Preface

This book comprises 50 class-tested lectures which both the authors have given to engineering and mathematics major students under the titles *Boundary Value Problems* and *Methods of Mathematical Physics* at various institutions all over the globe over a period of almost 35 years. The main topics covered in these lectures are power series solutions, special functions, boundary value problems for ordinary differential equations, Sturm–Liouville problems, regular and singular perturbation techniques, Fourier series expansion, partial differential equations, Fourier series solutions to initial-boundary value problems, and Fourier and Laplace transform techniques. The prerequisite for this book is calculus, so it can be used for a senior undergraduate course. It should also be suitable for a beginning graduate course because, in undergraduate courses, students do not have any exposure to various intricate concepts, perhaps due to an inadequate level of mathematical sophistication. The content in a particular lecture, together with the problems therein, provides fairly adequate coverage of the topic under study. These lectures have been delivered in one year courses and provide flexibility in the choice of material for a particular one-semester course. Throughout this book, the mathematical concepts have been explained very carefully in the simplest possible terms, and illustrated by a number of complete workout examples. Like any other mathematical book, it does contain some theorems and their proofs.

A detailed description of the topics covered in this book is as follows: In Lecture 1 we find explicit solutions of the first-order linear differential equations with variable coefficients, second-order homogeneous differential equations with constant coefficients, and second-order Cauchy–Euler differential equations. In Lecture 2 we show that if one solution of the homogeneous second-order differential equation with variable coefficients is known, then its second solution can be obtained rather easily. Here we also demonstrate the method of variation of parameters to construct the solutions of nonhomogeneous second-order differential equations.

In Lecture 3 we provide some basic concepts which are required to construct power series solutions to differential equations with variable coefficients. Here through various examples we also explain ordinary, regular singular, and irregular singular points of a given differential equation. In Lecture 4 first we prove a theorem which provides sufficient conditions so that the solutions of second-order linear differential equations can be expressed as power series at an ordinary point, and then construct power series solutions of Airy, Hermite, and Chebyshev differential equations. These equations occupy a central position in mathematical physics, engineering, and approximation theory. In Lectures 5 and 6 we demonstrate the method of Frobenius to construct the power series solutions of second-order linear differential equations at a regular singular point. Here we prove a general result which provides three possible different forms of the power series solution. We illustrate this result through several examples, including Laguerre's equation, which arises in quantum mechanics. In Lecture 7 we study Legendre's differential equation, which arises in problems such as the flow of an ideal fluid past a sphere, the determination of the electric field due to a charged sphere, and the determination of the temperature distribution in a sphere given its surface temperature. Here we also develop the polynomial solution of the Legendre differential equation. In Lecture 8 we study polynomial solutions of the Chebyshev, Hermite, and Laguerre differential equations. In Lecture 9 we construct series solutions of Bessel's differential equation, which first appeared in the works of Euler and Bernoulli. Since many problems of mathematical physics reduce to the Bessel equation, we investigate it in somewhat more detail. In Lecture 10 we develop series solutions of the hypergeometric differential equation, which finds applications in several problems of mathematical physics, quantum mechanics, and fluid dynamics.

Mathematical problems describing real world situations often have solutions which are not even continuous. Thus, to analyze such problems we need to work in a set which is bigger than the set of continuous functions. In Lecture 11 we introduce the sets of piecewise continuous and piecewise smooth functions, which are quite adequate to deal with a wide variety of applied problems. Here we also define periodic functions, and introduce even and odd extensions. In Lectures 12 and 13 we introduce orthogonality of functions and show that the Legendre, Chebyshev, Hermite, and Laguerre polynomials and Bessel functions are orthogonal. Here we also prove some fundamental properties about the zeros of orthogonal polynomials.

In Lecture 14 we introduce boundary value problems for second-order ordinary differential equations and provide a necessary and sufficient condition for the existence and uniqueness of their solutions. In Lecture 15 we formulate some boundary value problems with engineering applications, and show that often solutions of these problems can be written in terms of Bessel functions. In Lecture 16 we introduce Green's functions of homogeneous boundary value problems and show that the solution of a given nonhomogeneous boundary value problem can be explicitly expressed in terms of Green's function of the corresponding homogeneous equation.

In Lecture 17 we discuss the regular perturbation technique which relates the unknown solution of a given initial value problem to the known solutions of the infinite initial value problems. In many practical problems one often meets cases where the methods of regular perturbations cannot be applied. In the literature such problems are known as singular perturbation problems. In Lecture 18 we explain the methodology of singular perturbation technique with the help of some examples.

If the coefficients of the homogeneous differential equation and/or of the boundary conditions depend on a parameter, then one of the pioneer problems of mathematical physics is to determine the values of the parameter (eigenvalues) for which nontrivial solutions (eigenfunctions) exist. In Lecture 19 we explain some of the essential ideas involved in this vast field, which is continuously growing.

In Lectures 20 and 21 we show that the sets of orthogonal polynomials and functions we have provided in earlier lectures can be used effectively as the basis in the expansions of general functions. This in particular leads to Fourier's cosine, sine, trigonometric, Legendre, Chebyshev, Hermite and Bessel series. In Lectures 22 and 23 we examine pointwise convergence, uniform convergence, and the convergence in the mean of the Fourier series of a given function. Here the importance of Bessel's inequality and Parseval's equality are also discussed. In Lecture 24 we use Fourier series expansions to find periodic particular solutions of nonhomogeneous differential equations, and solutions of nonhomogeneous self-adjoint differential equations satisfying homogeneous boundary conditions, which leads to the well-known Fredholm's alternative.

In Lecture 25 we introduce partial differential equations and explain several concepts through elementary examples. Here we also provide the most fundamental classification of second-order linear equations in two independent variables. In Lecture 26 we study simultaneous differential equations, which play an important role in the theory of partial differential equations. Then we consider quasilinear partial differential equations of the Lagrange type and show that such equations can be solved rather easily, provided we can find solutions of related simultaneous differential equations. Finally, we explain a general method to find solutions of nonlinear first-order partial differential equations which is due to Charpit. In Lecture 27 we show that like ordinary differential equations, partial differential equations with constant coefficients can be solved explicitly. We begin with homogeneous second-order differential equations involving only second-order terms, and then show how the operator method can be used to solve some particular nonhomogeneous differential equations. Then, we extend the method to general second and higher order partial differential equations. In Lecture 28 we show that coordinate transformations can be employed successfully to reduce second-order linear partial differential equations to some standard forms, which are known as canonical forms. These transformed equations sometimes can be solved rather easily. Here the concept of characteristic of second-order partial differential equations plays an important role.

The method of separation of variables involves a solution which breaks up into a product of functions each of which contains only one of the variables. This widely used method for finding solutions of linear homogeneous partial differential equations we explain through several simple examples in Lecture 29. In Lecture 30 we derive the one-dimensional heat equation and formulate initial-boundary value problems, which involve the heat equation, the initial condition, and homogeneous and nonhomogeneous boundary conditions. Then we use the method of separation of variables to find the Fourier series solutions to these problems. In Lecture 31 we construct the Fourier series solution of the heat equation with Robin's boundary conditions. In Lecture 32 we provide two different derivations of the one-dimensional wave equation, formulate an initial-boundary value problem, and find its Fourier series solution. In Lecture 33 we continue using the method of separation of variables to find Fourier series solutions to some other initial-boundary value problems related to one-dimensional wave equation. In Lecture 34 we give a derivation of the two-dimensional Laplace equation, formulate the Dirichlet problem on a rectangle, and find its Fourier series solution. In Lecture 35 we discuss the steady-state heat flow problem in a disk. For this, we consider the Laplace equation in polar coordinates and find its Fourier series solution. In Lecture 36 we use the method of separation of variables to find the temperature distribution of rectangular and circular plates in the transient state. Again using the method of separation of variables, in Lecture 37 we find vertical displacements of thin membranes occupying rectangular and circular regions. The three-dimensional Laplace equation occurs in problems such as gravitation, steady-state temperature, electrostatic potential, magnetostatics, fluid flow, and so on. In Lecture 38 we find the Fourier series solution of the Laplace equation in a three-dimensional box and in a circular cylinder. In Lecture 39 we use the method of separation of variables to find the Fourier series solutions of the Laplace equation in and outside a given sphere. Here, we also discuss briefly Poisson's integral formulas. In Lecture 40 we demonstrate how the method of separation of variables can be employed to solve nonhomogeneous problems.

The Fourier integral is a natural extension of Fourier trigonometric series in the sense that it represents a piecewise smooth function whose domain is semi-infinite or infinite. In Lecture 41 we develop the Fourier integral with an intuitive approach and then discuss Fourier cosine and sine integrals which are extensions of Fourier cosine and sine series, respectively. This leads to Fourier cosine and sine transform pairs. In Lecture 42 we introduce the complex Fourier integral and the Fourier transform pair and find the Fourier transform of the derivative of a function. Then, we state and prove the Fourier convolution theorem, which is an important result. In Lectures 43 and 44 we consider problems in infinite domains which can be effectively solved by finding the Fourier transform, or the Fourier sine or cosine transform of the unknown function. For such problems usually the method of separation of variables does not work because the Fourier series are not adequate to yield complete solutions. We illustrate the method by considering several examples, and obtain the famous Gauss–Weierstrass, d'Alembert's, and Poisson's integral formulas.

In Lecture 45 we introduce some basic concepts of Laplace transform theory, whereas in Lecture 46 we prove several theorems which facilitate the computation of Laplace transforms. The method of Laplace transforms has the advantage of directly giving the solutions of differential equations with given initial and boundary conditions without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready table of Laplace transforms reduces the problem of solving differential equations to mere algebraic manipulations. In Lectures 47 and 48 we employ the Laplace transform technique to find solutions of ordinary and partial differential equations, respectively. Here we also develop the famous Duhamel's formula.

A given problem consisting of a partial differential equation in a domain with a set of initial and/or boundary conditions is said to be well-posed if it has a unique solution which is stable. In Lecture 49 we demonstrate that problems considered in earlier lectures are well-posed. Finally, in Lecture 50 we prove a few theorems which verify that the series or integral form of the solutions we have obtained in earlier lectures are actually the solutions of the problems considered.

Two types of exercises are included in the book, those which illustrate the general theory, and others designed to fill out text material. These exercises form an integral part of the book, and every reader is urged to attempt most, if not all of them. For the convenience of the reader we have provided answers or hints to almost all the exercises.

In writing a book of this nature no originality can be claimed, only a humble attempt has been made to present the subject as simply, clearly, and accurately as possible. It is earnestly hoped that *Ordinary and Partial Differential Equations* will serve an inquisitive reader as a starting point in this rich, vast, and ever-expanding field of knowledge.

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Lecture 2
Second-Order Differential Equations

Generally, second-order differential equations with variable coefficients cannot be solved in terms of the known functions. In this lecture we shall show that if one solution of the homogeneous equation is known, then its second solution can be obtained rather easily. Further, by employing the method of variation of parameters, the general solution of the nonhomogeneous equation can be constructed provided two solutions of the corresponding homogeneous equation are known.

Homogeneous equations. For the homogeneous linear DE of second-order with variable coefficients

$$y'' + p_1(x)y' + p_2(x)y = 0, \tag{2.1}$$

where $p_1(x)$ and $p_2(x)$ are continuous in J, there does not exist any method to solve it. However, the following results are well-known.

Theorem 2.1. There exist exactly two solutions $y_1(x)$ and $y_2(x)$ of (2.1) which are linearly independent (essentially different) in J, i.e., there does not exist a constant c such that $y_1(x) = cy_2(x)$ for all $x \in J$.

Theorem 2.2. Two solutions $y_1(x)$ and $y_2(x)$ of (2.1) are linearly independent in J if and only if their *Wronskian* defined by

$$W(x) = W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \tag{2.2}$$

is different from zero for some $x = x_0$ in J.

Theorem 2.3. For the Wronskian defined in (2.2) the following Abel's identity holds:

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p_1(t)dt\right), \quad x_0 \in J. \tag{2.3}$$

Thus, if Wronskian is zero at some $x_0 \in J$, then it is zero for all $x \in J$.

Theorem 2.4. If $y_1(x)$ and $y_2(x)$ are solutions of (2.1) and c_1 and c_2 are arbitrary constants, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of (2.1).

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Further, if $y_1(x)$ and $y_2(x)$ are linearly independent, then any solution $y(x)$ of (2.1) can be written as $y(x) = \overline{c}_1 y_1(x) + \overline{c}_2 y_2(x)$, where $\overline{c}_1$ and $\overline{c}_2$ are suitable constants.

Now we shall show that, if one solution $y_1(x)$ of (2.1) is known (by some clever method) then we can employ *variation of parameters* to find the second solution of (2.1). For this, we let $y(x) = u(x)y_1(x)$ and substitute this in (2.1), to get

$$(uy_1)'' + p_1(uy_1)' + p_2(uy_1) = 0,$$

or

$$u''y_1 + 2u'y_1' + uy_1'' + p_1u'y_1 + p_1uy_1' + p_2uy_1 = 0,$$

or

$$u''y_1 + (2y_1' + p_1y_1)u' + (y_1'' + p_1y_1' + p_2y_1)u = 0.$$

However, since y_1 is a solution of (2.1), the above equation with $v = u'$ is the same as

$$y_1v' + (2y_1' + p_1y_1)v = 0, \tag{2.4}$$

which is a first-order equation, and it can be solved easily provided $y_1 \neq 0$ in J. Indeed, multiplying (2.4) by y_1, we find

$$(y_1^2v' + 2y_1'y_1v) + p_1y_1^2v = 0,$$

which is the same as

$$(y_1^2v)' + p_1(y_1^2v) = 0;$$

and hence

$$y_1^2v = c\exp\left(-\int^x p_1(t)dt\right),$$

or, on taking $c = 1$,

$$v(x) = \frac{1}{y_1^2(x)}\exp\left(-\int^x p_1(t)dt\right).$$

Hence, the second solution of (2.1) is

$$y_2(x) = y_1(x)\int^x \frac{1}{y_1^2(t)}\exp\left(-\int^t p_1(s)ds\right)dt. \tag{2.5}$$

Example 2.1. It is easy to verify that $y_1(x) = x^2$ is a solution of the DE

$$x^2y'' - 2xy' + 2y = 0, \quad x \neq 0.$$

For the second solution we use (2.5), to obtain

$$y_2(x) = x^2\int^x \frac{1}{t^4}\exp\left(-\int^t \left(-\frac{2s}{s^2}\right)ds\right)dt = x^2\int^x \frac{1}{t^4}t^2dt = -x.$$

We note that the substitution $w = y'/y$ converts (2.1) into a first-order nonlinear DE

$$w' + p_1(x)w + p_2(x) + w^2 = 0. \tag{2.6}$$

This DE is called *Riccati's equation.* In general it is not integrable, but if a particular solution, say, $w_1(x)$ is known, then by the substitution $z = w - w_1(x)$ it can be reduced to Bernoulli's equation (see Problem 1.6). In fact, we have

$$z' + w_1'(x) + p_1(x)(z + w_1(x)) + p_2(x) + (z + w_1(x))^2 = 0,$$

which is the same as

$$z' + (p_1(x) + 2w_1(x))z + z^2 = 0. \tag{2.7}$$

Since this equation can be solved easily to obtain $z(x)$, the solution of (2.6) takes the form $w(x) = w_1(x) + z(x)$.

Example 2.2. It is easy to verify that $w_1(x) = x$ is a particular solution of the Riccati equation

$$w' = 1 + x^2 - 2xw + w^2.$$

The substitution $z = w - x$ in this equation gives the Bernoulli equation

$$z' = z^2,$$

whose general solution is $z(x) = 1/(c-x),\ x \neq c$. Thus, the general solution of the given Riccati's equation is $w(x) = x + 1/(c - x),\ x \neq c$.

Nonhomogeneous equations. Now we shall find a particular solution of the nonhomogeneous equation

$$y'' + p_1(x)y' + p_2(x)y = r(x). \tag{2.8}$$

For this also we shall apply the method of *variation of parameters.* Let $y_1(x)$ and $y_2(x)$ be two solutions of (2.1). We assume $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$ is a solution of (2.8). Note that $c_1(x)$ and $c_2(x)$ are two unknown functions, so we can have two sets of conditions which determine $c_1(x)$ and $c_2(x)$. Since

$$y' = c_1y_1' + c_2y_2' + c_1'y_1 + c_2'y_2$$

as a first condition we assume that

$$c_1'y_1 + c_2'y_2 = 0. \tag{2.9}$$

Thus, we have

$$y' = c_1y_1' + c_2y_2'$$

and on differentiation

$$y'' = c_1 y_1'' + c_2 y_2'' + c_1' y_1' + c_2' y_2'.$$

Substituting these in (2.8), we get

$$c_1(y_1'' + p_1 y_1' + p_2 y_1) + c_2(y_2'' + p_1 y_2' + p_2 y_2) + (c_1' y_1' + c_2' y_2') = r(x).$$

Clearly, this equation, in view of $y_1(x)$ and $y_2(x)$ being solutions of (2.1), is the same as

$$c_1' y_1' + c_2' y_2' = r(x). \tag{2.10}$$

Solving (2.9), (2.10), we find

$$c_1' = -\frac{r(x)y_2(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}}, \quad c_2' = \frac{r(x)y_1(x)}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}};$$

and hence a particular solution of (2.8) is

$$\begin{aligned} y_p(x) &= c_1(x)y_1(x) + c_2(x)y_2(x) \\ &= -y_1(x)\int^x \frac{r(t)y_2(t)}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}dt + y_2(x)\int^x \frac{r(t)y_1(t)}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}}dt \\ &= \int^x H(x,t)r(t)dt, \end{aligned} \tag{2.11}$$

where

$$H(x,t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} \Bigg/ \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}. \tag{2.12}$$

Thus, the general solution of (2.8) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x). \tag{2.13}$$

The following properties of the function $H(x,t)$ are immediate:

(i). $H(x,t)$ is defined for all $(x,t) \in J \times J$;

(ii). $\partial^j H(x,t)/\partial x^j,\ j = 0, 1, 2$ are continuous for all $(x,t) \in J \times J$;

(iii). for each fixed $t \in J$ the function $z(x) = H(x,t)$ is a solution of the homogeneous DE (2.1) satisfying $z(t) = 0,\ z'(t) = 1$; and

(iv). the function

$$v(x) = \int_{x_0}^x H(x,t)r(t)dt$$

is a particular solution of the nonhomogeneous DE (2.8) satisfying $y(x_0) = y'(x_0) = 0$.

Example 2.3. Consider the DE

$$y'' + y = \cot x.$$

For the corresponding homogeneous DE $y'' + y = 0$, $\sin x$ and $\cos x$ are solutions. Thus, its general solution can be written as

$$\begin{aligned}
y(x) &= c_1 \sin x + c_2 \cos x + \int^x \frac{\begin{vmatrix} \sin t & \cos t \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix}} \frac{\cos t}{\sin t} dt \\
&= c_1 \sin x + c_2 \cos x - \int^x (\sin t \cos x - \sin x \cos t) \frac{\cos t}{\sin t} dt \\
&= c_1 \sin x + c_2 \cos x - \cos x \sin x + \sin x \int^x \frac{1 - \sin^2 t}{\sin t} dt \\
&= c_1 \sin x + c_2 \cos x - \cos x \sin x - \sin x \int^x \sin t dt + \sin x \int^x \frac{1}{\sin t} dt \\
&= c_1 \sin x + c_2 \cos x + \sin x \int^x \frac{\operatorname{cosec} t(\operatorname{cosec} t - \cot t)}{(\operatorname{cosec} t - \cot t)} dt \\
&= c_1 \sin x + c_2 \cos x + \sin x \ \ln[\operatorname{cosec} x - \cot x].
\end{aligned}$$

Finally, we remark that if the functions $p_1(x)$, $p_2(x)$ and $r(x)$ are continuous on J and $x_0 \in J$, then the DE (2.8) together with the *initial conditions*

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{2.14}$$

has a unique solution. The problem (2.8), (2.14) is called an *initial value problem*. Note that in (2.14) conditions are prescribed at the same point, namely, x_0.

Problems

2.1. Given the solution $y_1(x)$, find the second solution of the following DEs:

(i) $(x^2 - x)y'' + (3x - 1)y' + y = 0 \quad (x \neq 0, 1), \quad y_1(x) = (x - 1)^{-1}$

(ii) $x(x - 2)y'' + 2(x - 1)y' - 2y = 0 \quad (x \neq 0, 2), \quad y_1(x) = (1 - x)$

(iii) $xy'' - y' - 4x^3 y = 0 \quad (x \neq 0), \quad y_1(x) = \exp(x^2)$

(iv) $(1 - x^2)y'' - 2xy' + 2y = 0 \quad (|x| < 1), \quad y_1(x) = x.$

2.2. The differential equation

$$xy'' - (x+n)y' + ny = 0$$

is interesting because it has an exponential solution and a polynomial solution.

(i) Verify that one solution is $y_1(x) = e^x$.

(ii) Show that the second solution has the form $y_2(x) = ce^x \int^x t^n e^{-t} dt$. Further, show that with $c = -1/n!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Note that $y_2(x)$ is the first $n+1$ terms of the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$.

2.3. The differential equation

$$y'' + \delta(xy' + y) = 0$$

occurs in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution. Find its second solution.

2.4. Let $y_1(x) \neq 0$ and $y_2(x)$ be two linearly independent solutions of the DE (2.1). Show that $y(x) = y_2(x)/y_1(x)$ is a nonconstant solution of the DE

$$y_1(x)y'' + (2y_1'(x) + p_1(x)y_1(x))y' = 0.$$

2.5. Let the function $p_1(x)$ be differentiable in J. Show that the substitution $y(x) = z(x)\exp\left(-\frac{1}{2}\int^x p_1(t)dt\right)$ transforms (2.1) to the differential equation

$$z'' + \left(p_2(x) - \frac{1}{2}p_1'(x) - \frac{1}{4}p_1^2(x)\right)z = 0.$$

In particular show that the substitution $y(x) = z(x)/\sqrt{x}$ transforms *Bessel's DE*

$$x^2y'' + xy' + (x^2 - a^2)y = 0, \tag{2.15}$$

where a is a constant (parameter), into a simple DE

$$z'' + \left(1 + \frac{1-4a^2}{4x^2}\right)z = 0. \tag{2.16}$$

2.6. Let $v(x)$ be the solution of the initial value problem

$$y'' + p_1y' + p_2y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

where p_1 and p_2 are constants. Show that the function

$$y(x) = \int_{x_0}^{x} v(x-t)r(t)dt$$

is the solution of the nonhomogeneous DE

$$y'' + p_1 y' + p_2 y = r(x)$$

satisfying $y(x_0) = y'(x_0) = 0$.

2.7. Find general solutions of the following nonhomogeneous DEs:

(i) $y'' + 4y = \sin 2x$
(ii) $y'' + 4y' + 3y = e^{-3x}$
(iii) $y'' + 5y' + 4y = e^{-4x}$.

2.8. Verify that $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions of

$$x^3 y'' + x^2 y' - xy = 0.$$

Use this information and the variation of parameters method to find the general solution of

$$x^3 y'' + x^2 y' - xy = x/(1+x).$$

Answers or Hints

2.1. (i) $\ln x/(x-1)$ (ii) $(1/2)(1-x)\ln[(x-2)/x]-1$ (iii) e^{-x^2} (iv)$(x/2)\times \ln[(1+x)/(1-x)] - 1$.

2.2. (i) Verify directly (ii) Use (2.5).

2.3. $e^{-\delta x^2/2}\int^x e^{\delta t^2/2}dt$.

2.4. Use $y_2(x) = y_1(x)y(x)$ and the fact that $y_1(x)$ and $y_2(x)$ are solutions.

2.5. Verify directly.

2.6. Use Leibniz's formula:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,t)dt = f(x,\beta(x))\frac{d\beta}{dx} - f(x,\alpha(x))\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,t)dt.$$

2.7. (i) $c_1\cos 2x + c_2 \sin 2x - \frac{1}{4}x\cos 2x$ (ii) $c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2}xe^{-3x}$ (iii) $c_1 e^{-x} + c_2 e^{-4x} - \frac{1}{3}xe^{-4x}$.

2.8. $c_1 x + (c_2/x) + (1/2)[(x-(1/x))\ln(1+x) - x\ln x - 1]$.