Preface

The adjective 'granular' is attributed to materials when they are made of sets of unfastened discrete solid particles (granules) of a size larger than one micron, a length scale above which thermal agitation is negligible. In fact, the dominant energy scale in granular materials is the one of a single grain under gravity. Granular matter is common and we meet it everyday. Examples range from the dust settled on the books of our libraries, to the sand in the desert, to the meal used in cooking, itself obtained from grain, often stored in silos. Granular matter displays a variety of peculiarities that distinguish it from other substances studied by condensed matter physics and renders its overall mathematical modeling arduous. In a review paper of 1999 [dG] P.G. de Gennes writes: "granular matter is a new type of condensed matter, as fundamental as a liquid or a solid and showing in fact two states: one fluidlike, one solidlike. But there is as yet no consensus on the description of these two states!"

Almost all preconceptions on which the standard theory of continua is based seem to fail. The standard concept that the material element is well identified (even in the statistical conception common in gas dynamics) fails and, with it, the current mathematical picture assigning to it a precise placement. Even useful results of the standard kinetic theory of gases can be called upon confidently, in general. The populations of grains are far less profuse than the molecular ones in gases and far more crowded. The constraints imposed by grains on one another are generally too conspicuous to rigidify the lot. Also, boundary conditions are far from the simple classical scheme suggested by the divergence theorem and need separate critical modeling.

Heaps of granules do not sustain tension unless (at least a small) cohesion is present. They are, in general, in anisotropic metastable states. Such states last indefinitely unless external perturbations occur. Contrary to common solids and fluids, no thermal average among nearby states arises (see [JNB]). Interactions between granules are exerted through contacts occurring along graphs with topology depending on the way in which granules are packaged, on the distribution of the sizes of the granules themselves, on the boundary

conditions and, above all, on the sources of external disturbances. Subsets of granules may self-organize in order to sustain and distribute tensions along arcs: When granules are stored in a silo the pressure at the bottom of the silo does not increase indefinitely as the height of the stored material grows, rather it reaches a maximum value if the lateral walls of the silo are sufficiently tall. If the silo is shaken and grains with different sizes are stored within it, size segregation occurs instead of mixing. The phenomenon does not fit the entropic effects that one recognizes in standard liquids. Both traveling and standing waves accrue in the surface layer of grains, and slip appears along the walls of the silo. It is not clear yet how inelastic interactions and local disorder or segregation contribute to the acoustic propagation.

Layer dynamics is present also in 'avalanches'. The example is the addition of grains to a heap from the top: the surface layer moves, the core persists. The description of the connection between the surface flow and the core at rest forces one to account for the transition between two phases, if one wants to propose a global picture of the phenomenon. As for the contact stresses in granular materials at rest, even for rapid flows the stresses induced by the collisions depend on the local numerosity of granules, in other words on the 'degree of clustering'. Segregation in dense granular flows also has an influence. The overall mechanical behavior seems to be history-dependent.

Plastic effects may be prominent in the quasi-static regime: shear bands appear and inelastic deformations may be accumulated by cyclic loading processes. Inelastic collisions usually play a decisive role in dynamics, as in the fall of avalanches and in the walk of desert dunes. Chaotic agitation of granules leads them far from thermodynamic equilibrium, so fluctuations may be prominent. Microscopic slip friction between granules induces relaxation analogous to that of some solids with complex microstructure. Macroscopic friction is induced by earthquakes which may induce fluidization of granular matter.

Critical reviews such as [JNB], [dG], [K], and [AT] provide an adequate description of phenomena occurring in granular matter. They give a picture of the scenario.

A typical approach to the dynamics of polydisperse granular systems is based on the 'inelastic' Boltzmann equation which also provides the starting point for a plethora of hydrodynamic models. Assumptions on the types of contacts occurring must be chosen carefully: relative rotations may need to be accounted for, depending on the circumstances. Closures are obtained by assuming specific forms of the distribution function. A critical review of the results along these lines can be found in [V].

Even in a disperse state, a granular flow is dissipative as a consequence of the presence of inelastic collisions. Absence of equilibrium is the source of difficulties when one applies the Chapman-Enskog perturbative method to derive a (hydrodynamic) continuum picture from the kinetic description based on Boltzmann equation. Since collisions are inelastic, a double expansion in Knudsen number K and degree of inelasticity ϵ has to be used. Moreover, since

$\epsilon \propto K^3$ in the steady state, the expansion has to be extended up to Burnett or super-Burnett regimes to assure consistency, the latter regime being of degree $O\left(K^3\right)$. The appearance of unphysical instabilities under short-wave perturbations, typical of Burnett and super-Burnett regimes, requires viscosity regularizations of the field equations or the use of other possible techniques.

Attempts have been made to construct continuum models of granular matter from first principles, without resorting to kinetic justifications based on Boltzmann equation, especially in statics. Non-trivial difficulties arise, as already mentioned: at a gross scale some standard paradigms of traditional continuum mechanics need to be modified accurately. We have already mentioned for example the loss of the possibility of identifying perfectly a generic material element (i.e. even to define it). This difficulty is generated by the occurrence of segregation and also by the general lack of coherence due to the absence of cohesion between neighboring granules. Each material element must be then considered as an open system (as in [M]) from which granules may migrate from it to neighboring ones. Standard balances of interactions have to be then supplemented by an equation ruling the rate of local numerosity of granules. Such a rate, which is the rate of migration of granules, generates loss of information about the local texture of the granular matter, and increases the configurational entropy, although segregation is opposite to the entropic mixing (as mentioned above).

The definition of measures of deformation and stresses in terms of granular geometries and grain-to-grain interactions may be non-trivial. As regards the standard stress tensor, for example, a typical definition is made by the sum of the tensor product between the intergranular force and the vector indicating in a local frame the contact point between neighboring granules, a sum extended to all granules in contact with a given granule. However, although such a definition has a physical meaning, one does not know point by point (or better, element by element) the local distribution of granules, their geometry and the type of contact which may be inelastic and not even punctual. All these information characterize each model that can be constructed and have also constitutive nature.

By looking at deeper details, it is natural to consider a granular material as a complex body (in the sense of a body in which the material texture influences strongly the gross behavior). So, multifield descriptions of granular matter need to be called upon, as explained in some chapters of this book.

In any case, no accepted general consensus about the overall description of granular bodies exists. Such an absence of a unified description of the mechanics of the granular matter and the lack of preference for one or another approach, for aesthetic or experimental reasons, has pushed us to collect advances in the various prominent directions of research. Our aim is to furnish a panorama clarifying the state of current researches, solving problems, discussing critically points of view and opening new questions. Contributions range, in fact, from the kinetic approaches to granular flows to the continuum description of static and quasi-static behaviors. A non-trivial tentative of a

connection between the kinetic approach and a continuum modeling based on first principles is also present (see Chap. 4).

At the present state of knowledge and in the absence of a unitary point of view, one can only say that the variety and the peculiarities of behavior of granular matter render arduous the task of its overall mathematical modeling. Mathematical and physical questions of an intricate nature appear and tools additional to the traditional ones are needed. Some of these questions and tools are discussed in the subsequent chapters.

Motivated by experimental results on shear bands due to (unstable) plastic behavior of granular bodies, in Chap. 1, *Joe D. Goddard* discusses critically various techniques for the homogenization of granular media in a quasi-static regime, media that are seen here as multipolar continua in the sense of the mechanics of complex bodies. An energy-based method of homogenization is proposed as an improvement on previous approaches.

If one describes granular flows from the point of view of the kinetic theory, inelastic collisions play a role as mentioned above. The Maxwell model of binary collisions is a typical scheme adopted and is based on the assumption that the collision frequency is independent of the velocity of colliding particles. Such a model can be translated from rarefied gases to the description of granular flows. In Chap. 2, *Alexander V. Bobylev*, *Carlo Cercignani* and *Irene Martinez Gamba* discuss, from a general point of view, variants of the Maxwell model of pairwise interactions and establish their key properties that lead to self-similar asymptotics. Existence and uniqueness issues are also analyzed.

The approach based on the dissipative Boltzmann equation is further analyzed in Chap. 3 by *Giuseppe Toscani*. Two paths toward the hydrodynamic limit are discussed. They account for two different methodologies for the closure of macroscopic equations: (i) low inelasticity in the system, namely a perturbation in a precise sense that allows the local resort to Maxwellian functions, (ii) small spatial variations implying the use of a homogeneous cooling state.

One of the editors (**GC**) proposes in Chap. 4 (a chapter already mentioned above) further results on an earlier proposal for fast sparse flows. Complex bodies with kinetic substructure are considered. They are bodies in which each material element is a system in continuous agitation and are called *pseudofluids*. Granular flows fall naturally in this framework. The maelstrom within each material element and its influence on the neighboring fellows is governed by peculiar hydrodynamic balances: they are offspring of the microscopic interactions between granules.

The recursive analysis of higher momenta of the distribution function, even beyond Grad's 13 moments, has been the basic source of Extended Thermodynamics. The results on monoatomic gases provided by such an approach suggest its use in modeling granular flows. After summarizing the main modeling issues of Extended Thermodynamics, in Chap. 5 *Tommaso Ruggeri* analyzes the resulting hyperbolic system and discusses global existence questions and stability of constant state on the basis of Kawashima condition.

The construction of hydrodynamic models from first principles can find appropriate suggestions from detailed numerical simulations performed by looking directly at single granules and their contact interactions. The molecular dynamic approach is pursued in Chap. 6 by *Ramon García-Rojo*, *Sean McNamara* and *Hans J. Herrmann* who analyze (amid possible choices) the persistent elastic–plastic strain accumulation in compacted granular soils under cyclic stress conditions.

Since driven sets of granules (for example confined in a box) are in essence systems very far from thermodynamic equilibrium, as mentioned above, the effects of fluctuations are in general significant. Analysis of the injected power fluctuations is presented by *Alain Barrat*, *Andrea Puglisi*, *Emmanuel Trizac*, *Paolo Visco* and *Frederic van Wijland* in Chap. 7, a chapter divided into two parts: The first part deals with the way in which the probability density function of the fluctuations of the total energy is related to the characterization of energy correlations for both boundary and homogeneous driving. The second part contains an interpretation of some numerical and experimental results that seem to invalidate Gallavotti–Cohen symmetry [GC]. Such results appear contradictory to common analyses that seem to satisfy Gallavotti–Cohen fluctuation relation. By means of Lebowitz–Spohn approach to Markov processes, an approach applied to the inelastic Boltzmann equation, a functional satisfying a fluctuation relation is also introduced.

The last two editors collect their contributions in Chaps. 8 and 9. In particular, in Chap. 8 **PG** analyzes in the continuum limit the thermodynamics of a granular material modeled as a complex body endowed with a microstructure which is constrained and/or latent in the sense introduced by Capriz [C]. The consequences of grain rotations are described together with effects like dilatancy.

Finally, in Chap. 9, **PMM** considers granular matter in slow motion and describes it as a two-scale complex body for which each material element is considered as a grand-canonical ensemble of granules. The evolution equation of the numerosity of grains in each material element is derived in terms of grain-to-grain interactions.

February, 2008   Gianfranco Capriz
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References

[AT]   Aranson, I.S., Tsimring, L.S.: Patterns and collective behavior in granular media: Theoretical concepts. Rev. Modern Phys., **78**, 641–692 (2006)

[C]   Capriz, G.: Continua with latent microstructure. Arch. Rational Mech. Anal., **90**, 43–56 (1985)

[dG]   de Gennes, P.G.: Granular matter, a tentative view. Rev. Modern Phys., **71**, S375–S382 (1999)

[G]   Goldhirsch, J.: Kinetic and continuum descriptions of granular flows. In: Aref, H., Philips, J.W. (eds) Mechanics for a New Millenium. Philips Edts., Kluwer, Dordrecht, 345–358 (2001)

[GC]   Gallavotti, G., Cohen, E.G.D.: Dynamical ensembles in nonequilibrium statistical mechanics. Phys. Rev. Letters, **74**, 2694–2698 (1995)

[K]   Kadanoff, L.P.: Built upon sand: Theoretical ideas inspired by granular flows. Rev. Modern Phys., **71**, 435–444 (1999)

[JNB]   Jaeger, H.M., Nagel, S.R., Behringer, R.P.: Granular solids, liquids, and gases. Rev. Modern Phys., **68**, 1259–1273 (1996)

[M]   Mariano, P.M.: Migration of substructures in complex fluids. J. Phys. A: Math. Teoretical, **38**, 6823–6839 (2005)

[V]   Villani, C.: Mathematics of granular materials. J. Stat. Phys., **124**, 781–822 (2007)

Generalized Kinetic Maxwell Type Models of Granular Gases

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Summary. In this chapter we consider generalizations of kinetic granular gas models given by Boltzmann equations of Maxwell type. These type of models for non-linear elastic or inelastic interactions, have many applications in physics, dynamics of granular gases, economy, etc. We present the problem and develop its form in the space of characteristic functions, i.e., Fourier transforms of probability measures, from a very general point of view, including those with arbitrary polynomial non-linearities and in any dimension space. We find a whole class of generalized Maxwell models that satisfy properties that characterize the existence and asymptotic of dynamically scaled or self-similar solutions, often referred as *homogeneous cooling states*. Of particular interest is a concept interpreted as an operator generalization of usual Lipschitz conditions which allows to describe the behavior of solutions to the corresponding initial value problem. In particular, we present, in the most general case, existence of self similar solutions and study, in the sense of probability measures, the convergence of dynamically scaled solutions associated with the Cauchy problem to those self-similar solutions, as time goes to infinity. In addition we show that the properties of these self-similar solutions lead to non classical equilibrium stable states exhibiting power tails. These results apply to different specific problems related to the Boltzmann equation (with elastic and inelastic interactions) and show that all physically relevant properties of solutions follow directly from the general theory developed in this presentation.

1 Introduction

It has been noticed in recent years that a significant non-trivial physical phenomena in granular gases can be described mathematically by dissipative Boltzmann type equations, as can be seen in [17] for a review in the area. As motivated by this particular phenomena of energy dissipation at the kinetic level, we consider in this chapter the Boltzmann equation for non-linear interactions of Maxwell type and some generalizations of such models.

The classical conservative (elastic) Boltzmann equation with the Maxwell-type interactions is well-studied in the literature (see [5, 14] and references therein). Roughly speaking, this is a mathematical model of a rarefied gas with binary collisions such that the collision frequency is independent of the velocities of colliding particles, and even though the intermolecular potentials are not of those corresponding to hard sphere interactions, still these models provide a very rich inside to the understanding of kinetic evolution of gases.

Recently, Boltzmann equations of Maxwell type were introduced for models of granular gases were introduced in [7] in three dimensions, and a bit earlier in [3] for in one dimension case. Soon after that, these models became very popular among the community studying granular gases (see, for example, the book [13] and references therein). There are two obvious reasons for such studies The first one is that the inelastic Maxwell–Boltzmann equation can be essentially simplified by the Fourier transform similarly as done for the elastic case, where its study becomes more transparent [6, 7]. The second reason is motivated by the special phenomena associated with *homogeneous cooling* behavior, i.e., solutions to the spatially homogeneous inelastic Maxwell–Boltzmann equation have a non-trivial self-similar asymptotics, and in addition, the corresponding self-similar solution has a power-like tail for large velocities. The latter property was conjectured in [16] and later proved in [9, 11]. This is a rather surprising fact, since the Boltzmann equation for hard spheres inelastic interactions has been shown to have self similar solutions with all moments bounded and large energy tails decaying exponentially. The conjecture of self-similar (or homogeneous cooling) states for such model of Maxwell type interactions was initially based on an exact one-dimensional solution constructed in [1]. It is remarkable that such an asymptotics is absent in the elastic case (as the elastic Boltzmann equation has *too many conservation laws*). Later, the self-similar asymptotics was proved in the elastic case for initial data with infinite energy [8] by using another mathematical tools compared to [9] and [12].

Surprisingly, the recently published exact self-similar solutions [12] for elastic Maxwell type model for a slow down process, derived as a formal asymptotic limit of a mixture, also is shown to have power-like tails. This fact definitely suggests that self-similar asymptotics are related to total energy dissipation rather than local dissipative interactions. As an illustration to this fact, we mention some recent publications [2, 15], where one-dimensional Maxwell-type models were introduced for non-standard applications such as models in economics and social interactions, where also self-similar asymptotics and power-like tail asymptotic states were found.

Thus all the above discussed models describe qualitatively different processes in physics or even in economics, however their solutions have a lot in common from mathematical point of view. It is also clear that some further generalizations are possible: one can, for example, include in the model multiple (not just binary) interactions still assuming the constant

(Maxwell-type) rate of interactions. Will the multi-linear models have similar properties? The answer is *yes*, as we shall see below.

Thus, it becomes clear that there must be some general mathematical properties of Maxwell models, which, in turn, can explain properties of any particular model. That is to say there must be just one *main theorem*, from which one can deduce all above discussed facts and their possible generalizations. Our goal is to consider Maxwell models from very general point of view and to establish their key properties that lead to the self-similar asymptotics.

All the results presented in this chapter are mathematically rigorous. Their full proofs can be found in [10].

After this introduction, we introduce in Sect. 2 three specific Maxwell models of the Boltzmann equation: (A) classical (elastic) Boltzmann equation; (B) the model (A) in the presence of thermostat; (C) inelastic Boltzmann equation for Maxwell type interactions. Then, in Sect. 3, we perform the Fourier transform and introduce an equation that includes all the three models as particular cases. A further generalization is done in Sect. 4, where the concept of generalized multi-linear Maxwell model (in the Fourier space) is introduced. Such models and their generalizations are studied in detail in Sects. 5 and 6. The most important for our approach concept of L-Lipschitz nonlinear operator is explained in Sect. 4. It is shown (Theorem 4.2) that all multi-linear Maxwell models satisfy the L-Lipschitz condition. This property of the models constitutes a basis for the general theory.

The existence and uniqueness of solutions to the initial value problem is stated in Sect. 5.1 (Theorem 5.2). Then, in Sect. 5.2, we present and study the large time asymptotics under very general conditions that are fulfilled, in particular, for all our models. It is shown that *L-Lipschitz* condition leads to self-similar asymptotics, provided the corresponding self-similar solution does exist. The existence and uniqueness of such self-similar solutions is stated in Sect. 5.3 (Theorem 5.12). This theorem can be considered, to some extent, as the *main theorem* for general Maxwell-type models. Then, in Sect. 5.4, we go back to multi-linear models of Sect. 4 and study more specific properties of their self-similar solutions.

We explain in Sect. 6 how to use our theory for applications to any specific model: it is shown that the results can be expressed in terms of just one function $\mu(p)$, $p > 0$, that depends on spectral properties of the specific model. General properties (positivity, power-like tails, etc.) self-similar solutions are studied in Sect. 6.1 and 6.2. It includes also the case of one-dimensional models, where the Laplace (instead of Fourier) transform is used. In Sect. 6.3, we formulate, in the unified statement (Theorem 11.1), the main properties of Maxwell models (A), (B) and (C) of the Boltzmann equation. This result is, in particular, an essential improvement of earlier results of [7] for the model (A) and quite new for the model (B).

Applications to one-dimensional models are also briefly discussed at the end of Sect. 6.3.

2 Maxwell Models of the Boltzmann Equation

We consider a spatially homogeneous rarefied d-dimensional gas $(d = 2, 3, \ldots)$ of particles having a unit mass. Let $f(v, t)$, where $v \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ denote respectively the velocity and time variables, be a one-particle distribution function with usual normalization

$$\int_{\mathbb{R}^d} dv\, f(v, t) = 1. \tag{1}$$

Then $f(v, t)$ has an obvious meaning of a time-dependent probability density in $\mathbb{R}^d$. We assume that the collision frequency is independent of the velocities of the colliding particles (Maxwell-type interactions). We discuss three different physical models (A), (B) and (C).

(A) Classical Maxwell gas (elastic collisions). In this case $f(v, t)$ satisfies the usual Boltzmann equation

$$f_t = Q(f, f) = \int_{\mathbb{R}^d \times S^{d-1}} dw\, d\omega\, g(\frac{u \cdot \omega}{|u|})[f(v')f(w') - f(v)f(w)], \tag{2}$$

where the exchange of velocities after a collision are given by

$$v' = \frac{1}{2}(v + w + |u|\omega), \quad \text{and} \quad w' = \frac{1}{2}(v + w - |u|\omega),$$

where $u = v - w$ is the relative velocity and $\Omega \in S^{d-1}$. For the sake of brevity we shall consider below the model non-negative collision kernels $g(s)$ such that $g(s)$ is integrable on $[-1, 1]$. The argument t of $f(v, t)$ and similar functions is often omitted below (as in (2)).

(B) Elastic model with a thermostat. This case corresponds to model (A) in the presence of a thermostat that consists of Maxwell particles with mass $m > 0$ having the Maxwellian distribution

$$M(v) = (\frac{2\pi T}{m})^{-d/2} \exp(-\frac{m|v|^2}{2T}) \tag{3}$$

with a constant temperature $T > 0$. Then the evolution equation for $f(x, t)$ becomes

$$f_t = Q(f, f) + \theta \int dw\, d\omega\, g(\frac{u \cdot \omega}{|u|})[f(v')M(w') - f(v)M(w)], \tag{4}$$

where $\theta > 0$ is a coupling constant, and the exchange of velocities is now

$$v' = \frac{v + m(w + |u|\omega)}{1 + m}, \quad \text{and} \quad w' = \frac{v + mw - |u|\omega}{1 + m},$$

with $u = v - w$ the relative velocity, and $\omega \in S^{d-1}$.

Equation (4) was derived in [12] as a certain limiting case of a binary mixture of weakly interacting Maxwell gases.

(C) Maxwell model for inelastic particles. We consider this model in the form given in [9]. Then the inelastic Boltzmann equation in the weak form reads

$$\frac{\partial}{\partial t}(f, \psi) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}} dv\, dw\, d\omega\, f(v)f(w)\frac{|u \cdot \omega|}{|u|}[\psi(v') - \psi(v)], \qquad (5)$$

where $\psi(v)$ is a bounded and continuous test function,

$$(f, \psi) = \int_{\mathbb{R}^d} dv\, f(v,t)\psi(v),\ u = v - w,\ \omega \in S^{d-1},\ v' = v - \frac{1+e}{2}(u \cdot \omega)\omega,\ (6)$$

the constant parameter $0 < e \leq 1$ denotes the restitution coefficient. Note that the model (C) with $e = 1$ is equivalent to the model (A) with some kernel $g(s)$.

All three models can be simplified (in the mathematical sense) by taking the Fourier transform.

We denote
$$\hat{f}(k,t) = \mathcal{F}[f] = (f, e^{-ik \cdot v}), \qquad k \in \mathbb{R}^d, \qquad (7)$$

and obtain (by using the same trick as in [6] for the model (A)) for all three models the following equations:

(A) $\qquad \hat{f}_t = \widehat{Q}(\hat{f}, \hat{f}) = \int_{S^{d-1}} d\omega\, g(\frac{k \cdot \omega}{|k|})[\hat{f}(k_+)\hat{f}(k_-) - \hat{f}(k)\hat{f}(0)],$

$$(8)$$

where $k_\pm = \frac{1}{2}(k \pm |k|\omega)$, $\omega \in S^{d-1}$, $\hat{f}(0) = 1$.

(B) $\qquad \hat{f}_t = \widehat{Q}(\hat{f}, \hat{f}) + \theta \int_{S^{d-1}} d\omega\, g(\frac{k \cdot \omega}{|k|})[\hat{f}(k_+)\widehat{M}(k_-) - \hat{f}(k)\widehat{M}(0)],$

$$(9)$$

where $\widehat{M}(k) = e^{-\frac{T|k|^2}{2m}}$, $k_+ = \frac{k + m|k|\omega}{1+m}$, $k_- = k - k_+$, $\omega \in S^{d-1}$, $\hat{f}(0) = 1$.

(C) $\qquad \hat{f}_t = \int_{S^{d-1}} d\omega\, \frac{|k \cdot \omega|}{|k|}[\hat{f}(k_+)\hat{f}(k_-) - \hat{f}(k)\hat{f}(0)], \qquad (10)$

where $\hat{f}(0) = 1$, $k_+ = \frac{1+e}{2}(k \cdot \omega)\omega$, $k_- = k - k_+$, with $\omega \in S^{d-1}$ is the direction containing the two centers of the particles at the time of the interaction. Equivalently, one may alternative write $k_- = \frac{1+e}{4}(k - |k|n\tilde{\omega})$, and $k_+ = k - k_-$, where now $\tilde{\omega} \in S^{d-1}$ is the direction of the post collisional relative velocity, and the term $\frac{|k \cdot \omega|}{|k|} dw$ is replaced by a function $g(\frac{k \cdot \tilde{\omega}}{|k|})d\tilde{\omega}$.

Case (B) can be simplified by the substitution

$$\hat{f}(k,t) = \widetilde{\widetilde{f}}(k,t) \exp[-\frac{T|k|^2}{2}], \qquad (11)$$

leading, omitting tildes, to the equation

$$(B') \qquad \hat{f}_t = \widehat{Q}(\hat{f}, \hat{f}) + \theta \int_{S^{d-1}} d\omega \, g(\frac{k \cdot \omega}{|k|})[\hat{f}(\frac{k + m|k|\omega}{1 + m}) - \hat{f}(k)], \qquad (12)$$

i.e., the model for (B) with $T = 0$, or equivalently a linear collisional term the background singular distribution. Therefore, we shall consider below just the case (B'), assuming nevertheless that $\hat{f}(k, t)$ in (12) is the Fourier transform (7) of a probability density $f(v, t)$.

3 Isotropic Maxwell Model in the Fourier Representation

We shall see that these three models (A), (B) and (C) admit a class of isotropic solutions with distribution functions $f = f(|v|, t)$. Indeed, according to (7) we look for solutions $\hat{f} = \hat{f}(|k|, t)$ to the corresponding isotropic Fourier transformed problem, given by

$$x = |k|^2, \quad \varphi(x, t) = \hat{f}(|k|, t) = \mathcal{F}[f(|v|, t)], \qquad (13)$$

where $\varphi(x, t)$ solves the following initial value problem

$$\varphi_t = \int_0^1 ds \, G(s) \left\{ \varphi[a(s)x]\varphi[b(s)x] - \varphi(x) \right\} +$$
$$+ \int_0^1 ds H(s) \left\{ \varphi[c(s)x] - \varphi(x) \right\}, \qquad (14)$$
$$\varphi_{t=0} = \varphi_0(x), \qquad \varphi(0, t) = 1,$$

where $a(s)$, $b(s)$, $c(s)$ are non-negative continuous functions on $[0, 1]$, whereas $G(s)$ and $H(s)$ are generalized non-negative functions such that

$$\int_0^1 ds \, G(s) < \infty, \qquad \int_0^1 ds \, H(s) < \infty. \qquad (15)$$

Thus, we do not exclude such functions as $G = \delta(s - s_0)$, $0 < s_0 < 1$, etc. We shall see below that, for isotropic solutions (13), each of the three equations (8), (10), (12) is a particular case of (14).

Let us first consider (8) with $\hat{f}(k, t) = \varphi(x, t)$ in the notation (13). In that case

$$|k_\pm|^2 = |k|^2 \frac{1 \pm (\omega_0 \cdot \omega)}{2}, \qquad \omega_0 = \frac{k}{|k|} \in S^{d-1}, \, d = 2, \ldots,$$

and the integral in (8) reads

$$\int_{S^{d-1}} d\omega \, g(\omega_0 \cdot \omega) \varphi \left[x \frac{1 + \omega_0 \cdot \omega}{2} \right] \varphi \left[x \frac{1 - \omega_0 \cdot \omega}{2} \right]. \qquad (16)$$

It is easy to verify the identity

$$\int_{S^{d-1}} d\omega \, F(\omega \cdot \omega_0) = |S^{d-2}| \int_{-1}^{1} dz \, F(z)(1-z^2)^{\frac{d-3}{2}}, \qquad (17)$$

where $|S^{d-2}|$ denotes the "area" of the unit sphere in $\mathbb{R}^{d-1}$ for $d \geq 3$ and $|S^0| = 2$. The identity (17) holds for any function $F(z)$ provided the integral as defined in the right-hand side of (17) exists.

The integral (16) now reads

$$|S^{d-2}| \int_{-1}^{1} dz \, g(z)(1-z^2)^{\frac{d-3}{2}} \varphi(x \frac{1+z}{2}) \varphi(x \frac{1-z}{2}) =$$

$$= \int_{0}^{1} ds \, G(s)\varphi(sx)\varphi[(1-s)x],$$

where

$$G(s) = 2^{d-2}|S^{d-2}|g(1-2s)[s(1-s)]^{\frac{d-3}{2}}, \qquad d = 2, 3, \dots. \qquad (18)$$

Hence, in this case we obtain (14), where

(A) $$a(s) = s, \qquad b(s) = 1-s, \qquad H(s) = 0, \qquad (19)$$

$G(s)$ is given in (18).

Two other models (B$'$) and (C), described by (12), (10) respectively, can be considered quite similarly. In both cases we obtain (14), where

(B$'$) $$a(s) = s, \quad b(s) = 1 - s, \quad c(s) = 1 - \frac{4m}{(1+m)^2}s,$$

$$H(s) = \theta G(s), \qquad (20)$$

$G(s)$ is given in (18):

(C) $$a(s) = \frac{(1+e)^2}{4}s, \quad b(s) = 1 - \frac{(1+e)(3-e)}{4}s,$$

$$H(s) = 0, \quad G(s) = |S^{d-2}|(1-s)^{\frac{d-3}{2}}. \qquad (21)$$

Hence, all three models are described by (14) where $0 < a(s), b(s), c(s) \leq 1$ are non-negative linear functions. One can also find in recent publications some other useful equations that can be reduced after Fourier or Laplace transformations to (14) (see, for example, [2, 15] that correspond to the case $G = \delta(s - s_0)$, $H = 0$).

Equation (14) with $H(s) = 0$ first appeared in its general form in [9] in connection with models (A) and (C). The consideration of the problem of self-similar asymptotics for (14) in that paper made it quite clear that the most important properties of "physical" solutions depend very weakly on the specific functions $G(s)$, $a(s)$ and $b(s)$.

4 Models with Multiple Interactions

We present now a general framework to study solutions to the type of problems introduced in the previous section.

We assume, without loss of generality, (scaling transformations $\tilde{t} = \alpha t$, $\alpha = \text{const.}$) that

$$\int_0^1 ds\, [G(s) + H(s)] = 1 \tag{22}$$

in (14). Then (14) can be considered as a particular case of the following equation for a function $u(x, t)$

$$u_t + u = \Gamma(u), \qquad x \geq 0,\ t \geq 0, \tag{23}$$

where

$$\Gamma(u) = \sum_{n=1}^N \alpha_n \Gamma^{(n)}(u), \quad \sum_{n=1}^N \alpha_n = 1,\ \alpha_n \geq 0,$$

$$\Gamma^{(n)}(u) = \int_0^\infty da_1 \ldots \int_0^\infty da_n\, A_n(a_1, \ldots, a_n) \prod_{k=1}^n u(a_k x), \quad n = 1, \ldots, N. \tag{24}$$

We assume that

$$A_n(a) = A_n(a_1, \ldots, a_n) \geq 0, \qquad \int_0^\infty da_1 \ldots \int_0^\infty da_n\, A(a_1, \ldots, a_n) = 1, \tag{25}$$

where $A_n(a) = A_n(a_1, \ldots, a_n)$ is a generalized density of a probability measure in $\mathbb{R}_+^n$ for any $n = 1, \ldots, N$. We also assume that all $A_n(a)$ have a compact support, i.e.,

$$A_n(a_1, \ldots, a_n) \equiv 0 \ \text{ if } \ \sum_{k=1}^n a_k^2 > R^2, \qquad n = 1, \ldots, N, \tag{26}$$

for sufficiently large $0 < R < \infty$.

Equation (14) is a particular case of (23) with

$$N = 2, \quad \alpha_1 = \int_0^1 ds\, H(s), \quad \alpha_2 = \int_0^1 ds\, G(s)$$

$$A_1(a_1) = \frac{1}{\alpha_1} \int_0^1 ds\, H(s)\delta[a_1 - c(s)] \tag{27}$$

$$A_2(a_1, a_2) = \frac{1}{\alpha_2} \int_0^1 ds\, G(s)\delta[a_1 - a(s)]\delta[a_2 - b(s)].$$

It is clear that (23) can be considered as a generalized Fourier transformed isotropic Maxwell model with multiple interactions provided $u(0, t) = 1$, the case $N = \infty$ in (24) can be treated in the same way.

4.1 Statement of the General Problem

The general problem we consider below can be formulated in the following way. We study the initial value problem

$$u_t + u = \Gamma(u), \quad u_{|t=0} = u_0(x), \quad x \geq 0, \quad t \geq 0, \tag{28}$$

in the Banach space $B = C(\mathbb{R}_+)$ of continuous functions $u(x)$ with the norm

$$\|u\| = \sup_{x \geq 0} |u(x)|. \tag{29}$$

It is usually assumed that $\|u_0\| \leq 1$ and that the operator Γ is given by (24). On the other hand, there are just a few properties of $\Gamma(u)$ that are essential for existence, uniqueness and large time asymptotics of the solution $u(x, t)$ of the problem (28). Therefore, in many cases the results can be applied to more general classes of operators Γ in (28) and more general functional space, for example $B = C(\mathbb{R}^d)$ (anisotropic models). That is why we study below the class (24) of operators Γ as the most important example, but simultaneously indicate which properties of Γ are relevant in each case. In particular, most of the results of Sects. 4–6 do not use a specific form (24) of Γ and, in fact, are valid for a more general class of operators.

Following this way of study, we first consider the problem (28) with Γ given by (24) and point out the most important properties of Γ.

We simplify notations and omit in most of the cases below the argument x of the function $u(x, t)$. The notation $u(t)$ (instead of $u(x, t)$) means then the function of the real variable $t \geq 0$ with values in the space $B = C(\mathbb{R}_+)$.

Remark 1. We shall omit below the argument $x \in \mathbb{R}_+$ of functions $u(x)$, $v(x)$, etc., in all cases when this does not cause a misunderstanding. In particular, inequalities of the kind $|u| \leq |v|$ should be understood as a point-wise control in absolute value, i.e., "$|u(x)| \leq |v(x)|$ for any $x \geq 0$" and so on.

We first start by giving the following general definition for operators acting on a unit ball of a Banach space B denoted by

$$U = \{u \in B : \|u\| \leq 1\} \tag{30}$$

Definition 1. *The operator $\Gamma = \Gamma(u)$ is called an L-Lipschitz operator if there exists a linear bounded operator $L : B \to B$ such that the inequality*

$$|\Gamma(u_1) - \Gamma(u_2)| \leq L(|u_1 - u_2|) \tag{31}$$

holds for any pair of functions $u_{1,2}$ in U.

Remark 2. Note that the L-Lipschitz condition (31) holds, by definition, at any point $x \in \mathbb{R}_+$ (or $x \in \mathbb{R}^d$ if $B = C(\mathbb{R}^d)$). Thus, condition (31) is much stronger than the classical Lipschitz condition

$$\|\Gamma(u_1) - \Gamma(u_2)\| < C\|u_1 - u_2\| \quad \text{if} \quad u_{1,2} \in U \tag{32}$$

which obviously follows from (31) with the constant $C = \|L\|_B$, the norm of the operator L in the space of bounded operators acting in B. In other words, the terminology "L-Lipschitz condition" means the point-wise Lipschitz condition with respect to an specific linear operator L.

We assume, without loss of generality, that the kernels $A_n(a_1, \ldots, a_n)$ in (24) are symmetric with respect to any permutation of the arguments $(a_1, \ldots, a_n)$, $n = 2, 3, \ldots, N$.

The next lemma states that the operator $\Gamma(u)$ defined in (24), which satisfies $\Gamma(1) = 1$ (mass conservation) and maps U into itself, satisfies an L-Lipschitz condition, where the linear operator L is the one given by the linearization of Γ near the unity. See [10] for its proof.

Theorem 1. *The operator $\Gamma(u)$ defined in (24) maps U into itself and satisfies the L-Lipschitz condition (31), where the linear operator L is given by*

$$Lu = \int_0^\infty da\, K(a) u(ax), \tag{33}$$

with

$$K(a) = \sum_{n=1}^N n\alpha_n K_n(a),$$

$$\text{where} \quad K_n(a) = \int_0^\infty da_2 \ldots \int_0^\infty da_n\, A_n(a, a_2, \ldots, a_n) \quad \text{and} \quad \sum_{n=1}^N \alpha_n = 1. \tag{34}$$

for symmetric kernels $A_n(a, a_2, \ldots, a_n), n = 2, \ldots N$.

And the following corollary holds.

Corollary 1. *The Lipschitz condition (32) is fulfilled for $\Gamma(u)$ given in (24) with the constant*

$$C = \|L\| = \sum_{n=1}^N n\alpha_n, \qquad \sum_{n=1}^\infty \alpha_n = 1, \tag{35}$$

where $\|L\|$ is the norm of L in B.

It can also be shown that the L-Lipschitz condition holds in $B = C(\mathbb{R}^d)$ for "gain-operators" in Fourier transformed Boltzmann equations (8), (9) and (10).

5 The General Problem in Fourier Representation

5.1 Existence and Uniqueness of Solutions

It is possible to show, with minimal requirements, the existence and uniqueness results associated with the initial value problem (28) in the Banach space $(B, \|\cdot\|)$, where the norm associated to B is defined in (29). In fact, this existence and uniqueness result is an application of the classical Picard iteration scheme and holds for any operator Γ which satisfies the usual Lipschitz condition (32) and transforms the unit ball U into itself. The proof of all statements below can be found in [10].

Lemma 1 (Picard Iteration scheme). *The operator $\Gamma(u)$ maps U into itself and satisfies the L-Lipschitz condition (32), then the initial value problem (28) with arbitrary $u_0 \in U$ has a unique solution $u(t)$ such that $u(t) \in U$ for any $t \geq 0$.*

Next, we observe that the L-Lipschitz condition yields an estimate for the difference of any two solutions of the problems in terms of their initial states. This is a key fact in the development of the further studies of the self similar asymptotics.

Theorem 2. *Consider the Cauchy problem (28) with $\|u_0\| \leq 1$ and assume that the operator $\Gamma : B \to B$*

(a) Maps the closed unit ball $U \subset B$ to itself, and
(b) Satisfies a L-Lipschitz condition (31) for some positive bounded linear operator $L : B \to B$.

Then

(i) There exists a unique solution $u(t)$ of the problem (28) such that $\|u(t)\| \leq 1$ for any $t \geq 0$;
(ii) Any two solutions $u(t)$ and $w(t)$ of problem (28) with initial data in the unit ball U satisfy the inequality

$$|u(t) - w(t)| \leq \exp\{t(L-1)\}(|u_0 - w_0|). \tag{36}$$

Note that under the same conditions as in Theorem 1 the operator Γ given in (24) satisfies necessary conditions for the Theorem 2.

We remind to the reader that the initial value problem (28) appeared as a generalization of the initial value problem (14) for a characteristic function $\varphi(x,t)$, i.e., for the Fourier transform of a probability measure (see (13), (1)). It is important therefore to show that the solution $u(x,t)$ of the problem (28) is a characteristic function for any $t > 0$ provided this is so for $t = 0$, which is addressed in the following statement.

Lemma 2. *Let $U' \subset U \subset B$ be any closed convex subset of the unit ball U (i.e., $u = (1-\theta)u_1 + \theta u_2 \in U'$ for any $u_{1,2} \in U'$ and $\theta \in [0,1]$). If $u_0 \in U'$ in (28) and U is replaced by U' in the condition (1) of Theorem 2, the theorem holds and $u(t) \in U'$ for any $t \geq 0$.*

Remark 3. It is well-known (see, for example, the textbook [18]) that the set $U' \subset U$ of Fourier transforms of probability measures in $\mathbb{R}^d$ (Laplace transforms in the case of $\mathbb{R}_+$) is convex and closed with respect to uniform convergence. On the other hand, it is easy to verify that the inclusion $\Gamma(U') \subset U'$, where Γ is given in (24), holds in both cases of Fourier and Laplace transforms. Hence, all results obtained for (23), (24) can be interpreted in terms of "physical" (positive and satisfying the condition (1)) solutions of corresponding Boltzmann-like equations with multi-linear structure of any order.

We also point out that all results of this section remain valid for operators Γ satisfying conditions (24), with a more general condition such as

$$\sum_{n=1}^{N} \alpha_n \leq 1, \qquad \alpha_n \geq 0, \tag{37}$$

so that $\Gamma(1) < 1$ and so the mass may not be conserved. The only difference in this case is that the operator L satisfying conditions (33), (34) is not a linearization of $\Gamma(u)$ near the unity, but nevertheless Theorem 1 remains true. The inequality (37) is typical for Fourier (Laplace) transformed Smoluchowski-type equations where the total number of particles is decreasing in time (see [21, 22] for related work).

In the next three sections we study in more detail the solutions to the initial value problem (28)–(29) constructed in Theorem 2 and, in particular, their long time behavior, existence, uniqueness, and properties of the self-similar solutions.

5.2 Large Time Asymptotics

The long time asymptotics results are a consequence of some very general properties of operators Γ, namely, that Γ maps the unit ball U of the Banach space $B = C(\mathbb{R}_+)$ into itself, Γ is an L-Lipschitz operator (i.e., satisfies (31)) and that Γ is invariant under dilations.

These three properties are sufficient to study self-similar solutions and large time asymptotic behavior for the solution to the Cauchy problem (28) in the unit ball U of the Banach space $C(\mathbb{R}_+)$.

Main properties of the operator Γ :

(a) Γ maps the unit ball U of the Banach space $B = C(\mathbb{R}_+)$ into itself, that is

$$\|\Gamma(u)\| \leq 1 \quad \text{for any} \quad u \in C(\mathbb{R}_+) \quad \text{such that} \quad \|u\| \leq 1. \tag{38}$$

(b) Γ is an L-Lipschitz operator (i.e., satisfies (31)) with L from (33), i.e.,

$$|\Gamma(u_1) - \Gamma(u_2)|(x) \leq L(|u_1 - u_2|)(x) = \int_0^\infty da K(a)|u_1(ax) - u_2(ax)|, \tag{39}$$

for $K(a) \geq 0$, for all $x \geq 0$ and for any two functions $u_{1,2} \in C(\mathbb{R}_+)$ such that $\|u_{1,2}\| \leq 1$.

(c) Γ is invariant under dilations:

$$e^{\tau \mathcal{D}} \Gamma(u) = \Gamma(e^{\tau \mathcal{D}} u), \quad \mathcal{D} = x \frac{\partial}{\partial x}, \quad e^{\tau \mathcal{D}} u(x) = u(xe^\tau), \quad \tau \in \mathbb{R}. \tag{40}$$

No specific information about Γ beyond these three conditions will be used in this section.

It was already shown in Theorem 2 that the conditions (a) and (b) guarantee existence and uniqueness of the solution $u(x,t)$ to the initial value problem (28)–(29). The property (b) yields the estimate (36) that is very important for large time asymptotics, as we shall see below. The property (c) suggests a special class of self-similar solutions to (28).

We recall the usual meaning of the notation $y = O(x^p)$ (often used below): $y = O(x^p)$ if and only if there exists a positive constant C such that

$$|y(x)| \leq Cx^p \quad \text{for any} \quad x \geq 0. \tag{41}$$

In order to study long time stability properties to solutions whose initial data differs in terms of $O(x^p)$, we will need some spectral properties of the linear operator L.

Definition 2. *Let L be the positive linear operator given in (33), (34), then*

$$Lx^p = \lambda(p)x^p, \qquad 0 < \lambda(p) = \int_0^\infty da\, K(a)a^p < \infty, \quad p \geq 0, \tag{42}$$

and the spectral function $\mu(p)$ is defined by

$$\mu(p) = \frac{\lambda(p) - 1}{p}. \tag{43}$$

An immediate consequence of properties (a) and (b), as stated in (39), is that one can obtain a criterion for a point-wise in x estimate of the difference of two solutions to the initial value problem (28) yielding decay properties depending on the spectrum of L, as the following statement and its corollary assert.

Lemma 3. *Let $u_{1,2}(x,t)$ be any two classical solutions of the problem (28) with initial data satisfying the conditions*

$$|u_{1,2}(x,0)| \leq 1, \qquad |u_1(x,0) - u_2(x,0)| \leq C\, x^p, \quad x \geq 0 \qquad (44)$$

for some positive constant C and p. Then

$$|u_1(x,t) - u_2(x,t)| \leq Cx^p\, e^{-t(1-\lambda(p))}, \qquad for\ all\ \ t \geq 0 \qquad (45)$$

Corollary 2. *The minimal constant C for which condition (44) is satisfied is*

$$C_0 = \sup_{x \geq 0} \frac{|u_1(x,0) - u_2(x,0)|}{x^p} = \left\| \frac{|u_1(x,0) - u_2(x,0)|}{x^p} \right\|, \qquad (46)$$

and the following estimate holds

$$\left\| \frac{u_1(x,t) - u_2(x,t)|}{x^p} \right\| \leq e^{-t(1-\lambda(p))} \left\| \frac{u_1(x,0) - u_2(x,0)|}{x^p} \right\| \qquad (47)$$

for any $p > 0$.

A result similar to Lemma 3 was first obtained in [9] for the inelastic Boltzmann equation whose Fourier transform is given in example (C), (10). Its corollary in the form similar to (47) for (10) was stated later in [11] and was interpreted there as "the contraction property of the Boltzmann operator" (note that the left-hand side of (47) can be understood as a distance between two solutions). Independently of the terminology. the key reason for estimates (45)–(47) is the Lipschitz property of the operator Γ. It is remarkable that the large time asymptotics of $u(x,t)$, satisfying the problem (28) with such Γ, can be explicitly expressed through spectral characteristics of the linear operator L.

In order to study the large time asymptotics of $u(x,t)$ in more detail we distinguish two different kinds of asymptotic behavior:

(1) Convergence to stationary solutions
(2) Convergence to self-similar solutions provided the condition (c), of the main properties on Γ, is satisfied

The case (1) is relatively simple. Any stationary solution $\bar{u}(x)$ of the problem (28) satisfies the equation

$$\Gamma(\bar{u}) = \bar{u}, \qquad \bar{u} \in C(\mathbb{R}_+), \ \ \|\bar{u}\| \leq 1. \qquad (48)$$

If the stationary solution $\bar{u}(x)$ does exists (note, for example, that $\Gamma(0) = 0$ and $\Gamma(1) = 1$ for Γ given in (24)) then the large time asymptotics of some classes of initial data $u_0(x)$ in (28) can be studied directly on the basis of Lemma 3. It is enough to assume that $|u_0(x) - \bar{u}(x)|$ satisfies (44) with p such that $\lambda(p) < 1$. Then $u(x,t) \to \bar{u}(x)$ as $t \to \infty$, for any $x \geq 0$.

This simple consideration, however, does not answer at least two questions:

(A) What happens with $u(x,t)$ if the inequality (44) for $|u_0(x) - \bar{u}(x)|$ is satisfied with such p that $\lambda(p) > 1$?

(B) What happens with $u(x,t)$ for large x (note that the estimate (45) becomes trivial if $x \to \infty$).

In order to address these questions we consider a special class of solutions of (28), the so-called self-similar solutions. Indeed the property (c) of Γ shows that (28) admits a class of formal solutions $u_s(x,t) = w(x\,e^{\mu_* t})$ with some real μ_*. It is convenient for our goals to use a terminology that slightly differs from the usual one.

Definition 3. *The function $w(x)$ is called a self-similar solution associated with the initial value problem* (28) *if it satisfies the problem*

$$\mu_* \mathcal{D}w + w = \Gamma(w), \qquad \|w\| \le 1, \tag{49}$$

in the notation of (40), (24).

The convergence of solutions $u(x,t)$ of the initial value problem (28) to a stationary solution $\bar{u}(x)$ can be considered as a special case of the self-similar asymptotics with $\mu_* = 0$.

Under the assumption that self-similar solutions exists (the existence is proved in the next section), we state the fundamental result on the convergence of solutions $u(x,t)$ of the initial value problem (28) to self-similar ones (sometimes called in the literature *self-similar stability*).

Lemma 4. *We assume that*

(i) For some $\mu_* \in \mathbb{R}$, there exists a classical (continuously differentiable if $\mu_* \ne 0$) solution $w(x)$ of (49) *such that $\|w\| \le 1$;*

(ii) The initial data $u(x,0) = u_0$ in the problem (28) *satisfies*

$$u_0 = w + O(x^p), \qquad \|u_0\| \le 1, \ \text{for} \ \ p > 0 \ \ \text{such that} \ \mu(p) < \mu_*, \tag{50}$$

where $\mu(p)$ defined in (43) *is the spectral function associated to the operator L.*

Then

$$|u(xe^{-\mu_* t}, t) - w(x)| = O(x^p)e^{-pt(\mu_* - \mu(p))} \tag{51}$$

and therefore

$$\lim_{t \to \infty} u(xe^{-\mu_* t}, t) = w(x), \qquad x \ge 0. \tag{52}$$

Remark 4. Lemma 4 shows how to find a domain of attraction of any self-similar solution provided the self-similar solution is itself known. It is remarkable that the domain of attraction can be expressed in terms of just the *spectral function $\mu(p)$, $p > 0$,* defined in (43), associated with the linear operator L for which the operator Γ satisfies the L-Lipschitz condition.

Generally speaking, the equality (52) can be also fulfilled for some other values of p with $\mu(p) > \mu_*$ in (50), but, at least, it always holds if $\mu(p) < \mu_*$.

We shall need some properties of the *spectral function* $\mu(p)$. Having in mind further applications, we formulate these properties in terms of the operator Γ given in (24), though they depend only on $K(a)$ in (42)

Lemma 5. *The spectral function $\mu(p)$ has the following properties:*

(i) It is positive and unbounded as $p \to 0^+$, with asymptotic behavior given by

$$\mu(p) \approx \frac{\lambda(0) - 1}{p}, \qquad p \to 0, \tag{53}$$

where, for Γ from (24)

$$\lambda(0) = \int_0^\infty da\, K(a) = \sum_{n=1}^N \alpha_n n \geq 1, \qquad \sum_{n=1}^N \alpha_n = 1,\ \alpha_n \geq 0, \tag{54}$$

and therefore $\lambda(0) = 1$ if and only if the operator Γ (24) is linear ($N = 1$);

(ii) In the case of a multi-linear Γ operator, there is not more than one point $0 < p_0 < \infty$, where the spectral function $\mu(p)$ achieves its minimum, that is, $\mu'(p_0) = \frac{d\mu}{dp}(p_0) = 0$, with $\mu(p_0) \leq \mu(p)$ for any $p > 0$, provided $N \geq 2$ and $\alpha_N > 0$.

Remark 5. From now on, we shall always assume below that the operator Γ from (24) is multi-linear. Otherwise it is easy to see that the problem (49) has no solutions (the condition $\|w\| \leq 1$ is important!) except for the trivial ones $w = 0, 1$.

The following corollaries are readily obtained from Lemma 5 part (ii) and its proof.

Corollary 3. *For the case of a non-linear Γ operator, i.e., $N \geq 2$, the spectral function $\mu(p)$ is always monotone decreasing in the interval $(0, p_0)$, and $\mu(p) \geq \mu(p_0)$ for $0 < p < p_0$. This implies that there exists a unique inverse function $\mathrm{p}(\mu) : (\mu(p_0), +\infty) \to (0, p_0)$, monotone decreasing in its domain of definition.*

Corollary 4. *There are precisely four different kinds of qualitative behavior of $\mu(p)$ shown on Fig. 1.*

Proof. There are two options: $\mu(p)$ is monotone decreasing function (Fig. 1a) or $\mu(p)$ has a minimum at $p = p_0$ (Fig. 1b–d). In case Fig. 1a $\mu(p) > 0$ for all $p > 0$ since $\mu(p) > 1/p$. The asymptotics of $\lambda(p)$ (42) is clear:

$$(1) \qquad \lambda(p) \xrightarrow[p\to\infty]{} \lambda_\infty \in \mathbb{R}_+ \ \text{ if } \ \int_{1^+}^\infty da\, K(a) = 0; \tag{55}$$

$$(2) \qquad \lambda(p) \xrightarrow[p\to\infty]{} \infty \ \text{ if } \ \int_{1^+}^\infty da\, K(a) > 0. \tag{56}$$

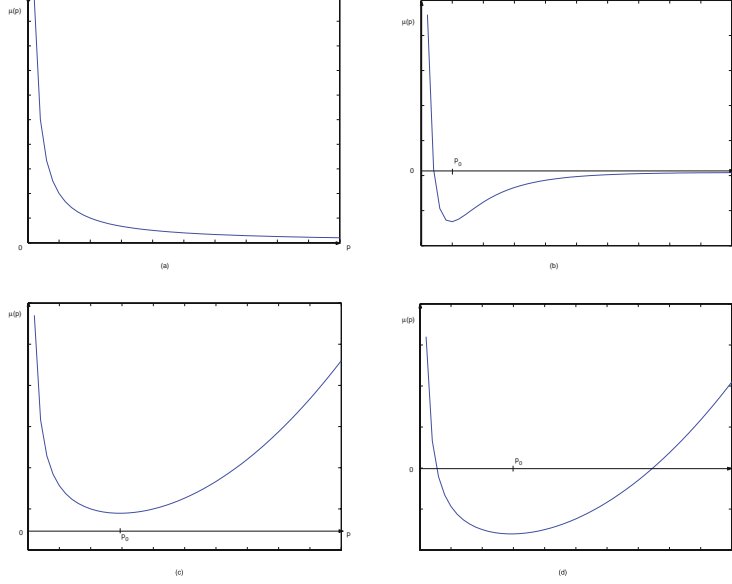**Fig. 1.** Possible profiles of the spectral function $\mu(p)$

In the case (1) when $\mu(p) \to \infty$ as $p \to 0$, two possible pictures (with and without minimum) are shown on Fig. 1b and Fig. 1a, respectively. In case (2), from (42) it is clear that $\lambda(p)$ grows exponentially for large p, therefore $\mu(p) \to \infty$ as $p \to \infty$. Then the minimum always exists and we can distinguish two cases: $\mu(p_0) < 0$ (Fig. 1d) and $\mu(p_0) > 0$ (Fig. 1c).

We note that, for Maxwell models (A), (B), (C) of Boltzmann equation (Sects. 2 and 3), only cases (a) and (b) of Fig. 1 can be possible (actually this is the case (b)) since the condition (55) holds. Figure 1 gives a clear graphic representation of the domains of attraction of self-similar solutions (Lemma 4): it is sufficient to draw the line $\mu(p) = \mu_* = constant$, and to consider a p such that the graph of $\mu(p)$ lies below this line.

Therefore, the following corollary follows directly from the properties of the spectral function $\mu(p)$, as characterized by the behaviors in Fig. 1, where we assume that $\mu(p_0) = 0$ for $p_0 = \infty$, for the case shown on Fig. 1a.

Corollary 5. *Any self-similar solution $u_s(x,t) = w(xe^{\mu_* t})$ with $\mu(p_0) < \mu_* < \infty$ has a non-empty domain of attraction, where p_0 is the unique (minimum) critical point of the spectral function $\mu(p)$.*

Proof. We use Lemma 4 part (ii) on any initial state $u_0 = w + O(x^p)$ with $p > 0$ such that $\mu(p_0) \le \mu(p) < \mu_*$. In particular, (51) and (52) show that the domain of attraction of $w(xe^{\mu_* t})$ contains any solution to the initial value problem (28) with the initial state as above.

It is clear that the inequalities of the kind $u_1 - u_2 = O(x^p)$ for any $p > 0$ such that $\mu(p) < \mu_*$, for any fixed $\mu_* \geq \mu(p_0)$ play an important role. We can use specific properties of $\mu(p)$ in order to express such inequalities in more convenient form.

Lemma 6. *For any given $\mu_* \in (\mu(p_0), \infty)$ and $u_{1,2}(x)$ such that $\|u_{1,2}\| < \infty$, the following two statements are equivalent:*

(i) There exists $p > 0$ such that

$$u_1 - u_2 = O(x^p), \qquad \text{with} \ \ \mu(p) < \mu_*. \tag{57}$$

(ii) There exists $\varepsilon > 0$ such that

$$u_1 - u_2 = O(x^{\mathrm{p}(\mu_*)+\varepsilon}), \qquad \text{with} \ \ p(\mu_*) < p_0, \tag{58}$$

where $p(\mu)$ is the inverse to $\mu(p)$ function, as defined in Corollary 3.

Finally, to conclude this section, we show a general property of the initial value problem (28) for any non-linear Γ operator satisfying conditions (a) and (b) given in (38) and (39) respectively. This property gives the control to the point-wise difference of any two rescaled solutions to (28) in the unit sphere of B , whose initial states differ by $O(x^p)$. It is formulated as follows.

Lemma 7. *Consider the problem* (28)*, where Γ satisfies the conditions (a) and (b). Let $u_{1,2}(x,t)$ are two solutions satisfying the initial conditions $u_{1,2}(x,0) = u_0^{1,2}(x)$ such that*

$$\|u_0^{1,2}\| \leq 1, \quad u_0^1 - u_0^2 = O(x^p), \quad p > 0. \tag{59}$$

then, for any real μ_*,

$$\Delta_{\mu_*}(x,t) = u_1(xe^{-\mu_* t}, t) - u_2(xe^{-\mu_* t}, t) = O(x^p)e^{-pt[\mu_* - \mu(p)]} \tag{60}$$

and therefore

$$\lim_{t \to \infty} \Delta_{\mu_*}(x,t) = 0, \qquad x \geq 0, \tag{61}$$

for any $\mu_* > \mu(p)$.

Remark 6. There is an important point to understand here. Lemmas 3 and 4 hold for any operator Γ that satisfies just the two properties (a) and (b) stated in (38) and (39). It says that, in some sense, a distance between any two solutions with initial conditions satisfying (59) tends to zero as $t \to \infty$. Such terminology and corresponding distances were introduced for specific forms of Maxwell–Boltzmann models in [4, 19]. It should be pointed out, however, that this *contraction property* may not say much about large time asymptotics of $u(x,t)$, unless the corresponding self-similar solutions are known, for which the operator Γ must be invariant under dilations (so it satisfies also property (c) as well, as stated in (40)). In such case one can use estimate (61) to deduce the convergence in the form (52), (53).

Therefore one must study the problem of existence of self-similar solutions, which is considered in the next section.

5.3 Existence of Self-Similar Solutions

We develop now a criteria for existence, uniqueness and self-similar asymptotics to the problem (49) for any operator Γ that satisfies conditions (a), (b) and (c) from Sect. 6, with the corresponding spectral function $\mu(p)$ defined in (43).

Theorem 3 below shows the criteria for existence and uniqueness of self-similar solutions for any operator Γ that satisfies just conditions (a) and (b). Then Theorem 4 follows, showing a general criteria to self-similar asymptotics for the problem (28) for any operator Γ that satisfying conditions (a), (b) and (c).

We consider (49) written in the form

$$\mu x w'(x) + w(x) = g(x), \qquad g = \Gamma(w), \ \mu \in \mathbb{R}, \tag{62}$$

and, assuming that $\|w\| < \infty$, transform this equation to the integral form. It is easy to verify that the resulting integral equation reads

$$w(x) = \int_0^1 d\tau \, g(x\tau^\mu). \tag{63}$$

By means of an iteration scheme, the following result can be proved.

Theorem 3. *Consider* (62) *with arbitrary* $\mu \in \mathbb{R}$ *and the operator* Γ *satisfying the conditions (a) and (b) from Sect. 6. Assume that there exists a continuous function* $w_0(x)$, $x \geq 0$, *such that*

(i) $\|w_0\| \leq 1$ *and*

(ii)

$$\int_0^1 d\tau \, g_0(x\tau^\mu) = w_0(x) + O(x^p), \qquad g_0 = \Gamma(w_0), \tag{64}$$

with some $p > 0$ *satisfying the inequality*

$$\mu(p) = \frac{1}{p}\Big[\int_0^\infty da \, K(a)a^p - 1\Big] < \mu. \tag{65}$$

Then there exists a classical solution $w(x)$ *of* (62). *The solution is unique in the class of continuous functions satisfying conditions*

$$\|w\| \leq 1, \qquad w(x) = w_0(x) + O(x^{p_1}), \tag{66}$$

with any p_1 *such that* $\mu(p_1) < \mu$.

Combining Lemmas 1 and 4, the following general statement related to the self-similar asymptotics for the problem (28) is obtained.

Theorem 4. *Let $u(x,t)$ be a solution of the problem (28) with $\|u_0\| \leq 1$ and Γ satisfying the conditions (a), (b), (c) from Sect. 6. Let $\mu(p)$ denote the spectral function (65) having its minimum (infimum) at $p = p_0$ (see Fig. 1), the case $p_0 = \infty$ is also included. We assume that there exists $p \in (0, p_0)$ and $0 < \varepsilon < p_0 - p$ such that*

$$\int_0^1 d\tau \, g_0(x\tau^{\mu(p)}) = u_0(x) + 0(x^{p+\varepsilon}), \qquad g_0 = \Gamma(u_0), \quad \varepsilon > 0. \tag{67}$$

Then

(i) There exists a unique solution $w(x)$ of (62) with $\mu = \mu(p)$ such that

$$\|w\| \leq 1, \qquad w(x) = u_0(x) + O(x^{p+\varepsilon}), \tag{68}$$

(ii)

$$\lim_{t\to\infty} u(x\, e^{-\mu(p)t}, t) = w(x), \qquad x \geq 0, \tag{69}$$

where the convergence is uniform on any bounded interval in $\mathbb{R}_+$ and

$$u(x\, e^{-\mu(p)t}, t) - w(x) = O(x^{p+\varepsilon} e^{-\beta(p,\varepsilon)t}), \tag{70}$$

with $\beta(p, \varepsilon) = (p + \varepsilon)(\mu(p) - \mu(p + \varepsilon)) > 0$.

Hence, a general criterion (68) is obtained for the self-similar asymptotics of $u(x,t)$ with a given initial condition $u_0(x)$. The criterion can be applied to the problem (28) with any operator Γ satisfying conditions (a), (b), (c) from Sect. 5.2. The specific class (24) of operators Γ is studied in Sect. 6. We shall see below that the condition (68) can be essentially simplified for such operators.

5.4 Properties of Self-Similar Solutions

We now apply the general theory (in particular, Theorem 4) to the particular case of the multi-linear operators Γ considered in Sect. 4, where their corresponding spectral function $\mu(p)$ satisfies (65), (34) whose behavior corresponds to Fig. 1. We also show that $p_0 = \min_{p>0} \mu(p) > 1$ is a necessary condition for self-similar asymptotics.

In addition, Theorem 5 establish sufficient conditions for which self-similar solutions of problem (62) will lead to well defined self-similar solutions (distribution functions) of the original problem after taking the inverse Fourier transform.

We consider the integral equation (63) written as

$$w = \Gamma_\mu(w) = \int_0^1 dt\, g(xt^\mu), \qquad g = \Gamma(w), \ \mu \in \mathbb{R}. \tag{71}$$

The following two properties of $w(x)$ that are independent of the specific form (24) of Γ.

Lemma 8.

(i) If there exist a closed subset $U' \subset U$ of the unit ball U in B, such that $\Gamma_\mu(U') \subset U'$ for any $\mu \in \mathbb{R}$, and for some function $w_0 \in U'$ the conditions of Theorem 3 are satisfied, then $w \in U'$.

(ii) If the conditions of Theorem 3 for Γ are satisfied and, in addition, $\Gamma(1) = 1$, then the solution $w_* = 1$ of (71) is unique in the class of functions $w(x)$ satisfying the condition

$$w(x) = 1 + O(x^p), \qquad \mu(p) < \mu. \tag{72}$$

We observe that the statement (ii) can be interpreted as a necessary condition for existence of non-trivial ($w \neq$ const.) solutions of (71): if there exists a non-trivial solution $w(x)$ of (71), where $\Gamma(1) = 1$, such that

$$\|w\| = 1, \qquad w = 1 + O(x^p), \quad p > 0, \qquad \text{then} \quad \mu \le \mu(p). \tag{73}$$

We recall that $\mu(p)$ satisfies the inequality $\mu(p) \ge \mu(p_0)$ (Fig. 1).

If $p \ge p_0$ (provided $p_0 < \infty$) in (73), then there are no non-trivial solutions with $\mu > \mu(p_0)$.

On the other hand, possible solutions with $\mu \le \mu(p_0)$ (even if they exist) are irrelevant for the problem (28) since they have an empty domain of attraction (Lemma 4).

Therefore we always assume below that $\mu > \mu(p_0)$ and, consequently, $p \in (0, p_0)$ in (73).

Let us consider now the specific class (24)–(25) of operators Γ, with functions $u(x)$ satisfying the condition $u(0) = 1$. Then, $u(0, t) = 1$ for the solution $u(x, t)$ of the problem (28).

In addition, the operators (24) are invariant under dilation transformations (40) (property (c), Sect. 6). Therefore, the problem (28) with the initial condition $u_0(x)$ satisfying

$$u(0) = 1, \quad \|u_0\| = 1; \qquad u_0(x) = 1 - \beta x^p + \cdots, \quad x \to 0, \tag{74}$$

can be always reduced to the case $\beta = 1$ by transformation $x' = x\beta^{1/p}$.

Moreover, the whole class of operators (24) with different kernels $A_n(a_1, \ldots, a_n)$, $n = 1, 2, \ldots$, is invariant under transformations $\tilde{x} = x^p$, $p > 0$. The result of such transformation of Γ is another operator $\widetilde{\Gamma}$ of the same class (24) with kernels $\tilde{A}_n(a_1, \ldots, a_n)$.

Therefore we fix the initial condition (74) with $\beta = 1$ and transform the function (74) and the (28) to new variables $\tilde{x} = x^p$. Then we omit the tildes and reduce the problem (28), with initial condition (74) to the case $\beta = 1$, $p = 1$. We study this case in detail and formulate afterward the results in terms of initial variables.

Next, we assume a bit more about the asymptotics of the initial data $u_0(x)$ for small x, namely

$$\|u_0\| = 1, \quad u_0(x) = 1 - x + O(x^{1+\varepsilon}), \quad x \to 0, \tag{75}$$

with some $\varepsilon > 0$.

Then, our goal now is to apply the general theory (in particular, Theorem 4 and criterion (67)) to this particular case. We assume that the spectral function $\mu(p)$ given by (65), (34), corresponds to one of the four cases shown on Fig. 1 with $p_0 > 1$.

Let us take a typical function $u_0 = e^{-x}$ satisfying (75) and apply the criterion (67), from Theorem 4 or, equivalently, look for such $p > 0$ that (67) is satisfied. That is, find possibles values of $p > 0$ such that

$$\Gamma_{\mu(p)}(e^{-x}) - e^{-x} = 0(x^{p+\varepsilon}), \tag{76}$$

in the notation of (71).

It is important to observe that now the spectral function $\mu(p)$ is closely connected with the operator Γ (see (65) and (34)), since this was not assumed in the general theory of Sects. 4–7. This connection leads to much more specific results, than, for example, the general Theorems 3, and 4.

The properties of self-similar solutions to problem (62) for $p_0 > 1$, and consequently, for

$$\mu(p) \geq \mu(p_0) > -\frac{1}{p_0} > -1, \tag{77}$$

can be obtained from the structure of $\Gamma_\mu(e^{-x})$ for any $\mu > -1$ using its explicit formula ((65) and (34)). In particular, for

$$\Gamma_\mu(e^{-x}) = \sum_{n=1}^{N} \alpha_n \int_{\mathbb{R}_+^n} da_1 \ldots da_n A_n(a_1, \ldots, a_n) I_\mu \Big[x \sum_{k=1}^{n} a_k \Big], \tag{78}$$

where

$$I_\mu(y) = \int_0^1 dt \, e^{-yt^\mu}, \qquad \mu \in \mathbb{R}, \ y > 0, \qquad \sum_{n=1}^{N} \alpha_n = 1. \tag{79}$$

the following two statements can be proven.

Lemma 9. *The condition (76) is fulfilled if and only if $p \leq 1$, and therefore $\mu(p) \geq \mu(1)$ whenever $p_0 > 1$ with $\mu(p_0) = \min_{p>0} \mu(p)$.*

Theorem 5. *The limiting function $w(x)$ constructed by the iteration scheme to prove existence in the solution w in Theorem 3 satisfies (71) with $\mu = \mu(1)$, where Γ is given in (24), $\mu(p)$ is defined in (65), (34). Then, the following conditions are fulfilled for $w(x)$:*

(1) It satisfies

$$0 \leq < w(x) \leq 1, \qquad with \ w(0) = 1 \ \ and \ \ w'(0) = -1, \qquad (80)$$

$$w'(x) \leq 0, \qquad |w'(x)| \leq 1, \qquad and \qquad w(x) = e^{-x} + O(x^{\pi(\mu)}), \quad (81)$$

with

$$\pi(\mu) = \begin{cases} 2 & if \ \mu > -\frac{1}{2}, \\ 2 - \varepsilon \quad for \ any \ \varepsilon > 0 & if \ \mu = \frac{1}{2}, \\ \frac{1}{|\mu|} & if \ -1 < \mu < -\frac{1}{2}. \end{cases}$$

(2) Further

$$e^{-x} \leq w(x) \leq 1, \qquad \lim_{x \to \infty} w(x) = 0, \qquad and \qquad (82)$$

(3) There exists a generalized non-negative function $R(\tau)$, $\tau \geq 0$, such that

$$w(x) = \int_0^\infty d\tau \, R(\tau) e^{-\tau x} \qquad \int_0^\infty d\tau \, R(\tau) = \int_0^\infty d\tau \, R(\tau)\tau = 1. \quad (83)$$

The integral representation (83) is important for properties of corresponding distribution functions satisfying Boltzmann-type equations. Now it is easy to return to initial variables with u_0 given in (74) and to describe the complete picture of the self-similar relaxation for the problem (28).

6 Main Results for Maxwell Models with Multiple Interactions

6.1 Self-Similar Asymptotics

We apply now the results of Sect. 6 to the specific case when the Cauchy problem (28) with a fixed operator Γ (24) corresponds to the Fourier transform problem for Maxwell models with multiple interactions. In particular we study the time evolution of $u_0(x)$ satisfying the conditions

$$\|u_0\| = 1; \quad u_0 = 1 - x^p + O(x^{p+\varepsilon}), \ x \to 0, \qquad (84)$$

with some positive p and ε. Then, from Theorems 3 and 4, there exists a unique classical solution $u(x, t)$ of the problem (28), (84) such that, for all $t \geq 0$,

$$\|u(\cdot, t)\| = 1; \qquad u(x, t) = 1 + O(x^p), \ x \to 0. \qquad (85)$$

First, consider the linearized operator L given in (33)–(34) and construct the spectral function $\mu(p)$ given in (65) which will be of one of four kinds described qualitatively on Fig. 1.

Second, find the value $p_0 > 0$ that corresponds to minimum (infimum) of $\mu(p)$. Note that $p_0 = \infty$ just for the case described on Fig. 1a, otherwise $0 < p_0 < \infty$. Compare p_0 with the value p from (84). If $p < p_0$ then the problem (28), (84) has a self-similar asymptotics (see below).

In particular, two different cases are possible: (1) $p \geq p_0$ provided $p_0 < \infty$; (2) $0 < p < p_0$. In the first case a behavior of $u(x,t)$ for large t may depend strictly on initial conditions.

Depending on how p compares with p_0, we can obtain. We again use Lemma 7 with $u_1 = u$ and $u_2 = u_s = \psi(xe^{\mu(p)t})$ and obtain for the solution $u(x,t)$ of the problem (28), (84):

$$\lim_{t \to \infty} u(xe^{-\mu t}, t) = \begin{cases} 1 & \text{if } \mu > \mu(p) \\ \psi(x) & \text{if } \mu = \mu(p) \\ 0 & \text{if } \mu(p) > \mu > \mu(p + \delta), \end{cases} \qquad (86)$$

with sufficiently small $\delta > 0$.

We see that $\psi(x) = w(x^p)$, where $w(x)$ has all properties described in Theorem 5. The equalities (86) explain the exact meaning of the approximate identity,

$$u(x,t) \approx \psi(xe^{\mu(p)t}), \qquad t \to \infty, \ xe^{\mu(p)t} = \text{const.}, \qquad (87)$$

that we call self-similar asymptotics.

In particular, the following statement holds.

Proposition 1. *The solution $u(x,t)$ of the problem (28), (84), with Γ given in (24), satisfies either one of the following limiting identities:*

(1)

$$\lim_{t \to \infty} u(xe^{-\mu t}, t) = 1, \qquad x \geq 0, \qquad (88)$$

for any $\mu > \mu(p_0)$. if $p \geq p_0$ for the initial data (84),
(2) Equation (86) provided $0 < p < p_0$.

The convergence in (88), (86) is uniform on any bounded interval $0 \leq x \leq R$, and

$$u(xe^{\mu(p)t}, t) - \psi(x) = O(x^{p+\varepsilon})e^{-\beta(p,\varepsilon)t}, \qquad \beta(p,\varepsilon) = (p+\varepsilon)\left(\mu(p) - \mu(p+\varepsilon)\right),$$

for $0 < p < p_0$ and $0 < \varepsilon < p_0 - p$.

Remark 7. There is a connection between self-similar asymptotics and non-linear wave propagation. It is easy to see that self-similar asymptotics becomes more transparent in logaritheoremic variables

$$y = \ln x, \quad u(x,t) = \hat{u}(y,t), \quad \psi(x,t) = \hat{\psi}(y,t)$$

Thus, (87) becomes

$$\hat{u}(y,t) \approx \hat{\psi}(y + \mu(p)t), \qquad t \to \infty, \quad y + \mu(p)t = \text{const.}, \qquad (89)$$

hence, the self-similar solutions are simply nonlinear waves (note that $\psi(-\infty) = 1$, $\psi(+\infty) = 0$) propagating with constant velocities $c_p = -\mu(p)$ to the right if $c_p > 0$ or to the left if $c_p < 0$. If $c_p > 0$ then the value $u(-\infty, t) = 1$

is transported to any given point $y \in \mathbb{R}$ when $t \to \infty$. If $c_p < 0$ then the profile of the wave looks more naturally for the functions $\tilde{u} = 1 - \hat{u}$, $\tilde{\psi} = 1 - \psi$.

We conclude that (28) can be considered in some sense as the equation for nonlinear waves. The self-similar asymptotics (89) means a formation of the traveling wave with a universal profile for a broad class of initial conditions. This is a purely non-linear phenomenon, it is easy to see that such asymptotics cannot occur in the particular case ($N = 1$ in (24)) of the linear operator Γ.

6.2 Distribution Functions, Moments and Power-Like Tails

We have described above the general picture of behavior of the solutions $u(x, t)$ to the problem (28), (84). On the other hand, (28) (in particular, its special case (14)) was obtained as the Fourier transform of the kinetic equation. Therefore we need to study in more detail the corresponding distribution functions.

Set $u_0(x)$ in the problem (28) to be an isotropic characteristic function of a probability measure in $\mathbb{R}^d$, i.e.,

$$u_0(x) = \mathcal{F}[f_0] = \int_{\mathbb{R}^d} dv\, f_0(|v|)e^{-ik\cdot v}, \qquad k \in \mathbb{R}^d,\ x = |k|^2, \qquad (90)$$

where f_0 is a generalized positive function normalized such that $u_0(0) = 1$ (distribution function). Let U be a closed unit ball in the $B = C(\mathbb{R}_+)$ as defined in (30).

Then, we can apply all results of Sect. 5, and conclude that there exists a distribution function $f(v, t)$, $v \in \mathbb{R}^d$, satisfying (1), such that

$$u(x, t) = \mathcal{F}[f(\cdot, t)], \qquad x = |k|^2, \qquad (91)$$

for any $t \geq 0$, and a similar conclusion can be obtain if we assume the Laplace (instead of Fourier) transform in (90).

Then there exists a distribution function $f(v, t)$, $v > 0$, such that

$$u(x, t) = \mathcal{L}[f(\cdot, t)] = \int_0^\infty dv\, f(v, t)e^{-xv}, \qquad u(0, t) = 1,\ x \geq 0,\ t \geq 0, \quad (92)$$

where $u(x, t)$ is the solution of the problem (28) constructed in Theorem 2 and Lemma 2.

The approximate equation (87) in terms of distribution functions (91) reads

$$f(|v|, t) \simeq e^{-\frac{d}{2}\mu(p)\, t} F_p(|v|e^{-\frac{1}{2}\mu(p)\, t}), \quad t \to \infty,\ |v|e^{-\frac{1}{2}\mu(p)\, t} = \text{const.}, \quad (93)$$

where $F_p(|v|)$ is a distribution function such that

$$\psi_p(x) = \mathcal{F}[F_p], \qquad x = |k|^2, \qquad (94)$$

with ψ_p given by

$$u_s(x, t) = \psi(xe^{\mu(p)t}) \tag{95}$$

(the notation ψ_p is used in order to stress that ψ defined in (95), depends on p). The factor $1/2$ in (93) is due to the notation $x = |k|^2$. Similarly, for the Laplace transform, we obtain

$$f(v, t) \simeq e^{-\mu(p)t}\Phi_p(ve^{-\mu(p)t}), \qquad t \to \infty, \ ve^{-\mu(p)t} = \text{const.}, \tag{96}$$

where

$$\psi_p(x) = \mathcal{L}[\Phi_p]. \tag{97}$$

The positivity and some other properties of $F_p(|v|)$ follow from the fact that $\psi_p(x) = w_p(x^p)$, where $w_p(x)$ satisfies Theorem 5. Hence

$$\psi_p(x) = \int_0^\infty d\tau \, R_p(\tau)e^{-\tau x^p}, \qquad \int_0^\infty d\tau \, R_p(\tau) = \int_0^\infty d\tau \, R_p(\tau)\tau = 1, \tag{98}$$

where $R_p(\tau)$, $\tau \geq 0$, is a non-negative generalized function (of course, both ψ_p and R_p depend on p).

In particular we can conclude that the self-similar asymptotics (93) for any initial data $f_0 \geq 0$ occurs if $p_0 > 1$, otherwise it occurs for $p \in (0, p_0) \subset (0, 1)$. Therefore, for any spectral function $\mu(p)$ (Fig. 1), the approximate equality (93) holds for sufficiently small $0 < p \leq 1$. In addition,

$$m_2 = \int_{\mathbb{R}^d} dv \, f_0(|v|)|v|^2 < \infty \ \text{if} \ p = 1$$

and $m_2 = \infty$ if $p < 1$. Similar conclusions can be made for the Laplace transforms.

The positivity of $F(|v|)$ in (94)–(97) follows from the integral representation (98) with $p \leq 1$, since it is well-known that

$$\mathcal{F}^{-1}(e^{-|k|^{2p}}) > 0, \qquad \mathcal{L}^{-1}(e^{-x^{2p}}) > 0$$

for any $0 < p \leq 1$ (the so-called infinitely divisible distributions [18]). Thus, (98) explains the connection of the self-similar solutions of generalized Maxwell models with infinitely divisible distributions.

Using standard formulas for the inverse Fourier (Laplace) transforms, denote by ($d = 1, 2, \ldots$ is fixed)

$$M_p(|v|) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk e^{-|k|^{2p}+ik\cdot v},$$

$$N_p(v) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dx \, e^{-x^p+xv}, \qquad 0 < p \leq 1, \tag{99}$$

we obtain the self-similar solutions (distribution functions) are given in (96), (98) (right-hand sides), by

$$F_p(|v|) = \int_0^\infty d\tau\, R_p(\tau) \tau^{-\frac{d}{2p}} M_p(|v|\tau^{-\frac{1}{2p}}),$$

$$\Phi_p(v) = \int_0^\infty d\tau\, R_p(\tau) \tau^{-\frac{1}{p}} N_p(v\tau^{-\frac{1}{p}}), \qquad v \geq 0,\ 0 < p \leq 1. \tag{100}$$

Note that $M_1(|v|)$ is the standard Maxwellian in $\mathbb{R}^d$. The functions $N_p(v)$ (99) are studied in detail in the literature [18, 20]. Thus, for given $0 < p \leq 1$, the kernel $R(\tau)$, $\tau \geq 0$, is the only unknown function that is needed to describe the distribution functions $F(|v|)$ and $\Phi(v)$. See [10] for a study of $R(\tau)$ in more detail.

Now, from (34), (42), and recalling $\mu = \mu(1)$, we can show the moments equation can be written in the form

$$(s\mu(1) - \lambda(s) + 1)m_s = \sum_{n=2}^{N} \alpha_n I_n(s), \tag{101}$$

where

$$I_n(s) = \int_{\mathbb{R}_+^n} da_1 \dots da_n A(a_1, \dots, a_n) \int_{\mathbb{R}_+^n} d\tau_1 \dots, d\tau_n\, g_n^{(s)}(a_1\tau_1, \dots, a_n\tau_n) \prod_{j=1}^{n} R(\tau_j)$$

$$g_n^{(s)}(y_1, \dots, y_n) = \left(\sum_{k=1}^{n} y_k \right)^s - \sum_{k=1}^{n} y_k^s, \qquad n = 1, 2, \dots. \tag{102}$$

and, due to the properties of $R(\tau)$, one gets $g_1^{(s)} = 0$ for any $s \geq 0$ and $m_0 = m_1 = 1$.

Our aim is to study the moments m_s defined in (92), for $s > 1$, on the basis of (101). The approach is similar to the one used in [15] for a simplified version of (101) with $N = 2$. The main results are formulated below in terms of the spectral function $\mu(p)$ (see Fig. 1) under assumption that $p_0 > 1$.

Proposition 2.

(i) If the equation $\mu(s) = \mu(1)$ has the only solution $s = 1$, then $m_s < \infty$ for any $s > 0$.

(ii) If this equation has two solutions $s = 1$ and $s = s_* > 1$, then $m_s < \infty$ for $s < s_*$ and $m_s = \infty$ for $s > s_*$.

(iii) $m_{s_*} < \infty$ only if $I_n(s_*) = 0$ in (101) for all $n = 2, \dots, N$.

Now we can draw some conclusions concerning the moments of the distribution functions (100) as follows. Denote by

$$m_s(\Phi_p) = \int_0^\infty dv\, \Phi_p(v)v^s, \qquad m_s(R_p) = \int_0^\infty d\tau\, R_p(\tau)\tau^s,$$

$$m_{2s}(F_p) = \int_{\mathbb{R}^d} dv\, F_p(|v|)|v|^{2s}, \qquad s > 0,\ 0 < p \le 1,$$

and use similar notations for $N_p(v)$ and $M_p(|v|)$ in (100). Then, by formal integration of (100), we obtain

$$m_s(\Phi_p) = m_s(N_p)m_{s/p}(R_p)$$

$$m_{2s}(F_p) = m_{2s}(M_p)m_{s/p}(R_p),$$

where M_p and N_p are given in (99) and we show that finite only for $s < p < 1$.

In the remaining case $p = 1$ *all* moments of functions

$$M_1(|v|) = (4\pi)^{-d/2}\exp\left[-\frac{|v|^2}{4}\right], \qquad v \in \mathbb{R}^d;$$

$$N_1(v) = \delta(v - 1), \qquad v \in \mathbb{R}_+,$$

are finite. Therefore, everything depends on moments of R_1 in with $p = 1$, so it only needs to apply Proposition 2.

In particular, the following statement holds for the moments of the distribution functions (93), (96).

Proposition 3.

(i) If $0 < p < 1$, then $m_{2s}(F_p)$ and $m_s(\Phi_p)$ are finite if and only if $0 < s < p$.
(ii) If $p = 1$, then Proposition 2 holds for $m_s = m_{2s}(F_1)$ and for $m_s = m_s(\Phi_1)$.

Remark 8. Proposition 3 can be interpreted in other words: the distribution functions $F_p(|v|)$ and $\Phi_p(v)$, $0 < p \le 1$, can have finite moments of all orders in the only case when two conditions are satisfied

(1) $p = 1$, and
(2) The equation $\mu(s) = \mu(1)$ (see Fig. 1) has the unique solution $s = 1$.

In all other cases, the maximal order s of finite moments $m_{2s}(F_p)$ and $m_s(\Phi_p)$ is bounded.

This fact means that the distribution functions F_p and Φ_p have power-like tails.

6.3 Applications to the Conservative or Dissipative Boltzmann Equation

We recall the three specific Maxwell models (A), (B), (C) of the Boltzmann equation from Sect. 2. Our goal in this section is to study isotropic solutions $f(|v|, t)$, $v \in \mathbb{R}^d$, of (2), (4), and (5) respectively. All three cases are considered below from a unified point of view. First we perform the Fourier transform and denote

$$u(x,t) = \mathcal{F}[f(|v|,t)] = \int_{\mathbb{R}^d} dv\, f(|v|,t) e^{-ik\cdot v}, \quad x = |k|^2, \quad u(0,t) = 1. \quad (103)$$

It was already said at the beginning of Sect. 4 that $u(x,t)$ satisfies (in all three cases) (23), where $N = 2$ and all notations are given in (27), (14), (18)–(21). Hence, all results of our general theory are applicable to these specific models. In all three cases (A), (B), (C) we assume that the initial distribution function

$$f(|v|,0) = f_0(|v|) \geq 0, \qquad \int_{\mathbb{R}^d} dv\, f_0(|v|) = 1, \qquad (104)$$

and the corresponding characteristic function

$$u(0,t) = u_0(x) = \mathcal{F}[f_0(|v|)], \quad x = |k|^2, \qquad (105)$$

are given. Moreover, let $u_0(x)$ be such that

$$u_0(x) = 1 - \alpha x^p + O(x^{p+\varepsilon}), \qquad x \to 0, \qquad 0 < p \leq 1, \qquad (106)$$

with some $\alpha > 0$ and $\varepsilon > 0$. We distinguish below the initial data with finite energy (second moment)

$$E_0 = \int_{\mathbb{R}^d} dv\, |v|^2 f_0(|v|) < \infty \qquad (107)$$

implies $p = 1$ in (106) and the in-data with infinite energy $E_0 = \infty$. If $p < 1$ in (106) then

$$m_q^{(0)} = \int_{\mathbb{R}^d} dv\, f_0(|v|) |v|^{2q} < \infty \qquad (108)$$

only for $q \leq p < 1$ (see [18, 20]).

Also, the case $p > 1$ in (106) is not possible for $f_0(|v|) \geq 0$.

In addition, note that the coefficient $\alpha > 0$ in (106) can always be changed to $\alpha = 1$ by the scaling transformation $\tilde{x} = \alpha^{1/p} x$. Then, without loss of generality, we set $\alpha = 1$ in (106).

Since it is known that the operator $\Gamma(u)$ in all three cases belongs to the class (24), we can apply Theorem 5 and state that self-similar solutions of (23) are given by

$$u_s(x,t) = \Psi(x\, e^{\mu(p)t}), \quad \Psi(x) = w(x^p), \qquad (109)$$

where $w(x)$ is given in Theorem 5 and $0 < p < p_0$ (the spectral function $\mu(p)$, defined in (43), and its critical point p_0 depends on the specific model.)

According to Sects. 6.1–6.2, we just need to find the spectral function $\mu(p)$. In order to do this we first define the linearized operator $L = \Gamma'(1)$ for $\Gamma(u)$ given in (24), (27). One should be careful at this point since $A_2(a_1, a_2)$ in (27) is not symmetric and therefore (32) cannot be used. A straight-forward computation leads to

$$Lu(x) = \int_0^1 ds\, G(s)(u(a(s)x) + u(b(s)x)) + \int_0^1 ds H(s) u(c(s)x), \qquad (110)$$

in the notation (18)–(21). Then, the eigenvalue $\lambda(p)$ is given by

$$Lx^p = \lambda(p)\, x^p \quad \text{which implies}$$

$$\lambda(p) = \int_0^1 ds\, G(s) \left\{ (a(s))^p + (b(s))^p \right\} + \int_0^1 ds H(s)(c(s))^p, \qquad (111)$$

and the spectral function (43) reads

$$\mu(p) = \frac{\lambda(p) - 1}{p}. \qquad (112)$$

Note that the normalization (22) is assumed.

At that point we consider the three models (A), (B), (C) separately and apply (111) and (112) to each case.

(A) Elastic Boltzmann Equation (2) in $\mathbb{R}^d, d \geq 2$. By using (18), (19), and (22) we obtain

$$\lambda(p) = \int_0^1 ds\, G(s)(s^p + (1 - s)^p), \qquad G(s) = A_d\, g(1 - 2s)[s(1 - s)]^{\frac{d-3}{2}}, \qquad (113)$$

where the normalization constant A_d is such that (22) is satisfied with $H = 0$. Then

$$\mu(p) = \frac{1}{p} \int_0^1 ds\, G(s)(s^p + (1 - s)^p - 1), \quad p > 0. \qquad (114)$$

It is easy to verify that $p\, \mu(p) \to 1$ as $p \to 0$, $\mu(p) \to 0$ as $p \to \infty$, and

$$\mu(p) > 0 \ \text{ if } p < 1; \qquad \mu(p) < 0 \ \text{ if } p > 1;$$

$$\mu(1) = 0, \qquad \mu(2) = \mu(3) = -\int_0^1 ds\, G(s)\, s(1 - s). \qquad (115)$$

Hence, $\mu(p)$ in this case is similar to the function shown on Fig. 1b with $2 < p_0 < 3$ and such that $\mu(1) = 0$. Then the self-similar asymptotics hold all $0 < p < 1$.

(B) Elastic Boltzmann Equation in the presence of a thermostat (4) in $\mathbb{R}^d$, $d \geq 2$. We consider just the case of a cold thermostat with $T = 0$ in (14), since the general case $T > 0$ can be considered after that with the help of (11). Again, by using (18), (19), and (22) we obtain

$$
\lambda(p) = \int_0^1 ds\, G(s)(s^p + (1-s)^p) + \theta \int_0^1 ds\, G(s)(1 - \frac{4m}{(1+m)^2})^p,
$$

$$
G(s) = \frac{1}{1+\theta} A_d\, g(1-2s)[s(1-s)]^{\frac{d-3}{2}},
$$

(116)

with the same constant A_d as in (113). Then

$$
\mu(p) = \frac{1}{p} \int_0^1 ds\, G(s)(\, s^p + (1-s)^p - \theta(1 - \beta s)^p - (1+\theta)),
$$

$$
\beta = \frac{4m}{(1+m)^2}, \quad p > 0,
$$

(117)

and therefore, as in the previous case (A), $p\,\mu(p) \to 1$ as $p \to 0$, $\mu(p) \to 0$ as $p \to \infty$, with

$$
\mu(1) = -\theta\,\beta \int_0^1 ds\, G(s)\, s.
$$

(118)

which again verifies that $\mu(p)$ is of the same kind as in the elastic case (A) and shown on Fig. 1b. A position of the critical point p_0 such that $\mu'(p_0) = 0$ (see Fig. 1b) depends on θ. It is important to distinguish two cases: (1) $p_0 > 1$ and (2) $p_0 < 1$. In case (1) any non-negative initial data (104) has the self-similar asymptotics. In case (2) such asymptotics holds just for in-data with infinity energy satisfying (106) with some $p < p_0 < 1$. A simple criterion to separate the two cases follows directly from Fig. 1b: it is enough to check the sign of $\mu'(1)$. If

$$
\mu'(1) = \lambda'(1) - \lambda(1) + 1 < 0
$$

(119)

in the notation of (116), then $p_0 > 1$ and the self-similar asymptotics hold for any non-negative initial data.

The inequality (119) is equivalent to the following condition on the positive coupling constant θ

$$
0 < \theta < \theta_* = -\frac{\int_0^1 ds\, G(s)\, (\, s \log s + (1-s)\log(1-s))}{\int_0^1 ds\, G(s)\, (\,\beta\, s + (1-\beta s)\log(1-\beta s))}.
$$

(120)

The right-hand side of this inequality is positive and independent on the normalization of $G(s)$, therefore it does not depend on θ (see (117). We note that a new class of exact self-similar solutions to (4) with finite energy was recently found in [12] for $\beta = 1, \theta = 4/3$ and $G(s) = const$. A simple calculation of the integrals in (120) shows that $\theta_* = 2$ in that case, therefore the criterion (119)

is fulfilled for the exact solutions from [12] and they are asymptotic for a wide class of initial data with finite energy. Similar conclusions can be made in the same way about exact positive self-similar solutions with infinite energy constructed in [12]. Note that the inequality (119) shows the non-linear character of the self-similar asymptotics: it holds unless the linear term in (4) is "too large".

(C) Inelastic Boltzmann Equation (5) in $\mathbb{R}^d$. Equations (21) and (22) lead to

$$\lambda(p) = \int_0^1 ds\, G(s)((a\,s)^p + (1 - b\,s)^p),$$

where

$$G(s) = C_d\,(1 - s)^{\frac{d-3}{2}}, \qquad a = \frac{(1 + e)^2}{4}, \qquad b = \frac{(1 + e)(3 - e)}{4},$$

with such constant C_d that (22) with $H = 0$ is fulfilled. Hence

$$\mu(p) = \frac{1}{p} \int_0^1 ds\, G(s)((a\,s)^p + (1 - b\,s)^p - 1), \qquad p > 0, \qquad (121)$$

and once more as in the previous two cases, $p\,\mu(p) \to 1$ as $p \to 0$, $\mu(p) \to 0$ as $p \to \infty$, with now

$$\mu(1) = -\frac{1 - e^2}{4} \int_0^1 ds\, G(s)\, s.$$

Thus, the same considerations lead to the shape of $\mu(p)$ shown in Fig. 1b. The inequality (119) with $\lambda(p)$ given in (121) was proved in [11] (see (4.26) of [11], where the notation is slightly different from ours). Hence, the inelastic Boltzmann equation (5) has self-similar asymptotics for any restitution coefficient $0 < e < 1$ and any non-negative initial data.

Hence, the spectral function $\mu(p)$ in all three cases above is such that $p_0 > 1$ provided the inequality (120) holds for the model (B).

Therefore, according to our general theory, all "physical" initial conditions (104) satisfying (106) with any $0 < p \le 1$ lead to self-similar asymptotics. Hence, the main properties of the solutions $f(v, t)$ are qualitatively similar for all three models (A), (B) and (C), and can be described in one unified statement: Theorem 6 below.

Before we formulate such general statement, it is worth to clarify one point related to a special value $0 < p_1 \le 1$ such that $\mu(p_1) = 0$. The reader can see that on Fig. 1b that the unique root of this equation exits for all models (A), (B), (C) since $\mu(1) = 0$ in the case (A) (energy conservation), and $\mu(1) < 0$ in cases (B) and (C) (energy dissipation). If $p = p_1$ in (106) then the self-similar solution (109) is simply a stationary solution of (23). Thus, the time relaxation to the non-trivial $(u \ne 0, 1)$ stationary solution is automatically included in Theorem 6 as a particular case of self-similar asymptotics.

Thus we consider simultaneously (2), (4), (5), with the initial condition (104) such that (106) is satisfied with some $0 < p \leq 1$, $\varepsilon > 0$ and $\alpha = 1$. We also assume that $T = 0$ in (4) and the coupling parameter $\theta > 0$ satisfies the condition (120).

In the following Theorem 6, the solution $f(|v|, t)$ is understood in each case as a generalized density of probability measure in $\mathbb{R}^d$ and the convergence $f_n \to f$ in the sense of weak convergence of probability measures.

Theorem 6. *The following two statements hold*

(i) *There exists a unique (in the class of probability measures) solution $f(|v|, t)$ to each of (2), (4), (5) satisfying the initial condition (104). The solution $f(|v|, t)$ has self-similar asymptotics in the following sense:*
For any given $0 < p \leq 1$ in (106) there exits a unique non-negative self-similar solution

$$f_s^{(p)}(|v|, t) = e^{-\frac{d}{2}\mu(p)\,t} F_p(|v| e^{-\frac{1}{2}\mu(p)\,t}), \tag{122}$$

such that

$$e^{\frac{d}{2}\mu(p)\,t} f(|v| e^{-\frac{1}{2}\mu(p)\,t}, t) \;\to_{t\to\infty}\; F_p(|v|), \tag{123}$$

where $\mu(p)$ is given in (114), (117), (121), respectively, for each of the three models.

(ii) *Except for the special case of the Maxwellian*

$$F_1(|v|) = M(|v|) = (4\pi)^{-d/2} e^{-\frac{|v|^2}{4}} \tag{124}$$

for (2) with $p = 1$ in (106) (note that $\mu(1) = 0$ in this case), the function $F_p(|v|)$ does not have finite moments of all orders. If $0 < p < 1$, then

$$m_q = \int_{\mathbb{R}^d} dv\, F_p(|v|) |v|^{2q} < \infty \qquad only \; for \; 0 < q < p. \tag{125}$$

If $p = 1$ in the case of (4), (5), then $m_q < \infty$ only for $0 < q < p_*$, where $p_* > 1$ is the unique maximal root of the equation $\mu(p_*) = \mu(1)$, with $\mu(p)$ given in (106), (121) respectively.

In addition, we also we obtain the following corollary.

Corollary 6. *Under the same conditions of Theorem 6, the following two statements hold.*

(i) *The rate of convergence in (123) is characterized in terms of the corresponding characteristic functions in Proposition 1.*

(ii) *The function $F_p(|v|)$ admits the integral representation (100) through infinitely divisible distributions (99).*

Proof. It is enough to note that all results of Sects. 6.1 and 6.2 are valid, in particular, for (2), (4), (5).

Finally, we mention that the statement similar to Theorem 6, can be easily derived from general results of Sect. 6.1 in the case of one-dimensional Maxwell models introduced in [2, 15] for applications to economy models (Pareto tails, etc.). The only difference is that the "kinetic" equation can be transformed to its canonical form (23)–(24) by the Laplace transform and that the spectral function $\mu(p)$ can have in this case any of the four kind of behaviors shown in Fig. 1. The only remaining problem for any such one-dimensional models is to study them for their specific function $\mu(p)$, and then to apply Propositions 1, 2 and 3.

Thus, the general theory developed in this chapter is applicable to all existing multi-dimensional isotropic Maxwell models and to one-dimensional models as well.

Acknowledgement

References

1. A. Baldassarri, U.M.B. Marconi, and A. Puglisi; Influence of correlations on the velocity statistics of scalar granular gases, Europhys. Lett., 58 (2002), pp. 14–20.
2. Ben-Abraham D., Ben-Naim E., Lindenberg K., Rosas A.; Self-similarity in random collision processes, Phys. Review E, 68, R050103 (2003).
3. Ben-Naim E., Krapivski P.; Multiscaling in inelastic collisions, Phys. Rev. E, R5–R8 (2000).
4. Bisi M., Carrillo J.A., Toscani G.; Decay rates in probability metrics towards homogeneous cooling states for the inelastic Maxwell model, *J. Stat. Phys.* 118 (2005), no. 1–2, 301–331.
5. A.V. Bobylev; The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. Mathematical Physics Reviews, Vol. 7, 111–233, Soviet Sci. Rev. Sect. C Math. Phys. Rev., 7, Harwood Academic Publ., Chur, (1988).
6. A.V. Bobylev; The Fourier transform method for the Boltzmann equation for Maxwell molecules, Sov. Phys. Dokl. 20: 820–822 (1976).
7. A.V. Bobylev, J.A. Carrillo, and I.M. Gamba; On some properties of kinetic and hydrodynamic equations for inelastic interactions. J. Statist. Phys. 98 (2000), no. 3–4, 743–773.
8. A.V. Bobylev and C. Cercignani; Self-similar solutions of the Boltzmann equation and their applications, J. Statist. Phys. 106 (2002), 1039–1071.

9. A.V. Bobylev and C. Cercignani; Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions, J. Statist. Phys. 110 (2003), 333–375.

10. A.V. Bobylev, C. Cercignani, and I.M. Gamba; On the self-similar asymptotics for generalized non-linear kinetic Maxwell models, To appear in Comm. Math. Physics. `http://arxiv.org/abs/math-ph/0608035` (2006).

11. A.V. Bobylev, C. Cercignani, and G. Toscani; Proof of an asymptotic property of self-similar solutions of the Boltzmann equation for granular materials, J. Statist. Phys. 111 (2003), 403–417.

12. A.V. Bobylev and I.M. Gamba; Boltzmann equations for mixtures of Maxwell gases: exact solutions and power like tails, *J. Stat. Phys.* 124, no. 2–4, 497–516. (2006).

13. N. Brilliantov and T. Poschel (Eds.); Granular Gas Dynamics, Springer, Berlin, 2003.

14. C. Cercignani; The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules, *Sov. Sci. Rev. C. Math. Phys.*, **7**, 111–233 (1988).

15. S. Cordier, L. Pareschi, and G. Toscani; On a kinetic model for a simple market economy. J. Statist. Phys. 120 (2005), 253–277.

16. M.H. Ernst and R.Brito; Scaling solutions of inelastic Boltzmann equations with overpopulated high energy tails; J. Stat. Phys. 109 (2002), 407–432.

17. M. Ernst, E. Trizac, and A. Barrat, Granular gases: Dynamics and collective effects, Journal of Physics C: Condensed Matter. condmat-0411435. (2004).

18. W. Feller; An Introduction to Probability Theory and Applications, Vol. 2, Wiley, N.-Y., 1971.

19. E. Gabetta, G. Toscani, and W. Wennberg; Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation, J. Stat. Phys. 81:901 (1995).

20. E. Lukacs; Characteristic Functions, Griffin, London 1970.

21. G. Menon and R. Pego; Approach to self-similarity in Smoluchowski's coagulation equations. Comm. Pure Appl. Math. 57 (2004), no. 9, 1197–1232.

22. G. Menon and R. Pego; Dynamical scaling in Smoluchowski's coagulation equations: uniform convergence, no. 5, 1629–1651, SIAM Applied Math, 2005.