Preface

In the past decades now a famous class of evolution equations has been discovered and intensively studied, a class including the nowadays celebrated *Korteweg-de Vries* equation, *sine-Gordon* equation, *nonlinear Schrödinger* equation, etc. The equations from this class are known also as *the soliton equations* or equations solvable by the so- called *Inverse Scattering Transform Method*. They possess a number of interesting properties, probably the most interesting from the geometric point of view of being that most of them are *Liouville integrable* Hamiltonian systems. Because of the importance of the soliton equations, a dozen monographs have been devoted to them. However, the great variety of approaches to the soliton equations has led to the paradoxical situation that specialists in the same field sometimes understand each other with difficulties. We discovered it ourselves several years ago during a number of discussions the three of us had. Even though by friendship binds us, we could not collaborate as well as we wanted to, since our individual approach to the field of integrable systems (finite and infinite dimensional) is quite different. We have become aware that things natural in one approach are difficult to understand for people using other approaches, though the objects are the same, in our case – *the Recursion (generating) Operators* and their applications to finite and infinite dimensional (not necessarily integrable) Hamiltonian systems. Since even between us, in order to overcome our differences, we needed some serious efforts, we decided that it was time to bring together the analytic and geometric aspects, if not of the theory of the soliton equations (this would be too ambitious) but at least the analytic and the geometric aspects of the so-called Recursion Operators, which are among the powerful tools for the study of soliton equations. We had to do it in such a way, that a specialist in one of the approaches can read and understand the value of the other approach. However, the material we started to collect soon began growing rapidly, and we realized that a book should be written on this topic. The realization of the book project took longer than we expected – more than six years. But now we are happy that we are able to present a text which in our opinion reflects our original ideas.

The book has two parts, the first is dedicated to the analytic approach to the Recursion operators, the second, to the geometric nature of these operators, that is, to their interpretation as mixed tensor fields with special geometric properties over the manifold of potentials.

As we mentioned, we expect that the book will be useful to specialists in the Recursion Operator approach to the soliton equations. However, with an intent to target a larger audience, we have included some other important topics, such as the construction of the soliton solutions, for example. We have tried to develop the material in such a way that the book proves useful for graduate students who want to enter this interesting field of research.

The present book is based on some material that has become already classical, as well as on some of our works. The last few have been written in collaboration with many other friends and colleagues, namely:

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The Lax Representation and the AKNS Approach

In the present Chapter, we outline the famous AKNS approach [1] to the integrable equations. This approach soon became popular, because it provided a simple and effective tool for deriving NLEE allowing Lax representation.

In the first three sections of this Chapter, we show how the AKNS method allows one to derive the family of integrable NLEE related to the Zakharov-Shabat system $L(\lambda)$ and their Lax representations $[L(\lambda), M(\lambda)] = 0$. Assuming $M(\lambda)$ to be polynomial in λ, we derive the recursion procedure for calculating the coefficients of $M(\lambda)$ in terms of $q(x)$ and its derivatives. These relations are solved in compact form using the recursion (generating) operator Λ, which plays a fundamental role in the theory of NLEE. Our derivation is slightly different from that of AKNS in the sense that it is gauge covariant. The advantage of such formulation will become clear in Chap. 8, where we treat the gauge-equivalent NLEE. In the last two sections of this Chapter, we outline two of the natural generalizations of the AKNS approach.

2.1 The Lax Representation in the AKNS Approach

By definition, one can apply the ISM to a given NLEE only if it allows the so-called Lax representation:

$$iL_t = [L, A] \ . \tag{2.1}$$

In his original paper [2] Lax has chosen L to be the Sturm-Liouville operator:

$$-\frac{d^2\psi}{dx^2} + (v(x,t) - k^2)\psi(x,t,k) = 0 \ . \tag{2.2}$$

Taking A as the third-order ordinary differential operator:

$$A\psi \equiv 4\frac{d^3\psi}{dx^3} - 6v(x,t)\frac{d\psi}{dx} - 3\frac{d(v(x,t))}{dx}\psi(x,t,k) \tag{2.3}$$

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Lax proved that (2.1) is satisfied if and only if $v(x,t)$ satisfies the KdV equation:

$$v_t + v_{xxx} + 6v_x v(x,t) = 0 \,. \tag{2.4}$$

Zakharov and Shabat were the first to realize that it is useful to consider Lax representations (2.1) with L-operators more general than (2.2). In [3], they considered as Lax operator the system:

$$L\chi \equiv \left(i\frac{d}{dx} + U(x,t,\lambda) \right) \chi(x,t,\lambda) = 0 \,, \tag{2.5a}$$

$$U(x,t,\lambda) = q(x,t) - \lambda\sigma_3 \,, \tag{2.5b}$$

$$q(x,t) = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix} \tag{2.5c}$$

with $q^+ = (q^-)^* = u(x,t)$ which is the ZS system. Then they constructed explicitly a 2×2 matrix operator A such that the Lax representation (2.1) became equivalent to the NLS equation for $u(x,t)$.

Below, we shall use the AKNS approach [1, 4], which is technically more convenient. In it, we rewrite (2.1) as the compatibility condition:

$$[L(\lambda), M(\lambda)] = 0 \,. \tag{2.6}$$

of two linear operators, whose potentials depend nontrivially on the spectral parameter λ. The λ-dependence is chosen explicitly, and as a rule it is taken to be polynomial or rational in λ. Then the M–operator takes the form:

$$M \equiv i\frac{d}{dt} + V(x,t,\lambda) \,, \tag{2.7}$$

where $V(x,t,\lambda)$ has a prescribed dependence on λ (say, a polynomial one). We also require that the condition (2.6) holds identically with respect to λ. As we shall explain below, this gives us the possibility to express the coefficients of $V(x,t,\lambda)$ in terms of the potential $q(x,t)$ of L.

The compatibility condition (2.6) can be understood also as the zero curvature condition for some connection defined on a conveniently chosen fiber bundle.

Let $\chi(x,t,\lambda)$ be a fundamental solution of L, i.e. this is a matrix-valued function whose determinant does not vanish:

$$L(\lambda)\chi(x,t,\lambda) = 0, \qquad \det \chi(x,t,\lambda) \neq 0 \,. \tag{2.8}$$

From the compatibility condition (2.6) there follows:

$$\begin{aligned} [L(\lambda), M(\lambda)]\chi(x,t,\lambda) &\equiv L(\lambda)M(\lambda)\chi(x,t,\lambda) - M(\lambda)L(\lambda)\chi(x,t,\lambda) \\ &= L(\lambda)M(\lambda)\chi(x,t,\lambda) = 0 \,. \end{aligned} \tag{2.9}$$

i.e. if $\chi(x,t,\lambda)$ is a fundamental solution of $L(\lambda)$, then $M(\lambda)\chi(x,t,\lambda)$ is also a fundamental solution of $L(\lambda)$. From the general theory of ordinary differential operators, it is known that every two fundamental solutions of a given

ODE must be linearly related. Therefore, there exist an x-independent matrix $C(\lambda, t)$ such that

$$M(\lambda)\chi(x,t,\lambda) = \chi(x,t,\lambda)C(\lambda,t) \ . \tag{2.10}$$

In Sect. 2.3 below, we shall analyze in greater detail the convenient choices for $C(\lambda, t)$; as a rule we shall assume it is t-independent. Here, we just remark that the compatibility condition (2.6) holds true for any $C(\lambda, t)$.

Thus, as M operator we choose in agreement with (2.7) and (2.10):

$$M\chi \equiv \left(i\frac{d}{dt} + V(x,t,\lambda) \right) \chi(x,t,\lambda) = \chi(x,t,\lambda)C(\lambda) \tag{2.11}$$

where $V(x,t,\lambda)$ is a polynomial of order N in λ

$$V(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} V_k(x,t) \ . \tag{2.12}$$

Let us outline the AKNS approach. To this end, we insert the expression (2.11) into (2.6) and equate to zero the coefficients in front of the positive powers of λ. This gives:

$$[V_0(x,t), \sigma_3] = 0 \ , \tag{2.13a}$$

$$i\frac{dV_k}{dx} + [q, V_k(x,t)] - [\sigma_3, V_{k+1}(x,t)] = 0 \ , \tag{2.13b}$$

for $k = 0, 1, \dots, N-1$ and the λ–independent term gives:

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N}{\partial x} + [q(x,t), V_N(x,t)] = 0 \ . \tag{2.13c}$$

The (2.13) with $k = 1$ will be treated as the initial condition for the recurrent relations, which allow one to express subsequently the coefficients $V_k(x,t)$ in (2.12) through $q(x,t)$ and its x–derivatives. Thus, (2.13c) finally turns into an NLEE for the off–diagonal matrix $q(x,t)$ or into a system of NLEE for the coefficient functions $q^{\pm}(x,t)$.

Let us list some of the specific choices for the M-operator, which lead to integrable equations.

If we choose:

$$V(x,t,\lambda) = -i\sigma_3 q_x - q^+ q^- \sigma_3 - 2\lambda q(x,t) + 2\lambda^2 \sigma_3 \ , \tag{2.14}$$

we easily find that

(1) The coefficients in front of the positive powers of λ in the compatibility condition (2.6) vanish identically;
(2) The term independent of λ in (2.6) leads to:

$$-iq_t + \sigma_3 q_{xx} + 2q^+ q^- \sigma_3 q(x,t) = 0 \ . \tag{2.15}$$

Thus, it becomes obvious that the choice of L (2.5) and M (2.11), (2.14) in the Lax representation (2.6) is equivalent to the system (2.15), which generalizes the NLS equation.

The next example is related to the KdV and mKdV equations. In both cases $V(x,t,\lambda)$ is a cubic polynomial of λ:

$$\begin{aligned} &V_0 = -4\sigma_3, \qquad V_1 = 4q(x,t), \qquad V_2 = 2q^+q^-\sigma_3 + 2i\sigma_3 q_x \ , \\ &V_3 = -i(q^+q^-_x - q^-q^+_x)\sigma_3 - q_{xx} - 2q^+q^-q(x,t) \ , \end{aligned} \tag{2.16}$$

The compatibility condition (2.6) in this case leads to the following system of NLEE for $q^{\pm}(x,t)$:

$$\begin{aligned} \frac{\partial q^+}{\partial t} + \frac{\partial^3 q^+}{\partial x^3} + 6q^+q^-(x,t)\frac{\partial q^+}{\partial x} = 0 \ , \\ \frac{\partial q^-}{\partial t} + \frac{\partial^3 q^-}{\partial x^3} + 6q^-q^+(x,t)\frac{\partial q^-}{\partial x} = 0 \ , \end{aligned} \tag{2.17}$$

One can obtain two important soliton equations by imposing proper constraints (involutions) on $q^{\pm}(x,t)$. Indeed, choosing $q^+ = v(x,t)$, $q^- = 1$, we see that the system (2.17) reduces to the KdV equation:

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + 6\frac{\partial v}{\partial x}v(x,t) = 0 \ , \tag{2.18}$$

Similarly, imposing the involution $q^+ = \kappa q^- = p(x,t)$, where $p(x,t)$ can be viewed also as a real-valued function, we obtain the modified KdV (mKdV) equation

$$\frac{\partial p}{\partial t} + \frac{\partial^3 p}{\partial x^3} + 6\frac{\partial p}{\partial x}p^2(x,t) = 0 \ . \tag{2.19}$$

The last example is connected with the s-G equation:

$$w_{xt} + \gamma \sin 2w(x,t) = 0 \ . \tag{2.20}$$

In this case $V(x,t,\lambda)$ has the form:

$$V(x,t,\lambda) = \frac{\gamma}{2\lambda}\left(\cos 2w(x,t)\,\sigma_3 - \sin 2w(x,t)\,\sigma_1\right) \ , \tag{2.21}$$

where $q^{\pm}$ are expressed through the real valued-function $w(x,t)$ as follows:

$$q^+(x,t) = -q^-(x,t) = -iw_x(x,t) \ . \tag{2.22}$$

If instead of (2.21) we use:

$$V(x,t,\lambda) = \frac{\gamma}{2\lambda}\left(\cosh 2w(x,t)\,\sigma_3 + \sinh 2w(x,t)\,\sigma_2\right) \ , \tag{2.23}$$

the compatibility condition (2.6) leads to the so-called sinh–Gordon equation:

$$\frac{\partial^2 w}{\partial x \partial t} + \gamma \sinh 2w(x,t) = 0 \ . \tag{2.24}$$

2.2 The Recursion Operators and the NLEE

Following the ideas of AKNS, we shall solve the recursion relations (2.13b) with generic initial conditions, i.e. for arbitrary choice of N and

$$V_0 = c_0\sigma_3, \qquad c_0 = \text{const} . \tag{2.25}$$

The analysis of these relations involves the splitting off of each $V_k(x,t)$ into diagonal and off-diagonal parts. This corresponds to splitting of the algebra $\mathfrak{g} = sl(2)$ into a direct sum $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ of linear subspaces, corresponding to the kernel and the image of ad_{σ_3} considered as operator on $sl(2)$. Therefore, $\mathfrak{g}^{(0)}$ consists of all diagonal 2×2 matrices with vanishing trace, while $\mathfrak{g}^{(1)}$ contains all off-diagonal matrices. Such splitting has the grading property:

$$[X^{(0)}, Y^{(0)}] = 0, \qquad [X^{(0)}, Y^{(1)}] \in \mathfrak{g}^{(1)}, \qquad [X^{(1)}, Y^{(1)}] \in \mathfrak{g}^{(0)} , \tag{2.26}$$

where $X^{(0)}$, $Y^{(0)}$ and $X^{(1)}$, $Y^{(1)}$ are arbitrary elements of $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$, respectively. We shall make use of the projectors onto $\mathfrak{g}^{(1)}$ defined by the above splitting:

$$\pi_0 \cdot \equiv \frac{1}{4}[\sigma_3, [\sigma_3, \,\cdot\,]] . \tag{2.27}$$

Applied to any traceless 2×2 matrix X, π_0 projects out its diagonal part:

$$\pi_0 X \equiv X - X^{\mathrm{d}} = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \in \mathfrak{g}^{(1)} , \tag{2.28}$$

and

$$(\mathbb{1} - \pi_0)X = X^{\mathrm{d}} = X_{11}\sigma_3 , \tag{2.29}$$

Each $V_k(x,t)$ can be split into:

$$V_k(x,t) = w_k(x,t)\sigma_3 + V_k^{\mathrm{f}}(x,t) , \tag{2.30}$$

where

$$V_k^{\mathrm{f}}(x,t) = \pi_0 V_k(x,t) , \tag{2.31a}$$

$$w_k(x,t) = \frac{1}{2}\mathrm{tr}\,(V_k(x,t)\sigma_3) . \tag{2.31b}$$

We start by the relation (2.13b) with $k = 0$:

$$i\frac{dc_0}{dx}\sigma_3 + [q(x,t), \sigma_3] - [\sigma_3, V_1(x,t)] = 0 . \tag{2.32}$$

The diagonal term here is the one proportional to dc_0/dx. It vanishes with c_0 as a constant. The two off-diagonal terms in (2.32) give us:

$$V_1^{\mathrm{f}}(x,t) = -c_0 q(x,t) \; . \tag{2.33}$$

For generic k, we extract first the diagonal part by multiplying (2.13b) by σ_3 and taking the trace. Using (2.31) we find:

$$i\frac{dw_k}{dx} + \frac{1}{2}\mathrm{tr}\,(\sigma_3[q(x,t), V_k(x,t)]) = 0 \; . \tag{2.34}$$

Note that in the second term of (2.34) only the off-diagonal part of V_k contributes. Thus (2.34) relates w_k and V_k^{f}. Integrating it we get:

$$w_k(x,t) = c_k + \frac{i}{2}\int_{\pm\infty}^{x} dy\,\mathrm{tr}\,(\sigma_3[q(y,t), V_k^{\mathrm{f}}(y,t)]) \; , \tag{2.35}$$

where c_k is an integration constant. Next the off-diagonal part of (2.13b) gives:

$$i\frac{dV_k^{\mathrm{f}}}{dx} + [q(x,t), \sigma_3]w_k(x,t) = [\sigma_3, V_{k+1}^{\mathrm{f}}(x,t)] \; . \tag{2.36}$$

It remains to apply $\frac{1}{4}[\sigma_3, \cdot]$ to both sides of (2.36) and to make use of (2.27) and (2.35) to find:

$$\begin{aligned} V_{k+1}^{\mathrm{f}}(x,t) &= \frac{i}{4}\left[\sigma_3, \frac{dV_k^{\mathrm{f}}}{dx}\right] - \frac{1}{4}\left[\sigma_3, [\sigma_3, q(x,t)]\right] w_k(x,t) \\ &= \frac{i}{4}\left[\sigma_3, \frac{dV_k^{\mathrm{f}}}{dx}\right] - \frac{i}{2}q(x,t)\int_{\pm\infty}^{x} dy\,\mathrm{tr}\,(\sigma_3[q(y,t), V_k^{\mathrm{f}}(y,t)]) \\ &\quad - c_k q(x,t) \; . \end{aligned} \tag{2.37}$$

Therefore the recurrent relation (2.13) now can be rewritten in the following compact form:

$$V_{k+1}^{\mathrm{f}}(x,t) = \Lambda_\pm V_k^{\mathrm{f}}(x,t) - c_k q(x,t) \; , \tag{2.38a}$$

$$V_1(x,t) = -c_0 q(x,t) \; , \tag{2.38b}$$

where by $\Lambda_\pm$ we have denoted the recursion operators:

$$\Lambda_\pm X \equiv \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] - \frac{i}{2}q(x,t)\int_{\pm\infty}^{x} dy\,\mathrm{tr}\,(\sigma_3[q(y,t), X(y,t)]) \; . \tag{2.39}$$

As we shall see in the next chapters, these operators play an important role in the theory of the NLEE. Here, we shall use them to write down the solution of the recurrent relations in the following compact form:

$$V_k^{\mathrm{f}}(x,t) = -\sum_{p=0}^{k-1} c_p \Lambda_\pm^{k-p-1} q(x,t) \; , \tag{2.40a}$$

$$w_k(x,t) = c_k - \frac{i}{2}\sum_{p=0}^{k-1} c_p \times \int_{\pm\infty}^{x} dy \operatorname{tr}\left(\sigma_3 \left[q(y,t), \Lambda_{\pm}^{k-p-1} q(y,t)\right]\right) . \tag{2.40b}$$

We shall show below that, although the operators $\Lambda_{\pm}$ are integro-differential applying their positive powers to $q(x,t)$, we always get expressions, which are local in $q(x,t)$, i.e. depend only on q and its x-derivatives.

The explicit solution of the recursion relation (2.13b) allows us now to describe the class of all NLEE, which can be solved applying the ISM to the ZS system. To do this, we have to insert the expression for $V_N(x,t)$ from (2.40) into (2.13) and to separate again the diagonal and the off-diagonal parts in it. The diagonal part gives us the necessary expression for $w_N(x,t)$ as an integral containing $q(x,t)$ and $V_N^{\rm f}(x,t)$, i.e. we get (2.35) with $k = N$. The off-diagonal part leads to the following NLEE:

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N}{\partial x} + [q(x,t), \sigma_3] w_N(x,t) = 0 . \tag{2.41}$$

Now, we apply to both sides $-\frac{1}{4}[\sigma_3, \cdot]$ and using (2.35) find:

$$\frac{i}{4}\left[\sigma_3, \frac{\partial q}{\partial t}\right] - \Lambda_{\pm} V_N^{\rm f}(x,t) + c_N q(x,t) = 0 , \tag{2.42}$$

or

$$\frac{i}{4}\left[\sigma_3, \frac{\partial q}{\partial t}\right] + f(\Lambda_{\pm}) q(x,t) = 0 , \tag{2.43}$$

where $f(\lambda)$ is the polynomial:

$$f(\lambda) = \sum_{p=0}^{N} c_p \lambda^{N-p} . \tag{2.44}$$

In the form (2.43), the NLEE is quite analogous to the generic partial differential equation with constant coefficients (1.30) in Chap. 1. Indeed, since $q(x,t)$ is an off-diagonal matrix, then $[\sigma_3, q_t] = 2\sigma_3 q_t$ and that makes the (1.30) and (2.43) quite analogous; one just has instead of $f(D_0)$, $f(\Lambda_{\pm})$.

Our main aim will be to prove that this analogy is not coincidental, and its roots are in the spectral decompositions of the recursion operators.

2.3 Evolution of the Scattering Data

We introduced already some NLEE having Lax representation. In this subsection, we shall explain the idea of what earlier was called a type of "change of variables", which linearizes the NLEE. To this end, we shall use the ZS system (2.5) with a complex-valued potential $q(x,t)$.

We shall suppose also that the potential $q(x,t)$ depends on the additional parameter t in such a way that its coefficients $q^{\pm}(x,t)$ satisfy one of the above mentioned NLEE. Another important choice consists in fixing up the class of functions, to which the potential belongs. Here, and in what follows, we assume that $q(x,t)$ belongs to the space $\mathcal{M}$ of off-diagonal 2×2 matrix-valued complex functions of Schwartz-type; i.e. it is an infinitely differentiable function tending to 0 for $|x|\to\infty$ faster than any negative power of x. We also assume that these properties are fulfilled for all values of t.

Note that the ZS system can be viewed formally as a quantum mechanical problem for the scattering of a "plane wave" on the "potential" $q(x,t)$. This "scattering" will be used, however, as a technical tool and will not be assigned any real physical meaning. Nevertheless, we shall make use of the well-developed theory for solving the direct and inverse scattering problems in quantum mechanics, which can easily be generalized to complex-valued "potentials." Thus, we shall omit the quotation marks as we use the standard terminology.

We recall some well-known facts from the theory of the linear differential equations. By $\chi(x,t,\lambda)$, we shall denote a matrix-valued solution of (2.5). Since $\operatorname{tr} U(x,t,\lambda) = 0$, then $\det\chi(x,t,\lambda)$ does not depend on x. $\chi(x,t,\lambda)$ is called a *fundamental solution* if its determinant does not vanish, i.e. $\det\chi(x,t,\lambda)\neq 0$.

Any fundamental solution of (2.5) can be fixed up uniquely by specifying its value at a given point $x = x_0$. Another important property of the linear systems in general and of the ZS system in particular is that any two fundamental solutions must be linearly related; see (2.47) below.

A special role in the direct and inverse scattering theory for the ZS system is played by the so-called Jost solutions $\psi(x,t,\lambda)$ and $\phi(x,t,\lambda)$. They are special fundamental solutions of (2.5) introduced by fixing up their asymptotics for $x\to\infty$ (or to $x\to-\infty$) to be plane waves:

$$\lim_{x\to\infty} \exp(i\lambda\sigma_3 x)\psi(x,t,\lambda) = \mathbb{1}, \qquad \lambda\in\mathbb{R} \tag{2.45a}$$

$$\lim_{x\to-\infty} \exp(i\lambda\sigma_3 x)\phi(x,t,\lambda) = \mathbb{1}, \qquad \lambda\in\mathbb{R}\,. \tag{2.45b}$$

By plane wave above, we mean the matrix-valued function $\exp(-i\lambda x\sigma_3)$ for real values of the spectral parameter λ; obviously it is a solution of (2.5) for the asymptotic value of the potential $q(x,t)=0$.

In the special cases in (2.45), x_0 is taken to be ∞ and $-\infty$ correspondingly. Both solutions have determinants equal to 1:

$$\det\psi(x,t,\lambda) = \det\phi(x,t,\lambda) = 1, \qquad \lambda\in\mathbb{R}\,. \tag{2.46}$$

so they are fundamental, and they must be linearly related. This means that there exist the so-called scattering matrix $T(t,\lambda)$ such that

$$\phi(x,t,\lambda) = \psi(x,t,\lambda)T(t,\lambda), \qquad \lambda\in\mathbb{R}\,. \tag{2.47}$$

Let us denote the entries of the scattering matrix $T(t,\lambda)$ by:

$$T(t,\lambda) = \begin{pmatrix} a^+(\lambda) & -b^-(t,\lambda) \\ b^+(t,\lambda) & a^-(\lambda) \end{pmatrix} . \tag{2.48}$$

From (2.46) and (2.47) it follows that

$$\det T(t,\lambda) \equiv a^+(\lambda)a^-(\lambda) + b^+(t,\lambda)b^-(t,\lambda) = 1, \qquad \lambda \in \mathbb{R} . \tag{2.49}$$

This is known as the "unitarity" condition for the scattering matrix $T(t,\lambda)$.

Next, we derive the corresponding evolution of the scattering matrix $T(t,\lambda)$. To this end, we make use of the explicit form of the M-operator (2.11) derived in the previous section with conveniently chosen $C(\lambda)$. Consider (2.11) with $\chi = \phi(x,t,\lambda)$:

$$M\phi \equiv \left(i\frac{d}{dt} + V_N(x,t,\lambda) \right) \phi(x,t,\lambda) = \phi(x,t,\lambda)C(\lambda) , \tag{2.50}$$

multiply it on the left by $\exp(i\lambda\sigma_3 x)$ and take the limit $x \to -\infty$. Assuming that the asymptotics of the Jost solution $\phi(x,t,\lambda)$ for $x \to -\infty$ in (2.45a) is valid for all t, we get:

$$\begin{aligned} \lim_{x\to-\infty} e^{i\lambda\sigma_3 x} V_-(x,t,\lambda)\phi(x,t,\lambda) &\equiv \lim_{x\to-\infty} V_N(x,t,\lambda) \\ &= f(\lambda)\sigma_3 \\ &= C(\lambda) \end{aligned} \tag{2.51}$$

Thus, we find that $C(\lambda)$ can be directly related to the dispersion law of the NLEE:

$$C(\lambda) = f(\lambda)\sigma_3 . \tag{2.52}$$

In the limit $x \to \infty$, in view of (2.47) we get:

$$\left(i\frac{dT}{dt} + \lim_{x\to\infty} V_N(x,t,\lambda)T(t,\lambda) \right) = T(t,\lambda)C(\lambda) . \tag{2.53}$$

With (2.52), (2.56), we find that the scattering matrix $T(t,\lambda)$ satisfies the following *linear* evolution equation:

$$i\frac{dT}{dt} + f(\lambda)[\sigma_3, T(t,\lambda)] = 0 . \tag{2.54}$$

Written in terms of the entries of $T(\lambda)$ (2.54), the evolution takes the form of linear equations:

$$i\frac{da^\pm}{dt} = 0, \qquad i\frac{db^\pm}{dt} \mp 2f(\lambda)b^\pm(t,\lambda) = 0 \tag{2.55}$$

that can be easily solved for any choice of the *dispersion law* $f(\lambda)$.

The same results can be derived by taking $\chi = \psi(x,t,\lambda)$ and considering the limits $x\pm \to \infty$. Thus we established that

$$V_+(\lambda) = V_-(\lambda) = C(\lambda) = f(\lambda)\sigma_3, \qquad V_\pm(\lambda) = \lim_{x\to\pm\infty} V_N(x,t,\lambda) \tag{2.56}$$

But (2.6) means also that $q(x,t)$ satisfies the NLEE (2.43). Therefore, we outlined the proof of the following

Theorem 2.1 ([1]). *If $q(x,t) \in \mathcal{M}$ and satisfies the NLEE (2.43), then the scattering matrix $T(t,\lambda)$ satisfies the linear evolution equation (2.54).*

Thus the dispersion law $f(\lambda)$ of the corresponding NLEE determines both the NLEE itself through (2.43) and the evolution of the scattering data through (2.54) or (2.55).

Calculating the limits $V_\pm(\lambda)$ from the explicit expressions for $V(x,t,\lambda)$ corresponding to the NLS, KdV and s-G equations we get:

$$f_{\rm NLS}(\lambda) = -2\lambda^2, \qquad f_{\rm KdV}(\lambda) = -4\lambda^3, \qquad f_{\text{s-G}}(\lambda) = \frac{\gamma}{2\lambda}\,. \tag{2.57}$$

The two functions $a^\pm(\lambda)$ are in fact t–independent. This means that if we expand them in asymptotic series in λ their expansion coefficients also will be t–independent, i.e. they will be integrals of motion for the corresponding NLEE. In what follows, we treat $a^\pm(\lambda)$ as generating functionals of the integrals of motion of the NLEE.

2.4 Generalizations of the AKNS Method I

The AKNS method can be applied also to special multicomponent generalizations of the NLS type equations. One way to do this is to apply it to the block-matrix generalization of the Zakharov-Shabat system.

$$\boldsymbol{L}\boldsymbol{\chi} \equiv \left(i\frac{d}{dx} + \boldsymbol{U}(x,t,\lambda)\right) \boldsymbol{\chi}(x,t,\lambda) = 0\,, \tag{2.58a}$$

$$\boldsymbol{U}(x,t,\lambda) = \boldsymbol{q}(x,t) - \lambda\boldsymbol{\sigma}\,, \tag{2.58b}$$

$$\boldsymbol{q}(x,t) = \begin{pmatrix} 0 & \boldsymbol{q}^+ \\ \boldsymbol{q}^- & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma} = \frac{2}{s+p}\begin{pmatrix} p\mathbb{1}_s & 0 \\ 0 & -s\mathbb{1}_p \end{pmatrix}, \tag{2.58c}$$

where $\boldsymbol{q}^+(x,t)$ and $(\boldsymbol{q}^-)^T(x,t)$ are rectangular $s\times p$ matrix-valued functions, $\mathbb{1}_s$ and $\mathbb{1}_p$ are the unit matrices of dimension s and p, $s+p=n$.

As M operator we choose:

$$\boldsymbol{M}\boldsymbol{\chi} \equiv \left(i\frac{d}{dt} + \boldsymbol{V}(x,t,\lambda)\right) \boldsymbol{\chi}(x,t,\lambda) = \boldsymbol{\chi}(x,t,\lambda)\boldsymbol{C}(\lambda) \tag{2.59}$$

where $\boldsymbol{V}(x,t,\lambda)$ is a polynomial of order N in λ

$$\boldsymbol{V}(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} \boldsymbol{V}_k(x,t) \ , \tag{2.60}$$

$$\boldsymbol{C}(\lambda) = \lim_{x\to\infty} \boldsymbol{V}(x,t,\lambda) = \lim_{x\to-\infty} \boldsymbol{V}(x,t,\lambda) \ . \tag{2.61}$$

The compatibility condition $[\boldsymbol{L},\boldsymbol{M}] = 0$ holds true for any choice of the matrix $\boldsymbol{C}(\lambda)$. Now $\boldsymbol{U}(x,t,\lambda)$ and $\boldsymbol{V}(x,t,\lambda)$ are elements (of special form) of the algebra $sl(n)$. Since this condition must hold identically with respect to λ, we equate to zero the coefficients in front of all powers of λ with the result:

$$[\boldsymbol{V}_0(x,t), \boldsymbol{\sigma}] = 0 \ , \tag{2.62a}$$

$$i\frac{d\boldsymbol{V}_k}{dx} + [\boldsymbol{q}(x,t), \boldsymbol{V}_k(x,t)] - [\boldsymbol{\sigma}, \boldsymbol{V}_{k+1}(x,t)] = 0 \ , \tag{2.62b}$$

for $k = 0, 1, \ldots, N-1$. The λ–independent term provides the corresponding multicomponent NLEE:

$$-i\frac{\partial \boldsymbol{q}}{\partial t} + i\frac{\partial \boldsymbol{V}_N}{\partial x} + [\boldsymbol{q}(x,t), \boldsymbol{V}_N(x,t)] = 0 \ . \tag{2.62c}$$

These relations again can be viewed as recursion relations, allowing to determine $\boldsymbol{V}_k(x,t)$ in terms of $\boldsymbol{q}(x,t)$ and its derivatives. Generalizing the AKNS approach, we split each $\boldsymbol{V}_k(x,t)$ into block-diagonal and block-off-diagonal parts. This corresponds to splitting of the algebra $\mathfrak{g} = sl(n)$ into a direct sum $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ of linear subspaces, corresponding to the kernel and the image of the operator $\mathrm{ad}_{\boldsymbol{\sigma}}$ on $sl(n)$. Since the nonvanishing eigenvalues of $\mathrm{ad}_{\boldsymbol{\sigma}}$ are equal to ± 2, the projector $\boldsymbol{\pi}_0$ onto $\mathfrak{g}^{(1)}$ takes the form:

$$\boldsymbol{\pi}_0 \cdot \equiv \frac{1}{4}[\boldsymbol{\sigma}, [\boldsymbol{\sigma}, \cdot\,]] \ . \tag{2.63}$$

Applied to any $n \times n$ matrix $\boldsymbol{X}$ it projects out its block-diagonal part:

$$\boldsymbol{\pi}_0 \boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}^{(0)} = \begin{pmatrix} 0 & \boldsymbol{X}_{12} \\ \boldsymbol{X}_{21} & 0 \end{pmatrix} \ , \tag{2.64}$$

The projector onto $\mathfrak{g}^{(0)}$ is given by:

$$(\mathbb{1} - \boldsymbol{\pi}_0)\boldsymbol{X} = \boldsymbol{X}^{(0)} = \begin{pmatrix} \boldsymbol{X}_{11} & 0 \\ 0 & \boldsymbol{X}_{22} \end{pmatrix}, \qquad \mathrm{tr}\, \boldsymbol{X}^{(0)} = 0 \ . \tag{2.65}$$

Therefore, $\mathfrak{g}^{(0)}$ consists of all block-diagonal matrices (2.65) with vanishing trace $\mathrm{tr}\, \boldsymbol{X}_{11} + \mathrm{tr}\, \boldsymbol{X}_{22} = 0$, while $\mathfrak{g}^{(1)}$ contains all block-off-diagonal matrices. Such splitting also has the grading property:

$$[\boldsymbol{X}^{(0)}, \boldsymbol{Y}^{(0)}] = 0, \qquad [\boldsymbol{X}^{(0)}, \boldsymbol{Y}^{(1)}] \in \mathfrak{g}^{(1)}, \qquad [\boldsymbol{X}^{(1)}, \boldsymbol{Y}^{(1)}] \in \mathfrak{g}^{(0)} \ , \tag{2.66}$$

where $\boldsymbol{X}^{(i)}$, $\boldsymbol{Y}^{(i)}$ are arbitrary elements of $\mathfrak{g}^{(i)}$, $i = 1, 2$. Each $\boldsymbol{V}_k(x,t)$ can be split into:

$$\boldsymbol{V}_k(x,t) = \boldsymbol{w}_k(x,t) + \boldsymbol{V}_k^{\mathrm{f}}(x,t) \ , \tag{2.67}$$

where

$$\boldsymbol{V}_k^{\mathrm{f}}(x,t) = \boldsymbol{\pi}_0 \boldsymbol{V}_k(x,t) \ , \tag{2.68a}$$

$$\boldsymbol{w}_k(x,t) = (\mathbb{1}_n - \boldsymbol{\pi}_0)\boldsymbol{V}_k(x,t) \ . \tag{2.68b}$$

Then (2.62a) means that $\boldsymbol{V}_0^{\mathrm{f}}(x,t) = 0$, i.e. $\boldsymbol{V}_0(x,t) = \boldsymbol{w}_0(x,t)$. From the block-diagonal part of (2.62b) with $k = 1$ we conclude that

$$\frac{d\boldsymbol{w}_0}{dx} = 0 \ , \tag{2.69}$$

i.e. we assume that $\boldsymbol{w}_0 = \mathrm{const} \in \mathfrak{g}^{(0)}$. The block-off-diagonal part of (2.62b) with $k = 1$ is equivalent to:

$$\boldsymbol{V}_1^{\mathrm{f}}(x,t) = \mathrm{ad}_{\boldsymbol{\sigma}}^{-1}[\boldsymbol{q}(x,t), \boldsymbol{w}_0] \ . \tag{2.70}$$

For $k > 1$, we again use the same splitting with the results:

$$i\frac{d\boldsymbol{w}_k}{dx} + [\boldsymbol{q}(x,t), \boldsymbol{V}_k^{\mathrm{f}}(x,t)] = 0 \ . \tag{2.71}$$

Thus, (2.71) establishes a relation between $\boldsymbol{w}_k$ and $\boldsymbol{V}_k^{\mathrm{f}}$. Integrating it we get:

$$\boldsymbol{w}_k(x,t) = \boldsymbol{w}_k^0 + i\int_{\pm\infty}^{x} dy\, [\boldsymbol{q}(y,t), \boldsymbol{V}_k^{\mathrm{f}}(y,t)] \ , \tag{2.72}$$

where $\boldsymbol{w}_k^0 \in \mathfrak{g}^{(0)}$ is a matrix-valued integration constant.

Next, the block-off-diagonal part of (2.62b) gives:

$$i\frac{d\boldsymbol{V}_k^{\mathrm{f}}}{dx} + [\boldsymbol{q}(x,t), \boldsymbol{w}_k(x,t)] = [\boldsymbol{\sigma}, \boldsymbol{V}_{k+1}^{\mathrm{f}}(x,t)] \ . \tag{2.73}$$

It remains to apply $[\boldsymbol{\sigma}, \cdot]$ to both sides of (2.73) and to make use of (2.63) and (2.72) to find:

$$\begin{aligned}
\boldsymbol{V}_{k+1}^{\mathrm{f}}(x,t) &= \frac{i}{4}\left[\boldsymbol{\sigma}, \frac{d\boldsymbol{V}_k^{\mathrm{f}}}{dx}\right] - \frac{1}{4}\left[\boldsymbol{\sigma}, [\boldsymbol{w}_k(x,t), \boldsymbol{q}(x,t)]\right] \ , \\
&= \frac{i}{4}\left[\boldsymbol{\sigma}, \frac{d\boldsymbol{V}_k^{\mathrm{f}}}{dx}\right] + \frac{i}{4}\left[\boldsymbol{\sigma}, \left[\boldsymbol{q}(x,t)\int_{\pm\infty}^{x} dy\, [\boldsymbol{q}(y,t), \boldsymbol{V}_k^{\mathrm{f}}(y,t)]\right]\right] \\
&\quad - \frac{1}{4}\left[\boldsymbol{\sigma}, [\boldsymbol{w}_k^0, \boldsymbol{q}(x,t)]\right] \ .
\end{aligned} \tag{2.74}$$

Thus, the recurrent relation (2.62) acquires the following compact form:

$$\boldsymbol{V}^{\mathrm{f}}_{k+1}(x,t) = \boldsymbol{\Lambda}_{\pm}\boldsymbol{V}^{\mathrm{f}}_{k}(x,t) - \frac{1}{4}\left[\boldsymbol{\sigma}, [\boldsymbol{w}^0_k, \boldsymbol{q}(x,t)]\right] , \tag{2.75a}$$

$$\boldsymbol{V}_1(x,t) = -\mathrm{ad}_{\boldsymbol{\sigma}}^{-1}[\boldsymbol{w}^0_k, \boldsymbol{q}(x,t)] , \tag{2.75b}$$

where by $\boldsymbol{\Lambda}_{\pm}$ we have denoted the recursion operators:

$$\boldsymbol{\Lambda}_{\pm}\boldsymbol{X} \equiv \frac{i}{4}\left[\boldsymbol{\sigma}, \frac{d\boldsymbol{X}}{dx}\right] + \frac{i}{4}\left[\boldsymbol{\sigma}, \left[\boldsymbol{q}(x,t), \int_{\pm\infty}^{x} dy\, [\boldsymbol{q}(y,t), \boldsymbol{X}(y,t)]\right]\right] . \tag{2.76}$$

The formal solution to this recurrent relations is given by:

$$\boldsymbol{V}^{\mathrm{f}}_{k+1}(x,t) = -\frac{1}{4}\sum_{p=0}^{k} \boldsymbol{\Lambda}_{\pm}^{k-p}\left[\boldsymbol{\sigma}, \left[\boldsymbol{w}^0_p, \boldsymbol{q}(x,t)\right]\right] . \tag{2.77}$$

Applying the same reasoning to the λ-independent term in the compatibility condition, we get the explicit form for the multicomponent NLS-type (MNLS-type) equations:

$$\frac{i}{4}\left[\boldsymbol{\sigma}, \frac{\partial \boldsymbol{q}}{\partial t}\right] - \boldsymbol{\Lambda}_{\pm}\boldsymbol{V}^{\mathrm{f}}_{N}(x,t) + \frac{1}{4}\left[\boldsymbol{\sigma}, \left[\boldsymbol{w}^0_N, \boldsymbol{q}(x,t)\right]\right] = 0 . \tag{2.78}$$

It remains to insert the solution (2.77) for $\boldsymbol{V}^{\mathrm{f}}_N(x,t)$ into (2.78) to get these NLEE in terms of the recursion operators $\boldsymbol{\Lambda}_{\pm}$:

$$\frac{i}{4}\left[\boldsymbol{\sigma}, \frac{\partial \boldsymbol{q}}{\partial t}\right] + \frac{1}{4}\sum_{p=0}^{N} \boldsymbol{\Lambda}_{\pm}^{N-p}\left[\boldsymbol{\sigma}, \left[\boldsymbol{w}^0_p, \boldsymbol{q}(x,t)\right]\right] = 0 . \tag{2.79}$$

Obviously the multicomponent analog of the dispersion law for the NLEE (2.79) is provided by the matrix-valued polynomial $\boldsymbol{f}(\lambda)$:

$$\boldsymbol{f}(\lambda) = \sum_{p=0}^{N} \lambda^{N-p}\boldsymbol{w}^0_p \in \mathfrak{g}^{(0)} . \tag{2.80}$$

Let us list several important examples of MNLS-type equations.

The Manakov model [5] originally was obtained by taking $N = 2$, $\boldsymbol{f}_{\mathrm{Man}}(\lambda) = -2\lambda^2\boldsymbol{\sigma}$ with $s = 1$, $p = 2$; then $\boldsymbol{q}^{+} = (\boldsymbol{q}^{-})^{\dagger}$ is a two-component vector $\mathbf{u}(x,t)$ satisfying:

$$i\mathbf{u}_t + \mathbf{u}_{xx} + (\mathbf{u}^{\dagger}, \mathbf{u})\mathbf{u}(x,t) = 0, \qquad \mathbf{u} = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} . \tag{2.81}$$

It became famous due to its numerous applications in nonlinear optics [6, 7, 8, 9, 10].

Of course, one can consider a generalization of the Manakov model with p-component vectors, $p > 2$. Also the use of an involution of the form $\boldsymbol{q}^+ = B_0(\boldsymbol{q}^-)^\dagger$, where $B_0 = \operatorname{diag}(\epsilon_1, \dots, \epsilon_p)$ with $\epsilon_j = \pm 1$ leads to another version of the Manakov model:

$$i\mathbf{u}_t + \mathbf{u}_{xx} + (\mathbf{u}^\dagger, B_0\mathbf{u})\mathbf{u}(x,t) = 0, \qquad \mathbf{u} = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_p(x,t) \end{pmatrix} . \tag{2.82}$$

Matrix NLS models. The above two models and all other multicomponent generalizations of the MNLS equation are particular cases of the system:

$$i\frac{\partial \boldsymbol{q}^+}{\partial t} + \frac{\partial^2 \boldsymbol{q}^+}{\partial x^2} + 2\boldsymbol{q}^+\boldsymbol{q}^-\boldsymbol{q}^+(x,t) = 0 , \tag{2.83a}$$

$$-i\frac{\partial \boldsymbol{q}^-}{\partial t} + \frac{\partial^2 \boldsymbol{q}^-}{\partial x^2} + 2\boldsymbol{q}^-\boldsymbol{q}^+\boldsymbol{q}^-(x,t) = 0 , \tag{2.83b}$$

or in matrix form:

$$\frac{i}{2}\left[\boldsymbol{\sigma}, \frac{\partial \boldsymbol{q}}{\partial t}\right] + \frac{\partial^2 \boldsymbol{q}}{\partial x^2} + 2\boldsymbol{q}^3(x,t) = 0 . \tag{2.84}$$

The dispersion law of this equation is

$$\boldsymbol{f}_{\mathrm{MNLS}}(\lambda) = -2\lambda^2\boldsymbol{\sigma} . \tag{2.85}$$

Let us impose on $\boldsymbol{q}(x,t)$ the condition:

$$\boldsymbol{q}(x,t) = B\boldsymbol{q}^\dagger(x,t)B^{-1}, \qquad B = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \in \mathfrak{g}^{(0)} , \tag{2.86}$$
$$B_0 = \operatorname{diag}(\epsilon_1, \dots, \epsilon_s), \qquad B_1 = \operatorname{diag}(\eta_1, \dots, \eta_p) ,$$

where $\epsilon_j = \pm 1$ and $\eta_s = \pm 1$. Each specific choice of the sets of ϵ_j and η_s provides an allowed involution of the system (2.83). The involution (2.86) means that the block matrices $\boldsymbol{q}^\pm(x,t)$ are related by:

$$\boldsymbol{q}^+(x,t) = \boldsymbol{r}(x,t), \qquad \boldsymbol{q}^-(x,t) = B_0\boldsymbol{r}^\dagger(x,t)B_1 \tag{2.87}$$

Inserting (2.87) into (2.83), we easily find that the second equation (2.83b) can be obtained from the first one (2.83a) with hermitian conjugation. As a result, we get the following matrix NLS equation:

$$i\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial^2 \boldsymbol{r}}{\partial x^2} + 2\boldsymbol{r}B_0\boldsymbol{r}^\dagger\boldsymbol{r}(x,t) = 0 . \tag{2.88}$$

Vector and matrix mKdV models. The well-known mKdV equation (2.18) is characterized by dispersion law, which is cubic in λ; see (2.16). We also choose here $s = p$, i.e. $n = 2p$.

Choosing in (2.79) $\boldsymbol{f}_{\mathrm{mKdV}} = -4\lambda^3\boldsymbol{\sigma}$, we obtain the following multicomponent generalization of the system (2.17):

$$\frac{\partial \boldsymbol{q}^+}{\partial t} + \frac{\partial^3 \boldsymbol{q}^+}{\partial x^3} + 3\boldsymbol{q}^+\boldsymbol{q}^-(x,t)\frac{\partial \boldsymbol{q}^+}{\partial x} + 3\frac{\partial \boldsymbol{q}^+}{\partial x}\boldsymbol{q}^-\boldsymbol{q}^+(x,t) = 0 \,, \tag{2.89a}$$

$$\frac{\partial \boldsymbol{q}^-}{\partial t} + \frac{\partial^3 \boldsymbol{q}^-}{\partial x^3} + 3\boldsymbol{q}^-\boldsymbol{q}^+(x,t)\frac{\partial \boldsymbol{q}^-}{\partial x} + 3\frac{\partial \boldsymbol{q}^-}{\partial x}\boldsymbol{q}^+\boldsymbol{q}^-(x,t) = 0 \,, \tag{2.89b}$$

The multicomponent mKdV equation is obtained from the system (2.89) imposing the involution:

$$B\boldsymbol{q}^*(x,t)B^{-1} = -\boldsymbol{q}(x,t), \qquad B = \begin{pmatrix} 0 & B_2 \\ B_2^{-1} & 0 \end{pmatrix} , \tag{2.90}$$

This choice of B satisfies $B^2 = \mathbb{1}$, i.e. the constraint (2.90) is an involution. If we denote $\boldsymbol{q}^+(x,t) = \boldsymbol{r}(x,t)$ then we have:

$$\boldsymbol{q}^+(x,t) = \boldsymbol{r}(x,t), \qquad \boldsymbol{q}^-(x,t) = -B_2^{-1}\boldsymbol{r}^*(x,t)B_2^{-1} \,. \tag{2.91}$$

Then the system (2.89) becomes equivalent to:

$$\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial^3 \boldsymbol{r}}{\partial x^3} - 3\boldsymbol{r}B_2\boldsymbol{r}^*B_2\frac{\partial \boldsymbol{r}}{\partial x} - 3\frac{\partial \boldsymbol{r}}{\partial x}B_2\boldsymbol{r}^*B_2\boldsymbol{r}(x,t) = 0 \,, \tag{2.92}$$

for the complex-valued $p\times p$-matrix function $\boldsymbol{r}(x,t)$. If we choose $B_2 = \mathbb{1}_p$, we get another version of the multicomponent mKdV equation:

$$\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial^3 \boldsymbol{r}}{\partial x^3} - 3\frac{\partial \boldsymbol{r}}{\partial x}\boldsymbol{r}^*\boldsymbol{r}(x,t) - 3\boldsymbol{r}\boldsymbol{r}^*(x,t)\frac{\partial \boldsymbol{r}}{\partial x} = 0 \,. \tag{2.93}$$

Imposing additional involution, we can make $\boldsymbol{r}(x,t)$ either real-valued $p\times p$ matrix or purely imaginary one.

In order to solve these multicomponent generalizations of the NLS and mKdV equations, we need to develop the direct and inverse scattering theory for the block-matrix Zakharov-Shabat system (2.58a). Its Jost solutions are also introduced by fixing up their asymptotics for $x \to \infty$ (or to $x \to -\infty$) to be plane waves, that is, we require that $\boldsymbol{\psi}(x,t,\lambda)$ and $\boldsymbol{\phi}(x,t,\lambda)$ be fundamental solution of $\boldsymbol{L}$ satisfying:

$$\lim_{x\to\infty} \exp(i\lambda\boldsymbol{\sigma}x)\boldsymbol{\psi}(x,t,\lambda) = \mathbb{1}, \qquad \lambda \in \mathbb{R} \tag{2.94a}$$

$$\lim_{x\to-\infty} \exp(i\lambda\boldsymbol{\sigma}x)\boldsymbol{\phi}(x,t,\lambda) = \mathbb{1}, \qquad \lambda \in \mathbb{R} \,. \tag{2.94b}$$

Note that these definitions of the Jost solutions are compatible with the $\boldsymbol{M}$-operator in the form (2.59) with the special choice (2.58a) for $\boldsymbol{C}(\lambda)$.

One can check that

$$\det\psi(x,t,\lambda) = \det\phi(x,t,\lambda) = 1, \qquad \lambda \in \mathbb{R} \,. \tag{2.95}$$

so they are fundamental, and they must be linearly related by the scattering matrix $\boldsymbol{T}(t,\lambda)$:

$$\boldsymbol{\phi}(x,t,\lambda) = \boldsymbol{\psi}(x,t,\lambda)\boldsymbol{T}(t,\lambda), \qquad \lambda \in \mathbb{R} . \tag{2.96}$$

It is natural that the scattering matrix $\boldsymbol{T}(t,\lambda)$ will have the same type of block-matrix structure as $\boldsymbol{U}(x,t,\lambda)$:

$$\boldsymbol{T}(t,\lambda) = \begin{pmatrix} \boldsymbol{a}^+(t,\lambda) & -\boldsymbol{b}^-(t,\lambda) \\ \boldsymbol{b}^+(t,\lambda) & \boldsymbol{a}^-(t,\lambda) \end{pmatrix} . \tag{2.97}$$

From (2.95) and (2.96), it follows that the generalization of the "unitarity" condition (2.49) is:

$$\det \boldsymbol{T}(t,\lambda) = 1, \qquad \lambda \in \mathbb{R} . \tag{2.98}$$

Next we conclude that

$$\boldsymbol{V}_+(\lambda) = \boldsymbol{V}_-(\lambda) = \boldsymbol{C}(\lambda) = \boldsymbol{f}(\lambda), \qquad V_\pm(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda) \tag{2.99}$$

so $\boldsymbol{T}(t,\lambda)$ must satisfy the following linear evolution equation:

$$i\frac{d\boldsymbol{T}}{dt} + [\boldsymbol{f}(\lambda), \boldsymbol{T}(t,\lambda)] = 0 . \tag{2.100}$$

In the special case, when $\boldsymbol{f}(\lambda) = f(\lambda)\boldsymbol{\sigma}$ from (2.97) and (2.100) we find:

$$i\frac{d\boldsymbol{a}^\pm}{dt} = 0, \qquad i\frac{d\boldsymbol{b}^\pm}{dt} \mp 2f(\lambda)\boldsymbol{b}^\pm(t,\lambda) = 0 \tag{2.101}$$

that can be easily solved for any $f(\lambda)$.

Thus, we outlined the proof of the following generalization of Theorem 2.1:

Theorem 2.2 ([11, 12]). *If $\boldsymbol{q}(x,t)$ satisfies the NLEE (2.78), then the scattering matrix $\boldsymbol{T}(t,\lambda)$ satisfies the linear evolution equation (2.100).*

Remark 2.3. Not all MNLS equations are local. Only equations, whose Hamiltonians are from the principal series, i.e. ones whose dispersion laws are of the form $\boldsymbol{f}(\lambda) = f(\lambda)\boldsymbol{\sigma}$ are local. Such equations are superintegrable: They have more generating functionals of integrals of motion than are necessary for integrability. These functionals are not all in involutions. Due to this, boomerons and trappons are possible [13, 14].

2.5 Generalizations of the AKNS Method II

The AKNS method can be applied also to Lax operators generalizing the Zakharov-Shabat system to the following first-order $n \times n$ system:

$$L_{\mathrm{g}}\chi_{\mathrm{g}} \equiv \left(i\frac{d}{dx} + U_{\mathrm{g}}(x,t,\lambda) \right) \chi_{\mathrm{g}}(x,t,\lambda) = 0 \; , \tag{2.102a}$$

$$U_{\mathrm{g}}(x,t,\lambda) = q(x,t) - \lambda J \; , \tag{2.102b}$$

$$q(x,t) = \begin{pmatrix} 0 & q_{12} & \dots & q_{1n-1} & q_{1n} \\ q_{12} & 0 & \dots & q_{2n-1} & q_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n1} & q_{n2} & \dots & q_{n-1n} & 0 \end{pmatrix} , \tag{2.102c}$$

$$J = \mathrm{diag}\,(a_1, a_2, \dots, a_{n-1}, a_n), \qquad \mathrm{tr}\, J = 0 \; . \tag{2.102d}$$

The second operator in the Lax representation is also a first-order $n \times n$ matrix-valued operator:

$$M_{\mathrm{g}}\chi_{\mathrm{g}} \equiv \left(i\frac{d}{dt} + V_{\mathrm{g}}(x,t,\lambda) \right) \chi_{\mathrm{g}}(x,t,\lambda) = \chi_{\mathrm{g}}(x,t,\lambda) C_{\mathrm{g}}(\lambda) \tag{2.103}$$

where $V_{\mathrm{g}}(x,t,\lambda)$ is a polynomial of order N in λ

$$V_{\mathrm{g}}(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} V_k(x,t) \; . \tag{2.104}$$

Here and below, we shall use the same letter for the potential $q(x,t)$ and for the coefficients $V_k(x,t)$, remembering that now they are $n \times n$ matrices.

The recurrent relations (2.13) now are modified into:

$$[V_0(x,t), J] = 0 \; , \tag{2.105a}$$

$$i\frac{dV_k}{dx} + [q, V_k(x,t)] - [J, V_{k+1}(x,t)] = 0 \; , \tag{2.105b}$$

for $k = 0, 1, \dots, N-1$ and the λ–independent term gives the corresponding NLEEs:

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N}{\partial x} + [q(x,t), V_N(x,t)] = 0 \; . \tag{2.105c}$$

Before proceeding with solving the recurrent relations (2.105), we shall fix up the gauge of the Lax operator L_{g}, taking J to be a constant diagonal matrix. We assume that J has n different real eigenvalues. Without loss of generality we can consider them ordered:

$$J = \mathrm{diag}\,(a_1, a_2, \dots, a_n), \qquad a_1 > a_2 > \dots > a_n \; , \tag{2.106}$$

and $\mathrm{tr}\, J = 0$. Applying a convenient gauge transformation commuting with J we can always achieve that

$$q(x,t) = [J, \widetilde{q}(x,t)], \qquad \text{i.e.} \qquad q_{jj} = 0 \; . \tag{2.107}$$

In analogy with the analysis of the previous sections, we again will need to split off each $V_k(x,t)$ into diagonal and off-diagonal parts. Now the corresponding algebra $\mathfrak{g} = sl(n)$ is split into a direct sum $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(\mathrm{f})}$ of linear subspaces, corresponding to the kernel and the image of ad_J.

Obviously, if the eigenvalues of J are all different then the kernel $\mathfrak{g}^{(0)}$ of ad_J will consist of the diagonal matrices, or more precisely, of the Cartan subalgebra of $sl(n)$. In contrast, with the $sl(2)$-case such splitting satisfies only two of the properties in (2.26), namely,

$$[X^{(0)}, Y^{(0)}] = 0, \qquad [X^{(0)}, Y^{(1)}] \in \mathfrak{g}^{(1)} , \tag{2.108}$$

whereas $[X^{(1)}, Y^{(1)}] \notin \mathfrak{g}^{(0)}$; such commutators contain both diagonal and off-diagonal parts. The operator ad_J is well defined on the whole algebra $\mathfrak{g}$, while its inverse is well defined only on $\mathfrak{g}^{(1)}$. In components we have:

$$([J,X])_{jk} = (a_j - a_k)X_{jk}, \qquad \left(\mathrm{ad}_J^{-1} Y^{(\mathrm{f})}\right)_{jk} = \frac{Y_{jk}^{(\mathrm{f})}}{a_j - a_k} , \tag{2.109}$$

where by definition $Y^{(\mathrm{f})}$ is off-diagonal, $Y_{jj}^{(\mathrm{f})} = 0$. The analog of the projector π_J is given by:

$$\pi_J \cdot \equiv \mathrm{ad}_J^{-1} [J, \cdot\,] . \tag{2.110}$$

Applied to any $n \times n$ matrix X, it projects it out onto its off-diagonal part:

$$\pi_J X = X - X^{(0)} = \begin{pmatrix} 0 & X_{12} & \dots & X_{1n-1} & X_{1n} \\ X_{21} & 0 & \dots & X_{2n-1} & X_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{n-1,1} & X_{n-1,2} & \dots & 0 & X_{n-1,n} \\ X_{n,1} & X_{n,2} & \dots & X_{n,n-1} & 0 \end{pmatrix} , \tag{2.111}$$

Then the projector onto $\mathfrak{g}^{(0)}$ is:

$$(\mathbb{1} - \pi_J)X = X^{(0)} = \begin{pmatrix} X_{11} & 0 & \dots & 0 & 0 \\ 0 & X_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & X_{n-1,n-1} & 0 \\ 0 & 0 & \dots & 0 & X_{n,n} \end{pmatrix} . \tag{2.112}$$

Below, in this and the next subsections, we shall use the following generalization of the Condition **C1** (see page 71 below): $q(x,t)$ belongs to the space $\mathcal{M}_J$ of $n \times n$ if $q(x,t) \equiv \pi_J q(x,t)$, and its matrix elements are complex Schwartz–type functions. Each $V_k(x,t)$ can be split into:

$$V_k(x,t) = V_k^{(0)}(x,t) + V_k^{\mathrm{f}}(x,t) , \tag{2.113}$$

where

$$V_k^{(0)}(x,t, \equiv (\mathbb{1}-\pi_J)V_k(x,t) = w_k(x,t) \in \mathfrak{g}^{(0)} \;, \tag{2.114a}$$
$$V_k^{\mathrm{f}}(x,t) = \pi_J V_k(x,t) \;. \tag{2.114b}$$

From (2.105a), we immediately get that $V_0^{\mathrm{f}} = 0$, i.e.:

$$V_0(x,t) = w_0(x,t) \in \mathfrak{g}^{(0)} \;. \tag{2.115a}$$

Next, we consider (2.105b) with $k = 0$. Projecting it onto $\mathfrak{g}^{(0)}$ we obtain:

$$\frac{\partial w_0}{\partial x} = 0 \;. \tag{2.115b}$$

Therefore, in what follows we can assume that

$$V_0(x,t) \equiv w_0^0 \in \mathfrak{g}^{(0)} \tag{2.115c}$$

is a constant diagonal matrix. The off-diagonal part of (2.105b) with $k = 0$

$$[q(x,t), w_0^0] - [J, V_1(x,t)] = 0 \;, \tag{2.116a}$$

allows one to determine only the off-diagonal part of $V_1(x,t)$:

$$V_1^{\mathrm{f}}(x,t) = -\mathrm{ad}\,_J^{-1}\left[w_0^0, q(x,t)\right] \;. \tag{2.116b}$$

Analogously, for $k = 1$ (2.105b) gives:

$$i\frac{\partial w_1}{\partial x} + (\mathbb{1}-\pi_J)[q(x,t), V_1^{\mathrm{f}}(x,t)] = 0 \;, \tag{2.117a}$$
$$i\frac{\partial V_1^{\mathrm{f}}}{\partial x} + \pi_J\left([q(x,t), V_1^{\mathrm{f}}(x,t)]\right) = [J, V_2^{\mathrm{f}}(x,t)] \;. \tag{2.117b}$$

Inserting (2.116b) into (2.117a), we find

$$\frac{\partial w_1}{\partial x} = 0 \;, \tag{2.118}$$

i.e. we can assume that $w_1 = w_1^0 = \mathrm{const}$. Equation (2.117b) leads to:

$$V_2^{\mathrm{f}}(x,t) = \mathrm{ad}\,_J^{-1}\left(i\frac{\partial v_1^{\mathrm{f}}}{\partial x} + \pi_J\left([q(x,t), V_1^{\mathrm{f}}]\right)\right) \;. \tag{2.119}$$

For $k > 1$ we get in a similar way:

$$i\frac{\partial w_k}{\partial x} + (\mathbb{1}-\pi_J)[q(x,t), V_k^{\mathrm{f}}(x,t)] = 0 \;, \tag{2.120a}$$

$$i\frac{\partial V_k^{\mathrm{f}}}{\partial x} + \pi_J\left([q(x,t), V_k^{\mathrm{f}}(x,t)]\right) + [q(x,t), w_k(x,t)] = [J, V_{k+1}^{\mathrm{f}}(x,t)] \;. \tag{2.120b}$$

Formally integrating (2.120a), we can express $w_k(x,t)$ through $V_k^{\mathrm{f}}(x,t)$ by:

$$w_k(x,t) = w_k^0 + i\int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y,t), V_k^{\mathrm{f}}(y,t)]\ , \tag{2.121}$$

where $w_k^0 \in \mathfrak{g}^{(0)}$ are constant diagonal matrices. We insert it into (2.120b) with the result:

$$\begin{aligned} &i\frac{\partial V_k^{\mathrm{f}}}{\partial x} + \pi_J[q(x,t), V_k^{\mathrm{f}}] + [q(x,t), w_k^0] \\ &\quad + i\pi_J\left[q(x,t), \int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y,t), V_k^{\mathrm{f}}(y,t)]\right] = [J, V_{k+1}^{\mathrm{f}}]. \end{aligned} \tag{2.122}$$

Here, and below, we shall use the fact that $\pi_J[q(x,t), w_k^0] \equiv [q(x,t), w_k^0]$.

Applying to both sides of (2.122) ad_J^{-1} we get:

$$V_{k+1}^{\mathrm{f}} = \Lambda_\pm V_k^{\mathrm{f}} + \mathrm{ad}_J^{-1}[q(x,t), w_k^0]\ , \tag{2.123}$$

where the recursion operators $\Lambda_\pm$ are defined by:

$$\begin{aligned} \Lambda_\pm X \equiv \mathrm{ad}_J^{-1}\Bigg\{ &i\frac{\partial X}{\partial x} + \pi_J[q(x), X(x)] \\ &+ i\pi_J\left[q(x), \int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y), X(y)]\right]\Bigg\}\ . \end{aligned} \tag{2.124}$$

Note that the structure of $\Lambda_\pm$ ensures that if $X \in \mathfrak{g}^{(1)}$ then also $\Lambda_\pm X \in \mathfrak{g}^{(1)}$.

Thus, we have cast the recursion relations (2.105) in the form (2.123); (2.116b) must be viewed as the initial condition for them. Its formal solution can be written down in compact form as:

$$V_{k+1}^{\mathrm{f}} = -\sum_{s=0}^{k} \Lambda_\pm^s \mathrm{ad}_J^{-1}[w_{k-s}^0, q(x)]\ . \tag{2.125}$$

It remains to repeat the "splitting" procedure also to the NLEEs (2.105c):

$$i\frac{\partial w_N}{\partial x} + (\mathbb{1} - \pi_J)[q(x,t), V_N^{\mathrm{f}}(x,t)] = 0\ , \tag{2.126a}$$

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N^{\mathrm{f}}}{\partial x} + \pi_J\left([q(x,t), V_N^{\mathrm{f}}(x,t)]\right) + [q(x,t), w_N(x,t)] = 0\ , \tag{2.126b}$$

with the result

$$w_N(x,t) = w_N^0 + i\int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y,t), V_N^{\mathrm{f}}(y,t)]\ , \tag{2.127}$$

and applying the operator ad_J^{-1} to both sides of (2.127) we get:

$$-i\mathrm{ad}\,_J^{-1}\frac{\partial q}{\partial t} + \Lambda_\pm V_N^{\mathrm{f}} - \mathrm{ad}\,_J^{-1}[w_N^0, q(x,t)] = 0 \,. \tag{2.128}$$

This is the generic form of the NLEE solvable by the ISM applied to L_{g} (2.102). Using (2.125), we can write it down in compact form:

$$i\mathrm{ad}\,_J^{-1}\frac{\partial q}{\partial t} + \sum_{s=0}^{N} \Lambda_\pm^s \mathrm{ad}\,_J^{-1}[w_{N-s}^0, q(x,t)] = 0 \,. \tag{2.129}$$

Note that in solving the recurrent relations we obtained $N+1$ integration constants w_k^0. These constant diagonal matrices determine the function:

$$f_{\mathrm{g}}(\lambda) = \sum_{s=0}^{N} w_{N-s}^0 \lambda^s \,, \tag{2.130}$$

which is the proper generalization of the dispersion law $f(\lambda)$. Indeed, one can check that $f_{\mathrm{g}}(\lambda)$ determine the evolution of the scattering matrix of L_{g}.

The simplest of these NLEE is obtained already for $N=1$. This is the famous N-wave equation:

$$\begin{aligned} &i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[I, \frac{\partial Q}{\partial x}\right] + [[I, Q(x,t)], [J, Q(x,t)]] = 0 \,, \\ &Q(x,t) = \mathrm{ad}\,_J^{-1} q(x,t) \in \mathfrak{g}^{(1)}, \end{aligned} \tag{2.131}$$

where $I = w_0^0 \in \mathfrak{g}^{(0)}$. Its dispersion law is linear in λ:

$$f_{\mathrm{Nw}}(\lambda) = \lambda I \,. \tag{2.132}$$

The scattering matrix $T_{\mathrm{g}}(\lambda, t)$ and the Jost solutions of L_{g} are natural generalizations of the ones for the Zakharov-Shabat system L. They are defined by:

$$\lim_{x\to\infty} \psi_{\mathrm{g}}(x,t,\lambda) e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi_{\mathrm{g}}(x,t,\lambda) e^{i\lambda Jx} = \mathbb{1} \,, \tag{2.133a}$$

$$T_{\mathrm{g}}(\lambda, t) = \hat{\psi}_{\mathrm{g}}(x,t,\lambda)\phi_{\mathrm{g}}(x,t,\lambda) \,. \tag{2.133b}$$

The detailed investigation of the direct and inverse scattering problems for L_{g} comes out of the scope of the present Chapter. Here, we shall just derive the t-dependence of $T_{\mathrm{g}}(\lambda, t)$ using the Lax representation (2.102). We fix up $C_{\mathrm{g}}(\lambda)$ in the right hand-side of (2.103) in such a way that the definitions of the Jost solutions (2.133a) are valid for all time t, i.e.:

$$\begin{aligned} C_{\mathrm{g}}(\lambda) &\equiv \lim_{x\to\pm\infty} V_{\mathrm{g}}(x,t,\lambda) \\ &= \sum_{s=0}^{N} w_{N-s}^0 \lambda^s = f_{\mathrm{g}}(\lambda) \,. \end{aligned} \tag{2.134}$$

Then, we recall that the Jost solution $\phi_{\rm g}(x,t\lambda)$ must satisfy (2.103) and consider its limits for $x \to -\infty$ and $x \to \infty$. In view of our choice for $C_{\rm g}(\lambda)$ (2.134), the first limit becomes the identity $0 = 0$. Doing the second limit, we make use of (2.133b): $\phi_{\rm g}(x,t\lambda) = \psi_{\rm g}(x,t\lambda)T_{\rm g}(t,\lambda)$ and get:

$$i\frac{dT_{\rm g}}{dt} + [f_{\rm g}(\lambda), T_{\rm g}(t,\lambda)] = 0\ . \tag{2.135}$$

This result can be formulated as the following generalization of theorem

Theorem 2.3 ([15]). *Let $q(x,t) \in \mathcal{M}_J$ and satisfies the NLEE (2.129) then the scattering matrix $T_{\rm g}(t,\lambda)$ satisfies the linear evolution equation (2.135).*

Thus, we demonstrated the analogy between the NLEE related to the ZS system (2.43), and their generalizations (2.79) and (2.129), and the generic partial differential equation with constant coefficients (1.30) in Chap. 1. The polynomials $f(\lambda)$ determine the dispersion laws of these equations. In the case of the NLEE, the derivative operator $\frac{1}{i}\frac{\partial}{\partial x}$ has been replaced by the corresponding recursion operator $\Lambda_{\pm}$. This analogy is not coincidental, and its roots are in the spectral decompositions of the recursion operators.

2.6 Comments and Bibliographical Review

1. The KdV, NLS, MKdV, s-G, N-wave equations are only several of the NLEE that are integrable and have a wide range of applications in physics. In fact, they describe different regimes of wave-wave interactions, which do not depend on the physical origin of the waves. This explains their universality [16, 17]. Here, we give a short list of monographs and review papers [4, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] in which these problems are analyzed and which contain the necessary references.
2. The fact that to each Lax operator L, one can relate a hierarchy of solvable NLEE became obvious in 1974 after the AKNS Chapter [1]. In it, they proposed a modification of the Lax approach, which simplified substantially the derivation of the relevant NLEE. The AKNS scheme, formulated initially for the ZS system, substantially simplified finding new "higher" NLEE related to a given Lax operator L and reduced it to the solving of a set of recurrent relations. They constructed also the recursion operators $\Lambda_{\pm}$, which solves the recurrent relations and plays fundamental role in deriving the properties of the NLEE. Another important fact discovered by AKNS [1] was the importance of the Wronskian relations and the squared solutions of L in studying the mapping between the potential $q(x,t)$ of L and the scattering data of L. They revealed that the squared solutions are eigenfunctions of the recursion operators $\Lambda_{\pm}$ and may be viewed as

natural generalizations of the usual exponentials. As a consequence, the ISM can be viewed as a generalized Fourier transform. In order to establish this fact rigorously, one needs to prove that the squared solutions are complete sets of functions in the space of allowed potentials $\mathcal{M}$ of L. In 1976, Kaup [38] formulated the completeness relation for the squared solutions of the ZS system. Later in 1979, Kaup and Newell [39] derived the fundamental properties of the NLS hierarchy through the recursion operators $\Lambda_{\pm}$ using the completeness property of its eigenfunctions – the squared solutions.

At about the same time, Khristov and one of the authors of the present monograph (VSG) [40, 41], independently of Kaup and Newell, proposed a rigorous proof of Kaup's compeleteness relation and applied it to the theory of the NLS-type equations. It was shown also that the completeness relation of the squared solutions can be viewed as the spectral decomposition of the recursion operator Λ.

Besides, in [40, 41] it was proved that the "products" of solutions of two different ZS systems also satisfy a completeness relation. These products of solutions are eigenfunctions of the operators $\boldsymbol{\Lambda}_{\pm}$ generalizing $\Lambda_{\pm}$ and generating the Bäcklund transformations of the NLEE. These results extend the results of Calogero and Degasperis [11, 12, 42]. The same type of results have been derived also for the Sturm-Liouville problem [43, 44, 45], for the ZS system with periodic boundary conditions [45, 46, 47, 48] and for the Sturm-Liouville problem on the semiaxis [49, 50].

3. The AKNS paper stimulated a number of other scientists [11, 12, 13, 14, 27, 39, 40, 41, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118]. In 1976, Calogero and Degasperis [11, 12] proposed generalized Wronskian identities to describe the class of Bäcklund transformations for the NLS-type NLEE.
4. The necessity to consider Lax operators generalizing the ZS system naturally called for generalizations of the AKNS approach and for the explicit derivation of the corresponding recursion operators Λ. Here, we list some of the best known ones:
   - the ZS system in the pole gauge [71, 81, 82, 118];
   - $n \times n$ ZS system [27, 119, 120, 121];
   - ZS system related to symmetric spaces [7, 9, 72, 122, 123];
   - The natural generalizations of the Zakharov-Shabat system to simple Lie algebras of rank higher than one:

$$i\frac{d\psi}{dx} + (Q(x,t) - \lambda J)\,\psi(x,t,\lambda) = 0, \qquad Q(x,t) = [J, Q'(x,t)]\,, \tag{2.136}$$

     where $Q'(x,t)$ takes values in the simple Lie algebra $\mathfrak{g}$, and J is a constant element of some fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ see [9, 124, 125]

and [51, 74, 75, 84, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151];
- Polynomial generalizations of the Zakharov-Shabat system to simple Lie algebras of higher rank:

$$i\frac{d\psi}{dx} + \left(\sum_{k=1}^{n} \lambda^{n-k} Q_k(x,t) - \lambda^n J \right) \psi(x,t,\lambda) = 0 \ , \tag{2.137}$$

where $Q_{n-1}(x,t) = [J, Q'(x,t)]$ and $Q_k(x,t)$ take values in the simple Lie algebra $\mathfrak{g}$, and J is a constant element of some fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The best known examples of this form are related to the $sl(2)$ algebra [70, 78, 114, 115, 121, 152, 153, 154, 155, 156, 157, 158]; for the bundles (pencils) with qubic and higher powers in λ see [70, 152, 156, 157, 159, 160]). They can be extended also to algebras of higher ranks, as well as to nonvanishing boundary conditions case [161];
- Gelfand–Dickey–Zakharov–Shabat problem [162];
- to the difference version of ZS system known as the Ablowitz-Ladik system [79, 163, 164, 165], their gauge equivalent ones [166], and their multicomponent generalizations [37, 76];
- for ZS system with periodic boundary conditions, see [47, 167, 168, 169, 170] and the numerous references therein. Another important class of boundary conditions, whose treatment requires a number of additional constructions are the constant boundary conditions; see [161, 171, 172].
- for ZS system with elliptic dependence on λ see [107, 173].

5. For a number of important choices of $L(\lambda)$, to the best of our knowledge, the derivation of the AKNS scheme has not yet been done, and the corresponding recursion operators Λ are not yet known. This refers to the cases in which $L(\lambda)$ is rational function of λ [174, 175].
6. The formal approach to the recursion operators and NLEE is outlined in a series of papers [127, 176]; see also [177];
7. The so-called U-V-systems were introduced by Zakharov and Mikhailov [174, 175] where

$$L(\lambda) = i\frac{d}{dx} + U(x,t,\lambda), \qquad M(\lambda) = i\frac{d}{dt} + V(x,t,\lambda) \ , \tag{2.138}$$

and $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are rational functions of λ taking values in $\mathfrak{g}$. Such Lax pairs allow to solve the principal chiral field equation in $1+1$ dimensions, as well as a number of fermionic models in field theory;
8. U-V-systems with elliptic λ-dependence were used to solve the Landau-Lifshitz equations and its generalizations related to the $sl(n)$-algebras [173, 178].

For the last two items, the recursion operator is known only for the simplest case of rational U-V-system relevant for the principal chiral field [61] and for the $sl(2)$-Landau-Lifshitz equation [179, 180].

9. Quite different from the operators in [179, 180] is the recursion operator found for the Landau-Lifshitz equation with Lax pairs using some deformations of the algebra $so(4)$, see [181].
10. Discrete systems such as the Ablowitz-Ladik system [182] and its multicomponent generalizations have been treated along the same lines in [76, 79, 183, 166].

References

1. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.*, 53: 249–315, 1974.
2. P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math*, 21:467–490, 1968.
3. V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP*, 34:62–69, 1972.
4. H. Segur and M. J. Ablowitz. *Solitons and the Inverse Scattering Transform.* Society for Industrial & Applied Mathematics, Philadelphia, PA 1981.
5. S. V. Manakov. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. *Sov. Phys. JETP*, 38:248–253, 1974.
6. S. V. Manakov. On the theory of two-dimensional stationary self-focusing of electromagnetic waves. *J. Theor. Math. Phys.*, 65(2):172–179, 1976.
7. A. P. Fordy and P. P. Kulish. Nonlinear Schrödinger equations and simple Lie algebras. *Commun. Math. Phys.*, 89(3):427–443, 1983.
8. V. E. Zakharov and E. I. Schulman. To the integrability of the system of two coupled nonlinear Schrödinger equations. *Physica*, 4:270–274, 1982.
9. V. S. Gerdjikov. Complete integrability, gauge equivalence and Lax representations of the inhomogeneous nonlinear evolution equations. *Theor. Math. Phys.*, 92:374–386, 1992.
10. V. S. Gerdjikov. Basic aspects of soliton theory. In Mladenov, I. M. and Hirshfeld, A. C., editor, *Geometry, Integrability and Quantization*, pages 78–125. Softex, Sofia, 2005.
11. F. Calogero and A. Degasperis. Nonlinear evolution equations solvable by the inverse spectral transform. I. *Nuovo Cimento B*, 32(2):1–54, 1976.
12. F. Calogero and A. Degasperis. Nonlinear evolution equations solvable by the inverse spectral transform. II. *Nuovo Cimento B*, 39(1):1–54, 1976.
13. F. Calogero and A. Degasperis. Coupled nonlinear evolution equations solvable via the inverse spectral transform, and solitons that come back: The boomeron. *Lett. Nuovo Cimento*, 16:425–433, 1976.
14. F. Calogero and A. Degasperis. Bäcklund transformations, nonlinear superposition principle, multisoliton solutions and conserved quantities for the ‘boomeron’ nonlinear evolution equation. *Lett. Nuovo Cimento*, 16:434–438, 1976.

15. V. E. Zakharov and S. V. Manakov. The theory of resonant interactions of wave packets in nonlinear media. *Zh. Eksp. Teor. Fiz*, 69(5), 1975.
16. F. Calogero. Universality and integrability of the nonlinear evolution PDEs describing N-wave interactions. *J. Math. Phys.*, 30:28, 1989.
17. F. Calogero. Why are certain nonlinear PDEs both widely applicable and integrable. *What is Integrability?*, pages 1–62, 1991.
18. J. L. Lamb Jr. Analytical description of ultra-short optical pulse propagation in a resonant medium. *Rev. Mod. Phys.*, 43:99–124, 1971.
19. A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin. The soliton: A new concept in applied science. *Proc. IEEE*, 61(10):1443–1483, 1973.
20. J. Moser, editor. *Integrable Systems of Nonlinear Evolution Equations and Dynamical Systems. Theory and Applications.* Springer-Verlag, New York, 1975.
21. F. Calogero, editor. *Nonlinear Evolution Equations Solvable by the Spectral Transform*, volume 26 of *Res. Notes in Math.* Pitman, London, 1978.
22. R. K. Bullough and P. J. Caudrey, editors. *Solitons*. Springer, Berlin, 1980.
23. S. V. Manakov. The inverse scattering transform for the time-dependent Schrödinger soliton theory. *Physica D: Nonl. Phen.*, 3D(1–2):1–438, 1981.
24. Y. Kodama, M. J. Ablowitz, and J. Satsuma. Direct and inverse scattering problems of the nonlinear intermediate long wave equation. *J. Math. Phys.*, 23:564, 1982.
25. V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. I. Pitaevskii. *Theory of Solitons: The Inverse Scattering Method*. Plenum, New York, 1984.
26. A. C. Newell. *Solitons in Mathematics and Physics*. Regional Conf. Ser. in Appl. Math. Philadelphia, 1985.
27. Konopelchenko, B. G.: *Nonlinear Integrable Equations. Recursion Operators, Group Theoretical and Hamiltonian Structures of Soliton Equations*. Lect. Notes Phys. **270**. Springer, Berlin (1987)
28. C. Desem. PhD thesis, University of New South Wales, Kensington, New South Wales, Australia, 1987.
29. Y. S. Kivshar and B. A. Malomed. Dynamics of solitons in nearly integrable systems. *Rev. Mod. Phys.*, 61(4):763–915, 1989.
30. M. Toda. *Theory of Nonlinear Lattices*. Springer-Verlag, Berlin, 1989.
31. E. E. Infeld and G. Rowlands. *Nonlinear Waves, Solitons and Chaos*. Cambridge University Press, Cambridge, 1990.
32. M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, volume 149 of *London Mathematical Society Lecture Notes Series*. Cambridge University Press, Cambridge, 1991.
33. O. I. Bogoyavlensky. *Inverting Solitons. Nonlinear Integrable Equations.* Nauka, Moscow, 1991.
34. A. C. Scott. Davydovs soliton. *Phys. Rep.*, 217(1):1–67, 1992.
35. A. Hasegawa and Y. Kodama. *Solitons in Optical Communications*. Oxford University Press, New York, 1995.
36. G. R. Agrawal. *Nonlinear Fiber Optics*. Elsevier, Amsterdam, 2001.
37. M. J. Ablowitz, A. D. Trubatch, and B. Prinari. *Discrete and Continuous Nonlinear Schrodinger Systems*. Cambridge University Press, Cambridge, 2003.
38. D. J. Kaup. Closure of the squared Zakharov–Shabat eigenstates. *J. Math. Anal. Appl*, 54(3):849–864, 1976.
39. D. J. Kaup and A. C. Newell. Soliton equations, singular dispersion relations and moving eigenvalues. *Adv. Math*, 31:67–100, 1979.

40. V. S. Gerdjikov and E. K. Khristov. On the expansions over the products of solutions of two Dirac systems. *Mat. Zametki*, 28:501–512, 1980. (in Russian).
41. V. S. Gerdjikov and E. K. Khristov. On the evolution equations, solvable by the inverse problem method. I. Spectral theory. *Bulg. J. Phys*, 7:28–31, 1980. (in Russian).
42. F. Calogero and A. Degasperis. *Spectral Transform and Solitons. I. Tools to Solve and Investigate Nonlinear Evolution Equations*, volume 144 of *Studies in Mathematics and Its Applications, 13. Lecture Notes in Computer Science.* North-Holland Publishing Co., Amsterdam, New York, 1982.
43. V. A. Arkad'ev, A. K. Pogrebkov, and M. K. Polivanov. Expansions with respect to squares, symplectic and Poisson structures associated with the Sturm-Liouville problem. I. *Theor. Math. Phys.*, 72(3):909–920, 1987.
44. V. A. Arkad'ev, A. K. Pogrebkov, and M. K. Polivanov. Expansions with respect to squares, symplectic and Poisson structures associated with the Sturm-Liouville problem. II. *Theor. Math. Phys.*, 75(2):448–460, 1988.
45. I. D. Iliev, E. Kh. Christov, and K. P. Kirchev. *Spectral Methods in Soliton Equations*, volume 73 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* John Wiley & Sons, New York, 1991.
46. E. Kh. Khristov. On an application of Crum–Krein transform to expansions in products of solutions of two SturmLiouville equations. *J. Math. Phys.*, 40:3162–3174, 1990.
47. P. L. Christiansen, J. C. Eilbeck, V. Z. Enol'skii, and N. A. Kostov. Quasi-periodic solutions of the coupled nonlinear Schrödinger equations. *Proc. R. Soc. Lond. A*, 456:2263–2281, 2000.
48. V. B. Daskalov and E. K. Khristov. Explicit formulae for the inverse problem for the regular Dirac operator. *Inverse Probl.*, 16(1):247–258, 2000.
49. E. Kh. Khristov. On the Λ-operators associated with two SturmLiouville problems on the semi-axis. *Inverse Probl.*, 14:647–660, 1998.
50. M. Boiti, J. Leon, and F. Pempinelli. Solution of the boundary value problem for the integrable discrete SRS system on the semi-line. *J. Phys. A*, 32(6): 927–943, 1999.
51. M. J. Bergwelt and A. P. E. ten Kroode. Diferential-difference AKNS equations and homogeneous Heisenberg algebras. *J. Math. Phys.*, 28(2):302–306, 1987.
52. R. K. Dodd and R. K. Bullough. The generalized Marchenko equation and the canonical structure of the AKNS–ZS inverse method. *Physica Scripta*, 20(3–4):514–530, 1979.
53. B. G. Konopelchenko. *Introduction to Multidimensional Integrable Equations. The Inverse Spectral Transfrom in* $2+1$ *Dimensions.* Plenum Press, New York and London, 1992.
54. M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.*, 16:598, 1975.
55. M. J. Ablowitz and J. F. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.*, 17:1011, 1976.
56. DJ Kaup, A. Reiman, and A. Bers. Space-time evolution of nonlinear three-wave interactions. I. Interaction in a homogeneous medium. *Rev. Mod. Phys.*, 51(2):275–309, 1979.
57. M. Boiti, J. Leon, and F. Pempinelli. A recursive generation of local higher-order sine-Gordon equations and their Bäcklund transformation. *J. Math. Phys.*, 25(6):1725–1734, 1984.

58. M. Boiti, J. Leon, and F. Pempinelli. Solution of the Cauchy problem for a generalized sine-Gordon equation. *J. Math. Phys.*, 26:270, 1985.
59. M. Boiti, F. Pempinelli, and G. Z. Tu. Canonical structure of soliton equations via isospectral eigenvalue problems. *Nouvo Cimento B*, 73(2):231–265, 1984.
60. M. Boiti and F. Pempinelli. Some integrable finite-dimensional systems and their continuous counterparts. *Inverse Probl.*, 13(4):919–937, 1997.
61. M. Bruschi, D. Levi, and O. Ragnisco. The chiral field hierarchy. *Phys. Lett. A*, 88(8):379–382, 1982.
62. M. Bruschi, S. V. Manakov, O. Ragnisco, and D. Levi. The nonabelian Toda latticediscrete analogue of the matrix Schrodinger equation. *J. Math. Phys*, 21:2749–53, 1980.
63. M. Bruschi and O. Ragnisco. Nonlinear differential-difference equations, associated Bäcklund transformations and Lax technique. *J. Phys. A: Math. Gen.*, 14(5):1075–1081, 1981.
64. M. Bruschi and O. Ragnisco. Nonlinear evolution equations associated with the chiral field spectral problem. *Nuovo Cimento B*, B88(2):119–139, 1985.
65. F. Kako and N. Mugibayashi. Complete integrability of general nonlinear differential-difference equations solvable by the inverse method. I. *Prog. Theor. Phys.*, 60(4):975–984, 1978.
66. P. J. Caudrey. The inverse problem for the third order equation $u_{xxx}+q(x)u_x+r(x)u=-i\zeta^3 u$. *Phys. Lett. A*, 79(4):264–268, 1980.
67. P. J. Caudrey. The inverse problem for a general $N \times N$ spectral equation. *Physica D: Nonl. Phen.*, 6(1):51–66, 1982.
68. S. C. Chiu and J. F. Ladik. Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique. *J. Math. Phys.*, 18:690, 1977.
69. A. S. Fokas and P. M. Santini. The recursion operator of the Kadomtzev–Petviashvili equation and the squared eigenfunctions of the Schrodinger operator. Clarksson College of Technology preprint, 1985.
70. I. T. Gadjiev, V. S. Gerdjikov, and M. I. Ivanov. Hamiltonian structures of the nonlinear evolution equations related to the polynomial bundle. *Notes LOMI Sci.*, 120:55–68, 1982.
71. V. S. Gerdjikov. Completely integrable Hamiltonian systems and the classical r-matrices. *In the series "Mathematical methods of the theoretical physics". Lectures for young scientists, Editorial house of the Bulgarian acad. sci. (In Bulgarian)*, 6:1–60, 1990.
72. V. S. Gerdjikov. The generalized Zakharov–Shabat system and the soliton perturbations. *Theor. Math. Phys.*, 99(2):593–598, 1994. translated from: Teoret. Mat. Fiz. **99** (1994), no. 2, 292–299 (Russian).
73. V. S. Gerdjikov and G. G. Grahovski. Reductions and real forms of Hamiltonian systems related to N-wave type equations. *Balkan Phys. Lett. BPL (Proc. Suppl.)*, BPU-4:531–534, 2000.
74. V. S. Gerdjikov, G. G. Grahovski, and N. A. Kostov. Reductions of N-wave interactions related to low-rank simple Lie algebras: I. Z_2-reductions. *J. Physics A: Math. Gen.*, 34(44):9425–9461, 2001.
75. V. S. Gerdjikov, G. G. Grahovski, R. I. Ivanov, and N. A. Kostov. N-wave interactions related to simple Lie algebras. *Inverse Probl.*, 17:999–1015, 2001.
76. V. S. Gerdjikov and M. I. Ivanov. Hamiltonian structure of multicomponent nonlinear Schrödinger equations in difference form. *Theor. Math. Phys.*, 52(1):676–685, 1982.

77. V. S. Gerdjikov and M. I. Ivanov. Expansions over the squared solutions and the inhomogeneous nonlinear Schrödinger equation. *Inverse Probl.*, 8(6):831–847, 1992.
78. V. S. Gerdjikov, M. I. Ivanov, and P. P. Kulish. Quadratic bundle and nonlinear evolution equations. *Theor. Math. Phys*, 44:342–357, 1980.
79. V. S. Gerdjikov, M. I. Ivanov, and P. P. Kulish. Expansions over the squared-solutions and difference evolution equations. *J. Math. Phys.*, 25:25, 1984.
80. V. S. Gerdjikov and E. K. Khristov. On the evolution equations solvable with the inverse scattering problem. II. Hamiltonian structures and Bäcklund transformations. *Bulg. J. Phys.*, 7(2):119–133, 1980. (in Russian).
81. V. S. Gerdjikov and A. B. Yanovski. Gauge covariant formulation of the generating operator. 1. The Zakharov–Shabat system. *Phys. Lett. A*, 103(5):232–236, 1984.
82. V. S. Gerdjikov and A. B. Yanovski. Gauge covariant formulation of the generating operator. 2. Systems on homogeneous spaces. *Phys. Lett. A*, 110(2):53–58, 1985.
83. V. S. Gerdjikov and A. B. Yanovski. The generating operator and the locality of the conserved densities for the Zakharov–Shabat system . JINR communication P5–85–505, Dubna, 1985.
84. V. S. Gerdjikov and A. B. Yanovski. Completeness of the eigenfunctions for the Caudrey–Beals–Coifman system. *J. Math. Phys.*, 35:3687–3721, 1994.
85. S. Ghosh. Soliton solutions, Liouville integrability and gauge equivalence of Sasa-Satsuma equation. *J. Math. Phys.*, 40(4):1993, 1999.
86. D. J. Kaup. The three-wave interaction – a nondispersive phenomenon. *Stud. Appl. Math*, 55(9), 1976.
87. D. J. Kaup. The solution of the general initial value problem for the full three dimensional three-wave resonant interaction. *Physica D: Nonl. Phen.*, 3(1–2):374–395, 1981.
88. D. J. Kaup. The squared eigenstates of the sine-Gordon eigenvalue problem. *J. Math. Phys.*, 25:2467, 1984.
89. T. Kawata and H. Inoue. Exact solution of the derivative nonlinear Schrödinger equation under the non-vanishing condition. *J. Phys. Soc. Japan*, 44(6):1968–1976, 1978.
90. I. T. Khabibulin. The inverse scattering problem for difference equations. *Dokl. Acad. Nauk SSSR*, 249:67–70, 1979.
91. E. Kh. Khristov. Spectral properties of operators generated by KdV equations. *Diff. Eqs.*, 19(9):1168–1177, 1980.
92. B. G. Konopelchenko. On the structure of equations integrable by the arbitrary-order linear spectral problem. *J. Phys. A: Math. Gen.*, 14(6):1237–1259, 1981.
93. B. G. Konopelchenko. General structure of nonlinear evolution equations in 1+2 dimensions integrable by the two-dimensional Gelfand–Dickey–Zakharov–Shabat spectral problem and their transformation properties. *Commun. Math. Phys.*, 88(4):531–549, 1983.
94. B. G. Konopelchenko. Hamiltonian structure of the general integrable equations under reductions. *Physica D: Nonl. Phen.*, 15D(3):305–334, 1985.
95. B. G. Konopelchenko and V. G. Dubrovski. General N-th order differential spectral problem: General structure of the integrable equations, nonuniqueness of the recursion operator and gauge invariance. *Ann. Phys.*, 156(2):256–302, 1984.

96. V. V. Konotop and V. E. Vekslerchik. Direct perturbation theory for dark solitons. *Phys. Rev. E*, 49(3):2397–2407, 1994.
97. P. P. Kulish. Multicomponent nonlinear Schrödinger equations with graded matrices. *Sov. Phys. Doklady*, 25:912, 1980.
98. P. P. Kulish. Action-angle variables for a multicomponent nonlinear Schrödiger equation. *J. Sov. Math*, 705–713, 1985.
99. A. Kundu. Landau–Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations. *J. Math. Phys.*, 25:3433, 1984.
100. A. Kundu. Unifying scheme for generating discrete integrable systems including inhomogeneous and hybrid models. *J. Math. Phys.*, 44:4589, 2003.
101. A. Kundu and B. B. Mallick. Classical and quantum integrability of a derivative nonlinear Schrödinger model related to quantum group. *J. Math. Phys.*, 34:1052, 1993.
102. E. A. Kuznetsov and A. V. Mikhailov. On the complete integrability of the two-dimensional classical Thirring model. *Theor. Math. Phys.*, 30(3):193–200, 1977.
103. E. A. Kuznetsov, A. V. Mikhailov, and I. A. Shimokhin. Nonlinear interaction of solitons and radiation. *Physica D*, 87(1–4):201–215, 1994.
104. V. G. Makhankov and V. K. Fedyanin. Non-linear effects in quasi-one-dimensional models of condensed matter theory. *Phys. Rep.*, 104(1):1–86, 1984.
105. A. C. Newell. The inverse scattering transform. *Topics in Current Physics. Solitons.* ed. by R. Bullogh and P. Caudrey, Springer, Berlin, 1978.
106. P. M. Santini, M. J. Ablowitz, and A. S. Fokas. On the initial value problem for a class of nonlinear integral evolution equations including the sine–Hilbert equation. *J. Math. Phys.*, 28:2310, 1987.
107. Y. N. Sidorenko. Elliptic bundles and generating operators. *Zapiski Nauchn. Semin. LOMI*, 161:76–87, 1987.
108. L. A. Takhtadjan. Hamiltonian systems connected with the Dirac equation. *J. Sov. Math.*, 8(2):219–228, 1973.
109. L. A. Takhtadjan. Exact theory of propagation of ultrashort optical pulses in two-level media. *J. Exp. Theor. Phys.*, 39(2):228–233, 1974.
110. L. A. Takhtadjan and L. D. Faddeev. Essentially nonlinear one-dimensional model of classical field theory. *Theor. Math. Phys*, 21:1046–1057, 1974.
111. N. Y. Reshetikhin and L. D. Faddeev. Hamiltonian structures for integrable models of field theory. *Theor. Math. Phys.*, 56(3):847–862, 1983.
112. L. A. Takhtadjan and L. D. Faddeev. Simple relation between the geometric and Hamiltonian formulation of integrable nonlinear equations. *Sci. Notes of LOMI Seminars*, 115:264–273, 1982.
113. L. A. Takhtadjan and L. D. Faddeev. Hamiltonian system related to the equation $u_{\xi,\eta} + \sin u = 0$. *Sci. Notes of LOMI Seminars*, 142:254–266, 1976.
114. Y. Vaklev. Gauge transformations for the quadratic bundle. *J. Math. Phys.*, 30:1744–1755, 1989.
115. Y. Vaklev. Some soliton solutions for the quadratic bundle. *J. Math. Phys.*, 33:4111–4115, 1992.
116. Y. Vaklev. Soliton solutions and gauge equivalence for the problem of ZakharovShabat and its generalizations. *J. Math. Phys.*, 37:1393–1413, 1992.
117. F. Calogero, A. Degasperis, and J. Xiaoda. Nonlinear Schrödinger-type equations from multiscale reduction of PDEs. I. Systematic derivation. *J. Math. Phys.*, 41:6399, 2000.

118. V. S. Gerdjikov and A. B. Yanovski. Gauge covariant theory of the generating operator. I. *Commun. Math. Phys.*, 103(4):549–568, 1986.
119. I. Miodek. IST-solvable nonlinear evolution equations and existence an extension of Laxs method. *J. Math. Phys.*, 19:19, 1978.
120. V. S. Gerdjikov and P. P. Kulish. The generating operator for $n \times n$ linear system. *Physica D: Nonl. Phen.*, 3D(3):549–564, 1981.
121. I. B. Formusatik and Konopelchenko B. G. On the structure of the nonlinear evolution equations integrable by the $\mathbf{Z}_2$-graded quadratic bundle. *J. Phys. A: Math. Gen.*, 15(6):2017–2040, 1982.
122. V. S. Gerdjikov and P. P. Kulish. On the multicomponent nonlinear Schrödinger equation in the case of non-vanishing boundary conditions. *Sci. Notes of LOMI Seminars*, 131:34–46, 1983.
123. J. Langer and R. Perline. Geometric realizations of Fordy–Kulish nonlinear Schrödinger systems. *Pac. J. Math.*, 195:157–178, 2000.
124. V. S. Gerdjikov. On the spectral theory of the integro–ifferential operator generating nonlinear evolution equations. *Lett. Math. Phys*, 6:315–324, 1982.
125. V. S. Gerdjikov. Generalised Fourier transforms for the soliton equations. Gauge covariant formulation. *Inverse Probl.*, 2(1):51–74, 1986.
126. A. Rogers. Graded manifolds, supermanifolds and infinite-dimensional Grassmann algebras. *Commun. Math. Phys.*, 105:375–384, 1968.
127. I. M. Gel'fand and L. A. Dickey. Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation. *Russ. Math. Surveys*, 30:77, 1975.
128. M. A. Olshanetsky and A. M. Perelomov. Completely integrable Hamiltonian systems connected with semisimple Lie algebras. *Inventiones Mathematicae*, 37(2):93–108, 1976.
129. A. G. Reyman and M. A. Semenov-Tian-Shansky. Reduction of Hamiltonian systems, affine Lie algebras and Lax equations I. *Inventiones Mathematicae*, 54(1):81–100, 1979.
130. A. G. Reyman and M. A. Semenov-Tian-Shansky. Reduction of Hamiltonian systems, affine Lie algebras and Lax equations II. *Inventiones Mathematicae*, 63(3):423–432, 1981.
131. S. A. Bulgadaev. Two-dimensional integrable field theories connected with simple Lie algebras. *Phys. Lett. B*, 96(1-2):151–153, 1980.
132. A. G. Reiman and M. A. Semenov-Tyan-Shanskii. The jets algebra and nonlinear partial differential equations. *Dokl. Akad. Nauk SSSR*, 251(6):1310–1314, 1980.
133. A. G. Reyman. Integrable Hamiltonian systems connected with graded Lie algebras. *J. Sov. Math*, 19:1507–1545, 1982.
134. H. Flaschka, A. C. Newell, and T. Ratiu. Kac-Moody Lie algebras and soliton equations. II. Lax equations associated with $A_1^{(1)}$. *Physica D: Nonl. Phen.*, 9(3):300–323, 1983.
135. A. N. Leznov. The inverse scattering method in a form invariant with respect to representations of the internal symmetry algebra. *Theor. Math. Phys.*, 58(1):103–106, 1984.
136. A. N. Leznov and M. V. Saveliev. Nonlinear equations and graded Lie algebras. *Sov. Prob. Mat. Mat. Anal.*, 22:101–136, 1980.
137. A. Weinstein. *Poisson structures and Lie algebras*, pages 421–434. The mathematical heritage of É. Cartan, Astérisque. Lyon, numero hors serie edition, 1985.

138. J. M. Maillet. Kac-Moody algebra and extended Yang–Baxter relations in the $O(N)$ non-linear sigma-model. *Phys. Lett. B*, 162(1–3):137–142, 1985.
139. B. G. Konopelchenko and V. G. Dubrovsky. Bäcklund–Calogero group and general form of integrable equations for the two-dimensional Gelfand–Dikij-Zakharov–Shabat problem bilocal approach. *Physica D: Nonl. Phen.*, 16(1): 79–98, 1985.
140. B. A. Kupershmidt. Super Korteweg-de Vries equations associated to super extensions of the Virasoro algebra. *Phys. Lett. A*, 109(9):417–423, 1985.
141. D. Olive and N. Turok. The Toda lattice field theory hierarchies and zero-curvature conditions in Kac-Moody algebras. *Nucl. Phys. B*, 265(3):469–484, 1986.
142. V. S. Gerdjikov. The Zakharov–Shabat dressing method and the representation theory of the semisimple Lie algebras. *Physics Lett. A*, 126(3):184–188, 1987.
143. A. G. Reyman. Integrable Hamiltonian systems related to affine Lie algebras. *Zapiski LOMI*, 95:3–54, 1980.
144. L. D. Faddeev, N. Y. Reshetikhin, and L. Takhtadjan. Quantization of Lie groups and Lie algebras. *Algebra Anal.*, 1:178, 1989.
145. M. R. Adams, J. Harnad, and J. Hurtubise. Darboux coordinates and Liouville-Arnold integration in loop algebras. *Commun. Math. Phys.*, 155(2):385–413, 1993.
146. S. Zhang. Classical Yang–Baxter equation and low-dimensional triangular Lie bi-algebras. *Phys. Lett. A*, 246:71–81, 1998.
147. M. R. Adams, J. Harnad, and J. Hurtubise. Darboux coordinates on coadjoint orbits of Lie algebras. *Lett. Math. Phys.*, 40:41–57, 1997.
148. A. Kundu. Algebraic approach in unifying quantum integrable models. *Phys. Rev. Lett.*, 82(20):3936–3939, 1999.
149. T. Skrypnik. 'Doubled' generalized Landau–Lifshitz hierarchies and special quasigraded algebras. *J. Math. Phys.*, 37:7755–7768, 2004.
150. S. Lombardo and A. V. Mikhailov. Reduction group and automorphic Lie algebras. *Commun. Math. Phys.*, 258:179–202, 2005.
151. V. S. Gerdjikov, N. A. Kostov, and T. I. Valchev. N-wave equations with orthogonal algebras: Z_2 and $Z_2 \times Z_2$ reductions and soliton solutions. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 3, 2007.
152. L. Martinez Alonso. Schrödinger spectral problems with energy-dependent potentials as sources of nonlinear Hamiltonian evolution equations. *J. Math. Phys.*, 21(9):2342–2349, 1980.
153. V. S. Gerdjikov and M. I. Ivanov. A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures. *Russ. Bulg. J. Phys. 10*, 130–143, 1983.
154. I. V. Barashenkov and B. S. Getmanov. Multisoliton solutions in the scheme for unified description of integrable relativistic massive fields. Non-degenerate sl(2,C)-case. *Commun. Math. Phys.*, 112(3):423–446, 1987.
155. A. Kundu. Exact solutions to higher-order nonlinear equations through gauge transformation. *Physica D*, 25(1–3):399–406, 1987.
156. E. Fan. A family of completely integrable multi-Hamiltonian systems explicitly related to some celebrated equations. *J. Math. Phys.*, 42:4327–4344, 2001.
157. E. Fan. A Liouville integrable Hamiltonian system associated with a generalized Kaup–Newell spectral problem. *Physica A: Stat. Mech. Appl.*, 301 (1–4):105–113, 2001.

158. Z. S. Feng and X. H. Wang. Explicit exact solitary wave solutions for the Kundu equation and the derivative Schrödinger equation. *Physica Scripta*, 64(1):7–14, 2001.
159. A. P. Fordy. Derivative nonlinear Schrödinger equations and Hermitian symmetric spaces. *J. Phys. A: Math. Gen.*, 17(6):1235–1245, 1984.
160. E. Fan. Integrable evolution systems based on Gerdjikov–Ivanov equations, bi-Hamiltonian structure, finite-dimensional integrable systems and N-fold Darboux transformation. *J. Math. Phys.*, 41:7769, 2000.
161. V. S. Gerdjikov and P. P. Kulish. Completely integrable Hamiltonian systems related to the non-self-adjoint Dirac operator. *Bulg. J. Phys.*, 5(4):337–349, 1978. (In Russian).
162. V. G. Dubrovsky and Konopelchenko B. G. Bäcklund–Calogero group and general form of integrable equations for the 2-dimensional Gelfand–Dickey–Zakharov–Shabat problem. Bi-local approach. *Physica D: Nonl. Phen.*, 16D(1):79–98, 1985.
163. J. F. Ladik and S. C. Chiu. Solutions of nonlinear network equations by the inverse scattering method. *J. Math. Phys.*, 18:701, 1977.
164. F. Kako and N. Mugibayashi. Complete integrability of general nonlinear differential-difference equations solvable by the inverse method. II. *Prog. Theor. Phys.*, 61(3):776–790, 1979.
165. Y. Ishimori. A Relationship between the Ablowitz–Kaup–Newell–Segur and Wadati–Konno–Ichikawa schemes of the inverse scattering method. *J. Phys. Soc. Japan*, 51(9):3036–3041, 1982.
166. V. S. Gerdjikov, M. I. Ivanov, and Y. S. Vaklev. Gauge transformations and generating operators for the discrete Zakharov–Shabat system. *Inverse Probl.*, 2(4):413–432, 1986.
167. E. D. Belokolos, A. I. Bobenko, V. Z. Enolskii, A. R. Its, and V. B Matveev. *Algebro-Geometric Approach to Nonlinear Integrable Equations.* Springer-Verlag, New York, 1994.
168. A. M. Kamchatnov. New approach to periodic solutions of integrable equations and nonlinear theory of modulational instability. *Phys. Rep.*, 286(4):199–270, 1997.
169. A. M. Kamchatnov. *Nonlinear Periodic Waves and Their Modulations. An Introductory Course.* World Scientific, Singapore, 2000.
170. F. Gesztesy and H. Holden. *Soliton Equations and Their Algebro-Geometric Solutions.* Cambridge University Press, Cambridge, 2003.
171. V. S. Gerdjikov and P. P. Kulish. Derivation of the Bäcklund transformation in the formalism of the inverse scattering problem. *Theoreticheskaya i Mathematicheskaya Fizika*, 39(1):69–74, 1979.
172. V. S. Gerdjikov. Selected aspects of soliton theory. Constant boundary conditions. In Gerdjikov, V. and Tsvetkov, M., editor, *Prof. G. Manev's Legacy in Contemporary Aspects of Astronomy, Gravitational and Theoretical Physics*, pages 277–290. Heron Press Ltd, Sofia, 2005.
173. E. K. Sklyanin. On complete integrability of the Landau–Lifshitz equation. *Preprint LOMI E-3-79, Leningrad*, 1979.
174. V. E. Zakharov and A. V. Mikhailov. Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method. *Sov. Phys. -JETP*, 47(6), 1978.

175. V. E. Zakharov and A. V. Mikhailov. On the integrability of classical spinor models in two-dimensional space-time. *Commun. Math. Phys.*, 74(1):21–40, 1980.
176. I. M. Gelfand and L. A. Dickey. Asymptotic behavior of the resolvent of Sturm-Liouville equations, and the algebra of the Korteweg-de Vries equations. *Funct. Anal. Appl.*, 10:13–29, 1976.
177. M. Błaszak. *Multi-Hamiltonian Theory of Dynamical Systems.* Springer-Verlag, Berlin, Heidelberg, New-York, 1998.
178. I. V. Cherednik. *Contemporary Problems in Mathematics*, pages 176–219. VINITI, Moscow, 1980.
179. E. Barouch, A. S. Fokas, and V. G. Papageorgiou. Algorithmic construction of the recursion operatiors of Toda and Landau–Lifshitz equation. Potsdam, NY, 1987.
180. E. Barouch, A. S. Fokas, and V. G. Papageorgiou. The bi-Hamiltonian formulation of the Landau–Lifshiftz equation. *J. Math. Phys.*, 29:2628, 1988.
181. A. B. Yanovski. Recursion operators and bi-Hamiltonian formulations of the Landau–Lifshitz equation hierarchies. *J. Phys. A: Math. Gen.*, 39(10): 2409–2433, 2006.
182. M. J. Ablowitz. Lectures on the inverse scattering transform. *Stud. Appl. Math.*, 58(1):17–94, 1978.
183. M. Bruschi, O. Ragnisco, and D. Levi. Evolution equations associated with the discrete analog of the matrix Schrödinger spectral problem solvable by the inverse spectral transform. *J. Math. Phys.*, 22:2463, 1981.