CHAPTER 2

PARTIAL ALGEBRAS

§13 and §16 contain the elements of the theory of partial algebras. §14 and §15 are rather technical; the reader is advised to omit the proofs at the first reading. §17 and §18 give the characterization theorem of congruence lattices; the reader should omit these sections completely at first reading. Since §17 and §18 contain a long series of results, it is useful to cover them first without reading the proofs. These two sections were included to show the usefulness of partial algebras.

§13. BASIC NOTIONS

Let us recall that a *partial algebra* $\mathfrak{A}$ is a pair $\langle A; F\rangle$ where A is a nonvoid set and F is a collection of partial operations on A. We will always assume that F is well ordered, $F=\langle f_0, f_1, \cdots, f_\gamma, \cdots\rangle_{\gamma<o(\tau)}$. The *type* τ of the partial algebra $\mathfrak{A}$ is defined in the same way as for algebras.

Two partial algebras $\mathfrak{A}$, $\mathfrak{B}$ of the same type τ are *isomorphic* if there exists a 1–1 mapping φ of A onto B such that $f_\gamma(a_0, \cdots, a_{n_\gamma-1})$ exists if and only if $f_\gamma(a_0\varphi, \cdots, a_{n_\gamma-1}\varphi)$ exists and

$$f_\gamma(a_0, \cdots. a_{n_\gamma-1})\varphi = f_\gamma(a_0\varphi, \cdots, a_{n_\gamma-1}\varphi).$$

The first question that arises is why we consider partial algebras in the study of algebras. Our most important motivation is the following: Consider an algebra $\mathfrak{A}$ and a nonvoid subset B of A. Restrict all the operations to B in the following way: Let $f_\gamma \in F$, $b_0, \cdots, b_{n_\gamma-1} \in B$; if $f_\gamma(b_0, \cdots, b_{n_\gamma-1}) \in B$, then we do not change $f_\gamma(b_0, \cdots, b_{n_\gamma-1})$. However, if $f_\gamma(b_0, \cdots, b_{n_\gamma-1}) \notin B$, we will say that $f_\gamma(b_0, \cdots, b_{n_\gamma-1})$ is *not* defined. We will denote by $\mathfrak{B}=\langle B; F\rangle$ the system that arises.

In spite of the fact that we started out with an algebra, $\mathfrak{B}$ is only a partial algebra unless B is closed under *all* the operations.

Thus we can say that the language of partial algebras is the natural one if we want to talk about subsets of an algebra and the properties of operations on these subsets even if the subsets are not closed under all the operations. The question we are now going to settle is a very simple one. Is the concept of partial algebras too general from this point of view?

Theorem 1. *Let $\mathfrak{B}$ be a partial algebra. Then there exists an algebra $\mathfrak{A}$ and $A_1 \subseteq A$ such that*

$$\mathfrak{B} \cong \langle A_1; F\rangle.$$

Proof. Construct A as $B \cup \{p\}$ $(p \notin B)$. If $f_\gamma(a_0, \cdots, a_{n_\gamma-1}) = a$ in $\mathfrak{B}$, keep it. Otherwise, let $f_\gamma(a_0, \cdots, a_{n_\gamma-1}) = p$. Take $A_1 = B$; the rest is trivial.

For algebras, there is only one reasonable way to define the concepts of subalgebra, homomorphism, and congruence relation. For partial algebras we will define three different types of subalgebra, three types of homomorphism, and two types of congruence relation. In many papers, the authors select one of each (probably based on the assumption that if there was one good concept for algebras then there is only one good concept for partial algebras) and give the reasons for their choices. In the author's opinion, all these concepts have their merits and drawbacks, and each particular situation determines which one should be used.

First we define the three subalgebra concepts.

Let $\mathfrak{A}$ be a partial algebra and let $\varnothing \neq B \subseteq A$. We say that $\mathfrak{B}$ is a *subalgebra* of $\mathfrak{A}$ if it is closed under all operations in $\mathfrak{A}$, i.e., if $b_0, \cdots, b_{n_\gamma-1} \in B$ and $f_\gamma(b_0, \cdots, b_{n_\gamma-1})$ is defined in $\mathfrak{A}$, then

$$f_\gamma(b_0, \cdots, b_{n_\gamma-1}) \in B.$$

In this case,

$$D(f_\gamma, \mathfrak{A}) \cap B^{n_\gamma} = D(f_\gamma, \mathfrak{B}) \qquad \text{for } \gamma < o(\tau),$$

where $D(f_\gamma, \mathfrak{A})$ and $D(f_\gamma, \mathfrak{B})$ denote the domain of f_γ in $\mathfrak{A}$, and in $\mathfrak{B}$, respectively.

In the case of algebras, the new notion of subalgebra is the same as the old one.

We shall now describe other ways of obtaining partial algebras from a given one.

Consider a partial algebra $\mathfrak{A}$ and let $\varnothing \neq B \subseteq A$. For every $\gamma < o(\tau)$ we define f_γ on B as follows: $f_\gamma(b_0, \cdots, b_{n_\gamma-1})$ is defined for $b_0, \cdots, b_{n_\gamma-1}$ and equals b if and only if $f_\gamma(b_0, \cdots, b_{n-1})$ is defined in $\mathfrak{A}$ and $f_\gamma(b_0, \cdots, b_{n_\gamma-1}) = b \in B$ in $\mathfrak{A}$. Thus for $\mathfrak{B} = \langle B; F\rangle$ we have that

$$D(f_\gamma, \mathfrak{B}) = \{\langle b_0, \cdots, b_{n_\gamma-1}\rangle \mid \langle b_0, \cdots, b_{n_\gamma-1}\rangle \in D(f_\gamma, \mathfrak{A}) \cap B^{n_\gamma} \text{ and } f_\gamma(b_0, \cdots, b_{n_\gamma-1}) \in B\}.$$

In this case, we say that $\mathfrak{B}$ is a *relative subalgebra* of $\mathfrak{A}$, and $\mathfrak{A}$ an *extension* of $\mathfrak{B}$. We will use the convention that if we write, "let $\mathfrak{A}$ be a partial algebra, $B \subseteq A$, then the partial algebra $\mathfrak{B} \cdots$", then $\mathfrak{B}$ always means the relative subalgebra determined by B. Observe that a subalgebra $\mathfrak{B}$ of a partial algebra $\mathfrak{A}$ is only a partial algebra, and that a subalgebra $\mathfrak{B}$ is a relative subalgebra of $\mathfrak{A}$ with $D(f_\gamma, \mathfrak{B}) = D(f_\gamma, \mathfrak{A}) \cap B^{n_\gamma}$, for $\gamma < o(\tau)$.

To introduce the third kind of subalgebra, we will have to be somewhat more careful about our notation. Let $\mathfrak{A}$ be a partial algebra and $\varnothing \neq B \subseteq A$. Suppose we have partial operations f_γ' defined on B such that if $f_\gamma'(b_0, \cdots, b_{n_\gamma - 1}) = b$, then $f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) = b$. Let

$$F' = \langle f_0', f_1', \cdots, f_\gamma', \cdots \rangle_{\gamma < o(\tau)}.$$

Then we say that $\mathfrak{B}_1 = \langle B; F' \rangle$ is a *weak subalgebra* of $\mathfrak{A}$. In this case,

$$D(f_\gamma', \mathfrak{B}_1) \subseteq D(f_\gamma, \mathfrak{B}) \subseteq D(f_\gamma, \mathfrak{A}).$$

Note that we could not use the notation $\langle B; F \rangle$ in this case because this would suggest that the partial operations on B are the restrictions of the partial operations on A which is not at all the case.

Next we define three notions of homomorphism.

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are partial algebras. $\varphi: A \to B$ is called a *homomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$ if whenever $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ is defined, then so is $f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi)$ and

$$f_\gamma(a_0, \cdots, a_{n_\gamma - 1})\varphi = f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi).$$

By the definition of homomorphism, if f_γ can be performed on some elements of A, then f_γ can be performed on their images. A homomorphism is called full if the only partial operations which can be performed on the image are the ones that follow from the definition of homomorphism.

Formally, the homomorphism φ of $\mathfrak{A}$ into $\mathfrak{B}$ is a *full homomorphism* if

$$f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi) = a\varphi, \ a_0, \cdots, a_{n_\gamma - 1}, a \in A$$

imply that there exist $b_0, \cdots, b_{n_\gamma - 1}, b \in A$ with $b_0\varphi = a_0\varphi, \cdots, b_{n_\gamma - 1}\varphi = a_{n_\gamma - 1}\varphi$, $b\varphi = a\varphi$ and $f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) = b$.

A *strong homomorphism* φ is a homomorphism such that $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ is defined in $\mathfrak{A}$ if and only if $f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi)$ is defined in $\mathfrak{B}$.

Every strong homomorphism is thus a full homomorphism, but the converse is false. Every full homomorphism is a homomorphism, and the converse is again false. In the case of algebras, all three concepts are equivalent to the concept of a homomorphism of an algebra.

Let φ be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, $C = A\varphi$, and $\mathfrak{C}$ the corresponding relative subalgebra of $\mathfrak{B}$. If φ is an isomorphism of $\mathfrak{A}$ and $\mathfrak{C}$, then φ is called an *embedding of* $\mathfrak{A}$ into $\mathfrak{B}$.

We shall now discuss congruence relations.

Given a partial algebra $\mathfrak{A}$ and Θ, an equivalence relation, Θ is called a *congruence relation* if we have:

(SP) If $a_i \equiv b_i(\Theta)$ and if $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ and $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ are *both* defined, then

$$f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \cdots, b_{n_\gamma - 1})(\Theta).$$

A congruence relation Θ on $\mathfrak{A}$ is called *strong* if whenever $a_i \equiv b_i(\Theta)$, $0 \leqq i < n_\gamma$, and $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ exists, then $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ also exists.

The following four lemmas connect up the above defined concepts.

Lemma 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras and let φ be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Let ε_φ be the equivalence relation induced by φ. Then ε_φ is a congruence relation.*

Proof. Suppose that $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ and $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ are both defined and that $a_i \equiv b_i(\varepsilon_\varphi)$. Since $a_i \equiv b_i(\varepsilon_\varphi)$ is equivalent to $a_i\varphi = b_i\varphi$, we have that

$$\begin{aligned} f_\gamma(a_0, \cdots, a_{n_\gamma - 1})\varphi &= f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi) = f_\gamma(b_0\varphi, \cdots, b_{n_\gamma - 1}\varphi) \\ &= f_\gamma(b_0, \cdots, b_{n_\gamma - 1})\varphi, \end{aligned}$$

so that

$$f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \cdots, b_{n_\gamma - 1})(\varepsilon_\varphi).$$

Lemma 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras and let φ be a strong homomorphism; then ε_φ is a strong congruence relation.*

Proof. It suffices to verify that if $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ is defined and $a_i \equiv b_i(\varepsilon_\varphi)$, then $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is also defined.

Since $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$ is defined and φ is a homomorphism, we have that $f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi)$ is also defined and so

$$f_\gamma(a_0, \cdots, a_{n_\gamma - 1})\varphi = f_\gamma(a_0\varphi, \cdots, a_{n_\gamma - 1}\varphi) = f_\gamma(b_0\varphi, \cdots, b_{n_\gamma - 1}\varphi).$$

By the definition of strong homomorphism, $f_\gamma(b_0\varphi, \cdots, b_{n_\gamma - 1}\varphi)$ is defined if and only if $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is defined; thus $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is defined.

To prove the converse of Lemmas 1 and 2 we need to define a quotient partial algebra.

Let $\mathfrak{A}$ be a partial algebra and let Θ be a congruence relation of $\mathfrak{A}$. We define the *quotient partial algebra* $\mathfrak{A}/\Theta = \langle A/\Theta;\ F\rangle$ as follows:

If $b_0, \cdots, b_{n_\gamma - 1} \in A/\Theta$, then $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is defined to be equal to b

if and only if *there exist* $a_i \in A$ and $a \in A$ such that $b_i = [a_i]\Theta$, $b = [a]\Theta$ and $f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) = a$.

Lemma 3. *Let* $\mathfrak{A}$ *be a partial algebra and* Θ *a congruence relation of* $\mathfrak{A}$. *Then the mapping* $\varphi: a \to [a]\Theta$ *is a full homomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{A}/\Theta = \langle A/\Theta; F\rangle$ *and* $\varepsilon_\varphi = \Theta$.

Proof. The proof follows directly from the definition.

Lemma 4. *Let* $\mathfrak{A}$ *be a partial algebra and* Θ *a strong congruence relation of* $\mathfrak{A}$. *Then the mapping* $\varphi: a \to [a]\Theta$ *is a strong homomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{A}/\Theta$ *and* $\varepsilon_\varphi = \Theta$.

Proof. Again, by the definitions.

Summarizing, we have the following theorem.

Theorem 2. *Under the correspondence* $\varphi \to \varepsilon_\varphi$ *homomorphisms correspond to congruence relations on the one hand and strong homomorphisms correspond to strong congruence relations on the other hand.*

There is no such concept as "full congruence relation", which would correspond to full homomorphism, since "φ is full" means a relationship between $\mathfrak{A}$ and $\mathfrak{B}$ and is not a property of ε_φ.

As we explained at the beginning of this section, we develop the theory of partial algebras in order to obtain a theory to use when considering the properties of an operation on a subset of an algebra. Therefore, if $\mathfrak{A}$ is an algebra, $\varnothing \neq B \subseteq A$ and Θ is a congruence relation of $\mathfrak{A}$, then it is quite natural to require that Θ_B be a congruence relation of the partial algebra $\mathfrak{B}$, and every congruence relation of $\mathfrak{B}$ can be so obtained from some algebra $\mathfrak{A}$. Our next theorem states that the notion of congruence relation as defined above does exactly this.

Theorem 3. *Let* $\mathfrak{B}$ *be a partial algebra and let* Θ *be a congruence relation of* $\mathfrak{B}$. *Then there exists an algebra* $\mathfrak{A}$ *which is an extension of* $\mathfrak{B}$, *and a congruence relation* Φ *of* $\mathfrak{A}$ *such that* $\Phi_B = \Theta$.

Theorem 3 will be proved in §14 and §15 in a much stronger form. It was proved in another form by G. Grätzer and E. T. Schmidt [2]. A similar characterization of strong congruence relations will be given in §16. A very simple direct proof of Theorem 3 is given in G. Grätzer and G. H. Wenzel [1].

§14. POLYNOMIAL SYMBOLS OVER A PARTIAL ALGEBRA†

Let τ be a fixed type of partial algebras. The polynomial symbols $\mathbf{P}^{(\alpha)}(\tau)$ are defined the same as they were for algebras. In this case, an α-ary polynomial symbol does not always induce a mapping of A^α into A, if $\mathfrak{A}$ is a partial algebra. However, some of them do; this will be clear from the following definition.

Definition 1. *Let $\mathfrak{A}$ be a partial algebra of type τ, $a_0, \cdots, a_\gamma, \cdots \in A$, $\gamma < \alpha$, $\mathbf{p} \in \mathbf{P}^{(\alpha)}(\tau)$. Then $p(a_0, \cdots, a_\gamma, \cdots)$ is defined and equals $a \in A$ if and only if it follows from the following rules:*

(i) *If $\mathbf{p} = \mathbf{x}_\delta$, for $\delta < \alpha$, then $p(a_0, \cdots, a_\gamma, \cdots) = a_\delta$;*

(ii) *if $p_i(a_0, \cdots)$ are defined and $p_i(a_0, \cdots) = b_i$ $(0 \leqq i < n_\gamma)$, $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is defined and $\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\gamma - 1})$, then $p(a_0, \cdots)$ is defined and*

$$p(a_0, \cdots) = f_\gamma(b_0, \cdots, b_{n_\gamma - 1}).$$

The basic difficulty which arises is that if we take

$$\bar{a} = \langle a_0, \cdots, a_\gamma, \cdots \rangle_{\gamma < \alpha}$$

$(a_\gamma \in A)$ where $\mathfrak{A}$ is a partial algebra, then the congruence relation $\Theta_{\bar{a}}$ of $\mathfrak{P}^{(\alpha)}(\tau)$ cannot be defined as in Theorem 8.2. As a matter of fact, it can be defined that way if and only if the a_γ generate a subalgebra which is an algebra.

Our main result in this section is the following theorem.

Theorem 1. *Let $\mathfrak{A}$ be a partial algebra, $\bar{a} \in A^\alpha$, $\bar{a} = \langle a_0, \cdots, a_\gamma, \cdots \rangle$. Define a binary relation $\Theta_{\bar{a}}$ on $\mathbf{P}^{(\alpha)}(\tau)$ as follows:*

$\mathbf{p} \equiv \mathbf{q}(\Theta_{\bar{a}})$ if and only if there exist $\mathbf{r} \in \mathbf{P}^{(k)}(\tau)$, $\mathbf{p}_i, \mathbf{q}_i \in \mathbf{P}^{(\alpha)}(\tau)$ $(0 \leqq i < k)$ such that $p_i(a_0, \cdots, a_\gamma, \cdots)$ and $q_i(a_0, \cdots, a_\gamma, \cdots)$ exist and

$$p_i(a_0, \cdots, a_\gamma, \cdots) = q_i(a_0, \cdots, a_\gamma, \cdots)$$

and

$$\mathbf{p} = r(\mathbf{p}_0, \cdots, \mathbf{p}_{k-1}),$$
$$\mathbf{q} = r(\mathbf{q}_0, \cdots, \mathbf{q}_{k-1}).$$

Then $\Theta_{\bar{a}}$ is a congruence relation of $\mathfrak{P}^{(\alpha)}(\tau)$.

Remark. If we want to find a congruence relation Θ of $\mathfrak{P}^{(\alpha)}(\tau)$ such that $p(a_0, \cdots, a_\gamma, \cdots) = q(a_0, \cdots, a_\gamma, \cdots)$ implies $\mathbf{p} \equiv \mathbf{q}(\Theta)$, then it is obvious that our $\Theta_{\bar{a}}$ is contained in Θ. One does not expect, however, that $\Theta_{\bar{a}}$ is

† The results of this section are taken from G. Grätzer [13].

transitive. Thus the natural statement would be that the smallest such congruence relation is the transitive extension of $\Theta_{\bar{a}}$.

Proof. $\Theta_{\bar{a}}$ is reflexive; indeed, let $\mathbf{p} \in \mathbf{P}^{(\alpha)}(\tau)$; then, by Lemma 8.5′,

$$\mathbf{p} = p(\mathbf{x}_0, \cdots, \mathbf{x}_\gamma, \cdots).$$

By Lemma 8.6,

$$p(\mathbf{x}_0, \cdots, \mathbf{x}_\gamma, \cdots) = r(\mathbf{x}_{\gamma_0}, \cdots, \mathbf{x}_{\gamma_{k-1}}),$$

for some $\mathbf{r} \in \mathbf{P}^{(k)}(\tau)$. Thus

$$\mathbf{p} = r(\mathbf{x}_{\gamma_0}, \cdots, \mathbf{x}_{\gamma_{k-1}});$$

since $x_{\gamma_i}(a_0, \cdots, a_\gamma, \cdots)$ always exists, this verifies that $\mathbf{p} \equiv \mathbf{p}(\Theta_{\bar{a}})$.

It is trivial that $\Theta_{\bar{a}}$ is symmetric. To prove the substitution property, let $\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\gamma - 1})$, $\mathbf{q} = \mathbf{f}_\gamma(\mathbf{q}_0, \cdots, \mathbf{q}_{n_\gamma - 1})$ and

$$\mathbf{p}_i \equiv \mathbf{q}_i(\Theta_{\bar{a}}), \qquad 0 \leqq i < n_\gamma.$$

Then

$$\mathbf{p}_i = r_i(\mathbf{p}_0{}^i, \cdots, \mathbf{p}^i_{n_i - 1}),$$
$$\mathbf{q}_i = r_i(\mathbf{q}_0{}^i, \cdots, \mathbf{q}^i_{n_i - 1}),$$

and $p_j{}^i(a_0, \cdots, a_\gamma, \cdots)$, $q_j{}^i(a_0, \cdots, a_\gamma, \cdots)$ exist and

$$p_j{}^i(a_0, \cdots, a_\gamma, \cdots) = q_j{}^i(a_0, \cdots, a_\gamma, \cdots).$$

Set $n = n_0 + n_1 + \cdots + n_{n_\gamma - 1}$. By the second part of Lemma 8.6, for $0 \leqq i < n_\gamma$ there exists an n-ary polynomial symbol $\mathbf{r}_i'$, such that

$$r_i(b_0, \cdots, b_{n_i - 1}) = r_i'(c_0, \cdots, c_{n_0 + \cdots + n_{i-1} - 1}, b_0, \cdots, b_{n_i - 1}, c_{n_0 + \cdots + n_i}, \cdots, c_{n-1})$$

for any values b_j and c_j. Thus we have that

$$\mathbf{p}_i = r_i'(\mathbf{p}_0{}^0, \cdots, \mathbf{p}^0_{n_0 - 1}, \cdots, \mathbf{p}_0^{n_\gamma - 1}, \cdots, \mathbf{p}^{n_\gamma - 1}_{n(n_\gamma - 1) - 1})$$

for all $0 \leqq i < n_\gamma$.

Set

$$\mathbf{r} = \mathbf{f}_\gamma(\mathbf{r}_0', \cdots, \mathbf{r}'_{n_\gamma - 1}).$$

Then

$$r(\mathbf{p}_0{}^0, \cdots, \mathbf{p}^0_{n_0 - 1}, \mathbf{p}_0{}^1, \cdots, \mathbf{p}^1_{n_1 - 1}, \cdots, \mathbf{p}_0^{n_\gamma - 1}, \cdots, \mathbf{p}^{n_\gamma - 1}_{n(n_\gamma - 1) - 1}) = \mathbf{p},$$
$$r(\mathbf{q}_0{}^0, \cdots, \mathbf{q}^0_{n_0 - 1}, \cdots, \mathbf{q}_0^{n_\gamma - 1}, \cdots, \mathbf{q}^{n_\gamma - 1}_{n(n_\gamma - 1) - 1}) = \mathbf{q},$$

establishing that

$$\mathbf{p} \equiv \mathbf{q}(\Theta_{\bar{a}}).$$

which was to be proved.

To establish the transitivity of $\Theta_{\bar{a}}$, we need a lemma.

Lemma 1†. *Let* $\mathbf{p}=\mathbf{f}_\gamma(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\gamma-1})$ *and* $\mathbf{q}=\mathbf{f}_\delta(\mathbf{q}_0, \cdots, \mathbf{q}_{n_\delta-1})$. *Then* $\mathbf{p}\equiv\mathbf{q}(\Theta_{\bar{a}})$, *if and only if either* $p(\bar{a})$ *and* $q(\bar{a})$ *exist and* $p(\bar{a})=q(\bar{a})$, *or* $\gamma=\delta$ and $\mathbf{p}_i\equiv\mathbf{q}_i(\Theta_{\bar{a}})$. *Moreover, if* $\mathbf{p}\equiv\mathbf{q}(\Theta_{\bar{a}})$ *and* $p(\bar{a})$ *and* $q(\bar{a})$ *exist, then* $p(\bar{a})=q(\bar{a})$.

Proof. Let us assume that $p(\bar{a})$ does not exist. By the definition of $\Theta_{\bar{a}}$, $\mathbf{p}$ and $\mathbf{q}$ have representations of the form

$$\mathbf{p} = r(\mathbf{p}_0', \cdots, \mathbf{p}_{k-1}'),$$
$$\mathbf{q} = r(\mathbf{q}_0', \cdots, \mathbf{q}_{k-1}'),$$

where $p_i'(\bar{a})=q_i'(\bar{a})$, $0\leqq i<k$ and $\mathbf{r}\in\mathbf{P}^{(k)}(\tau)$. Since $p(\bar{a})$ does not exist, $\mathbf{r}\neq\mathbf{x}_i$ for $0\leqq i<k$, and so

$$\mathbf{r} = \mathbf{f}_\nu(\mathbf{r}_0, \cdots, \mathbf{r}_{n_\nu-1}).$$

Therefore,

$$\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\gamma-1}) = \mathbf{f}_\nu(r_0(\mathbf{p}_0', \cdots, \mathbf{p}_{k-1}'), \cdots, r_{n_\nu-1}(\mathbf{p}_0', \cdots, \mathbf{p}_{k-1}'))$$

and

$$\mathbf{q} = \mathbf{f}_\delta(\mathbf{q}_0, \cdots, \mathbf{q}_{n_\delta-1}) = \mathbf{f}_\nu(r_0(\mathbf{q}_0', \cdots, \mathbf{q}_{k-1}'), \cdots, r_{n_\nu-1}(\mathbf{q}_0', \cdots, \mathbf{q}_{k-1}')).$$

Thus $\gamma=\nu$ and $\delta=\nu$ and so $\gamma=\delta$. From the equalities given above we conclude that

$$\mathbf{p}_i = r_i(\mathbf{p}_0', \cdots, \mathbf{p}_{k-1}')$$

and

$$\mathbf{q}_i = r_i(\mathbf{q}_0', \cdots, \mathbf{q}_{k-1}')$$

for $i=0, \cdots, k-1$. Since $\mathbf{p}_i'\equiv\mathbf{q}_i'(\Theta_{\bar{a}})$ for $0\leqq i<k$ and $\Theta_{\bar{a}}$ has (SP), we conclude that

$$\mathbf{p}_i \equiv \mathbf{q}_i(\Theta_{\bar{a}}), \qquad i = 0, \cdots, k-1,$$

which was to be proved. The other statements of Lemma 1 are trivial.

Now we return to the proof of transitivity of the $\Theta_{\bar{a}}$. Let $\mathbf{q}\equiv\mathbf{p}(\Theta_{\bar{a}})$ and $\mathbf{p}\equiv\mathbf{r}(\Theta_{\bar{a}})$. It follows from the definition of $\Theta_{\bar{a}}$, that if $q(\bar{a})$ exists, then $p(\bar{a})$ and $r(\bar{a})$ exist and $q(\bar{a})=p(\bar{a})=r(\bar{a})$, hence $\mathbf{q}\equiv\mathbf{r}(\Theta_{\bar{a}})$.

Let us assume now that $q(\bar{a})$ does not exist. Then $p(\bar{a})$ and $r(\bar{a})$ do not exist. Let n be the maximum of the ranks of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. We prove the transitivity by induction on n. If $n=2$, we get a contradiction to the assumption that $q(\bar{a})$ does not exist. Let us assume that the transitivity has been proven for maximum rank $<n$, and apply Lemma 1 to the two congruences.

† This lemma and the conclusion of the proof of Theorem 1 are due to G. H. Wenzel; the original proof was much longer.

We get that

$$\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\gamma - 1}),$$
$$\mathbf{q} = \mathbf{f}_\gamma(\mathbf{q}_0, \cdots, \mathbf{q}_{n_\gamma - 1}),$$
$$\mathbf{r} = \mathbf{f}_\gamma(\mathbf{r}_0, \cdots, \mathbf{r}_{n_\gamma - 1}),$$

and $\mathbf{q}_i \equiv \mathbf{p}_i(\Theta_{\bar{a}})$, $\mathbf{p}_i \equiv \mathbf{r}_i(\Theta_{\bar{a}})$ for $i=0, \cdots, n_{\gamma-1}$. Since for a fixed i, the maximum of the ranks of $\mathbf{q}_i$, $\mathbf{p}_i$, $\mathbf{r}_i$ is less than n, we get $\mathbf{q}_i \equiv \mathbf{r}_i(\Theta_{\bar{a}})$, and so by (SP), $\mathbf{q} \equiv \mathbf{r}(\Theta_{\bar{a}})$. This completes the proof of Theorem 1.

Let $\mathfrak{A}$ be a partial algebra, $\bar{a} = \langle a_0, \cdots, a_\gamma, \cdots \rangle_{\gamma < \alpha}$, and assume that each element of A occurs once and only once in this sequence. We consider the quotient algebra $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ and we denote by A^* the set of elements of the form $[\mathbf{x}_\gamma]\Theta_{\bar{a}}$.

Theorem 2. *The relative subalgebra* $\mathfrak{A}^* = \langle A^*; F \rangle$ *of* $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ *is isomorphic to* $\mathfrak{A}$, *and the correspondence*

$$\varphi\colon a_\gamma \to [\mathbf{x}_\gamma]\Theta_{\bar{a}}$$

is an isomorphism between $\mathfrak{A}$ *and* $\mathfrak{A}^*$.

Proof. As the first step, we prove that

$$[\mathbf{x}_\gamma]\Theta_{\bar{a}} = [\mathbf{x}_\delta]\Theta_{\bar{a}}$$

if and only if $\gamma = \delta$.

Assume that $[\mathbf{x}_\gamma]\Theta_{\bar{a}} = [\mathbf{x}_\delta]\Theta_{\bar{a}}$, that is,

$$\mathbf{x}_\gamma \equiv \mathbf{x}_\delta(\Theta_{\bar{a}}).$$

Then by Lemma 1, $x_\gamma(\bar{a}) = x_\delta(\bar{a})$, that is, $a_\gamma = a_\delta$, and so $\gamma = \delta$.

Thus, we have proved that the mapping φ is 1–1; φ is obviously onto.

To conclude the proof of Theorem 2, we must verify that

$$f_\gamma(a_{\delta_0}, \cdots, a_{\delta_{n_\gamma - 1}}) = a_\delta \tag{1}$$

if and only if

$$f_\gamma([\mathbf{x}_{\delta_0}]\Theta_{\bar{a}}, \cdots, [\mathbf{x}_{\delta_{n\gamma-1}}]\Theta_{\bar{a}}) = [\mathbf{x}_\delta]\Theta_{\bar{a}}. \tag{2}$$

(2) is equivalent to

$$f_\gamma(\mathbf{x}_{\delta_0}, \cdots, \mathbf{x}_{\delta_{n\gamma-1}}) \equiv \mathbf{x}_\delta(\Theta_{\bar{a}}). \tag{3}$$

Using the same argument as we used for the congruence

$$\mathbf{x}_\gamma \equiv \mathbf{x}_\delta(\Theta_{\bar{a}})$$

above, we can prove analogously that the two sides of (3) have only trivial representations and then the equivalence of (1) and (3) follows. This concludes the proof of Theorem 2.

Theorem 2 gives another proof of Theorem 13.1, namely, it gives an embedding of a partial algebra $\mathfrak{A}$ into an algebra. While Theorem 13.1 gives the most economical construction, Theorem 2 gives the least economical one, that is, $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ is the largest algebra into which $\mathfrak{A}$ can be embedded, such that the image of $\mathfrak{A}$ is a generating set.

We conclude this section by describing the structure of the algebra $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$.

First we define certain subsets $A_{\langle n,\gamma\rangle}$ and $A'_{\langle n,\gamma\rangle}$ $(0 \leqq n < \omega, 0 \leqq \gamma < o(\tau))$ of this algebra as follows:

$$A'_{\langle 0,0\rangle} = A^*,$$

where A^* was defined before Theorem 2;

$$A_{\langle n,\delta\rangle} = A'_{\langle n,\delta\rangle} \cup \{f_\delta(b_0, \cdots, b_{n_\delta - 1}) \mid b_0, \cdots, b_{n_\delta - 1} \in A'_{\langle n,\delta\rangle}\};$$
$$A'_{\langle n,\delta\rangle} = \bigcup (A_{\langle m,\gamma\rangle} \mid \langle m, \gamma\rangle < \langle n, \delta\rangle), \quad \text{if} \quad \langle n, \delta\rangle \neq \langle 0, 0\rangle,$$

where $\langle m, \gamma\rangle < \langle n, \delta\rangle$ means that $m < n$ or $m = n$ and $\gamma < \delta$ (thus the $\langle m, \gamma\rangle$ form a well-ordered set of order type $\omega \cdot o(\tau)$).

Lemma 2. *The following equality holds:*

$$\mathbf{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}} = \bigcup (A_{\langle n,\delta\rangle} \mid 0 \leqq n < \omega, 0 \leqq \delta < o(\tau)).$$

Proof. The following inclusions are trivial, by the definitions of $A_{\langle n,\gamma\rangle}$ and $A'_{\langle n,\gamma\rangle}$:

$$A_{\langle n,\gamma\rangle} \subseteq A'_{\langle n,\delta\rangle} \subseteq A_{\langle n,\delta\rangle} \quad \text{if} \quad \gamma < \delta,$$
$$A_{\langle n,\gamma\rangle} \subseteq A'_{\langle m,\delta\rangle} \subseteq A_{\langle m,\delta\rangle} \quad \text{if} \quad n < m.$$

Take $\mathbf{p} \in \mathbf{P}^{(\alpha)}(\tau)$. We will prove by induction on the rank of $\mathbf{p}$ that

$$[\mathbf{p}]\Theta_{\bar{a}} \in A_{\langle n,\delta\rangle} \tag{4}$$

for some $n < \omega$ and $\delta < o(\tau)$. If $\mathbf{p} = \mathbf{x}_\gamma$, then $[\mathbf{x}_\gamma]\Theta_{\bar{a}} \in A'_{\langle 0,0\rangle}$. Let

$$\mathbf{p} = \mathbf{f}_\gamma(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\gamma - 1}),$$

and assume that (4) holds for each $\mathbf{p}_i$, that is,

$$[\mathbf{p}_i]\Theta_{\bar{a}} \in A_{\langle n_i,\delta_i\rangle}.$$

We set

$$n = \max(n_0, \cdots, n_{n_\gamma - 1})$$

and

$$\delta = \max(\delta_0, \cdots, \delta_{n_\gamma - 1}).$$

Then

$$A_{\langle n_i,\delta_i\rangle} \subseteq A_{\langle n,\delta\rangle} \subseteq A'_{\langle n+1,0\rangle} \subseteq A'_{\langle n+1,\gamma\rangle}.$$

Thus,

$$[\mathbf{p}]\Theta_{\bar{a}} = f_\gamma([\mathbf{p}_0]\Theta_{\bar{a}}, \cdots, [\mathbf{p}_{n_\gamma - 1}]\Theta_{\bar{a}}) \in A_{\langle n+1, \gamma\rangle},$$

which was to be proved.

To get our final result in this section, we introduce the following notation.

Definition 2. *Let $\mathfrak{B}$ be a partial algebra, $X \subseteq B$ and*

$$Y = X \cup \{f_\gamma(x_0, \cdots, x_{n_\gamma - 1}) \mid x_i \in X\}, \qquad \gamma < o(\tau).$$

We will write

$$Y = X[f_\gamma]$$

if $f_\gamma(x_0, \cdots, x_{n_\gamma-1}) = f_\delta(x_0', \cdots, x'_{n_\delta - 1}) \in Y - X$, $\delta < o(\tau)$ imply that

$$\gamma = \delta, x_0 = x_0', \cdots, x_{n_\gamma - 1} = x'_{n_\gamma - 1},$$

and if whenever $\{x_0, \cdots, x_{n_\rho - 1}\} \nsubseteq X$, then $f_\rho(x_0, \cdots, x_{n_\rho - 1})$ does not exist in Y, for any $\rho < o(\tau)$. If $\mathfrak{A}$ and $\mathfrak{B}$ are partial algebras, $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{B}$ and $B = A[f_\gamma]$, then we will write $\mathfrak{B} = \mathfrak{A}[f_\gamma]$.

Lemma 3. $A_{\langle n, \gamma\rangle} = A'_{\langle n, \gamma\rangle}[f_\gamma]$.

Proof. We start with the following observation which follows immediately from the definition of $\Theta_{\bar{a}}$:

(*) For any $\mathbf{p} \in \mathbf{P}^{(\alpha)}(\tau)$, $p(\bar{a})$ is defined if and only if $[\mathbf{p}]\Theta_{\bar{a}} \in A^*$.

Now to prove Lemma 3 we first observe that the first requirement of Definition 2 follows trivially from Lemma 1 and (*). Now assume that $\{a_0, \cdots, a_{n_\delta - 1}\} \nsubseteq A'_{\langle n, \gamma\rangle}$, but that $f_\delta(a_0, \cdots, a_{n_\delta - 1})$ exists in $A_{\langle n, \gamma\rangle}$. By Lemma 1 and (*) we get that $f_\delta(a_0, \cdots, a_{n_\delta - 1}) \in A'_{\langle n, \gamma\rangle}$. Let $a_i = [\mathbf{p}_i]\Theta_{\bar{a}}$, $\mathbf{p} = \mathbf{f}_\delta(\mathbf{p}_0, \cdots, \mathbf{p}_{n_\delta - 1})$. Since $[\mathbf{p}]\Theta_{\bar{a}} \in A'_{\langle n, \gamma\rangle}$, we have that

$$[\mathbf{p}]\Theta_{\bar{a}} \in A_{\langle m, \lambda\rangle}$$

for some smallest $\langle m, \lambda\rangle < \langle n, \gamma\rangle$. By (*) and the assumption that $\{a_0, \cdots, a_{n_\delta - 1}\} \nsubseteq A'_{\langle n, \gamma\rangle}$, we have that $\langle m, \lambda\rangle \neq \langle 0, 0\rangle$ so

$$[\mathbf{p}]\Theta_{\bar{a}} \in A_{\langle m, \lambda\rangle} - A'_{\langle m, \lambda\rangle}.$$

Hence $\mathbf{p} \equiv f_\lambda(\mathbf{q}_0, \cdots, \mathbf{q}_{n_\lambda - 1})$ for some $[\mathbf{q}_i]\Theta_{\bar{a}} \in A'_{\langle m, \lambda\rangle}$, which implies by Lemma 1 and (*) that $\lambda = \delta$ and $\mathbf{q}_i \equiv \mathbf{p}_i(\Theta_{\bar{a}})$. Thus $a_i = [\mathbf{p}_i]\Theta_{\bar{a}} \in A'_{\langle m, \lambda\rangle} \subseteq A'_{\langle n, \gamma\rangle}$, a contradiction. This completes the proof of Lemma 3.

We now summarize what we have proved so far concerning the structure of $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$:

Theorem 3. *$\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ contains a relative subalgebra $\mathfrak{A}^*$ isomorphic to the partial algebra $\mathfrak{A}$; if we start with A^* and we perform two kinds of constructions,*

(i) *taking the set union of the previously constructed sets,*
(ii) *constructing $X[f_\gamma]$ from X,*

then we get a transfinite sequence of increasing subsets of $\mathbf{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ such that the union of all these subsets is the whole set.

It is obvious from Theorem 3 that $\mathfrak{B}=\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ has the following properties:

(α) $\mathfrak{B}$ has a relative subalgebra $\mathfrak{A}^+$ isomorphic to $\mathfrak{A}$ and A^+ generates $\mathfrak{B}$;

(β) if $f_\gamma(b_0, \cdots, b_{n_\gamma-1})=f_\delta(b_0', \cdots, b'_{n_\delta-1}) \notin A^+$, then $\gamma=\delta$ and $b_0=b_0', \cdots, b_{n_\gamma-1}=b'_{n_\gamma-1}$;

(γ) if $f_\gamma(b_0, \cdots, b_{n_\gamma-1}) \in A^+$, then $b_0, \cdots, b_{n_\gamma-1} \in A^+$.

Theorem 4†. *Conditions* (α)–(γ) *characterize* $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ *up to isomorphism.*

Proof. Let $\mathfrak{B}$ satisfy (α)–(γ). Then $B_{\langle n,\gamma\rangle}$ and $B'_{\langle n,\gamma\rangle}$ can be defined in $\mathfrak{B}$ as $A_{\langle n,\gamma\rangle}$ and $A'_{\langle n,\gamma\rangle}$ were defined in $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$, respectively.

Let $\varphi'_{\langle 0,0\rangle}$ be an isomorphism between $\mathfrak{A}^+$ and $\mathfrak{A}^*$. If $\varphi_{\langle m,\delta\rangle}$ is defined for all $\langle m, \delta\rangle < \langle n, \gamma\rangle$, set

$$\varphi'_{\langle n,\gamma\rangle} = \bigcup (\varphi_{\langle m,\delta\rangle} \mid \langle m, \delta\rangle < \langle n, \gamma\rangle).$$

Then $\varphi'_{\langle n,\gamma\rangle}$ will map $B'_{\langle n,\gamma\rangle}$ into $A'_{\langle n,\gamma\rangle}$, and it is 1–1 and onto. If $x \in B_{\langle n,\gamma\rangle}=B'_{\langle n,\gamma\rangle}[f_\gamma]$, then $x=f_\gamma(x_0, \cdots, x_{n_\gamma-1})$, where $x_0, \cdots, x_{n_\gamma-1}$ are uniquely determined elements of $B'_{\langle n,\gamma\rangle}$. Set

$$x\varphi_{\langle n,\gamma\rangle} = f_\gamma(x_0\varphi'_{\langle n,\gamma\rangle}, \cdots, x_{n_\gamma-1}\varphi'_{\langle n,\gamma\rangle}).$$

Then

$$\varphi = \bigcup (\varphi_{\langle n,\gamma\rangle} \mid n < \omega, \gamma < o(\tau))$$

will be the required isomorphism. The easy details are left to the reader.

It should be noted that if $\mathfrak{A}=\langle\{0\}; '\rangle$, $\mathfrak{A}$ is of type $\langle 1\rangle$, and $D(', \mathfrak{A})=\varnothing$, then ($\alpha$)–($\gamma$) is the usual Peano axiom system of natural numbers. If $\mathfrak{A}$ is arbitrary with $D(f_\gamma, \mathfrak{A})=\varnothing$ for all $\gamma<o(\tau)$, then $\Theta_{\bar{a}}=\omega$, and thus (α)–(γ)

† J. Schmidt (oral communication).

characterize $\mathfrak{P}^{(\alpha)}(\tau)$ up to isomorphism. In this special case, algebras satisfying (α)–(γ) are called *absolutely free algebras* or *Peano algebras* in the literature.

§15. EXTENSION OF CONGRUENCE RELATIONS

In this section we will prove a strong version of Theorem 13.3. Using the notation of §14, we proved that $\mathfrak{A}$ and $\mathfrak{A}^*$ are isomorphic (Theorem 14.2). Let us identify these two partial algebras; then we can say that $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ is an algebra which contains $\mathfrak{A}$ as a relative subalgebra.

Theorem 1. *Let Θ be a congruence relation of $\mathfrak{A}$. There exists a congruence relation $\bar{\Theta}$ of $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ such that $\bar{\Theta}_A = \Theta$.*

According to Theorem 14.3, it suffices to prove the following two lemmas.

Lemma 1. *Let $\mathfrak{A}$ be a partial algebra, $A = \bigcup (X_\gamma \mid \gamma < \alpha)$, and $X_{\gamma_0} \subseteq X_{\gamma_1}$ if $\gamma_0 < \gamma_1$.*

Let Θ^γ be a congruence relation of $\mathfrak{X}_\gamma$ such that

$$\Theta^{\gamma_1}_{X_{\gamma_0}} = \Theta^{\gamma_0}$$

if $\gamma_0 < \gamma_1$.

Then there exists a congruence relation Θ of $\mathfrak{A}$ such that

$$\Theta_{X_\gamma} = \Theta^\gamma$$

for each $\gamma < \alpha$.

Lemma 2. *Let $\mathfrak{A}$ be a partial algebra and $\mathfrak{B}$ a relative subalgebra of $\mathfrak{A}$. Assume that $\mathfrak{A} = \mathfrak{B}[f_\gamma]$ for some $\gamma < o(\tau)$. Then to every congruence relation Θ of $\mathfrak{B}$ there corresponds a congruence relation $\bar{\Theta}$ of $\mathfrak{A}$ such that $\bar{\Theta}_B = \Theta$.*

Remark. Let us note that Theorem 1 is stronger than Theorem 13.3 since we extended $\mathfrak{A}$ to an algebra such that *every* congruence relation of $\mathfrak{A}$ can be extended—not merely a given one.

Theorem 1 was first given in G. Grätzer and E. T. Schmidt [2], but in a weaker version; namely, in that paper it was proved that every partial algebra can be extended to an algebra which satisfies the requirements of Theorem 1 but it was not proved that this algebra can be represented as $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$. As a matter of fact, that version follows directly from Lemmas 1 and 2; for that we do not need the investigations of §14 at all. A minor difference is that in that paper a third construction was also needed to get the algebra (besides the constructions given by Lemmas 1 and 2), but it is easy to see that it can be eliminated.

Proof of Lemma 1. Set

$$\Theta = \bigcup (\Theta^\gamma \mid \gamma < \alpha).$$

It is routine to check that Θ is a congruence relation. As an illustration, we prove the transitivity of Θ.

Let $a \equiv b(\Theta)$ and $b \equiv c(\Theta)$. Then $\langle a, b\rangle, \langle b, c\rangle \in \bigcup (\Theta^\gamma \mid \gamma < \alpha)$. Therefore, $\langle a, b\rangle \in \Theta^{\gamma_0}$, $\langle b, c\rangle \in \Theta^{\gamma_1}$. Suppose, for instance, that $\gamma_0 \leqq \gamma_1$. Then

$$\langle a, b\rangle, \langle b, c\rangle \in \Theta^{\gamma_1}$$

and thus by the transitivity of Θ^{γ_1}, $\langle a, c\rangle \in \Theta^{\gamma_1}$. The proof of reflexivity, symmetry, and the substitution property is similar.

Finally, let us compute Θ_{X_γ};

$$\begin{aligned}
\Theta_{X_\gamma} &= \Theta \cap (X_\gamma \times X_\gamma) = \bigcup (\Theta^\delta \mid \delta < \alpha) \cap (X_\gamma \times X_\gamma) \\
&= \bigcup (\Theta^\delta \cap (X_\gamma \times X_\gamma) \mid \delta < \alpha) \\
&= \bigcup (\Theta^\delta \cap (X_\gamma \times X_\gamma) \mid \gamma \leqq \delta < \alpha) \\
&= \bigcup (\Theta^\delta_{X_\gamma} \mid \gamma \leqq \delta < \alpha) \\
&= \bigcup (\Theta^\gamma \mid \gamma \leqq \delta < \alpha) \\
&= \Theta^\gamma,
\end{aligned}$$

which was to be proved.

Lemma 3. *Under the conditions of Lemma 2, for a fixed Θ, define a relation Φ on $\mathfrak{A}$ as follows:*

(i) $a \equiv b(\Phi)$, $a, b \in B$ *if and only if* $a \equiv b(\Theta)$;

(ii) $a \equiv b(\Phi)$, $a \in B, b \notin B$ $(b = f_\gamma(x_0, \cdots, x_{n_\gamma - 1}))$ *if and only if there exists a* $u = f_\gamma(y_0, \cdots, y_{n_\gamma - 1}) \in B$ *such that* $a \equiv u(\Theta)$, $x_i \equiv y_i(\Theta)$, $0 \leqq i < n_\gamma$; *and the symmetric condition holds for* $a \notin B$, $b \in B$;

(iii) $a \equiv b(\Phi)$, $a, b \notin B$ $(a = f_\gamma(x_0, \cdots, x_{n_\gamma - 1}),\ b = f_\gamma(y_0, \cdots, y_{n_\gamma - 1}))$ *if and only if*

(iii$_1$) $x_i \equiv y_i(\Theta)$, $0 \leqq i < n_\gamma$, *or*

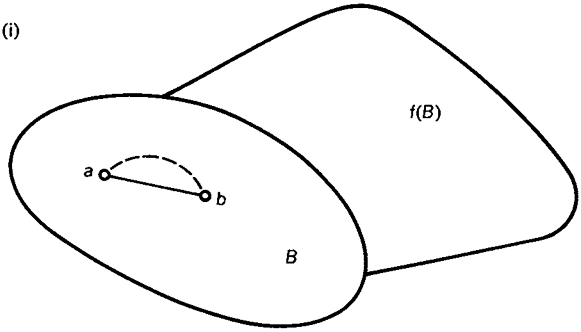
(iii$_2$) *there exist* $u = f_\gamma(u_0, \cdots, u_{n_\gamma - 1}) \in B$, $v = f_\gamma(v_0, \cdots, v_{n_\gamma - 1}) \in B$ *such that* $x_i \equiv u_i(\Theta)$, $v_i \equiv y_i(\Theta)$, $0 \leqq i < n_\gamma$, *and* $u \equiv v(\Theta)$.

Then Φ is a congruence relation of $\mathfrak{A}$.

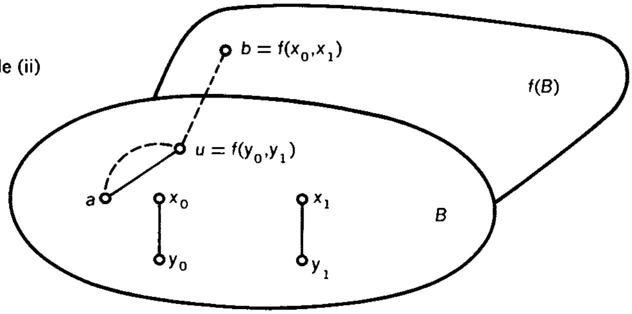
Let us note that Lemma 3 implies Lemma 2 since $\Phi_B = \Theta$ is equivalent to (i).

The following diagrams illustrate rules (i)–(iii), in case $f_\gamma = f$ is binary. Dotted lines denote congruence modulo Φ and solid lines denote congruence modulo Θ.

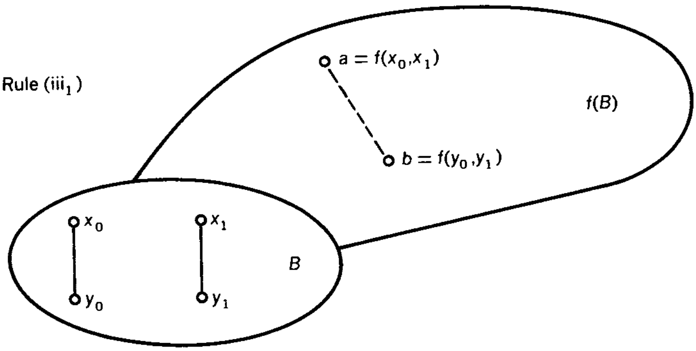
Rule (i)



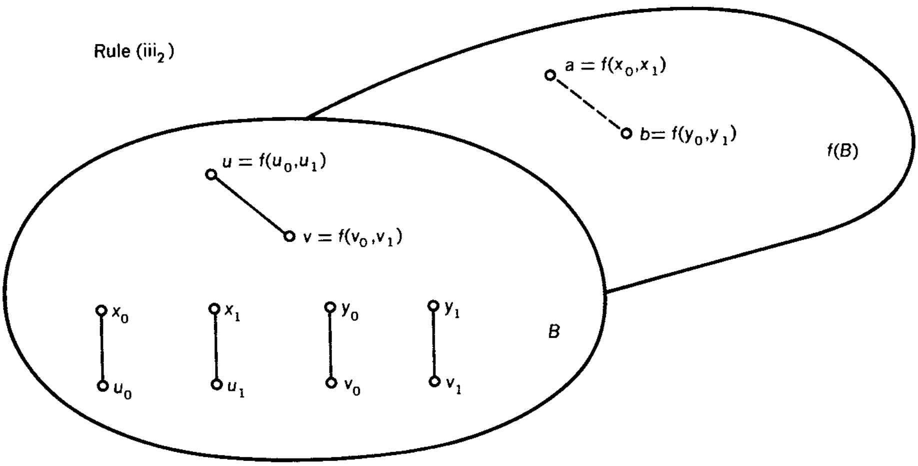
Rule (ii)



Rule (iii_1)



Rule (iii_2)



Proof of Lemma 3. Φ is reflexive since $a \equiv a(\Phi)$ follows from (i) if $a \in B$ and $a \equiv a(\Phi)$ follows from (iii_1) if $a \notin B$. Since all conditions are symmetric, Φ is symmetric. To prove the substitution property, assume that

$$a_i \equiv b_i(\Phi), \qquad 0 \leqq i < n_\delta,$$

and suppose that

$$f_\delta(a_0, \cdots, a_{n_\delta - 1}) \quad \text{and} \quad f_\delta(b_0, \cdots, b_{n_\delta - 1})$$

exist. If $\delta \neq \gamma$, then this implies

$$a_0, \cdots, a_{n_\delta - 1}, b_0, \cdots, b_{n_\delta - 1}, f_\delta(a_0, \cdots, a_{n_\delta - 1}), f_\delta(b_0, \cdots, b_{n_\delta - 1}) \in B;$$

thus, by (i), $a_i \equiv b_i(\Theta)$ and so $f_\delta(a_0, \cdots, a_{n_\delta - 1}) \equiv f_\delta(b_0, \cdots, b_{n_\delta - 1})(\Theta)$ which, by (i), implies the same congruence modulo Φ.

If $\gamma = \delta$, then $a_i, b_i \in B$ and $a_i \equiv b_i(\Theta)$. Then we get the congruence

$$f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \cdots, b_{n_\gamma - 1})\ (\Phi)$$

by (i) if $f_\gamma(a_0, \cdots, a_{n_\gamma - 1}), f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) \in B$; by ($\text{iii}_1$) if $f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$, $f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) \notin B$; and if $f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \in B$, $f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) \notin B$ (and in the symmetric case), then we have to use rule (ii) with

$$u = f_\gamma(a_0, \cdots, a_{n_\gamma - 1}).$$

All that remains is to prove the transitivity of Φ. To simplify the computations, let $f = f_\gamma$ be a binary partial operation, as in the diagrams. Assume that $a \equiv b(\Phi)$, $b \equiv c(\Phi)$. We will distinguish eight cases according to the positions of a, b, c with respect to B.

(1) $a, b, c \in B$. Then, by (i), $a \equiv b(\Theta)$, $b \equiv c(\Theta)$. Thus, $a \equiv c(\Theta)$ and, by (i), this implies $a \equiv c(\Phi)$.

(2) $a, b \in B$, $c \notin B$, $c = f(c_0, c_1)$. Then by (i) and (ii), $a \equiv b(\Theta)$ and there exists $u = f(u_0, u_1) \in B$ such that $c_0 \equiv u_0(\Theta)$, $c_1 \equiv u_1(\Theta)$, and $b \equiv u(\Theta)$. Then $a \equiv u(\Theta)$ and thus (ii) implies $a \equiv c(\Phi)$, using the auxiliary element u.

(3) $a \in B$, $b \notin B$, $c \in B$, $b = f(b_0, b_1)$. Then by (ii) there exist $u = f(u_0, u_1) \in B$, $v = f(v_0, v_1) \in B$ such that $a \equiv u(\Theta)$, $u_0 \equiv b_0(\Theta)$, $u_1 \equiv b_1(\Theta)$, and $b_0 \equiv v_0(\Theta)$, $b_1 \equiv v_1(\Theta)$, and $v \equiv c(\Theta)$. Then $u_0 \equiv v_0(\Theta)$ and $u_1 \equiv v_1(\Theta)$; thus, $u = f(u_0, u_1) \equiv f(v_0, v_1) = v(\Theta)$. Thus $a \equiv u \equiv v \equiv c(\Theta)$ which implies $a \equiv c(\Theta)$, and by (i) we obtain $a \equiv c(\Phi)$.

(4) $a \in B$, $b \notin B$, $c \notin B$, $b = f(b_0, b_1)$, $c = f(c_0, c_1)$. Then by (ii) there exists $u = f(u_0, u_1) \in B$ such that $a \equiv u(\Theta)$, $u_0 \equiv b_0(\Theta)$, $u_1 \equiv b_1(\Theta)$. We distinguish two cases according to $b \equiv c(\Phi)$ by (iii_1) or (iii_2):

(4_1) $b_0 \equiv c_0(\Theta)$, $b_1 \equiv c_1(\Theta)$.

(4_2) There exist $v = f(v_0, v_1) \in B$, $w = f(w_0, w_1) \in B$ such that $b_0 \equiv v_0(\Theta)$, $b_1 \equiv v_1(\Theta)$, $w_0 \equiv c_0(\Theta)$, $w_1 \equiv c_1(\Theta)$ and $v \equiv w(\Theta)$.

In the first case, (4_1), $u_0 \equiv c_0(\Theta)$ and $u_1 \equiv c_1(\Theta)$ and thus by (ii) we get $a \equiv c(\Phi)$, using the auxiliary element u.

In the second case, (4_2), $u_0 \equiv v_0(\Theta)$, $u_1 \equiv v_1(\Theta)$ and thus $u = f(u_0, u_1) \equiv f(v_0, v_1) = v(\Theta)$. Therefore $a \equiv u \equiv v \equiv w(\Theta)$ and so $a \equiv w(\Theta)$. Thus by (ii) we get $a \equiv c(\Phi)$, using the auxiliary element w.

(5) $a \notin B$, $b \in B$, $c \in B$. The proof is similar to that of (2).

(6) $a \notin B$, $b \in B$, $c \notin B$, $a = f(a_0, a_1)$, $c = f(c_0, c_1)$. Then, applying (ii) twice, we get the existence of $u = f(u_0, u_1) \in B$ and of $v = f(v_0, v_1) \in B$ such that $b \equiv u(\Theta)$, $u_0 \equiv a_0(\Theta)$, $u_1 \equiv a_1(\Theta)$ and $b \equiv v(\Theta)$, $v_0 \equiv c_0(\Theta)$, $v_1 \equiv c_1(\Theta)$. Then $u \equiv v(\Theta)$ and thus $a \equiv c(\Phi)$ by (iii_2), using the auxiliary elements u and v.

(7) $a \notin B$, $b \notin B$, $c \in B$. The proof is similar to that of (4).

(8) $a, b, c \notin B$, $a = f(a_0, a_1)$, $b = f(b_0, b_1)$, $c = f(c_0, c_1)$.

We have four subcases to distinguish, according to which of (iii_1) and (iii_2) give us $a \equiv b(\Phi)$ and $b \equiv c(\Phi)$.

(8_1) We apply (iii_1) twice. Then $a_0 \equiv b_0(\Theta)$, $a_1 \equiv b_1(\Theta)$, $b_0 \equiv c_0(\Theta)$, $b_1 \equiv c_1(\Theta)$; thus we get $a \equiv c(\Phi)$ by (iii_1).

(8_2) We first apply (iii_1) and then (iii_2). Then $a_0 \equiv b_0(\Theta)$, $a_1 \equiv b_1(\Theta)$, and there exist $u = f(u_0, u_1) \in B$, $v = f(v_0, v_1) \in B$ such that $b_0 \equiv u_0(\Theta)$, $b_1 \equiv u_1(\Theta)$, $v_0 \equiv c_0(\Theta)$, $v_1 \equiv c_1(\Theta)$ and $u \equiv v(\Theta)$. Then $a_0 \equiv u_0(\Theta)$, $a_1 \equiv u_1(\Theta)$; thus, by (iii_2) $a \equiv c(\Phi)$, using the auxiliary elements u and v.

(8_3) We first apply (iii_2) and then (iii_1). The proof is similar to (8_2).

(8_4) We apply (iii_2) twice. Then there exist $u = f(u_0, u_1) \in B$, $v = f(v_0, v_1) \in B$, $w = f(w_0, w_1) \in B$, $z = f(z_0, z_1) \in B$ such that $a_0 \equiv u_0(\Theta)$, $a_1 \equiv u_1(\Theta)$, $u \equiv v(\Theta)$, $v_0 \equiv b_0(\Theta)$, $v_1 \equiv b_1(\Theta)$, $b_0 \equiv w_0(\Theta)$, $b_1 \equiv w_1(\Theta)$, $w \equiv z(\Theta)$, $z_0 \equiv c_0(\Theta)$, $z_1 \equiv c_1(\Theta)$. Then $v_0 \equiv w_0(\Theta)$ and $v_1 \equiv w_1(\Theta)$, and so $v = f(v_0, v_1) \equiv f(w_0, w_1) = w(\Theta)$. Consequently, $u \equiv v \equiv w \equiv z(\Theta)$; that is, $u \equiv z(\Theta)$ and thus we get $a \equiv c(\Phi)$, using (iii_2) and the auxiliary elements u and z.

This completes the proof of Lemma 3.

To conclude this section, we give another version of Theorem 1.

Theorem 2. *Let $\mathfrak{A}$ be a partial algebra, Θ a congruence relation on $\mathfrak{A}$, and let $\bar{a} = \langle a_0, \cdots, a_\gamma, \cdots \rangle_{\gamma < \alpha}$ be a sequence of type α of elements of A, containing each element of A exactly once. Then there exists a congruence relation Φ of $\mathfrak{P}^{(\alpha)}(\tau)$ such that $\Phi \geqq \Theta_{\bar{a}}$ and $\mathbf{x}_\gamma \equiv \mathbf{x}_\delta(\Phi)$ if and only if $a_\gamma \equiv a_\delta(\Theta)$.*

Theorem 2 is simply Theorem 1 combined with the second isomorphism theorem (Theorem 11.4).

§16. SUBALGEBRAS AND HOMOMORPHISMS OF PARTIAL ALGEBRAS

In this section we will review some of the results of Chapter 1 within the framework of partial algebras.

Since the proofs in most cases remain the same we will just rephrase the results. Some further results will be reviewed in the Exercises.

Let $\mathfrak{A}$ be a partial algebra and let $\mathscr{S}(\mathfrak{A})$ denote the family of all subsets B such that $\langle B; F\rangle$ is a subalgebra of $\mathfrak{A}$ with the void set added if there are no nullary partial operations (defined in $\mathfrak{A}$). Then Theorem 9.1 remains true; in Lemma 9.3 we have to add the condition that $p(h_0, \cdots, h_{n-1})$ is defined and equals a. The only result which fails to hold for partial algebras is Lemma 9.1.

However, congruence relations of partial algebras behave differently from congruence relations of algebras.

Lemma 10.1 remains valid and we can add that it is valid not only for congruence relations, but also for strong congruence relations. Lemma 10.2 is in general false for partial algebras, but Corollary 3 of Lemma 10.2 and Lemma 10.3 are valid. Of course, we must change the proofs, since they cannot be referred to Lemma 10.2. Since we needed only Lemmas 10.1 and 10.3 to prove Theorems 10.1 and 10.2, they remain valid.

We now proceed to prove for partial algebras the converse of Theorem 10.2.

Theorem 1 (*G. Grätzer and E. T. Schmidt* [2]). *Let $\mathfrak{A}$ be a partial algebra and let $C(\mathfrak{A})$ denote the system of all congruence relations of $\mathfrak{A}$. Then $\mathfrak{C}(\mathfrak{A}) = \langle C(\mathfrak{A}); \leqq\rangle$ is an algebraic lattice. Conversely, if $\mathfrak{L}$ is an algebraic lattice, then it is isomorphic to some $\mathfrak{C}(\mathfrak{A})$.*

Proof. The first part of Theorem 1 is just a restatement of Theorem 10.2 for partial algebras. To prove the second statement, let $\mathfrak{L}$ be an algebraic lattice. Represent this algebraic lattice $\mathfrak{L}$ as $\mathfrak{I}(\mathfrak{S})$, the lattice of all ideals of a semilattice $\mathfrak{S} = \langle S; \vee\rangle$ with 0 (Theorem 6.3).

We construct the partial algebra as follows. Let $A = S$. For $a, b \in S$, define a binary partial operation f_{ab} so that $D(f_{ab}) = \{\langle a, b\rangle, \langle 0, 0\rangle\}$, $f_{ab}(a, b) = a \vee b, f_{ab}(0, 0) = 0$. Further, for every $a, b \in S$ such that $b \leqq a$ we define a unary partial operation g_{ab} so that $D(g_{ab}) = \{a, 0\}$, $g_{ab}(a) = b$, $g_{ab}(0) = 0$.

For every $a, b \in S$ such that $a \neq b$, define a unary partial operation h_{ab} such that $D(h_{ab}) = \{a, b\}$ and $h_{ab}(a) = a$, $h_{ab}(b) = 0$.

Consider the partial algebra $\mathfrak{A} = \langle A; F\rangle$, where F denotes the collection of all these partial operations.

Consider an ideal I of the semilattice $\mathfrak{S}$ and define a binary relation Θ_I on A as follows:

$$x \equiv y(\Theta_I) \quad \text{if and only if} \quad x = y \quad \text{or} \quad x, y \in I.$$

We shall now verify that Θ_I is a congruence relation of $\mathfrak{A}$. It is clear that Θ_I is reflexive, symmetric, and transitive.

To prove the substitution property for f_{ab}, assume that $x_0 \equiv y_0(\Theta_I)$ and $x_1 \equiv y_1(\Theta_I)$, and that $f_{ab}(x_0, x_1)$ and $f_{ab}(y_0, y_1)$ exist and $\langle x_0, y_0\rangle \neq \langle x_1, y_1\rangle$. Then $\langle x_0, x_1\rangle = \langle a, b\rangle$ and $\langle y_0, y_1\rangle = \langle 0, 0\rangle$ (or $\langle y_0, y_1\rangle = \langle a, b\rangle$ and $\langle x_0, x_1\rangle = \langle 0, 0\rangle$). Then the conditions mean that $a, b \in I$. By applying f_{ab}, we get $a \vee b \equiv 0(\Theta_I)$, which is true since $0, a \vee b \in I$.

Similarly, the substitution property for g_{ab} is satisfied since $a \in I$, $b \leqq a$ imply $b \in I$; the substitution property for h_{ab} is satisfied since $a \neq b$, $a \equiv b(\Theta_I)$ imply $a, 0 \in I$.

Thus we have proved that:

(i) Θ_I is a congruence relation.

The following statement is trivial:

(ii) $\Theta_I \leqq \Theta_J$ if and only if $I \subseteq J$.

(iii) Let Θ be any congruence relation on $\mathfrak{A}$ and define

$$I = \{x \mid x \equiv 0(\Theta)\}.$$

Then I is an ideal.

To prove (iii), let $a, b \in I$. This means that $a \equiv 0(\Theta)$, $b \equiv 0(\Theta)$. Therefore, $a \vee b = f_{ab}(a, b) \equiv f_{ab}(0, 0) = 0(\Theta)$ and so $a \vee b \in I$.

Let $a \in I$, $b \leqq a$; then $a \equiv 0(\Theta)$ and thus $b = g_{ab}(a) \equiv g_{ab}(0) = 0(\Theta)$ and so $b \in I$, which completes the proof of (iii).

(iv) Let Θ be a congruence relation, $I = \{x \mid x \equiv 0(\Theta)\}$. Then $\Theta = \Theta_I$.

$\Theta_I \leqq \Theta$ is trivial. To prove that $\Theta_I \geqq \Theta$, let $x \equiv y(\Theta)$, $x \neq y$. Then

$$x = h_{xy}(x) \equiv h_{xy}(y) = 0(\Theta),$$

that is, $x \in I$. Similarly, $y \in I$. Thus, $x \equiv y(\Theta_I)$.

Statements (i), (ii), (iii), (iv) prove that the correspondence $I \rightarrow \Theta_I$ is an isomorphism between $\mathfrak{I}(\mathfrak{S})$ and $\mathfrak{C}(\mathfrak{A})$, completing the proof of Theorem 1.

Now we consider the problem of defining the concept of a homomorphic image of a partial algebra. Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras, and let φ be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Then the relative subalgebra $\langle A\varphi; F\rangle$ of $\mathfrak{B}$ is not necessarily isomorphic to the quotient algebra $\langle A/\varepsilon_\varphi; F\rangle$, not even if φ is 1–1 and onto. Consider the following trivial example. Let $A=\{x\}$, $B=\{y\}$, $F=\{f\}$, $\tau=\langle 1\rangle$, $D(f, \mathfrak{A})=\varnothing$, $D(f, \mathfrak{B})=\{y\}$, and $f(y)=y$, $\varphi\colon x\to y$. Then φ is a 1–1 homomorphism of $\langle A; F\rangle$ onto $\langle B; F\rangle$ but $\langle A; F\rangle\not\cong\langle A\varphi; F\rangle$ since f is not defined in $\langle A; F\rangle$, whereas it is defined in $\langle A\varphi; F\rangle$. The reason for this is that only

$$D(f_\gamma, \mathfrak{A})\varphi \subseteq D(f_\gamma, \mathfrak{A}\varphi)$$

holds in general, and we do not always have equality. Therefore, we define $\mathfrak{B}$ to be a *homomorphic image* of $\mathfrak{A}$ if there exists a homomorphism $\varphi\colon A\to B$ which is onto and full.

Note that an isomorphism is always a full homomorphism.

Adopting this definition, we encounter no difficulty in proving the homomorphism theorem for full homomorphisms. Also, the isomorphism theorems carry over, without any difficulty, the first isomorphism theorem (Theorem 11.2) for strong congruences, and the second isomorphism theorem (Theorem 11.4) for all congruences.

We can then define *endomorphisms*, *full endomorphisms*, and *strong endomorphisms* and consider the sets

$$E(\mathfrak{A}),\quad E_F(\mathfrak{A}),\quad \text{and}\quad E_S(\mathfrak{A})$$

of all endomorphisms, full endomorphisms, and strong endomorphisms of the partial algebra $\mathfrak{A}$, respectively.

Then $E(\mathfrak{A})\supseteq E_F(\mathfrak{A})\supseteq E_S(\mathfrak{A})$.

Lemma 1. *$\langle E(\mathfrak{A}); \cdot\rangle$, $\langle E_F(\mathfrak{A}); \cdot\rangle$, and $\langle E_S(\mathfrak{A}); \cdot\rangle$ are semigroups with unit element and the first contains the second and third and the second contains the third as subsemigroups.*

Finally, we will prove an embedding theorem for partial algebras which is similar to Theorem 13.3 and which characterizes the strong congruence relations.

Theorem 2. *Let $\mathfrak{A}$ be a partial algebra and let Θ be a congruence relation of $\mathfrak{A}$. The congruence relation Θ is strong if and only if $\mathfrak{A}$ can be embedded in an algebra $\mathfrak{B}$ and Θ can be extended to a congruence relation $\overline{\Theta}$ of $\mathfrak{B}$ such that*

$$[a]\Theta = [a]\overline{\Theta} \quad \textit{for all} \quad a\in A.$$

The algebra $\mathfrak{B}$ can always be chosen as $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ (see Theorem 14.2).

Remark. This condition means that $\overline{\Theta}_A=\Theta$ and any equivalence class of Θ in A is also an equivalence class of $\overline{\Theta}$ in B. Theorem 2 was announced by G. Grätzer in the Notices Amer. Math. Soc. 13 (1966), p. 146. A direct proof of Theorem 2 without the last statement can be given using the construction of Theorem 13.1.

Proof. We first prove that if such an embedding exists, then Θ is strong. Recall that a congruence relation Θ is strong if whenever $f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \in A$ and $a_i \equiv b_i(\Theta)$, then $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is defined in $\mathfrak{A}$.

Since $f_\gamma(b_0, \cdots, b_{n_\gamma - 1})$ is always defined in $\mathfrak{B}$, all we have to prove is that it is in A. Set $a = f_\gamma(a_0, \cdots, a_{n_\gamma - 1})$; then by assumption $[a]\Theta = [a]\overline{\Theta}$.

Since $\overline{\Theta}$ is an extension of Θ, we have that $a_i \equiv b_i(\overline{\Theta})$ and thus

$$f_\gamma(a_0, \cdots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \cdots, b_{n_\gamma - 1})(\overline{\Theta}),$$

that is,

$$f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) \in [a]\overline{\Theta} = [a]\Theta \subseteq A.$$

Thus,

$$f_\gamma(b_0, \cdots, b_{n_\gamma - 1}) \in A,$$

which was to be proved.

Now assume that Θ is a strong congruence relation and put

$$\mathfrak{B} = \mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}.$$

We extend Θ to $\mathfrak{B}$ using Lemmas 15.1 and 15.3.

We prove that if we assume that Θ is a strong congruence relation, then $[a]\Theta = [a]\overline{\Theta}$ holds for $a \in A$.

Suppose that in Lemma 15.1, $\langle A_0; F\rangle$ is the partial algebra we start with and that we know that for each $\gamma < \alpha$,

$$[a]\Theta^0 = [a]\Theta^\gamma.$$

Then

$$\begin{aligned} [a]\overline{\Theta} &= \bigcup ([a]\Theta^\gamma \mid \gamma < \alpha) \\ &= \bigcup ([a]\Theta^0 \mid \gamma < \alpha) \\ &= [a]\Theta^0, \end{aligned}$$

so that this property is preserved under the construction of Lemma 15.1.

Now consider the construction in Lemma 15.3. Let† $a \in B$ and assume that $[a]\Theta \neq [a]\Phi$. Then there exists a $b \notin B$ such that $a \equiv b(\Phi)$. By Rule (ii) this means that $b = f(x_0, x_1)$ and that there exists a $u = f(y_0, y_1) \in B$ such that $a \equiv u(\Theta)$, $y_0 \equiv x_0(\Theta)$ and $y_1 \equiv x_1(\Theta)$. The last two congruences together with the existence of $f(y_0, y_1)$ imply (since Θ is strong) that $f(x_0, x_1)$ exists in B, that is, $b \in B$, which is a contradiction. This completes the proof of Theorem 2.

† We use the notation of Lemma 15.3.

§17. THE CHARACTERIZATION THEOREM OF CONGRUENCE LATTICES: PRELIMINARY CONSIDERATIONS

Let $\mathfrak{A}=\langle A;\ F\rangle$ be a unary partial algebra and let $\mathfrak{B}=\langle B;\ F\rangle$ denote the algebra $\mathfrak{P}^{(\alpha)}(\tau)/\Theta_{\bar{a}}$ of Theorem 14.2. $\mathfrak{B}$ contains $\mathfrak{A}$ as a relative subalgebra and A generates $\mathfrak{B}$. If g and h are unary operations, we will write $gh(x)$ for $g(h(x))$ and similarly for n unary operations. If $b \in B$, then we can always represent b in the form

$$(*) \qquad b = g_1 \cdots g_n(a), \qquad a \in A \quad \text{and} \quad g_i \in F^*$$

where $F^* = F \cup \{e\}$ and e is the identity function on A, that is, $e(a)=a$ for all $a \in A$.

A representation (*) of b is *reduced* provided $b \in A$ and the representation is $b=e(b)$, or $b \notin A$ and $a \notin D(g_n, \mathfrak{A})$.

It is obvious from Theorem 14.3 that every element of B has a reduced representation.

Lemma 1. *The reduced representation is unique, that is, if $g_1 \cdots g_r(a)$ and $h_1 \cdots h_s(a')$ are both reduced representations of $b \in B$, then $a=a'$, $r=s$, and $g_1=h_1, \cdots, g_r=h_r$.*

Proof. This follows easily from Theorem 14.4. A more direct proof is the following.

Let $b \in B$; then $b \in A_{\langle n,\gamma\rangle}$ for some $n<\omega$, $\gamma<o(\tau)$ (Lemma 14.2). We will prove the statement by transfinite induction on $\langle n, \gamma\rangle$. The statement is known for $A=A'_{\langle 0,0\rangle}$. Assume that it has been proved for all elements of $A_{\langle m,\delta\rangle}$ with $\langle m, \delta\rangle<\langle n, \gamma\rangle$ and let $b \in A_{\langle n,\gamma\rangle}$.

We can assume by the induction hypothesis that $b \notin A'_{\langle n,\gamma\rangle}$. Thus, if $b=g_1 \cdots g_r(a)$ is any reduced representation of b, then $g_1=f_\gamma$. Let $b=g_1' \cdots g_s'(a')$ be another reduced representation of b. Then, again, $g_1'=f_\gamma$. Thus, by Definition 14.2 and Lemma 14.3, $f_\gamma(g_2 \cdots g_r(a))=f_\gamma(g_2' \cdots g_s'(a'))$ if and only if $g_2 \cdots g_r(a)=g_2' \cdots g_s'(a')$. Now we can apply the induction hypothesis to this element. This completes the proof.

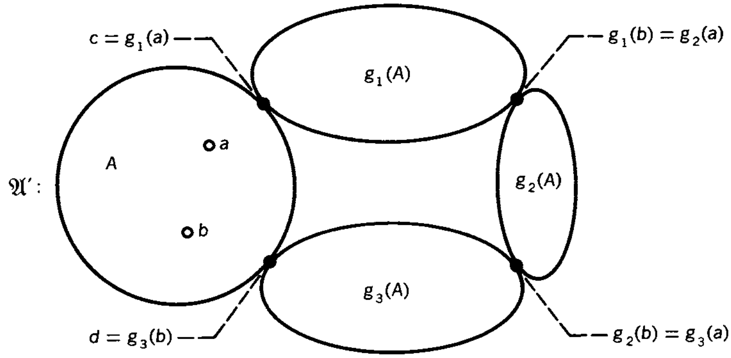
Summarizing, we have that every element of B has a reduced representation and equality of these representations is formal equality.

Let us assume that there are in F three unary partial operations g_1, g_2, and g_3 such that $D(g_1, \mathfrak{A})=\{a\}$, $D(g_3, \mathfrak{A})=\{b\}$, $D(g_2, \mathfrak{A})=\varnothing$, $g_1(a)=c$, $g_3(b)=d$, $a, b, c, d \in A$, and $a \neq b$. Form

$$A'' = A[g_1] \cup A[g_2] \cup A[g_3] \subseteq B.$$

We define in $\mathfrak{A}''=\langle A'';\ F\rangle$ a relation Φ: $x\equiv y(\Phi)$ if and only if $x=y$ or $x=g_1(b)$, $y=g_2(a)$ or $x=g_2(a)$, $y=g_1(b)$, or $x=g_2(b)$, $y=g_3(a)$ or $x=g_3(a)$,

$y = g_2(b)$. Obviously, Φ is a congruence relation. Set $\mathfrak{A}' = \mathfrak{A}''/\Phi$. By identifying $[x]\Phi$ with x, we get the diagram for $\mathfrak{A}'$. Note that $D(f_\gamma, \mathfrak{A}') = D(f_\gamma, \mathfrak{A})$ if $f_\gamma \neq g_i$ and $D(g_i, \mathfrak{A}') = A$, $i = 1, 2, 3$.



Let Θ be a congruence relation of $\mathfrak{A}$. Θ is *admissible* provided either $a \not\equiv b(\Theta)$ or $a \equiv b(\Theta)$ and $c \equiv d(\Theta)$.

Lemma 2. *Let Θ be a congruence relation of $\mathfrak{A}$. Then Θ can be extended to $\mathfrak{A}'$ if and only if Θ is admissible.*

Proof. Assume that Θ can be extended to $\mathfrak{A}'$, that is, there exists a congruence relation Φ of $\mathfrak{A}'$ such that $\Phi_A = \Theta$. If $a \equiv b(\Theta)$, then $a \equiv b(\Phi)$, and so $c = g_1(a) \equiv g_1(b) = g_2(a) \equiv g_2(b) = g_3(a) \equiv g_3(b) = d(\Phi)$, that is, $c \equiv d(\Theta)$, which was to be proved.

Assume that Θ is admissible. Define a binary relation Θ^* on A' as follows: $x \equiv y(\Theta^*)$ if $x, y \in A$ and $x \equiv y(\Theta)$, or $x, y \in g_i(A)$ for some i, $x = g_i(x')$, $y = g_i(y')$, $x', y' \in A$, and $x' \equiv y'(\Theta)$.

We claim that the transitive extension Φ of Θ^* is a congruence relation and $\Phi_A = \Theta$.

Let us agree that $g_0(x) = x$, for $x \in A$ and that the elements $g_i(a)$ and $g_i(b)$, $i = 1, 2, 3$ are called the *extreme elements* of A'. If $n \equiv i \pmod 4$, $0 \leqq i \leqq 3$, then $g_n(A)$ stands for $g_i(A)$. Then it is obvious that $g_i(A) \cap g_{i+1}(A)$ consists of one element which is an extreme element.

Since Θ is transitive on A, Θ^* is transitive on each $g_i(A)$. This implies that if $u, v \in A'$ and $u \equiv v(\Phi)$, then a nonredundant sequence $u = x_0, \cdots, x_n = v$ such that $x_{i-1} \equiv x_i(\Theta^*)$ consists of u and v and of extreme elements. Since an extreme element cannot occur twice in a nonredundant sequence, we deduce that $n \leqq 5$.

Suppose that $u, v \in g_i(A)$ and $u \equiv v(\Phi)$. Let $u = x_0, \cdots, x_n = v$ be a nonredundant sequence, as before. If $n \neq 1$, we may have $n = 3$, 4, or 5 (if u or v is an extreme element the cases $n = 3$ or $n = 4$ may occur). By possibly adding a slight redundancy, and by symmetry, we may assume that $n = 5$,

$x_1, x_4 \in g_i(A)$. Then $g_j(a) \equiv g_j(b)$ (Θ^*) for $j = 1, 2$, or 3, so $a \equiv b(\Theta)$ which implies that $g_i(a) \equiv g_i(b)$ for *all* $i = 1, 2, 3$ and $c \equiv d$ (Θ). Thus, we have $u \equiv x_1 \equiv x_4 \equiv v(\Theta^*)$, that is, $u \equiv v(\Theta^*)$.

This proves that $\Phi_{g_i(A)} = \Theta^*{}_{g_i(A)}$ and, in particular, $\Phi_A = \Theta$. It remains to prove that Φ is a congruence relation. Φ is obviously an equivalence relation. The substitution property for all $f_\gamma \neq g_i$ follows from $\Phi_A = \Theta$ and for the g_i from the definiton of Θ^*. This completes the proof of Lemma 2.

Corollary 1. *Let Θ be an admissible congruence relation of $\mathfrak{A}$ and Φ the smallest extension of Θ to $\mathfrak{A}'$. Then $u \equiv v(\Phi)$ if and only if, for some i, one of the following holds:*

(i) *$u, v \in g_i(A)$ and $u \equiv v(\Theta^*)$.*

(ii) *$u \in g_i(A)$, $v \in g_{i+1}(A)$ and for $\{x\} = g_i(A) \cap g_{i+1}(A)$, we have $u \equiv x(\Theta^*)$ and $x \equiv v(\Theta^*)$ (and the symmetric case).*

(iii) *$u \in g_i(A)$ and $v \in g_{i+2}(A)$ and for $\{x\} = g_i(A) \cap g_{i+1}(A)$ and $\{y\} = g_{i+1}(A) \cap g_{i+2}(A)$ we have $u \equiv x \equiv y \equiv v(\Theta^*)$, or the same condition for $\{x'\} = g_i(A) \cap g_{i-1}(A)$ and $\{y'\} = g_{i-1}(A) \cap g_{i-2}(A)$.*

Proof. We already know the cases (i) and (iii). To prove case (ii), it is enough to observe that in this case there are only two nonredundant sequences, namely, the one given in (ii) and $u, g_i(A) \cap g_{i-1}(A), g_{i-1}(A) \cap g_{i-2}(A), g_{i-2}(A) \cap g_{i-3}(a), v$. In the latter case, we will have $g_j(a) \equiv g_j(b)(\Theta^*)$ for $j = 1, 2$ or 3. Thus, $a \equiv b(\Theta)$ and all the extreme elements are congruent to one another and to u and v. In particular, $u \equiv x \equiv v(\Theta^*)$ for the x given in (ii).

If $u = x_0, \cdots, x_n = v$ and $x_{i-1} \equiv x_i(\Theta^*)$, then let us call this a *Θ-sequence connecting u and v.*

Corollary 2. *In cases* (i) *and* (ii), *the shortest Θ-sequence connecting u and v is unique; in case* (iii), *there are one or two shortest Θ-sequences.*

Lemma 3. *Let Θ be a congruence relation of $\mathfrak{A}$. Then there exists a smallest admissible congruence relation $\Theta^0 \geqq \Theta$.*

Proof. If $a \not\equiv b(\Theta)$, then $\Theta = \Theta^0$ and if $a \equiv b(\Theta)$, then $\Theta^0 = \Theta \vee \Theta(c, d)$.

If Θ is an admissible congruence relation of $\mathfrak{A}$, then $\bar{\Theta}$ will denote the smallest extension of Θ to $\mathfrak{A}$. Note that $\bar{\Theta}$ is described by Corollary 1 to Lemma 2.

Lemma 4. *Let $u, v \in A'$. Then there exists a smallest admissible congruence relation $\Phi(u, v)$ of $\mathfrak{A}$ such that $u \equiv v(\bar{\Phi}(u, v))$.*

Proof. We distinguish three cases as in Corollary 1 to Lemma 2. Let Θ be an admissible congruence relation such that $u \equiv v(\bar{\Theta})$.

(i) $u, v \in g_i(A)$, that is, $u = g_i(u')$ and $v = g_i(v')$, $u', v' \in A$. Then $u \equiv v(\bar{\Theta})$ if and only if $u \equiv v(\Theta^*)$, which is equivalent to $u' \equiv v'(\Theta)$, that is, $\Theta(u', v') \leqq \Theta$. This implies that in this case

$$\Phi(u, v) = (\Theta(u', v'))^0. \tag{1}$$

(ii) $u \in g_i(A), v \in g_{i+1}(A), u = g_i(u'), v = g_{i+1}(v')$. Let $\{x\} = g_i(A) \cap g_{i+1}(A)$ and $x = g_i(x') = g_{i+1}(x'')$. Then $u \equiv v(\bar{\Theta})$ if and only if $u \equiv x(\Theta^*)$ and $x \equiv v(\Theta^*)$, which implies that

$$\Phi(u, v) = (\Theta(u', x') \vee \Theta(x'', v'))^0. \tag{2}$$

(iii) $u \in g_i(A)$, $v \in g_{i+2}(A)$, $u = g_i(u')$, $v = g_{i+2}(v')$. We distinguish two subcases.

First, let $i = 0$ (the case $i = 2$ is similar). Then $u \equiv v(\bar{\Theta})$ if and only if $u \equiv c = g_1(a) \equiv g_1(b) = g_2(a) \equiv v(\Theta^*)$, or, $u \equiv d = g_3(b) \equiv g_3(a) = g_2(b) \equiv v(\Theta^*)$. Let

$$\Theta_1 = \Theta(u, c) \vee \Theta(a, b) \vee \Theta(a, v')$$

and

$$\Theta_2 = \Theta(u, d) \vee \Theta(a, b) \vee \Theta(b, v').$$

Then either $\Theta_1 \leqq \Theta$ or $\Theta_2 \leqq \Theta$. Thus, if we prove that $\Theta_1{}^0 = \Theta_2{}^0$, then $\Phi(u, v) = \Theta_1{}^0$ will be established. Observe that $a \equiv b(\Theta_1{}^0)$; thus, $c \equiv d(\Theta_1{}^0)$. Therefore, $d \equiv c \equiv u(\Theta_1{}^0)$; that is, $\Theta(u, d) \leqq \Theta_1{}^0$.

Since $\Theta(b, v') \leqq \Theta_1{}^0$, we have $\Theta_2 \leqq \Theta_1{}^0$. Similarly, $\Theta_1 \leqq \Theta_2{}^0$; thus, $\Theta_1{}^0 = \Theta_2{}^0$. Therefore, in this case,

$$\Phi(u, v) = (\Theta(u, c) \vee \Theta(a, b) \vee \Theta(a, v'))^0. \tag{3}$$

Second, let $i = 1$ (the case $i = 3$ is similar). Just as in the first subcase, we form the congruence relations $\Theta_1 = \Theta(u', b) \vee \Theta(a, b) \vee \Theta(a, v')$ and $\Theta_2 = \Theta(u', a) \vee \Theta(c, d) \vee \Theta(b, v')$ and again we have that $u \equiv v(\bar{\Theta})$ implies $\Theta_1 \leqq \Theta$ or $\Theta_2 \leqq \Theta$. We will establish $\Theta_2{}^0 \leqq \Theta_1{}^0$, which will prove $\Phi(u, v) = \Theta_2{}^0$.

Indeed, $u' \equiv b \equiv a(\Theta_1)$; thus, $\Theta(u', a) \leqq \Theta_1{}^0$. Since $\Theta_1{}^0$ is admissible and $a \equiv b(\Theta_1{}^0)$, we have $\Theta(c, d) \leqq \Theta_1{}^0$. Finally, $b \equiv a \equiv v'(\Theta_1{}^0)$; thus, $\Theta(b, v') \leqq \Theta_1{}^0$. Thus, $\Theta_2 \leqq \Theta_1{}^0$, which implies that $\Theta_2{}^0 \leqq \Theta_1{}^0$. Thus, in this case,

$$\Phi(u, v) = (\Theta(u', a) \vee \Theta(c, d) \vee \Theta(b, v'))^0. \tag{4}$$

This completes the proof of Lemma 4.

We will now generalize the results of Lemmas 2 through 4.

Consider a partial algebra $\mathfrak{S} = \langle S; F \rangle$, where

$$F = \{g_i{}^\lambda \mid \lambda \in \Lambda, i = 1, 2, 3\} \cup \{f_\sigma \mid \sigma \in \Omega\},$$

and $D(g_1^\lambda)=\{a^\lambda\}$, $D(g_3^\lambda)=\{b^\lambda\}$, $D(g_2^\lambda)=\varnothing$, $g_1^\lambda(a^\lambda)=c^\lambda$, $g_3^\lambda(b^\lambda)=d^\lambda$ and $D(f_\sigma)=S$. In other words, every partial operation is either a member of a pathological triplet, g_1, g_2, g_3 discussed above, or it is a unary operation. We call the congruence relation Θ of $\mathfrak{S}$ *admissible* if for any $\lambda\in\Lambda$, either $a^\lambda\not\equiv b^\lambda(\Theta)$ or $a^\lambda\equiv b^\lambda(\Theta)$ and $c^\lambda\equiv d^\lambda(\Theta)$. We assume that $a^\lambda\neq b^\lambda$ for $\lambda\in\Lambda$.

Lemma 3′. *Let Θ be a congruence relation of $\langle S; F\rangle$. Then there exists a smallest admissible congruence relation $\Theta^0\geqq\Theta$.*

Proof. Define $\Theta_0=\Theta$, $\Theta_{i+1}=\Theta_i\vee\bigvee(\Theta(c^\lambda, d^\lambda)\mid\lambda\in\Lambda$ and $a^\lambda\equiv b^\lambda(\Theta_i))$. It is routine to check that $\Theta^0=\bigvee(\Theta_i\mid i<\omega)$.

Let $\mathfrak{S}^\lambda$ be the partial algebra which is constructed from $\mathfrak{S}$ using g_1^λ, g_2^λ, and g_3^λ the same way as $\mathfrak{A}'$ was constructed from $\mathfrak{A}$ using g_1, g_2, and g_3. Assume that all the $\mathfrak{S}^\lambda$ are constructed in such a way that $S^\lambda\cap S^\nu=S$ if $\lambda,\nu\in\Lambda$, $\lambda\neq\nu$.

Define $S'=\bigcup(S^\lambda\mid\lambda\in\Lambda)$. Defining the operations on S' in the natural way, we get the partial algebra $\mathfrak{S}'$.

Let Θ be a congruence relation of $\mathfrak{S}$. It is obvious that if Θ can be extended to $\mathfrak{S}'$, then Θ is admissible. If Θ is admissible, then it has a smallest extension Φ_λ to $\langle S^\lambda; F\rangle$ by Lemma 2. (Note that we used the obvious fact that if Θ is admissible in the new sense, then it is admissible for any fixed $\lambda\in\Lambda$ in the old sense.)

We define a relation Φ on S' as follows: let $u\equiv v(\Phi)$ mean $u\equiv v(\Phi_\lambda)$ if $u, v\in S^\lambda$; if $u\in S^\lambda$ and $v\in S^\nu$, $\lambda,\nu\in\Lambda$, $\lambda\neq\nu$, then let $u\equiv v(\Phi)$ mean that there exists an $x\in S$ such that $u\equiv x(\Phi_\lambda)$ and $x\equiv u(\Phi_\nu)$. Φ is well defined because if $u, v\in S^\lambda$ and $u, v\in S^{\lambda'}$ with $\lambda\neq\lambda'$, $\lambda,\lambda'\in\Lambda$, then $u, v\in S^\lambda\cap S^{\lambda'}=S$. Since $(\Phi_\lambda)_S=(\Phi_{\lambda'})_S=\Theta$, we get that $u\equiv v(\Phi)$ means $u\equiv v(\Theta)$, which does not depend on λ. Φ is obviously reflexive and symmetric, and the substitution property follows from the simple observation that for any $u, v\in S'$ and operation f, if $f(u)$ and $f(v)$ are defined, then there exists a λ such that $u, v, f(u), f(v)\in S^\lambda$. Φ is also transitive. Indeed, let $u\in S^{\lambda_1}$, $v\in S^{\lambda_2}$, $w\in S^{\lambda_3}$, and $u\equiv v(\Phi)$, $v\equiv w(\Phi)$.

First, let $\lambda_1\neq\lambda_2$. Then there exists an $x\in S$ such that $u\equiv x(\Phi_{\lambda_1})$ and $x\equiv v(\Phi_{\lambda_2})$. If $\lambda_2=\lambda_3$, then $u\equiv x(\Phi_{\lambda_1})$ and $x\equiv w(\Phi_{\lambda_2})$, establishing $u\equiv w(\Phi)$. If $\lambda_2\neq\lambda_3$, then there exists a $y\in S$ such that $v\equiv y(\Phi_{\lambda_2})$ and $y\equiv w(\Phi_{\lambda_3})$. This implies that $x\equiv v\equiv y(\Phi_{\lambda_2})$ and since $x, y\in S$, we have $x\equiv y(\Theta)$. Consequently, $x\equiv y(\Phi_{\lambda_3})$. Thus, $x\equiv y\equiv w(\Phi_{\lambda_3})$. We proved that $u\equiv x(\Phi_{\lambda_1})$ and $x\equiv w(\Phi_{\lambda_3})$; thus, $u\equiv w(\Phi)$. The case $\lambda_1=\lambda_2$ can be discussed as was the case $\lambda_2=\lambda_3$.

By definition, Φ is an extension of Θ. It is also obvious that again Φ is nothing more than the transitive extension of Θ^*. (Θ^* is defined for $\mathfrak{S}'$ the same way as it was for $\mathfrak{A}'$.)

Theorem 1. *A congruence relation* Θ *of* $\mathfrak{S}$ *can be extended to* $\mathfrak{S}'$ *if and only if* Θ *is admissible. If* Θ *is admissible, the smallest extension of* Θ *to* $\mathfrak{S}'$ *is the transitive extension of* Θ^*. *Let* $u, v \in S'$. *Then there exists a smallest admissible congruence relation* $\Phi(u, v)$ *such that* $u \equiv v(\overline{\Phi}(u, v))$, *where* $\overline{\Phi}(u, v)$ *denotes the minimal extension of* $\Phi(u, v)$ *to* S'.

Proof. We have proved all but the last statement of Theorem 1. It has also been established for $u, v \in S^\lambda$ for some $\lambda \in \Lambda$.

To establish the last statement in the general case, it is useful to introduce the following terminology.

Let $u, v \in S'$ and let $\sigma: u = x_0, \cdots, x_n = v$ be a sequence of elements having the property that, for each i, x_{i-1} and $x_i \in g_j^\lambda(S)$ for some $\lambda \in \Lambda$ and $j = 1, 2, 3$. Then $x_{i-1} = g_j^\lambda(x'_{i-1})$ and $x_i = g_j^\lambda(x_i^*)$, where x'_{i-1} and x_i^* are uniquely determined elements of S. We form the congruence relation

$$(\bigvee (\Theta(x'_{i-1}, x_i^*) \mid i = 1, \cdots, n))^0$$

and we call this congruence relation Θ^σ, the *congruence relation associated with the sequence* σ. We will again call σ a Θ-*sequence* if $x_i \equiv x_{i+1}(\Theta^*)$ and σ is nonredundant. a is an *extreme element* of S' if it is an extreme element of some S^λ. It is obvious that all members of a Θ-sequence, except the first and last one, must be extreme elements; any two consecutive members are in some $g_i^\lambda(S)$; and excepting the first and last elements there are at most two consecutive extreme elements of S^λ in it; if any sequence σ has these properties, we will call it a *path*.

If Θ is an admissible congruence relation of $\mathfrak{S}$ and $\sigma : u = x_0, \cdots, x_n = v$ is a Θ-sequence connecting u and v, then $\Theta^\sigma \leqq \Theta$. Hence, to prove the existence of the smallest admissible Θ such that $u \equiv v(\overline{\Theta})$, we have to find all paths $\sigma_1, \cdots$ between u and v and we have to prove that there is a smallest congruence relation of the form Θ^{σ_i}.

Let T^λ denote the set $g_1^\lambda(S) \cup g_2^\lambda(S) \cup g_3^\lambda(S)$.

Now let $u \in S^\lambda$, $v \in S^\nu$, $u, v \notin S$, $\lambda \neq \nu$, and take a path σ connecting u and v. The sequence σ breaks up into three parts, σ_1 in T^λ, σ_2 in S, and σ_3 in T^ν; let $\sigma_1 : u = x_0, \cdots, u_0$; $\sigma_2 : u_0, v_0$; $\sigma_3 : v_0, \cdots, x_n = v$. Then u_0 is c^λ or d^λ and v_0 is c^ν or d^ν. If c^λ or d^λ is not u_0, then denote it by u_1 and similarly for v_1. Further, let σ_1' denote the path between u and u_1 which does not contain u_0.

(‡) If σ_1 contains two extreme elements, then for the sequence σ' which consists of σ_1'; u_1, v_0; and σ_3 we have $\Theta^{\sigma'} \leqq \Theta^\sigma$.

Indeed, by assumption, $g_i^\lambda(a)$ and $g_i^\lambda(b)$ are in σ_1; thus, $a^\lambda \equiv b^\lambda(\Theta^\sigma)$. Hence $c^\lambda \equiv d^\lambda(\Theta^\sigma)$, that is, $u_1 \equiv u_0(\Theta^\sigma)$ and $\Theta^{\sigma'_1} \leqq \Theta^\sigma$. This, of course, implies that $\Theta^{\sigma'} \leqq \Theta^\sigma$.

Therefore, we can find a σ connecting u and v such that $\Theta^\sigma \leqq \Theta^{\sigma'}$ for any path σ' connecting u and v, in the following way: if $u \in g_1^\lambda(S)$, then choose $u_0 = c^\lambda$; if $u \in g_3{}^\lambda(S)$, choose $u_0 = d^\lambda$; otherwise, let $u_0 = c^\lambda$ or d^λ. We choose v_0 similarly. Then let σ_1(resp. σ_3) be the path connecting u and u_0 (resp. v and v_0) and let σ equal $\sigma_1; u_0, v_0; \sigma_3$. This completes the proof of Theorem 1.

In the next step we want to extend the result of Theorem 1 to the algebra $\mathfrak{B}$ which we get from $\mathfrak{S}'$ by Theorem 14.2.

Lemma 5. *Every element $b \in B$, $b \notin S'$, has a representation of the form*

$$(**) \qquad b = h_1 \cdots h_n g_i{}^\lambda(a),$$

where $n \geqq 1$, $h_1, \cdots, h_n \in F$, and $a \in S$. If $a \neq a^\lambda$, $a \neq b^\lambda$, and $a \neq c^\nu$, $a \neq d^\nu$, for all $\nu \in \Lambda$, then the representation (**) is unique. In general, if $b = h_1{}' \cdots h_m{}' g_j{}^\nu(a')$ is another representation of b, then for some p with $0 \leqq p \leqq n$, $0 \leqq p \leqq m$ we have $h_t = h_t{}'$ for $t \leqq p$ and $h_{p+1} \cdots h_n g_i{}^\lambda(a) = h'_{p+1} \cdots h_m{}' g_j{}^\nu(a') \in S'$.

Proof. Trivial from Lemma 1 and the construction of S'.

Let $T_i{}^\lambda(h_1, \cdots, h_n)$ denote the set of all elements of the form

$$h_1 \cdots h_n g_i{}^\lambda(a)$$

for $a \in S$ and

$$T^\lambda(h_1, \cdots, h_n) = T_1{}^\lambda(h_1, \cdots, h_n) \cup T_2{}^\lambda(h_1, \cdots, h_n) \cup T_3{}^\lambda(h_1, \cdots, h_n).$$

In case $n = 0$, $T_i{}^\lambda$ will stand for $g_i{}^\lambda(S)$.

Corollary 1. *$T_i{}^\lambda(h_1, \cdots, h_n)$ and $T_{i+1}^\lambda(h_1, \cdots, h_n)$, $i = 1, 2$, have exactly one element in common, namely for $i = 1$, $h_1 \cdots h_n g_1{}^\lambda(b^\lambda) = h_1 \cdots h_n g_2{}^\lambda(a^\lambda)$, for $i = 2$, $h_1 \cdots h_n g_2{}^\lambda(b^\lambda) = h_1 \cdots h_n g_3{}^\lambda(a^\lambda)$.*

Corollary 2. *Let $b \in T_i{}^\lambda(h_1, \cdots, h_n)$; then b has one and only one representation of the form*

$$b = h_1 \cdots h_n g_i{}^\lambda(a), \qquad a \in S.$$

In other words, if we already know that $b \in T_i{}^\lambda(h_1, \cdots, h_n)$, then with *fixed* $h_1, \cdots, h_n$, λ and i in (**), a is uniquely determined.

Let us introduce the following notation:

$$S_0 = S, \cdots, S_n = \{h(x) \mid x \in S_{n-1}, h \in F^*\}.$$

Then $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots$ and

$$\bigcup (S_i \mid i = 1, 2, \cdots) = B.$$

Corollary 3. *$T^\lambda(h_1, \cdots, h_n)$ and S_n have one or two elements in common, namely, $h_1 \cdots h_n(c^\lambda) = h_1 \cdots h_n g_1{}^\lambda(a^\lambda)$ and $h_1 \cdots h_n(d^\lambda) = h_1 \cdots h_n g_3{}^\lambda(b^\lambda)$.*

Lemma 6. *The following equality holds:*

$$S_n = S_{n-1} \cup \bigcup (T^\lambda(h_1, \cdots, h_{n-1}) \mid \lambda \in \Lambda, h_1, \cdots, h_{n-1} \in F).$$

Proof. Observe that $S_1 = S \cup \bigcup (T^\lambda \mid \lambda \in \Lambda)$. Hence,

$$\begin{aligned} S_2 &= S_1 \cup \{h(x) \mid x \in S_1, h \in F\} \\ &= S_1 \cup \bigcup (\{h(x) \mid x \in T^\lambda\} \mid \lambda \in \Lambda, h \in F) \\ &= S_1 \cup \bigcup (h(T^\lambda) \mid \lambda \in \Lambda, h \in F) \\ &= S_1 \cup \bigcup (T^\lambda(h) \mid \lambda \in \Lambda, h \in F). \end{aligned}$$

This proves the statement for $n=2$. The proof of the general case is similar.

Next we define the relation Θ^* on B. Let Θ be an admissible congruence relation of $\mathfrak{S}$; let $u \equiv v(\Theta^*)$ if $u, v \in S$ and $u \equiv v(\Theta)$, or $u, v \in T_i^\lambda(h_1, \cdots, h_n)$ and $u' \equiv v'(\Theta)$, where u', v' are given by $u = h_1 \cdots h_n g_i^\lambda(u')$ and $v = h_1 \cdots h_n g_i^\lambda(v')$.

Then Θ^* is well defined; indeed, u and v uniquely determine u' and v' if $h_1, \cdots, h_n$, λ and i are fixed (Corollary 2 to Lemma 5). Furthermore, if $u, v \in T_i^\lambda(h_1, \cdots, h_n)$ and also $u, v \in T_j^\nu(g_1, \cdots, g_m)$, with $\lambda \neq \nu$ or $i \neq j$, then $u = v$, since if $u \neq v$, then one of the representations $u = h_1 \cdots h_n g_i^\lambda(u')$ or $v = h_1 \cdots h_n g_i^\lambda(v')$ is reduced.

Lemma 7. *Θ^* is reflexive and symmetric. It is transitive on S and on each $T_i^\lambda(h_1, \cdots, h_n)$. Finally, if $u \equiv v(\Theta^*)$, then $h(u) \equiv h(v)(\Theta^*)$ for any $h \in F$.*

Proof. All the statements are trivial since if $u \neq v$, $u, v \in T_i^\lambda(h_1, \cdots, h_n)$, then u and v uniquely determine n, $h_1, \cdots, h_n$, λ and i, and keeping these fixed u' and v' are unique.

Let Φ_n denote the transitive extension of Θ^* in S_n.

Lemma 8. *Φ_n is a congruence relation of $\mathfrak{S}_n = \langle S_n; F \rangle$. Furthermore, if $\overline{\Phi}_{n-1}$ denotes the minimal extension of Φ_{n-1} to $\mathfrak{S}_n$, then $\overline{\Phi}_{n-1} = \Phi_n$.*

Proof. The first statement of this lemma follows from the second statement since we know that Φ_0 is a congruence relation of $\mathfrak{S} = \mathfrak{S}_0$; thus, by the second statement, $\Phi_1 = \overline{\Phi}_0$ is a congruence relation of $\mathfrak{S}_1$, and so on.

We prove the second statement by induction on n.

$\overline{\Phi}_0 = \Phi_1$ was proved in Theorem 1.

Assume that $\overline{\Phi}_{k-1} = \Phi_k$ has already been proved for $k < n$. This implies that Φ_{n-1} is a congruence relation of $\mathfrak{S}_{n-1}$. It follows from Lemma 7 that

$\overline{\Phi}_{n-1} \geqq \Phi_n$. Finally, we prove that $\overline{\Phi}_{n-1} \leqq \Phi_n$. Let $u, v \in S_{n-1}[h] = S_{n-1} \cup \{h(x) \mid x \in S_{n-1}\}$. This notation is justified, since $S_{n-1} \cup \{h(x) \mid x \in S_{n-1}\}$ satisfies the requirements of Definition 14.2 by Lemma 5. Let Ψ_h denote the minimal extension of Φ_{n-1} to $S_{n-1}[h]$. We will prove that $u \equiv v(\Psi_h)$ implies $u \equiv v(\Phi_n)$. Lemma 8 follows from this since $\overline{\Phi}_{n-1}$ can be described in terms of Ψ_h in just the same way as Φ was described in terms of Φ_λ on page 104, and this description implies $\overline{\Phi}_{n-1} \leqq \Phi_n$.

So, let $u \equiv v(\Psi_h)$. Then by Lemma 15.3, we have to distinguish three cases:

(1) $u, v \in S_{n-1}$. Then $u \equiv v(\Phi_{n-1})$; thus, $u \equiv v(\Phi_n)$.

(2) $u \in S_{n-1}$, $v \notin S_{n-1}$. Then $v = h(v_1)$, and there exists a $w = h(w_1) \in S_{n-1}$ such that $u \equiv w(\Phi_{n-1})$ and $w_1 \equiv v_1(\Phi_{n-1})$. Thus, there exist sequences $u = x_0, \cdots, x_n = w$ and $w_1 = y_0, \cdots, y_m = v_1$ such that $x_{i-1} \equiv x_i(\Theta^*)$ and $y_{j-1} \equiv y_j(\Theta^*)$. By Lemma 7, $h(y_{j-1}) \equiv h(y_j)(\Theta^*)$; thus, the sequence $u = x_0, \cdots, x_n = w = h(w_1)$, $h(y_1), \cdots, h(y_m) = h(v_1) = v$ will establish that $u \equiv v(\Phi_n)$.

(3) $u, v \notin S_{n-1}$. Using the condition in Lemma 15.3 and Lemma 7, we get $u \equiv v(\Phi_n)$ in a manner similar to case (2). This completes the proof of Lemma 8.

Theorem 2. *Let $u, v \in B$. Then there exists a smallest admissible congruence relation Θ of $\mathfrak{S}$ such that $u \equiv v(\overline{\Theta})$, where $\overline{\Theta}$ denotes the smallest extension of Θ to B.*

Proof. We will use the following notation. If Θ is an admissible congruence relation of $\mathfrak{S}$, then Θ^n will denote the transitive extension of Θ^* in S_n. By Lemma 8, if $u, v \in S_n$, then $u \equiv v(\overline{\Theta})$ if and only if $u \equiv v(\Theta^n)$. Since for any $u, v \in B$ we have $u, v \in S_n$ for some n, Theorem 2 is equivalent to the following statement.

If $u, v \in S_n$, then there exists a smallest admissible congruence relation Θ such that $u \equiv v(\Theta^n)$.

We will prove this statement by induction on n. If $n = 1$, then this is simply Theorem 1. Assume that the statement has been proved for $n - 1$.

If $u \equiv v(\Theta^n)$, then there exists a sequence $\sigma : u = x_0, \cdots, x_m = v$ such that $x_{i-1} \equiv x_i(\Theta^*)$. By Corollary 2 to Lemma 5 and the definition of Θ^*, we can find elements x'_{i-1} and $x_i{}^*$ of S such that $x'_{i-1} \equiv x_i{}^*(\Theta)$ if and only if $x_{i-1} \equiv x_i(\Theta^*)$.

Thus, we can associate again with σ an admissible congruence relation Θ^σ and then necessarily $\Theta^\sigma \leqq \Theta$. Hence, again, we have only to find all paths $\sigma_1, \cdots$ connecting u and v and we have to prove that there exists a smallest congruence relation of the form Θ^{σ_i}.

Let $u \in T^\lambda(h_1, \cdots, h_{n-1})$; if $v \in T^\lambda(h_1, \cdots, h_{n-1})$, then we find Θ as in

Lemma 4. If $v \notin T^{\lambda}(h_1, \cdots, h_{n-1})$, then any path $u = x_0, \cdots, x_m = v$ breaks up into two parts $\sigma_1 : u = x_0, \cdots, u_0$ and $\sigma_2 : u_0, \cdots, x_m = v$, where

$$u_0 \in T^{\lambda}(h_1, \cdots, h_{n-1}) \cap S_{n-1},$$

that is, $u_0 = h_1 \cdots h_{n-1}(c^{\lambda})$ or $h_1 \cdots h_{n-1}(d^{\lambda})$. Hence, the principle (‡) of Theorem 1 applies in this case as well, that is, if the sequence σ_1 contains two extreme elements, then we take σ_1', the other nonredundant sequence between u and u_0, and the sequence σ', consisting of σ_1' and σ_2, will have the property that $\Theta^{\sigma'} \leqq \Theta^{\sigma}$. Thus, we can find the σ_i for which Θ^{σ_i} is minimal in the following manner. Let u_0 be that one of $h_1 \cdots h_{n-1}(c^{\lambda})$ and $h_1 \cdots h_{n-1}(d^{\lambda})$ for which $\sigma_1 : u, u_0$ is a sequence connecting u and u_0; if neither of them has this property, then u_0 is either of them. In this case, let σ_1 be the shortest path connecting u and u_0. If $v \in S_{n-1}$, we choose $v = v_0$. If $v \in T^{\nu}(k_1, \cdots, k_{n-1})$, $\nu \neq \lambda$, $v \notin S_{n-1}$, then we choose v_0 in the same manner as we have chosen u_0, and we define σ_3 the same way we defined σ_1. Since u_0 and v_0 are in S_{n-1}, there exists a smallest congruence relation Θ_1 such that $u_0 \equiv v_0(\Theta_1^{n-1})$. Let σ_2 be a nonredundant Θ_1-sequence which connects u_0 and v_0. Then the sequence σ which consists of σ_1, σ_2, and σ_3 will be the required sequence.

§18. THE CHARACTERIZATION THEOREM OF CONGRUENCE LATTICES

Theorem 1. *Let $\mathfrak{L}$ be an algebraic lattice. Then there exists a partial algebra $\mathfrak{B} = \langle B; F \rangle$ with the following properties:*

(i) *The congruence lattice of $\mathfrak{B}$ is isomorphic to $\mathfrak{L}$.*

(ii) *Every $f \in F$ is unary and f is either an operation or $D(f)$ consists of two elements.*

(iii) *B consists of all finite subsets of K containing 0, where K is the set of all compact elements of $\mathfrak{L}$.*

(iv) *Θ is a compact congruence relation of $\mathfrak{B}$ if and only if $\Theta = \Theta(a^*, \{0\})$, where $a^* = \{a, 0\}$, $a \in K$; the representation of Θ in this form is unique.*

Note that this result is a sharpening of Theorem 16.1. The proof is also quite similar.

Proof. Let K be given as in (iii). For $a \in K$, let us put $a^* = \{a, 0\}$; in particular, $0^* = \{0\}$.

We define B as the set of all finite subsets of K containing 0. Then $\langle B; \cup, \cap \rangle$ is a distributive lattice with 0^* as the zero element. It is also *relatively complemented*, which means that if $x \geqq y \geqq z$, then there exists a y_1 such that $y \cup y_1 = x$ and $y \cap y_1 = z$. This implies that there is a 1–1 correspondence between congruence relations and ideals; we obtain this correspondence by letting the congruence relation Θ correspond to the ideal $I_\Theta = \{x \mid x \equiv 0^*(\Theta)\}$. If I is an ideal, then $\Theta(I)$ will denote the congruence relation which corresponds to I. Let us define F to consist of the following operations and partial operations: for every $x \in B$, we define k_x and l_x by $k_x(y) = x \cup y$ and $l_x(y) = x \cap y$; for $a, b \in K$, $a \neq b$, $a \neq 0$, $b \neq 0$, we define g_{ab} by $D(g_{ab}) = \{\{a,b,0\},0^*\}$ and $g_{ab}(\{a,b,0\}) = (a \vee b)^*$, $g_{ab}(0^*) = 0^*$. Finally, for $a, b \in K$, $0 \neq b \leqq a$, we define h_{ab} by $D(h_{ab}) = \{a^*, 0^*\}$ and $h_{ab}(a^*) = b^*$, $h_{ab}(0^*) = 0^*$.

Let F denote the collection of all partial operations defined so far; let F_0 denote the collection of all operations k_x and l_x, and set $\mathfrak{B} = \langle B; F \rangle$.

A binary relation Θ is a congruence relation of $\langle B; \cup, \cap \rangle$ if and only if it is a congruence relation of $\langle B; F_0 \rangle$ (cf. Exercise 1.50). Thus, every congruence relation of $\mathfrak{B}$ is also a congruence relation of $\langle B; \cup, \cap \rangle$.

Let I be an ideal of $\langle K; \vee \rangle$ and let $\hat{I}$ denote the family of all finite subsets of I containing 0. Then $\hat{I}$ is an ideal of $\langle B; \cup, \cap \rangle$. Thus, $\hat{I}$ determines a congruence relation $\Theta(\hat{I})$. We claim that the mapping $I \to \Theta(\hat{I})$ is an isomorphism between the lattice of all ideals of $\langle K; \vee \rangle$ and the congruence lattice of $\langle B; F \rangle$. The details of the proof of this step are the same as those of Theorem 16.1, and so they can be omitted.

Now all the statements of Theorem 1 are clear; (iv) means that the compact elements correspond to the principal ideals.

In this section, let us call a partial algebra *regular* if it is of the type described on pages 103 and 104.

Lemma 1. *Let $\langle B; F' \rangle$ be a partial algebra satisfying* (ii) *of Theorem* 1. *Then there exists a regular partial algebra $\langle B; F_1 \rangle$ such that Θ is a congruence relation of $\langle B; F' \rangle$ if and only if Θ is an admissible congruence relation of $\langle B; F_1 \rangle$.*

Proof. Trivial. All we have to do is to replace every $f \in F'$ for which $D(f)$ consists of two elements a, b by three partial operations f_1, f_2, f_3 in the obvious manner.

Theorem 2. *Let $\mathfrak{A} = \langle A; F \rangle$ be a regular partial algebra having the property that if Θ is a compact congruence relation of $\mathfrak{A}$, then Θ^0 (the smallest admissible congruence relation containing Θ) is of the form $(\Theta(a, b))^0$ for some $a, b \in A$. Then there exists another regular partial algebra $\mathfrak{A}_1 = \langle A_1; F_1 \rangle$ such that the following conditions hold:*

(i) *$A \subseteq A_1$, $F \subseteq F_1$ and $\langle A; F \rangle$ is a relative subalgebra of $\langle A_1; F \rangle$.*

(ii) *Every $f \in F$ is fully defined on A_1.*

(iii) *Every admissible congruence relation Θ of $\langle A; F \rangle$ has one and only one extension $\overline{\Theta}$ to an admissible congruence relation of $\langle A_1; F_1 \rangle$.*

(iv) *Every admissible congruence relation* Φ *of* $\langle A_1; F_1\rangle$ *can be written in the form* $\Phi = \bar{\Theta}$ *for some admissible congruence relation* Θ *of* $\langle A; F\rangle$.

(v) *If* Θ *is a compact congruence relation of* $\langle A_1; F_1\rangle$, *then* Θ^0 *is of the form* $(\Theta(a, b))^0$ *for some* $a, b \in A_1$.

Proof. Let us construct the partial algebra $\langle A'; F\rangle$ as on page 104 and then let us consider the algebra $\langle A_1; F\rangle$ which we get from $\langle A'; F\rangle$ by Theorem 14.2. By Theorem 17.2, for $u, v \in A_1$, there exists a smallest admissible congruence relation Θ of $\langle A; F\rangle$ such that $u \equiv v(\bar{\Theta})$. This Θ was constructed as the least admissible congruence relation containing a compact congruence relation. Hence, by assumption

$$\Theta = (\Theta(a(u, v), b(u, v)))^0.$$

Of course, $a(u, v)$ and $b(u, v)$ are not necessarily unique but by the Axiom of Choice we can fix them.

For every $u, v \in A_1$, we define k_{uv} by $D(k_{uv}) = \{u, v\}$ and $k_{uv}(u) = a(u, v)$ $k_{uv}(v) = b(u, v)$. Let $F' = F \cup \{k_{uv} \mid u, v \in A_1\}$.

Then $\langle A_1; F'\rangle$ has the following properties:

(i′) $A \subseteq A_1$, $F \subseteq F'$, and $\langle A; F\rangle$ is a relative subalgebra of $\langle A_1; F\rangle$.

(ii′) Every $f \in F$ is fully defined on A_1.

(iii′) Every admissible congruence relation Θ of $\langle A; F\rangle$ has one and only one extension $\bar{\Theta}$ to a congruence relation of $\langle A_1; F'\rangle$.

(iv′) Every congruence relation Φ of $\langle A_1; F'\rangle$ can be written in the form $\Phi = \bar{\Theta}$ for some admissible congruence relation Θ of $\langle A; F\rangle$.

Of these, (i′) and (ii′) are trivial. To prove (iii′), first we note that by Theorem 17.1 and Theorem 15.1, every admissible congruence relation Θ of $\langle A; F\rangle$ can be extended to a congruence relation $\bar{\Theta}$ of $\langle A_1; F\rangle$. We claim that $\bar{\Theta}$ is a congruence relation of $\langle A_1; F'\rangle$, that is, the substitution property can be proved for the k_{uv}. In other words, $u \equiv v(\bar{\Theta})$ implies $a(u, v) \equiv b(u, v)(\bar{\Theta})$. Indeed, $u \equiv v(\bar{\Theta})$ implies that $\Theta \geqq \Phi(u, v) = (\Theta(a(u, v), b(u, v)))^0$, where $\Phi(u, v)$ denotes the smallest admissible congruence relation of $\langle A; F\rangle$ such that $u \equiv v(\bar{\Phi}(u, v))$. Hence, $a(u, v) \equiv b(u, v)(\Theta)$ and so $a(u, v) \equiv b(u, v)(\bar{\Theta})$.

To prove the uniqueness statement of (iii′), assume that Φ_1 and Φ_2 are both congruence relations of $\langle A_1; F'\rangle$ and that both are extensions of the admissible congruence relation Θ of $\langle A; F\rangle$. If $\Phi_1 \neq \Phi_2$, then there exist $u, v \in A_1$ such that $u \equiv v(\Phi_1)$ and $u \not\equiv v(\Phi_2)$ (or, symmetrically, $u \not\equiv v(\Phi_1)$ and $u \equiv v(\Phi_2)$). Since $u \equiv v(\Phi_1)$, we get $k_{uv}(u) \equiv k_{uv}(v)(\Phi_1)$; that is, $a(u, v) \equiv b(u, v)(\Phi_1)$. Thus, $a(u, v) \equiv b(u, v)(\Theta)$, that is, $\Theta \geqq \Phi(u, v)$. But we have that $\bar{\Theta} \leqq \Phi_2$, thus $u = v(\Phi_2)$, which is a contradiction.

(iv′) is trivial.

If we combine what we have proved so far with Lemma 1, we get the proof of (i)–(iv) of Theorem 2.

To prove (v), let Θ be a compact congruence relation of $\langle A_1; F'\rangle$, $\Theta = \bigvee (\Theta(u_i, v_i) \mid 0 \leqq i < n)$. Let Φ be a congruence relation of $\langle A; F\rangle$ defined by $\Phi = \bigvee (\Theta(a(u_i, v_i), b(u_i, v_i)) \mid 0 \leqq i < n)$. Then by assumption, $\Phi^0 = (\Theta(a, b))^0$, for some $a, b \in A$. Now it is easy to check that $\Theta = (\Theta(a, b))^0$ in $\langle A_1; F'\rangle$ implying (v).

Now we are ready to state and prove the characterization theorem for congruence lattices.

Theorem 3 (*G. Grätzer and E. T. Schmidt* [2]). *Let $\mathfrak{L}$ be an algebraic lattice. Then there exists an algebra $\mathfrak{A}$ whose congruence lattice is isomorphic to $\mathfrak{L}$.*

Proof. Consider the partial algebra $\langle B; F\rangle$ constructed in Theorem 1 and let $\langle B; F'\rangle = \langle A_0; F_0\rangle$ denote the regular partial algebra that we get from $\langle B; F\rangle$ by applying Lemma 1. By (iv) of Theorem 1, $\langle A_0; F_0\rangle$ satisfies the conditions of Theorem 2; hence, we can apply the construction of Theorem 2 and we get a regular partial algebra $\langle A_1; F_1\rangle$. By (v) of Theorem 2, $\langle A_1; F_1\rangle$ again satisfies the conditions of Theorem 2; hence, it can be applied again and we get the regular partial algebra $\langle A_2; F_2\rangle$. Proceeding thus, we construct $\langle A_n; F_n\rangle$ for every nonnegative integer n. Set $A = \bigcup (A_n \mid n < \omega)$ and $\bar{F} = \bigcup (F_n \mid n < \omega)$. We claim that $\langle A; \bar{F}\rangle$ is an algebra and its congruence lattice is isomorphic to $\mathfrak{L}$.

First we note that

$$B = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

and

$$F' = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots .$$

Let $f \in \bar{F}$ and let $a \in A$. Then $a \in A_n$ for some n and by (ii) of Theorem 2 we have that f is fully defined on A_n. Thus, $\langle A; \bar{F}\rangle$ is an algebra.

Finally, we observe that every admissible congruence relation of $\langle B; F'\rangle$ can be extended to a congruence relation of $\langle A; \bar{F}\rangle$ in one and only one way. Indeed, if Θ is an admissible congruence relation of $\langle B; F'\rangle$, then by Theorem 2 it has one and only one extension Θ_1 to $\langle A_1; F_1\rangle$, and so on. Let us define the congruence relation Θ_n of $\langle A_n; F_n\rangle$ as the only extension of Θ_{n-1} to $\langle A_n; F_n\rangle$.

Set $\Theta_\omega = \bigcup (\Theta_n \mid n < \omega)$. It is obvious that Θ_ω is a congruence relation of $\langle A; \bar{F}\rangle$. The uniqueness is also obvious since if Θ has two extensions Φ_1, Φ_2 to $\langle A; \bar{F}\rangle$, then the restriction of Φ_1 and Φ_2 to some A_n would also be different, contradicting (iii) of Theorem 2.

Thus, the congruence lattice of $\langle A; \bar{F}\rangle$ is isomorphic to the lattice of admissible congruence relations of $\langle B; F'\rangle$, which in turn by Lemma 1 is isomorphic to the lattice of congruence relations of $\mathfrak{B}$, which by Theorem 1 is isomorphic to $\mathfrak{L}$, and this is what we were required to prove.

The method of the last section can be summarized as follows: we want to construct an algebra $\mathfrak{A}$ having property P; it is easier to construct a partial algebra $\mathfrak{B}$ having P; $\mathfrak{B}$ generates an algebra $\bar{\mathfrak{A}}$, however $\bar{\mathfrak{A}}$ does not have P; introducing additional partial operations on $\bar{\mathfrak{A}}$ we make it into a partial algebra which has P; and so on $\cdots$; finally a "direct limit" is formed.

This method has been successfully used by others. For instance, A. A. Iskander [1] used this method to prove that for any algebraic lattice $\mathfrak{L}$ there exists an algebra $\mathfrak{A}$ such that $\mathfrak{L} \cong \langle \mathscr{S}(\mathfrak{A}^2); \subseteq\rangle$. See also G. Grätzer and W. A. Lampe [1].

EXERCISES

1. Characterize all partial algebras in which every relative subalgebra is a subalgebra.
2. Characterize all partial algebras in which every weak subalgebra is a relative subalgebra.
3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras and φ a full homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Prove that $\langle A\varphi; F\rangle$ is a subalgebra of $\mathfrak{B}$. Is the converse true?
4. Is it possible to distinguish within $\mathfrak{A}$ between congruence relations induced by homormorphisms and congruence relations induced by full homomorphisms?
5. Simplify Theorem 14.1 (that is, simplify the description of $\Theta_{\bar{a}}$) in case all partial operations are unary.
6. Let $\bar{a}$ be as in Theorem 14.1, and consider different representations of a polynomial symbol $\mathbf{p}$ in the form

   (*) $$\mathbf{p} = r(\mathbf{p}_0, \cdots, \mathbf{p}_{k-1}),$$

   where $\bar{a}$ can be substituted into $\mathbf{p}_i$. Is there a largest such representation (*) in the sense that if

   $$\mathbf{p} = r_1(\mathbf{p}'_0, \cdots, \mathbf{p}'_{n-1})$$

   is another such representation, then the $\mathbf{p}_i$ are polynomials of $\mathbf{p}'_0, \cdots, \mathbf{p}'_{n-1}$?
7. Prove that if $\langle n, \delta\rangle < \langle m, \lambda\rangle$, then in general

   $$A_{\langle n,\delta\rangle} \neq A_{\langle m,\lambda\rangle}.$$

8. Prove that for $\mathbf{p}_i \in \mathbf{P}^{(\alpha)}(\tau)$, $\{\mathbf{p}_0, \cdots, \mathbf{p}_{n-1}\}$ is $\mathscr{S}$-independent if and only if for $\mathbf{r}, \mathbf{s} \in \mathbf{P}^{(n)}(\tau)$, $r(\mathbf{p}_0, \cdots, \mathbf{p}_{n-1}) = s(\mathbf{p}_0, \cdots, \mathbf{p}_{n-1})$ implies $\mathbf{r} = \mathbf{s}$.

9. Prove that $\mathbf{p}_0, \cdots, \mathbf{p}_{n-1}$ is $\mathscr{S}$-independent if and only if $\langle[\mathbf{p}_0, \cdots, \mathbf{p}_{n-1}]; F\rangle$ is isomorphic to $\mathfrak{P}^{(n)}(\tau)$ and there is an isomorphism φ such that $\mathbf{p}_i\varphi = \mathbf{x}_i$.
10. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{P}^{(\alpha)}(\tau)$. Let us say that $\mathbf{p} \in B$ is irreducible in $\mathfrak{B}$ if $\mathbf{p} = r(\mathbf{p}_0, \cdots, \mathbf{p}_{n-1})$, $\mathbf{p}_0, \cdots, \mathbf{p}_{n-1} \in B$ implies $\mathbf{r} = \mathbf{x}_i$ and $\mathbf{p} = \mathbf{p}_i$. Prove that any sequence of irreducible polynomials is $\mathscr{S}$-independent.
11. Prove that every subalgebra of $\mathfrak{P}^{(\alpha)}(\tau)$ is isomorphic to some $\mathfrak{P}^{(\beta)}(\tau)$.
12. Let $\mathbf{p}$ be an n-ary polynomial symbol and let $\mathbf{q}_0, \cdots, \mathbf{q}_{n-1}$ be α-ary polynomial symbols. Let $\mathbf{p}(\mathbf{q}_0, \cdots, \mathbf{q}_{n-1})$ denote the α-ary polynomial symbol that we get from $\mathbf{p}$ by replacing every occurrence of $\mathbf{x}_i$ by $\mathbf{q}_i$. Prove that

$$\mathbf{p}(\mathbf{q}_0, \cdots, \mathbf{q}_{n-1}) = p(\mathbf{q}_0, \cdots, \mathbf{q}_{n-1}).$$

13. Prove Theorem 13.3 using only Lemmas 15.1 and 15.3.
14. Generalize Lemmas 7.3 and 7.4 for partial algebras.
15. Why does Lemma 8.4 fail for partial algebras?
16. Let Θ and Φ be congruence relations of the partial algebra $\mathfrak{A}$. Then $\Theta \vee \Phi$ is not necessarily a congruence relation of $\mathfrak{A}$ ($\vee$ is formed in $\mathfrak{E}(A)$).
17. Let $C_s(\mathfrak{A})$ denote the set of strong congruence relations of $\mathfrak{A}$. Show that $\mathfrak{C}_s(\mathfrak{A}) = \langle C_s(\mathfrak{A}); \leqq\rangle$ is a sublattice of $\mathfrak{E}(A)$.
18. Is $\mathfrak{C}(\mathfrak{A})$ a sublattice of $\mathfrak{E}(A)$? Is $\mathfrak{C}_s(\mathfrak{A})$ a sublattice of $\mathfrak{C}(\mathfrak{A})$?
19. Let $\mathfrak{A}$ and $\mathfrak{B}$ be partial algebras and φ a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. When is it possible to find algebras $\mathfrak{A}_1$ and $\mathfrak{B}_1$ such that $\mathfrak{A}$ is a relative subalgebra of $\mathfrak{A}_1$, $\mathfrak{B}$ is a relative subalgebra of $\mathfrak{B}_1$, and there exists a homomorphism ψ of $\mathfrak{A}_1$ into $\mathfrak{B}_1$ with $\psi_A = \varphi$?
20. Can you generalize Ex. 1.50 to partial algebras?
21. Prove that the description of $\Theta(a, b)$ (Theorem 10.3) does not hold for partial algebras.
22. Does Lemma 10.4 hold for partial algebras? Does it hold for strong congruence relations?
23. Prove the homomorphism theorem for full homomorphisms.
24. Under what conditions can we prove the isomorphism theorem for partial algebras? Prove the necessity of the conditions.
25. Define the concept of derived partial algebra and prove Theorem 12.1 for partial algebras.
26. Characterize those subsets B of $P(A \times A)$ for which there exists a partial algebra $\mathfrak{A} = \langle A; F\rangle$ with $B = C(\mathfrak{A})$.
27. In Lemma 15.3, is it true that for given $u, v \in A$, there exists a smallest congruence relation Θ of $\mathfrak{B}$ such that $u \equiv v(\bar{\Theta})$?
28. Let $\mathfrak{L}$ be an algebraic lattice. Show that there exists a set A such that $\mathfrak{L}$ is isomorphic to some complete sublattice of $\mathfrak{E}(A)$.
29. (P. M. Whitman) Show that every lattice can be embedded into some $\mathfrak{E}(A)$.
30. For every algebra $\langle A; F\rangle$ there exists an algebra $\langle A_1; F_1\rangle$ such that $A \subseteq A_1$, $F \subseteq F_1$, $\langle A; F\rangle$ is a subalgebra of $\langle A_1; F\rangle$ and

    (i) every congruence Θ of $\langle A; F\rangle$ can be extended to a congruence relation $\bar{\Theta}$ of $\langle A_1; F_1\rangle$;

    (ii) $\Theta \rightarrow \bar{\Theta}$ is an isomorphism between $\mathfrak{C}(\langle A; F\rangle)$ and $\mathfrak{C}(\langle A_1; F_1\rangle)$;

(iii) every compact congruence relation of $\langle A_1; F_1\rangle$ is principal. (G. Grätzer and E. T. Schmidt [1] and [2].)

31. Show that the results of §17 cannot be extended to nonunary algebras (Theorem 17.2 fails to hold, in fact the extension Φ_2 of Φ_1 from $\mathfrak{S}_1$ to $\mathfrak{S}_2$ does not necessarily have the property stated in Theorem 17.2).†
32. Let $\mathfrak{G}$ be a group. Find a simple algebra $\mathfrak{A}$ such that the automorphism group of $\mathfrak{A}$ is isomorphic to $\mathfrak{G}$.
33. Let $\mathfrak{L}$ be an algebraic lattice. Find an algebra $\mathfrak{A}$ such that the congruence lattice of $\mathfrak{A}$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{A}$ has *no* nontrivial automorphism (i.e., $\mathfrak{G}(\mathfrak{A})=1$).
34. $|C(\mathfrak{A})|=1$ implies $|G(\mathfrak{A})|=1$.
35. (W. A. Lampe) Let $\mathfrak{L}$ be an algebraic lattice in which there exists an element $a\neq 0$ such that $a\leqq \bigvee(x_i \mid i\in I)$ implies $a\leqq x_i$ for some $i\in I$. Then for any group $\mathfrak{G}$ there exists an algebra $\mathfrak{A}$ such that $\mathfrak{C}(\mathfrak{A})\cong\mathfrak{L}$ and $\mathfrak{G}(\mathfrak{A})\cong\mathfrak{G}$.
36. Let $\mathfrak{A}$ be an algebra of type τ generated by $H=\{h_\gamma \mid \gamma<\alpha\}$. There is an isomorphism φ between $\mathfrak{A}$ and $\mathfrak{P}^{(\alpha)}(\tau)$ such that $h_\gamma\varphi=\mathbf{x}_\gamma$ for $\gamma<\alpha$ if and only if one of the following conditions holds:

(i) for $\mathbf{p},\mathbf{q}\in\mathbf{P}^{(n)}(\tau)$, $n<\alpha$, $p(h_{\gamma_0},\cdots,h_{\gamma_{n-1}})=q(h_{\gamma_0},\cdots,h_{\gamma_{n-1}})$ and $\gamma_i\neq\gamma_j$ for $i\neq j$ imply $\mathbf{p}=\mathbf{q}$;

(ii) if $\mathfrak{B}$ is an algebra of type τ, $b_\gamma\in B$ for $\gamma<\alpha$, then there is a homomorphism ψ of $\mathfrak{A}$ into $\mathfrak{B}$ with $h_\gamma\psi=b_\gamma$, for $\gamma<\alpha$;

(iii) there exists a homomorphism ψ from $\mathfrak{A}$ into $\mathfrak{P}^{(\alpha)}(\tau)$ with $h_\gamma\psi=\mathbf{x}_\gamma$ for $\gamma<\alpha$.

37. Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras of type τ. Prove that $\mathfrak{A}$ and $\mathfrak{B}$ have up to isomorphism a common extension if and only if either there are no nullary operations or there are nullary operations and $\langle[\varnothing]_{\mathfrak{A}}; F\rangle\cong\langle[\varnothing]_{\mathfrak{B}}; F\rangle$.
38. (K. H. Diener‡) The following condition can be added to Ex. 36:

(iv) (ii) holds for every extension $\mathfrak{B}$ of $\mathfrak{A}$ and if there are nullary operations, $\langle[\varnothing]_{\mathfrak{A}}; F\rangle\cong\mathfrak{P}^{(0)}(\tau)$.

39. Generalize Ex. 37 for partial algebras.
40. Generalize Ex. 37 to any set of algebras (partial algebras).
41. Let $H,K\subseteq\{\mathbf{x}_\gamma \mid \gamma<\alpha\}\subseteq\mathbf{P}^{(\alpha)}(\tau)$. Prove that $H\cap K=\varnothing$ implies that $[H]\cap[K]=\varnothing$ if there are no nullary operations and $[H]\cap[K]=\mathbf{P}^{(0)}(\tau)$ otherwise.
42. Let $\mathfrak{A}$ be an infinitary partial algebra. Then there exists an infinitary algebra $\mathfrak{B}$ which contains $\mathfrak{A}$ as a relative subalgebra and has the property that every congruence relation of $\mathfrak{A}$ can be extended to $\mathfrak{B}$. (Generalize Lemmas 15.1–15.3.)
43. (W. A. Lampe) Let $\mathfrak{A}$ be an algebra, $\varphi\in E(\mathfrak{A})$, ρ_φ the right multiplication by φ on $E(\mathfrak{A})$, ε_φ and $\varepsilon_{\rho_\varphi}$ the equivalence relations induced by φ and ρ_φ on A and $E(\mathfrak{A})$ respectively. Then $\varepsilon_\varphi\to\varepsilon_{\rho_\varphi}$ is an order preserving map.

† This shows that the proofs of the Theorem of E. T. Schmidt [2], and Theorem 6 of G. Grätzer [8] are incorrect.

‡ See K. H. Diener and G. Grätzer [1].

44. (W. A. Lampe) If φ is a right-zero in $\mathfrak{E}(\mathfrak{A})$, then $\varepsilon_\varphi \geqq \varepsilon_\psi$ for all $\psi \in E(\mathfrak{A})$.
45. Let $\mathfrak{m}$ and $\mathfrak{n}$ be regular cardinals and $\mathfrak{m} < \mathfrak{n}$. Prove that every $\mathfrak{m}$-algebraic lattice is also $\mathfrak{n}$-algebraic and find an $\mathfrak{n}$-algebraic lattice which is not $\mathfrak{m}$-algebraic. (See Ex. 0.82.)
46. Describe those partial algebras $\mathfrak{A}$ in which all congruence relations are strong. ($D(f_\gamma, \mathfrak{A}) = \varnothing$ or A^{n_γ}.)
47. State and prove Theorem 11.4 for infinitary algebras.
48. An *endomorphism* of a relational system $\langle A; R\rangle$ is a mapping φ of A into A such that $r(a_0, a_1, \cdots)$ implies $r(a_0\varphi, a_1\varphi, \cdots)$ for all $r \in R$. For a unary algebra $\mathfrak{A}$ find a relational system $\langle A; R\rangle$ such that each $r \in R$ is binary and $\varphi: A \to A$ is an endomorphism of $\mathfrak{A}$ if and only if φ is an endomorphism of $\langle A; R\rangle$.
49. For every set A there exists a binary relation r such that the identity map is the only endomorphism of $\langle A; r\rangle$. (P. Vopěnka, A. Pultr and Z. Hedrlin, Comment. Math. Univ. Carolinae 6 (1965), 149–155). (Hint: for $|A| \geqq \aleph_0$ assume $A = \{\gamma \mid \gamma \leqq \delta + 1\}$ where δ is an initial ordinal. Define r by the following rules: (i) $0r2$; (ii) $\alpha r(\alpha+1)$, $\alpha \leqq \delta$; (iii) if β is a limit ordinal not cofinal with ω, then $\alpha r \beta$ if and only if α is a limit ordinal and $\alpha < \beta$; (iv) if α is a limit ordinal cofinal with ω, then $\alpha = \lim \alpha_n$, $\alpha_1 < \alpha_2 < \cdots$, and $\alpha_n = \bar{\alpha}_n + n$, where $\bar{\alpha}_n$ is a limit ordinal; set $\gamma r \alpha$ if and only if $\gamma = \alpha_n$ for some $n \geqq 2$; (v) $\alpha r(\delta+1)$ if and only if $\alpha = \delta$ or α is a nonlimit ordinal $\neq \delta + 1$.)
50. Let $\langle A; R\rangle$ be a relational system with all $r \in R$ binary. Find a binary relational system $\langle B; r\rangle$ whose endomorphism semigroup is isomorphic to the endomorphism semigroup of $\langle A; R\rangle$. (A. Pultr, Comment. Math. Univ. Carolinae 5 (1964), 227–239.) (Hint: Let $R = \{r_i \mid i \in I\}$. Set $B = A \cup \bigcup (r_i \times \{i\} \mid i \in I) \cup I \cup \{v_1, v_2, v_3, u_1, u_2\}$. Define r as follows: (i) r on I as in Ex. 49; (ii) $x_0 r \langle x_0, x_1, i\rangle r x_1$; (iii) $\langle x_0, x_1, i\rangle ri$ for $i \in I$; (iv) $v_1 r v_2 r v_3 r v_1$; (v) for $i \in I$, iru_2; (vi) $u_1 r u_2$ and $u_j r v_1$, $j = 1, 2$; (vii) for $x \in A$, xru_1.)
51. (Z. Hedrlin and A. Pultr [1]) In Theorem 12.3, $\mathfrak{A}$ can be chosen of type $\langle 1, 1\rangle$. (Hint: combine Theorem 12.3 with Ex. 48–50. In constructing $\mathfrak{A}$ from $\langle B; r\rangle$, the two unary operations should act as projection maps for r.)

PROBLEMS

11. Let $B \subseteq P(A \times A)$. When is it possible to find a partial algebra $\langle A; F\rangle$ with $B = C_s(\langle A; F\rangle)$? (See Ex. 17 and 18.)
12. Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be lattices. Under what conditions does there exist a partial algebra $\mathfrak{A}$ with $\mathfrak{C}(\mathfrak{A}) \cong \mathfrak{L}_1$ and $\mathfrak{C}_s(\mathfrak{A}) \cong \mathfrak{L}_2$? (See Ex. 17 and 18.)
13. Relate the following four classes of lattices:
L_0: the class of finite lattices; L_1: the class of lattices isomorphic to sublattices of finite partition lattices (i.e., lattices which are isomorphic to a sublattice of some $\langle \mathrm{Part}(A); \leqq\rangle$ for some finite set A); L_2: the class of lattices isomorphic to strong congruence lattices of finite partial algebras; L_3: the class of lattices isomorphic to congruence lattices of finite algebras.

14. Does Theorem 14.1 hold for infinitary partial algebras?
15. Let $B \subseteq P(A \times A)$. When is it possible to find an infinitary algebra $\mathfrak{A} = \langle A; F \rangle$ with $C(\mathfrak{A}) = B$? Characterize $\mathfrak{C}(\mathfrak{A})$.
16. Characterize $\langle \mathfrak{E}(\mathfrak{A}), \mathfrak{E}_F(\mathfrak{A}), \mathfrak{E}_S(\mathfrak{A}) \rangle$ as a triplet of semigroups. (See Lemma 16.1.)
17. Characterize the congruence lattices of algebras of finite type.
18. For an integer $n > 2$ characterize the algebraic lattices $\mathfrak{L}$ which can be represented as $\langle \mathscr{S}(\mathfrak{A}^n); \subseteq \rangle$ for some algebra $\mathfrak{A}$. (See the result mentioned on p. 113).
19. For a nonvoid set A, and integer $n > 1$, characterize those subsets $B \subseteq A^n$ for which $B = \mathscr{S}(\mathfrak{A}^n)$, for some algebra $\mathfrak{A} = \langle A; F \rangle$. (For $n = 1$ this was done in Theorems 9.1 and 9.2. In contrast with Problem 18, this is open also for $n = 2$.)
20. Develop properties of algebras whose automorphism groups are transitive doubly transitive, and so on. (See, e.g., G. Grätzer [2]).