Preface

This book grew out of lectures given at Kyushu University under the support of the Twenty-first Century COE Program "Development of Dynamical Mathematics with High Functionality" (Program Leader: Prof. Mitsuhiro Nakao). They were meant to serve as a primer to my book [Har5]. Indeed that book is very condense, and hard to read. We included however many new themes, such as the higher rank generalization of [Har5], and the fundamental semigroup. Since the audience consisted mainly of representation theorists, the focus shifted more into representation theory (hence less into geometry). We kept the lecture flair, sometimes explaining basic material in more detail, and sometimes only giving brief descriptions.

This book would have never come to life without the many efforts of Professor Masato Wakayama. The author thanks him also for his incredible hospitality. Thanks are also due to Yoshinori Yamasaki, who did an excellent job of writing down and typing the lectures into LaTeX.

July 2006 Haifa

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Markov Chains

Summary. In Sect. 2.1 we show there is a bijection between probability measures τ on the boundary space ∂X of a tree X, and Markov chain on X. For each point x on the tree, we consider the set of all the paths going through x and call it the interval $I(x)$. The interval splits into intervals $I(x')$ corresponding to each arrow $x \mapsto x'$, and we give this arrow the probability $\tau(I(x'))/\tau(I(x))$. The sum of the probability is equal to 1. This is a Markov chain. We then give a brief description in Sect. 2.2 of the boundary theory of general transient Markov chains. Let $X = \bigsqcup_n X_n$, $X_0 = \{x_0\}$ be the state space, $P : \bigsqcup_n X_n \times X_{n+1} \to [0, 1]$ the transition probability. Then we have

Probability measure	$\tau_n(x) = (P^*)^n \delta_{x_0}(x)$	$(x \in X_n)$,
Green kernel	$G(x, y) = P^{m-n}(x, y)$	$(x \in X_n, y \in X_m)$,
Martin kernel	$K(x, y) = \dfrac{G(x, y)}{G(x_0, y)}$.	

The Martin kernel gives a metric. The sequence $\{y_n\}$ is a Cauchy sequence if $\{K(x, y_n)\}$ is a Cauchy sequence of $\mathbb{R}$ for all x and $\{y_n\} \sim \{y'_n\}$ if $\{K(x, y_n)\} \sim \{K(x, y'_n)\}$. Then we obtain the compactification

$$\overline{X} = \{\text{Cauchy sequence of } X\}/\!\sim\ = X \sqcup \partial X.$$

Recall the theorem that every super-harmonic function f is equal to K_μ for some μ which is a probability measure on $X \sqcup \partial X$. Here a function f is called super-harmonic if $Pf \geq f$. If $Pf = f$, we call f a harmonic function and μ is a measure supported only on the boundary ∂X. The set $\mathrm{Harm}(X)$ of all harmonic functions on X is divided as

$$\mathrm{Harm}(X) = \mathrm{Harm}(X)_{\mathrm{ext}} \sqcup \mathrm{Harm}(X)_{\text{non-ext}}$$

and the boundary ∂X also decomposes as

$$\partial X = \partial X_{\mathrm{ext}} \sqcup \partial X_{\text{non-ext}}.$$

Here a point $y \in \partial X$ is called extream if $K_{\delta_y} = K(x,y)$ is extream harmonic function. Then there is one-to-one correspondence between the probability measures on ∂X_{ext} and the harmonic functions on X.

2.1 Markov Chain on Trees

2.1.1 Probability Measures on ∂X

Let X be a tree and $x_0 \in X$ the root. For $n \geq 0$, we denote by X_n the set

$$X_n := \{x \in X \,|\, d(x_0, x) = n\}.$$

Then X decomposes as the disjoint union of X_n; $X = \bigsqcup_{n \geq 0} X_n$. Note that $X_0 = \{x_0\}$ and X_n is a finite set. The boundary ∂X of X is defined by the inverse limit of sets X_n or as the collection of all paths starting from the root x_0,

$$\partial X := \varprojlim X_n = \left\{\tilde{x} = \{x_n\} \,\middle|\, x_n \in X_n,\ d(x_n, x_{n+1}) = 1\right\}.$$

For $x \in X_n$, we denote $I(x) \subset \partial X$, which is called the "interval" of x, by

$$I(x) := \left\{\tilde{x} = \{x_n\} \in \partial X \,|\, x_n = x\right\}$$

and give a topology in ∂X by regarding the family $\{I(x) \,|\, x \in X\}$ as open base of ∂X.

Let τ be a probability measure on the boundary ∂X. Then we obtain a function $\tau : X \to [0,1]$ defined by $\tau(x) := \tau(I(x))$ and it satisfies

$$\tau(x_0) = 1, \quad \tau(x) = \sum_{\substack{x' \in X_{n+1} \\ x \mapsto x'}} \tau(x') \quad (x \in X_n) \tag{2.1}$$

since $I(x_0) = \partial X$ and $I(x) = \bigsqcup_{x \mapsto x'} I(x')$. Here we write $x \mapsto x'$ instead of $d(x, x') = 1$. Conversely, let τ be a function on the tree X satisfying the condition (2.1). Let $\tau(I(x)) := \tau(x)$. Then τ gives a probability measure on ∂X since each open set of ∂X is expressed as the disjoint union of some intervals $I(x)$. Therefore we have the following one-to-one correspondence;

$$\mathfrak{M}_1(\partial X) \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{the function on } X \text{ satisfying} \\ \text{the condition (2.1)} \end{matrix} \right\}.$$

Here $\mathfrak{M}_1(Y)$ denotes the set of all probability measure on Y.

Now given such a τ, we define the probability of going from x to x' by $P(x \mapsto x') := \tau(x')/\tau(x)$. It is clear from (2.1) that

$$\sum_{\substack{x' \in X \\ x \mapsto x'}} P(x \mapsto x') = 1 \qquad (x \in X). \tag{2.2}$$

Hence we have a Markov chain (the condition (2.2) is called the Markov condition). Namely, we have a tree X, which is called the "state space", and the function

$$P : \bigsqcup_{n\geq 0} X_n \times X_{n+1} \longrightarrow [0,1]$$

satisfying the condition (2.2). We call such a function P the "transition probability". Conversely, if we are given a tree X and a function P satisfying the Markov condition, we can get a probability measure on ∂X as follows; For any $x \in X$, we have the unique path $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x$ from x_0 to x. Define the function $\tau : X \to [0,1]$ by

$$\tau(x) := P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x_n = x).$$

Then we have from (2.2) that

$$\begin{aligned}\sum_{x\mapsto x'} \tau(x') &= \sum_{x\mapsto x'} P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) P(x \mapsto x') \\ &= P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) \sum_{x\mapsto x'} P(x \mapsto x') \\ &= \tau(x).\end{aligned}$$

Hence the function $\tau(x)$ satisfies the condition (2.1) and $\tau(I(x)) := \tau(x)$ gives a probability measure on ∂X. We call τ the harmonic measure of P. Hence we obtain the following one-to-one correspondence;

$$\mathfrak{M}_1(\partial X) \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Markov chain on } X; \\ \text{transition probability } P \end{array} \right\}.$$

2.1.2 Hilbert Spaces

Let P be a transition probability and τ its harmonic measure on ∂X. Then we can obtain the probability measure τ_n on X_n by

$$\tau_n(x) = \tau(I(x)) := P(x_0 \mapsto x_1) \cdots P(x_{n-1} \mapsto x) \qquad (x \in X_n),$$

where $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_n = x$ is the unique path from x_0 to x. This can be also written as $\tau_n(x) = (P^*)^n \delta_{x_0}(x)$ where P^* is the adjoint of P and δ_{x_0} is the delta function at x_0 (see the next section). Hence, for all $n \geq 0$, we obtain the Hilbert space

$$H_n := \ell^2(X_n, \tau_n) = \big\{ f : X_n \to \mathbb{C} \,\big|\, ||f||_{H_n} < \infty \big\},$$

where $||f||_{H_n} := (f,f)_{H_n}^{1/2}$ and $(\cdot,\cdot)_{H_n}$ is the inner product of H_n defined by

$$(f,g)_{H_n} := \sum_{x\in X_n} f(x)\overline{g(x)}\tau_n(x).$$

For each $n \geq 0$, we have an embedding $H_n \hookrightarrow H_{n+1}$ defined by

$$H_n \ni \varphi \longmapsto \varphi' \in H_{n+1}; \qquad \varphi'(x') := \varphi(x),$$

where $x \in X_n$ is the unique element such that $x \mapsto x'$. This is an unitary embedding, that is, it preserves the inner product, and we hence identify H_n with a subspace of H_{n+1}. On the other hand we have the orthogonal projection from H_{n+1} onto the subspace H_n

$$H_{n+1} \ni \varphi' \longmapsto \varphi = P\varphi' \in H_n; \qquad P\varphi'(x) := \sum_{\substack{x' \in X_{n+1} \\ x \mapsto x'}} P(x \mapsto x')\varphi'(x').$$

In fact, we can easily show that $\varphi' - P\varphi' \in {H_n}^{\perp} := \{f \in H_{n+1} \,|\, (f,g)_{H_{n+1}} = 0$ for all $g \in H_n\}$.

Since we have a probability measure τ on ∂X, we have another Hilbert space

$$H := \ell^2(\partial X, \tau) = \big\{ f : \partial X \to \mathbb{C} \,\big|\, ||f||_H < \infty \big\},$$

where $||f||_H := (f,f)_H^{1/2}$ and $(\cdot,\cdot)_H$ is the inner product of H defined by

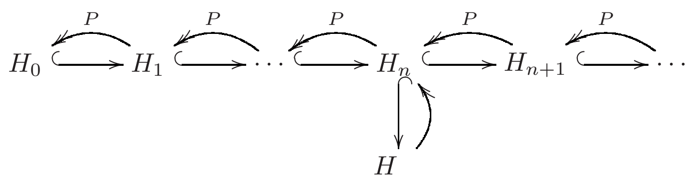
$$(f,g)_H := \int_{\partial X} f(\tilde{x})\overline{g(\tilde{x})}\tau(d\tilde{x}).$$

There is also an unitary embedding map $H_n \hookrightarrow H$ for all $n \geq 0$ defined by

$$H_n \ni \varphi \longmapsto \tilde{\varphi} \in H; \qquad \tilde{\varphi}(\tilde{x}) := \varphi(x_n)$$

with $\tilde{x} = \{x_n\}$ and this is an unitary embedding. The orthogonal projection from H onto H_n is given as follows;

$$H \ni \tilde{\varphi} \longmapsto \varphi \in H_n; \qquad \varphi(x_n) := \frac{1}{\tau_n(x_n)} \int_{I(x_n)} \tilde{\varphi}(\tilde{x})\tau(d\tilde{x}).$$



2.1.3 Symmetric *p*-Adic *β*-Chain

Let us describe the Markov chains associated to the p-adic trees and measures on them. We first give the symmetric β-chain on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ with β-measure. The set of all points on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ is identified with $X = \mathbb{N} \times \mathbb{N}$, the state space. In fact, let

$$X_n = \big\{ (i,j) \in \mathbb{N} \times \mathbb{N} \,\big|\, \max\{i,j\} = n \big\}.$$

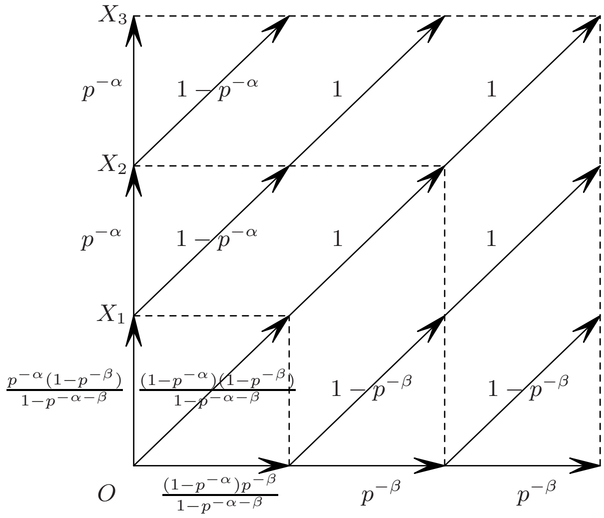


Fig. 2.1. Symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$

Then X_n can be identified with $\mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^*$ by the following correspondence;

$$X_n \ni (i,j) \longmapsto (p^{n-i} : p^{n-j}) \in \mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^*.$$

One can easily obtain the probability measure of each arrow (see Fig. 2.1).

Remember the projection from $\mathbb{P}^1(\mathbb{Q}_p)$ onto $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. If we want to know the probability measure of an arrow in the tree of $\mathbb{P}^1(\mathbb{Q}_p)$, we divide the probability of the projected arrow in $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ by the number of the arrow of $\mathbb{P}^1(\mathbb{Q}_p)$ corresponding to the given arrow in $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$. For example if $\alpha=\beta=1$, it it easy to see that the probability of each arrow is given as in Fig. 2.2 (for the case $p=3$). Note that if $\alpha=\beta=1$, the β-measure $\tau_p^{1,1}$ is the unique $PGL_2(\mathbb{Z}_p)$-invariant measure. In this case we call this the "random walk". Random means that the probability of each arrow is alway the same at any stage. But this is only $\alpha=\beta=1$.

2.1.4 Non-Symmetric p-Adic β-Chain

The symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ is still too complicated for us. We next consider the chain on the tree $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$. Since this is not symmetric, we call this non-symmetric β-chain. Note that the tree of $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$ is obtained by collapsing all of the paths corresponding to $(p^n:1)\mathbb{Z}_p^*$ for $n \geq 0$ of $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^*$ together. Let

$$X_n = \big\{(i,j) \in \mathbb{N}\times\mathbb{N} \,\big|\, i+j=n\big\}.$$

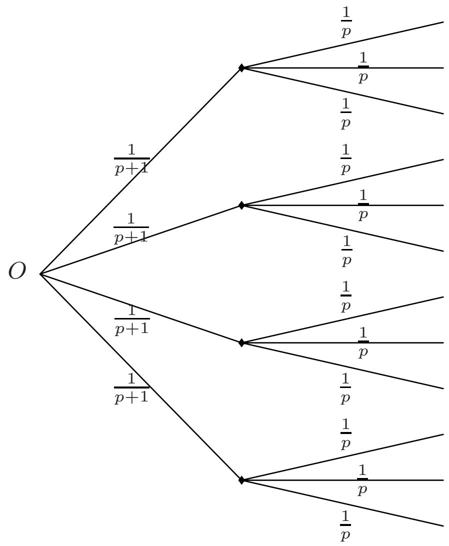


Fig. 2.2. Random walk on $\mathbb{P}^1(\mathbb{Q}_p)$

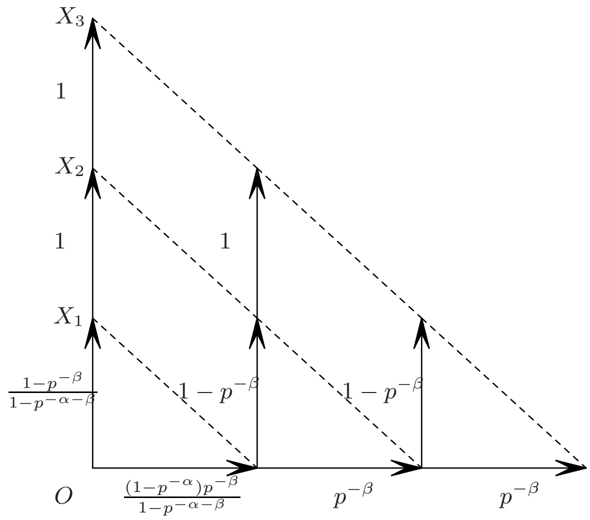


Fig. 2.3. Non-symmetric β-chain on $\mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$

We also regard $X = \mathbb{N} \times \mathbb{N}$ as the state space by the following correspondence;

$$X_n \ni (i,j) \longmapsto (1 : p^{n-j}) = (1 : p^i) \in \mathbb{P}^1(\mathbb{Z}/p^n)\big/(\mathbb{Z}/p^n)^* \ltimes (\mathbb{Z}/p^n).$$

The probability measure is also given in Fig. 2.3. We will concentrate on this chain because it is very simple and will expect a real analogue of the chain.

Notice that the dimension of the Hilbert space $H_n = \ell^2(X_n, \tau_n)$ is given by $\dim H_n = \# X_n = n+1$. Now H_n is embedding into H_{n+1} and the dimension grows by 1 at each stage. Therefore we conclude that there is a unique function $\varphi_n \neq 0$, up to constant multiplied in $H_n \cap (H_{n-1})^{\perp}$ and obtain the orthogonal decomposition $H_n = \mathbb{C}\varphi_n \oplus H_{n-1}$. Let us decide this function. First it is easy to see that

$$\varphi_1 = \mathbf{1}, \tag{2.3}$$

where $\mathbf{1}$ is the constant function. Next φ_1 is the function on $X_1 = \{(1,0),(0,1)\}$ and satisfies $(\varphi_1, \varphi_0)_{H_1} = 0$. Namely,

$$\varphi_1(1,0)\tau_1(1,0) + \varphi_1(0,1)\tau_1(0,1) = 0.$$

Since $\tau_1(1,0) = (1-p^{-\alpha})p^{-\beta}/(1-p^{-\alpha-\beta})$ and $\tau_1(0,1) = (1-p^{-\beta})/(1-p^{-\alpha-\beta})$, we conclude that

$$\varphi_1(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } (i,j) = (1,0), \\ -(1-p^{-\alpha}) & \text{if } (i,j) = (0,1). \end{cases} \tag{2.4}$$

Similar on the n-th set $X_n = \{(n,0),(n-1,1),\dots,(0,n)\}$ for $n \geq 2$, the function φ_n is given by

$$\varphi_n(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta n} & \text{if } (i,j) = (n,0), \\ -p^{\beta(n-1)} & \text{if } (i,j) = (n-1,0), \\ 0 & \text{if } 0 \leq i < n-1. \end{cases} \tag{2.5}$$

By the embedding $H_n \hookrightarrow H_{n+1}$, the function φ_n, which is an element of H_n, can be viewed also as the function on the following spaces H_N for $N > n$. Hence we also obtain the orthogonal decomposition of the N-th layer H_N from (2.3), (2.4) and (2.5);

$$H_N = \bigoplus_{0 \leq m \leq N} \mathbb{C}\varphi_{N,m},$$

where

$$\varphi_{N,0} = \mathbf{1},$$

$$\varphi_{N,1}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } 0 < i \leq N, \\ -(1-p^{-\alpha}) & \text{if } i = 0, \end{cases}$$

$$\varphi_{N,m}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta m} & \text{if } m-1 < i \leq N, \\ -p^{\beta(m-1)} & \text{if } i = m-1, \\ 0 & \text{if } 0 \leq i < m-1, \end{cases} \qquad (m \geq 2).$$

Remember that the function φ_n can be naturally viewed as the element of the boundary space $H = \ell^2(\partial X, \tau)$. Therefore the Hilbert space H is also written as the orthogonal direct sum over all $m \geq 0$;

$$H = \bigoplus_{m \geq 0} \mathbb{C}\varphi_m.$$

Identifying the boundary $\partial X = \mathbb{P}^1(\mathbb{Q}_p)/\mathbb{Z}_p^* \ltimes \mathbb{Z}_p$, we have

$$\begin{aligned}
\varphi_0 &= \mathbf{1}, \\
\varphi_1 &= (1 + p^{\beta} - p^{-\alpha})\phi_{p\mathbb{Z}_p} - (1 - p^{-\alpha})\mathbf{1}, \\
\varphi_m &= p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}\phi_{p^{m-1}\mathbb{Z}_p} \qquad (m \geq 2)
\end{aligned}$$

since, say for $m \geq 2$, $\varphi_m = (1 - p^{-\beta})p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}(\phi_{p^{m-1}\mathbb{Z}_p} - \phi_{p^m\mathbb{Z}_p})$.

We will denote in future H_N by $H_{p(N)}^{(\alpha)\beta}$. (The reason why we denote $(\alpha)\beta$ but not α, β is that it is not symmetric for α and β.) The boundary space H is also written as $H_p^{(\alpha)\beta}$. Further we denote the basis $\varphi_{N,m}$ of H_N by $\varphi_{p(N),m}^{(\alpha)\beta}$ and the basis φ_m of H by $\varphi_{p,m}^{(\alpha)\beta}$. We call $\varphi_{p(N),m}^{(\alpha)\beta}$ the p-Hahn basis (an analogue of the Hahn polynomial) and $\varphi_{p,m}^{(\alpha)\beta}$ the p-Jacobi basis (an analogue of the Jacobi polynomial).

2.1.5 p-Adic γ-Chain

Let us consider the γ-measure. Take $\alpha \to \infty$ in either the symmetric β-chain or non-symmetry β-chain. We get the following tree in Fig. 2.4, called the p-adic γ-chain.

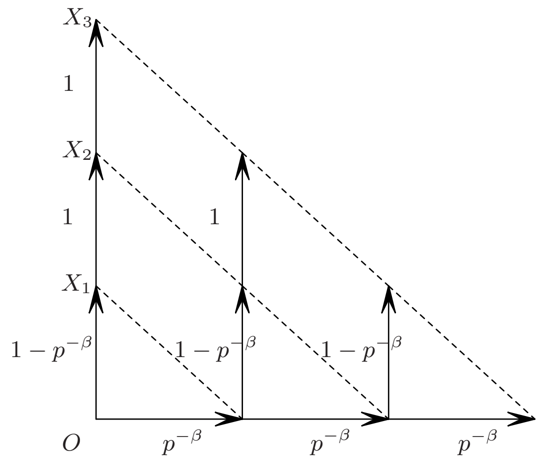


Fig. 2.4. γ-chain on $\mathbb{Z}_p/\mathbb{Z}_p^*$

Similarly we obtain the orthogonal decomposition of H_N and H;

$$H_N := H^{\beta}_{p(N)} = \bigoplus_{0\le m\le N} \mathbb{C}\varphi^{\beta}_{p(N),m},$$
$$H := H^{\beta}_{\mathbb{Z}_p} = \bigoplus_{m\ge 0} \mathbb{C}\varphi^{\beta}_{\mathbb{Z}_p,m},$$

where $\varphi^{\beta}_{p(N),m}$ (resp. $\varphi^{\beta}_{\mathbb{Z}_p,m}$) is the basis of H_N (resp. H) defined by

$$\varphi^{\beta}_{p(N),0} = \mathbf{1},$$
$$\varphi^{\beta}_{p(N),1}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta} & \text{if } 0 < i \le N, \\ -1 & \text{if } i = 0, \end{cases}$$
$$\varphi^{\beta}_{p(N),m}(i,j) = \begin{cases} (1-p^{-\beta})p^{\beta m} & \text{if } m-1 < i \le N, \\ -p^{\beta(m-1)} & \text{if } i = m-1, \\ 0 & \text{if } 0 \le i < m-1, \end{cases} \qquad (m \ge 2).$$

and

$$\varphi^{\beta}_{\mathbb{Z}_p,0} = \phi_{\mathbb{Z}_p},$$
$$\varphi^{\beta}_{\mathbb{Z}_p,m} = p^{\beta m}\phi_{p^m\mathbb{Z}_p} - p^{\beta(m-1)}\phi_{p^{m-1}\mathbb{Z}_p} \qquad (m \ge 1).$$

We call $\varphi^{\beta}_{\mathbb{Z}_p,m}$ the p-Laguerre basis, it is the analogue of the Laguerre polynomial.

Note that if $\beta = 1$, the γ-measure can be written as $\tau^{1}_{\mathbb{Z}_p} = \phi_{\mathbb{Z}_p}(x)|x|^{1}_{p}d^*x/\zeta_p(1) = dx$, where dx is the Haar measure of the additive group $\mathbb{Q}_p$ normalized to be a probability measure by $dx(\mathbb{Z}_p) = 1$. This show that $\tau^{1}_{\mathbb{Z}_p}$ is an "additive" measure. Hence the probability of each arrow in the tree of $\mathbb{Z}_p$ (which is over that of $\mathbb{Z}_p/\mathbb{Z}_p^*$) is given by $1/p$, therefore it is also random walk see Fig. 2.5 (for $p = 3$).

Notice also that if we take the limit $\beta \to \infty$, the γ-measure $\tau^{\beta}_{\mathbb{Z}_p}$ becomes the probability measure on $\mathbb{Z}_p^*$ since $\tau^{\beta}_{\mathbb{Z}_p}(x) \to 0$ for $x \in p\mathbb{Z}_p$. Further if $x \in \mathbb{Z}_p^*$, we have $\tau^{\beta}_{\mathbb{Z}_p}(x) = \phi_{\mathbb{Z}_p^*}(x)d^*x/\zeta_p(\beta) \to d^*x$ and this gives the "multiplicative" measure.

2.2 Markov Chain on Non-Trees

2.2.1 Non-Tree

Now let us consider the real analogue. We already obtain the real analogue of the measure on the boundary, the real analogue of the γ-measure and β-measure. Then what is the real analogue of the Markov chain? We usually

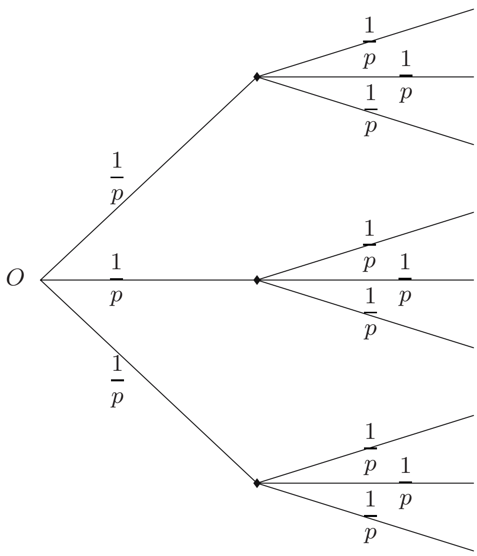


Fig. 2.5. Random walk on $\mathbb{Z}_p$

represent a real number as a path in a "tree". For example, in decimal expansion, each real number is identified with a path from the origin in the $10+1$ regular tree and we obtain $\mathbb{R}$, the set of all real numbers, as the boundary of the tree. Here we sometimes identify two paths, for instance, $1.0000\ldots$ is identified with $0.9999\ldots$. This shows that the boundary is not totally disconnected, hence this is a non-tree (for any tree, the boundary is always totally disconnected). In this section we study the Markov chain on non-trees, which can have continuous boundary.

2.2.2 Harmonic Functions

Let $X = \bigsqcup_{n\geq 0} X_n$, $X_0 = \{x_0\}$ and X_n be a finite set for all $n \geq 0$. We call X the state space. Let $P : \bigsqcup_{n\geq 0} X_n \times X_{n+1} \to [0,1]$ be a transition probability, that is, P satisfies

$$\sum_{x' \in X_{n+1}} P(x,x') = 1 \qquad (x \in X_n). \tag{2.6}$$

Then we says that we have a Markov chain. If for any $x \in X_n$ there exists a sequence $x_0, x_1, \ldots, x_n = x$ such that $x_j \in X_j$ and $P(x_j, x_{j+1}) > 0$, we say that x is reachable from x_0. We assume that every state $x \in X$ is reachable from x_0. The function P can be extended as a function on $X \times X$ by giving 0 if two points x, x' are not connected. Therefore we can regard P as a matrix over $X \times X$.

We also regard P as an operator which acts on $\ell^\infty(X)$, the space of all bounded function on X, as follows;

$$Pf(x) := \sum_{x' \in X} P(x, x') f(x')$$

It is easy to see

$$(i) \quad f \geq 0 \Longrightarrow Pf \geq 0,$$
$$(ii) \quad P\mathbf{1} = \mathbf{1}$$

from the Markov property (2.6).

We have the adjoint operator P^*, which acts on $\ell^1(X)$, defined by

$$P^*\mu(x') := \sum_{x \in X} \mu(x) P(x, x').$$

This operator satisfies

$$(i) \quad \mu \geq 0 \Longrightarrow P^*\mu \geq 0,$$
$$(ii) \quad \int_X P^*\mu = \int_X \mu = \sum_{x \in X} \mu(x).$$

The Laplacian Δ is given by the operator

$$\Delta := \mathbf{1} - P.$$

The function $f : X \to [0, \infty)$ is called harmonic if

$$\Delta f \equiv 0, \qquad f(x_0) = 1.$$

(Here the second condition is a normalization.) Note that the constant function $\mathbf{1}$ is clearly harmonic. Up to a constant multiplication, this is equivalent to the equation

$$f(x) = \sum_{x'} P(x, x') f(x')$$

We denote by $\mathrm{Harm}(X)$ the collection of all harmonic functions. Notice that $\mathrm{Harm}(X)$ is convex. Namely,

$$\begin{matrix} f_0, f_1 \in \mathrm{Harm}(X) \\ \lambda_0, \lambda_1 \geq 0,\ \lambda_0 + \lambda_1 = 1 \end{matrix} \Longrightarrow \lambda_0 f_0 + \lambda_1 f_1 \in \mathrm{Harm}(X).$$

The set $\mathrm{Harm}(X)$ is also compact for the topology of pointwise convergence. If we can take $\lambda_0, \lambda_1 > 0$, then such a function is called non-extremal and we let $\mathrm{Harm}(X)_{\text{non-ext}}$ be the set of all non-extremal harmonic function;

$$\mathrm{Harm}(X)_{\text{non-ext}} := \left\{ \lambda_0 f_0 + \lambda_1 f_1 \,\middle|\, f_0, f_1 \in \mathrm{Harm}(X),\ \lambda_0, \lambda_1 > 0, \lambda_0 + \lambda_1 = 1 \right\}.$$

The harmonic function is called extream if it is not non-extream and we denote by $\mathrm{Harm}(X)_{\text{ext}}$ the set of all extream harmonic functions. Then we obtain

$$\mathrm{Harm}(X) = \mathrm{Harm}(X)_{\text{non-ext}} \sqcup \mathrm{Harm}(X)_{\text{ext}}.$$

This is a basic decomposition of a convex set (see Fig. 2.6).

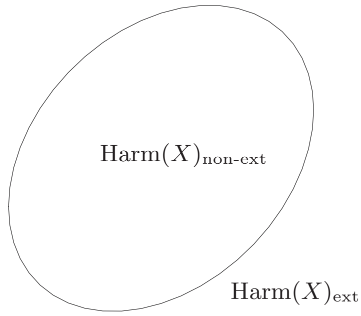


Fig. 2.6. Harm(X)

2.2.3 Martin Kernel

The Green kernel G is given by the operator

$$G := \Delta^{-1} = \sum_{m \geq 0} P^m.$$

If we view P as a matrix on $X \times X$, G can be expressed as follows; Since $P^m(x, y)$ is 0 unless $x \in X_n$ and $y \in X_{n+m}$ for some $n \in \mathbb{N}$, we have

$$G(x, y) = \sum_{x, x_1, \ldots, x_m = y} P(x, x_1) \cdots P(x_{m-1}, y)$$

where the sum is over all paths from x to y. Fix a point $y \in X$. Then the function $G(\cdot, y) : X \to [0, \infty)$ has finite support and is essentially harmonic except for the point $x = y$. Namely,

$$G(x, y) = \sum_{x \mapsto x'} P(x, x')G(x', y) \qquad (x \neq y).$$

If $x = y$, we have $G(y, y) = 1$ by the definition. Therefore we conclude that

$$\Delta G(\cdot, y) = \delta_{y, \cdot}.$$

We next define the Martin Kernel K by

$$K(x, y) := \frac{G(x, y)}{G(x_0, y)}.$$

Hence this function will also be harmonic outside of $x = y$ if we regard $K(x, y)$ as a function of x for a fixed $y \in X$. Note that

$$G(x_0, y) \geq G(x_0, x)G(x, y).$$

and we obtain the bound of the Martin Kernel;

$$K(x,y) \leq \frac{1}{G(x_0,x)}.$$

Now the Martin metric $d : X \times X \to [0,1]$ is defined by

$$d(y_1,y_2) := \sum_{n\geq 0} \frac{1}{2^{n+1}} \frac{1}{\#X_n} \sum_{x\in X_n} G(x_0,x)|K(x,y_1) - K(x,y_2)|.$$

The sequence $\{x_n\}$ is a Cauchy sequence with respect to the Martin metric if, for every $x \in X$, $\{K(x,x_n)\} \subset \mathbb{R}$ is a Cauchy sequence. We say that two such sequences $\{x_n\}$ and $\{x'_n\}$ are equivalent (we write simply $\{x_n\} \sim \{x'_n\}$) if $d(x_n,x'_n) \to 0$ as $n \to \infty$. This is equivalent to $\{K(x,x_n)\} \sim \{K(x,x'_n)\}$ for all $x \in X$. This clearly gives an equivalence relation on the set of all Cauchy sequences and we obtain

$$\overline{X} := \{\text{Cauchy sequences on } X\}/\sim.$$

This is a compactification of X. Actually, for $x \in X$, the constant sequence $\{x_n\}$ with $x_n = x$ for all $n \geq 0$ gives a Cauchy sequence, whence $X \subset \overline{X}$. We then obtain $\overline{X} = X \sqcup \partial X$ where $\partial X := \overline{X} \setminus X$.

The Martin kernel $K(x,y)$, which is defined on $X \times X$, is extended to $X \times \partial X$ as follows; For $x \in X$ and $\{x_n\}/\sim \in \partial X$, we define

$$K(x,\{x_n\}/\sim) := \lim_{n\to\infty} K(x,x_n).$$

(Since $\{K(x,x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, the limit exists.) This is well-defined. Fix a point $y = \{y_n\}/\sim \in \partial X$. Let us write $K\delta_y(x) = K(x,y)$. Then this is always Harmonic:

$$\sum_{x\mapsto x'} P(x,x')K(x',y) = K(x,y)$$

If we take $y_1 \neq y_2$, then we have $K\delta_{y_1} \neq K\delta_{y_2}$. More generally, for any probability measure μ on the boundary ∂X, the function

$$K_\mu(x) := \int_{\partial X} K(x,y)\mu(dy)$$

is always a harmonic function.

The main theorem of the potential theory is as follows:

Theorem 2.2.1. *For every harmonic function $f \in \mathrm{Harm}(X)$, there exists a probability measure $\mu \in \mathfrak{M}_1(\partial X)$ such that $f = K_\mu$.*

We here gives some remarks. The function f is called super harmonic if $Pf \geq f$. The proof of Theorem 2.2.1 goes via showing that every super harmonic function f is of the form $f = K_\mu$ where μ is a probability measure on $\overline{X} = X \sqcup \partial X$. Note that if $f \in \mathrm{Harm}(X)_{\mathrm{ext}}$, then the corresponding measure μ has support at one point. Therefore $f = K\delta_y$ for some $y \in \partial X$. We define

$$\partial X_{\mathrm{ext}} := \left\{ y \in \partial X \,\middle|\, K\delta_y \in \mathrm{Harm}(X)_{\mathrm{ext}} \right\},$$

In generally, we have $\partial X = \partial X_{\mathrm{ext}} \sqcup \partial X_{\text{non-ext}}$. For our case, we have $\partial X = \partial X_{\mathrm{ext}}$. Now if in Theorem 2.2.1 the probability measure μ is supported on the extream points, then it is unique. Therefore, for general Markov chain (on a non-tree), we obtain the following one-to-one correspondence;

$$\begin{aligned} \mathrm{Harm}(X) &\xleftrightarrow{1:1} \mathfrak{M}_1(\partial X_{\mathrm{ext}}) \\ K_\mu &\longleftrightarrow \mu \end{aligned}$$

This is the one-to-one correspondence stated at the beginning of this chapter.

In particular, the constant function $\mathbf{1}$ is always harmonic. The corresponding unique measure τ, supported at the extream points, is called the harmonic measure.