mathbb{R}^2$ and corresponds to the momentum variables dual to (x_1, x_2).

This is what is called in the physical literature the analysis in normal modes. The reader can for example look in the introductory chapter of [Cha] for a more heuristic explanation. This leads to an ordinary differential equation (in the x_3-variable) for v:

$$-(k_1^2 + k_2^2)(\rho^0 \gamma^2 v + (\rho^0)' g v) + \gamma^2 \frac{d}{dx_3} \rho^0 \frac{d}{dx_3} v = 0 \ . \tag{17}$$

Replacing x_3 by x ($x \in \mathbb{R}$) and dividing by $\gamma^2 k^2$ with

$$k^2 = k_1^2 + k_2^2 \ ,$$

we get:

$$\left[-\frac{1}{k^2} \frac{d}{dx} \rho^0 \frac{d}{dx} + \rho^0 + (\rho^0)' \frac{g}{\gamma^2} \right] v = 0 \ . \tag{18}$$

So we are interested in analyzing for which value of (γ, k) (with $\gamma > 0$) there exists a non trivial v satisfying (17).

The choice of $\gamma > 0$ corresponds to our interest for instability. Actually, we could have started by looking at possibly complex γ's but one immediately get as a necessary condition that γ^2 should be real and the pure imaginary γ's are not interesting for the problem.

2.2 Rayleigh–Taylor Mathematically

In the case of the Rayleigh–Taylor model, as we have seen in (18), the main point is to analyze as a function of $\delta \in \mathbb{R}$ the kernel in $L^2(\mathbb{R})$ of:

$$P(h,\delta) := -h^2 \frac{d}{dx}\varrho(x)\frac{d}{dx} + \varrho(x) + \delta\varrho'(x)\ . \tag{19}$$

Here $h > 0$ and $\varrho(x) \in C^\infty(\mathbb{R})$ satisfies:

$$\begin{array}{l} \lim_{x\to-\infty} \varrho(x) = \rho_- > 0\ , \\ \lim_{x\to+\infty} \varrho(x) = \rho_+ > 0\ , \end{array} \tag{20}$$

$$\varrho(x) > 0\ ,\ \forall x \in \mathbb{R}\ , \tag{21}$$

$$\rho_- \neq \rho_+\ , \tag{22}$$

$$\lim_{|x|\to+\infty} \varrho'(x) = 0\ . \tag{23}$$

We look at $h \to 0$ (see[1] [HelLaf1] for the case $h \to +\infty$). The problem comes from the analysis of the Euler equations in a gravity field. The physical parameters are the intensity g of the gravity, a wave number $k > 0$ and a parameter γ which measures the large time behavior of the solution. The mathematical problem is to determine a pair (u, γ) such that

$$P\left(\frac{1}{k}, \frac{g}{\gamma^2}\right) u = 0 \qquad \text{with } u \text{ non trivial}$$

This means that the link between the physical parameters (g, k, γ) and the mathematical parameters is:

$$\delta = \frac{g}{\gamma^2}\ ,\ h = \frac{1}{k}\ . \tag{24}$$

The physical situation leads to analyze the case $\delta g > 0$. This implies $\gamma^2 > 0$, and we choose $\gamma > 0$.

Note that the instability is only analyzed when

$$\rho_+ \neq \rho_-\ .$$

This implies that $\varrho'(x)$ is not identically 0.

The most physical case corresponds to:

$$\rho_- > \rho_+\ ,\ g > 0\ ,$$

so δ is positive and ϱ' is negative somewhere.

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[1] In this case the limiting model corresponds to $\rho = \rho_-$ for $x < 0$ and $\rho = \rho_+$ for $x > 0$.

Generally ϱ is assumed monotone but the semi-classical techniques are not limited to this case.

2.3 Elementary Spectral Theory

First we observe that there is no problem for defining the selfadjoint extension of $P(h,\delta)$ in $L^2(\mathbb{R})$ (which is unique starting from $C_0^\infty(\mathbb{R})$) and it is immediate that $P(h,0)$ is injective. More precisely, the bottom of its spectrum is strictly positive.

Definition 2.1. We call generalized spectrum of the family $P(h,\delta)$ the set of the δ's in $\mathbb{R}$ such that $P(h,\delta)$ is non injective.

The standard analysis of the solution at ∞ for ordinary differential equations shows that, for all δ, the dimension of $\ker P(h,\delta)$ is zero or one.

The next result is relatively well known (connected to the Birman–Schwinger principle [Si1]).

Proposition 2.1. *Under the previous assumptions and assuming in addition that ϱ' is not identically* 0*, then the generalized spectrum $P(h,\delta)$ is the union of two sequences (possibly empty or finite) δ_n^+ et δ_n^- s.t.:*

$$\begin{aligned} &0 < \delta_n^+ < \delta_{n+1}^+ \;, \\ &\lim_{n\to+\infty} \delta_n^+ = +\infty \;, \end{aligned} \tag{25}$$

$$\begin{aligned} &0 < -\delta_n^- < -\delta_{n+1}^- \;, \\ &\lim_{n\to+\infty} \delta_n^- = -\infty \;. \end{aligned} \tag{26}$$

Proof. If we observe that:

$$\ker P(h,\delta) \neq \{0\} \text{ iff } \ker(K(h) - \frac{1}{\delta}) \neq \{0\} \;, \tag{27}$$

where

$$K(h) = -P(h,0)^{-\frac{1}{2}}\varrho'(x)P(h,0)^{-\frac{1}{2}} \;. \tag{28}$$

the proof is immediately reduced to the standard result for $K(h)$, which is a compact selfadjoint operator.

For the compactness of $K(h)$, we can for example observe that the operator $P(h,0)^{-\frac{1}{2}}$ belongs to $\mathcal{L}(L^2(\mathbb{R}); H^1(\mathbb{R}))$ and that, under Assumption (23), the operator of multiplication by ρ' is compact from $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$.

Note that when $\varrho' < 0$, which is the simplest natural physical case, the operator $K(h)$ is positive.

Let us also mention an a priori “universal” estimate of [CCLaRa]. If u is, for some $\delta \neq 0$, in the kernel of $P(h,\delta)$, we get by taking the scalar product in L^2 by u:

$$\begin{aligned}\int_{-\infty}^{+\infty} \varrho(h^2 u'(x)^2 + u(x)^2)\, dx &= -\delta \int_{-\infty}^{+\infty} \varrho'(x) u(x)^2\, dx \\ &= 2\delta \int_{-\infty}^{+\infty} \varrho u(x) u'(x)\, dx\ .\end{aligned} \tag{29}$$

Using Cauchy–Schwarz, we get:

$$\int_{-\infty}^{+\infty} \varrho(x)(1 - \frac{|\delta|}{h})(u'(x)^2 + u(x)^2)\, dx \leq 0\ . \tag{30}$$

This implies

$$\ker P(h,\delta) = \{0\}\ ,\ \forall \delta \in]-h,h[\ . \tag{31}$$

Universal upper bound
We could have started from the operator:

$$-h^2 \varrho^{-\frac{1}{2}} \frac{d}{dx} \varrho \frac{d}{dx} \varrho^{-\frac{1}{2}} + 1 + \delta \frac{\varrho'(x)}{\varrho(x)}\ ,$$

which shows more clearly the role of the function ϱ'/ϱ.
One way is to change of functions introducing

$$u = \varrho(x)^{-\frac{1}{2}} v\ .$$

This shows also that if:

$$1 + \delta \frac{\varrho'(x)}{\varrho(x)} > 0\ ,\ \forall x \in \mathbb{R}\ , \tag{32}$$

then δ is not in the generalized spectrum.

Remark 2.1. The theory can be extended to the cases $\rho_+ = 0$ or $\rho_- = 0$, under Condition (34).

2.4 A Crash Course on h-Pseudodifferential Operators

At least if the profile ϱ is regular, the h-pseudodifferential calculus gives an easy way for getting the extremal eigenvalues of $K(h)$ in the semi-classical limit. Let us briefly describe this tool.

A family ($h \in]0,h_0]$) of h-pseudodifferential operators

$$A_h = \mathrm{Op}_h(a)\,,$$

associated to a symbol $(x,\xi) \mapsto a(x,\xi;h)$ is defined for $u \in \mathcal{S}(\mathbb{R}^m)$ by:

$$\begin{aligned}&(\mathrm{Op}_h(a)u)(x) = \\ &(2\pi h)^{-m} \int_{\mathbb{R}^m \times \mathbb{R}^m} \exp(\tfrac{i}{h}(x-y)\cdot\xi)\ a(\tfrac{x+y}{2},\xi;h)\, u(y)\, dy d\xi\ .\end{aligned} \tag{33}$$

The function a is called the Weyl symbol (or h-Weyl symbol if we want to recall the dependence on h) of A_h. We refer to the book of D. Robert [Rob] for a course on this theory which is specifically semi-classical (and to the course of N. Lerner [Le] in this volume[2]) and the assumptions which can be done on the symbols.

Here it is enough to consider as symbols C^∞ (with respect to the variables $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$) functions a s.t., for some given p, p', q and $h_0 > 0$, there exists, for all α and β in $\mathbb{N}^m$, constants $C_{\alpha,\beta}$ s.t., for all $h \in]0, h_0]$,

$$|D_x^\alpha D_\xi^\beta a(x,\xi;h)| \leq C_{\alpha,\beta}\, h^q\, \langle x \rangle^{p-|\alpha|} \langle \xi \rangle^{p'-|\beta|}\ .$$

When the symbol satisfies this condition, we write simply $a \in S^{(q,p,p')}$, and the corresponding operator $\mathrm{Op}_h(a)$ is said to belong to $\mathrm{Op}_h S^{(q,p,p')}$.

This class is an algebra by composition and the composition is just a multiplication for the principal symbols. Typically, if $a \in S^{(q,p,p')}$ and $b \in S^{(q_1,p_1,p_1')}$, then there exists c in $S^{(q+q_1,p+p_1,p'+p_1')}$ s.t.:

$$\mathrm{Op}_h(a) \circ \mathrm{Op}_h(b) = \mathrm{Op}_h(c)\ ,$$

and

$$c - ab \in S^{(q+q_1+1,p+p_1-1,p'+p_1'-1,)}\ .$$

This leads to the natural definition of "principal symbol". In the current situation, the symbol $a \in S^{q,p,p'}$ has more properties. It admits the formal expansion:

$$a(x,\xi;h) \sim h^q \sum_{j\geq 0} h^j a_j(x,\xi)\ ,$$

with:

$$a_j(x,\xi) \in S^{0,-j,-j}\ ,$$

and one has, for any $N > 0$, a good control of the remainders

$$r^N(x,\xi,h) := a(x,\xi;h) - h^q \sum_{0\leq j\leq N} h^j a_j(x,\xi)\ ,$$

in $S^{(q-N,-N,-N)}$.

The symbol $a_0(x,\xi)$ is called the principal symbol. The symbol $a_1(x,\xi)$ is called the subprincipal symbol. We note that the principal symbol is independent of the quantization (this is not the case for the subprincipal symbol).

We have natural continuity theorems (based on the Calderon–Vaillancourt Theorem) in $H^s(\mathbb{R}^m)$, where moreover the constants are controlled with respect to h.

In addition compact operators on $L^2(\mathbb{R}^n)$ can be recognized as the operators whose symbol in $S^{(0,0,0)}$ tends to 0 as $|x| + |\xi| \to +\infty$.

[2] N. Lerner has a slightly different convention for the quantization. But taking $h = \frac{1}{2\pi}$ in (33) leads to this convention.

Typically, an operator in $\mathrm{Op}_h S^{(q,p,p')}$ with $p < 0$ and $p' < 0$ is compact. The role of q is to give the size of the norm of the operator with respect to h.

Finally, let us briefly discuss invertibility. As the principal symbol of an operator (sat in $\mathrm{Op}_h(S^{0,0,0})$), is invertible (=elliptic), one can inverse the operator for h small enough. This is indeed very simple. If B_h is the operator of h-Weyl symbol $\frac{1}{a_0}$, then the calculus gives that:

$$B_h\, A_h = I + hR_h$$

with $R_h \in \mathrm{Op}_h(S^{(0,-1,-1)})$.
Then the uniform control in $\mathcal{L}(L^2)$ of R_h gives the invertibility of $(I+hR_h)$ in $\mathcal{L}(L^2)$ and hence the invertibility of A_h. For the invertibility, modulo $\mathcal{O}(h^\infty)$, one can also inverse $(I + hR_h)$ by using the Neumann series:

$$(I + hR_h)^{-1} \sim \sum_{j\geq 0} (-1)^j h^j (R_h)^j\ .$$

2.5 Application for Rayleigh–Taylor: Semi-Classical Analysis for $K(h)$

Under strong assumptions on ϱ, one can use the previous h- pseudodifferential calculus. We assume:

$$|D_x^\alpha \varrho(x)| \leq C_\alpha \varrho(x) \langle x\rangle^{-|\alpha|}\ . \tag{34}$$

This assumption permits to see that:

$$K(h) = -(-h^2 \frac{d}{dx}\varrho\frac{d}{dx} + \varrho)^{-\frac{1}{2}} \varrho'(x) (-h^2 \frac{d}{dx}\varrho\frac{d}{dx} + \varrho)^{-\frac{1}{2}} \tag{35}$$

is an h-pseudodifferential operator. More precisely it belongs to $\mathrm{Op}_h S^{(0,0,0)}$. The operator $K(h)$ appears indeed as the composition of three h-pseudo-differential operators $(-h^2 \frac{d}{dx}\rho\frac{d}{dx}+\rho)^{-\frac{1}{2}}$, $-\rho'(x)$ and again $(-h^2\frac{d}{dx}\rho\frac{d}{dx}+\rho)^{-\frac{1}{2}}$.

So the h-pseudodifferential calculus gives that it is an h-pseudodifferential operator.
The principal symbol of $K(h)$ is

$$(x,\xi) \mapsto p(x,\xi) = -(\xi^2+1)^{-1} \frac{\varrho'(x)}{\varrho(x)}\ . \tag{36}$$

For the analysis of the extremal eigenvalues, we have first to determine the extrema of this symbol. If these extrema are non degenerate then we can apply the harmonic approximation as in [HelSj1]. The tunneling effect together with the decay of the eigenfunctions can also be analyzed (see [BrHe], [HePa]). There is indeed a natural extension of Agmon Estimates

for h-pseudodifferential operators whose symbol admit an holomorphic extension in suitable bands $|\mathrm{Im}\,\xi| \leq R$ in the ξ variable.
This leads to the following computations. We get

$$\begin{aligned}\frac{\partial p}{\partial \xi}(x,\xi) &= 2\xi \frac{\varrho'(x)}{\varrho(x)}(\xi^2+1)^{-2} ,\\ \frac{\partial p}{\partial x}(x,\xi) &= -(\varrho''(x)\varrho(x) - \varrho'(x)^2)(\varrho(x))^{-2}(\xi^2+1)^{-1} .\end{aligned}$$

The condition $\varrho'(x) = 0$ should be excluded because it does not correspond to an extremum of $p(x,\xi)$. So we get:

$$\xi = 0\ ;\ \varrho''(x)\varrho(x) - \varrho'(x)^2 = 0 .$$

This corresponds to the condition that x_0 is a critical point of the map $x \mapsto -\varrho'(x)/\varrho(x)$.
It remains to verify that the extrema are non degenerate. We obtain at a critical point $(x_0, 0)$:

$$\begin{aligned}\frac{\partial^2 p}{\partial \xi^2}(x_0,0) &= +2\varrho'(x_0)/\varrho(x_0)\\ \frac{\partial^2 p}{\partial \xi\,\partial x}(x_0,0) &= 0\\ \frac{\partial^2 p}{\partial x^2}(x_0,0) &= -\frac{\varrho'''(x_0)\varrho(x_0) - \varrho'(x_0)\varrho''(x_0)}{\varrho(x_0)^2}\end{aligned}$$

It is then easy to determine if $(x_0, 0)$ corresponds to:

- A minimum of p,
  if $\varrho'(x_0)/\varrho(x_0) > 0$
  and $\varrho'''(x_0)\varrho(x_0) - \varrho'(x_0)\varrho''(x_0)) < 0$.
- A maximum of p,
  if $\varrho'(x_0)/\varrho(x_0) < 0$
  and $\varrho'''(x_0)\varrho(x_0) - \varrho'(x_0)\varrho''(x_0) > 0$.

When $\rho' < 0$ and $\rho > 0$, then the maxima of the symbol correspond to $\xi = 0$ and to the x's such that $-\frac{\varrho'}{\rho}$ is maximal.

We recall that the simplest physical situation corresponds to $\varrho'(x) < 0$. In this case we have only maxima, which actually are the points of interest if looking for largest eigenvalue.

2.6 Harmonic Approximation

If we are interested in the largest eigenvalue of $K(h)$ a very general theory has been developed (of course for Schrödinger, but also for more general h-pseudodifferential operators).

We just sketch what corresponds to the first approximation. We have just to consider the following harmonic operator associated to a point $(x_0, 0)$ corresponding to a maximum of p, and to consider the spectrum of

$$p(x_0,0)+h\left(\frac{1}{2}\frac{\partial^2 p}{\partial\xi^2}(x_0,0)D_y^2+\frac{1}{2}\frac{\partial^2 p}{\partial x^2}(x_0,0)y^2\right)+hp_1(x_0,0)\ ,$$

where p_1 is the subprincipal Weyl symbol of $K(h)$, which actually is 0.

This operator is consequently

$$-\frac{\varrho'(x_0)}{\varrho(x_0}(1-hD_y^2)-h\frac{\varrho'''(x_0)\varrho(x_0)-\varrho'(x_0)\varrho''(x_0)}{2\varrho(x_0)^2}y^2$$

The largest eigenvalue of this operator (which is semi-bounded from above!) is explicitly known and gives the existence of an eigenvalue for $K(h)$ (with some error $\mathcal{O}(h^{\frac{3}{2}})$).

If there are more than one critical maximum point for p, the largest eigenvalue of $K(h)$ is well approximated by the largest (over the maxima of p) of the largest eigenvalue of the approximating harmonic oscillators.

2.7 Instability of Rayleigh–Taylor: An Elementary Approach via WKB Constructions

We present here what simple constructions of WKB solutions can give for the model of Rayleigh–Taylor. A very detailed analysis have been given in [HelLaf1] extending previous works by Cherfils, Lafitte, Raviart [CCLaRa]. Here we present a simpler analysis but this will only give conditions under which one can construct approximate solutions in the kernel of $P(h,\delta)$.

In the semi-classical situation, we look for a solution in the form

$$u(x,h)=a(x,h)\exp-\frac{\varphi(x)}{h} \tag{37}$$

near some point x_0 (to be determined!) with

$$a(x,h)\sim\sum_{j\geq 0}h^j a_j(x)\ , \tag{38}$$

$$\delta(h)\sim\sum_j h^j\delta_j \tag{39}$$

such that

$$\exp\frac{\varphi}{h}\cdot P(h,\delta(h))\cdot u(h)\sim 0\ . \tag{40}$$

Here "$\sim$0" means that the right-hand side should be $\mathcal{O}(h^\infty)$.

Concretely, we expand $\exp\frac{\varphi}{h}\cdot P(h,\delta(h))\cdot u(h)$ in powers of h and express the cancellation of each coefficient of h^j.

We get as first eikonal equation

$$-\varrho(x)\varphi'(x)^2 + \varrho(x) + \delta_0\varrho'(x) = 0 \,. \tag{41}$$

In order to have an (exponentially) localized (as $h \to 0$) in a neighborhood of x_0, it is natural to impose the condition that φ admits a minimum at x_0. So the first condition is:

$$\varphi'(x_0) = 0 \,. \tag{42}$$

This leads as a first necessary condition to

$$\varrho(x_0) + \delta_0\varrho'(x_0) = 0 \,. \tag{43}$$

A second necessary condition is obtained by differentiating the eikonal equation:

$$-\varrho'(x)\varphi'(x)^2 - 2\varrho(x)\varphi'(x)\varphi''(x) + \varrho'(x) + \delta_0\varrho''(x) = 0 \,.$$

This gives at x_0:

$$\varrho'(x_0) + \delta_0\varrho''(x_0) = 0 \,. \tag{44}$$

We are asking for a non-degenerate minimum of φ at x_0.
Differentiating two times the eikonal equation, we obtain:

$$-2\varrho(x_0)(\varphi''(x_0))^2 + \varrho''(x_0) + \delta_0\varrho'''(x_0) = 0 \tag{45}$$

which implies

$$\varrho''(x_0) + \delta_0\varrho'''(x_0) > 0 \,. \tag{46}$$

We recover the condition obtained in the previous analysis.

Till now, we just looked for a phase. The next step is to determine the amplitude. The coefficient δ_1 will be determined by looking at the first transport equation:

$$\begin{aligned} &2\varrho(x)\varphi'(x)a_0'(x) + \varrho'(x)\varphi'(x)a_0(x) \\ &\quad + \varrho(x)\varphi''(x)a_0(x) + \delta_1\varrho'(x)a_0(x) = 0 \,. \end{aligned} \tag{47}$$

If we impose the condition[3]

$$a_0(x_0) = 1 \,,$$

a necessary (and actually sufficient) condition for solving is:

$$\varrho(x_0)\varphi''(x_0) + \delta_1\varrho'(x_0) = 0 \,. \tag{48}$$

We then obtain a_0 by simple integration:

$$a_0'(x)/a_0(x) = (\varrho'(x)\varphi'(x) + \varrho\varphi''(x) + \delta_1\varrho'(x)) \,/\, (2\varrho(x)\varphi'(x)) \,.$$

[3] This condition corresponds to the idea that we look for the ground state, hence non vanishing.

The condition (48) permits indeed to extend the right-hand side as a C^∞ function and we get explicitly:

$$a_0(x) = \exp \int_{x_0}^{x} (\varrho'(\tau)\varphi'(\tau) + \varrho\varphi''(\tau) + \delta_1 \varrho'(\tau)) \ / \ (2\varrho(\tau)\varphi'(\tau)) \ d\tau \ . \quad (49)$$

It is then not difficult to iterate at any order the construction: At each step the cancellation of the coefficient of h^j in the expansion of $\exp \frac{\varphi}{h} \cdot P(h, \delta(h)) \cdot u(h)$ permits to determine δ_j and to find $a_{j-1}(x)$, with, for $j \geq 2$, the initial condition

$$a_{j-1}(x_0) = 0 \ .$$

We have now constructed a formal solution. Let us recall now how one can associate to this formal expansion an explicit realization. The first idea is to consider a finite sum. We let $\delta^N(h) = \sum_{j=0}^N \delta_j h^j$ and introduce $a^N(x,h) = \sum_{j=0}^N h^j a_j(x)$ which is well defined in the neighborhood of x_0.

We then introduce a cut-off χ which localizes in a neighborhood of x_0. We then let

$$u_\chi^N(x,h) = \chi(x) a^N(x,h) \exp -(\varphi(x)/h) \ .$$

Computing $P(h, \delta^N(h)) u_\chi^N(x,h)$, we find:

$$\begin{aligned} &P(h, \delta^N(h)) u_\chi^N(x,h) \\ &\quad = (\chi h^N r_N(x,h) + \tilde{\chi}(x) b_0(x,h)) \exp - \tfrac{\varphi(x)}{h} \ , \end{aligned} \quad (50)$$

where $\tilde{\chi}$ is C^∞, with a support disjoint of x_0. Here it is important to observe that $\exp - \frac{\varphi(x)}{h}$ is exponentially small on the support of $\tilde{\chi}$ (here we have used that φ has a local minimum at x_0).

What can we deduce from this construction? Under the previous assumptions, $P(h,\delta)$ is selfadjoint and we can deduce that, in an interval $]-Ch^N, +Ch^N[$, the spectrum of $P(h, \delta^N(h))$ is not empty for h small enough. Assumption (20) permits also to say that near 0 the spectrum is discrete.

This is not the complete answer to our question. But this strongly suggests the existence, close to $\delta^N(h))$ (modulo $\mathcal{O}(h^N)$) of an effective $\delta(h)$ such that $P(h, \delta(h)$ has a non zero kernel. Note that the answer to this last question is easier when ϱ is strictly monotone. Note that the question is more delicate as for example $\rho_- = 0$. The essential spectrum of $P(h,\delta)$ contains indeed 0. The previous analysis (see [HelLaf1]) based on the h-pseudodifferential calculus avoids this difficulty (finally artificial) if $\frac{\varrho'}{\varrho} \to 0$ at ∞ and if ϱ is regular.

Three remarks for ending this first lecture:

- One can take $N = +\infty$ by using a summation procedure à la Borel.
  The Borel Lemma says that for a given sequence of reals α_n $(n \in \mathbb{N}))$ one can always find a C^∞ function $h \mapsto f(h)$ admitting $\sum_n \alpha_n h^n$ as Taylor expansion at 0.

Here we need a version with parameters, but we can define some realizations of $\sum_{j=0}^{\infty} \delta_j h^j$ and $\sum_{j=0}^{\infty} h^j a_j(x)$, permitting to replace the remainder $\mathcal{O}(h^N)$ by $\mathcal{O}(h^\infty)$.

- With more work, one can also hope a result in the analytic category by using the notion of analytic symbol introduced by J. Sjöstrand.
  We should assume in this case that the function $x \mapsto \varrho(x)$ is analytic.
  We warn the reader that this does not mean that the above formal sums become convergent. This simply means that one can prove that, in a fixed complex neighborhood of x_0, $|a_j(x)|$ is bounded by $C^{j+1} j!$ and that we have similar estimates for the sequence $(\delta_j)_{j\in\mathbb{N}}$ (cf. the works by J. Sjöstrand [Sj1], Helffer–Sjöstrand [HelSj1], Klein–Schwarz [KlSc90]). This simply means that, by a "finite" tricky summation ($N(h) = \frac{C_0}{h}$ depending on h), one gets the existence of $\epsilon_0 > 0$, such that:
$$P(h, \delta^{N(h)}(h)) u_\chi^N(x,h) = \mathcal{O}(\exp -\frac{\epsilon_0}{h}) \exp -\frac{\varphi(x)}{h} \,. \tag{51}$$
- Here we have used the self-adjointness property for getting information on the spectrum. We will now see in the next lecture that for more complicate models the selfadjoint character of the problem disappears.

3 Lecture 2: Towards Non Self-Adjoint Models

3.1 Instability for Kelvin–Helmholtz I: Physical Origin

As a motivation, we will start with a generalization of the Kelvin–Helmholtz model. We refer to Chap. XI in Chandrasekhar's book [Cha] for a complete exposition of the origin of the model. This is a generalization [CCLa] of the classical Kelvin–Helmholtz instability which appears when two fluids move with different parallel velocities on each side of an interface.

When linearizing along the stationary solution $(\varrho_0, \mathbf{u_0}, p_0)$ for a given density ϱ_0 and a given (this time not zero) velocity u_0 (see (3)), where u_0 is the first component of $\mathbf{u_0}$, and following what we have done for Rayleigh–Taylor, we get the following one-dimensional question.

Can we analyze in function of the parameters (k_1, k_2, g, γ, k) with $k^2 = k_1^2 + k_2^2$, if the operator
$$\begin{aligned}&\mathcal{P}_{KH}(\gamma, k_1, k_2, g) := \\ &\quad -\tfrac{d}{dx}\varrho_0 \tfrac{d}{dx}(\gamma + ik_1 u_0(x))^2 + k^2 \varrho_0(x)(\gamma + ik_1 u_0(x))^2 \\ &\qquad -ik_1(\gamma + ik_1 u_0(x)) \tfrac{d}{dx}\varrho_0 u_0'(x) + gk^2 \varrho_0'((x)\end{aligned}$$
is approximately injective (say for large values of k).

Like in Rayleigh–Taylor which corresponds to $k_1=0$ (or actually to $u_0=0$), our semi-classical parameter will be $h = \frac{1}{k}$. The parameter $\gamma = \Gamma_0 + i\Gamma_1$ is not necessarily real but we are interested in approximate null solutions for which Γ_0 is as large as possible (or complementarily) to show that Γ_0 should necessarily remain bounded in the regime k large.

So we divide by k^2 in the equation above and meet the following semi-classical operator:

$$\begin{aligned}\mathcal{P}_k(x, hD_x) := & -h\tfrac{d}{dx}\varrho_0 h\tfrac{d}{dx}(\gamma + ik_1u_0(x))^2 \\ & +\varrho_0(x)(\gamma + ik_1u_0(x))^2 \\ & -ihk_1(\gamma + ik_1u_0(x))h\tfrac{d}{dx}\varrho_0 u_0'(x) \\ & +g\varrho_0'(x)\ .\end{aligned}$$

So in this regime k_1 is fixed such that $|k_1| \leq k = \frac{1}{h}$. This last inequality will not be a restriction in the semi-classical regime.

Semi-classically, the principal symbol is given by

$$p_0(x, \xi) := \varrho_0(1 + \xi^2)(\gamma + ik_1u_0(x))^2 + g\varrho_0'(x)\ . \tag{52}$$

This symbol is not real, hence the associated operator is clearly not symmetric and cannot be extended as a selfadjoint operator. Our aim is to describe a rather systematic strategy for constructing approximate null solutions or to decide that we can not construct such solutions. This question is naturally related to the notion of pseudo-spectra for families (depending in particular on h but also on other parameters) and adapted to the analysis of h-pseudodifferential operators. This is what we will explain now before to treat the various physical examples including this one.

3.2 Around the ϵ-Pseudo-Spectrum

Definition 3.1. If A is a closed operator with dense domain $D(A)$ in an Hilbert space $\mathcal{H}$, the ϵ-pseudospectrum $\sigma_\epsilon(A)$ of A is defined by

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} \mid ||(zI - A)^{-1}|| \geq \frac{1}{\epsilon}\}.$$

We take the convention that $||(zI - A)^{-1}|| = +\infty$ if $z \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A, so it is clear that we always have:

$$\sigma(A) \subset \sigma_\epsilon(A)\ .$$

When A is selfadjoint (or more generally normal), $\sigma_\epsilon(A)$ satisfies, by the Spectral Theorem

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \mid d(z, \sigma(A)) \leq \epsilon\}\ .$$

So this is only in the case of non self-adjoint operators that this new concept (first appearing in numerical analysis, see Trefethen [Tref]) becomes interesting.

Although formulated in a rather abstract way, the following result by Roch–Silbermann [RoSi] explains rather well to what corresponds the pseudo-spectrum

$$\sigma_\epsilon(A) = \bigcup_{\{\delta A \in \mathcal{L}(\mathcal{H}) \text{ s. t. } ||\delta A||_{\mathcal{L}(\mathcal{H})} \leq \epsilon\}} \sigma(A + \delta A) \; .$$

In other words, z is in the ϵ-pseudo-spectrum of A if z is in the spectrum of some perturbation $A+\delta A$ of A with $||\delta A|| \leq \epsilon$. This is indeed a natural notion thinking of the fact that the models we are analyzing are only approximations of the real problem and of the fact that the numerical analysis of the model goes through the analysis of explicitly computable approximate problems.

3.3 Around the h-Family-Pseudospectrum

We are mainly interested in the semiclassical version of this concept attached to a family (indexed by $h \in]0, h_0]$) of operators A_h. Here we are inspired by various presentations of the subject including [Sj2], [DeSjZw] and [Pra3], without to necessary follow their terminology.

For a given $\mu \geq 0$, the h-family-pseudospectrum of index μ of a family A_h (indexed by $h \in]0, h_0]$) (of closed operators with a dense domain $D(A_h)$ in a fixed Hilbert Space $\mathcal{H}$) is defined by

$$\begin{aligned} &\Psi_\mu((A_h)) \\ &\quad := \{z \in \mathbb{C} \mid \forall C > 0, \forall h_0 > 0 \, s.t. \, \exists h \in]0, h_0], \\ &\qquad\qquad ||(A_h - z)^{-1}|| \geq C\, h^{-\mu}\} \; . \end{aligned} \tag{53}$$

We can then define

$$\Psi_\infty((A_h)) = \bigcap_{\mu \geq 0} \Psi_\mu((A_h)) \; . \tag{54}$$

May be it is easier to understand the quantifiers by observing that the h-family pseudoresolvent set corresponds to the z such that $\exists C > 0$ and $h_0 > 0$ such that $\forall h \in]0, h_0]$

$$||(A_h - z)^{-1}|| \leq C\, h^{-\mu} \; .$$

If one thinks of applications to Physics, these concepts are more stable by perturbation than the corresponding notion of spectrum and they are for this reason particularly relevant in the non self-adjoint case. Practically, one will exhibit the existence of this h-family-pseudo-spectrum by constructing quasimodes or approximate solutions. This leads to another natural definition.

For a given $\mu \geq 0$, the h-family-quasispectrum of index μ of the family A_h is defined by

$$\begin{aligned} &\psi_\mu((A_h)) \\ &\quad := \{z \in \mathbb{C} \mid \forall C > 0, \forall h_0 > 0\, s.t.\, \exists h \in]0, h_0], \\ &\qquad\qquad \exists u_h \in D(A_h) \setminus \{0\}\, s.t. \\ &\qquad\qquad\quad ||(A_h - z)u_h|| \leq C\, h^\mu ||u_h||\} \,. \end{aligned} \tag{55}$$

We can then define

$$\psi_\infty((A_h)) = \bigcap_{\mu \geq 0} \psi_\mu((A_h)) \,. \tag{56}$$

The main point is then that

$$\psi_\mu((A_h)) \subset \Psi_\mu((A_h)) \,.$$

Note that the converse is not true (see the discussion in [Pra3]) in general.

We will be particularly interested in using these tools when A_h is actually an h-pseudodifferential operator.

The elliptic theory (with suitable conditions at ∞) for h-pseudodifferential operators says for example that

Proposition 3.1 (see the book of D. Robert).
If $z \notin \Sigma(p)$, where

$$\Sigma(p) := \{\lambda \in \mathbb{C}\,, \mid \exists (x_n, \xi_n)\, s.t.\, \lambda = \lim_{n \to +\infty} p(x_n, \xi_n)\} \,, \tag{57}$$

then $z \notin \Psi_\mu(\ Op_h(p))$.

This will actually also be true for any $A_h = \ \mathrm{Op}_h(p_h)$, for which the principal symbol of A_h is p.

The proof is very easy once an h-pseudodifferential calculus has been constructed. It is enough to use $\mathrm{Op}_h((p-z)^{-1})$ as first approximate inverse and then to use a Neumann series. The reader can look at the end of Sect. 2.4 for more details.

So the first natural thing to do when analyzing the h-pseudospectrum of the family is to analyze the numerical range $\Sigma(p)$ of its principal symbol.

3.4 The Davies Example by Hand

We present a variant of the proof of the generalization, by Pravda-Starov [Pra1], of the Davies result on the h-family pseudo-spectrum for the Schrödinger operator

$$A_h := -h^2 \frac{d^2}{dx^2} + V(x) \,.$$

This proof is inspired by similar proofs in [HelLaf2, Mar].

Remark 3.1. Davies treats a particular case by hand. Then Zworski observes that it can be interpreted as a semi-classical version of a result for operators of principal type (Hörmander [Ho1], [Ho2], Duistermaat–Sjöstrand [DuSj]). This was pushed further by Dencker–Sjöstrand–Zworski [DeSjZw], N. Lerner (together with collaborators) (see in [Le] and references therein), Pravda-Starov [Pra1].

One should of course compare with the selfadjoint result at the bottom of the well but here what is crucial is the non-selfadjointness!!

Theorem 3.1 (Davies–Pravda). *Let us assume that there exist x_0 and z such that*

$$z - V(x_0) \in \mathbb{R}^+ \,, \tag{58}$$

and such that, for an even $k \geq 0$,

$$Im\, V^{(j)}(x_0) = 0 \,, \forall j \leq k \,, \tag{59}$$

and

$$Im\, V^{(k+1)}(x_0) \neq 0 \,. \tag{60}$$

Then $z \in \psi_\infty((A_h))$.

Some Elementary Proof by a WKB Construction

The crucial point is that there exists $\xi_0 > 0$ such that

$$\xi_0^2 + V(x_0) = z \,.$$

In other words, there exists (x_0, ξ_0) such that $p(x_0, \xi_0) = z$. Hence, $z \in \Sigma(p)$ as defined in (57) and we are not at the boundary of $\Sigma(p)$.

Following the construction described in the first Lecture (see (37)–(40)), we look for a solution in the form

$$u(x,h) = a(x,h) \exp -\frac{\varphi(x)}{h} \tag{61}$$

near x_0 with

$$a(x,h) \sim \sum_{j\geq 0} h^j a_j(x) \,, \tag{62}$$

such that

$$\exp \frac{\varphi}{h}(A_h - z_0) u(\cdot\,; h) \sim 0 \,. \tag{63}$$

Let us emphasize that (conversely to what was done in the analysis of the Rayleigh–Taylor model) we keep z_0 fixed and did not look for an expansion $z(h) \sim \sum_{j\geq 0} z_j h^j$.

Expanding in powers of h and expressing the cancellation of each coefficient of h^ℓ, we first get an eikonal equation. The phase φ (appearing in (61)) should satisfy (we can after a change of notations assume that $z = 0$:

$$-\varphi'(x)^2 + V(x) = 0 \;, \tag{64}$$

where V satisfies by assumption $\operatorname{Re} V(x_0) < 0$, (59) and (60).
The existence of $\varphi(x)$, with $\varphi(x_0) = 0$ and $\varphi'(x_0) = i\xi_0$ is evident. So the important point, in order to have an approximate eigenfunction which is localized at x_0, is to verify that $\operatorname{Re}\varphi$ has actually a local minimum at x_0.
Taking the real and imaginary parts in (64), we get

$$-\operatorname{Re}\varphi'(x)^2 + \operatorname{Im}\varphi'(x)^2 + \operatorname{Re} V(x) = 0 \;, \tag{65}$$

and

$$-2\operatorname{Re}\varphi'(x)\cdot\operatorname{Im}\varphi'(x) + \operatorname{Im} V(x) = 0 \;, \tag{66}$$

in a neighborhood of x_0.
In particular, this implies at x_0

$$\operatorname{Re}\varphi'(x_0) = 0,\ \xi_0^2 = \operatorname{Im}\varphi'(x_0)^2 = -\operatorname{Re} V(x_0)\;.$$

What we now need is to verify that the first non zero derivative of $\operatorname{Re}\varphi$ at x_0 is even and strictly positive.
We start from

$$\operatorname{Re}\varphi'(x) = \frac{\operatorname{Im} V(x)}{2\operatorname{Im}\varphi'(x)}\;.$$

But it is immediate from the assumptions that

$$\operatorname{Re}\varphi^{(j)}(x_0) = 0\;,\quad \text{for } j \le k+1\;,$$

and

$$\operatorname{Re}\varphi^{(k+2)}(x_0) = \frac{\operatorname{Im} V^{(k+1)}(x_0)}{2\operatorname{Im}\varphi'(x_0)}\;.$$

We can now choose the sign of ξ_0 in order to have

$$\operatorname{Re}\varphi^{(k+2)}(x_0) > 0\;.$$

Due to the fact that $(\partial_\xi p)(x_0,\xi_0) = \xi_0 \neq 0$, the solution of the transport equations does not create problems like in the case of Rayleigh–Taylor and we can construct a solution $u_h = a(x,h)\exp -\frac{\varphi(x)}{h}$ in the neighborhood of x_0. Let us briefly show how to treat the cancellation of the coefficient of h which leads to the so-called first transport equation. This equation reads

$$2\varphi'(x)a_0'(x) + \varphi''(x)a_0(x) = 0\;, \tag{67}$$

with as initial condition

$$a_0(x_0) = 1\;.$$

But $\varphi'(x_0) = i\xi_0 \neq 0$, so it is immediate to find in a neighborhood of x_0 the main amplitude a_0 by

$$a_0(x) = \exp -\frac{1}{2}\left(\int_{x_0}^{x} \frac{\varphi''(\tau)}{\varphi'(\tau)}\, d\tau\right) .$$

The next equation has the same structure as in (67) except that there is a r.h.s. This equation reads

$$2\varphi'(x)a_1'(x) + \varphi''(x)a_1(x) = a_o''(x) , \tag{68}$$

with as initial condition

$$a_1(x_0) = 0 ,$$

and has again a unique explicit solution. More generally all the successive equations read

$$2\varphi'(x)a_j'(x) + \varphi''(x)a_j(x) = a_{j-1}''(x) , \tag{69}$$

with as initial condition

$$a_j(x_0) = 0 ,$$

and can be solve by recursion for $j \geq 2$.

Remark 3.2. K. Pravda-Starov constructs a solution in the form $\exp -\frac{\varphi(x,h)}{h}$ with $\varphi(x;h) \sim \sum_j h^j \varphi_j(x)$ but this is not really different when working with a groundstate which is supposed to have no zero.

Remark 3.3. Note that if $z \notin \overline{\Sigma(p)}$, then the elliptic theory says that it is impossible to construct an approximate solution, so it leaves open only the points at the boundary of $\Sigma(p)$.

3.5 Kelvin–Helmholtz II: Mathematical Analysis

We now come back to our motivating model and see if the ideas behind the treatment of Davies example are efficient.
Note also that our question is a little different and could be reformulated as: *For which values of the parameters is* 0 *in the h-family pseudospectrum of the family (with $h = \frac{1}{k}$)?*

So we have to analyze if 0 belongs to $\Sigma(p_0)$, where p_0 was defined in (52). We just do the local analysis (the analysis of the ellipticity at ∞ should be interesting to do). According to (52), we have:

$$\begin{aligned} \operatorname{Re} p_0(x,\xi) &= \varrho_0(x)(\xi^2+1)(\Gamma_0^2 - (k_1 u_0(x) + \Gamma_1)^2) + g\varrho_0'(x) , \\ \operatorname{Im} p_0(x,\xi) &= 2\varrho_0(x)(\xi^2+1)\Gamma_0(k_1 u_0(x) + \Gamma_1) . \end{aligned} \tag{70}$$

Assuming that

$$\Gamma_0 \neq 0 , \tag{71}$$

and that

$$\varrho_0(x) > 0\,,\ \forall x \in \mathbb{R}\,, \tag{72}$$

we observe that

$$\operatorname{Im} p_0(x,\xi) = 0 \ \text{ iff } k_1 u_0(x) + \Gamma_1 = 0\ .$$

When this condition is satisfied, we get

$$\operatorname{Re} p_0(x,\xi) = \varrho_0(x)(\xi^2+1)\Gamma_0^2 + g\varrho_0'(x)\ .$$

If

$$\varrho_0' < 0\,,\ \text{ on } \mathbb{R}\,, \tag{73}$$

then we see ($g > 0$), that, if

$$\Gamma_0^2 > g \max_x -\frac{\varrho_0'(x)}{\varrho_0(x)}\ ,$$

then the principal symbol is elliptic.
Hence no local approximate null solution can be constructed. 0 does not belong to the h-family-pseudospectrum of the operator.

We also observe that this condition is the same as for Rayleigh–Taylor (see for example (32), with in mind (24))!

Conversely, when

$$\Gamma_0^2 < g \max_x -\frac{\varrho_0'(x)}{\varrho_0(x)}\ ,$$

one can, for any x_0 such that

$$-g\frac{\varrho_0'(x_0)}{\varrho_0(x_0)} > \Gamma_0^2\ ,$$

find some $\xi_0 \neq 0$ such that

$$\Gamma_0^2(1+\xi_0^2) = -g\frac{\varrho_0'(x_0)}{\varrho_0(x_0)}\ .$$

We are now looking on the condition under which the operator A_h, which is not elliptic at (x_0,ξ_0) which determines the parameter Γ_1 by,

$$\Gamma_1 = -k_1 u_0(x_0)\ ,$$

is not subelliptic at this point (we will explain later in the next lecture (Theorem 4.1) what we can do in this case).

The computation of the bracket of $\operatorname{Re} p_0$ and $\operatorname{Im} p_0$ gives

$$\{\operatorname{Re} p_0, \operatorname{Im} p_0\}(x_0,\xi_0) = 4k_1\xi_0\varrho_0(x_0)^2 u_0'(x_0)\Gamma_0^3\ . \tag{74}$$

So it is immediate by playing with the sign of k_1 (or of ξ_0) to get the condition (75) satisfied if $u_0'(x_0) \neq 0$.

A detailed analysis of what is going on for $\gamma = \Gamma_0 + i\Gamma_1$ with Γ_0 close to $\widetilde{\Gamma}_0$ with

$$\widetilde{\Gamma}_0^2 = g \max_x -\frac{\varrho_0'(x)}{\varrho_0(x)}$$

should surely be interesting. The techniques presented at the end of the last lecture will be helpful.

Here the simplest toy model should be

$$h^2 D_x^2 + ik_1 x \ ,$$

the complex Airy operator, which is for $k_1 \neq 0$ a particular case of Davies example and can be also analyzed close to 0 by Dencker–Sjöstrand–Zworski result.

Let us explain more in detail how we guess this model. We do not try to be rigorous. For convenience we assume that ϱ' is strictly negative so the associated $K(h)$ (see (28)) appearing in the treatment of the Rayleigh–Taylor model is positive. At least locally near a maximum of $x \mapsto -\frac{\varrho'(x)}{\varrho(x)}$, one can (this is an interesting exercise in semi-classical analysis) modulo $\mathcal{O}(h^\infty)$ rewrite our problem of research of approximate null solutions in looking for which values of γ, the operator

$$\sqrt{K(h)} - ik_1 u_1(x) + hp_1(x, hD_x, h, k_1, \gamma)) - \gamma$$

has approximate null solutions.

There is a technique (functional calculus of Helffer–Robert ([Rob] and references therein) or direct approach for the square root) for recognizing $f(K(h))$ as an h-pseudodifferential operator if f is regular. In our case, one can use a C^∞-positive function coinciding with $\sqrt{t}$ on $[2\epsilon_0, +\infty[$ and equal to a strictly positive constant for $t \in]-\infty, \epsilon_0]$.

If we forget the dependence on γ in p_1, we are facing a very standard question of h-family-pseudospectrum.

The question becomes simply:

Is γ in the pseudospectrum of

$$\sqrt{K(h)} - ik_1 u_1(x) + hp_1(x, hD_x, h, k_1, \gamma))?$$

Taking the harmonic approximation of $\sqrt{K(h)}$ at a point where the principal symbol of $\sqrt{K(h)}$ (which is the square root of the principal symbol of $K(h)$) and the linear approximation of u_1 at x_0 leads (up to the constants) to the toy model.

3.6 Other Toy Models

Other toy models have been analyzed in detail. Let us mention

$$h^2 D_x^2 + ihD_x + x^2\ ,$$

whose symbol is $p(x,\xi) = \xi^2 + i\xi + x^2$ (See [DeSjZw], p. 3).

The spectrum is easy to determine as given by the sequence $\frac{1}{4} + (2n+1)h$ ($n \in \mathbb{N}$), the corresponding eigenfunctions being directly related with the Hermite functions. This permits to diagonalize the operator BUT in a non orthonormal basis.

The h-family pseudospectrum is given by the numerical range of the principal symbol of the operator:

$$\Sigma(p) = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z|^2 \leq \mathrm{Re}\, z\}\ .$$

More generally the h-family pseudospectrum of the Schrödinger operators $-h^2\Delta + V(x)$, with V quadratic has been analyzed in great detail in the PhD thesis of Pravda-Starov [Pra3].

Other models appear in connection with the analysis of the resolvent of the Fokker–Planck operator (see Risken (for the quadratic case), [Ris], Hérau–Nier [HerNi], Helffer–Nier [HelNi], Hérau–Sjöstrand–Stolk [HerSjSt]) or for other models (See Hager [Ha] and works in progress from Hager–Sjöstrand).

4 Lecture 3: On Semi-Classical Subellipticity

4.1 Introduction

The references for this lecture are papers by Davies [Da2], Zworski [Zw], Dencker–Sjöstrand–Zworski [DeSjZw], Lerner [Le] (and references therein).

We would like to show how the microlocal techniques (suitably adapted to the semi-classical context) permit to recover or complete the previous results. We will see in the last lecture how one can also analyze the transition between the elliptic region and the non elliptic one. We have already seen that many results of non-existence of approximate null solutions are just the consequence of "elliptic" semi-classical results. As a second step, we can look if, at non-elliptic points, some subellipticity condition is satisfied, starting by $\frac{1}{2}$-semi-classical subellipticity. This would again imply the same type of results.

Conversely, if the operator is not subelliptic, one can try to construct directly WKB solutions in the form $a(x,h) \exp -\frac{\varphi(x)}{h}$ with φ admitting a minimum at some point x_0 or to apply more general theorems in semi-classical analysis. We start in the next subsection by a typical result of the last alternative.

4.2 Non Subellipticity: Generic Result

The main relevant theorem in our context can be stated in the following way (see [DeSjZw]). One considers an h-pseudodifferential $A_h := a(x, hD_x)$ with principal symbol a_0 and one is looking for a simple criterion under which 0 belongs to the h family pseudospectrum of A_h.

Theorem 4.1. *Let us assume that at a point (x_0, ξ_0), we have*

$$a_0(x_0, \xi_0) = 0 \ , \ \{Re\ a_0, Im\ a_0\}(x_0, \xi_0) < 0 \ . \tag{75}$$

Then there exists an L^2-normalized solution u_h, whose h-wave front is (x_0, ξ_0), and such that (x_0, ξ_0) is not in the h-wave front of $A_h u_h$.

We recall that, for a bounded family of L^2 functions v_h, we say that a point (y, η) *is not* in the h-wave front set,[4] if there exists a C_0^∞ function χ equal to 1 in the neighborhood of y, such that $(\mathcal{F}_h \chi v_h)(\xi) := h^{-\frac{n}{2}} \widehat{\chi v_h}(\xi/h) = \mathcal{O}(h^\infty)$ in a neighborhood of η.

Another (equivalent) definition is to use the Fourier–Bros–Iagolnitzer (which will be familiar to the users of the Gabor transform) as intensively developed by J. Sjöstrand [DiSj].

We say that (x_0, ξ_0) is not in the h-Wave front set of a bounded family u_h in L^2 if the function

$$(x, \xi) \mapsto h^{-\frac{3n}{4}} \int \exp \frac{i}{h}(x - y) \cdot \xi \ \exp - \frac{(x - y)^2}{2h} \ u_h(y)\, dy \ ,$$

is $\mathcal{O}(h^\infty)$ in some (h-independent) neighborhood of (x_0, ξ_0).

Applications
Let us see what this theorem say for the two examples we have already met: the Davies example and the Kelvin–Helmholtz example.
In the first case, we have

$$\operatorname{Re} a_0(x, \xi) = \xi^2 + \operatorname{Re} V(x) - \operatorname{Re} z_0 \ , \ \operatorname{Im} a_0(x, \xi) = \operatorname{Im} V(x) - \operatorname{Im} z_0 \ . \tag{76}$$

The Poisson Bracket at (x_0, ξ_0) is

$$\{\operatorname{Re} a_0, \operatorname{Im} a_0\}(x_0, \xi_0) = 2\xi_0 \operatorname{Im} V'(x_0) \ , \tag{77}$$

and we recall that $\xi_0 \neq 0$ with ξ_0^2 determined. So if $\operatorname{Im} V'(x_0) \neq 0$, (which corresponds to $k = 0$ in Davies–Pravda theorem), the non-subelliptic theorem applies for the right choice of the sign x_0.

In the second case, we send back the reader to Formula (74).

[4] Another terminology used for example in [Rob] is to speak of frequency set.

4.3 Link with the Standard Non-Hypoellipticity Results for Operators of Principal Type

In the theory of Partial Differential Equations, Theorem 4.1 corresponds to a result of non-hypoellipticity. The basic simplest model is $D_x + ixD_t$, which is known to be non hypoelliptic microlocally at $(0,0)$ in the direction $(0,-1)$. Hence it is not hypoelliptic. But one should keep in mind that the link between the two problems is microlocal. As already explained in the lectures by N. Lerner [Le] (see also [Trev]), the link between the two theories is through the partial Fourier transform in the t-variable. For an operator in the form $D_x+ib(x)D_t$, we first get the family in τ, $D_x+ib(x)\tau$, that we have to analyze for $|\tau|$ large. With $h = \frac{1}{|\tau|}$, we get two semi-classical families of operators to analyze $hD_x \pm ib(x)$, each one corresponding to a microlocal analysis in the direction $(0,1)$ or $(0,-1)$.

4.4 Elementary Proof for the Non-Subelliptic Model

We give an elementary proof (cf. [Mar]) under the additional assumption that

$$a_0(x, i\xi) \in \mathbb{R}\ ,\ \forall (x,\xi) \in \mathbb{R}^2\ , \tag{78}$$

which appears to be satisfied for the two last physical models, which will be analyzed in the next section, but is not satisfied for the Davies example and the Kelvin–Helmholtz model.
In this case, we define the real symbol

$$q_0(x,\xi) = a_0(x, i\xi)\ ,\ \forall (x,\xi) \in \mathbb{R}^2\ ,$$

and we look for a point $(x_0, 0)$ such that

$$q_0(x_0, 0) = 0\ ,$$

and for a non negative real phase φ defined in a neighborhood of x_0 such that $\varphi(x_0) = 0$ admitting at x_0 a local minimum and solution of

$$q_0(x, \varphi'(x)) = 0\ . \tag{79}$$

Under the condition that $\partial_\xi q_0(x_0, 0)$ it is immediate to find φ by the implicit function theorem.

The first natural condition for having a minimum is then to see under which condition one has

$$\varphi''(x_0) > 0\ .$$

Differentiating the eikonal equation (79), we obtain

$$(\partial_x q_0)(x,\varphi'(x)) + (\partial_\xi q_0)(x,\varphi'(x))\varphi''(x) = 0\ ,$$

hence

$$\varphi''(x_0) = -\frac{\partial_x q_0(x_0,0)}{\partial_\xi q_0(x_0,0)}\ .$$

So we are done if the r.h.s. is strictly positive:

$$-\frac{\partial_x q_0(x_0,0)}{\partial_\xi q_0(x_0,0)} > 0\ . \tag{80}$$

Let us now control that this condition can be recognized as the condition of the theorem.
From the relations

$$\partial_x q_0(x,\xi) = \partial_x a_0(x,i\xi)\ ,\ \partial_\xi q_0(x,\xi) = i\partial_\xi a_0(x,i\xi)\ ,$$

we get at any point $(x,0)$:

$$\begin{array}{ll} \partial_x \mathrm{Im}\, a_0(x,0) = 0\ , & \partial_\xi \mathrm{Re}\, a_0(x,0) = 0 \\ \partial_x \mathrm{Re}\, a_0(x,0 = \partial_x q_0(x,0)\ , & \partial_\xi \mathrm{Im}\, a_0(x,0) = -\partial_\xi q_0(x,0)\ . \end{array}$$

So this gives the relation:

$$\{\mathrm{Re}\, a_0, \mathrm{Im}\, a_0\}(x,0) = \partial_x q_0(x_0,0)\partial_\xi q_0(x_0,0)\ ,$$

and the result becomes clear.

The second step is to construct a quasimode in the form

$$u_h := b(x,h)\exp -\frac{\varphi(x)}{h}\ ,$$

with

$$b(x,h) \sim \sum_{j\geq 0} b_j(x)h^j\ .$$

The equation for b_0 reads

$$(\partial_\xi q_0)(x,\varphi'(x))b_0'(x) + \left(\frac{\varphi''(x)}{2}(\partial_\xi^2 q_0)(x,\varphi'(x)) + q_1(x,\varphi'(x))\right) b_0(x) = 0\ ,$$

where q_1 is the “subprincipal” symbol. One can always solve this equation with $b_0(x_0) = 1$ (see (67)).

Remark 4.1. When the first Poisson bracket of a_0 and $\overline{a_0}$ is 0 (which is equivalent to $\partial_x q_0(x,0) = 0$), one can find a criterion involving higher order brackets. See [Pra3], [Mar] and the standard results on subelliptic operators obtained in the seventies.

We are in a particular case of the following more general situation. We look for solutions of $a(x, hD_x)u_h = \mathcal{O}(h^\infty)$ which are localized in a neighborhood of a point (x_0, ξ_0) such that

$$a_0(x_0, \xi_0) - z = 0 \ , \ (\partial_\xi a_0)(x_0, \xi_0) \neq 0 \ .$$

In addition, we have

$$-i(\mathrm{ad}\, a_0)^k(\{a_0, \overline{a}_0\})(x_0, \xi_0) = 0 \ ,$$

for $k < k_0$ and

$$-i(\mathrm{ad}\, a_0)^{k_0}(\{a_0, \overline{a}_0\})(x_0, \xi_0) > 0 \ ,$$

where $\mathrm{ad}\, p$ is the operator of commutation

$$(\mathrm{ad}\, p)q = \{p, q\} \ .$$

This time we have to take a complex phase.

4.5 $\frac{1}{2}$ Semi-Classical Subellipticity

When the principal symbol is not elliptic, the best we can hope is a subelliptic result. The next theorem corresponds to the first (and the most generic) result of this type.

Theorem 4.2 ($\frac{1}{2}$-Subellipticity). *If $(u_h)_{h\in]0,h_0]}$ is an L^2 normalized solution in the domain of A_h such that $A_h u_h = \mathcal{O}(h^\infty)$, then if for some (x_0, ξ_0) we have*

$$a_0(x_0, \xi_0) = 0 \ , \ \{Re\ a_0, Im\ a_0\}(x_0, \xi_0) > 0 \ ,$$

then (x_0, ξ_0) does not belong to the h-wave front set of the family u_h.

Remark 4.2. In PDE theory this corresponds to the simplest result of microlocal hypoellipticity. The basic simplest model is $D_x + ixD_t$, which is known to be hypoelliptic (with loss of $\frac{1}{2}$ derivatives microlocally at $(0, 0)$ in the direction $(0, 1)$).

We will come back later in the last lecture to high order subellipticity.

Remark 4.3. Note that the elliptic theory simply says that if $z \notin \Sigma(p)$, then z is not in the pseudospectrum of $-h^2\Delta + V$. So what remains is simply a more precise analysis at $\partial\Sigma(p)$.

About the Proof

We refer to the lectures of N. Lerner [Le]. Let us just sketch the semi-classical proof. If we write

$$A_h = B_h + iC_h \ ,$$

with B_h and C_h selfadjoint respectively of principal symbol $\mathrm{Re}\, a_0$ and $\mathrm{Im}\, a_0$, the basic point is that

$$A_h^* A_h = B_h^2 + C_h^2 + i[B_h, C_h] \ ,$$

and to observe that $\frac{i}{h}[B_h, C_h]$ is positive elliptic at the points where A_h is not elliptic.
We can use rather weak forms of the Garding inequality. We refer to the lectures of N. Lerner ([Le]) for discussions around this point and the Fefferman–Phong inequality.

Remarks 4.3

- *Here we gave the impression that everything is done globally but let us now emphasize that one has to do very often the argument microlocally.*
- *Note that we do not really need this result. In the case of the symbol appearing in Kelvin–Helmholtz model the sign of the Poisson bracket at* (x_0, ξ_0) *is opposite to the sign at* $(x_0, -\xi_0)$.
  This will not be the case for the two next models for which we will have $\xi_0 = 0$ *at the non-elliptic points.*

5 Lecture 4: Other Non Self-Adjoint Models Coming from Hydrodynamics

5.1 Introduction

The two next models are deduced from the mass conservation and the momentum conservation equation of the Euler equation, and differ through the modelling of the energy equation. For simplicity the systems are written in $\mathbb{R}^2_{\tilde{x},\tilde{y}} \times \mathbb{R}_t$ (instead of $\mathbb{R}^3_{\tilde{x},\tilde{y},\tilde{z}} \times \mathbb{R}_t$).
The density of the fluid satisfies, for some strictly positive constant $\rho_a > 0$,

$$\rho(\tilde{x}, \tilde{y}) \to \rho_a \quad \text{when } \tilde{x} \to +\infty \ ,$$

and the velocity of the fluid satisfies, for some $V_a > 0$,

$$\mathbf{U} := (u, v) \to (-V_a, 0) \quad \text{when } \tilde{x} \to +\infty \ .$$

ρ_a is the density of the ablated fluid and V_a the modulus of the velocity of the ablated fluid.

The *Rayleigh model with convection* assumes that the perturbation of the velocity is incompressible. This means that there exists a function $\mathbf{U}_0(\tilde{x})$, called the convective velocity, such that

$$\text{div } (\mathbf{U} - \mathbf{U}_0) = 0 \ .$$

The system will be denoted by (RC) and writes

$$(RC) \qquad \begin{cases} \partial_t \rho + \partial_{\tilde{x}}(\rho u) + \partial_{\tilde{y}}(\rho v) = 0 \ , \\ \partial_t(\rho u) + \partial_{\tilde{x}}(\rho u^2 + p) + \partial_{\tilde{y}}(\rho u v) = -\rho g \ , \\ \partial_t(\rho v) + \partial_{\tilde{x}}(\rho u v) + \partial_{\tilde{y}}(\rho v^2 + p) = 0 \ , \\ \text{div } (\mathbf{U} - \mathbf{U}_0) = 0 \ , \end{cases}$$

where the unknowns are the density ρ, the velocity (u, v) and the pressure p.

The *ablation front model* uses an energy equation with heat conduction. The enthalpy is defined by

$$h = C_p T \ , \tag{81}$$

With $T(t, \tilde{x}, \tilde{y})$ denoting the temperature of the fluid (at a point $\tilde{x}, \tilde{y}$ and a time t) and C_p being a constant characterizing the calorific capacity of the fluid, the enthalpy satisfies the equation:

$$\rho(\partial_t + \mathbf{U} \cdot \nabla)h - (\partial_t + \mathbf{U} \cdot \nabla)p = -\text{div}\, \mathbf{J}_q \tag{82}$$

Here $\mathbf{J}_q$ is the heat flux given by the Fourier conduction law

$$\mathbf{J}_q = -\lambda(T)\nabla T \ .$$

In this law, $\lambda(T)$ is proportional to a power of the temperature, that is satisfying, for some constants $\kappa > 0$ and $\nu > 0$,

$$\lambda(T) = \kappa T^\nu \ .$$

Note that these formulas assume that $T > 0$ and consequently, with p related with T as below in (83) to the condition $p > 0$. The parameter ν is called the conduction index.

We now write the perfect gas relation

$$p = \rho T(C_p - C_v) \ , \tag{83}$$

where C_v is the calorific capacity at constant volume. C_p/C_v is 5/3. Starting from (82) and then using (81), (83) and the first equation in (RC), we get:

$$C_p\, \rho T \,\text{div } \mathbf{U} + C_v(\partial_t + \mathbf{U} \cdot \nabla)\rho T + \text{div } \mathbf{J}_q = 0 \ . \tag{84}$$

We shall not analyze this model, in particular because this model has no stationary solution. So the physicists use other models for which we can just explain (without being in any way rigorous) how they can be obtained.

5.2 Quasi-Isobaric Model (Kull and Anisimov)

The starting point consists in replacing the perfect gas relation by the relation:

$$\rho T = D_0 \ , \tag{85}$$

where D_0 is a constant.
Implementing (85) in (84) gives:

$$D_0 C_p \operatorname{div} \mathbf{U} \, + \, \operatorname{div} \mathbf{J}_q = 0 \ .$$

This constant is identified through the hypothesis that $T \to T_a$, $T_a > 0$, when $\tilde{x}$ goes to $+\infty$ (temperature of the ablated fluid).

Hence

$$D_0 = \rho_a T_a \quad \text{and} \ T = \frac{\rho_a T_a}{\rho} \ .$$

For a derivation of this model, see [KullA], [Go,Mas,La3]. A similar model arises also in the Low Mach approximation (see [Li]).
The system of equations writes

$$(KA) \quad \begin{cases} \partial_t \rho + \partial_{\tilde{x}}(\rho u) + \partial_{\tilde{y}}(\rho v) = 0 \ , \\ \partial_t(\rho u) + \partial_{\tilde{x}}(\rho u^2 + p) + \partial_{\tilde{y}}(\rho u v) = -\rho g \ , \\ \partial_t(\rho v) + \partial_{\tilde{x}}(\rho u v) + \partial_{\tilde{y}}(\rho v^2 + p) = 0 \ , \\ \operatorname{div} \left(\mathbf{U} - \frac{\kappa}{C_p \rho_a} T_a^\nu (\frac{\rho_a}{\rho})^\nu \nabla \frac{\rho_a}{\rho}\right) = 0 \ , \end{cases}$$

where the unknowns are the functions $(t, \tilde{x}, \tilde{y}) \mapsto (\rho, u, v, p)$.
Of course we can recover T by the equation $\rho T = \rho_a T_a$, but in this approximation, we will no more impose that the perfect gas relation is satisfied when pursuing the analysis. So the solution of (KA) will not be satisfied with p constant as we could have thought by combining previous equations.

5.3 Stationary Laminar Solution

Both systems are studied around a stationary laminar (independent of $\tilde{y}$ and t) solution of the equations.

For the system (RC), we are given an arbitrary convective velocity $\mathbf{U_0}$, and for the system (KA) it is deduced from the energy equation. In both cases a reference length L_0 plays an important role (for defining in which asymptotic regime we are).

For the system of Rayleigh with convection,

$$\mathbf{U}_0(\tilde{x}) = (\tilde{u}_0(\tilde{x}), 0) \ ,$$

with

$$\tilde{u}_0(\tilde{x}) = u_0(\frac{\tilde{x}}{L_0}) \,.$$

For the ablation front model,

$$L_0 = \kappa \frac{T_a^{\nu+1}}{C_p \rho_a V_a} \,.$$

We use the rescaled variable

$$x := \frac{\tilde{x}}{L_0} \,.$$

The stationary laminar solution is given by

$$(\tilde{x}, \tilde{y}) \mapsto (\tilde{\rho}_0(\tilde{x}), \tilde{u}_0(\tilde{x}), 0, \tilde{p}_0(\tilde{x}))$$

with

$$\tilde{\rho}_0(\tilde{x}) = \rho_0(\tfrac{\tilde{x}}{L_0}) \,,\, \tilde{p}_0(\tilde{x}) = p_0(\tfrac{\tilde{x}}{L_0}) \,.$$

Here ρ_0, u_0, p_0 are functions on $\mathbb{R}$

$$\begin{cases} \rho_0(x) u_0(x) = -\rho_a V_a \,, \\ \frac{d}{dx}\big(\rho_0(x)\, u_0(x)^2 \,+\, p_0(x)\big) = -\rho_0(x) g L_0 \,. \end{cases}$$

Note that p_0 is determined modulo a constant C_0 by:

$$\rho_0(x)\, u_0(x)^2 \,+\, p_0(x) = -g L_0 \int_0^x \rho_0(t) dt + C_0 \,.$$

Finally, we introduce the adimensionalized density profile $\varrho(x)$ which is the function

$$\varrho(x) = \frac{\rho_0(x)}{\rho_a} \,.$$

5.4 From the Physical Parameters to the Relevant Mathematical Parameters

Following [CCLaRa], we can now associate with the physical parameters, g, L_0, V_a, k, the parameters

$$\alpha = \frac{\sqrt{gk} L_0}{V_a} \,,\, \beta = V_a \sqrt{\frac{k}{g}} \,,$$

and the relevant constants of this study (the constant σ_c stands for the Rayleigh with convection model and the constant σ_a is characteristic of the ablation front model)

$$h = \frac{1}{kL_0} = \frac{1}{\alpha\beta} \ , \ \sigma_c = \frac{h^{\frac{1}{2}}}{\beta} \ , \ \sigma_a = \frac{h^2}{\beta^2} \ .$$

These constants are linked to the reduced wave number

$$\varepsilon = kL_0 \ ,$$

and the Froude number,

$$F_r = \frac{V_a^2}{gL_0} \ .$$

They are linked to α and β through

$$F_r = \frac{\beta}{\alpha} \ , \ \varepsilon = \alpha\beta \ .$$

From the growth rate $\bar{\gamma}$, we deduce two dimensionless growth rates

$$\gamma = \frac{\bar{\gamma}}{\sqrt{gk}} \ ,$$

and

$$\Gamma = \frac{\bar{\gamma}}{kV_a} = \frac{\gamma}{\beta} \ . \tag{86}$$

The growth rate γ is the growth rate generally used in the classical Rayleigh–Taylor analysis, and the growth rate Γ is the one relevant in the semiclassical regime, that we study here.

As a conclusion, *Semi-classical analysis can be applied when the Froude Number is small enough.*

5.5 The Convection Velocity Model

In our rescaled variable x, the linearized system writes (with $q_4 = r_4\varrho - q_1$):

$$(LRC)\begin{cases} \frac{dq_1}{dx} + \alpha\gamma(\varrho^2 r_4 - \varrho q_1) - \alpha\beta\varrho q_3 = 0 \ , \\ \frac{dq_2}{dx} + \alpha\gamma q_1 + \alpha\beta q_3 + \frac{\alpha}{\beta}(\varrho^2 r_4 - \varrho q_1) = 0 \ , \\ \frac{dq_3}{dx} - \alpha\beta(q_2 + \frac{2q_1+q_4}{\varrho}) - \alpha\gamma\varrho q_3 = 0 \ , \\ \frac{dr_4}{dx} - \alpha\beta q_3 = 0 \ . \end{cases}$$

Here (q_1, q_2, q_3, q_4) correspond to infinitesimal variation of the new unknowns $(\rho u, \rho u^2 + p, \rho uv, u)$.

This system rewrites, with $d_h = h\frac{d}{dx}$,

$$d_h \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ r_4 \end{pmatrix} + \begin{pmatrix} -\Gamma\varrho & 0 & -\varrho & \Gamma\varrho^2 \\ \Gamma - \frac{\varrho}{\beta^2} & 0 & 1 & \frac{\varrho^2}{\beta^2} \\ -\frac{1}{\varrho} & -1 & -\Gamma\varrho & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ r_4 \end{pmatrix} = 0 \,.$$

The main point is now that we can reduce the analysis of the system to the analysis of one equation.

Proposition 5.1. *The C^4-valued function (q_1, q_2, q_3, r_4) is a solution of the linearized system (LRC) if and only if $q_4 := r_4\varrho - q_1$ belongs to the kernel of the operator (ELRC),*

$$\begin{aligned} &\mathcal{P}_c(x, \tfrac{1}{i}h\tfrac{d}{dx}, h, \sigma_c, \Gamma) \\ &:= d_h[(d_h - \Gamma\varrho)(d_h(\tfrac{1}{\varrho'}(d_h - \Gamma\varrho))) - \tfrac{2}{\varrho'}(d_h - \Gamma\varrho) + \tfrac{h}{\varrho}] \\ &\quad +\sigma_c^2\varrho + d_h(\tfrac{1}{\varrho'}(d_h - \Gamma\varrho)) + \Gamma(\tfrac{\varrho}{\varrho'}(d_h - \Gamma\varrho) - h) \,. \end{aligned}$$

Here the interesting point is that we have only two effective parameters (h, σ_c) which will make the discussion about various asymptotic regimes easier. The semi-classical regime will correspond to fix $\sigma_c > 0$ and to analyze the question when $h \to 0$.

The semi-classical principal symbol is

$$(x, \xi) \mapsto \mathcal{P}_c^0(x, \xi) := -\frac{1}{\varrho'}(i\xi - \Gamma\varrho)^2(\xi^2 + 1) + \varrho\sigma_c^2 \,. \tag{87}$$

Assumption 1
The profile ϱ satisfies:

1. $\varrho \in C^\infty(\mathbb{R};]0, 1[)$
2. $\lim_{x\to-\infty} \varrho(x) = \varrho_- \geq 0$
3. $\lim_{x\to+\infty} \varrho(x) = \varrho_+ = 1$
4. $\varrho' > 0$
5. $\lim_{|x|\to+\infty} \frac{\varrho'(x)}{\varrho(x)} = 0$

Remark 5.1. The reader should be aware that, in comparison with the two first models, we have changed the convention in order to be coherent to the reference [HelLaf2] in which the reader can find additional details.

Assumption 2
The maximum of $\frac{\varrho'}{\varrho}$ is attained at a unique x_{max}:

$$0 < \frac{\varrho'}{\varrho}(x_{max}) := (\vartheta_c^{max})^2 \,,$$

and the map $x \mapsto \frac{\varrho'(x)}{\varrho(x)}$ is strictly increasing over $]-\infty, x_{max}[$ and then strictly decreasing over $]x_{max}, +\infty[$.

Local Ellipticity Condition
The imaginary part of the symbol is

$$\operatorname{Im}\mathcal{P}_c^0(x,\xi) = \frac{2\xi}{\varrho'(x)}\, \Gamma\, \varrho(x)(\xi^2+1)\ .$$

It is non zero except for

$$\xi = 0\ .$$

Looking at the real part restricted to $\xi = 0$, we obtain that

$$\operatorname{Re}\mathcal{P}_c^0(x,0) = -\Gamma^2 \frac{\varrho^2(x)}{\varrho'(x)} + \varrho(x)\sigma_c^2\ .$$

This leads us to the following local ellipticity condition:

$$\frac{\Gamma}{\sigma_c} > \vartheta_c^{max}\ .$$

5.6 The Model for the Ablation Regime

Similarly, the linearization of the system (KA) leads to the following system

$$(LKA) \quad d_h\mathbf{q} + M_0(\varrho(x))\mathbf{q} = 0\ ,$$

where

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_4 \\ q_5 \end{pmatrix}.$$

and the matrix is

$$M_0(\varrho) = \begin{pmatrix} 0 & 0 & \varrho & h\Gamma\varrho^{\nu+2} & 0 \\ \Gamma & 0 & -1 & \frac{h}{\beta^2}\varrho^{\nu+2} & 0 \\ \frac{2}{\varrho} & 1 & -\Gamma\varrho & h\varrho^{\nu} & 0 \\ \frac{1}{\varrho} & 0 & 0 & h\varrho^{\nu} & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

Proposition 5.2. *The C^5-valued function $\mathbf{q}$ is a solution of (LKA) if and only if its fourth component p_4 is in the kernel of the operator (ELKA):*

$$
\begin{aligned}
&\mathcal{P}_a(x, \tfrac{1}{i} d_h, h, \sigma_a, \Gamma) := \\
&\qquad \big[d_h(d_h - \Gamma\varrho)d_h - (d_h - \Gamma\varrho)\big] \times \\
&\qquad \times \tfrac{\varrho}{\varrho'} \big[d_h(d_h + h\varrho^\nu) - 1 - h\Gamma\varrho^{\nu+1}\big] \\
&\qquad + h\big(d_h\,(d_h - \Gamma\varrho)\,(d_h(d_h + h\varrho^\nu) - 1)\big) \\
&\qquad + h(d_h^2 - 1) + \sigma_a\,\varrho^{\nu+2}\,.
\end{aligned}
$$

The principal symbol (in the semi-classical sense) is

$$
\mathcal{P}_a^0(x, \xi, \sigma_a, \Gamma) = \frac{\varrho(x)}{\varrho'(x)}(i\xi - \Gamma\varrho(x))(\xi^2 + 1)^2 + \sigma_a \varrho(x)^{\nu+2}\,. \tag{88}
$$

The analysis of the zeroes of the symbol is similar to the other model. We have:

$$
\operatorname{Re}\mathcal{P}_a^0(x, \xi, \sigma_a, \Gamma) = \frac{\varrho(x)}{\varrho'(x)}(-\Gamma\varrho(x))(\xi^2 + 1)^2 + \sigma_a \varrho(x)^{\nu+2}\,,
$$

and

$$
\operatorname{Im}\mathcal{P}_a^0(x, \xi, \sigma_a, \Gamma) = \frac{\varrho(x)}{\varrho'(x)}\xi(\xi^2 + 1)^2\,.
$$

The zero set of $\operatorname{Im}\mathcal{P}_a^0$ is in $\{\xi = 0\}$ and:

$$
\operatorname{Re}\mathcal{P}_a^0(x, 0, \sigma_a, \Gamma) = \frac{\varrho(x)}{\varrho'(x)}(-\Gamma\varrho(x)) + \sigma_a \varrho(x)^{\nu+2}\,,
$$

which leads to the analysis of the solutions of:

$$
\sigma_a \varrho(x)^\nu \varrho'(x) = \Gamma
$$

or

$$
\sigma_a \varrho(x)^{2\nu+1}(1 - \varrho(x)) = \Gamma\,.
$$

Hence we have first to analyze the variation of the function:

$$
[0, 1] \ni t \mapsto \theta(t) := (1 - t)t^{2\nu+1}\,. \tag{89}
$$

If $\nu > 0$, θ is an application from $]0, 1[$ onto $]0, \vartheta_a^{max}]$, with

$$
\vartheta_a^{max} = \frac{(2\nu + 1)^{2\nu+1}}{(2\nu + 2)^{2\nu+2}}\,. \tag{90}
$$

$$
0 < \vartheta_a^{max} < 1\,,
$$

and the maximum in $]0, 1[$ is obtained at

$$
t_a^{max} = \frac{2\nu + 1}{2\nu + 2}\,.
$$

For $L \in]0, \vartheta_a^{max}[$, two solutions of $\theta(t) = L$, satisfying:

$$0 < t_-(L) < t_a^{max} < t_+(L) \ .$$

$x \mapsto \varrho(x)$ is a bijection of $\mathbb{R}$ onto $]0,1[$.

For any $L \in]0, \vartheta_a^{max}[$, there exist two points $x_\pm(L)$ such that

$$\varrho(x_\pm(L)) = t_\pm(L) \ ,$$

and consequently

$$\theta(\varrho(x_\pm(L))) = L \ .$$

We note also that, when $\xi = 0$,

$$(\partial \mathcal{P}_a^0 / \partial \xi)(x, 0) = i \frac{\varrho(x)}{\varrho'(x)} \neq 0 \ ,$$

which shows that $\mathcal{P}_a^0$ is also of principal type.
Finally when $\frac{\Gamma}{\sigma_a} > \vartheta_a^{max}$, is satisfied, one gets the local ellipticity of the symbol $\mathcal{P}_a^0$.

5.7 Semi-Classical Regimes for the Ablation Models

Let us emphasize at this stage the analogies between the three last physical models. As in the case of the Kelvin–Helmholtz model, two different "effective" parameters have been exhibited corresponding to each situation of the convective velocity problem (parameter denoted by σ_c) and in the ablation front problem (parameter denoted by σ_a), together with h. Both problems lead to a h-differential equation on one of the unknowns, and consist in finding a function $u(x,h)$ such that

$$\mathcal{P}_p(x, \frac{1}{i} h \frac{d}{dx}, h, \sigma_p, \Gamma) u = 0 \ ,$$

where $\mathcal{P}_p$ is a fifth or fourth order h-differential operator. The main results will take the following form:

Under suitable relations on the reference density profile at $\tilde{x} \to \pm\infty$, then, if

$$\Gamma \in]0, \vartheta_p^{max} \sigma_p[\ ,$$

then 0 *belongs to the h-family-pseudospectrum of*

$$\mathcal{P}_p(x, \frac{1}{i} h \frac{d}{dx}, h, \sigma_p, \Gamma) \ .$$

More precisely there exists $x_p(\Gamma, \sigma_p)$ such that there exists a WKB solution of

$$\mathcal{P}_p u = \mathcal{O}(h^\infty)$$

localized in the neighborhood of the point $x_p(\Gamma, \sigma_p)$.

Note that in the three models there is no quantization of Γ. The result is with this respect quite different from the solution of the problem linked with pure Rayleigh–Taylor instability.

The assumptions are essentially optimal in this semi-classical regime: *Under the same assumptions on the density profile, and, for $\Gamma > \vartheta_p^{max} \sigma_p$, no approximate (in the WKB sense) bounded solution can be constructed, if h is small enough.*

This was a consequence of the ellipticity of the operator for this regime of operators. Let us now look at what is obtained by application of Theorem 4.1.

5.7.1 Application to the (ELRC) Model

We start from $a_0 = \mathcal{Q}_c^0$:

$$a_0(x, \xi) = (\xi + i\Gamma\varrho)^2(\xi^2 + 1) + \varrho\varrho'\sigma_c^2 \; .$$

We obtain

$$\operatorname{Re} a_0(x, \xi) = (\xi^2 - \Gamma^2\varrho^2)(\xi^2 + 1) + \varrho\varrho'\sigma_c^2 \; ,$$

and

$$\operatorname{Im} a_0(x, \xi) = 2\Gamma\varrho\xi(\xi^2 + 1) \; .$$

Let us compute the Poisson bracket at $(x_c, 0)$

$$\begin{aligned} &\{\operatorname{Re} a_0, \operatorname{Im} a_0\}(x_c, 0) \\ &\quad = -2\Gamma\varrho(x_c)[-2\Gamma^2\varrho(x_c)\varrho'(x_c) + \sigma_c^2(\varrho\varrho')'(x_c)] \; , \end{aligned}$$

which is effectively strictly negative and Theorem 4.1 can be applied.

5.7.2 Application to the (ELKA) Model

The principal symbol is here:

$$\mathcal{P}_a^0(x, \xi) = \frac{\varrho(x)}{\varrho'(x)}(i\xi - \Gamma\varrho(x))(\xi^2 + 1)^2 + \sigma_a\varrho^{\nu+2} \; . \tag{91}$$

Because we are interested in null solutions, it is equivalent to apply the criterion for

$$a_0(x, \xi) = (i\xi - \Gamma\varrho(x))(\xi^2 + 1)^2 + \sigma_a\varrho(x)^{2\nu+2}(1 - \varrho(x)) \; .$$

We get

$$\mathrm{Re}\ a_0 = -\Gamma\varrho(x)(\xi^2+1)^2 + \sigma_a\varrho(x)^{2\nu+2}(1-\varrho(x))\ ,$$

and

$$\mathrm{Im}\ a_0 = \xi(\xi^2+1)^2\ .$$

A point in $a_0^{-1}(0)$ should satisfy $\xi = 0$, and for the real part:

$$-\Gamma\varrho(x_0) + \sigma_a\varrho(x_0)^{2\nu+1}(1-\varrho(x_0)) = 0\ .$$

Let us compute the Poisson bracket at $(x_0, 0)$:

$$\begin{aligned}\{\mathrm{Re}\ a_0, \mathrm{Im}\ a_0\}(x_0,\xi_0) = \Gamma\varrho'(x_0)&\\ -\sigma_a(2\nu+2)\varrho'(x_0)\varrho^{2\nu+1}(x_0)&\\ +\sigma_a(2\nu+3)\varrho'(x_0)\varrho^{2\nu+2}(x_0)\ .&\end{aligned}$$

Dividing by $\varrho'(x_0)$ (which is positive), we get that this bracket is negative if:

$$\begin{aligned}\frac{\Gamma}{\sigma_a} &< (2\nu+2)\varrho^{2\nu+1}(x_0)\ -\ (2\nu+3)\varrho^{2\nu+2}(x_0)\\ &= \varrho^{2\nu+1}(x_0)\left((2\nu+2)-(2\nu+3)\varrho(x_0)\right)\ .\end{aligned} \tag{92}$$

Hence Theorem 4.1 can be applied if this last condition is verified.

5.8 Subellipticity II: At the Boundary of $\Sigma(a_0)$

In the case of our example the neighborhood of the maximal Γ, for which one can construct quasimodes can be analyzed by analyzing the iterated brackets. One can then apply the results, which were recalled in [DeSjZw] which are related to the much older theory of the subelliptic operators (see [Ho3] and references therein). More recent work have been performed by N. Lerner (See his lectures in this conference) and by K. Pravda-Starov in his quite recent PhD [Pra2].

The theorem in [DeSjZw] reads:

Theorem 5.1. *We assume that a_0 is a C^∞ bounded function together with all its derivatives and that our operator is an h-pseudodifferential operator with principal symbol $(x,\xi)\mapsto a_0(x,\xi)$. Then if $z_0\in\partial\Sigma(a_0)$ is of finite type for a_0 of order $k\geq 1$, then k is even and there exists $C>0$ such that, for h small enough,*

$$||(A(h)-z_0)^{-1}|| \leq C\, h^{-\frac{k}{k+1}}\ . \tag{93}$$

Here $\Sigma(a_0)$ is the closure of the numerical range of a_0.

The condition that a_0 is of finite type for the value z_0 is that a_0 is of principal type (i.e. $\nabla_{x,\xi}a_0(x,\xi)\neq 0$) at any point (x,ξ) such that $a_0(x,\xi) = z_0$ and that at these points there is at least one non zero (possibly iterated) bracket of $\mathrm{Re}\, a_0$ and $\mathrm{Im}\, a_0$.

Remarks 5.2

- *The authors in [DeSjZw] mention that one can reduce more general cases to this one by use of the functional calculus. This can be verified more directly in our case.*
- *In the case of (ELRC), it is enough to compose on the left by $(I - h^2\Delta)^{-2}$. In the second case, the situation is a little more delicate. See [HelLaf2].*

Let us show how this theorem can be applied in this case, with $k = 2$.

Application to (ELRC) Model
Coming back to this model, we first observe that

$$\{\mathrm{Re}\, a_0\, , \mathrm{Im}\, a_0\}(x, \xi) = -2\Gamma\varrho[-2\Gamma^2\varrho\varrho' + \sigma_c^2(\varrho\varrho')'] + \mathcal{O}(\xi^2)\ , \tag{94}$$

When

$$\Gamma = \Gamma_c := \vartheta_c^{max}\sigma_c\ , \tag{95}$$

we can verify that

$$a_0(x_c, 0) = 0\ ,\ \{\mathrm{Re}\, a_0\, , \mathrm{Im}\, a_0\}(x_c, 0) = 0\ ,$$

and that, under the additional assumption that the point x_c is a non degenerate maximum of $\frac{\varrho'}{\varrho}$,

$$\{\mathrm{Im}\, a_0, \{\mathrm{Re}\, a_0, \mathrm{Im}\, a_0\}\}(x_0, 0) \neq 0\ . \tag{96}$$

This implies that the operator is of type 2.

Application to the (ELKA) Model
We consider, after a small change, as principal symbol the function:

$$(x, \xi) \mapsto -\Gamma\varrho(x) + \sigma_a\varrho(x)^{2\nu+2}(1 - \varrho(x))(1 + \xi^2)^{-2} + i\xi\ . \tag{97}$$

Here we choose $\Gamma/\sigma_a = \vartheta_a^{max}$, where ϑ_a^{max} is defined in (90). The Poisson bracket $\{\mathrm{Re}\, a_0, \mathrm{Im}\, a_0\}$ vanishes at $(x_0, 0)$, where x_0 is the point such as $\varrho(x_0) = \frac{2\nu+1}{2\nu+2}$. Now the computation of the first iterated bracket gives

$$\begin{aligned} &\{\mathrm{Im}\, a_0\, , \{\mathrm{Im}\, a_0\, , \mathrm{Re}\, a_0\}\}(x_0, 0) \\ &\quad = (2\nu + 1)\varrho'(x_0)^2\varrho(x_0)^{2\nu} \neq 0\ . \end{aligned} \tag{98}$$

As in the case of the ellipticity zone, one can eliminate the problem at ∞.

Remark 5.2. The Dencker–Sjöstrand–Zworski Theorem shows that there exists $C > 0$ and h_0 such that, when Γ belongs to $]\Gamma_p - Ch^{\frac{2}{3}}, \Gamma_p]$ and $h \in]0, h_0]$, then no approximate solution in the kernel of $\mathcal{P}_p(x, \frac{1}{i}\, d_h, h, \sigma_p, \Gamma)$ exists.

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