Preface

In the theory of locally compact topological groups, the aspects and notions from abstract group theory have conquered a meaningful place from the beginning (see New Bibliography in [44] and, e.g. [41–43]). Imposing group-theoretical conditions on the closed connected subgroups of a topological group has always been the way to develop the theory of locally compact groups along the lines of the theory of abstract groups.

Despite the fact that the class of algebraic groups has become a classical object in the mathematics of the last decades, most of the attention was concentrated on reductive algebraic groups. For an affine connected solvable algebraic group G, the theorem of Lie–Kolchin has been considered as definitive for the structure of G, whereas for connected non-affine groups, the attention turns to the analytic and homological aspects of these groups, which are quasi-projective varieties (cf. [79, 80, 89]). Complex Lie groups and algebraic groups as linear groups are an old theme of group theory, but connectedness of subgroups does not play a crucial rôle in this approach, as can be seen in [97]. Non-linear complex commutative Lie groups are a main subject of complex analysis (cf. [1, 7]).

In these notes we want to include systematically algebraic groups, as well as real and complex Lie groups, in the frame of our investigation. Although affine algebraic groups over fields of characteristic zero are related to linear Lie groups (cf. [11–13]), the theorems depending on the group topology differ (cf. e.g. Remark 5.3.6). For algebraic groups we want to stress the differences between algebraic groups over a field of characteristic $p > 0$ and over fields of characteristic zero.

One essential task of group theory is the description of a given group by its composition of more elementary groups. There are two kinds of most elementary Lie groups and algebraic groups. One class is formed by such groups that have a dense cyclic subgroup; such groups are commutative. In the class of locally compact groups these groups are determined in [15], in the class of algebraic groups (over a not necessarily algebraically closed field) they are classified in [24]. The other kind of elementary groups are those which have a chain as lattice of their subgroups. In the class of finite groups they are cyclic groups of prime power order. In Lie groups and in algebraic groups the lattice of closed connected subgroups is a chain precisely in the following cases. If such groups do not have dimension one then in the class of Lie groups they are Shafarevich extensions of simple complex tori (cf. [7], Chapter 1, Section 6). In the class of algebraic groups over fields of characteristic zero they are either simple abelian varieties or extensions of a one-dimensional affine group by a simple abelian variety (cf. Theorem 4.1.3). In the class of algebraic groups over fields of positive characteristic the situation is much more complicated. Besides simple abelian varieties and extensions of a one-dimensional torus by a simple abelian variety there are also affine algebraic groups having a chain as lattice of connected closed subgroups. If they are commutative then they are Witt groups (cf. [89], Chapter 7, Sections 8 and 10), if they are not commutative then they form a very rich family of unipotent groups as our work shows.

Already J. Dieudonné was interested in groups having a chain as their lattice of closed connected subgroups; for such groups we introduce in this book the term *chain*. Namely, in [19], Section 7, he deals with non-commutative two-dimensional groups of this type and remarks that they are counter-examples to conjectures derived from the case of characteristic zero. In general, one reason for the importance of chains is the fact that a precise knowledge of them is indispensable for group theoretical investigations referring to the lattice of connected subgroups. Since the lattices of connected subgroups of the preimage and the image of an algebraic or topological epimorphism with finite kernel are isomorphic (see [71], Lemma 1.3, p. 256), we consider groups related in this way as equivalent and use for them the term isogenous. More precisely, we use the notion of isogeny for algebraic and topological groups in the sense of [77], p. 417 (see also Section 2.1). This aspect motivated us to open, up to isogeny, the door to the exotic world of algebraic non-commutative chains which consists of unipotent chains since a reduction of algebraic chains to unipotent groups can be easily achieved. Even though over fields of positive characteristic not every connected algebraic group is generated by chains (see Remark 4.1.5), our work documents that they are the fundamental ingredients of unipotent groups over fields of positive characteristic.

Already the unipotent chains of nilpotency class two are difficult to treat. Namely, for a complete classification of them one needs a classification of all non-commutative extensions G of n-dimensional Witt groups by m-dimensional Witt groups such that G has an n-dimensional commutator subgroup. Despite these great obstacles, using the classification of two-dimensional unipotent groups given in [18], II, § 3, 4.6, p. 197, we could concretely determine (through hard and involved computations) all unipotent chains of dimension three over perfect fields of characteristic greater than two. In particular we obtain that any three-dimensional unipotent non-commutative chain over a perfect field of characteristic greater than two has nilpotency class two, its commutator subgroup has dimension one and the center is two-dimensional. An induction yields that for any unipotent chain over a perfect field of characteristic greater than two, the commutator subgroup has co-dimension at least two and the center has dimension at least two (cf. Corollary 4.2.10). Moreover, the knowledge of three-dimensional chains allows us to classify, up to isogenies, all three-dimensional unipotent groups over perfect fields of characteristic greater than two (cf. Section 6). These results demonstrate the richness of examples in dimension three compared with dimension two. The plethora of unipotent k-groups over non-perfect fields and of dimension less or equal two (see [51]) justifies our restriction to algebraic groups over perfect fields.

Using the regular factor systems determining Witt groups as extensions of Witt groups by Witt groups (cf. [18], V, § 1, 1.4, p. 542 or [102]) and [18], II, § 3, 4.6, p. 197, we obtain a classification of unipotent chains over perfect fields of positive characteristic having a one-dimensional commutator subgroup (cf. Theorem 4.3.1).

As Remark 4.1.6 and Example 4.3.4 show, there are in any dimension unipotent chains of nilpotency class two having two-dimensional and three-dimensional commutator subgroup. But the involved structure of these extensions gives no hope that a complete classification of unipotent chains of nilpotency class two having a commutator subgroup of dimension greater than one could be achieved.

The classification of chains having a one-dimensional commutator subgroup yields a classification of connected algebraic groups G over perfect fields of characteristic $p > 2$ such that G has a central subgroup of co-dimension one. These groups have a representation as an almost direct product of a commutative group and a group which is a direct group of chains with amalgamated factor group (cf. Theorem 4.3.12). Moreover, we prove that in an algebraic group G having a central maximal connected subgroup the commutator subgroup G' is a (central) vector group. Conversely, if G' is a central vector group and $G/\mathfrak{z}G$ is isogenous to a Witt group, then the center $\mathfrak{z}G$ has co-dimension one in G (cf. Theorem 3.2.8). In contrast to this, a non-commutative algebraic group over a field of characteristic zero cannot have its center of co-dimension one.

Our investigations on chains G with one-dimensional commutator subgroup G' yield conditions under which an automorphism of the factor group G/G' can be extended to an automorphism of G. Using these results we can illustrate that the non-commutative chains are much more rigid than Witt groups. Namely, any connected algebraic group of algebraic automorphisms of a non-commutative unipotent chain of dimension greater than two is unipotent (cf. Corollary 4.3.11).

The lattice of normal connected algebraic subgroups of unipotent algebraic groups for which the nilpotency class is equal to their dimension n forms a chain of length n (cf. Proposition 3.1.13). Such unipotent groups occur only over fields of positive characteristic and play an opposite rôle to the one of chains which cannot have maximal nilpotency class (see Corollary 4.2.10). In Section 3.1 we introduce for any n a significant class of unipotent groups

$\mathfrak{J}_n(\alpha)$ of dimension n and nilpotency class n, characterise these groups as linear groups and study their structure. These groups show that a group with maximal nilpotency class can have a trivial adjoint representation (cf. Remark 3.1.6). At various places we use the groups $\mathfrak{J}_n(\alpha)$ as a source of counter-examples to find the limits of our theorems.

The nilpotency class of n-dimensional algebraic groups over fields of characteristic zero as well as n-dimensional real or complex Lie groups is at most $n-1$. The Lie algebras corresponding to these groups of maximal nilpotency class are called *filiform Lie algebras* and form a class thoroughly studied for thirty years (cf. [36, 37]). The filiform groups, i.e. the groups having filiform Lie algebras, play in our results on groups in characteristic zero the same rôle as the unipotent chains in positive characteristic.

The simple structure of the lattice of connected subgroups of an algebraic or analytic chain motivated us to study to which extent individual properties of chains restrict the structure of algebraic and analytic groups. Most of these properties remain invariant under isogenies.

In Section 5.2 we investigate connected algebraic groups and connected Lie groups having exactly one maximal connected closed subgroup (*uni-maximal* groups) as well as connected algebraic groups and connected Lie groups having exactly one minimal connected closed subgroup (*uni-minimal* groups). The description of non-affine algebraic groups, respectively complex Lie groups, which are uni-minimal or uni-maximal easily reduces to extensions of affine groups of dimension at most one by abelian varieties, respectively to toroidal groups (cf. Theorem 5.2.7 and Proposition 5.2.4). Connected affine algebraic groups which are uni-minimal or uni-maximal and have dimension greater than one are unipotent algebraic groups over fields of positive characteristic (cf. Proposition 5.2.6).

A non-commutative connected unipotent algebraic group G is uni-maximal if and only if the commutator subgroup of every proper connected algebraic subgroup of G is smaller than the commutator subgroup of G (cf. Theorem 5.2.17). Any group in which the commutator subgroup is a maximal connected subgroup is uni-maximal; in particular the unipotent algebraic groups over fields of positive characteristic having maximal nilpotency class are of such type (cf. Section 3). But we construct in Remark 3.1.12 also numerous examples of uni-maximal algebraic groups in which the commutator subgroup is not maximal. Moreover, any non-commutative connected three-dimensional unipotent algebraic group over a field of positive characteristic which is not a product of two non-commutative chains is uni-maximal (see Theorem 6.4.5).

Uni-minimal non-commutative groups G turn out to be products of chains, where at most one factor C has dimension greater than two; if C is not commutative then the commutator subgroup of G coincides with the commutator subgroup of C (cf. Theorem 5.2.34). This result shows that the structure of uni-minimal groups is less complicated than the structure of uni-maximal groups.

In Corollary 5.2.35 we prove that the conditions to be uni-minimal and uni-maximal are strong enough to characterise the chains over fields of characteristic greater than two. Also the condition that in algebraic groups over fields of characteristic greater than two every proper algebraic subgroup is a chain characterises the chains up to two exceptions of small dimension (cf. Theorem 5.2.30). Moreover, a connected affine algebraic group over a field of arbitrary prime characteristic, containing a chain M as a maximal connected algebraic subgroup, is either a chain or a product of M with a chain of dimension at most two (cf. Theorem 5.2.40).

In chains with a one-dimensional commutator subgroup, any connected algebraic subgroup as well as any proper epimorphic image is commutative. In general however, for algebraic groups over fields of positive characteristic none of these two conditions is sufficient for a concrete description (cf. Corollary 5.2.23 and Proposition 5.2.38). In contrast to this, for real or complex Lie groups, for formal groups and for algebraic groups over fields of characteristic zero the assumption of commutativity of all proper connected subgroups as well as the dual condition of commutativity of all proper epimorphic images is strong enough for a classification. A powerful tool to achieve this goal is the classification of Lie algebras with one of these two properties. The Lie algebras in which every subalgebra is commutative have been studied thoroughly for thirty years (cf. [21, 30–32]). If G is a non-commutative connected affine algebraic group over a field of characteristic zero such that any connected algebraic subgroup is commutative, then G is at most three-dimensional (cf. Proposition 5.3.4). For formal groups we find an analogous situation (cf. Proposition 5.3.5). In contrast to this there exist real and complex Lie groups of any dimension having only commutative proper connected subgroups, they are precisely the extra-special real or complex Lie groups (see Remark 5.3.6). A connected non-simple non-commutative affine algebraic group of dimension at least three over a field of characteristic zero such that every epimorphic image of G is commutative is a Heisenberg group (cf. Corollary 5.3.9 and Proposition 5.3.11). A connected real or complex non-simple non-commutative Lie group of dimension greater than three having only commutative proper epimorphic images is an extra-special complex Lie group having as center a simple complex torus of dimension at least two.

An affine chain of dimension n has exactly one connected algebraic subgroup for any dimension $d \leq n$ and any two epimorphic images of the same dimension are isogenous. Investigating these two properties for connected algebraic groups, respectively for real or complex Lie groups, we call any such group *aligned* if any two proper connected closed subgroups of the same dimension are isomorphic, respectively *co-aligned* if all epimorphic images of the same dimension are isogenous. For algebraic groups over fields of positive characteristic the properties to be aligned and co-aligned are too weak to obtain a concrete description for such groups (see Section 5.7 and Theorem 6.4.9). Also for connected algebraic groups over fields k of characteristic zero and for real or complex Lie groups, the condition to be co-aligned alone is not strong enough to obtain a reasonable description for groups having this property. In particular there is a rich family of nilpotent co-aligned algebraic k-groups as well as of nilpotent co-aligned real or complex Lie groups (see Remark 5.5.5). Only if we assume that these affine groups have nilpotency class two and k is algebraically closed we obtain Heisenberg groups (cf. Proposition 5.5.4).

In the class of solvable non-nilpotent connected affine algebraic groups G of dimension greater than three over algebraically closed fields of characteristic zero such that the unipotent radical of G is commutative, the co-aligned groups are precisely those for which the lattice of connected algebraic subgroups forms a projective geometry (see Theorem 5.5.16 and [71], Lemma 4.7, p. 262). If the unipotent radical U has nilpotency class two and G has dimension greater than four, then the property to be co-aligned characterises the semi-direct products of Heisenberg groups H with a one-dimensional torus T acting on H such that any closed connected subgroup of H is normalized but not centralised by T (see Theorem 5.5.21). However, the filiform groups admitting a non-trivial action of a one-dimensional torus show that a classification of co-aligned solvable non-nilpotent affine algebraic groups with unipotent radical of nilpotency class greater than two is not accessible. The same results hold for solvable non-nilpotent connected linear complex Lie groups of dimension grater than three, respectively four (cf. Theorem 5.5.16 and Theorem 5.5.21).

For co-aligned real Lie groups we meet the same difficulties as for algebraic groups and hence we arrive at a classification only for special subclasses. A connected non-commutative solvable real Lie group G of dimension greater than six having commutative commutator subgroup and containing non-trivial compact elements is co-aligned if and only if G is the direct product of a torus of dimension at most one and a semi-direct product of an even-dimensional vector group V with a one-dimensional torus such that any irreducible subspace of V has dimension two (see Proposition 5.5.23). A connected solvable real Lie group G such that the commutator subgroup G' is not commutative and the factor group G/G' has a non-trivial compact subgroup of dimension ≥ 2 is co-aligned if and only if G is a semi-direct product of two Heisenberg groups with amalgamated centre by a two-dimensional torus (cf. Proposition 5.5.27).

In contrast to the condition to be co-aligned, the property to be aligned is strong. This is documented by the fact that a non-commutative connected affine algebraic group over a field of characteristic zero or a linear complex Lie group is aligned if and only if it is unipotent and has dimension three (see Theorem 5.4.4 and Proposition 5.4.9). Moreover, the classification of the three-dimensional unipotent algebraic groups in Chapter 6 yields that a three-dimensional non-commutative connected unipotent algebraic group G over a perfect field of characteristic $p > 2$ which is aligned is uni-maximal (cf. Theorem 6.4.7). Furthermore, if G is an aligned uni-minimal group then G is a chain (cf. Corollary 6.4.8).

For non-linear complex Lie groups the condition to be aligned creates a situation which is more complicated (cf. Example 5.4.10). However, a non-commutative connected real Lie group of dimension $n \geq 4$ is aligned if and only if it is locally isomorphic to one of the following compact Lie groups: $SO_2(\mathbb{R}) \times SO_3(\mathbb{R})$, $SO_3(\mathbb{R}) \times SO_3(\mathbb{R})$, $SU_3(\mathbb{C}, 0)$, $SO_5(\mathbb{R})$ and the 14-dimensional exceptional Lie group G_2 (cf. Theorem 5.4.8).

In Section 5.7 we characterise chains by the fact that they have only few non-isogenous factors. Namely a connected unipotent algebraic group is a chain if and only if it has only finitely many connected algebraic subgroups (cf. Theorem 5.7.1). This result allows far-reaching generalisations. For instance, a non-commutative unipotent algebraic group G is a chain if and only if every epimorphic image of G is isogenous to a subgroup of G and any two connected algebraic subgroups of G of the same dimension are isogenous (cf. Corollary 5.7.4). The dual conditions also give a characterisation of non-commutative unipotent chains (cf. Corollary 5.7.8).

In the theory of abstract groups, a group is called hamiltonian or sometimes a Dedekind group if all of its subgroups are normal. For algebraic groups this condition applied to all algebraic subgroups would not be interesting (see Theorem 7.2.1). Hence we say that a connected algebraic group is *hamiltonian* if all its connected algebraic subgroups are normal. For connected algebraic groups over a field of characteristic zero also this definition is too strong. Namely, we show that any connected algebraic k-group over a field k of characteristic zero such that any connected k-subgroup is normal is commutative (cf. Theorem 7.2.10). But the situation changes drastically if we consider hamiltonian groups over fields of positive characteristic. Any non-commutative chain, more generally any uni-minimal connected algebraical group, is hamiltonian (see Theorem 7.2.19). Other examples of connected hamiltonian algebraic groups are the groups in which the centre has co-dimension one. These groups in addition are *quasi-commutative*, i.e. algebraic groups where every commutative connected algebraic subgroup is central. We remark that non-commutative, but quasi-commutative algebraic groups exist only over fields of positive characteristic; they have nilpotency class two (Proposition 3.2.26).

Quite often replacing the condition of normality for certain subgroups by the condition of quasi-normality one obtains for abstract groups results of the same significance (see [93]). If G is an algebraic group and Q is a connected algebraic subgroup of G, then there are two natural possibilities to say that Q is quasi-normal. The stronger version is to demand that $QX = XQ$ for any algebraic subgroup of G. But with respect to this definition we can prove a sharper version of Theorem 1 in [87] : A connected algebraic k-subgroup P of a connected affine algebraic group G defined over an infinite perfect field k such that $PH = HP$ for any k-closed subgroup H of G is normal in G (see Theorem 7.1.1). Because of this result we call a connected algebraic subgroup Q of an algebraic group G *quasi-normal* if it is permutable with every connected algebraic subgroup of G. Quasi-normal, but not normal algebraic subgroups exist only in algebraic groups G over fields of positive characteristic (see Corollary 7.1.5). Essentially they are contained in the unipotent radical of G (see Corollary 7.1.8). Moreover, there are non-commutative algebraic groups over fields of positive characteristic in which every algebraic subgroup is quasi-normal. Among these groups that we call *quasi-hamiltonian* there are groups which are not hamiltonian (e.g. Example 7.1.18 and Remark 7.1.20). A consequence of Corollary 7.1.8 is the fact that every connected quasi-hamiltonian algebraic group is nilpotent (cf. Theorem 7.1.11).

In Sections 7.1 and 7.2 we give many examples of quasi-hamiltonian and hamiltonian algebraic groups which are neither chains nor uni-minimal (see e.g. Example 7.2.18 and Remark 7.2.20). A big class of such groups is formed by the algebraic groups such that the factor group over their centers is a chain (see Theorem 7.1.12). Moreover, in these sections we describe some product constructions to obtain hamiltonian groups from given ones, e.g. from chains, and discuss which limitations occur.

A subclass of the class of hamiltonian algebraic groups are those groups in which every connected algebraic subgroup is characteristic; we call these groups *super-hamiltonian*. A classification of super-hamiltonian algebraic groups G is easy if G is a direct product of chains (see Corollaries 7.4.2, 7.4.3 and Proposition 7.4.4). However, the decision whether a product of chains is super-hamiltonian is difficult if the factors do not intersect trivially.

The experience from algebraic groups over fields of characteristic zero, from abelian varieties and from finite groups makes it surprising that over fields of prime characteristic there are many examples of three-dimensional non-commutative connected unipotent algebraic groups which are super-hamiltonian. In Section 7.4 we use our classification of three-dimensional connected unipotent groups over perfect fields of characteristic greater than two to decide which of these groups are super-hamiltonian. Although non-hamiltonian three-dimensional unipotent groups G exist if and only if the centre of G is one-dimensional, super-hamiltonian three-dimensional unipotent groups G do not exist only under severe restrictions on the centre, the commutator subgroup G' and the factor group G/G' (see Propositions 7.2.6 and 7.4.6).

Connected quasi-normal subgroups of connected affine algebraic groups over fields of characteristic zero, respectively of connected Lie groups, are treated in [87], respectively [86]. A connected closed subgroup Q of a topological group G is defined to be quasi-normal if it is *topologically permutable* with any closed subgroup of G, i.e. if the sets QP and PQ have the same closure for any closed subgroup P of G (see [52]). This fact motivated us to seek in Section 7.3 a unified method for the study of quasi-normal subgroups in topological and algebraic groups G. It turns out that a unified treatment is possible using a suitable closure operator on the set of subgroups of G (cf. Definition 7.3.4). This procedure allows to prove that in a connected real or complex Lie group G, any connected closed subgroup which is topologically permutable with every closed connected subgroup of G must be normal in G (see Theorem 7.3.5). In the case of real or complex Lie groups this is a positive answer to a conjecture in [53]. Moreover, as a consequence we obtain that every connected real or complex Lie group in which every connected closed subgroup is topologically permutable with any other connected closed subgroup must be commutative (see Corollary 7.3.7). Our point of view has also the advantage that p-adic Lie groups are included in the considerations, provided we modify topological permutability to locally topological permutability (cf. Definition 7.3.10). Using this we can for instance show that any p-adic Lie group G (over an ultrametric field of characteristic zero) contains an open commutative subgroup if any family of subgroups which corresponds to some subalgebra in the Lie algebra of G is locally permutable with every other such family (cf. Theorem 7.3.14).

In our work we extend all results about affine algebraic groups, respectively about linear complex Lie groups, to non-affine algebraic groups, respectively to non-linear complex Lie groups. To do this we use for algebraic groups a series of well-known results of M. Rosenlicht (cf. Section 1.3) and for complex Lie groups some theorems on complex tori and toroidal groups (cf. Section 1.1). Moreover, for algebraic groups we discuss rationality questions and try to generalize our results to algebraic k-groups.

In Chapter 1 we collect known results on real and complex Lie groups, formal groups, p-adic groups and algebraic groups as far as they are needed for our investigations.

The main part of Section 2.1 is devoted to the theory of extensions of algebraic groups, in view of our use of regular factor systems. In particular, we need to know sufficient conditions for an algebraic k-group G, which is an extension of an algebraic k-group A by an algebraic k-group B, to be k-isomorphic to a k-group defined on the k-variety $B \times A$. Although such a representation of G is not always possible (see e.g. Remarks 2.1.4 and 2.1.5), it exists if the normal subgroup A is k-split and unipotent and B is affine (cf. [82], Theorem 1, p. 99). Another aim of Section 2.1 is to discuss relations between factor systems and isogenies. If there exists an isogeny from a connected commutative unipotent group G_1 onto G_2, then an isogeny from G_2 onto G_1 exists as well (see [89], Proposition 10, p. 176). One could ask if this holds for *unipotent* connected non-commutative algebraic groups. Example 2.1.14 answers this question negatively. We establish in Proposition 2.1.12 necessary and sufficient conditions for the existence of an isogeny between central extensions of A_1 by B_1 and A_2 by B_2 which extends two given isogenies between A_1 and A_2 and between B_1 and B_2. In Proposition 2.1.13 we give sufficient conditions for central extensions of a Witt group or of a vector group by an algebraic group to be isogenous in sense of [77], p. 417 (see also Section 2.1). Moreover, for two-dimensional non-commutative unipotent groups over perfect fields of characteristic $p > 2$ and of exponent p we give another procedure to obtain, up to a coboundary, all other factor systems from a suitable one (see Proposition 2.1.9).

In Section 2.2 we deal with commutative extensions of commutative connected complex Lie groups. Since any such group is (holomorphically) isomorphic to the direct product of a linear torus $(\mathbb{C}^*)^m$, a vector group $\mathbb{C}^l$ and a toroidal group X, the theory of commutative extensions of commutative connected complex Lie groups reduces to the case of extensions which are toroidal groups. The complex n-dimensional toroidal groups X are completely described by means of period matrices, the columns of which are vectors of the lattice Λ determining X. If the $\mathbb{R}$-span of the lattice Λ has dimension $2n$, then X is a compact group, namely a complex torus. The description of complex tori by period matrices given in [7], Chapter 1, suggests to us a generalization to toroidal groups. Using period matrices, necessary and sufficient conditions for two toroidal groups to be isogenous, respectively isomorphic, are obtained. Propositions 2.2.2 and 2.2.3 allow us to recognize suitable closed subgroups and factor groups of a toroidal group within the corresponding period matrix. Moreover, a non-compact toroidal group can contain more than one maximal closed complex linear torus (cf. Example 2.2.4). Using these results we can concretely decide under which circumstances commutative holomorphic extensions of toroidal groups by toroidal groups split, and we show that, analogously to the case of complex tori, Shafarevich extensions of toroidal groups by toroidal groups exist.

If one considers, as for connected algebraic groups, connected real and complex Lie groups in which every connected closed subgroup is characteristic, then the Shafarevich extensions are prominent examples of super-hamiltonian Lie groups. If a super-hamiltonian Lie group has dimension greater than two, then it is a commutative complex Lie group which is a closed subgroup of the direct product having as factors a one-dimensional vector group, a one-dimensional linear torus and a non-trivial toroidal group which is super-hamiltonian. To settle the question when in a toroidal group X any connected closed subgroup is characteristic is easy if X is the direct product of Shafarevich extensions of a simple torus by a simple torus. In this case X is super-hamiltonian if and only if there is no non-trivial homomorphism between two distinct direct factors of X. In general, however, only a thorough analysis of the period matrix determining X allows to decide whether X is super-hamiltonian or not (see Proposition 7.4.11, Example 7.4.12 and Proposition 7.4.13).

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2

Extensions

2.1 Extensions of Unipotent Groups and Isogenies

Let G_1 and G_2 be connected algebraic k-groups. A k-homomorphism $\eta : G_1 \longrightarrow G_2$ is a k-*isogeny* if it is an epimorphism with finite kernel. By [77], Corollary p. 412, any element of the kernel of a separable k-isogeny is defined over the separable closure $\mathsf{k_s}$ of k. The groups SL_2 and PSL_2 give an example for the fact that the existence of an isogeny is not a symmetric relation. But if G_1 and G_2 are connected commutative unipotent algebraic group and $\eta : G_1 \longrightarrow G_2$ is an isogeny, then there always exists an isogeny $\theta : G_2 \longrightarrow G_1$ (see [89], Proposition 10, p. 176). The equivalence relation generated by isogenies can be defined as follows: G_1 and G_2 are *isogenous* if there exists a group G_3 and two isogenies $\eta_i : G_3 \longrightarrow G_i$ $(i = 1, 2)$, see [77], section 3, p. 417.

A main result in the context of commutative unipotent algebraic groups, defined over a perfect field k, states that any such connected k-group is k-isogenous to a direct product of Witt groups $\mathfrak{W}_n$ (cf. [18], 6.11 p. 595). The class of Witt groups $\mathfrak{W}_n$ is thoroughly described in [18], Chapitre V, or in [89], VII, 8. The class of unipotent algebraic groups isogenous to a Witt group is therefore the class of commutative unipotent chains in positive characteristic.

In this section we treat the theory of extensions of algebraic groups, as developed in a series of papers by Weil [98], Rosenlicht [77,79,82,83], Serre [89]. The purpose is to describe an algebraic group G if a normal connected algebraic subgroup A and the factor group G/A are given.

A bit more general than the direct product are the *direct product with amalgamated central subgroup* $G = G_1 \curlyvee G_2$ and the *direct product with amalgamated factor group* $G = G_1 \curlywedge G_2$, defined as follows:

2.1.1 Definition. *Let G_1, G_2 be connected algebraic groups, let $Z_i \in G_i$ be isogenous connected central algebraic subgroups and let $\mu_i : H \longrightarrow Z_i$ be two isogenies. The factor group $(G_1 \times G_2)/\Delta$, where*

$$\Delta = \{(g_1, g_2^{-1}) \in G_1 \times G_2 : \; g_i = \mu_i(x) \text{ for } i = 1, 2 \quad \text{with } x \in H\}$$

is called the direct product with amalgamated central subgroup *and is denoted by* $(G_1 \curlyvee G_2)_H$ *or simply by* $G_1 \curlyvee G_2$*, if* H *is understood.*

2.1.2 Definition. *Let* G_1, G_2 *be connected algebraic groups, let* $N_i \in G_i$ *be connected normal algebraic subgroups such that* G_1/N_1 *is isogenous to* G_2/N_2 *and let* $\mu_i : H \longrightarrow G_i/N_i$ *be two isogenies. The subgroup*

$$(G_1 \curlywedge G_2)_H = \{(g_1, g_2) \in G_1 \times G_2 : \, g_i N_i = \mu_i(x) \text{ for } i = 1, 2 \quad \text{with } x \in H\}$$

of $G_1 \times G_2$ *is called the* direct product with amalgamated factor group *and is denoted by* $(G_1 \curlywedge G_2)_H$ *or simply by* $G_1 \curlywedge G_2$*, if* H *is understood.*

The following characterisation of the above products follows by standard arguments.

2.1.3 Proposition. *The connected algebraic group* G *contains two connected algebraic subgroups* G_1, G_2 *such that* $G = G_1 G_2$*,* $[G_1, G_2] = 1$ *and* $(G_1 \cap G_2)^\circ = H$ *if and only if* G *is isogenous to the direct product with amalgamated central subgroup* $(\overline{G}_1 \curlyvee \overline{G}_2)_{\overline{H}}$*, where* $\overline{G}_i$ *is isogenous to* G_i *and* $\overline{H}$ *is isogenous to* H*.*

The connected algebraic group G *contains two connected normal algebraic subgroups* N_1, N_2 *such that the homomorphism* $\pi : G \longrightarrow G/N_1 \times G/N_2$*,* $\pi(x) = (xN_1, xN_2)$ *is an isogeny if and only if* G *is isogenous to the direct product with amalgamated factor group* $(\overline{G}_1 \curlywedge \overline{G}_2)_H$*, where* $\overline{G}_i$ *is isogenous to* G/N_i *and* H *is isogenous to* $G/(N_1 N_2)$*.* □

Now we summarize some results concerning the theory of extensions of algebraic groups. These are slightly different from the ones in the general case of abstract groups, especially in the matter of the field of definition and in the non-affine case. One of the purposes is to show that the description of an algebraic group by means of coordinate functions cries for questions of separability of the field of definition. At the end of this section we will therefore abandon the attempt of describing algebraic chains in the context of algebraic k-groups for a general field k and we will assume (mainly) that k is perfect. The principal reference are the papers [82] and [77] of Rosenlicht.

Let A and B be two connected algebraic k-groups, with A commutative and *affine*, and let $\phi : B \times B \longrightarrow A$ be a k-rational regular *factor system*, i.e. an everywhere defined k-rational map satisfying the equation $\delta^2\phi = 0$, where one defines

$$\delta^2\phi : \, (x, y, z) \mapsto \phi(x, y) + \phi(xy, z) - \phi(y, z) - \phi(x, yz). \tag{2.1}$$

The following multiplication

$$(b_1, a_1)(b_2, a_2) = (b_1 b_2, a_1 + a_2 + \phi(b_1, b_2)) \tag{2.2}$$

makes $B \times A$ an algebraic k-group G_ϕ, where $1 \times A$ is a central algebraic k-subgroup of G_ϕ, the factor group $G_\phi/(1 \times A)$ is k-isomorphic to B and it is possible to establish an exact sequence

$$1 \longrightarrow A \xrightarrow{\imath} G_\phi \xrightarrow{\pi} B \longrightarrow 1 \tag{2.3}$$

of separable k-homomorphisms (cf. [77], Theorem 4, p. 413), where $\imath(a) = (1, a - \phi(1,1))$ and $\pi(b,a) = b$. Since $\imath$ and π are separable, it is possible to identify A with the k-subgroup $1 \times A$ of G_ϕ and B with the factor group $G_\phi/(1 \times A)$. We say that G_ϕ is an *explicit central extension* of A by B, emphasizing that A is, up to a separable k-isomorphism, a *central* k-subgroup of G_ϕ. The set $\mathsf{C}^2_\mathsf{k}(B,A)$ of all k-rational regular factor systems from $B \times B$ to A is a commutative group with respect to the addition of maps. For any k-rational regular map $\psi : B \longrightarrow A$, the map

$$\delta^1\psi : (x,y) \mapsto -\psi(y) + \psi(xy) - \psi(x)$$

is a k-rational regular factor system, usually called trivial. The trivial k-rational regular factor systems form a subgroup $\mathsf{B}^2_\mathsf{k}(B,A)$ of $\mathsf{C}^2_\mathsf{k}(B,A)$ and the factor group $\mathsf{C}^2_\mathsf{k}(B,A)/\mathsf{B}^2_\mathsf{k}(B,A)$ is usually denoted by $\mathsf{H}^2_\mathsf{k}(B,A)$.

Two explicit central extensions G_{ϕ_1}, G_{ϕ_2} of A by B, given by ϕ_1, ϕ_2, respectively, are k-*equivalent* if there exists a rational k-isomorphism $\gamma : G_{\phi_1} \longrightarrow G_{\phi_2}$ such that the diagram

$$\begin{array}{ccccccc} 1 \longrightarrow & A & \longrightarrow G_{\phi_1} & \longrightarrow & B & \longrightarrow 1 \\ & id_A\downarrow & \gamma\downarrow & & id_B\downarrow & \\ 1 \longrightarrow & A & \longrightarrow G_{\phi_2} & \longrightarrow & B & \longrightarrow 1 \end{array}$$

is commutative (see [50], 6.10, p. 363ff). This happens if and only if $(\phi_1 - \phi_2) \in \mathsf{B}^2_\mathsf{k}(B,A)$, hence two extensions G_{ϕ_1} and G_{ϕ_2} are equivalent if and only if ϕ_1 and ϕ_2 differ by a trivial factor system. In particular, the extension defined by a trivial factor system $\phi \in \mathsf{B}^2_\mathsf{k}(B,A)$ is equivalent to the direct product $G_\varphi = A \times B$, which corresponds to the factor system $\varphi(x,y) = 0$ for all $x, y \in B$. In this case we say that the extension *splits*.

Thus we have a bijection between the classes of equivalent explicit central extensions and the classes of $\mathsf{H}^2_\mathsf{k}(B,A)$. (This bijection is indeed an isomorphism of groups, if one defines a group multiplication on the set of extensions G_ϕ, as described by the well-known methods of Baer [5]).

Now we turn to the problem of describing a connected algebraic k-group G by means of factor systems, once we know a connected central affine k-subgroup A and the corresponding factor group $B = G/A$.

Let G be a connected algebraic k-group and let A be a connected central *affine* algebraic k-subgroup of G. Then the embedding of A in G and the canonical projection π of G onto $B = G/A$ give an exact sequence

$$1 \longrightarrow A \longrightarrow G \xrightarrow{\pi} B \longrightarrow 1$$

of separable rational k-homomorphisms (see [77], Theorem 4, p. 413). The question whether the group G is birationally k-isomorphic to an explicit central extension G_ϕ for a suitable k-rational regular factor system ϕ is answered by the existence of a k-rational regular cross section. A rational *cross section* is a rational map from B into G such that $\pi\sigma = id$. Note that a rational cross section is not necessarily a regular one, indeed it could be defined only on an open dense subset of B.

Let σ be a k-rational regular cross section. Since $\pi\sigma = id$, one has that $\delta^1\sigma$ is a k-rational regular function from $B \times B$ to A. As A is a central subgroup of G, writing the multiplication in G additively we have

$$[\sigma(y) - \sigma(xy) + \sigma(x)] + [\sigma(z) - \sigma(xyz) + \sigma(xy)]+$$

$$-[\sigma(z) - \sigma(yz) + \sigma(y)] - [\sigma(yz) - \sigma(xyz) + \sigma(x)] =$$

$$[\sigma(y) - \sigma(xy) + \sigma(x)] + [\sigma(z) - \sigma(xyz) + \sigma(xy)]+$$

$$[-\sigma(y) + \sigma(yz) - \sigma(z)] + [-\sigma(x) + \sigma(xyz) - \sigma(yz)] =$$

$$[\sigma(y) - \sigma(xy) + \sigma(x)] + [-\sigma(x) + \sigma(xyz) - \sigma(yz)]+$$

$$[\sigma(z) - \sigma(xyz) + \sigma(xy)] + [-\sigma(y) + \sigma(yz) - \sigma(z)] =$$

$$\sigma(y) - \sigma(xy) + \sigma(xyz) - \sigma(yz)+$$

$$[\sigma(z) - \sigma(xyz) + \sigma(xy)] + [-\sigma(y) + \sigma(yz) - \sigma(z)] =$$

$$\sigma(y)+[\sigma(z)-\sigma(xyz)+\sigma(xy)]-\sigma(xy)+\sigma(xyz)-\sigma(yz)+[-\sigma(y)+\sigma(yz)-\sigma(z)] =$$

$$\sigma(y) + \sigma(z) - \sigma(yz) + [-\sigma(y) + \sigma(yz) - \sigma(z)] =$$

$$\sigma(y) + [-\sigma(y) + \sigma(yz) - \sigma(z)] + \sigma(z) - \sigma(yz) = 0.$$

Therefore $\phi = -\delta^1\sigma$ is a k-rational regular factor system which makes G birationally k-isomorphic to the explicit central extension G_ϕ defined by (2.3). In fact, define a k-rational regular map $\rho : G_\phi \longrightarrow G$ by $\rho(b, a) = \sigma(b)a$. This turns out to be a birational isomorphism, the inverse of which is $\rho^{-1}(g) = (\pi(g), g(\sigma\pi(g))^{-1})$.

For any other k-rational regular cross section τ we have, by definition, $\pi\sigma = \pi\tau = id$. This forces $(\sigma - \tau)(B) \subseteq \ker \pi$, hence $\sigma - \tau : B \longrightarrow A$ and $\delta^1(\sigma-\tau)$ is a trivial factor system. Thus the factor systems $\delta^1\sigma$ and $\delta^1\tau$ defined by two cross sections give equivalent extensions, and, up to exchanging σ with $\sigma' : x \mapsto \sigma(x)\sigma(1)^{-1}$, we can always assume that $\sigma(1) = 1$. Moreover, a group G having a k-rational regular cross section σ characterises a unique class of equivalent factor systems of $\mathsf{H}^2_{\mathsf{k}}(B, A)$, and G is birationally k-isomorphic to the direct product $A \times B$ if and only if there exists a k-rational regular section $\sigma : B \longrightarrow G$ which is a homomorphism (injective since $\pi\sigma = id$).

2.1.4 Remark. As shown, the possibility that G is birationally k-isomorphic to a suitable explicit central extension G_ϕ depends on the existence of a k-rational regular cross section. In [77], Theorem 10, p. 426 (see also [83]), it is proved that, if A is k-split, then a k-rational cross section σ exists. In general, however, σ is not a regular map. In this case $\phi = -\delta^1\sigma$ is a k-rational, but not regular, factor system, defining, according to Weil's construction in [98], a *pre-group* $G(\phi)$ which is not a group, the law of composition being not defined everywhere. However, a birational map exists which transforms the law of composition of $G(\phi)$ into the one of G. (cf. [98], Théorème p. 375 or [101], Théorème 15, p. 136, [89], Lemme 8, p. 89). The group G is therefore not explicitly described, since the factor system giving $G(\phi)$ is not defined everywhere. A natural candidate to describe this situation is a *toroidal group* in the sense of Rosenlicht [80], i.e. a connected algebraic group containing no unipotent element. Tori, abelian varieties and algebraic groups with a torus as the maximal connected affine subgroup are all toroidal. By [80], Theorem 2, p. 986, any regular map $\phi : V \times W \longrightarrow A$, where V and W are varieties and A is a toroidal group, has the shape $\phi(v, w) = \phi_1(v) + \phi_2(w)$ for suitable regular mappings $\phi_1 : V \longrightarrow A$ and $\phi_2 : W \longrightarrow A$. If ϕ is a k-rational regular factor system from $B \times B$ to A, where B is an algebraic group, then

$$0 = \phi(v, w) + \phi(vw, z) - \phi(w, z) - \phi(v, wz) =$$

$$\phi_1(v) + \phi_2(w) + \phi_1(vw) + \phi_2(z) - \phi_1(w) - \phi_2(z) - \phi_1(v) - \phi_2(wz)$$

for all $v, w, z \in B$, and from this it follows that ϕ_1 and ϕ_2 are constant maps. This shows that there exist no regular non-trivial factor systems into a toroidal group.

Let for instance E be a smooth elliptic curve defined over a finite field and let $J_{\mathfrak{m}}$ be the generalized Jacobian of E defined, according to [76], Theorem 7, p. 518 in § 3, by the modulus $\mathfrak{m} = (M)+(N)$, where M, N are two distinct non-zero points of E. This means that $J_{\mathfrak{m}}$ is a connected commutative algebraic group having no non-trivial affine image and containing a one-dimensional torus T as maximal affine subgroup such that $J_{\mathfrak{m}}/T$ is isomorphic to E. By Proposition 1.3.3 the group $J_{\mathfrak{m}}$ cannot be defined over a finite field.

According to [17], Theorem 5, one can define on the set $T \times E$ a pre-group operation by

$$(k_1, P_1) + (k_2, P_2) = (k_1 \cdot k_2 \cdot \phi(P_1, P_2), P_1 + P_2)$$

putting

$$\phi(P_1, P_2) = \frac{\ell_{P_1,P_2}(M)}{\ell_{P_1+P_2,O}(M)} \cdot \frac{\ell_{P_1+P_2,O}(N)}{\ell_{P_1,P_2}(N)}$$

where $\ell_{P,Q}(X) = 0$ is the equation of the line through P and Q (tangent at E if $P = Q$) and O is the zero of E. A birational map exists which transforms the law of composition of the pre-group $T \times E$ into the one of $J_{\mathfrak{m}}$, but we observe that ϕ is defined only if $P_1, P_2, \pm(P_1 + P_2) \notin \{M, N\}$. Therefore this is an example of an extension of the one-dimensional torus T by the elliptic curve E defined by a rational factor system, which cannot be defined by a regular factor system. □

2.1.5 Remark. In [82], Theorem 1, p. 99 or Corollary 1, p. 100, Rosenlicht shows that sufficient conditions for the existence of k-rational regular cross sections are that A is k-split and unipotent and that B is affine. In contrast to this, let G be a connected nilpotent linear algebraic group defined over a separably algebraically closed non-perfect field k such that its unipotent part G_u is not defined over k. The maximal torus T of G is defined over k and G is a T-principal fiber space over G/T. Then there is no regular cross section $\sigma : G/T \to G$ that is defined over k (see [82], p. 100).

Now we give a concrete example for the situation taking for $T = \mathbf{G}_m$ a one-dimensional torus defined over a non-perfect field k of characteristic $p > 0$. Let E be a purely inseparable extension of k of degree $[\mathsf{E} : \mathsf{k}] = n = p^t$. We consider the connected commutative k-group ΠT of [91], Section 12.4, that we recalled in Remark 1.3.9. The group ΠT has dimension n and contains T up to a birational k-isomorphism. There exists a surjective E-homomorphism $\rho : \Pi T \longrightarrow T$, the kernel $\ker \rho$ of which is the unipotent radical of ΠT, is connected, has dimension $n - 1$ and does not contain non-trivial algebraic k-subgroups of ΠT. In particular, $\ker \pi$ is not defined over k. As a connected commutative algebraic group, over E the group ΠT is isomorphic to the direct sum of its unipotent radical ΠT_u with its maximal torus T (see [8] Theorem 10.6, p. 137). Assume by contradiction that the exact sequence

$$1 \longrightarrow T \longrightarrow \Pi T \xrightarrow{\pi} \Pi T/T \longrightarrow 1$$

has a k-rational regular cross section. Then there exists also a k-regular cross section σ with $0 = \sigma\pi(0)$, and the map $\psi : g \mapsto g(\sigma\pi(g))^{-1}$ is a morphism $\Pi T \to T$ sending 0 into 0. Hence ψ is a rational homomorphism, which is separable since ψ is the identity on T (cf. [45], Theorem, p. 44) and defined over k. This implies that its kernel ΠT_u is defined over k, which is a contradiction. □

Before leaving the questions of rationality and turning to the connections between factor systems and isogenies, we want to remark once more that the existence of a k-rational regular factor system is guaranteed if A is a unipotent group defined over a perfect field k and B is affine.

For any factor system $\phi \in \mathsf{C}^2(B, A)$ there is precisely one factor system ϕ_0, equivalent to ϕ, satisfying $\phi_0(1, 1) = 0$. In the group G_{ϕ_0}, equivalent to G_ϕ, we have the useful identity $(b, a) = (b, 0)(1, a)$. Given a factor system $\phi \in \mathsf{C}^2(B, A)$, one can construct others in the following way. If $f : A \longrightarrow A$ and

$g : B \longrightarrow B$ are rational epimorphisms we define $f\phi$ by $f\phi\,(x,y) = f(\phi(x,y))$ and we define ϕg by $\phi g\,(x,y) = \phi(g(x), g(y))$. The maps $f\phi$ and ϕg are factor systems and we have:

$$f(\phi + \psi) = f\phi + f\psi, \quad (\phi + \psi)g = \phi g + \psi g, \quad (f\phi)g = f(\phi g).$$

Moreover, we get an induced epimorphism $\hat{f}$ from G_ϕ onto $G_{f\phi}$ and an induced epimorphism $\hat{g}$ from $G_{\phi g}$ onto G_ϕ by:

$$\hat{f}(b,a) = (b, f(a)) \quad \text{and} \quad \hat{g}(b,a) = (g(b), a).$$

We note that f (respectively g) is an isogeny if and only if $\hat{f}$ (respectively $\hat{g}$) is one.

One has $f\mathsf{B}^2(B,A) \subseteq \mathsf{B}^2(B,A)$ and $\mathsf{B}^2(B,A)g \subseteq \mathsf{B}^2(B,A)$. Thus we obtain actions of the rational endomorphisms of A, respectively B, on $\mathsf{H}^2(B,A)$. We denote by $[\phi]$ the coset $\phi + \mathsf{B}^2(B,A)$ and by $G_{[\phi]}$ the set of extensions equivalent to G_ϕ. For rational endomorphisms $f : A \longrightarrow A$, $g : B \longrightarrow B$ and $[\phi] \in \mathsf{H}^2(A,B)$, one has the actions given by $f \cdot [\phi] = [f\phi]$, $\quad [\phi] \cdot g = [\phi g]$.

As the group A is commutative, the set $\mathsf{End}(A)$ of rational endomorphisms of A is a ring and the action of $\mathsf{End}(A)$ on $\mathsf{H}^2(B,A)$ just defined makes $\mathsf{H}^2(B,A)$ an $\mathsf{End}(A)$-module. It must be observed, however, that in general $\mathsf{H}^2(B,A)$ is not an $\mathsf{End}(B)$-module, even if B is commutative, because in general the element $\phi(g_1 + g_2) - \phi g_1 - \phi g_2$ does not belong to $\mathsf{B}^2(B,A)$, as the following Remark shows.

2.1.6 Remark. To see concretely that the right action of $\mathsf{End}(B)$ on $\mathsf{H}^2(B,A)$ does not define a module structure, let $A = B = \mathbf{G}_a$ be the connected unipotent one-dimensional additive group, over a perfect field of characteristic p. It is shown in [18], II, § 3, 4.6, that $\mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$ is a free left $\mathsf{End}(\mathbf{G}_a)$-module, having the following family of polynomials as a basis (modulo $\mathsf{B}^2(\mathbf{G}_a, \mathbf{G}_a)$):

$$\Phi_1(x,y) = \textstyle\sum_i \frac{(p-1)!}{i!(p-i)!}\, x^i y^{p-i};$$
$$\eta_j(x,y) = xy^{p^j} \quad (j = 1, 2, \cdots).$$

Put $g_1(t) = t$ and $g_2(t) = t^p$ and consider the factor system $\theta = \eta_1(g_1 + g_2) - (\eta_1 g_1 + \eta_1 g_2)$. Then we have

$$\theta(x,y) = (x + x^p)(y + y^p)^p - xy^p - x^p y^{p^2} = x^p y^p + xy^{p^2}.$$

Since $\theta(x,y) \neq \theta(y,x)$, we infer that $\theta \notin \mathsf{B}^2(B,A)$, i.e. $\mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$ is not a right $\mathsf{End}(\mathbf{G}_a)$-module. Moreover, if $p > 2$ then we have $x^p y^p = \frac{1}{2}[(x+y)^{2p} - x^{2p} - y^{2p}] \in \mathsf{B}^2(B,A)$, thus θ is equivalent to η_2. On the other hand, for $p = 2$ we have $x^2 y^2 = \Phi_1(x,y)^2$, thus θ is equivalent to $\Phi_1^2 + \eta_2$. □

2.1.7 Remark. The above computation shows that in characteristic 2 it can happen that the factor set $\eta_k g$ does not belong to the *left* $\mathsf{End}(A)$-submodule

M generated by the set $\{\eta_j : j = 1, 2, \cdots\}$. In odd characteristic this is not possible, because any group G_{η_j} has exponent p, whereas for $\phi \notin \mathsf{M}$ the group G_ϕ has exponent p^2. More details on the right action of $\mathsf{End}(B)$ on $\mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$ can be found in Proposition 2.1.9. □

2.1.8 Remark. For the basis element Φ_1 and an arbitrary p-polynomial $g(t) = \sum_i a_i t^{p^i}$ it is easy to check that

$$[\Phi_1 g] = [\tilde{g}\Phi_1],$$

where $\tilde{g}(t) = \sum_i a_i^p t^{p^i}$. Therefore the submodule of $\mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$ consisting of symmetric factor systems is a two-sided module over the ring $\mathsf{End}(\mathbf{G}_a)$, and this is basic for the fact that, given an isogeny $\gamma_1 : G_1 \longrightarrow G_2$ of two-dimensional commutative unipotent algebraic groups, one finds an isogeny $\gamma_2 : G_2 \longrightarrow G_1$ (see Proposition 2.1.12 *(i)* or [89], § VII, n. 10). More generally this is possible by the same reason for n-dimensional commutative unipotent algebraic groups. But it is by no means possible for non-commutative factor systems, as Example 2.1.14 shows.

If however we restrict our attention to the subspace generated a monomial $g(t) = at^{p^k}$ we obtain

$$\eta_j g(x, y) = ax^{p^k} \cdot (ay^{p^k})^{p^j} = a^{1+p^j}(xy^{p^j})^{p^k} = a^{1+p^j}(\eta_j(x, y))^{p^k}.$$

Putting $\tilde{g}_j(t) = a^{1+p^j} t^{p^k}$ we get $\eta_j g = \tilde{g}_j \eta_j$. □

In the following proposition we give a general formula for the factor systems $\eta_j \in \mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$, a special case of which has been used in the above Remark 2.1.6. We recall that the ring $\mathsf{End}_\mathsf{k}(\mathbf{G}_a)$ of k-endomorphisms of the additive group $\mathbf{G}_a$ is isomorphic to the non-commutative ring $\mathsf{k}[\mathbf{F}]$ of p-polynomials, where $\mathbf{F}$ is the Frobenius homomorphism and

$$\sum_i \alpha_i \mathbf{F}^i : x \mapsto \sum_i \alpha_i x^{p^i}.$$

The following proposition shows that in odd characteristic any factor system can be derived by Φ_1 and η_1 only.

2.1.9 Proposition. *If the characteristic of the ground field is greater than 2, then in the free left* $\mathsf{End}(\mathbf{G}_a)$*-module* $\mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$ *we have*

$$[\eta_{2k}] = \sum_{i=0}^{k-1} \mathbf{F}^i[\eta_1(1 + \mathbf{F}^{2(k-i)-1})] - \left(\sum_{i=0}^{2k-1} \mathbf{F}^i\right)[\eta_1]$$

$$[\eta_{2k+1}] = \sum_{i=0}^{k} \mathbf{F}^i[\eta_1(1 + \mathbf{F}^{2(k-i)})] + \mathbf{F}^k[\eta_1] - \sum_{i=1}^{k-1} \mathbf{F}^i[\eta_1] - \sum_{i=1}^{k} \mathbf{F}^{k+i}[\eta_1].$$

Proof. Put $\alpha(x) = \frac{1}{2}x^{2p}$, hence $\delta^1\alpha(x,y) = \frac{1}{2}((x+y)^{2p} - x^{2p} - y^{2p}) = x^p y^p$. The assertion follows from the fact that

$$\eta_2 = \eta_1(1+\mathbf{F}) - (1+\mathbf{F})\eta_1 + \delta^1\alpha$$

whereas, for any $k > 1$, we have

$$\eta_k(1+\mathbf{F}) = (1+\mathbf{F})\eta_k + \eta_{k+1} + \mathbf{F}\eta_{k-1}.$$

□

2.1.10 Corollary. *If the characteristic of the ground field is greater than 2, for any* $\varphi \in \mathsf{H}^2(\mathbf{G}_a, \mathbf{G}_a)$ *there exist* $f_0, f_1, \cdots f_n, g_1, \cdots, g_n \in \mathsf{k}[\mathbf{F}]$ *such that*

$$\varphi = f_0\Phi_1 + \sum_{k=1}^{n} f_k \eta_1 g_k.$$

□

The following Remark 2.1.11, which makes Propositions 2.1.12 and 2.1.13 particularly meaningful, plays a certain rôle in Section 4.2.

2.1.11 Remark. We illustrate here the fact that the functor $\mathsf{H}^2(B, A)$ is contra-variant in B and co-variant in A in the special case of isogenies. The arguments and the notations are essentially those of [89], VII, 1. p. 164-165.

1) For any explicit central extension G_{ϕ_1}

$$1 \longrightarrow A_1 \longrightarrow G_{\phi_1} \longrightarrow B \longrightarrow 1$$

of A_1 by B, defined by the factor system $\phi_1 : B \times B \longrightarrow A_1$, and any isogeny $\alpha : A_1 \longrightarrow A_2$ there exists a unique (up to equivalence) explicit central extension G_{ϕ_2}

$$1 \longrightarrow A_2 \longrightarrow G_{\phi_2} \longrightarrow B \longrightarrow 1$$

and an isogeny $\alpha_* : G_{\phi_1} \longrightarrow G_{\phi_2}$, such that the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_1 & \longrightarrow & G_{\phi_1} & \longrightarrow & B & \longrightarrow & 1 \\ & & \alpha\downarrow & & \alpha_*\downarrow & & id_B\downarrow & & \\ 1 & \longrightarrow & A_2 & \longrightarrow & G_{\phi_2} & \longrightarrow & B & \longrightarrow & 1. \end{array}$$

Explicitly we have $[\phi_2] = \alpha[\phi_1]$ and α_* is defined by $\alpha_*(x,y) = (x, \alpha(y))$. Moreover, the group G_{ϕ_2} is the factor group of $G_{\phi_1} \times A_2$ modulo the algebraic subgroup $\Delta = \{(-a, \alpha(a)) : a \in A_1\}$.

2) For any explicit central extension G_{ϕ_1}

$$1 \longrightarrow A \longrightarrow G_{\phi_1} \xrightarrow{\pi} B_1 \longrightarrow 1$$

of A by B_1, defined by the factor system $\phi_1 : B_1 \times B_1 \longrightarrow A$, and any isogeny $\beta : B_2 \longrightarrow B_1$ there exists a unique explicit central extension G_{ϕ_2}

$$1 \longrightarrow A \longrightarrow G_{\phi_2} \xrightarrow{\pi} B_2 \longrightarrow 1$$

and an isogeny $\beta^* : G_{\phi_2} \longrightarrow G_{\phi_1}$, such that the following diagram commutes

$$\begin{array}{ccccccc} 1 \longrightarrow & A & \longrightarrow & G_{\phi_2} & \xrightarrow{\pi} & B_2 & \longrightarrow 1 \\ & id_A\downarrow & & \beta^*\downarrow & & \beta\downarrow & \\ 1 \longrightarrow & A & \longrightarrow & G_{\phi_1} & \xrightarrow{\pi} & B_1 & \longrightarrow 1. \end{array}$$

Explicitly we have $[\phi_2] = [\phi_1]\beta$ and the isogeny β^* is defined by $\beta^*(x, y) = (\beta(x), y)$. Moreover, the group G_{ϕ_2} is the algebraic subgroup of the direct product $B_2 \times G_{\phi_1}$ defined by

$$G_{\phi_2} = \{(b, g) \in B_2 \times G_{\phi_1} : \beta(b) = \pi(g)\}.$$

□

In the case where G_{ϕ_1} and G_{ϕ_2} are two central extensions

$$1 \longrightarrow A_1 \longrightarrow G_{\phi_1} \longrightarrow B_1 \longrightarrow 1$$

$$1 \longrightarrow A_2 \longrightarrow G_{\phi_2} \longrightarrow B_2 \longrightarrow 1$$

we have:

2.1.12 Proposition. *Let G_{ϕ_1} (respectively G_{ϕ_2}) be a central extension of the algebraic affine group A_1 (respectively A_2) by the (not necessarily commutative) algebraic group B_1 (respectively B_2).*

(i) There is an isogeny $i : G_{\phi_1} \longrightarrow G_{\phi_2}$ such that $i(A_1) = A_2$ if and only if there exist isogenies $f : A_1 \longrightarrow A_2$ and $g : B_1 \longrightarrow B_2$ with $f[\phi_1] = [\phi_2]g$.

(ii) If there are isogenies $f : A_1 \longrightarrow A_2$ and $g : B_2 \longrightarrow B_1$ such that

$$[\phi_2] = f[\phi_1]g \tag{2.4}$$

then the groups G_{ϕ_1} and G_{ϕ_2} are isogenous.

Proof. (i) Let $i : G_{\phi_1} \longrightarrow G_{\phi_2}$ be an isogeny such that $i(A_1) = A_2$. Then i is given by

$$i(x_0, x_1) = (g(x_0), f(x_1) + h(x_0)),$$

where $f : A_1 \longrightarrow A_2$, $g : B_1 \longrightarrow B_2$ are isogenies and $h : B_1 \longrightarrow A_2$ is a rational regular map satisfying the equation $f\phi_1 = \phi_2 g + \delta^1 h$.

Conversely, given isogenies g, f as above and a rational regular map $h : B_1 \longrightarrow A_2$ satisfying $f\phi_1 = \phi_2 g + \delta^1 h$, the mapping $i : G_{\phi_1} \longrightarrow G_{\phi_2}$

$$(x_0, x_1) \mapsto (g(x_0), f(x_1) + h(x_0))$$

is an isogeny, since

$$\mathsf{ker}(i) = \{(b, a) : b \in \mathsf{ker}(g), f(a) + h(b) = 0\}$$

is finite, and $i(A_1)$ is clearly equal to A_2.

(ii) By *(i)* we find that G_{ϕ_1} is isogenous to $G_{f_1\phi_1}$ which in turn is isogenous to $G_{f_1\phi_1 g_1} = G_{\phi_2}$. □

The next proposition shows the crucial rôle played by the Ore condition in the context of extensions of algebraic groups and isogenies.

2.1.13 Proposition. *Let A (respectively A_1, A_2) be either the Witt group $\mathfrak{W}_m$ or the vector group $(\mathbf{G}_a)^m$. Let G_ψ (respectively G_{ϕ_1}, G_{ϕ_2}) be a central extension of A (respectively A_1, A_2) by the (not necessarily commutative) algebraic group B (respectively B_1, B_2). If $\eta_i : G_\psi \longrightarrow G_{\phi_i}$ are isogenies with $\eta_i(A) = A_i$, then there exist $h_i : A_i \longrightarrow A$ and $g_i : B \longrightarrow B_i$ such that $h_2[\phi_2]g_2 = h_1[\phi_1]g_1$.*

Proof. By *(i)* there exist isogenies $f_i : A \longrightarrow A_i$ and $g_i : B \longrightarrow B_i$, $(i = 1, 2)$, such that $f_1[\psi] = [\phi_1]g_1$ and $f_2[\psi] = [\phi_2]g_2$. It is shown in [18], V, § 3, 6.9, p. 593, that in the semigroup of isogenies of a Witt group the Ore condition holds. For a vector group this follows from [54], § 10, p. 313. Hence can find two isogenies $h_i : A_i \longrightarrow A$ such that $h_1 f_1 = h_2 f_2$ and we obtain $h_2[\phi_2]g_2 = h_1[\phi_1]g_1$. □

We have already mentioned that the groups SL_2 and PSL_2 show that the existence of an isogeny is not a symmetric relation. However, if there exists an isogeny from a connected commutative unipotent group G_1 onto G_2 then an isogeny from G_2 onto G_1 exists as well (see [89], Proposition 10, p. 176). Already for unipotent connected non-commutative algebraic groups this is not any more the case as the following example shows.

2.1.14 Example. In Remark 2.1.6 we denoted by $\eta_1 : \mathbf{G}_a \times \mathbf{G}_a \longrightarrow \mathbf{G}_a$ the factor system defined by $\eta_1(x, y) = xy^p$. As soon as the p-polynomial g is not monomial, the factor system $\eta_1 g$ is no longer contained in the left $\mathsf{End}(\mathbf{G}_a)$-submodule generated by η_1. Therefore a necessary condition to have the equality $f[\eta_1] = [\eta_1]g$, for some p-polynomial f, is that g is monomial, that is $g = a\mathbf{F}^k$ (see Remark 2.1.8). But in this case we have $\eta_1 g = \tilde{g}\eta_1$, where $\tilde{g} = a^{1+p}\mathbf{F}^k$. This shows that, if f is a p-polynomial which is not a monomial, it cannot happen that $f[\eta_1] = [\eta_1]g$, because the left $\mathsf{End}(\mathbf{G}_a)$-submodule generated by η_1 is free. By Proposition 2.1.12 *(i)*, there cannot exist an isogeny from $G_{f[\eta_1]}$ to $G_{[\eta_1]}$ whereas by the same Proposition we have an isogeny from $G_{[\eta_1]}$ to $G_{f[\eta_1]}$.

□

2.1.15 Remark. Non-central extensions of a group A by a group B are in general described by an action of B as a non-trivial group of automorphisms of A

$$a \mapsto a^b \qquad (a \in A, b \in B),$$

and a mapping $F : B \times B \longrightarrow A$, satisfying

$$F(b_1 b_2, b_3) \cdot F(b_1, b_2)^{b_3} = F(b_1, b_2 b_3) \cdot F(b_2, b_3). \tag{2.5}$$

For the trivial action of B on a commutative group A, the equation (2.5) just reduces to the functional equation of a factor system describing a central extension, as in Section 2.1. With a slight abuse, we call a mapping F satisfying (2.5) a factor system. If A and B are algebraic groups, it is necessary to assume that the factor system F and all the automorphisms $a \mapsto a^b$ for all $b \in B$ are rational maps, in order to have the extension of A by B as an algebraic group.

Let B_α be the central extension

$$1 \longrightarrow B_1 \longrightarrow B_\alpha \longrightarrow B_2 \longrightarrow 1$$

defined on $B_2 \times B_1$ by the product

$$(b_0, b_1)(b_0', b_1') = (b_0 \cdot b_0', b_1 + b_1' + \alpha(b_0, b_0'))$$

and let G_ϕ be the central extension

$$1 \longrightarrow A \longrightarrow G_\phi \longrightarrow B_\alpha \longrightarrow 1$$

defined on $B \times A$ by the product

$$((b_0, b_1), a) \cdot ((b_0', b_1'), a') = ((b_0 \cdot b_0', b_1 + b_1' + \alpha(b_0, b_0')!), a + a' + \phi(b_0, b_1, b_0', b_1')). \tag{2.6}$$

Let $H = \{(b_0, b_1, a) \in G_\phi :\ b_0 = 1\}$. Under the assumption that $[G, H] \leq A$ we want to find the factor system γ corresponding to the section $\tau : B_2 \longrightarrow G_\phi$, $\tau(b_0) = (b_0, 0, 0)$ of the non-central extension

$$1 \longrightarrow H \longrightarrow G_\phi \longrightarrow B_2 \longrightarrow 1$$

and we want to compare this factor system with ϕ. With the same argument mentioned in Section 2.1 for central extension, one can easily see that such a factor system $\gamma = (\gamma_1, \gamma_2) : B_2 \times B_2 \longrightarrow H$ is $\gamma = -\delta^1 \tau$. (It is remarkable that the effects of changing the section for a non-central extension are not those of adding a trivial factor system, because in this case $\delta^1(\tau - \tau') \neq \delta^1 \tau - \delta^1 \tau'$. For a concrete example see the proof of Theorem 6.4.7). A direct computation shows now that

$$\gamma(b_0, b_0') = (b_0 \cdot b_0', 0, 0)^{-1} \cdot (b_0, 0, 0) \cdot (b_0', 0, 0) = (1, \alpha(b_0, b_0'), \beta(b_0, b_0'))$$

where α is the map appearing in (2.6) and

$$\beta(b_0, b_0') = -\phi\big(b_0 b_0', 0; (b_0 b_0')^{-1}, -\alpha(b_0 b_0', (b_0 b_0')^{-1})\big) +$$

$$\phi(b_0, 0; b_0', 0) + \phi\big((b_0 b_0')^{-1}, -\alpha(b_0 b_0', (b_0 b_0')^{-1}); b_0 b_0', \alpha(b_0, b_0')\big).$$

Therefore the group G_ϕ is isomorphic to the group defined on $B_2 \times H$ by the multiplication

$$((b_0, b_1), a) \cdot ((b_0', b_1'), a') = (b_0, 0, 0) \cdot (1, b_1, a) \cdot (b_0', 0, 0) \cdot (1, b_1', a') =$$
$$(b_0, 0, 0) \cdot (b_0', 0, 0) \cdot (1, b_1, a)^{(b_0,0,0)} \cdot (1, b_1', a') =$$
$$(b_0 b_0', 0, 0) \cdot (1, \alpha(b_0, b_0'), \beta(b_0, b_0')) \cdot (1, b_1, a + \sigma_{b_0}(b_1)) \cdot (1, b_1', a') =$$
$$(b_0 b_0', b_1 + b_1' + \alpha(b_0, b_0'), a + a' + \rho(b_0, b_1, b_0', b_1'))$$

where

$$\rho(b_0, b_1, b_0', b_1') = \sigma_{b_0}(b_1) + \beta(b_0, b_0') + \phi(1, b_1, 1, b_1') + \phi(1, \alpha(b_0, b_0'), 1, b_1 + b_1')$$

whereas for any $b_0 \in B_2$ and for any $(1, b_1, a) \in H$ the map $\sigma_{b_0} : B_1 \longrightarrow A$ is a homomorphism such that $(1, b_1, a)^{b_0} = (1, b_1, a + \sigma_{b_0}(b_1))$.

Comparing the representation given by γ with the one given by ϕ we find the remarkable fact that $\gamma_1 = \alpha$ whereas ρ is in general different from ϕ. □

The universal covering $\mathbb{C}^n$ of an arbitrary connected commutative complex Lie group G is a decisive tool for the description of homomorphisms and extensions of connected commutative complex Lie groups. It plays a similar rôle as the Witt group $\mathfrak{W}_n$ for connected commutative unipotent groups. For non-commutative unipotent groups unfortunately no similar tool is available.

2.2 Extensions of Commutative Lie Groups

Since any commutative connected complex Lie group is (holomorphically) isomorphic to the direct product of a linear torus $(\mathbb{C}^*)^m$, a vector group $\mathbb{C}^l$ and a toroidal group X, the theory of commutative extensions of such Lie groups reduces to the case of extensions which are toroidal groups. These groups play a similar rôle as the connected algebraic group $G = D(G)$ with no non-trivial affine epimorphic image.

Homomorphisms and extensions of complex tori X are completely described in [7], Ch. 1, Section 5, by means of *period matrices*, the columns of which are the vectors of the lattice Λ of a suitable representation of $X = \mathbb{C}^n/\Lambda$. This method works also for connected commutative complex Lie groups $G = \mathbb{C}^n/\Lambda$ such that the complex rank of Λ is n, which we will treat now.

Let $X = \mathbb{C}^n/\Lambda$ be a connected commutative complex Lie group. If the complex rank of Λ is $m < n$, then Λ is contained in a complex subspace V of dimension m of $\mathbb{C}^n$. Up to a change of basis and a canonical identification of V with $\mathbb{C}^m$, we can see then that X is isomorphic to $\mathbb{C}^{n-m} \oplus \mathbb{C}^m/\Lambda$. From now on we assume therefore that the complex rank of Λ is n, and we say that such groups have *maximal complex rank*. Let the real rank of Λ be $n+q$, where $0 \leq q \leq n$.

Up to a change of basis we can assume that $\Lambda = \mathbb{Z}^n \oplus \Gamma$. The corresponding column matrix is

$$P = (I_n, G) = \begin{pmatrix} I_q & 0 & \widehat{T} \\ 0 & I_{n-q} & \widetilde{T} \end{pmatrix} \in M_{n,n+q}(\mathbb{C})$$

where the columns of $G = \begin{pmatrix} \widehat{T} \\ \widetilde{T} \end{pmatrix}$ are $\mathbb{R}$-independent generators of Γ.

In accordance to [7], p. 2, we call P the *period matrix* of X. The imaginary part of G has real rank q, because the columns of P are $\mathbb{R}$-independent. Up to a permutation of the vectors of the basis we can assume that the imaginary part of $\widehat{T}$ is invertible.

For $q = 0$ we have $\Lambda = \mathbb{Z}^n$, hence the group $X = \mathbb{C}^n/\mathbb{Z}^n \cong (\mathbb{C}^*)^n$ is a linear torus, whereas for $q = n$ the group X is a complex torus by definition ([7], p. 1). According to [1], 1.1.11, p. 9, if $P = (I_n\ G)$ is the matrix of a $\mathbb{R}$-basis of the lattice Λ, the group $\mathbb{C}^n/\Lambda$ is toroidal if and only the following *irrationality condition* holds:

$$\text{for any non-zero } \mathbf{v} \in \mathbb{Z}^n \text{ the vector } \mathbf{v}G \text{ is never contained in } \mathbb{Z}^q. \tag{2.7}$$

Homomorphisms of connected commutative complex Lie groups of maximal complex rank can be described in terms of period matrices. In fact, a homomorphism $f : X_1 = \mathbb{C}^{n_1}/\Lambda_1 \longrightarrow X_2 = \mathbb{C}^{n_2}/\Lambda_2$ lifts to a unique homomorphism $\hat{f} : \mathbb{C}^{n_1} \longrightarrow \mathbb{C}^{n_2}$ of $\mathbb{C}$-vector spaces such that $\hat{f}(\Lambda_1) \leq \Lambda_2$. This lifting defines therefore two homomorphisms

$$\rho_a : \mathsf{Hom}(X_1, X_2) \longrightarrow \mathsf{Hom}(\mathbb{C}^{n_1}, \mathbb{C}^{n_2}) \cong M_{n_2,n_1}(\mathbb{C})$$

$$\rho_r : \mathsf{Hom}(X_1, X_2) \longrightarrow \mathsf{Hom}(\Lambda_1, \Lambda_2) \cong M_{n_2+q_2,n_1+q_1}(\mathbb{Z})$$

such that

$$\rho_a(f)P_1 = P_2\rho_r(f), \tag{2.8}$$

where P_i is a period matrix of X_i $(i = 1, 2)$ and where we have identified $\rho_a(f)$ and $\rho_r(f)$ with the matrices corresponding to the chosen basis of $\mathbb{C}^{n_i}$. The homomorphisms ρ_a and ρ_r are called the *analytic* and the *rational representation* of $\mathsf{Hom}(X_1, X_2)$ and the equations in (2.8) are called *Hurwitz relations* (cf. [1], p. 8).

2.2.1 Proposition. *A homomorphism $f : X_1 \longrightarrow X_2$ is an isogeny if and only if $\rho_a(f)$ and $\rho_r(f)$ are square matrices with non-zero determinant. In this case there exists an isogeny $g : X_2 \longrightarrow X_1$ with $fg = l\, id_{X_2}$ and $gf = l\, id_{X_1}$ where $l = |\rho_r(f)|$. In particular, the isogeny f is an isomorphism if and only if $|\rho_r(f)| = \pm 1$.*

Proof. If f is an isogeny, then $\hat{f} = \rho_a(f)$ is bijective for dimensional reasons, hence $|\rho_a(f)| \neq 0$. If we put $\Gamma = \hat{f}^{-1}(\Lambda_2)$, then $\Lambda_1 \leq \Gamma$ and Γ/Λ_1 is the kernel of the isogeny f. If the real rank of Λ_2 were greater than the real rank of Λ_1, then Γ/Λ_1 would be infinite. As $\rho_a(f)P_1 = P_2\rho_r(f)$ we find $\Lambda_1 = \Gamma\rho_r(f)$. Hence we have: 1) the real rank of Λ_1 is not greater than the real rank of Λ_2, since Λ_2 has the same real rank as Γ, 2) $\rho_r(f)$ is a square matrix with non-zero determinant.

Conversely, if $\rho_a(f)$ and $\rho_r(f)$ are square matrices with non-zero determinant, then f is surjective, its kernel is discrete and Λ_1 and $\Gamma = \hat{f}^{-1}(\Lambda_2)$ have the same real rank. As the factor group Γ/Λ_1 is the kernel of f, it has to be finite, proving that f is an isogeny.

Finally, let $l = |\rho_r(f)|$ and let $R = l\rho_r(f)^{-1}$, hence R has integral entries. Since $l\rho_a(f)^{-1}P_2 = P_1R$ we can define a homomorphism $g : X_2 \longrightarrow X_1$ such that $\rho_a(g) = l\rho_a(f)^{-1}$ and $\rho_r(g) = R$. Since $\rho_a(g)$ and $\rho_r(g)$ are square matrices with non-zero determinant, the homomorphism g is an isogeny and it is easy to see that $fg = l\, id_{X_2}$ and $gf = l\, id_{X_1}$.

In particular, if $l = \pm 1$ the isogeny f is an isomorphism. Conversely, if f is an isomorphism, then the rational representation $\rho_r(f) : \Lambda_1 \longrightarrow \Lambda_2$ is an isomorphism of lattices having $\rho_r(f^{-1})$ as the inverse, hence $|\rho_r(f)| = \pm 1$. □

Now we want to study closed subgroups and factor groups of toroidal groups as well as holomorphic commutative extensions of toroidal groups by toroidal groups.

2.2.2 Proposition. *Let $X \cong \mathbb{C}^n/\Lambda$ be a connected commutative complex Lie group of maximal rank n. For any k-dimensional connected closed commutative complex subgroup $X_1 = \mathbb{C}^k/\Lambda_1$ of maximal rank k of X there exists a period matrix P such that*

$$P = \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix}$$

where P_1 is a period matrix of X_1 and P_2 is a period matrix of the factor group X/X_1.

Proof. Let P_1 be a period matrix of the closed subgroup $X_1 \cong \mathbb{C}^k/\Lambda_1$ of X. As X_1 is a closed subgroup of X we can construct an exact sequence

$$0 \longrightarrow \mathbb{C}^k/\Lambda_1 \overset{\imath}{\longrightarrow} \mathbb{C}^n/\Lambda \overset{\pi}{\longrightarrow} \mathbb{C}^{n-k}/\Lambda_2 \longrightarrow 0,$$

where Λ_2 is a lattice corresponding to the connected complex commutative Lie group $X/X_1 \cong \mathbb{C}^{n-k}/\Lambda_2$. Consider the linear maps $\hat{\imath} = \rho_a(\imath)$ and $\hat{\pi} = \rho_a(\pi)$. As $\ker \imath = \hat{\imath}^{-1}(\Lambda)/\Lambda_1$ and $\imath$ is injective we have $\hat{\imath}^{-1}(\Lambda) = \Lambda_1$ from which it follows that also $\hat{\imath}$ is injective. Furthermore by the relation $\ker \pi = \hat{\pi}^{-1}(\Lambda_2)/\Lambda = \imath(\mathbb{C}^k/\Lambda_1) = (\hat{\imath}(\mathbb{C}^k) + \Lambda)/\Lambda$ we have $\hat{\pi}^{-1}(\Lambda_2) = \hat{\imath}(\mathbb{C}^k) + \Lambda$, which yields $\ker \hat{\pi} = \hat{\imath}(\mathbb{C}^k)$. Consequently $\hat{\imath}$ and $\hat{\pi}$ define an exact sequence

$$0 \longrightarrow \mathbb{C}^k \overset{\hat{\imath}}{\longrightarrow} \mathbb{C}^n \overset{\hat{\pi}}{\longrightarrow} \mathbb{C}^{n-k} \longrightarrow 0.$$

By the relation $\hat{\pi}^{-1}(\Lambda_2) = \hat{\imath}(\mathbb{C}^k) + \Lambda$ we have that the homomorphism $\hat{\pi}_{|\Lambda} : \Lambda \longrightarrow \Lambda_2$ is surjective and $\ker \hat{\pi}_{|\Lambda} = \hat{\imath}(\Lambda_1)$. This defines an exact sequence

$$0 \longrightarrow \Lambda_1 \xrightarrow{\hat{\imath}} \Lambda \xrightarrow{\hat{\pi}} \Lambda_2 \longrightarrow 0.$$

Since Λ_2 is a free commutative group we get $\Lambda = \imath(\Lambda_1) \oplus \Gamma$ where $\Gamma \cong \Lambda_2$. Up to a change of basis of the spaces $\mathbb{C}^k$, $\mathbb{C}^n$ and $\mathbb{C}^{n-k}$ we can assume that $\hat{\imath} = \rho_a(\imath) = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$, $\hat{\pi} = \rho_a(\pi) = (0\ I_{n-k})$, and we can choose a period matrix P of X such that $P = \begin{pmatrix} P_1 & \Sigma \\ 0 & A \end{pmatrix}$, where the columns of the matrix $\begin{pmatrix} \Sigma \\ A \end{pmatrix}$ are $\mathbb{R}$-independent $\mathbb{Z}$-generators of Γ. Furthermore, fixing a period matrix P_2 of X/X_1, up to a change of generators of Γ we can assume that $\rho_r(\pi) = (0\ I_{n+q-k-q_1})$, where $n+q$ (respectively n_1+q_1) is the real rank of X (respectively of X_1). Now, by the Hurwitz relations we have

$$(0\ I_{n-k}) \begin{pmatrix} P_1 & \Sigma \\ 0 & A \end{pmatrix} = P_2 (0\ I_{n+q-k-q_1})$$

from which it follows that $A = P_2$. □

Now we look for closed linear subtori of a connected commutative complex Lie group $X \cong \mathbb{C}^n/\Lambda$ of maximal rank n with period matrix $P = (I_n\ G)$. Denote by $H = H(l_1, \cdots, l_{n-m})$ the m-dimensional subspace of $\mathbb{C}^n$ defined by

$$H = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : z_{l_k} = 0 \text{ for } l_k \in \{1, \cdots, n\} \text{ and } k = 1, \cdots, n-m\}$$

and let $C_H(P)$ be the matrix obtained from P in the following way: we cancel in P any row with exception of those labeled by $l_1, \cdots, l_{n-m}$ as well as any of the first n columns with exception of those labeled by $l_1, \cdots, l_{n-m}$. Clearly $C_H(P) = (I_{n-m}\ G')$, with $G' \in M_{n-m,q}(\mathbb{C})$.

2.2.3 Proposition. *Let $X \cong \mathbb{C}^n/\Lambda$ be a connected commutative complex Lie group of maximal rank n and let $P = (I_n\ G)$ be a period matrix of X. If the columns of $C_H(P)$ are $\mathbb{R}$-independent, then $X_1 = (H+\Lambda)/\Lambda$ is a closed linear subtorus of X.*

Proof. Let $X_2 \cong \mathbb{C}^{n-m}/\Lambda_2$ be the connected commutative complex Lie group of maximal rank $n-m$ having $C_H(P)$ as a period matrix and let $\hat{f} : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-m}$ be the homomorphism defined by $\hat{f}(z_1, \cdots, z_n) = (z_{l_1}, \cdots, z_{l_{n-m}})$. Since $\hat{f}(\Lambda) \leq \Lambda_2$, a homomorphism $f : X \longrightarrow X_2$ is induced such that X_1 is the kernel. This proves that X_1 is a closed subgroup. In order to prove that X_1 is a linear torus we show that $H \cap \Lambda$ has real rank m. This follows from the fact that the columns of $C_H(P)$ are $\mathbb{R}$-independent, hence no non-trivial linear combination of the columns $l_1, \cdots, l_{n-m}$ of the matrix P with integral (or even real) coefficients enters in H. □

2.2.4 Remark. The above proposition shows that a toroidal group X with period matrix

$$P = (I_n \; G) = \begin{pmatrix} I_q & 0 & \widehat{T} \\ 0 & I_{n-q} & \widetilde{T} \end{pmatrix} \in M_{n,n+q}(\mathbb{C})$$

(such that the imaginary part of $\widehat{T}$ is invertible) contains a closed linear subtorus L of dimension $n-q$ corresponding to the submatrix $P_1 = I_{n-q}$, because the submatrix $C_H(P) = (I_q \; \widehat{T})$ is the period matrix of a complex torus. Hence L is a maximal closed linear subtorus of X. For instance, in the three-dimensional toroidal group X having

$$P = \begin{pmatrix} 1 & 0 & 0 & i & i \\ 0 & 1 & 0 & i\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & i\sqrt{2} \end{pmatrix}$$

as a period matrix, the three subgroups $H(2,3)$, $H(1,3)$ and $H(1,2)$ are one-dimensional maximal closed linear subtori. Thus X is a $\mathbb{C}^*$-fiber bundle over the complex tori defined by the period matrices

$$C_{H(2,3)} = \begin{pmatrix} 1 & 0 & i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{2} \end{pmatrix}, \qquad C_{H(1,3)} = \begin{pmatrix} 1 & 0 & i & i \\ 0 & 1 & 0 & i\sqrt{2} \end{pmatrix},$$

$$C_{H(1,2)} = \begin{pmatrix} 1 & 0 & i & i \\ 0 & 1 & i\sqrt{2} & 0 \end{pmatrix}$$

. □

Let $X_1 = \mathbb{C}^{n_1}/\Lambda_1, X_2 = \mathbb{C}^{n_2}/\Lambda_2$ be connected commutative complex Lie groups of maximal ranks n_1, n_2 and let P_1, P_2 be the corresponding period matrices. Let

$$0 \longrightarrow X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow 0$$

be an exact sequence of connected commutative complex Lie groups. By Proposition 2.2.2 we find a basis such that the corresponding period matrix is

$$P = \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix} \in M_{n,n+q_1+q_2}(\mathbb{C}).$$

Conversely to each matrix of this form there corresponds a toroidal group X containing a closed subgroup X_1 having P_1 as a period matrix and such that X/X_1 is isomorphic to a toroidal group X_2 having P_2 as a period matrix. In fact, X_1 is the kernel of the homomorphism $f : X \longrightarrow X_2$ which lifts to $\hat{f}(z_1, \cdots, z_n) = (z_{n_1+1}, \cdots, z_n)$.

As a consequence of Hurwitz relations we find that $P = \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix}$ and $Q = \begin{pmatrix} P_1 & \Sigma' \\ 0 & P_2 \end{pmatrix}$ define equivalent extensions if and only if a matrix $A \in M_{n_1,n_2}(\mathbb{C})$ and a matrix $M \in M_{n_1+q_1,n_2+q_2}(\mathbb{Z})$ exists such that

$$\begin{pmatrix} I_{n_1} & A \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} P_1 & \Sigma' \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} I_{n_1+q_1} & M \\ 0 & I_{n_2+q_2} \end{pmatrix}.$$

The period matrix $P = \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix}$ defines therefore a split extension of X_1 by X_2 if and only if $\Sigma = P_1 M - A P_2$ with $A \in M_{n_1,n_2}(\mathbb{C})$ and $M \in M_{n_1+q_1,n_2+q_2}(\mathbb{Z})$. Moreover, if the period matrix $P = \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix}$ is such that $\Sigma = P_1 M - A P_2$ with $M \in M_{n_1+q_1,n_2+q_2}(\mathbb{Q})$, then P defines an extension of X_1 by X_2 which is isogenous to a split one. An isogeny $f : X_1 \to X_2$ is given by $\rho_a(f) = \begin{pmatrix} lI_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix}$ and $\rho_r(f) = \begin{pmatrix} lI_{n_1+q_1} & 0 \\ 0 & I_{n_2+q_2} \end{pmatrix}$, where $l \in \mathbb{Z}$ is such that lM has integral entries.

Hence we have the following

2.2.5 Proposition. *Let X_1, X_2 be connected commutative complex Lie groups of maximal rank n_1, n_2 and let P_1, P_2 be the corresponding period matrices. The period matrix*

$$P = \begin{pmatrix} P_1 & \Sigma \\ 0 & P_2 \end{pmatrix} \in M_{n,n+q_1+q_2}(\mathbb{C}) \qquad (n = n_1 + n_2)$$

defines an extension of X_1 by X_2 which is isogenous to a split one, via an isogeny f such that $\rho_a(f) = \begin{pmatrix} lI_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix}$ and $\rho_r(f) = \begin{pmatrix} lI_{n_1+q_1} & 0 \\ 0 & I_{n_2+q_2} \end{pmatrix}$, if and only if $\Sigma = P_1 M - A P_2$ with $A \in M_{n_1,n_2}(\mathbb{C})$ and $M \in M_{n_1+q_1,n_2+q_2}(\mathbb{Q})$, where $l \in \mathbb{Z}$ is such that lM has integral entries. □

Extensions of complex tori X_1 and X_2 which are not isogenous to a split analytic extension $X_1 \oplus X_2$ are called *Shafarevich extensions* in [7], Ch. 1, § 6, p. 23. Hence it seems for us to be natural to call also non-split analytic extensions of a toroidal group by a toroidal group Shafarevich extensions.

If X_1 and X_2 are abelian varieties, Shafarevich extensions of X_1 by X_2 are not abelian varieties and hence provide a wide class of non-projective complex tori, since an abelian variety is a complex torus admitting a holomorphic embedding into some projective space (cf. [7], p. xiii).