Preface

The third C.I.M.E. Session 'Symplectic 4-Manifolds and Algebraic Surfaces' took place from September 2 to September 10, 2003 in the customary beautiful location of the Grand Hotel San Michele, Cetraro, Cosenza.

The present volume contains the text of the five series of lectures, which were delivered during the course.

There were also some very interesting seminar lectures during the course. They are of a more specialized nature and are not reproduced here.

The lectures survey recent and important advances in symplectic and differential topology of 4-manifolds and algebraic surfaces.

Relations with real algebraic geometry have been treated only in part in the course by Catanese, and much more in the seminars by Frediani and Welschinger. Indeed, this and other very interesting topics of current research could only be treated rather quickly, in view of the vastness of the central theme of the School, the study of differential, symplectic and complex structures on even dimensional, and especially four-dimensional, manifolds.

The course had at least a double valency: on the one hand the introduction of new methods, for instance symplectic geometry, for the study of moduli spaces of complex structures on algebraic surfaces. On the other hand, the use of algebraic surfaces as concrete models for the investigation of symplectic topology in dimension 4, and for laying down a research plan based on the analogies with the surface classification.

One concrete example of the synergy of these two viewpoints is given for instance by the study of partial compactifications of moduli spaces of singular surfaces, which led to the construction of symplectomorphisms through surgeries associated with smoothings of singularities.

Let us now try to describe the contents of the courses and the interwoven thread which relates them to one another, thus making this volume a coherent exposition of an active field of current research.

As it is well known, every projective variety inherits from the ambient space the Fubini-Study Kähler form, and in this way one obtains the most natural examples of symplectic structures. Even if one wants to consider more general symplectic manifolds, for use in the theory of dynamical systems, or just for the sake of classification, the relation with complex manifolds theory is always present.

In fact, a symplectic manifold always admits almost-complex structures, and the Kähler condition has as an analogue the condition that the almost-complex structure be compatible with the given symplectic structure. Even if almost-complex structures are not integrable (i.e., there are no local holomorphic coordinates), nevertheless one can still consider maps from a complex curve to the given almost-complex manifold whose derivative is complex linear.

One of the fundamental ideas, due to Gromov, is to use such maps to study the topology of symplectic manifolds.

These maps are called pseudoholomorphic curves, and the key point is that the corresponding generalized Cauchy–Riemann equations in the nonintegrable case do not substantially differ from the classical case (analyticity, removable singularities,...). The great advantage is, however, that while complex structures may remain 'nongeneric' even after deformation, a generic almost-complex structure really has an apt behaviour for transversality questions.

The study of pseudoholomorphic curves leads to important invariants, the so-called Gromov–Witten invariants, which have become an active research subject since late 1993 and were also treated in the seminar by Pandharipande.

If we start from a complex manifold, there remains, however, a basic question: if we take symplectic curves, how much do these differ from the holomorphic curves pertaining to the initial complex structure? Can they be deformed isotopically to holomorphic curves? Its analogue in algebraic geometry is to classify holomorphic curves under algebraic equivalence. In general, there are complex manifolds which contain symplectic curves, not isotopic to holomorphic ones. However, if the underlying symplectic manifold is an algebraic surface with positive first Chern class, it is expected that any symplectic curve be isotopic to a holomorphic curve.

This is the so-called symplectic isotopy problem, one of the fundamental problems in the study of symplectic 4-manifolds. It has many topological consequences in symplectic geometry. For instance, the solution to this symplectic isotopy problem provides a very effective way of classifying simply-connected symplectic 4-manifolds. Significant progress has been made. The central topic of the course by Siebert and Tian was the study of the symplectic isotopy problem, describing main tools and recent progress. The course has a pretty strong analytic flavour because of the nonlinearity of the Cauchy–Riemann equations in the nonintegrable case. Some applications to symplectic 4-manifolds were also discussed, in particular, that any genus two symplectic Lefschetz fibration under some mild non-degeneracy conditions is equivalent to a holomorphic surface. This result ties in with what we might call the 'dual' approach.

This is the approach taken by Donaldson for the study of symplectic 4-manifolds. Donaldson was able to extend the algebro-geometric concept of Lefschetz pencils to the case of symplectic manifolds. Even if for any generic almost-complex structure one cannot find holomorphic functions (even locally); one can nevertheless find smooth functions and sections of line bundles whose antiholomorphic derivative ($\bar{\partial} f$), even if not zero, is still much smaller than the holomorphic derivative ∂f. This condition produces the same type of topological behaviour as the one possessed by holomorphic functions, and the functions satisfying it are called approximately holomorphic functions (respectively, sections).

In this way, Donaldson was able to extend the algebro-geometric concept of Lefschetz pencils to the case of symplectic manifolds.

The topic of Lefschetz pencils has occupied a central role in several courses, one by Auroux and Smith, one by Seidel, and one by Catanese.

In fact, one main use of Lefschetz pencils is, from the results of Kas and Gompf, to encode the differential topology and the symplectic topology of the fibred manifold into a factorization inside the mapping class group, with factors which are (positive) Dehn twists.

While the course of Auroux and Smith used symplectic Lefschetz pencils to study topological invariants of symplectic 4-manifolds and the differences between the world of symplectic 4-manifolds and that of complex surfaces, in the course by Catanese, Lefschetz fibrations were used to describe recent work done in collaboration with Wajnryb to prove explicit diffeomorphisms of certain simply connected algebraic surfaces which are not deformation equivalent.

An important ingredient here is a detailed knowledge of the mapping class group of the fibres, which are compact complex curves of genus at least two.

Applications to higher dimensional symplectic varieties, and to Mirror symmetry, were discussed in the course by Seidel, dedicated to the symplectic mapping class group of 4-manifolds, and to Dehn twists in dimension 4. The resulting picture of symplectic monodromy is surprising. In fact, Seidel shows that the natural homomorphism of the symplectic to the differential mapping class group may not be injective, and moreover reveals a delicate deformation behaviour: there are symplectomorphisms which are not isotopic to the identity for some special symplectic structure, but become isotopic after a small deformation of the symplectic structure.

Lefschetz pencils are the 'generic' maps with target the complex projective line: the course by Auroux and Smith went all the way to consider a generalization of the classical algebro-geometric concept of 'generic multiple planes'.

This notion was extended to the symplectic case, again via approximately holomorphic sections, by Auroux and Katzarkov, and the course discusses the geometric invariants of a symplectic structure (which are deformation invariant) that can be extracted in this way.

This research interest is related to an old problem, posed by Boris Moishezon, namely, whether one can distinguish connected components of moduli spaces of surfaces of general type via these invariants (essentially, fundamental groups of complements of branch curves).

The courses by Catanese and Manetti are devoted instead to a similar question, a conjecture raised by Friedman and Morgan in the 1980s on the grounds of gauge theoretic speculations.

This conjecture is summarized by the acronym def=diff, and stated that diffeomorphic algebraic surfaces should be deformation equivalent. The course by Catanese reports briefly on the first nontrivial counterexamples, obtained by Manetti in the interesting case of surfaces of general type, and on the 'trivial' counterexamples, obtained independently by several authors, where a surface S is not deformation equivalent to the complex conjugate surface.

The focus in both courses is set on the simplest and strongest counterexamples, the so-called 'abc' surfaces, which are simply connected, and for which Catanese and Wajnryb showed that the diffeomorphism type is determined by the integers (a+c) and b. The course by Manetti focuses on the deformation theoretic and degeneration aspects (especially, smoothings of singularities), which up to now were scattered in a long series of articles by Catanese and Manetti. Catanese's course has a broader content and includes also other introductory facts on surfaces of general type and on singularities.

We would like to point out that, in spite of tremendous progress in 4-manifold topology, starting with Michael Freedman's solution of the problem of understanding the topology of simply connected 4-manifolds up to homeomorphism, and Simon Donaldson's gauge theoretic discoveries that smooth and topological structures differ drastically in dimension 4, the differential and symplectic topology even of simply connected symplectic 4-manifolds and algebraic surfaces is still a deep mystery.

Modern approaches to the study of symplectic 4-manifolds and algebraic surfaces combine a wide range of techniques and sources of inspiration. Gauge theory, symplectic geometry, pseudoholomorphic curves, singularity theory, moduli spaces, braid groups, monodromy, in addition to classical topology and algebraic geometry, combine to make this one of the most vibrant and active areas of research in mathematics.

Some keywords for the present volume are therefore pseudoholomorphic curves, algebraic and symplectic Lefschetz pencils, Dehn twists and monodromy, symplectic invariants, deformation theory and singularities, classification and moduli spaces of algebraic surfaces of general type, applications to mirror symmetry.

It is our hope that these texts will be useful to people working in related areas of mathematics and will become standard references on these topics.

We take this opportunity to thank the C.I.M.E. foundation for making the event possible, the authors for their hard work, the other lecturers for their interesting contributions, and the participants of the conference for their lively interest and enthusiastic collaboration.

We would also like to take this opportunity to thank once more the other authors for their work and apologize for the delay in publication of the volume.

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Differentiable and Deformation Type of Algebraic Surfaces, Real and Symplectic Structures

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Che differenza c'e' fra il palo, il Paolo e la banana?
(G. Lastrucci, Firenze, 9/11/62)

1 Introduction

As already announced in Cetraro at the beginning of the C.I.M.E. course, we deflected from the broader target 'Classification and deformation types of complex and real manifolds', planned and announced originally.

First of all, the lectures actually delivered focused on the intersection of the above vast area with the theme of the School, 'Algebraic surfaces and symplectic 4-manifolds'.

Hence the title of the Lecture Notes has been changed accordingly.

Moreover, the Enriques classification of real algebraic surfaces is not touched upon here, and complex conjugation and real structures appear mostly through their relation to deformation types of complex manifolds, and in particular through their relation with strong and weak rigidity theorems.

In some sense then this course is a continuation of the C.I.M.E. course I held some 20 years ago in Montecatini [Cat88], about 'Moduli of algebraic surfaces'.

But whereas those Lecture Notes had an initial part of considerable length which was meant to be a general introduction to complex deformation theory, here the main results of deformation theory which we need are only stated.

Nevertheless, because the topic can be of interest not only to algebraic geometers, but also to people working in differential or symplectic topology, we decided to start dedicating the first lecture to recalling basic notions concerning projective and Kähler manifolds. Especially, we recall the main principles of classification theory, and state the Enriques classification of algebraic surfaces of special type.

Since surfaces of general type and their moduli spaces are a major theme for us here, it seemed worthwhile to recall in detail in lecture two the structure of their canonical models, in particular of their singularities, the socalled Rational Double Points, or Kleinian quotient singularities. The rest of lecture two is devoted to proving Bombieri's theorem on pluricanonical embeddings, to the analysis of other quotient singularities, and to the deformation equivalence relation (showing that two minimal models are deformation equivalent iff the respective canonical models are). Bombieri's theorem is proven in every detail for the case of an ample canonical divisor, with the hope that some similar result may soon be proven also in the symplectic case.

In lecture three we show first that deformation equivalence implies diffeomorphism, and then, using a result concerning symplectic approximations of projective varieties with isolated singularities and Moser's theorem, we show that a surfaces of general type has a 'canonical symplectic structure', i.e., a symplectic structure whose class is the class of the canonical divisor, and which is unique up to symplectomorphism.

In lecture three and the following ones we thus enter 'in medias res', since one of the main problems that we discuss in these Lecture Notes is the comparison of differentiable and deformation type of minimal surfaces of general type, keeping also in consideration the canonical symplectic structure (unique up to symplectomorphism and invariant for smooth deformation) which these surfaces possess.

We present several counterexamples to the DEF = DIFF speculation of Friedman and Morgan [F-M88] that deformation type and diffeomorphism type should coincide for complex algebraic surfaces. The first ones were obtained by Manetti [Man01], and exhibit non simply connected surfaces which are pairwise not deformation equivalent. We were later able to show that they are canonically symplectomorphic (see [Cat02] and also [Cat06]). An account of these results is to be found in Chap. 6, which is an extra chapter with title 'Epilogue' (we hope however that this title may soon turn out to be inappropriate in view of future further developments).

In lecture 4, after discussing some classical results (like the theorem of Castelnuovo and De Franchis) and some 'semi-classical' results (by the author) concerning the topological characterization of irrational pencils on Kähler manifolds and algebraic surfaces, we discuss orbifold fundamental groups and triangle covers.

We use the above results to describe varieties isogenous to a product. These yield several examples of surfaces not deformation equivalent to their complex conjugate surface. We describe in particular the examples by the present author [Cat03], by Bauer–Catanese–Grunewald [BCG05], and then the ones by Kharlamov–Kulikov [KK02] which yield ball quotients. In this lecture we discuss complex conjugation and real structures, starting from elementary examples and ending with a survey of recent results and with open problems on the theory of 'Beauville surfaces'.

The beginning of lecture 5 is again rather elementary, it discusses connected sums and other surgeries, like fibre sums, and recalls basic definitions and results on braid groups, mapping class groups and Hurwitz equivalence.

After recalling the theory of Lefschetz pencils, especially the differentiable viewpoint introduced by Kas [Kas80], we recall Freedman's basic results on the topology of simply connected compact (oriented) fourmanifolds (see [F-Q90]).

We finally devote ourselves to our main objects of investigation, namely, the socalled '(abc)-surfaces' (introduced in [Cat02]), which are simply connected. We finish Lecture 5 explaining our joint work with Wajnryb [CW04] dedicated to the proof that these last surfaces are diffeomorphic to each other when the two integers b and $a+c$ are fixed.

In Chap. 6 we sketch the proof that these, under suitable numerical conditions, are not deformation equivalent. A result which is only very slightly weaker is explained in the Lecture Notes by Manetti, but with many more details; needless to say, we hope that the combined synergy of the two Lecture Notes may turn out to be very useful for the reader in order to appreciate the long chain of arguments leading to the theorem that the abc-surfaces give us the simply connected counterexamples to a weaker version of the DEF= DIFF question raised by Friedman and Morgan in [F-M88].

An interesting question left open (in spite of previous optimism) concerns the canonical symplectomorphism of the (abc)-surfaces. We discuss this and other problems, related to the connected components of moduli spaces of surfaces of general type, and to the corresponding symplectic structures, again in Chap. 6.

The present text not only expands the contents of the five lectures actually held in Cetraro. Indeed, since otherwise we would not have reached a satisfactory target, we added the extra Chap. 6.

As we already mentioned, since the course by Manetti does not explain the construction of his examples (which are here called Manetti surfaces), we give a very brief overview of the construction, and sketch a proof of the canonical symplectomorphism of these examples.

2 Lecture 1: Projective and Kähler Manifolds, the Enriques Classification, Construction Techniques

2.1 Projective Manifolds, Kähler and Symplectic Structures

The basic interplay between complex algebraic geometry, theory of complex manifolds, and theory of real symplectic manifolds starts with projective manifolds.

We consider a closed connected $\mathbb{C}$-submanifold $X^n \subset \mathbb{P}^N := \mathbb{P}^N_{\mathbb{C}}$.

This means that, around each point $p \in X$, there is a neighbourhood U_p of p and a permutation of the homogeneous coordinates such that, setting

$$x_0 = 1,\ x' := (x_1, \dots x_n),\ x'' := (x_{n+1}, \dots x_N),$$

the intersection $X \cap U_p$ coincides with the graph of a holomorphic map Ψ:

$$X \cap U_p = \{(x', x'') \in U_p | x'' = \Psi(x')\}.$$

We can moreover assume, after a linear change of the homogeneous coordinates, that the Taylor development of Ψ starts with a second order term (i.e., p is the point $(1, 0, \dots 0)$ and the projective tangent space to X at p is the complex subspace $\{x'' = 0\}$.

Definition 2.1 *The Fubini-Study form is the differential 2-form*

$$\omega_{FS} := \frac{i}{2\pi} \partial\overline{\partial} log|z|^2,$$

where z is the homogeneous coordinate vector representing a point of $\mathbb{P}^N$.

In fact the above 2-form on $\mathbb{C}^{N+1} \backslash \{0\}$ is invariant:

(1) For the action of $\mathbb{U}(N, \mathbb{C})$ on homogeneous coordinate vectors

(2) For multiplication of the vector z by a nonzero holomorphic scalar function $f(z)$ (z and $f(z)z$ represent the same point in $\mathbb{P}^N$), hence

(3) ω_{FS} descends to a differential form on $\mathbb{P}^N$ (being $\mathbb{C}^*$-invariant)

The restriction ω of the Fubini-Study form to a submanifold X of $\mathbb{P}^n$ makes the pair (X, ω) a Kähler manifold according to the following

Definition 2.2 *A pair (X, ω) of a complex manifold X, and a real differential 2-form ω is called a* Kähler pair *if*

(i) ω is closed ($d\omega = 0$)

(ii) ω is of type (1,1) $\Leftrightarrow$ for each pair of tangent vectors v, w one has (J being the operator on complex tangent vectors given by multiplication by $i = \sqrt{-1}$),

$$\omega(Jv, Jw) = \omega(v, w)$$

(iii) the associated Hermitian form is strictly positive definite $\Leftrightarrow$ the real symmetric bilinear form $\omega(v, Jw)$ is positive definite

The previous definition becomes clearer if one recalls the following easy bilinear algebra lemma.

Lemma 2.3 *Let V be a complex vector space, and H a Hermitian form. Then, decomposing H in real and imaginary part,*

$$H = S + \sqrt{-1}A,$$

we have that S is symmetric, A is alternating, $S(u, v) = A(u, Jv)$ and $A(Ju, Jv) = A(u, v)$.

Conversely, given a real bilinear and alternating form A , A is the imaginary part of a Hermitian form $H(u, v) = A(u, Jv) + \sqrt{-1}A(u, v)$ if and only if A satisfies the socalled first Riemann bilinear relation*:*

$$A(Ju, Jv) = A(u, v).$$

Observe that property (iii) implies that ω is nondegenerate (if in the previous lemma S is positive definite, then A is nondegenerate), thus a Kähler pair yields a symplectic manifold according to the standard definition

Definition 2.4 *A pair (X, ω) consisting of a real manifold X, and a real differential 2-form ω is called a* symplectic pair *if*

(i) ω is a symplectic form, i.e., ω is closed ($d\omega = 0$) and ω is nondegenerate at each point (thus X has even dimension).

A symplectic pair (X, ω) is said to be integral *iff the De Rham cohomology class of ω comes from $H^2(X, \mathbb{Z})$, or, equivalently, there is a complex line bundle L on X such that ω is a first Chern form of L.*

An almost complex structure *J on X is a differentiable endomorphism of the real tangent bundle of X satisfying $J^2 = -Id$. It is said to be*

(ii) compatible with ω *if*

$$\omega(Jv, Jw) = \omega(v, w),$$

(iii) tame *if the quadratic form $\omega(v, Jv)$ is strictly positive definite.*

Finally, a symplectic manifold is a manifold admitting a symplectic form ω.

Observe that compatibility and tameness are the symplectic geometry translation of the two classical Riemann bilinear relations which ensure the existence of a hermitian form, respectively the fact that the latter is positive definite: the point of view changes mainly in the order of the choice for J, resp. ω.

Definition 2.5 *A submanifold Y of a symplectic pair (X, ω) is a* symplectic submanifold *if $\omega|_Y$ is nondegenerate.*

Let (X', ω') be another symplectic pair. A diffeomorphism $f : X \to X'$ is said to be a symplectomorphism *if $f^*(\omega') = \omega$.*

Thus, unlike the Kähler property for complex submanifolds, the symplectic property is not automatically inherited by submanifolds of even real dimension.

A first intuition about symplectic submanifolds is given by the following result, which holds more generally on any Kähler manifold, and says that a good differentiable approximation of a complex submanifold is a symplectic submanifold.

Lemma 2.6 *Let $W \subset \mathbb{P}^N$ be a differentiable submanifold of even dimension ($\dim W = 2n$), and assume that the tangent space of W is 'close to be complex' in the sense that for each vector v tangent to W there is another vector v' tangent to W such that*

$$Jv = v' + u, |u| < |v|.$$

Then the restriction to W of the Fubini Study form ω_{FS} makes W a symplectic submanifold of $\mathbb{P}^N$.

Proof. Let A be the symplectic form on projective space, so that for each vector v tangent to W we have:

$|v|^2 = A(v, Jv) = A(v, v') + A(v, u)$.

Since $|A(v, u)| < |v|^2$, $A(v, v') \neq 0$ and A restricts to a nondegenerate form. $\square$

The above intuition does not hold globally, since it was observed by Thurston [Thur76] that there are symplectic complex manifolds which are not Kähler. The first example of this situation was indeed given by Kodaira [Kod66] who described the socalled Kodaira surfaces $\mathbb{C}^2/\Gamma$, which are principal holomorphic bundles with base and fibre an elliptic curve (they are not Kähler since their first Betti number equals 3). Many more examples have been given later on.

To close the circle between the several notions, there is the following characterization of a Kähler manifold (the full statement is very often referred to as 'folklore', but it follows from the statements contained in Theorem 3.13, page 74 of [Vois02], and Proposition 4.A.8, page 210 of [Huy05]).

Kähler manifolds Theorem *Let (X, ω)be a symplectic pair, and let J be an almost complex structure which is compatible and tame for ω. Let $g(u, v) := \omega(u, Jv)$ be the associated Riemannian metric. Then J is parallel for the Levi Civita connection of g (i.e., its covariant derivative is zero in each direction) if and only if J is integrable (i.e., it yields a complex structure) and ω is a Kähler form.*

Returning to the Fubini-Study form, it has an important normalization property, namely, if we consider a linear subspace $\mathbb{P}^m \subset \mathbb{P}^N$ (it does not matter which one, by the unitary invariance mentioned in (1) above), then integration in pluripolar coordinates yields

$$\int_{\mathbb{P}^m} \frac{1}{m!}\omega_{FS}^m = 1.$$

The above equation, together with Stokes' Lemma, and a multilinear algebra calculation for which we refer for instance to Mumford's book [Mum76] imply

Wirtinger's Theorem *Let $X := X^n$ be a complex submanifold of $\mathbb{P}^N$. Then X is a volume minimizing submanifold for the n-dimensional Riemannian volume function of submanifolds M of real dimension $2n$,*

$$vol(M) := \int dVol_{FS},$$

where $dVol_{FS} = \sqrt{det(g_{ij})(x)}\ |dx|$ is the volume measure of the Riemannian metric $g_{ij}(x)$ associated to the Fubini Study form. Moreover, the global volume of X equals a positive integer, called the degree *of X.*

The previous situation is indeed quite more general:

Let (X, ω) be a symplectic manifold, and let Y be an oriented submanifold of even dimension $= 2m$: then the global symplectic volume of Y

$vol(Y) := \int_Y \frac{1}{n!}\omega^m$ depends only on the homology class of Y, and will be an integer if the pair (X,ω) is integral (i.e., if the De Rham class of ω comes from $H^2(X,\mathbb{Z})$).

If moreover X is Kähler, and Y is a complex submanifold, then Y has a natural orientation, and one has the

Basic principle of Kähler geometry: *Let Y be a compact submanifold of a Kähler manifold X: then $vol(Y) := \int_Y \omega^m > 0$, and in particular the cohomology class of Y in $H^{2m}(X,\mathbb{Z})$ is nontrivial.*

The main point of the basic principle is that the integrand of $vol(Y) := \int_Y \omega^m$ is pointwise positive, because of condition (iii). So we see that a similar principle holds more generally if we have a symplectic manifold X and a compact submanifold Y admitting an almost complex structure compatible and tame for the restriction of ω to Y.

Wirtinger's theorem and the following theorem of Chow provide the link with algebraic geometry mentioned in the beginning.

Chow's Theorem *Let $X := X^n$ be a (connected) complex submanifold of $\mathbb{P}^N$. Then X is an algebraic variety, i.e., X is the locus of zeros of a homogeneous prime ideal $\mathcal{P}$ of the polynomial ring $\mathbb{C}[x_0, \dots x_N]$.*

We would now like to show how Chow's theorem is a consequence of another result:

Siegel's Theorem *Let $X := X^n$ be a compact (connected) complex manifold of (complex) dimension n. Then the field $\mathbb{C}^{Mer}(X)$ of meromorphic functions on X is finitely generated, and its transcendence degree over $\mathbb{C}$ is at most n.*

The above was proven by Siegel just using the lemma of Schwarz and an appropriate choice of a finite cover of a compact complex manifold made by polycylinder charts (see [Sieg73], or [Corn76]).

Idea of proof of Chow's theorem.

Let $p \in X$ and take coordinates as in 2.1: then we have an injection $\mathbb{C}(x_1, \dots x_n) \hookrightarrow \mathbb{C}^{Mer}(X)$, thus $\mathbb{C}^{Mer}(X)$ has transcendency degree n by Siegel's theorem.

Let Z be the Zariski closure of X: this means that Z is the set of zeros of the homogeneous ideal $\mathcal{I}_X \subset \mathbb{C}[x_0, \dots x_N]$ generated by the homogeneous polynomials vanishing on X.

Since X is connected, it follows right away, going to nonhomogeneous coordinates and using that the ring of holomorphic functions on a connected open set is an integral domain, that the ideal $\mathcal{I}_X = \mathcal{I}_Z$ is a prime ideal.

We consider then the homogeneous coordinate ring $\mathbb{C}[Z]:=\mathbb{C}[x_0, \dots x_N]/\mathcal{I}_X$ and the field of rational functions $\mathbb{C}(Z)$, the field of the fractions of the integral domain $\mathbb{C}[Z]$ which are homogeneous of degree 0. We observe that we have an injection $\mathbb{C}(Z) \hookrightarrow \mathbb{C}^{Mer}(X)$.

Therefore $\mathbb{C}(x_1, \dots x_n) \hookrightarrow \mathbb{C}(Z) \hookrightarrow \mathbb{C}^{Mer}(X)$. Thus the field of rational functions $\mathbb{C}(Z)$ has transcendency degree n and Z is an irreducible algebraic subvariety of $\mathbb{P}^N$ of dimension n. Since the smooth locus $Z^* := Z \setminus Sing(Z)$ is dense in Z for the Hausdorff topology, is connected, and contains X, it follows that $X = Z$. $\square$

The above theorem extends to the singular case: a closed complex analytic subspace of $\mathbb{P}^N$ is also a closed set in the Zariski topology, i.e., a closed algebraic set.

We have seen in the course of the proof that the dimension of an irreducible projective variety is given by the transcendency degree over $\mathbb{C}$ of the field $\mathbb{C}(Z)$ (which, by a further extension of Chow's theorem, equals $\mathbb{C}^{Mer}(Z)$).

The degree of Z is then defined through the

Emmy Noether Normalization Lemma. *Let Z be an irreducible subvariety of $\mathbb{P}^N$ of dimension n: then for general choice of independent linear forms $(x_0, \dots x_n)$ one has that the homogeneous coordinate ring of Z, $\mathbb{C}[Z] := \mathbb{C}[x_0, \dots x_N]/\mathcal{I}_Z$ is an integral extension of $\mathbb{C}[x_0, \dots x_n]$. One can view $\mathbb{C}[Z]$ as a torsion free $\mathbb{C}[x_0, \dots x_n]$-module, and its rank is called the degree d of Z.*

The geometrical consequences of Noether's normalization are (see [Shaf74]):

- The linear projection with centre $L := \{x | x_0 = \dots x_n = 0)$, $\pi_L : \mathbb{P}^N \backslash L \to \mathbb{P}^n$ is defined on Z since $Z \cap L = \emptyset$, and $\pi := \pi|_L : X \to \mathbb{P}^n$ is surjective and finite.
- For $y \in \mathbb{P}^n$, the finite set $\pi^{-1}(y)$ has cardinality at most d, and equality holds for y in a Zariski open set $U \subset \mathbb{P}^n$.

The link between the volume theoretic and the algebraic notion of degree is easily obtained via the Noether projection π_L.

In fact, the formula $(x_0, x', x'') \to (x_0, x', (1-t)x'')$ provides a homotopy between the identity map of Z and a covering of $\mathbb{P}^n$ of degree d, by which it follows that $\int_{Z^*} \omega_{FS}^n$ converges and equals precisely d.

We end this subsection by fixing the standard notation: for X a projective variety, and x a point in X we denote by $\mathcal{O}_{X,x}$ the local ring of algebraic functions on X regular in x, i.e.,

$$\mathcal{O}_{X,x} := \{f \in \mathbb{C}(X) | \exists a, b \in \mathbb{C}[X], \text{homogeneous}, s.t. f = a/b \text{ and } b(x) \neq 0\}.$$

This local ring is contained in the local ring of restrictions of local holomorphic functions from $\mathbb{P}^N$, which we denote by $\mathcal{O}^h_{X,x}$.

The pair $\mathcal{O}_{X,x} \subset \mathcal{O}^h_{X,x}$ is a faithfully flat ring extension, according to the standard

Definition 2.7 *A ring extension $A \to B$ is said to be* flat, *respectively* faithfully flat, *if the following property holds: a complex of A-modules (M_i, d_i) is exact only if (respectively, if and only if) $(M_i \otimes_A B, d_i \otimes_A B)$ is exact.*

This basic algebraic property underlies the so called (see [Gaga55-6]).

G.A.G.A. Principle. *Given a projective variety, and a coherent (algebraic) $\mathcal{O}_X$-sheaf $\mathcal{F}$, let $\mathcal{F}^h := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}^h_X$ be the corresponding holomorphic coherent sheaf: then one has a natural isomorphism of cohomology groups*

$$H^i(X_{Zar}, \mathcal{F}) \cong H^i(X_{Haus}, \mathcal{F}^h),$$

where the left hand side stands for Čech cohomology taken in the Zariski topology, the right hand side stands for Čech cohomology taken in the Hausdorff topology. The same holds replacing $\mathcal{F}$ by $\mathcal{O}_X^*$.

Due to the GAGA principle, we shall sometimes make some abuse of notation, and simply write, given a divisor D on X, $H^i(X, D)$ instead of $H^i(X, \mathcal{O}_X(D))$.

2.2 The Birational Equivalence of Algebraic Varieties

A rational map of a (projective) variety $\phi : X \dashrightarrow \mathbb{P}^N$ is given through N rational functions $\phi_1, \dots \phi_N$.

Taking a common multiple s_0 of the denominators b_j of $\phi_j = a_j/b_j$, we can write $\phi_j = s_j/s_0$, and write $\phi = (s_0, \dots s_N)$, where the s_j's are all homogeneous of the same degree, whence they define a graded homomorphism $\phi^* : \mathbb{C}[\mathbb{P}^N] \to \mathbb{C}[X]$.

The kernel of ϕ^* is a prime ideal, and its zero locus, denote it by Y, is called the image of ϕ, and we say that X dominates Y.

One says that ϕ is a morphism in p if there is such a representation $\phi = (s_0, \dots s_N)$ such that some $s_j(p) \neq 0$. One can see that there is a maximal open set $U \subset X$ such that ϕ is a morphism on U, and that $Y = \overline{\phi(U)}$.

If the local rings $\mathcal{O}_{X,x}$ are factorial, in particular if X is smooth, then one can take at each point x relatively prime elements a_j, b_j, let s_0 be the least common multiple of the denominators, and it follows then that the *Indeterminacy Locus* $X \backslash U$ is a closed set of codimension at least 2. In particular, every rational map of a smooth curve is a morphism.

Definition 2.8 *Two algebraic varieties X, Y are said to be* birational *iff their function fields $\mathbb{C}(X), \mathbb{C}(Y)$ are isomorphic, equivalently if there are two dominant rational maps $\phi : X \dashrightarrow Y, \psi : Y \dashrightarrow X$, which are inverse to each other. If $\phi, \psi = \phi^{-1}$ are morphisms, then X and Y are said to be* isomorphic.

By Chow's theorem, biholomorphism and isomorphism is the same notion for projective varieties (this ceases to be true in the non compact case, cf. [Ser59]).

Over the complex numbers, we have [Hir64].

Hironaka's theorem on resolution of singularities. *Every projective variety is birational to a smooth projective variety.*

As we already remarked, two birationally equivalent curves are isomorphic, whereas for a smooth surface S, and a point $p \in S$, one may consider the blow-up of the point p, $\pi : \hat{S} \to S$. $\hat{S}$ is obtained glueing together $S \backslash \{p\}$ with the closure of the projection with centre p, $\pi_p : S \backslash \{p\} \to \mathbb{P}^{N-1}$. One can moreover show that $\hat{S}$ is projective. The result of blow up is that the point p is replaced by the projectivization of the tangent plane to S at p, which is a curve $E \cong \mathbb{P}^1$, with normal sheaf $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. In other words, the selfintersection of E, i.e., the degree of the normal bundle of E, is -1, and we simply say that E is an *Exceptional curve of the I Kind*.

Theorem of Castelnuovo and Enriques. *Assume that a smooth projective surface Y contains an irreducible curve $E \cong \mathbb{P}^1$ with selfintersection $E^2 = -1$: then there is a birational morphism $f : Y \to S$ which is isomorphic to the blow up $\pi : \hat{S} \to S$ of a point p (in particular E is the only curve contracted to a point by f).*

The previous theorem justifies the following

Definition 2.9 *A smooth projective surface is said to be* minimal *if it does not contain any exceptional curve of the I kind.*

One shows then that every birational transformation is a composition of blow ups and of inverses of blow ups, and each surface X is birational to a smooth minimal surface S. This surface S is unique, up to isomorphism, if X is not ruled (i.e., not birational to a product $C \times \mathbb{P}^1$), by the classical

Theorem of Castelnuovo. *Two birational minimal models S, S' are isomorphic unless they are birationally* ruled, *i.e., birational to a product $C \times \mathbb{P}^1$, where C is a smooth projective curve. In the ruled case, either $S \cong \mathbb{P}^2$, or S is isomorphic to the projectivization $\mathbb{P}(V)$ of a rank 2 vector bundle V on C.*

Recall now that a variety X is smooth if and only if the sheaf of differential forms Ω^1_X is locally free, and locally generated by $dx_1, \dots dx_n$, if $x_1, \dots x_n$ yield local holomorphic coordinates.

The vector bundle (locally free sheaf) Ω^1_X and its associated bundles provide birational invariants in view of the classical [B-H75].

Kähler's lemma. *Let $f : X^n \dashrightarrow Y^m$ be a dominant rational map between smooth projective varieties of respective dimensions n, m. Then one has injective pull back linear maps $H^0(Y, {\Omega^1_Y}^{\otimes r}) \to H^0(X, {\Omega^1_X}^{\otimes r})$. Hence the vector spaces $H^0(X, {\Omega^1_X}^{\otimes r_1} \otimes \dots \otimes {\Omega^n_X}^{\otimes r_n})$ are birational invariants.*

Of particular importance is the top exterior power $\Omega^n_X = \Lambda^n(\Omega^1_X)$, which is locally free of rank 1, thus can be written as $\mathcal{O}_X(K_X)$ for a suitable Cartier divisor K_X, called the canonical divisor, and well defined only up to linear equivalence.

Definition 2.10 *The* ith pluriirregularity *of a smooth projective variety X is the dimension $h^{0,i} := dim(H^i(X, \mathcal{O}_X))$, which by Hodge Theory equals $dim(H^0(X, \Omega^i_X))$. The* mth plurigenus P_m *is instead the dimension $P_m(X) := dim(H^0(X, {\Omega^n_X}^{\otimes m})) = h^0(X, mK_X)$.*

A finer birational invariant is the canonical ring of X.

Definition 2.11 *The* canonical ring *of a smooth projective variety X is the graded ring*

$$\mathcal{R}(X) := \bigoplus_{m=0}^{\infty} H^0(X, mK_X).$$

If $\mathcal{R}(X) = \mathbb{C}$ *one defines* $Kod(X) = -\infty$*, otherwise the* Kodaira dimension *of* X *is defined as the transcendence degree over* $\mathbb{C}$ *of the* canonical subfield of $\mathbb{C}(X)$*, given by the field* $\mathcal{Q}(X)$ *of homogeneous fractions of degree zero of* $\mathcal{R}(X)$*.*

X *is said to be of* general type *if its Kodaira dimension is maximal (i.e., equal to the dimension* n *of* X*).*

As observed in [Andr73] $\mathcal{Q}(X)$ is algebraically closed inside $\mathbb{C}(X)$, thus one obtains that X is of general type if and only if there is a positive integer m such that $H^0(X, mK_X)$ yields a birational map onto its image Σ_m.

One of the more crucial questions in classification theory is whether the canonical ring of a variety of general type is finitely generated, the answer being affirmative [Mum62, Mori88] for dimension $n \leq 3$.[1]

2.3 The Enriques Classification: An Outline

The main discrete invariant of smooth projective curves C is the *genus* $g(C) := h^0(K_C) = h^1(\mathcal{O}_C)$.

It determines easily the Kodaira dimension, and the Enriques classification of curves is the subdivision:

- $Kod(C) = -\infty \Leftrightarrow g(C) = 0 \Leftrightarrow C \cong \mathbb{P}^1$
- $Kod(C) = 0 \Leftrightarrow g(C) = 1 \Leftrightarrow C \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, with $\tau \in \mathbb{C}, Im(\tau) > 0 \Leftrightarrow C$ is an elliptic curve
- $Kod(C) = 1 \Leftrightarrow g(C) \geq 2 \Leftrightarrow C$ is of general type

Before giving the Enriques classification of projective surfaces over the complex numbers, it is convenient to discuss further the birational invariants of surfaces.

Remark 2.12 *An important birational invariant of smooth varieties* X *is the fundamental group* $\pi_1(X)$*.*

For surfaces, the most important invariants are:

- *The* irregularity $q := h^1(\mathcal{O}_X)$
- *The* geometric genus $p_g := P_1 := h^0(X, K_X)$*, which for surfaces combines with the irregularity to give the* holomorphic Euler–Poincaré characteristic $\chi(S) := \chi(\mathcal{O}_S) := 1 - q + p_g$
- *The* bigenus $P_2 := h^0(X, 2K_X)$ *and especially the* twelfth plurigenus $P_{12} := h^0(X, 12K_X)$

If S *is a non ruled minimal surface, then also the following are birational invariants:*

[1] The question seems to have been settled for varieties of general type, and with a positive answer.

- *The selfintersection of a canonical divisor* K_S^2, *equal to* $c_1(S)^2$
- *The* topological Euler number $e(S)$, *equal to* $c_2(S)$ *by the Poincaré Hopf theorem, and which by* Noether's theorem *can also be expressed as*
$$e(S) = 12\chi(S) - K_S^2 = 12(1 - q + p_g) - K_S^2$$
- *The* topological index $\sigma(S)$ *(the index of the quadratic form* $q_S : H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to \mathbb{Z}$*), which, by the* Hodge index theorem, *satisfies the equality*
$$\sigma(S) = \frac{1}{3}(K_S^2 - 2e(S))$$
- *In particular, all the Betti numbers* $b_i(S)$
- *The positivity* $b^+(S)$ *and the negativity* $b^-(S)$ *of* q_S *(recall that* $b^+(S) + b^-(S) = b_2(S)$*)*

The Enriques classification of complex algebraic surfaces gives a very simple description of the surfaces with nonpositive Kodaira dimension:

- S is a ruled surface of irregularity $g \iff$:
  $\iff$: S is birational to a product $C_g \times \mathbb{P}^1$, where C_g has genus $g \iff$
  $\iff P_{12}(S) = 0, q(S) = g \iff$
  $\iff Kod(S) = -\infty$, $q(S) = g$
- S has $Kod(S) = 0 \iff P_{12}(S) = 1$

There are four classes of such surfaces with $Kod(S) = 0$:

- Tori $\iff P_1(S) = 1, q(S) = 2$
- K3 surfaces $\iff P_1(S) = 1, q(S) = 0$
- Enriques surfaces $\iff P_1(S) = 0, q(S) = 0$, $P_2(S) = 1$
- Hyperelliptic surfaces $\iff P_{12}(S) = 1, q(S) = 1$

Next come the surfaces with strictly positive Kodaira dimension:

- S is a properly elliptic surface $\iff$:
  $\iff$: $P_{12}(S) > 1$, and $H^0(12K_S)$ yields a map to a curve with fibres elliptic curves $\iff$
  $\iff$ S has $Kod(S) = 1 \iff$
  $\iff$ assuming that S is minimal: $P_{12}(S) > 1$ and $K_S^2 = 0$
- S is a surface of general type $\iff$:
  $\iff$: S has $Kod(S) = 2 \iff$
  $\iff P_{12}(S) > 1$, and $H^0(12K_S)$ yields a birational map onto its image $\Sigma_{12} \iff$
  $\iff$ assuming that S is minimal: $P_{12}(S) > 1$ and $K_S^2 \geq 1$

2.4 Some Constructions of Projective Varieties

Goal of this subsection is first of all to illustrate concretely the meaning of the concept 'varieties of general type'. This means, roughly speaking, that if we have a construction of varieties of a fixed dimension involving some integer parameters, most of the time we get varieties of general type when these parameters are all sufficiently large.

[1] *Products.*

Given projective varieties $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, their product $X \times Y$ is also projective. This is an easy consequence of the fact that the product $\mathbb{P}^n \times \mathbb{P}^m$ admits the Segre embedding in $\mathbb{P}^{mn+n+m} \cong \mathbb{P}(Mat(n+1, m+1))$ onto the subspace of rank one matrices, given by the morphism $(x, y) \rightarrow x \cdot {}^t y$.

[2] *Complete intersections.*

Given a smooth variety X, and divisors $D_1 = \{f_1 = 0\}, \ldots, D_r = \{f_r = 0\}$ on X, their intersection $Y = D_1 \cap \cdots \cap D_r$ is said to be a complete intersection if Y has codimension r in X. If Y is smooth, or, more generally, reduced, locally its ideal is generated by the local equations of the D_i's ($\mathcal{I}_Y = (f_1, \ldots f_r)$).

Y tends to inherit much from the geometry of X, for instance, if $X = \mathbb{P}^N$ and Y is smooth of dimension $N - r \geq 2$, then Y is simply connected by the theorem of Lefschetz.

[3] *Finite coverings according to Riemann, Grauert and Remmert.*

Assume that Y is a normal variety (this means that each local ring $\mathcal{O}_{X,x}$ is integrally closed in the function field $\mathbb{C}(X)$), and that B is a closed subvariety of Y (the letter B stands for branch locus).

Then there is (cf. [GR58]) a correspondence between

[3a] subgroups $\Gamma \subset \pi_1(Y \backslash B)$ of finite index, and

[3b] pairs (X, f) of a normal variety X and a finite map $f : X \rightarrow Y$ which, when restricted to $X \backslash f^{-1}(B)$, is a local biholomorphism and a topological covering space of $Y \backslash B$.

The datum of the covering is equivalent to the datum of the sheaf of $\mathcal{O}_Y$-algebras $f_* \mathcal{O}_X$. As an $\mathcal{O}_Y$-module $f_* \mathcal{O}_X$ is locally free if and only if f is *flat* (this means that, $\forall x \in X$, $\mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X,x}$ is flat), and this is indeed the case when f is finite and Y is smooth of dimension 2.

[4] *Finite Galois coverings.*

Although this is just a special case of the previous one, namely when Γ is a normal subgroup with factor group $G := \pi_1(Y \backslash B)/\Gamma$, in the more special case (cf. [Par91]) where G is Abelian and Y is smooth, one can give explicit equations for the covering. This is due to the fact that all irreducible representations of an abelian group are 1-dimensional, so we are in the *split case* where $f_* \mathcal{O}_X$ is a direct sum of invertible sheaves.

The easiest example is the one of

[4a] Simple cyclic coverings of degree n.

In this case there is

(i) an invertible sheaf $\mathcal{O}_Y(L)$ such that

$$f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L) \oplus \cdots \oplus \mathcal{O}_Y(-(n-1)L).$$

(ii) A section $0 \neq \sigma \in H^0(\mathcal{O}_Y(nL))$ such that X is the divisor, in the geometric line bundle $\mathbb{L}$ whose sheaf of regular sections is $\mathcal{O}_Y(L)$, given by the equation $z^n = \sigma(y)$.

Here, z is the never vanishing section of $p^*(\mathcal{O}_Y(L))$ giving a tautological linear form on the fibres of $\mathbb{L}$: in other words, one has an open cover U_α of Y which is trivializing for $\mathcal{O}_Y(L)$, and X is obtained by glueing together the local equations $z_\alpha^n = \sigma_\alpha(y)$, since $z_\alpha = g_{\alpha,\beta}(y) z_\beta$, $\sigma_\alpha(y) = g_{\alpha,\beta}(y)^n \sigma_\beta(y)$.

One has as branch locus $B = \Delta := \{\sigma = 0\}$, at least if one disregards the multiplicity (indeed $B = (n-1)\Delta$). Assume Y is smooth: then X is smooth iff Δ is smooth, and, via direct image, all the calculations of cohomology groups of basic sheaves on X are reduced to calculations for corresponding sheaves on Y. For instance, since $K_X = f^*(K_Y + (n-1)L)$, one has:

$$f_*(\mathcal{O}_X(K_X)) = \mathcal{O}_Y(K_Y) \oplus \mathcal{O}_Y(K_Y + L) \oplus \cdots \oplus \mathcal{O}_Y(K_Y + (n-1)L)$$

(the order is exactly as above according to the characters of the cyclic group).

We see in particular that X is of general type if L is sufficiently positive.

[4b] Simple iterated cyclic coverings.

Suppose that we take a simple cyclic covering $f : Y_1 \to Y$ as above, corresponding to the pair (L, σ), and we want to consider again a simple cyclic covering of Y_1. A small calculation shows that it is not so easy to describe $H^1(\mathcal{O}^*_{Y_1})$ in terms of the triple (Y, L, σ); but in any case $H^1(\mathcal{O}^*_{Y_1}) \supset H^1(\mathcal{O}^*_Y)$. Thus one defines an *iterated simple cyclic covering* as the composition of a chain of simple cyclic coverings $f_i : Y_{i+1} \to Y_i$, $i = 0, \dots k-1$ (thus $X := Y_k$, $Y := Y_0$) such that at each step the divisor L_i is the pull back of a divisor on $Y = Y_0$.

In the case of iterated double coverings, considered in [Man97], we have at each step $(z_i)^2 = \sigma_i$ and each σ_i is written as $\sigma_i = b_{i,0} + b_{i,1} z_1 + b_{i,2} z_2 + \cdots + b_{i,1,\dots i-1} z_1 \dots z_{i-1}$, where, for $j_1 < j_2 \cdots < j_h$, we are given a section $b_{i,j_1,\dots j_h} \in H^0(Y, \mathcal{O}_Y(2L_i - L_{j_1} - \cdots - L_{j_h}))$.

In principle, it looks like one could describe the Galois covers with solvable Galois group G by considering iterated cyclic coverings, and then imposing the Galois condition. But this does not work without resorting to more complicated cyclic covers and to special geometry.

[4c] Bidouble covers (Galois with group $(\mathbb{Z}/2)^2$).

The *simple bidouble covers* are simply the fibre product of two double covers, thus here X is the complete intersection of the following two divisors

$$z^2 = \sigma_0, \ w^2 = s_0$$

in the vector bundle $\mathbb{L} \oplus \mathbb{M}$.

These are the examples we shall mostly consider.

More generally, a bidouble cover of a smooth variety Y occurs [Cat84] as the subvariety X of the direct sum of three line bundles $\mathbb{L}_1 \oplus \mathbb{L}_2 \oplus \mathbb{L}_3$, given by equations

$$Rank \begin{pmatrix} x_1 & w_3 & w_2 \\ w_3 & x_2 & w_1 \\ w_2 & w_1 & x_3 \end{pmatrix} = 1 \qquad (*)$$

Here, we have three Cartier divisors $D_j = div(x_j)$ on Y and three line bundles $\mathbb{L}_i$, with fibre coordinate w_i, such that the following linear equivalences hold on Y,

$$L_i + D_i \equiv L_j + L_k,$$

for each permutation (i, j, k) of $(1, 2, 3)$.

One has: $f_* \mathcal{O}_X = \mathcal{O}_Y \bigoplus (\oplus_i \mathcal{O}_Y(-L_i))$.

Assume in addition that Y is a smooth variety, then:

- X is normal if and only if the divisors D_j are reduced and have no common components.
- X is smooth if and only if the divisors D_j are smooth, they do not have a common intersection and have pairwise transversal intersections.
- X is Cohen–Macaulay and for its dualizing sheaf ω_X (which, if Y is normal, equals the sheaf of Zariski differentials that we shall discuss later) we have $f_*\omega_X = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \omega_Y) = \omega_Y \bigoplus (\oplus_i \omega_Y(L_i))$.

[5] Natural deformations.

One should in general consider Galois covers as 'special varieties'.

For instance, if we have a line bundle $\mathbb{L}$ on Y, we consider in it the divisor X described by an equation

$$z^n + a_2 z^{n-2} + \dots a_{n-1} z + a_n = 0, \; for \; a_i \in H^0(Y, \mathcal{O}_Y(iL)).$$

It is clear that we obtain a simple cyclic cover if we set $a_n = -\sigma_0$, and, for $j \neq n$, we set $a_j = 0$.

The family of above divisors (note that we may assume $a_1 = 0$ after performing a Tschirnhausen transformation) is called the family of *natural deformations* of a simple cyclic cover.

One can define more generally a similar concept for any Abelian covering. In particular, for simple bidouble covers, we have the following family of natural deformations

$$z^2 = \sigma_0(y) + w\sigma_1(y), \; w^2 = s_0(y) + z s_1(y),$$

where $\sigma_0 \in H^0(Y, \mathcal{O}_Y(2L)), \sigma_1 \in H^0(Y, \mathcal{O}_Y(2L - M)), s_0 \in H^0(Y, \mathcal{O}_Y(2M))$ $s_1 \in H^0(Y, \mathcal{O}_Y(2M - L))$.

[6] Quotients.

In general, given an action of a finite group G on the function field $\mathbb{C}(X)$ of a variety X, one can always take the birational quotient, corresponding to the invariant subfield $\mathbb{C}(X)^G$.

Assume that $X \subset \mathbb{P}^N$ is a projective variety and that we have a finite group $G \subset \mathbb{P}GL(N+1, \mathbb{C})$, such that $g(X) = X, \; \forall g \in G$.

We want then to construct a biregular quotient X/G with a projection morphism $\pi : X \to X/G$.

For each point $x \in X$ consider a hyperplane H such that $H \cap Gx = \emptyset$, and let $U := X \backslash (\cup_{g \in G} \; g(H))$.

U is an invariant affine subset, and we consider on the quotient set U/G the ring of invariant polynomials $\mathbb{C}[U]^G$, which is finitely generated since we are in characteristic zero and we have a projector onto the subspace of invariants.

It follows that if X is normal, then also X/G is normal, and moreover projective since there are very ample g-invariant divisors on X.

If X is smooth, one has that X/G is smooth if

(1) G acts freely or, more generally, if and only if

(2) For each point $p \in X$, the stabilizer subgroup $G_p := \{g | g(p) = p\}$ is generated by pseudoreflections (theorem of Chevalley, cf. for instance [Dolg82]).

To explain the meaning of a *pseudoreflection*, observe that, if $p \in X$ is a smooth point, by a theorem of Cartan [Car57], one can *linearize* the action of G_p, i.e., there exist local holomorphic coordinates $z_1, \dots z_n$ such that the action in these coordinates is linear. Thus, $g \in G_p$ acts by $z \to A(g)z$, and one says that g is a pseudoreflection if $A(g)$ (which is diagonalizable, having finite order) has $(n-1)$ eigenvalues equal to 1.

[7] *Rational Double Points = Kleinian singularities.*

These are exactly the quotients $Y = \mathbb{C}^2/G$ by the action of a finite group $G \subset SL(2, \mathbb{C})$. Since $A(g) \in SL(2, \mathbb{C})$ it follows that G contains no pseudoreflection, thus Y contains exactly one singular point p, image of the unique point with a nontrivial stabilizer, $0 \in \mathbb{C}^2$.

These singularities (Y, p) will play a prominent role in the next section.

In fact, one of their properties is due to the fact that the differential form $dz_1 \wedge dz_2$ is G-invariant (because $det(A(g)) = 1$), thus the sheaf Ω^2_Y is trivial on $Y \backslash \{p\}$.

Then the dualizing sheaf $\omega_Y = i_*(\Omega^2_{Y \backslash \{p\}})$ is also trivial.

3 Lecture 2: Surfaces of General Type and Their Canonical Models: Deformation Equivalence and Singularities

3.1 Rational Double Points

Let us take up again the Kleinian singularities introduced in the previous section

Definition 3.1 *A Kleinian singularity is a singularity (Y, p) analytically isomorphic to a quotient singularity $\mathbb{C}^n/G$ where G is a finite subgroup $G \subset SL(n, \mathbb{C})$.*

Example 3.2 *The surface singularity A_n corresponds to the cyclic group $\mu_n \cong \mathbb{Z}/n$ of nth roots of unity acting with characters 1 and $(n-1)$.*

I.e., $\zeta \in \mu_n$ acts by $\zeta(u, v) := (\zeta u, \zeta^{n-1} v)$, and the ring of invariants is the ring $\mathbb{C}[x, y, z]/(xy - z^n)$, where

$$x := u^n, \ y := v^n, z := uv.$$

Example 3.3 *One has more generally the cyclic quotient surface singularities corresponds to the cyclic group $\mu_n \cong \mathbb{Z}/n$ of nth roots of unity acting with characters a and b, which are denoted by $\frac{1}{n}(a,b)$.*

Here, $\zeta(u,v) := (\zeta^a u, \zeta^b v)$.

We compute the ring of invariants in the case $n = 4, a = b = 1$: the ring of invariants is generated by

$$y_0 := u^4, y_1 := u^3 v,\ y_2 := u^2 v^2,\ y_3 := uv^3, y_4 := v^4,$$

and the ring is $\mathbb{C}[y_0, \dots, y_4]/J$, where J is the ideal of 2×2 minors of the matrix $\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$, or equivalently of the matrix $\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix}$. The first realization of the ideal J corresponds to the identification of the singularity Y as the cone over a rational normal curve of degree 4 (in $\mathbb{P}^4$), while in the second Y is viewed as a linear section of the cone over the Veronese surface.

We observe that $2y_2$ and $y_0 + y_4$ give a map to $\mathbb{C}^2$ which is finite of degree 4. They are invariant for the group of order 16 generated by

$$(u,v) \mapsto (iu, iv),\ (u,v) \mapsto (iu, -iv),\ (u,v) \mapsto (v,u),$$

hence Y is a bidouble cover of $\mathbb{C}^2$ branched on three lines passing through the origin (cf. (*), we set $x_3 := x_1 - x_2$ and we choose as branch divisors $x_1, x_2, x_3 := x_1 - x_2$).

In dimension two, the classification of Kleinian singularities is a nice chapter of geometry ultimately going back to Thaetethus' Platonic solids. Let us briefly recall it.

First of all, by averaging the positive definite Hermitian product in $\mathbb{C}^n$, one finds that a finite subgroup $G \subset SL(n, \mathbb{C})$ is conjugate to a finite subgroup $G \subset SU(n, \mathbb{C})$. Composing the inclusion $G \subset SU(n, \mathbb{C})$ with the surjection $SU(n, \mathbb{C}) \to \mathbb{P}SU(n, \mathbb{C}) \cong SU(n, \mathbb{C})/\mu_n$ yields a finite group G' acting on $\mathbb{P}^{n-1}$.

Thus, for $n = 2$, we get $G' \subset \mathbb{P}SU(2, \mathbb{C}) \cong SO(3)$ acting on the Riemann sphere $\mathbb{P}^1 \cong S^2$.

The consideration of the Hurwitz formula for the quotient morphism $\pi : \mathbb{P}^1 \to \mathbb{P}^1/G'$, and the fact that $\mathbb{P}^1/G'$ is a smooth curve of genus 0, (hence $\mathbb{P}^1/G' \cong \mathbb{P}^1$) allows the classification of such groups G'.

Letting in fact $p_1, \dots p_k$ be the branch points of π, and $m_1, \dots m_k$ the respective multiplicities (equal to the order in G' of the element corresponding to the local monodromy), we have *Hurwitz's formula* (expressing the degree of the canonical divisor $K_{\mathbb{P}^1}$ as the sum of the degree of the pull back of $K_{\mathbb{P}^1}$ with the degree of the ramification divisor)

$$-2 = |G'|(-2 + \sum_{i=1}^{k} [1 - \frac{1}{m_i}]).$$

Each term in the square bracket is $\geq \frac{1}{2}$, and the left hand side is negative: hence $k \leq 3$.

The situation to classify is the datum of a ramified covering of $\mathbb{P}^1 \backslash \{p_1, \dots p_k\}$, Galois with group G'.

By the Riemann existence theorem, and since $\pi_1(\mathbb{P}^1 \backslash \{p_1, \dots p_k\})$ is the socalled infinite polygonal group $T(\infty^k) = T(\infty, \dots, \infty)$ generated by simple geometric loops $\alpha_1, \dots, \alpha_k$, satisfying the relation $\alpha_1 \cdot \dots \cdot \alpha_k = 1$, the datum of such a covering amounts to the datum of an epimorphism $\phi : T(\infty, \dots, \infty) \to G'$ such that, for each $i = 1, \dots, k$, $a_i := \phi(\alpha_i)$ is an element of order m_i.

The group $T(\infty^k)$ is trivial for $k = 1$, infinite cyclic for $k = 2$, in general a free group of rank $k - 1$.

Since $a_i := \phi(\alpha_i)$ is an element of order m_i, the epimorphism factors through the polygonal group

$$T(m_1, \dots, m_k) := \langle \alpha_1, \dots, \alpha_k | \alpha_1 \cdot \dots \cdot \alpha_k = \alpha_1^{m_1} = \dots = \alpha_k^{m_k} = 1 \rangle.$$

If $k = 2$, then we may assume $m_1 = m_2 = m$ and we have a cyclic subgroup G' of order m of $\mathbb{P}SU(2, \mathbb{C})$, which, up to conjugation, is generated by a transformation $\zeta(u, v) := (\zeta u, \zeta^{n-1} v)$, with ζ a primitive mth root of 1 for m odd, and a primitive 2mth root of 1 for m even. Thus, our group G is a cyclic group of order n, with $n = 2m$ for m even, and with $n = 2m$ or $n = m$ for m odd. G is generated by a transformation $\zeta(u, v) := (\zeta u, \zeta^{n-1} v)$ (with ζ a primitive nth root of 1), and we have the singularity A_n previously considered.

If $k = 3$, the only numerical solutions for the Hurwitz' formula are

$$m_1 = 2, m_2 = 2, m_3 = m \geq 2,$$

$$m_1 = 2, m_2 = 3, m_3 = 3, 4, 5.$$

Accordingly the order of the group G' equals $2m, 12, 24, 60$. Since m_3, for $m_3 \geq 3$, is not the least common multiple of m_1, m_2, the group G' is not abelian, and it follows (compare [Klein1884]) that G' is respectively isomorphic to $D_m, \mathcal{A}_4, \mathcal{S}_4, \mathcal{A}_5$.

Accordingly, since as above the lift of an element in G' of even order k has necessarily order $2k$, it follows that G is the full inverse image of G', and G is respectively called the binary dihedral group, the binary tetrahedral group, the binary octahedral group, the binary icosahedral group.

Felix Klein computed explicitly the ring of polynomial invariants for the action of G, showing that $\mathbb{C}[u, v]^G$ is a quotient ring $\mathbb{C}[x, y, z]/(z^2 - f(x, y))$, where

- $f(x, y) = x^2 + y^{n+1}$ for the A_n case
- $f(x, y) = y(x^2 + y^{n-2})$ for the D_n case ($n \geq 4$)
- $f(x, y) = x^3 + y^4$ for the E_6 case, when $G' \cong \mathcal{A}_4$
- $f(x, y) = y(x^2 + y^3)$ for the E_7 case, when $G' \cong \mathcal{S}_4$
- $f(x, y) = x^3 + y^5$ for the E_8 case, when $G' \cong \mathcal{A}_5$

We refer to [Durf79] for several equivalent characterizations of Rational Double points, another name for the Kleinian singularities. An important property (cf. [Reid80] and [Reid87]) is that these singularities may be resolved just by a sequence of point blow ups: in this procedure no points of higher multiplicity than 2 appear, whence it follows once more that the canonical divisor of the minimal resolution is the pull back of the canonical divisor of the singularity.

A simpler way to resolve these singularities (compare [BPV84], pages 86 and following) is to observe that they are expressed as double covers branched over the curve $f(x, y) = 0$. Then the standard method, explained in full generality by Horikawa in [Hor75] is to resolve the branch curve by point blow ups, and keeping as new branch curve at each step $B'' - 2D''$, where B'' is the total transform of the previous branch curve B, and D'' is the maximal effective divisor such that $B'' - 2D''$ is also effective. One obtains the following

Theorem 3.4 *The minimal resolution of a Rational Double Point has as exceptional divisor a finite union of curves $E_i \cong \mathbb{P}^1$, with selfintersection -2, intersecting pairwise transversally in at most one point, and moreover such that no three curves pass through one point. The dual graph of the singularity, whose vertices correspond to the components E_i, and whose edges connect E_i and E_j exactly when $E_i \cdot E_j = 1$, is a tree, which is a linear tree with $n - 1$ vertices exactly in the A_n case. In this way one obtains exactly all the Dynkin diagrams corresponding to the simple Lie algebras.*

Remark 3.5 *(i) See the forthcoming Theorem 3.9 for a list of these Dynkin diagrams.*

(ii) The relation to simple Lie algebras was clarified by Brieskorn in [Briesk71]: these singularities are obtained by intersecting the orbits of the coadjoint action with a three dimensional submanifold in general position.

We end this subsection with an important observation concerning the automorphisms of a Rational Double Point (X, x_0).

Let H be a finite group of automorphisms of the germ $(X, x_0) = (\mathbb{C}^2, 0)/G$.

Then the quotient $(X, x_0)/H$ is a quotient of $(\mathbb{C}^2, 0)$ by a group H' such that $H'/G \cong H$. Moreover, by the usual averaging trick (Cartan's lemma, see [Car57]) we may assume that $H' \subset GL(2, \mathbb{C})$. Therefore H' is contained in the normalizer N_G of G inside $GL(2, \mathbb{C})$. Obviously, N_G contains the centre $\mathbb{C}^*$ of $GL(2, \mathbb{C})$, and $\mathbb{C}^*$ acts on the graded ring $\mathbb{C}[x, y, z]/(z^2 - f(x, y))$ by multiplying homogeneous elements of degree d by t^d. Therefore H is a finite subgroup of the group H^* of graded automorphisms of the ring $\mathbb{C}[x, y, z]/(z^2 - f(x, y))$, which is determined as follows (compare [Cat87])

Theorem 3.6 *The group H^* of graded automorphisms of a RDP is:*

(1) $\mathbb{C}^*$ for E_8, E_7
(2) $\mathbb{C}^* \times \mathbb{Z}/2$ for $E_6, D_n (n \geq 5)$
(3) $\mathbb{C}^* \times \mathcal{S}_3$ for D_4

(4) $(\mathbb{C}^*)^2 \times \mathbb{Z}/2$ *for* $A_n (n \geq 2)$
(5) $GL(2, \mathbb{C})/\{\pm 1\}$ *for* A_1

Idea of proof . The case of A_1 is clear because $G = \{\pm 1\}$ is contained in the centre. In all the other cases, except D_4, y is the generator of smallest degree, therefore it is an eigenvector, and, up to using $\mathbb{C}^*$, we may assume that y is left invariant by an automorphism h. Some calculations allow to conclude that h is the identity in case (1), or the *trivial involution* $z \mapsto -z$ in case of E_6 and of D_n for n odd; while for D_n with n even the extra involution is $y \mapsto -y$.

Finally, for D_4, write the equation as $z^2 = y(x + iy)(x - iy)$ and permute the three lines which are the components of the branch locus. For A_n, one finds that the normalizer is the semidirect product of the diagonal torus with the involution given by $(u, v) \mapsto (v, u)$.

One may also derive the result from the symmetries of the Dynkin diagram. $\square$

3.2 Canonical Models of Surfaces of General Type

Assume now that S is a smooth minimal (projective) surface of general type.

We have (as an easy consequence of the Riemann Roch theorem) that S is minimal of general type if $K_S^2 > 0$ and K_S is nef (we recall that a divisor D is said to be *nef* if, for each irreducible curve C, we have $D \cdot C \geq 0$).

In fact, S is minimal of general type iff $K_S^2 > 0$ and K_S is nef. Since, if D is nef and, for $m > 0$, we write $|mD| = |M| + \Phi$ as the sum of its movable part and its fixed part, then $M^2 = m^2 D^2 - mD \cdot \Phi - M \cdot \Phi \leq m^2 D^2$. Hence, if $D^2 \leq 0$, the linear system $|mD|$ yields a rational map whose image has dimension at most 1.

Recall further that the Neron-Severi group $NS(S) = Div(S)/\sim$ is the group of divisors modulo numerical equivalence (D is numerically equivalent to 0, and we write $D \sim 0$, $\Leftrightarrow D \cdot C = 0$ for every irreducible curve C on S).

The Neron Severi group is a discrete subgroup of the vector space $H^1(\Omega_S^1)$, and indeed on a projective manifold Y it equals the intersection $(H^2(Y, \mathbb{Z})/\mathit{Torsion}) \cap H^{1,1}(Y)$.

By definition, the intersection form is non degenerate on the Neron Severi group, whose rank ρ is called the *Picard number*. But the Hodge index theorem implies the

Algebraic index theorem *The intersection form on* $NS(S)$ *has positivity index precisely* 1 *if* S *is an algebraic surface.*

The criterion of Nakai-Moishezon says that a divisor L on a surface S is ample if and only if $L^2 > 0$ and $L \cdot C > 0$ for each irreducible curve C on S. Hence:

The canonical divisor K_S *of a minimal surface of general type* S *is ample iff there does not exist an irreducible curve* C $(\neq 0)$ *on* S *with* $K \cdot C = 0$.

Remark 3.7 *Let S be a minimal surface of general type and C an irreducible curve on S with $K \cdot C = 0$. Then, by the index theorem, $C^2 < 0$ and by the adjunction formula we see that $2p(C) - 2 = K \cdot C + C^2 = C^2 < 0$.*

In general $p(C) := 1 - \chi(\mathcal{O}_C)$ is the arithmetic genus *of C, which is equal to the sum $p(C) = g(\tilde{C}) + \delta$ of the* geometric genus *of C, i.e., the genus of the normalization $p : \tilde{C} \to C$ of C, with the number δ of double points of C, defined as $\delta := h^0(p_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C)$.*

Therefore here $p(C) = 0$, so that $C \cong \mathbb{P}^1$, and $C^2 = -2$.

These curves are called (-2)-curves.

Thus K_S is not ample if and only if there exists a (-2)-curve on S. There is an upper bound for the number of these (-2)-curves.

Lemma 3.8 *Let $C_1, \dots, C_k$ be irreducible (-2)-curves on a minimal surface S of general type. We have:*

$$(\Sigma n_i C_i)^2 \leq 0,$$

and

$$(\Sigma n_i C_i)^2 = 0 \text{ if and only if } n_i = 0 \text{ for all } i.$$

Thus their images in the Neron-Severi group $NS(S)$ are independent and in particular $k \leq \rho - 1$ (ρ is the rank of $NS(S)$), and $k \leq h^1(\Omega^1_S) - 1$.

Proof. Let $\Sigma n_i C_i = C^+ - C^-$, ($C^+$ and C^- being effective divisors without common components) be the (unique) decomposition of $\Sigma n_i C_i$ in its positive and its negative part. Then $K \cdot C^+ = K \cdot C^- = 0$ and $C^+ \cdot C^- \geq 0$, whence $(C^+ - C^-)^2 = (C^+)^2 + (C^-)^2 - 2(C^+ \cdot C^-) \leq (C^+)^2 + (C^-)^2$. By the index theorem $(C^+)^2 + (C^-)^2$ is ≤ 0 and $= 0$ iff $C^+ = C^- = 0$. □

We can classify all possible configurations of (-2)-curves on a minimal surface S of general type by the following argument.

If C_1 and C_2 are two (-2)-curves on S, then:

$$0 > (C_1 + C_2)^2 = -4 + 2C_1 \cdot C_2,$$

hence $C_1 . C_2 \leq 1$, i.e., C_1 and C_2 intersect transversally in at most one point.

If C_1, C_2, C_3 are (-2)-curves on S, then again we have

$$0 > (C_1 + C_2 + C_3)^2 = 2(-3 + C_1 \cdot C_2 + C_1 \cdot C_3 + C_2 \cdot C_3).$$

Therefore no three curves meet in one point, nor do they form a triangle.

We associate to a configuration $\cup C_i$ of (-2)-curves on S its *Dynkin graph*: the vertices correspond to the (-2)-curves C_i, and two vertices (corresponding to C_i, C_j) are connected by an edge if and only if $C_i \cdot C_j = 1$.

Obviously the Dynkin graph of a configuration $\cup C_i$ is connected iff $\cup C_i$ is connected. So, let us assume that $\cup C_i$ is connected.

Theorem 3.9 *Let S be a minimal surface of general type and $\cup C_i$ a (connected) configuration of (-2)-curves on S. Then the associated (dual) Dynkin graph of $\cup C_i$ is one of those listed in Fig. 1.*

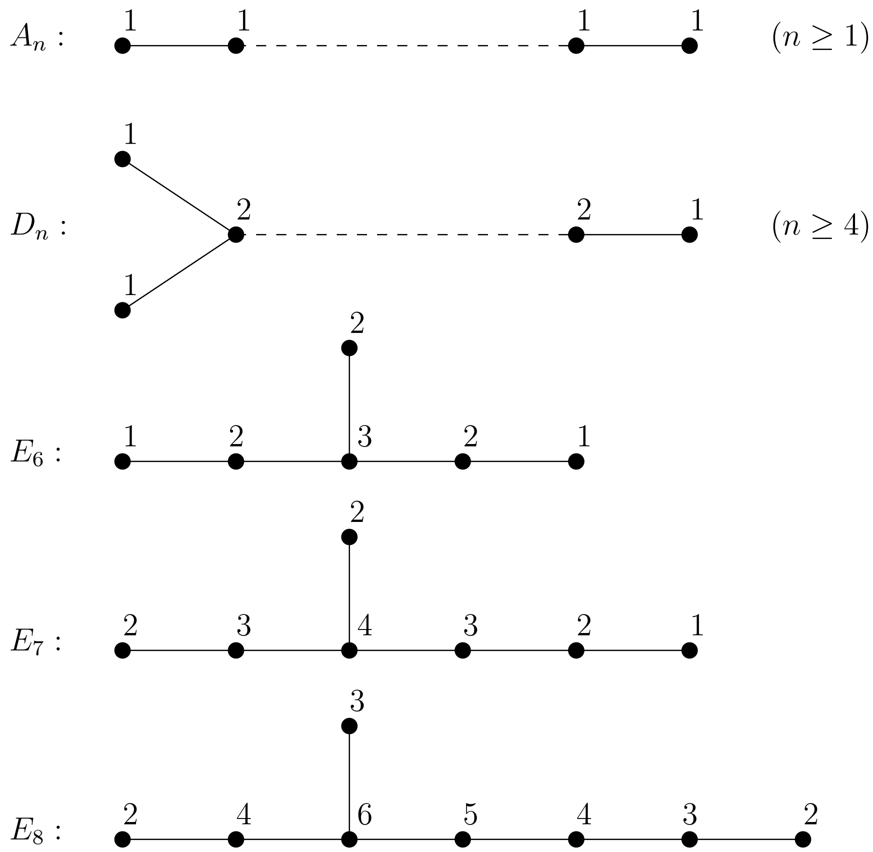

Fig. 1. The Dynkin-Diagrams of (-2)-curves configurations (the index n stands for the number of vertices, i.e. of curves). The labels for the vertices are the coefficients of the fundamental cycle

Remark 3.10 *The figure indicates also the weights n_i of the vertices of the respective trees. These weights correspond to a divisor, called fundamental cycle*

$$Z := \Sigma n_i C_i$$

defined (cf. [ArtM66]) by the properties

$$(**) \qquad Z \cdot C_i \leq 0 \text{ for all } \; i, \;\; Z^2 = -2, \text{ and } n_i > 0.$$

Idea of proof of 3.9. The simplest proof is obtained considering the above set of *Dynkin-Diagrams* $\mathcal{D} := \{A_n, D_n, E_6, E_7, E_8\}$ and the corresponding set of *Extended-Dynkin-Diagrams* $\tilde{\mathcal{D}} := \{\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$ which classify the *divisors of elliptic type* made of (-2)-curves and are listed in Fig. 2 (note that the divisors of elliptic type classify all the possible nonmultiple fibres

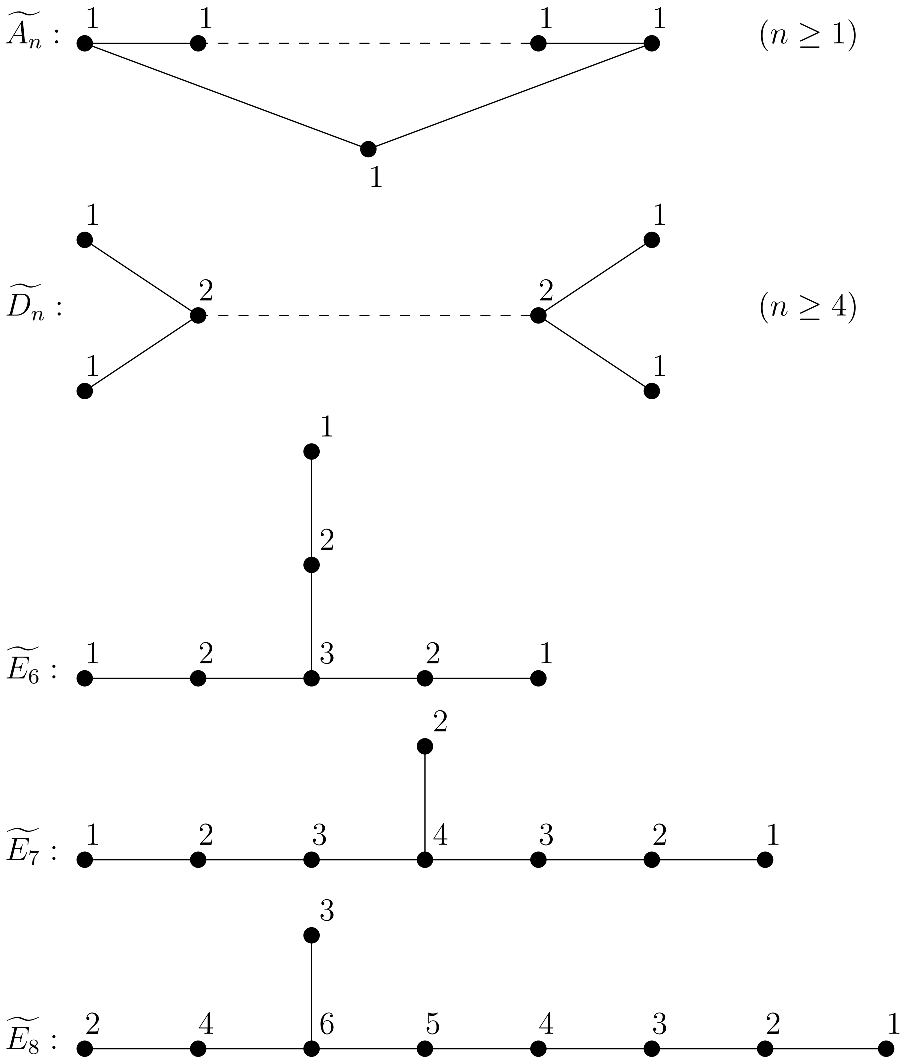

Fig. 2. The extended Dynkin-Diagrams of (-2)-curves configurations. The labels for the vertices are the coefficients of the divisor F of elliptic type

F of elliptic fibrations). Notice that each graph Γ in $\mathcal{D}$ is a subgraph of a corresponding graph $\tilde{\Gamma}$ in $\tilde{\mathcal{D}}$, obtained by adding exactly a (-2)-curve: $\Gamma = \tilde{\Gamma} - C_{\mathrm{end}}$. In this correspondence the fundamental cycle equals $Z = F - C_{\mathrm{end}}$ thus (**) is proven since $F \cdot C_i = 0$ for each i. Moreover, by Zariski's Lemma [BPV84] the intersection form on Γ is negative definite. If moreover Γ is a graph with a negative definite intersection form, then Γ does not contain as a subgraph a graph in $\tilde{\mathcal{D}}$, since $F^2 = 0$. The proof can now be easily concluded. $\square$

Artin [ArtM66] showed indeed that the above configurations can be holomorphically contracted to Rational Double Points, and that the fundamental cycle is indeed the inverse image of the maximal ideal in the local ring of the singularity. By applying these contractions to the minimal model S of a surface of general type one obtains in this way a normal surface X with Rational Double Points as singularities, called the *canonical model of S.*

We prefer however to sketch briefly how the canonical model is more directly obtained from the pluricanonical maps of S, and ultimately it can be defined as the Projective Spectrum (set of homogeneous prime ideals) of the canonical ring $\mathcal{R}(S)$. We need first of all Franchetta's theory of numerical connectedness.

Definition 3.11 *An effective divisor D is said to be m-connected if, each time we write $D = A + B$, with $A, B > 0$, then*

$$(*) \qquad A \cdot B \geq m.$$

Lemma 3.12 *Let D be a nef divisor on a smooth surface S, with $D^2 > 0$. Then, if D is effective, then D is 1-connected.*

Proof. Since D is nef,

$$A^2 + A \cdot B = D \cdot A \geq 0, B^2 + A \cdot B = D \cdot B \geq 0.$$

Assume $A \cdot B \leq 0$: then $A^2, B^2 \geq -(AB) \geq 0 \implies A^2 \cdot B^2 \geq (AB)^2$.

But, by the Index Theorem, $A^2 B^2 \leq (AB)^2$. Thus equality holds in the Index theorem $\iff \exists L$ such that $A \sim aL$, $B \sim bL$, $D \sim (a+b)L$. Moreover, since $D^2 > 0$ we have $L^2 \geq 1$, and we may assume $a, b > 0$ since A, B are effective. Thus $A \cdot B = a \cdot b \; L^2 \geq 1$, equality holding

$$\iff a = b = 1 (\implies D \sim 2L), L^2 = 1.$$

□

Remark 3.13 *Let $A \cdot B = 1$ and assume $A^2 B^2 < (AB)^2 \implies A^2 \cdot B^2 \leq 0$, but $A^2, B^2 \geq -1$. Thus, up to exchanging A and B, either $A^2 = 0$, and then $D \cdot A = 1, A^2 = 0$; or $A^2 > 0$, $B^2 = -1$, and then $D \cdot B = 0$, $B^2 = -1$.*

Hence the following

Corollary 3.14 *Let S be minimal of general type, $D \sim mK$, $m \geq 1$: then D is 2-connected except possibly if $K^2 = 1$, and $m = 2$, or $m = 1$, and $K \sim 2L$, $L^2 = 1$.*

Working a little more one finds

Proposition 3.15 *Let K be nef and big as before, $D \sim mK$ with $m \geq 2$. Then D is 3-connected except possibly if*

- $D = A + B, A^2 = -2,\ A \cdot K = 0\ (\implies A \cdot B = 2)$

- $m = 2, K^2 = 1, 2$
- $m = 3, K^2 = 1$

We use now the *Curve embedding Lemma* of [C-F-96], improved in [CFHR99] to the more general case of any curve C (i.e., a pure 1-dimensional scheme).

Lemma 3.16 *(Curve-embedding lemma) Let C be a curve contained in a smooth algebraic surface S, and let H be a divisor on C. Then H is very ample if, for each length 2 0-dimensional subscheme ζ of C and for each effective divisor $B \leq C$, we have*

$$\mathrm{Hom}(\mathcal{I}_\zeta, \omega_B(-H)) = 0.$$

In particular H is very ample on C if $\forall\ B \leq C$, $H \cdot B > 2p(B) - 2+$ length $\zeta \cap B$, where length $\zeta \cap B :=$colength $(\mathcal{I}_\zeta \mathcal{O}_B)$. A fortiori, H is very ample on C if, $\forall\ B \leq C$,

$$H \cdot B \geq 2p(B) + 1. \qquad (*)$$

Proof. It suffices to show the surjectivity $H^0(\mathcal{O}_C(H)) -\!\!>> H^0(\mathcal{O}_\zeta(H))$ $\forall$ such ζ. In fact, we can take either $\zeta = \{x, y\}$ [2 diff. points], or $\zeta = (x, \xi)$, ξ a tangent vector at x. The surjectivity is implied by $H^1(\mathcal{I}_\zeta \mathcal{O}_C(H)) = 0$.

By Serre-Grothendieck duality, and since $\omega_C = \mathcal{O}_C(K_S + C)$, we have, in case of nonvanishing, $0 \neq H^1(\mathcal{I}_\zeta \mathcal{O}_C(H))^\vee \cong \mathrm{Hom}(\mathcal{I}_\zeta \mathcal{O}_C(H), \mathcal{O}_C(K_S + C)) \ni \sigma \neq 0$.

Let Z be the maximal subdivisor of C such that σ vanishes on Z (i. e., $Z = \mathrm{div}(z)$, with $z|\sigma$) and let $B = V(\mathrm{Ann}(\sigma))$. Then $B + Z = C$ since, if $C = \{(\beta \cdot z) = 0\}$, $\mathrm{Ann}(\sigma) = (\beta)$.

Indeed, let $f \in \mathcal{I}_\zeta$ be a non zero divisor: then σ is identified with the rational function $\sigma = \frac{\sigma(f)}{f}$; we can lift everything to the local ring $\mathcal{O}_S$, then f is coprime with the equation $\gamma := (\beta z)$ of C, and $z = \mathrm{G.C.D.}(\sigma(f), \gamma)$. Clearly now $\mathrm{Ann}(\sigma) = \{u \mid u\sigma(f) \in (\beta z)\} = (\beta)$.

Hence σ induces

$$\hat{\sigma} := \frac{\sigma}{z} : \mathcal{I}_\zeta \mathcal{O}_B(H) \to \mathcal{O}_B(K_S + C - Z)$$

which is 'good' (i.e., it is injective and with finite cokernel), thus we get

$$0 \to \mathcal{I}_\zeta \mathcal{O}_B \xrightarrow{\hat{\sigma}} \mathcal{O}_B(K_B - H) \to \Delta \to 0$$

where supp(Δ) has dim $= 0$.
Then, taking the Euler Poincaré characteristics χ of the sheaves in question, we obtain

$$0 \leq \chi(\Delta) = \chi(\mathcal{O}_B(K_B - H)) - \chi(\mathcal{I}_\zeta \mathcal{O}_B) = -H \cdot B + 2p(B) - 2 + \mathit{length}(\zeta \cap B) < 0,$$

a contradiction. $\square$

The basic-strategy for the study of pluricanonical maps is then to find, for every length 2 subscheme of S, a divisor $C \in |(m-2)K|$ such that $\zeta \subset C$.

Since then, in characteristic $= 0$ we have the vanishing theorem [Ram72-4]

Theorem 3.17 (Kodaira, Mumford, Ramanujam) . *Let $L^2 > 0$ on S, L nef $\Longrightarrow H^i(-L) = 0, i \geq 1$. In particular, if S is minimal of general type, then $H^1(-K_S) = H^1(2K_S) = 0$.*

As shown by Ekedahl [Eke88] this vanishing theorem is false in positive characteristic, but only if char $= 2$, and for 2 very special cases of surfaces!

Corollary 3.18 *If $C \in |(m-2)K|$, then $H^0(\mathcal{O}_S(mK)) \twoheadrightarrow H^0(\mathcal{O}_C(mK))$. Therefore, $|mK_S|$ is very ample on S if $h^0((m-2)K_S) \geq 3$ and if the hypothesis on $H = mK_S$ in the curve embedding Lemma is verified for any $C \equiv (m-2)K$.*

We shall limit ourselves here to give the proof of a weaker version of Bombieri's theorem [Bom73]

Theorem on Pluricanonical-Embeddings. (Bombieri). (mK) is almost very ample (it embeds ζ except if $\exists\, B$ with $\zeta \subset B$, and $B \cdot K = 0$) if $m \geq 5$, $m = 4$ and $K^2 \geq 2$, $m = 3$, $p_g \geq 3$, $K^2 \geq 3$.

One first sees when $h^0((m-2)K) \geq 2$.

Lemma 3.19 *For $m \geq 3$ we have $h^0((m-2)K) \geq 3$ except if $m = 3$ $p_g \leq 2$, $m = 4$, $\chi = K^2 = 1$ (then $q = p_g = 0$) and ≥ 2 except if $m = 3$, $p_g \leq 1$.*

Proof. $p_g = H^0(K)$, so let us assume $m \geq 4$.

$$h^0((m-2)K) \geq \chi((m-2)K) \geq \chi + \frac{(m-2)(m-3)}{2}K^2.$$

Now, $\chi \geq 1$ and $K^2 \geq 1$, so we are done unless $m = 4$, $\chi = K^2 = 1$. □

The possibility that K_S may not be ample is contemplated in the following

Lemma 3.20 *Let $H = mK$, $B \leq C \equiv (m-2)K$ and assume $K \cdot B > 0$. Then*

$$H \cdot B \geq 2p(B) + 1 \text{ except possibly if}$$

(A) $\quad m = 4$ *and* $K^2 = 1$, *or* $m = 3$ *and* $K^2 \leq 2$.

Proof. Let $C = B + Z$ as above. Then we want

$$mK \cdot B \geq 2p(B) - 2 + 3 = (K+B) \cdot B + 3 = (K+C-Z) \cdot B + 3 = [(m-1)K - Z] \cdot B + 3,$$

i.e.,

$$K \cdot B + B \cdot Z \geq 3.$$

Since we assumed $K \cdot B \geq 1$, if $Z = 0$ we use $K^2 \geq 2$ if $m \geq 4$, and $K^2 \geq 3$ if $m = 3$, else it suffices to have $B \cdot Z \geq 2$, which is implied by the previous Corollary 3.14 (if $m = 3$, $B \sim Z \sim L$, $L^2 = 1$, then $K \cdot B = 2$). $\square$

Remark Note that then ζ is contracted iff $\exists\, B$ with $\zeta \subset B$, $K \cdot B = 0$! Thus, if there are no (-2) curves, the theorem says that we have an embedding of S. Else, we have a birational morphism which exactly contracts the fundamental cycles Z of S. To obtain the best technical result one has to replace the subscheme ζ by the subscheme $2Z$, and use that a fundamental cycle Z is 1-connected. We will not do it here, we simply refer to [CFHR99].

The following is the more precise theorem of Bombieri [Bom73]

Theorem 3.21 *Let S be a minimal surface of general type, and consider the linear system $|mK|$ for $m \geq 5$, for $m = 4$ when $K^2 \geq 2$, for $m = 3$ when $p_g \geq 3$, $K^2 \geq 3$.*

Then $|mK|$ yields a birational morphism onto its image, which is a normal surface X with at most Rational Double Points as singularities. For each singular point $p \in X$ the inverse image of the maximal ideal $\mathfrak{M}_p \subset \mathcal{O}_{X,p}$ is a fundamental cycle.

Here we sketch another way to look at the above surface X (called canonical model of S).

Proposition 3.22 *If S is a surface of general type the canonical ring $\mathcal{R}(S)$ is a graded $\mathbb{C}$-algebra of finite type.*

Proof. We choose a natural number such that $|mK|$ is without base points, and consider a pluricanonical morphism which is birational onto its image

$$\phi_m : S \to \Sigma_m = \Sigma \subset \mathbb{P}^N.$$

For $r = 0, \dots, m-1$, we set $\mathcal{F}_r := \phi_*(\mathcal{O}_S(rK))$.
The Serre correspondence (cf. [FAC55]) associates to $\mathcal{F}_r$ the module

$$M_r := \bigoplus_{i=1}^{\infty} H^0(\mathcal{F}_r(i)) = \bigoplus_{i=1}^{\infty} H^0(\phi_*(\mathcal{O}_S(rK))(i)) =$$

$$= \bigoplus_{i=1}^{\infty} H^0(\phi_*(\mathcal{O}_S((r+im)K))) = \bigoplus_{i=1}^{\infty} H^0(\mathcal{O}_S((r+im)K)) = \bigoplus_{i=1}^{\infty} \mathcal{R}_{r+\mathrm{im}}.$$

M_r is finitely generated over the ring $\mathcal{A} = \mathbb{C}[y_0, \dots, y_N]$, hence $\mathcal{R} = \bigoplus_{r=0}^{m-1} M_r$ is a finitely generated $\mathcal{A}$-module.

We consider the natural morphism $\alpha : \mathcal{A} \to \mathcal{R}$, $y_i \mapsto s_i \in \mathcal{R}_m$, (then the s_i generate a subring B of $\mathcal{R}$ which is a quotient of A). If $v_1, \dots, v_k$ generate $\mathcal{R}$ as a graded A-module, then $v_1, \dots, v_k$, $s_0, \dots, s_N$ generate $\mathcal{R}$ as a $\mathbb{C}$-algebra. $\square$

The relation between the *canonical ring* $\mathcal{R}(S, K_S)$ and the image of pluricanonical maps for $m \geq 5$ is then that $X = \mathrm{Proj}(\mathcal{R}(S, K_S))$.

In practice, since $\mathcal{R}$ is a finitely generated graded $\mathbb{C}$-algebra, generated by elements x_i of degree r_i, there is a surjective morphism

$$\lambda : \mathbb{C}[z_0, \dots, z_N] \twoheadrightarrow \mathcal{R},\ \lambda(z_i) = x_i.$$

If we decree that z_i has degree r_i, then λ is a graded surjective homomorphism of degree zero.

With this grading (where z_i has degree r_i) one defines (see [Dolg82]) the *weighted projective space* $\mathbb{P}(r_0, \dots r_n)$ as $\mathrm{Proj}(\mathbb{C}[z_0, \dots, z_N])$.

$\mathbb{P}(r_0, \dots r_n)$ is simply the quotient $:= \mathbb{C}^{N+1} - \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{N+1}$ in the following way:

$$t(z) = (z_0 t^{r_0}, \dots, z_N t^{r_N}).$$

The surjective homomorphism λ corresponds to an embedding of X into $\mathbb{P}(r_0, \dots r_n)$.

With the above notation, one can easily explain some classical examples which show that Bombieri's theorem is the best possible result.

Ex. 1: $m \geq 5$ is needed. Take a hypersurface $X_{10} \subset \mathbb{P}(1, 1, 2, 5)$ with Rational Double Points defined by a (weighted) homogeneous polynomial F_{10} of degree 10. Then $\omega_X = \mathcal{O}_X(10 - \Sigma e_i) = \mathcal{O}_X(1)$, $K_X^2 = 10/\prod e_i = 1$, and any m-canonical map with $m \leq 4$ is not birational.

In fact here the quotient ring $\mathbb{C}[y_0, y_1, x_3, z_5]/(F_{10})$, where $\deg y_i = 1, \deg x_3 = 2, \deg z_5 = 5$ is exactly the canonical ring $\mathcal{R}(S)$.

Ex. 2: $m = 3$, $K^2 = 2$ is also an exception.
Take $S = X_8 \subset \mathbb{P}(1, 1, 1, 4)$. Here S was classically described as a double cover $S \to \mathbb{P}^2$ branched on a curve B of degree 8 (since $F_8 = z^2 - f_8(x_0, x_1, x_2)$).

The canonical ring, since also here $\omega_S \cong \mathcal{O}_S(1)$, equals

$$\mathcal{R}(S) = \mathbb{C}[x_0, x_1, x_2, z]/(F_8).$$

Thus $p_g = 3$, $K^2 = 8/4 = 2$ but $|3K|$ factors through the double cover of $\mathbb{P}^2$.

3.3 Deformation Equivalence of Surfaces

The first important consequence of the theorem on pluricanonical embeddings is the finiteness, up to deformation, of the minimal surfaces S of general type with fixed invariants K^2 and χ.

In fact, their 5-canonical models Σ_5 are surfaces with Rational Double Points and of degree $25K^2$ in a fixed projective space $\mathbb{P}^N$, where $N + 1 = P_5 = h^0(5K_S) = \chi + 10K^2$.

In fact, the Hilbert polynomial of Σ_5 equals

$$P(m) := h^0(5mK_S) = \chi + \frac{1}{2}(5m - 1)5mK^2.$$

Grothendieck [Groth60] showed that there is

(i) An integer d and

(ii) A subscheme $\mathcal{H} = \mathcal{H}_P$ of the Grassmannian of codimension $P(d)$-subspaces of $H^0(\mathbb{P}^N, \mathcal{O}(d))$, called Hilbert scheme, such that

(iii) $\mathcal{H}$ parametrizes the degree d pieces $H^0(\mathcal{I}_\Sigma(d))$ of the homogeneous ideals of all the subschemes $\Sigma \subset \mathbb{P}^N$ having the given Hilbert polynomial P.

Inside $\mathcal{H}$ one has the open set

$$\mathcal{H}^0 := \{\Sigma | \Sigma \text{ is reduced with only R.D.P.'s as singularities}\}$$

and one defines

Definition 3.23 *The 5-pseudo moduli space of surfaces of general type with given invariants K^2, χ is the closed subset $\mathcal{H}_0 \subset \mathcal{H}^0$,*

$$\mathcal{H}_0(\chi, K^2) := \{\Sigma \in \mathcal{H}^0 | \omega_\Sigma^{\otimes 5} \cong \mathcal{O}_\Sigma(1)\}$$

Remark 3.24 *The group $\mathbb{P}GL(N+1, \mathbb{C})$ acts on $\mathcal{H}_0$ with finite stabilizers (corresponding to the groups of automorphisms of each surface) and the orbits correspond to the isomorphism classes of minimal surfaces of general type with invariants K^2, χ. A quotient by this action exists as a complex analytic space. Gieseker showed in [Gie77] that if one replaces the 5-canonical embedding by an m-canonical embedding with much higher m, then the corresponding quotient exists as a quasi-projective scheme.*

Since $\mathcal{H}_0$ is a quasi-projective scheme, it has a finite number of irreducible components (to be precise, these are the irreducible components of $(\mathcal{H}_0)_{red}$).

Definition 3.25 *The connected components of $\mathcal{H}_0(\chi, K^2)$ are called the* deformation types *of the surfaces of general type with given invariants K^2, χ.*

The above deformation types coincide with the equivalence classes for the relation of deformation equivalence (a more general definition introduced by Kodaira and Spencer), in view of the following:

Definition 3.26 *(1) A* deformation *of a compact complex space X is a pair consisting of*

(1.1) A flat morphism $\pi : \mathcal{X} \to T$ between connected complex spaces (i.e., $\pi^* : \mathcal{O}_{T,t} \to \mathcal{O}_{\mathcal{X},x}$ is a flat ring extension for each x with $\pi(x) = t$)

(1.2) An isomorphism $\psi : X \cong \pi^{-1}(t_0) := X_0$ of X with a fibre X_0 of π

(2) Two compact complex manifolds X, Y are said to be direct deformation equivalent *if there are a deformation $\pi : \mathcal{X} \to T$ of X with T irreducible and where all the fibres are smooth, and an isomorphism $\psi' : Y \cong \pi^{-1}(t_1) := X_1$ of Y with a fibre X_1 of π.*

(3) Two canonical models X, Y of surfaces of general type are said to be direct deformation equivalent *if there are a deformation $\pi : \mathcal{X} \to T$ of X where T is irreducible and where all the fibres have at most Rational Double*

Points as singularities , and an isomorphism $\psi' : Y \cong \pi^{-1}(t_1) := X_1$ *of* Y *with a fibre* X_1 *of* π.

(4) Deformation equivalence *is the equivalence relation generated by direct deformation equivalence.*

(5) A small deformation *is the germ* $\pi : (\mathcal{X}, X_0) \to (T, t_0)$ *of a deformation.*

(6) Given a deformation $\pi : \mathcal{X} \to T$ *and a morphism* $f : T' \to T$ *with* $f(t'_0) = t_0$*, the* pull-back $f^*(\mathcal{X})$ *is the fibre product* $\mathcal{X}' := \mathcal{X} \times_T T'$ *endowed with the projection onto the second factor* T' *(then* $X \cong X'_0$*).*

The two definitions (2) and (3) introduced above do not conflict with each other in view of the following

Theorem 3.27 *Given two minimal surfaces of general type* S, S' *and their respective canonical models* X, X'*, then*

S and S' are deformation equivalent (resp.: direct deformation equivalent) $\Leftrightarrow$ X and X' are deformation equivalent (resp.: direct deformation equivalent).

We shall highlight the idea of proof of the above proposition in the next subsection: we observe here that the proposition implies that the deformation equivalence classes of surfaces of general type correspond to the deformation types introduced above (the connected components of $\mathcal{H}_0$), since over $\mathcal{H}$ lies a natural family $\mathcal{X} \to \mathcal{H}$, $\mathcal{X} \subset \mathbb{P}^N \times \mathcal{H}$, and the fibres over $\mathcal{H}^0 \supset \mathcal{H}_0$ have at most RDP's as singularities.

A simple but powerful observation is that, in order to analyse deformation equivalence, one may restrict oneself to the case where $dim(T) = 1$: since two points in a complex space $T \subset \mathbb{C}^n$ belong to the same irreducible component of T if and only if they belong to an irreducible curve $T' \subset T$.

One may further reduce to the case where T is smooth simply by taking the normalization $T^0 \to T_{red} \to T$ of the reduction T_{red} of T, and taking the pull-back of the family to T^0.

This procedure is particularly appropriate in order to study the closure of subsets of the pseudomoduli space. But in order to show openness of certain subsets, the optimal strategy is to consider the small deformations of the canonical models (this is like Columbus' egg: the small deformations of the minimal models are sometimes too complicated to handle, as shown by Burns and Wahl [B-W74] already for surfaces in $\mathbb{P}^3$).

The basic tool is the generalization due to Grauert of Kuranishi's theorem ([Gra74], see also [Sern06], cor. 1.1.11 page 18, prop. 2.4.8 , page 70)

Theorem 3.28 (Grauert's Kuranishi type theorem for complex spaces) *Let* X *be a compact complex space: then*

(I) there is a semiuniversal deformation $\pi : (\mathcal{X}, X_0) \to (T, t_0)$ *of* X*, i.e., a deformation such that every other small deformation* $\pi' : (\mathcal{X}', X'_0) \to (T', t'_0)$ *is the pull-back of* π *for an appropriate morphism* $f : (T', t'_0) \to (T, t_0)$ *whose*

differential at t'_0 *is uniquely determined. (II)* (T, t_0) *is unique up to isomorphism, and is a germ of analytic subspace of the vector space* $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$, *inverse image of the origin under a local holomorphic map (called obstruction map and denoted by ob)* $ob : \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to T^2(X)$ *whose differential vanishes at the origin (the point corresponding to the point* t_0*).*

The obstruction space $T^2(X)$ *equals* $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ *if* X *is a local complete intersection.*

The theorem of Kuranishi [Kur62, Kur65] dealt with the case of compact complex manifolds, and in this case $\mathrm{Ext}^j(\Omega^1_X, \mathcal{O}_X) \cong H^j(X, \Theta_X)$, where $\Theta_X := Hom(\Omega^1_X, \mathcal{O}_X)$ is the sheaf of holomorphic vector fields. In this case the quadratic term in the Taylor development of *ob*, given by the cup product $H^1(X, \Theta_X) \times H^1(X, \Theta_X) \to H^2(X, \Theta_X)$, is easier to calculate.

3.4 Isolated Singularities, Simultaneous Resolution

The main reason in the last subsection to consider deformations of compact complex spaces was the aim to have a finite dimensional base T for the semiuniversal deformation (this would not have been the case in general).

Things work in a quite parallel way if one considers germs of isolated singularities of complex spaces (X, x_0). The definitions are quite similar, and there is an embedding $\mathcal{X} \to \mathbb{C}^n \times T$ such that π is induced by the second projection. There is again a completely similar general theorem by Grauert ([Gra72] and again see [Sern06], cor. 1.1.11 page 18, prop. 2.4.8 , page 70)

Theorem 3.29 (Grauert's theorem for deformations of isolated singularities) *Let* (X, x_0) *be a germ of an isolated singularity of a complex space: then*

(I) There is a semiuniversal deformation $\pi : (\mathcal{X}, X_0, x_0) \to (\mathbb{C}^n, 0) \times (T, t_0)$ *of* X*, i.e., a deformation such that every other small deformation* $\pi' : (\mathcal{X}', X'_0, x'_0) \to (\mathbb{C}^n, 0) \times (T', t'_0)$ *is the pull-back of* π *for an appropriate morphism* $f : (T', t'_0) \to (T, t_0)$ *whose differential at* t'_0 *is uniquely determined.*

(II) (T, t_0) *is unique up to isomorphism, and is a germ of analytic subspace of the vector space* $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X)$, *inverse image of the origin under a local holomorphic map (called obstruction map and denoted by ob)* $ob : \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to T^2(X)$ *whose differential vanishes at the origin (the point corresponding to the point* t_0*).*

The obstruction space $T^2(X)$ *equals* $\mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ *if* X *is a local complete intersection.*

One derives easily from the above a previous result of G. Tjurina concerning the deformations of isolated hypersurface singularities.

For, assume that $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ is the zero set of a holomorphic function f, $X = \{z | f(z) = 0\}$ and therefore, if $f_j = \frac{\partial f}{\partial z_j}$, the origin is the only point in the locus $\mathcal{S} = \{z | f_j(z) = 0 \ \forall j\}$.

We have then the exact sequence

$$0 \to \mathcal{O}_X \stackrel{(f_j)}{\to} \mathcal{O}_X^{n+1} \to \Omega_X^1 \to 0$$

which yields $\mathrm{Ext}^j(\Omega_X^1, \mathcal{O}_X) = 0$ for $j \geq 2$, and

$$\mathrm{Ext}^1(\Omega_X^1, \mathcal{O}_X) \cong \mathcal{O}_{\mathbb{C}^{n+1},0}/(f, f_1, \dots f_{n+1}) := T^1.$$

In this case the basis of the semiuniversal deformation is just the vector space T^1, called the *Tjurina Algebra*, and one obtains the following

Corollary 3.30 (Tjurina's deformation) *Given $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ an isolated hypersurface singularity $X = \{z | f(z) = 0\}$, let $g_1, \dots g_\tau$ be a basis of the Tjurina Algebra $T^1 = \mathcal{O}_{\mathbb{C}^{n+1},0}/(f, f_1, \dots f_{n+1})$ as a complex vector space.*

Then $\mathcal{X} \subset \mathbb{C}^{n+1} \times \mathbb{C}^\tau$, $\mathcal{X} := \{z | F(z,t) := f(z) + \sum_j t_j g_j(z) = 0\}$ is the semiuniversal deformation of $(X, 0)$.

A similar result holds more generally (with the same proof) when X is a complete intersection of r hypersurfaces $X = \{z | \phi_1(z) = \dots = \phi_r(z) = 0\}$, and then one has a semiuniversal deformation of the form $\mathcal{X} \subset \mathbb{C}^{n+1} \times \mathbb{C}^\tau$, $\mathcal{X} := \{z | F_i(z,t) := \phi_i(z) + \sum_j t_j G_{i,j}(z) = 0, i = 1, \dots r\}$.

In both cases the singularity admits a so-called *smoothing*, given by the Milnor fibre (cf. [Mil68])

Definition 3.31 *Given a hypersurface singularity $(X, 0)$, $X = \{z | f(z) = 0\}$, the Milnor fibre $\mathfrak{M}_{\delta,\epsilon}$ is the intersection of the hypersurface $\{z | f(z) = \epsilon\}$ with the ball $\overline{B(0,\delta)}$ with centre the origin and radius $\delta << 1$, when $|\epsilon| << \delta$.*

$\mathfrak{M} := \mathfrak{M}_{\delta,\epsilon}$ is a manifold with boundary whose diffeomorphism type is independent of ϵ, δ when $|\epsilon| << \delta << 1$.

More generally, for a complete intersection, the Milnor fibre is the intersection of the ball $\overline{B(0,\delta)}$ with centre the origin and radius $\delta << 1$ with a smooth level set $X_\epsilon := \{z | \phi_1(z) = \epsilon_1, \dots \phi_r(z) = \epsilon_r\}$.

Remark 3.32 *Milnor defined the Milnor fibre $\mathfrak{M}$ in a different way, as the intersection of the sphere $S(0,\delta)$ with centre the origin and radius $\delta << 1$ with the set $\{z | f(z) = \eta |f(z)|\}$, for $|\eta| = 1$.*

In this way the complement $S(0,\delta) \backslash X$ is fibred over S^1 with fibres diffeomorphic to the interiors of the Milnor fibres; using Morse theory Milnor showed that $\mathfrak{M}$ has the homotopy type of a bouquet of μ spheres of dimension n, where μ, called the Milnor number*, is defined as the dimension of the* Milnor algebra $M^1 = \mathcal{O}_{\mathbb{C}^{n+1},0}/(f_1, \dots f_{n+1})$ *as a complex vector space.*

The Milnor algebra and the Tjurina algebra coincide in the case of a weighted homogeneous singularity (this means that there are weights $m_0, \dots m_n$ such that f contains only monomials $z_0^{i_0} \dots z_n^{i_n}$ of weighted degree $\sum_j i_j m_j = d$), by Euler's rule $\sum_j m_j z_j f_j = df$.

This is the case, for instance, for the Rational Double Points, the singularities which occur on the canonical models of surfaces of general type. Moreover, for these, the Milnor number μ is easily seen to coincide with the index i in the label for the singularity (i.e., $i = n$ for an A_n-singularity), which in turn corresponds to the number of vertices of the corresponding Dynkin diagram.

Therefore, by the description we gave of the minimal resolution of singularities of a RDP, we see that this is also homotopy equivalent to a bouquet of μ spheres of dimension 2. This is in fact no accident, it is just a manifestation of the fact that there is a so-called simultaneous resolution of singularities (cf. [Tju70, Briesk68-b, Briesk71])

Theorem 3.33 (Simultaneous resolution according to Brieskorn and Tjurina) *Let $T := \mathbb{C}^\mu$ be the basis of the semiuniversal deformation of a Rational Double Point $(X, 0)$. Then there exists a finite ramified Galois cover $T' \to T$ such that the pull-back $\mathcal{X}' := \mathcal{X} \times_T T'$ admits a simultaneous resolution of singularities $p : \mathcal{S}' \to \mathcal{X}'$ (i.e., p is bimeromorphic and all the fibres of the composition $\mathcal{S}' \to \mathcal{X}' \to T'$ are smooth and equal, for t'_0, to the minimal resolution of singularities of $(X, 0)$.*

We shall give Tjurina' s proof for the case of A_n-singularities.

Proof. Assume that we have the A_n-singularity

$$\{(x, y, z) \in \mathbb{C}^3 | xy = z^{n+1}\}.$$

Then the semiuniversal deformation is given by

$$\mathcal{X} := \{((x, y, z), (a_2, \dots a_{n+1})) \in \mathbb{C}^3 \times \mathbb{C}^n | xy = z^{n+1} + a_2 z^{n-1} + \dots a_{n+1}\},$$

the family corresponding to the natural deformations of the simple cyclic covering.

We take a ramified Galois covering with group $\mathcal{S}_{n+1}$ corresponding to the splitting polynomial of the deformed degree $n + 1$ polynomial

$$\mathcal{X}' := \{((x, y, z), (\alpha_1, \dots \alpha_{n+1})) \in \mathbb{C}^3 \times \mathbb{C}^{n+1} | \sum_j \alpha_j = 0,\ xy = \prod_j (z - \alpha_j)\}.$$

One resolves the new family $\mathcal{X}'$ by defining $\phi_i : \mathcal{X}' \dashrightarrow \mathbb{P}^1$ as

$$\phi_i := (x, \prod_{j=1}^{i} (z - \alpha_j))$$

and then taking the closure of the graph of $\Phi := (\phi_1, \dots \phi_n) : \mathcal{X}' \dashrightarrow (\mathbb{P}^1)^n$. $\square$

We shall consider now in detail the case of a node, i.e., an A_1 singularity. This singularity and its simultaneous resolution was considered also in the course by Seidel, and will occur once more when dealing with Lefschetz pencils (but then in lower dimension).

Example 3.34 *Consider a node, i.e., an A_1 singularity.*

Here, we write $f = z^2 - x^2 - y^2$, and the total space of the semiuniversal deformation $\mathcal{X} = \{(x,y,z,t) | f - t = 0\} = \{(x,y,z,t) | z^2 - x^2 - y^2 = t\}$ is smooth. The base change $t = w^2$ produces a quadratic nondegenerate singularity at the origin for $\mathcal{X}' = \{(x,y,z,w) | z^2 - x^2 - y^2 = w^2\} = \{(x,y,z,w) | z^2 - x^2 = y^2 + w^2\}$.

The closure of the graph of $\psi := \frac{z-x}{w+iy} = \frac{w-iy}{z+x}$ yields a so-called small resolution, replacing the origin by a curve isomorphic to $\mathbb{P}^1$.

In the Arbeitstagung of 1958 Michael Atiyah made the observation that this procedure is nonunique, since one may also use the closure of the rational map $\tilde{\psi} := \frac{z-x}{w-iy} = \frac{w+iy}{z+x}$ to obtain another small resolution. An alternative way to compare the two resolutions is to blow up the origin, getting the big resolution (with exceptional set $\mathbb{P}^1 \times \mathbb{P}^1$) and view each of the two small resolutions as the contraction of one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$.

Atiyah showed in this way (see also [BPV84]) that the moduli space for K3 surfaces is non Hausdorff.

Remark 3.35 *The first proof of Theorem 3.33 was given by G. Tjurina. It had been observed that the Galois group G of the covering $T' \to T$ in the above theorem is the Weyl group corresponding to the Dynkin diagram of the singularity, defined as follows. If $\mathcal{G}$ is the simple algebraic group corresponding to the Dynkin diagram (see [Hum75]), and H is a Cartan subgroup, N_H its normalizer, then the Weyl group is the factor group $W := N_H/H$. For example, A_n corresponds to the group $SL(n+1, \mathbb{C})$, its Cartan subgroup is the subgroup of diagonal matrices, which is normalized by the symmetric group $\mathcal{S}_{n+1}$, and N_H is here a semidirect product of H with $\mathcal{S}_{n+1}$.*

As we already mentioned, E. Brieskorn [Briesk71] found a direct explanation of this interesting phenomenon, according to a conjecture of Grothendieck. He proved that an element $x \in \mathcal{G}$ is unipotent and subregular iff the morphism $\Psi : \mathcal{G} \to H/W$, sending x to the conjugacy class of its semisimple part x_s, factors around x as the composition of a submersion with the semiuniversal deformation of the corresponding RDP singularity.

With the aid of Theorem 3.33 we can now prove that deformation equivalence for minimal surfaces of general type is the same as restricted deformation equivalence for their canonical models (i.e., one allows only deformations whose fibres have at most canonical singularities).

Idea of the Proof of Theorem 3.27.

It suffices to observe that

(0) if we have a family $p \colon \mathcal{S} \to \Delta$ where $\Delta \subset \mathbb{C}$ is the unit disk, and the fibres are smooth surfaces, if the central fibre is minimal of general type, then so are all the others.

(1) If we have a family $p : \mathcal{S} \to \Delta$, where $\Delta \subset \mathbb{C}$ is the unit disk, and the fibres are smooth minimal surfaces of general type, then their canonical models form a flat family $\pi : \mathcal{X} \to \Delta$.

(2) If we have a flat family $\pi : \mathcal{X} \to \Delta$ whose fibres X_t have at most Rational Double Points and K_{X_t} is ample, then for each $t \in \Delta$ there is a ramified covering $f : (\Delta, 0) \to (\Delta, t)$ such that the pull back $f^*\mathcal{X}$ admits a simultaneous resolution.

(0) is a consequence of Kodaira's theorem on the stability of -1-curves by deformation (see [Kod63-b]) and of the two following facts:

(i) that a minimal surface S with $K_S^2 > 0$ is either of general type, or isomorphic to $\mathbb{P}^2$ or to a Segre-Hirzebruch surface $\mathbb{F}_n$ ($n \neq 1$, $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$)

(ii) that $\mathbb{P}^2$ is rigid (every deformation of $\mathbb{P}^2$ is a product), while $\mathbb{F}_n$ deforms only to $\mathbb{F}_m$, with $n \equiv m$ $(mod\ 2)$.

(2) is essentially the above quoted theorem, (1) is a consequence of Bombieri's theorem, since $p_*(\mathcal{O}_{\mathcal{X}}(5K_{\mathcal{X}})$ is generated by global sections and a trivialization of this sheaf provides a morphism $\phi : \mathcal{X} \to \Delta \times \mathbb{P}^N$ which induces the 5-canonical embedding on each fibre. □

We end this section by describing the results of Riemenschneider [Riem74] on the semiuniversal deformation of the quotient singularity $\frac{1}{4}(1,1)$ described in Example 3.3, and a generalization thereof.

More generally, Riemenschneider considers the singularity Y_{k+1}, a quotient singularity of the RDP (Rational Double Point) A_{2k+1} $\{uv - z^{2k+2} = 0\}$ by the involution multiplying (u, v, z) by -1. Indeed, this is a quotient singularity of type $\frac{1}{4k+4}(1, 2k+1)$, and the A_{2k+1} singularity is the quotient by the subgroup $2\mathbb{Z}/(4k+4)\mathbb{Z}$.

We use here the more general concept of Milnor fibre of a smoothing which the reader can find in Definition 4.5.

Theorem 3.36 (Riemenschneider) *The basis of the semiuniversal deformation of the singularity Y_{k+1}, quotient of the RDP A_{2k+1} by multiplication by -1, consists of two smooth components T_1, T_2 intersecting transversally. Both components yield smoothings, but only the smoothing over T_1 admits a simultaneous resolution. The Milnor fibre over T_1 has Milnor number $\mu = k+1$, the Milnor fibre over T_2 has Milnor number $\mu = k$.*

For the sake of simplicity, we shall explicitly describe the two families in the case $k = 0$ of the quotient singularity $\frac{1}{4}(1,1)$ described in Example 3.3. We use for this the two determinantal presentations of the singularity.

(1) View the singularity as $\mathbb{C}[y_0, \dots, y_4]/J$, where J is the ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_5 & y_6 & y_7 & y_4 \end{pmatrix}$ and by the three functions $f_i := y_i - y_{4+i}$, for $i = 1, 2, 3$ (geometrically, this amounts to viewing the rational normal curve of degree 4 as a linear section of the Segre four-fold $\mathbb{P}^1 \times \mathbb{P}^3$). We get the family T_1, with base $\mathbb{C}^3$, by changing the level sets of the three functions f_i , $f_i(y) = t_i$, for $t = (t_1, t_2, t_3) \in \mathbb{C}^3$.

(2) View the singularity as $\mathbb{C}[y_0, \dots, y_4]/I$, where I is the ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_5 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix}$ and by the function $f := y_5 - y_2$.

In this second realization the cone over a rational normal curve of degree 4 (in $\mathbb{P}^4$) is viewed as a linear section of the cone over the Veronese surface.

We get the family T_2, with base $\mathbb{C}$, by changing the level set of the function f, $y_5 - y_2 = t$, for $t \in \mathbb{C}$.

We see in the latter case that the Milnor fibre is just the complement to a smooth conic in the complex projective plane $\mathbb{P}^2$, therefore its Milnor number (equal by definition to the second Betti number) is equal to 0. Indeed the Milnor fibre is homotopically equivalent to the real projective plane, but this is better seen in another way which allows a great generalization.

In fact, as we already observed, the singularities Y_k are a special case $(n = 2, d = k + 1, a = 1)$ of the following

$$\textit{Cyclic quotient singularities } \frac{1}{dn^2}(1, dna - 1) = A_{dn-1}/\mu_n.$$

These are quotients of $\mathbb{C}^2$ by a cyclic group of order dn^2 acting with the indicated characters $(1, dna - 1)$, but can also be viewed as quotients of the Rational Double Point A_{dn-1} of equation $uv - z^{dn} = 0$ by the action of the group μ_n of n-roots of unity acting in the following way:

$$\xi \in \mu_n \text{ acts by} : (u, v, z) \to (\xi u, \xi^{-1} v, \xi^a z).$$

This quotient action gives rise to a quotient family $\mathcal{X} \to \mathbb{C}^d$, where $\mathcal{X} = \mathcal{Y}/\mu_n$, $\mathcal{Y}$ is the hypersurface in $\mathbb{C}^3 \times \mathbb{C}^d$ of equation

$$(***) \quad uv - z^{dn} = \Sigma_{k=0}^{d-1} t_k z^{kn}$$

and the action of μ_n is extended trivially on the factor $\mathbb{C}^d$.

We see in this way that the Milnor fibre is the quotient of the Milnor fibre of the Rational Double Point A_{dn-1} by a cyclic group of order n acting freely. In particular, in the case $n = 2, d = 1, a = 1$, it is homotopically equivalent to the quotient of S^2 by the antipodal map, and we get $\mathbb{P}^2_{\mathbb{R}}$.

Another important observation is that $\mathcal{Y}$, being a hypersurface, is Gorenstein (this means that the canonical sheaf $\omega_{\mathcal{Y}}$ is invertible). Hence, such a quotient $\mathcal{X} = \mathcal{Y}/\mu_n$ by an action which is unramified in codimension 1, is (by definition) $\mathbb{Q}$-Gorenstein.

Remark 3.37 *These smoothings were considered by Kollár and Shepherd Barron ([K-SB88], 3.7-3.8-3.9, cf. also [Man90]), who pointed out their relevance in the theory of compactifications of moduli spaces of surfaces, and showed that, conversely, any $\mathbb{Q}$-Gorenstein smoothing of a quotient singularity is induced by the above family (which has a smooth base, $\mathbb{C}^d$).*

Returning to the cyclic quotient singularity $\frac{1}{4}(1, 1)$, the first description that we gave of the $\mathbb{Q}$-Gorenstein smoothing (which does obviously not admit a simultaneous resolution since its Milnor number is 0) makes clear that an alternative way is to view the singularity (cf. Example 3.3) as a bidouble cover of the plane branched on three lines passing through the origin, and then this smoothing (T_2) is simply obtained by deforming these three lines till they meet in three distinct points.

4 Lecture 3: Deformation and Diffeomorphism, Canonical Symplectic Structure for Surfaces of General Type

Summarizing some of the facts we saw up to now, given a birational equivalence class of surfaces of general type, this class contains a unique (complete) smooth minimal surface S, called the *minimal model*, such that K_S is nef ($K_S \cdot C \geq 0$ for every effective curve C); and a unique surface X with at most Rational Double Points as singularities, and such that the invertible sheaf ω_X is ample, called the *canonical model*.

S is the minimal resolution of the singularities of X, and every pluricanonical map of S factors through the projection $\pi : S \to X$.

The basic numerical invariants of the birational class are $\chi := \chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) = 1 - q + p_g$ ($p_g = h^0(\mathcal{O}_S(K_S)) = h^0(\omega_X)$) and $K_S^2 = K_X^2$ (here K_X is a Cartier divisor such that $\omega_X \cong \mathcal{O}_X(K_X)$).

The totality of the canonical models of surfaces with fixed numerical invariants $\chi = x, K^2 = y$ are parametrized (not uniquely, because of the action of the projective group) by a quasi projective scheme $\mathcal{H}_0(x, y)$, which we called the *pseudo moduli space*.

The connected components of the pseudo moduli spaces $\mathcal{H}_0(x, y)$ are the deformation types of the surfaces of general type, and a basic question is whether one can find some invariant to distinguish these. While it is quite easy to find invariants for the irreducible components of the pseudo moduli space, just by using the geometry of the fibre surface over the generic point, it is less easy to produce effective invariants for the connected components. Up to now the most effective invariant to distinguish connected components has been the *divisibility index* r of the canonical class (r is the divisibility of $c_1(K_S)$ in $H^2(S, \mathbb{Z})$) (cf. [Cat86])

Moreover, as we shall try to illustrate more amply in the next lecture, there is another fundamental difference between the curve and the surface case. Given a curve, the genus g determines the topological type, the differentiable type, and the deformation type, and the moduli space $\mathfrak{M}_g$ is irreducible.

In the case of surfaces, the pseudo moduli space $\mathcal{H}_0(x, y)$ is defined over $\mathbb{Z}$, whence the absolute Galois group $Gal(\bar{\mathbb{Q}}, \mathbb{Q})$ operates on it. In fact, it operates by possibly changing the topology of the surfaces considered, in particular the fundamental group may change !

Therefore the algebro-geometric study of moduli spaces cannot be reduced only to the study of isomorphism classes of complex structures on a fixed differentiable manifold.

We shall now recall how the deformation type determines the differentiable type, and later we shall show that each surface of general type S has a symplectic structure (S, ω), unique up to symplectomorphism, such that the cohomology class of ω is the canonical class $c_1(K_S)$.

4.1 Deformation Implies Diffeomorphism

Even if well known, let us recall the theorem of Ehresmann [Ehr43]

Theorem 4.1 (Ehresmann) *Let $\pi : \mathcal{X} \to T$ be a proper submersion of differentiable manifolds with T connected: then π is a differentiable fibre bundle, in particular all the fibre manifolds X_t are diffeomorphic to each other.*

The idea of the proof is to endow $\mathcal{X}$ with a Riemannian metric, so that a local vector field ξ on the base T has a unique differentiable lifting which is orthogonal to the fibres. Then, in the case where T has dimension 1, one integrates the lifted vector field. The general case is proven by induction on $dim_{\mathbb{R}} T$.

The same argument allows a variant with boundary of Ehresmann's theorem

Lemma 4.2 *Let $\pi : \mathcal{M} \to T$ be a proper submersion where $\mathcal{M}$ is a differentiable manifold with boundary, such that also the restriction of π to $\partial\mathcal{M}$ is a submersion. Assume that T is a ball in $\mathbb{R}^n$, and assume that we are given a fixed trivialization ψ of a closed family $\mathcal{N} \to T$ of submanifolds with boundary. Then we can find a trivialization of $\pi : \mathcal{M} \to T$ which induces the given trivialization ψ.*

Proof. It suffices to take on $\mathcal{M}$ a Riemannian metric where the sections $\psi(p, T)$, for $p \in \mathcal{N}$, are orthogonal to the fibres of π. Then we use the customary proof of Ehresmann's theorem, integrating liftings orthogonal to the fibres of standard vector fields on T. $\square$

Ehresmann's theorem implies then the following

Proposition 4.3 *Let X, X' be two compact complex manifolds which are deformation equivalent. Then they are diffeomorphic by a diffeomorphism $\phi : X' \to X$ preserving the canonical class (i.e., such that $\phi^* c_1(K_X) = c_1(K_{X'})$).*

Proof. The result follows by induction once it is established for X, X' fibres of a family $\pi : \mathcal{X} \to \Delta$ over a 1-dimensional disk. Ehresmann's theorem provides a differentiable trivialization $\mathcal{X} \cong X \times \Delta$. Notice that, since the normal bundle to a fibre is trivial, the canonical divisor of a fibre K_{X_t} is the restriction of the canonical divisor $K_{\mathcal{X}}$ to X_t. It follows that the trivialization provides a diffeomorphism ϕ which preserves the canonical class. $\square$

Remark 4.4 *Indeed, by the results of Seiberg Witten theory, an arbitrary diffeomorphism between differentiable 4-manifolds carries $c_1(K_X)$ either to $c_1(K_{X'})$ or to $-c_1(K_{X'})$ (cf. [Wit94] or [Mor96]). Thus deformation equivalence imposes only ϵ more than diffeomorphism only.*

4.2 Symplectic Approximations of Projective Varieties with Isolated Singularities

The variant 4.2 of Ehresmann's theorem will now be first applied to the Milnor fibres of smoothings of isolated singularities.

Let (X, x_0) be the germ of an isolated singularity of a complex space, which is pure dimensional of dimension $n = dim_{\mathbb{C}} X$, assume $x_0 = 0 \in X \subset \mathbb{C}^{n+m}$, and consider as above the ball $\overline{B(x_0, \delta)}$ with centre the origin and radius δ. Then, for all $0 < \delta << 1$, the intersection $\mathcal{K}_0 := X \cap S(x_0, \delta)$, called the *link* of the singularity, is a smooth manifold of real dimension $2n - 1$.

Consider the semiuniversal deformation $\pi : (\mathcal{X}, X_0, x_0) \to (\mathbb{C}^{n+m}, 0) \times (T, t_0)$ of X and the family of singularity links $\mathcal{K} := \mathcal{X} \cap (S(x_0, \delta) \times (T, t_0))$. By a uniform continuity argument it follows that $\mathcal{K} \to T$ is a trivial bundle if we restrict T suitably around the origin t_0 (it is a differentiably trivial fibre bundle in the sense of stratified spaces, cf. [Math70]).

We can now introduce the concept of Milnor fibres of (X, x_0).

Definition 4.5 *Let (T, t_0) be the basis of the semiuniversal deformation of a germ of isolated singularity (X, x_0), and let $T = T_1 \cup \cdots \cup T_r$ be the decomposition of T into irreducible components. T_j is said to be a smoothing component if there is a $t \in T_j$ such that the corresponding fibre X_t is smooth. If T_j is a smoothing component, then the corresponding Milnor fibre is the intersection of the ball $\overline{B(x_0, \delta)}$ with the fibre X_t, for $t \in T_j$, $|t| < \eta << \delta << 1$.*

Whereas the singularity links form a trivial bundle, the Milnor fibres form only a differentiable bundle of manifolds with boundary over the open set $T_j^0 := \{t \in T_j, |t - t_0| < \eta| \ X_t \text{ is smooth}\}$.

Since however T_j is irreducible, T_j^0 is connected, and the Milnor fibre is unique up to smooth isotopy, in particular up to diffeomorphism.

We shall now apply again Lemma 4.2 in order to perform some surgeries to projective varieties with isolated singularities.

Theorem 4.6 *Let $X_0 \subset \mathbb{P}^N$ be a projective variety with isolated singularities admitting a smoothing component.*

Assume that for each singular point $x_h \in X$, we choose a smoothing component $T_{j(h)}$ in the basis of the semiuniversal deformation of the germ (X, x_h). Then (obtaining different results for each such choice) X can be approximated by symplectic submanifolds W_t of $\mathbb{P}^N$, which are diffeomorphic to the glueing of the 'exterior' of X_0 (the complement to the union $B = \cup_h B_h$ of suitable (Milnor) balls around the singular points) with the Milnor fibres $\mathcal{M}_h$, glued along the singularity links $\mathcal{K}_{h,0}$.

A pictorial view of the proof is contained in Fig. 3.

Proof.

First of all, for each singular point $x_h \in X$, we choose a holomorphic path $\Delta \to T_{j(h)}$ mapping 0 to the distinguished point corresponding to the germ (X, x_h), and with image of $\Delta \backslash 0$ inside the smoothing locus $T_{j(h)}^0 \cap \{t||t| < \eta\}$.

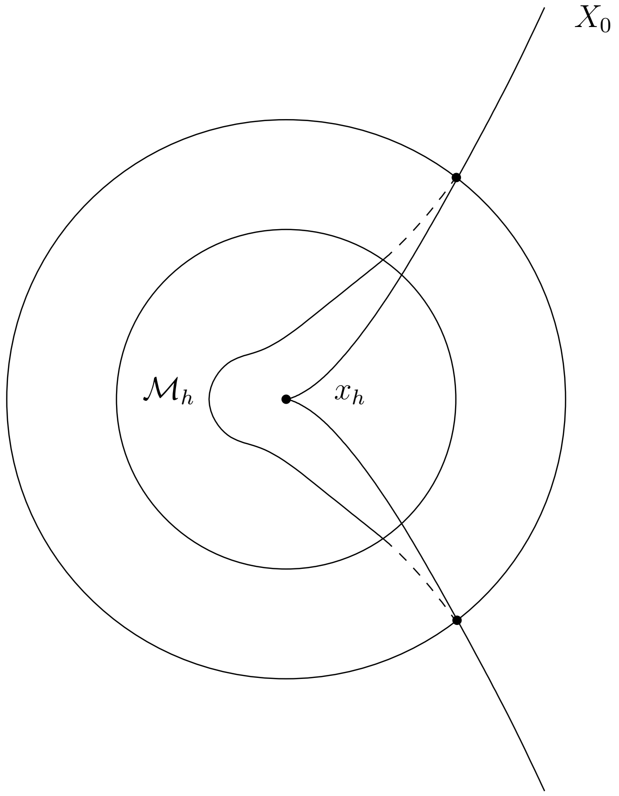

Fig. 3. Glueing the 'exterior' of X_0 (to the Milnor Ball around x_h) with a smaller Milnor fibre $\mathcal{M}_h$

We apply then Lemma 4.2 once more in order to thicken the trivialization of the singularity links to a closed tubular neighbourhood in the family $\mathcal{X}$.

Now, in order to simplify our notation, and without loss of generality, assume that X_0 has only one singular point x_0, and let $B := B(x_0, \delta)$ be a Milnor ball around the singularity. Moreover, for $t \neq 0, t \in \Delta \cap B(0, \eta)$ we consider the Milnor fibre $\mathcal{M}_{\delta,\eta}(t)$, whereas we have the two Milnor links

$$\mathcal{K}_0 := X_0 \cap S(x_0, \delta) \text{ and } \mathcal{K}_t := \mathcal{X}_t \cap S(x_0, \delta - \epsilon)$$

.

We can consider the Milnor collars $\mathcal{C}_0(\epsilon) := X_0 \cap (\overline{B(x_0, \delta)} \backslash B(x_0, \delta - \epsilon))$, and $\mathcal{C}_t(\epsilon) := \mathcal{X}_t \cap (\overline{B(x_0, \delta)} \backslash B(x_0, \delta - \epsilon))$.

The Milnor collars fill up a complex submanifold of dimension $dim X_0 + 1 := n + 1$ of $\mathbb{C}^{n+m} \times \Delta$.

We glue now $X \backslash B(x_0, \delta - \epsilon))$ and the Milnor fibre $\mathcal{M}_{\delta,\eta}(t)$ by identifying the Milnor collars $\mathcal{C}_0(\epsilon)$ and $\mathcal{C}_t(\epsilon)$.

We obtain in this way an abstract differentiable manifold W which is independent of t, but we want now to give an embedding $W \to W_t \subset \mathbb{C}^{n+m}$ such that $X \backslash B(x_0, \delta))$ maps through the identity, and the complement of the collar inside the Milnor fibre maps to $\mathcal{M}_{\delta,\eta}(t)$ via the restriction of the identity.

As for the collar $\mathcal{C}_0(\delta)$, its outer boundary will be mapped to $\mathcal{K}_0$, while its inner boundary will be mapped to $\mathcal{K}_t$ (i.e., we join the two different singularity links by a differentiable embedding of the abstract Milnor collar).

For $\eta << \delta$ the tangent spaces to the image of the abstract Milnor collar can be made very close to the tangent spaces of the Milnor collars $\mathcal{M}_{\delta,\epsilon}(t)$, and we can conclude the proof via Lemma 2.6. $\square$

The following well known theorem of Moser guarantees that, once the choice of a smoothing component is made for each $x_h \in Sing(X)$, then the approximating symplectic submanifold W_t is unique up to symplectomorphism.

Theorem 4.7 (Moser) *Let $\pi : \mathcal{X} \to T$ be a proper submersion of differentiable manifolds with T connected, and assume that we have a differentiable 2-form ω on $\mathcal{X}$ with the property that*

(*) $\forall t \in T$ $\omega_t := \omega|_{X_t}$ yields a symplectic structure on X_t whose class in $H^2(X_t, \mathbb{R})$ is locally constant on T (e.g., if it lies on $H^2(X_t, \mathbb{Z})$).

Then the symplectic manifolds (X_t, ω_t) are all symplectomorphic.

The unicity of the symplectic manifold W_t will play a crucial role in the next subsection.

4.3 Canonical Symplectic Structure for Varieties with Ample Canonical Class and Canonical Symplectic Structure for Surfaces of General Type

Theorem 4.8 *A minimal surface of general type S has a canonical symplectic structure, unique up to symplectomorphism, and stable by deformation, such that the class of the symplectic form is the class of the canonical sheaf $\Omega^2_S = \mathcal{O}_S(K_S)$. The same result holds for any projective smooth variety with ample canonical bundle.*

Proof.

Let V be a smooth projective variety of dimension n whose canonical divisor K_V is ample.

Then there is a positive integer m (depending only on n) such that mK_V is very ample (any $m \geq 5$ does by Bombieri's theorem in the case of surfaces, for higher dimension we can use Matsusaka's big theorem, cf. [Siu93] for an effective version).

Therefore the mth pluricanonical map $\phi_m := \phi_{|mK_V|}$ is an embedding of V in a projective space $\mathbb{P}^{P_m - 1}$, where $P_m := dim H^0(\mathcal{O}_V(mK_V))$.

We define then ω_m as follows: $\omega_m := \frac{1}{m}\phi_m^*(FS)$ (where FS is the Fubini-Study form $\frac{i}{2\pi}\partial\overline{\partial}log|z|^2)$, hence ω_m yields a symplectic form as desired.

One needs to show that the symplectomorphism class of (V, ω_m) is independent of m. To this purpose, suppose that the integer r has also the property that ϕ_r yields an embedding of V: the same holds also for rm, hence it suffices to show that (V, ω_m) and (V, ω_{mr}) are symplectomorphic.

To this purpose we use first the well known and easy fact that the pull back of the Fubini-Study form under the rth Veronese embedding v_r equals the rth multiple of the Fubini-Study form. Second, since $v_r \circ \phi_m$ is a linear projection of ϕ_{rm}, by Moser's Theorem follows the desired symplectomorphism. Moser's theorem implies also that if we have a deformation $\pi : \mathcal{V} \to T$ where T is connected and all the fibres have ample canonical divisor, then all the manifolds V_t, endowed with their canonical symplectic structure, are symplectomorphic.

Assume now that S is a minimal surface of general type and that K_S is not ample: then for any $m \geq 5$ (by Bombieri's cited theorem) ϕ_m yields an embedding of the canonical model X of S, which is obtained by contracting the finite number of smooth rational curves with selfintersection number $= -2$ to a finite number of Rational Double Point singularities. For these, the base of the semiuniversal deformation is smooth and yields a smoothing of the singularity.

By the quoted Theorem 3.33 on simultaneous resolution, it follows that

(1) S is diffeomorphic to any smoothing S' of X (but it can happen that X does not admit any global smoothing, as shown by many examples which one can find for instance in [Cat89]).

(2) S is diffeomorphic to the manifold obtained glueing the exterior $X \backslash B$ (B being the union of Milnor balls of radius δ around the singular points of X) together with the respective Milnor fibres, i.e., S is diffeomorphic to each of the symplectic submanifolds W of projective space which approximate the embedded canonical model X according to Theorem 4.6.

We already remarked that W is unique up to symplectomorphism, and this fact ensures that we have a unique canonical symplectic structure on S (up to symplectomorphism).

Clearly moreover, if X admits a global smoothing, we can then take S' sufficiently close to X as our approximation W. Then S' is a surface with ample canonical bundle, and, as we have seen, the symplectic structure induced by (a submultiple of) the Fubini Study form is the canonical symplectic structure.

The stability by deformation is again a consequence of Moser's theorem. □

4.4 Degenerations Preserving the Canonical Symplectic Structure

Assume once more that we consider the minimal surfaces S of general type with fixed invariants $\chi = x$ and $K^2 = y$, and their 5-canonical models Σ_5, which are surfaces with Rational Double Points and of degree $25K^2$ in a fixed projective space $\mathbb{P}^N$, where $N = \chi + 10K^2 - 1$.

The choice of S and of a projective basis for $\mathbb{P}H^0(5K_S)$ yields, as we saw, a point in the 5-pseudo moduli space of surfaces of general type with given invariants $\chi = x$ and $K^2 = y$, i. e., the locally closed set $\mathcal{H}_0(x,y)$ of the corresponding Hilbert scheme $\mathcal{H}$, which is the closed subset

$$\mathcal{H}_0(x,y) := \{\Sigma \in \mathcal{H}^0 | \omega_\Sigma^{\otimes 5} \cong \mathcal{O}_\Sigma(1)\}$$

of the open set

$$\mathcal{H}^0(x,y) := \{\Sigma | \Sigma \text{ is reduced with only R.D.P.}'\text{s as singularities}\}.$$

In fact, even if this pseudo moduli space is conceptually clear, it is computationally more complex than just an appropriate open subset of $\mathcal{H}^0(x,y)$, which we denote by $\mathcal{H}^{00}(x,y)$ and parametrizes triples

$$(S, L, \mathcal{B})$$

where

(i) S is a minimal surface of general type with fixed invariants $\chi = x$ and $K^2 = y$

(ii) $L \in Pic^0(S)$ is a topologically trivial holomorphic line bundle

(iii) $\mathcal{B}$ is a a projective basis for $\mathbb{P}H^0(5K_S + L)$.

To explain how to define $\mathcal{H}^{00}(x,y)$, let $\mathcal{H}^n(x,y) \subset \mathcal{H}^0(x,y)$ be the open set of surfaces Σ with $K_\Sigma^2 = y$. Let H be the hyperplane divisor, and observe that by the Riemann Roch theorem $P_\Sigma(m) = \chi(\mathcal{O}_\Sigma) + 1/2\, mH \cdot (mH - K_\Sigma)$, while by definition $P_\Sigma(m) = x + 1/2(5m-1)5my$. Hence, $H^2 = 25y$, $H \cdot K_\Sigma = 5y$, $\chi(\mathcal{O}_\Sigma) = x$, and by the Index theorem $K_\Sigma^2 \leq y$, equality holding if and only if $H \sim 5K_\Sigma$.

Since the group of linear equivalence classes of divisors which are numerically equivalent to zero is parametrized by $Pic^0(\Sigma) \times Tors(H^2(\Sigma, \mathbb{Z}))$, we get that the union of the connected components of $\mathcal{H}^n(x,y)$ containing $\mathcal{H}_0(x,y)$ yields an open set $\mathcal{H}^{00}(x,y)$ as described above.

Since $Pic^0(S)$ is a complex torus of dimension $q = h^1(\mathcal{O}_S)$, it follows that indeed there is a natural bijection, induced by inclusion, between irreducible (resp. connected) components of $\mathcal{H}_0(x,y)$ and of $\mathcal{H}^{00}(x,y)$. Moreover, $\mathcal{H}_0(x,y)$ and $\mathcal{H}^{00}(x,y)$ coincide when $q = 0$.

As we shall see, there are surfaces of general type which are diffeomorphic, or even canonically symplectomorphic, but which are not deformation equivalent.

Even if $\mathcal{H}^{00}(x,y)$ is highly disconnected, and not pure dimensional, one knows by a general result by Hartshorne [Hart66], that the Hilbert scheme $\mathcal{H}$ is connected, and one may therefore ask

(A) is $\overline{\mathcal{H}^{00}(x,y)}$ connected?

(B) which kind of singular surfaces does one have to consider in order to connect different components of $\mathcal{H}^{00}(x,y)$?

The latter question is particular significant, since first of all any projective variety admits a flat deformation to a scheme supported on the projective cone over its hyperplane section (iterating this procedure, one reduces to the so-called *stick figures*, which in this case would be supported on a finite union of planes. Second, because when going across badly singular surfaces, then the topology can change drastically (compare Example 5.12, page 329 of [K-SB88]).

We refer to [K-SB88] and to [Vieh95] for a theory of compactified moduli spaces of surfaces of general type. We would only like to mention that the theory describes certain classes of singular surfaces which are allowed, hence a certain open set in the Hilbert scheme $\mathcal{H}$.

One important question is, however, which degenerations of smooth surfaces do not change the canonical symplectomorphism class. In other words, which surgeries do not affect the canonical symplectic structure.

A positive result is the following theorem, which is used in order to show that the Manetti surfaces are canonically symplectomorphic (cf. [Cat06])

Theorem 4.9 *Let $\mathcal{X} \subset \mathbb{P}^N \times \Delta$ and $\mathcal{X}' \subset \mathbb{P}^N \times \Delta'$ be two flat families of normal surfaces over the disc of radius 2 in $\mathbb{C}$.*

Denote by $\pi : \mathcal{X} \to \Delta$ and by $\pi' : \mathcal{X}' \to \Delta$ the respective projections and make the following assumptions on the respective fibres of π, π':

(1) the central fibres X_0 and X'_0 are surfaces with cyclic quotient singularities and the two flat families yield $\mathbb{Q}$-Gorenstein smoothings of them.

(2) the other fibres X_t, X'_t, for $t, t' \neq 0$ are smooth.

Assume moreover that

(3) the central fibres X_0 and X'_0 are projectively equivalent to respective fibres ($X_0 \cong Y_0$ and $X'_0 \cong Y_1$) of an equisingular projective family $\mathcal{Y} \subset \mathbb{P}^N \times \Delta$ of surfaces.

Set $X := X_1$, $X' := X'_1$: then

(a) X and X' are diffeomorphic

(b) if FS denotes the symplectic form inherited from the Fubini-Study Kähler metric on $\mathbb{P}^N$, then the symplectic manifolds (X, FS) and (X', FS) are symplectomorphic.

The proof of the above is based on quite similar ideas to those of the proof of Theorem 4.6.

Remark 4.10 *Theorem 4.9 holds more generally for varieties of higher dimension with isolated singularities under the assumption that, for each singular point x_0 of X_0, letting $y_0(t)$ be the corresponding singularity of Y_t*

(i) $(X_0, x_0) \cong (Y_t, y_0(t))$

(ii) the two smoothings $\mathcal{X}, \mathcal{X}'$, correspond to paths in the same irreducible component of $Def(X_0, x_0)$.

5 Lecture 4: Irrational Pencils, Orbifold Fundamental Groups, and Surfaces Isogenous to a Product

In the previous lecture we considered the possible deformations and mild degenerations of surfaces of general type. In this lecture we want to consider a very explicit class of surfaces (and higher dimensional varieties), those which admit an unramified covering which is a product of curves (and are said to be *isogenous to a product*). For these one can reduce the description of the moduli space to the description of certain moduli spaces of curves with automorphisms.

Some of these varieties are rigid, i.e., they admit no nontrivial deformations; in any case these surfaces S have the weak rigidity property that any surface homeomorphic to them is deformation equivalent either to S or to the conjugate surface $\bar{S}$.

Moreover, it is quite interesting to see which is the action of complex conjugation on the moduli space: it turns out that it interchanges often two distinct connected components. In other words,there are surfaces such that the complex conjugate surface is not deformation equivalent to the surface itself (this phenomenon has been observed by several authors independently, cf. [F-M94] Theorem 7.16 and Corollary 7.17 on p. 208, completed in [Fried05] for elliptic surfaces, cf. [KK02,Cat03,BCG05] for the case of surfaces of general type). However, in this case we obtained surfaces which are diffeomorphic to each other, but only through a diffeomorphism not preserving the canonical class.

Other reasons to include these examples are not only their simplicity and beauty, but also the fact that these surfaces lend themself quite naturally to reveal the action of the Galois group $Gal(\overline{\mathbb{Q}}, \mathbb{Q})$ on moduli spaces.

In the next section we shall recall some basic results on fibred surfaces which are used to treat the class of surfaces isogenous to a product.

5.1 Theorem of Castelnuovo–De Franchis, Irrational Pencils and the Orbifold Fundamental Group

We recall some classical and some new results (see [Cat91] and [Cat03b] for more references)

Theorem 5.1 (Castelnuovo–de Franchis) *Let X be a compact Kähler manifold and $U \subset H^0(X, \Omega^1_X)$ be an isotropic subspace (for the wedge product) of dimension ≥ 2. Then there exists a fibration $f : X \to B$, where B is a curve, such that $U \subset f^*(H^0(B, \Omega^1_B))$ (in particular, the genus $g(B)$ of B is at least 2).*

Idea of proof

Let ω_1, ω_2 be two $\mathbb{C}$-linearly independent 1-forms $\in H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \omega_2 \equiv 0$. Then their ratio defines a nonconstant meromorphic function F with $\omega_1 = F\omega_2$.

After resolving the indeterminacy of the meromorphic map $F : X \dashrightarrow \mathbb{P}^1$ we get a morphism $\tilde{F} : \tilde{X} \to \mathbb{P}^1$ which does not need to have connected fibres, so we let $f : \tilde{X} \to B$ be its Stein factorization.

Since holomorphic forms are closed, $0 = d\omega_1 = dF \wedge \omega_2$ and the forms ω_j restrict to zero on the fibres of f. A small ramification calculation shows then that the two forms ω_j are pull back of holomorphic one forms on B, whence B has genus at least two. Since every map of $\mathbb{P}^1 \to B$ is constant, we see that f is indeed holomorphic on X itself. □

Definition 5.2 *Such a fibration f as above is called an* irrational pencil.

Using Hodge theory and the Künneth formula, the Castelnuovo–de Franchis theorem implies (see [Cat91]) the following

Theorem 5.3 *(Isotropic subspace theorem). (1) Let X be a compact Kähler manifold and $U \subset H^1(X, \mathbb{C})$ be an isotropic subspace of dimension ≥ 2. Then there exists an irrational pencil $f : X \to B$, such that $U \subset f^*(H^1(B, \mathbb{C}))$.*

(2) There is a 1-1 correspondence between irrational pencils $f : X \to B$, $g(B) = b \geq 2$, and subspaces $V = U \oplus \bar{U}$, where U is maximal isotropic of dimension b.

Proof.

(1) Using the fact that $H^1(X, \mathbb{C}) = H^0(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^1_X)}$ we write a basis of U as $(\phi_1 = \omega_1 + \overline{\eta_1}, \ldots, \phi_b = \omega_b + \overline{\eta_b})$.

Since again Hodge theory gives us the direct sum

$$H^2(X, \mathbb{C}) = H^0(X, \Omega^2_X) \oplus H^1(X, \Omega^1_X) \oplus \overline{H^0(X, \Omega^2_X)}$$

the isotropicity condition $\phi_i \wedge \phi_j = 0 \in H^2(X, \mathbb{C})$ reads:

$$\omega_i \wedge \omega_j \equiv 0, \eta_i \wedge \eta_j \equiv 0, \omega_i \wedge \overline{\eta_j} + \overline{\eta_i} \wedge \omega_j \equiv 0, \ \forall i, j.$$

The first two identities show that we are done if one can apply the theorem of Castelnuovo–de Franchis to the ω_j's, respectively to the η_j's , obtaining two irrational pencils $f : X \to B, f' : X \to B'$. In fact, if the image of $f \times f' : X \to B \times B'$ is a curve, then the main assertion is proven. Else, $f \times f'$ is surjective and the pull back f^* is injective. But then $\omega_i \wedge \overline{\eta_j} + \overline{\eta_i} \wedge \omega_j \equiv 0$ contradicts the Künneth formula.

Hence, there is only one case left to consider, namely that, say, all the ω_j's are $\mathbb{C}$-linearly dependent. Then we may assume $\omega_j \equiv 0, \ \forall \ j \geq 2$ and the above equation yields $\omega_1 \wedge \overline{\eta_j} = 0, \ \forall j \geq 2$. But then $\omega_1 \wedge \eta_j \equiv 0$, since if ξ is the Kähler form, $|\omega_1 \wedge \eta_j|^2 = \int_X \omega_1 \wedge \overline{\eta_j} \wedge \overline{\omega_1 \wedge \overline{\eta_j}} \wedge \xi^{n-2} = 0$.

(2) Follows easily from (1) as follows.

The correspondence is given by $f \mapsto V := f^*(H^1(B, \mathbb{C}))$.

In fact, since $f : X \to B$ is a continuous map which induces a surjection of fundamental groups, then the algebra homomorphism f^* is injective when restricted to $H^1(B, \mathbb{C})$ (this statement follows also without the Kähler hypothesis) and $f^*(H^1(B, \mathbb{C})) \subset H^1(X, \mathbb{C})$ contains many isotropic subspaces U of dimension b with $U \oplus \bar{U} = f^*(H^1(B, \mathbb{C}))$. If such subspace U is not maximal isotropic, then it is contained in U', which determines an irrational pencil f' to a curve B' of genus $> b$, and f factors through f' in view of the fact that every curve of positive genus is embedded in its Jacobian. But this contradicts the fact that f has connected fibres. □

To give an idea of the power of the above result, let us show how the following result due to Gromov ([Grom89], see also [Cat94] for details) follows as a simple consequence

Corollary 5.4 *Let X be a compact Kähler manifold and assume we have a surjective morphism $\pi_1(X) \to \mathit{\Gamma}$, where $\mathit{\Gamma}$ has a presentation with n generators, m relations, and with $n - m \geq 2$. Then there is an irrational pencil $f : X \to B$, such that $2g(B) \geq n - m$ and $H^1(\mathit{\Gamma}, \mathbb{C}) \subset f^*(H^1(B, \mathbb{C})$.*

Proof. By the argument we gave in (2) above, $H^1(\mathit{\Gamma}, \mathbb{C})$ injects into $H^1(X, \mathbb{C})$ and we claim that each vector v in $H^1(\mathit{\Gamma}, \mathbb{C})$ is contained in a nontrivial isotropic subspace. This follows because the classifying space $Y := K(\mathit{\Gamma}, 1)$ is obtained by attaching n 1-cells, m 2-cells, and then only cells of higher dimension. Hence $h^2(\mathit{\Gamma}, \mathbb{Q}) = h^2(Y, \mathbb{Q}) \leq m$, and $w \to w \wedge v$ has a kernel of dimension ≥ 2 on $H^1(\mathit{\Gamma}, \mathbb{C})$. The surjection $\pi_1(X) \to \mathit{\Gamma}$ induces a continuous map $F : X \to Y$, and each vector in the pull back of $H^1(\mathit{\Gamma}, \mathbb{C})$ is contained in a nontrivial maximal isotropic subspace, thus, by (2) above , in a subspace $V := f^*(H^1(B, \mathbb{C}))$ for a suitable irrational pencil f. Now, the corresponding subspaces V are defined over $\mathbb{Q}$ and $H^1(\mathit{\Gamma}, \mathbb{C})$ is contained in their union. Hence, by Baire's theorem, $H^1(\mathit{\Gamma}, \mathbb{C})$ is contained in one of them. □

In particular, Gromov's theorem applies to a surjection $\pi_1(X) \to \mathit{\Pi}_g$, where $g \geq 2$, and $\mathit{\Pi}_g$ is the fundamental group of a compact complex curve of genus g. But in general the genus b of the target curve B will not be equal to g, and we would like to detect b directly from the fundamental group $\pi_1(X)$. For this reason (and for others) we need to recall a concept introduced by Deligne and Mostow ([D-M93], see also [Cat00]) in order to extend to higher dimensions some standard arguments about Fuchsian groups.

Definition 5.5 *Let Y be a normal complex space and let D be a closed analytic set. Let $D_1, \dots, D_r$ be the divisorial (codimension 1) irreducible components of D, and attach to each D_j a positive integer $m_j > 1$.*

Then the orbifold fundamental group *$\pi_1^{orb}(Y \backslash D, (m_1, \dots m_r))$ is defined as the quotient of $\pi_1(Y \backslash (D_1 \cup \dots \cup D_r)$ by the subgroup normally generated by the $\{\gamma_1^{m_1}, \dots, \gamma_r^{m_r}\}$, where γ_i is a simple geometric loop around the divisor D_i (this means, γ_i is the conjugate via a simple path δ of a local loop γ which, in a local coordinate chart where Y is smooth and $D_i = \{(z) | z_1 = 0\}$, is given by $\gamma(\theta) := (exp(2\pi i\theta), 0, \dots 0)$, $\forall \theta \in [0, 1]$.*

We observe in fact that another choice for γ_i gives a conjugate element, so the group is well defined.

Example 5.6 *Let $Y = \mathbb{C}$, $D = \{0\}$: then $\pi_1^{orb}(\mathbb{C} \backslash \{0\}, m) \cong \mathbb{Z}/m$ and its subgroups correspond to the subgroups $H \subset \mathbb{Z}$ such that $H \supset m\mathbb{Z}$, i.e., $H = d\mathbb{Z}$, where d divides m.*

The above example fully illustrates the meaning of the orbifold fundamental group, once we use once more the well known theorem of Grauert and Remmert [GR58]

Remark 5.7 *There is a bijection between*

$$\text{Monodromies } \mu : \pi_1^{orb}(Y \backslash D, (m_1, \dots m_r)) \to \mathcal{S}(M)$$

and normal locally finite coverings $f : X \to Y$, with general fibre $\cong M$, and such that for each component R_i of $f^{-1}(D_i)$ the ramification index divides m_i.

We have moreover (see [Cat00]) the following

Proposition 5.8 *Let X be a complex manifold, and G a group of holomorphic automorphisms of X, acting properly discontinuously. Let D be the branch locus of $\pi : X \to Y := X/G$, and for each divisorial component D_i of D let m_i be the branching index. Then we have an exact sequence*

$$1 \to \pi_1(X) \to \pi_1^{orb}(Y \backslash D, (m_1, \dots m_r)) \to G \to 1.$$

Remark 5.9 *(I) In order to extend the above result to the case where X is only normal (then $Y := X/G$ is again normal), it suffices to define the orbifold fundamental group of a normal variety X as*

$$\pi_1^{orb}(X) := \pi_1(X \backslash Sing(X)).$$

(II) Taking the monodromy action of $\pi_1^{orb}(Y \backslash D, (m_1, \dots m_r))$ acting on itself by translations, we see that there exists a universal orbifold covering space $\overline{(Y \backslash D, (m_1, \dots m_r))}$ with a properly discontinuous action of $\pi_1^{orb}(Y \backslash D, (m_1, \dots m_r))$ having Y as quotient, and the prescribed ramification.

(III) Obviously the universal orbifold covering space $\overline{(Y \backslash D, (m_1, \dots m_r))}$ is (connected and) simply connected.

Example 5.10 *(a) Let Y be a compact complex curve of genus g, $D = \{p_1, \dots p_r\}$: then $\Gamma := \pi_1^{orb}(Y \backslash \{p_1, \dots p_r\}, (m_1, \dots m_r))$ has a presentation*

$$\Gamma := < \gamma_1, \dots, \gamma_r, \alpha_1, \beta_1, \dots \alpha_g, \beta_g | \gamma_1 \cdot \dots \cdot \gamma_r \cdot \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1, \gamma_j^{m_j} = 1 >$$

(b) Γ acts on a simply connected complex curve Σ, with $\Sigma/\Gamma \cong Y$. By the uniformization theorem $\Sigma \cong \mathbb{P}^1$ iff Σ is compact, i.e., iff Γ is finite (then $Y \cong \mathbb{P}^1$). If instead Γ is infinite, then there is a finite index subgroup Γ' acting freely on Σ. Then correspondingly we obtain $C' := \Sigma/\Gamma' \to Y$ a finite covering with prescribed ramification m_i at each point p_i.

Example 5.11 Triangle groups *We let $Y = \mathbb{P}^1$, $r = 3$, without loss of generality $D = \{\infty, 0, 1\}$. Then the orbifold fundamental group in this case reduces to the previously defined triangle group $T(m_1, m_2, m_3)$ which has a presentation*

$$T(m_1, m_2, m_3) := < \gamma_1, \gamma_2, \gamma_3 | \gamma_1 \cdot \gamma_2 \cdot \gamma_3 = 1, \gamma_1^{m_1} = 1, \gamma_2^{m_2} = 1, \gamma_3^{m_3} = 1 > .$$

The triangle group is said to be of elliptic type *iff* $\Sigma \cong \mathbb{P}^1$, *of* parabolic type *iff* $\Sigma \cong \mathbb{C}$, *of* hyperbolic type *iff* $\Sigma \cong \mathbb{H} := \{\tau | Im(\tau) > 0\}$.

It is classical (and we have already seen the first alternative as a consequence of Hurwitz' formula in lecture 2) that the three alternatives occur

- *Elliptic* $\Leftrightarrow \sum_i \frac{1}{m_i} > 1 \Leftrightarrow$ *(2,2,n) or (2,3,n)* $(n = 3, 4, 5)$
- *Parabolic* $\Leftrightarrow \sum_i \frac{1}{m_i} = 1 \Leftrightarrow$ *(3,3,3) or (2,3,6) or (2,4,4)*
- *Hyperbolic* $\Leftrightarrow \sum_i \frac{1}{m_i} < 1$

We restrict here to the condition $1 < m_i < \infty$, *else for instance there is also the parabolic case* $(2, 2, \infty)$, *where the uniformizing function is* $cos\colon \mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$.

The group $T(m_1, m_2, m_3)$, *which was described for the elliptic case in lecture 2, is in the parabolic case a semidirect product of the period lattice* Λ *of an elliptic curve by its group* μ_n *of linear automorphisms*

- *(3,3,3)* : $\Lambda = \mathbb{Z} \oplus \zeta_3\mathbb{Z}$, ζ_3 *a generator of* μ_3
- *(2,3,6)* : $\Lambda = \mathbb{Z} \oplus \zeta_3\mathbb{Z}$, $-\zeta_3$ *a generator of* μ_6
- *(2,4,4)* : $\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$, i *a generator of* μ_4.

There is a good reason to call the above 'triangle groups'. Look in fact at the ramified covering $f : \Sigma \to \mathbb{P}^1$, *branched in* $\{\infty, 0, 1\}$. *Complex conjugation on* $\mathbb{P}^1$ *lifts to the covering, as we shall see later in more detail. Consider then a connected component* Δ *of* $f^{-1}(\mathbb{H})$. *We claim that it is a triangle (in the corresponding geometry: elliptic, resp. Euclidean, respective hyperbolic) with angles* $\pi/m_1, \pi/m_2, \pi/m_3$.

In fact, take a lift of complex conjugation which is the identity on one of the three sides of Δ: *then it follows that this side is contained in the fixed locus of an antiholomorphic automorphism of* Σ, *and the assertion follows then easily.*

In terms of this triangle (which is unique up to automorphisms of Σ *in the elliptic and hyperbolic case) it turns out that the three generators of* $T(m_1, m_2, m_3)$ *are just rotations around the vertices of the triangle, while the triangle group* $T(m_1, m_2, m_3)$ *sits as a subgroup of index 2 inside the group generated by the reflections on the sides of the triangle.*

Let us leave for the moment aside the above concepts, which will be of the utmost importance in the forthcoming sections, and let us return to the irrational pencils.

Definition 5.12 *Let* X *be a compact Kähler manifold and assume we have a pencil* $f : X \to B$. *Assume that* $t_1, \dots t_r$ *are the points of* B *whose fibres* $F_i := f^{-1}(t_i)$ *are the multiple fibres of* f. *Denote by* m_i *the multiplicity of* F_i, *i.e., the G.C.D. of the multiplicities of the irreducible components of* F_i. *Then the* orbifold fundamental group of the fibration $\pi_1(f) := \pi_1(b, m_1, \dots m_r)$ *is defined as the quotient of* $\pi_1(B \backslash \{t_1, \dots t_r\})$ *by the subgroup normally generated by the* $\gamma_i^{m_i}$*'s, where* γ_i *is a geometric loop around* t_i.

The orbifold fundamental group is said to be of hyperbolic type *if the corresponding universal orbifold (ramified) covering of B is the upper half plane.*

The orbifold fundamental group of a fibration is a natural object in view of the following result (see [CKO03, Cat03b])

Proposition 5.13 *Given a fibration $f : X \to B$ of a compact Kähler manifold onto a compact complex curve B, we have the orbifold fundamental group exact sequence $\pi_1(F) \to \pi_1(X) \to \pi_1(b, m_1, \dots m_r) \to 0$, where F is a smooth fibre of f.*

The previous exact sequence leads to following result, which is a small generalization of Theorem 4.3. of [Cat03b] and a variant of several other results concerning fibrations onto curves (see [Cat00, Cat03b]), valid more generally for quasi-projective varieties (in this case the starting point is the closedness of logarithmic forms, proven by Deligne in [Del70], which is used in order to obtain extensions of the theorem of Castelnuovo and De Franchis to the non complete case, see [Bau97, Ara97]).

Theorem 5.14 *Let X be a compact Kähler manifold and let $(b, m_1, \dots m_r)$ be a hyperbolic type. Then there is a bijection between pencils $f : X \to B$ of type $(b, m_1, \dots m_r)$ and epimorphisms $\pi_1(X) \to \pi_1(b, m_1, \dots m_r)$ with finitely generated kernel.*

Proof. One direction follows right away from proposition 5.13, so assume that we are given such an epimorphism. Since $\pi_1(b, m_1, \dots m_r)$ is of hyperbolic type, it contains a normal subgroup H of finite index which is isomorphic to a fundamental group Π_g of a compact curve of genus $g \geq 2$.

Let H' be the pull back of H in $\pi_1(X)$ under the given surjection, and let $X' \to X$ the corresponding Galois cover, with Galois group $G \cong \pi_1(b, m_1, \dots m_r)/H$.

By the isotropic subspace theorem, there is an irrational pencil $f' : X' \to C$, where the genus of C is at least g, corresponding to the surjection $\psi : \pi_1(X') = H' \to H \cong \Pi_g$. The group G acts on X' leaving the associated cohomology subspace $({f'}^*(H^1(C, \mathbb{C}))$ invariant, whence G acts on C preserving the fibration, and we get a fibration $f : X \to B := C/G$.

By Theorem 4.3 of [Cat03b], since the kernel of ψ is finitely generated, it follows that $\psi = f'_* : \pi_1(X') \to \Pi_g = \pi_1(C)$. G operates freely on X' and effectively on C: indeed G acts nontrivially on Π_g by conjugation, since a hyperbolic group has trivial centre. Thus we get an action of $\pi_1(b, m_1, \dots m_r)$ on the upper half plane $\mathbb{H}$ whose quotient equals $C/G := B$, which has genus b.

We use now again a result from Theorem 4.3 of [Cat03b], namely, that f' has no multiple fibres. Since the projection $C \to B$ is branched in r points with ramification indices equal to $(m_1, \dots m_r)$, it follows immediately that the orbifold fundamental group of f is isomorphic to $\pi_1(b, m_1, \dots m_r)$. $\square$

Remark 5.15 *The crucial property of Fuchsian groups which is used in [Cat03b] is the so called NINF property, i.e., that every normal nontrivial subgroup of infinite index is not finitely generated. From this property follows that, given a fibration $f : X' \to C$, the kernel of $f_* : \pi_1(X') \to \pi_1(C)$ is finitely generated (in the hyperbolic case) if and only if there are no multiple fibres.*

5.2 Varieties Isogenous to a Product

Definition 5.16 *A complex algebraic variety X of dimension n is said to be* isogenous to a higher product *if and only if there is a finite étale cover $C_1 \times \dots C_n \to X$, where $C_1, \dots, C_n$ are compact Riemann surfaces of respective genera $g_i := g(C_i) \geq 2$.*

In fact, X is isogenous to a higher product if and only if there is a finite étale Galois cover of X isomorphic to a product of curves of genera at least two, ie., $X \cong (C_1 \times \dots C_n)/G$, where G is a finite group acting freely on $C_1 \times \dots C_n$.

Moreover, one can prove that there exists a unique minimal such Galois realization $X \cong (C_1 \times \dots C_n)/G$ (see [Cat00]).

In proving this plays a key role a slightly more general fact:

Remark 5.17 *The universal covering of a product of curves $C_1 \times \dots C_n$ of hyperbolic type as above is the polydisk $\mathbb{H}^n$.*

The group of automorphisms of $\mathbb{H}^n$ is a semidirect product of the normal subgroup $Aut(\mathbb{H})^n$ by the symmetric group $\mathcal{S}_n$ (cf. [Ves84] VIII, 1 pages 236–238). This result is a consequence of three basic facts:

(i) Using the subgroup $Aut(\mathbb{H})^n$ we may reduce to consider only automorphisms which leave the origin invariant

(ii) We use the Hurwitz trick to show that the tangent representation is faithful: if $g(z) = z + F_m(z) + \dots$ is the Taylor development at the origin and with mth order term $F_m(z) \neq 0$, then for the rth iterate of g we get $z \to z + rF_m(z) + \dots$, contradicting the Cauchy inequality for the rth iterate when $r \gg 0$

(iii) Using the circular invariance of the domain ($z \to \lambda z, |\lambda| = 1$), one sees that the automorphisms which leave the origin invariant are linear, since, if $g(0) = 0$, then $g(z)$ and $\lambda^{-1} g(\lambda z)$ have the same derivative at the origin, whence by ii) they are equal

A fortiori, the group of automorphisms of such a product, $Aut(C_1 \times \dots C_n)$ has as normal subgroup $Aut(C_1) \times \dots Aut(C_n)$, and with quotient group a subgroup of $\mathcal{S}_n$.

The above remark leads to the following

Definition 5.18 *A variety isogenous to a product is said to be* unmixed *if in its minimal realization $G \subset Aut(C_1) \times \dots Aut(C_n)$. If $n = 2$, the condition of minimality is equivalent to requiring that $G \to Aut(C_i)$ is injective for $i = 1, 2$.*

The characterization of varieties X isogenous to a (higher) product becomes simpler in the surface case. Hence, assume in the following $X = S$ to be a surface: then

Theorem 5.19 *(see [Cat00]). (a) A projective smooth surface is isogenous to a higher product if and only if the following two conditions are satisfied:*

(1) there is an exact sequence

$$1 \to \Pi_{g_1} \times \Pi_{g_2} \to \pi = \pi_1(S) \to G \to 1,$$

where G is a finite group and where Π_{g_i} denotes the fundamental group of a compact curve of genus $g_i \geq 2$;

(2) $e(S)(= c_2(S)) = \frac{4}{|G|}(g_1 - 1)(g_2 - 1)$.

(b) Any surface X with the same topological Euler number and the same fundamental group as S is diffeomorphic to S. The corresponding subset of the moduli space, $\mathfrak{M}_S^{top} = \mathfrak{M}_S^{diff}$, corresponding to surfaces orientedly homeomorphic, resp. orientedly diffeomorphic to S, is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.

In particular, if X is orientedly diffeomorphic to S, then X is deformation equivalent to S or to $\bar{S}$.

Sketch of the Proof.

The necessity of conditions (1) and (2) of (a) is clear, since there is an étale Galois cover of S which is a product, and then $e(S) \cdot |G| = e(C_1 \times C_2) = e(C_1) \cdot e(C_2) = 4(g_1 - 1)(g_2 - 1)$.

Conversely, take the étale Galois cover S' of S with group G corresponding to the exact sequence (1). We need to show that S' is isomorphic to a product.

By Theorem 5.14 the two projections of the direct product $\Pi_{g_1} \times \Pi_{g_2}$ yield two holomorphic maps to curves of respective genera g_1, g_2, hence we get a holomorphic map $F : S' \to C_1 \times C_2$, such that $f_j := p_j \circ F : S' \to C_j$ is a fibration. Let h_2 be the genus of the fibres of f_1: then since Π_{g_2} is a quotient of the fundamental group of the fibre, it follows right away that $h_2 \geq g_2$.

We use then the classical (cf. [BPV84], Proposition 11.4, page 97).

Theorem of Zeuthen-Segre *Let $f : S \to B$ be a fibration of an algebraic surface onto a curve of genus b, with fibres of genus g: then*

$$e(S) \geq 4(g - 1)(b - 1),$$

equality holding iff all the fibres are smooth, or , if $g = 1$, all the fibres are multiple of smooth curves.

Hence $e(S) \geq 4(g_1 - 1)(h_2 - 1) \geq (g_1 - 1)(g_2 - 1) = e(S)$, equality holds, $h_2 = g_2$, all the fibres are smooth and F is then an isomorphism.

Part (b): we consider first the unmixed case. This means that the group G does not mix the two factors, whence the individual subgroups Π_{g_i} are normal in $\pi_1(S)$, and moding out by the second of them one gets the exact sequence

$$1 \to \Pi_{g_1} \to \pi_1(S)/\Pi_{g_2} \to G \to 1,$$

which is easily seen to be the orbifold exact sequence for the quotient map $C_1 \to C_1/G$. This immediately shows that the differentiable structure of the action of G on the product $C_1 \times C_2$ is determined, hence also the differentiable structure of the quotient S is determined by the exact sequence (1) in 5.19.

We have now to choose complex structures on the respective manifolds C_i, which make the action of G holomorphic. Note that the choice of a complex structure implies the choice of an orientation, and that once we have fixed the isomorphism of the fundamental group of C_i with Π_{g_i} and we have chosen an orientation (one of the two generators of $H^2(\Pi_{g_i}, \mathbb{Z})$) we have a marked Riemann surface. Then the theory of Teichmüller spaces shows that the space of complex structures on a marked Riemann surface of genus $g \geq 2$ is a complex manifold $\mathcal{T}_g$ of dimension $3(g-1)$ diffeomorphic to a ball. The finite group G, whose differentiable action is specified, acts on $\mathcal{T}_g$, and the fixed point set equals the set of complex structures for which the action is holomorphic. The result follows then from Proposition 4.13 of [Cat00], which is a slight generalization of one of the solutions [Tro96] of the Nielsen realization problem.

Proposition 5.20 (Connectivity of Nielsen realization) *Given a differentiable action of a finite group G on a fixed oriented and marked Riemann surface of genus g, the fixed locus $Fix(G)$ of G on $\mathcal{T}_g$ is non empty, connected and indeed diffeomorphic to an euclidean space.*

Let us first explain why the above proposition implies part (b) of the theorem (in the unmixed case). Because the moduli space of such surfaces is then the image of a surjective holomorphic map from the union of two connected complex manifolds. We get two such manifolds because of the choice of orientations on both factors which together must give the fixed orientation on our algebraic surface. Now, if we change the choice of orientations, the only admissible choice is the one of reversing orientations on both factors, which is exactly the result of complex conjugation.

Idea of proof Let us now comment on the underlying idea for the above proposition: as already said, Teichmüller space $\mathcal{T}_g$ is diffeomorphic to an Euclidean space of dimension $6g-6$, and admits a Riemannian metric, the Weil-Petersson metric, concerning which Wolpert and Tromba proved the existence of a C^2-function f on $\mathcal{T}_g$ which is proper, G-invariant, non negative ($f \geq 0$), and finally such that f is strictly convex for the given metric (i.e., strictly convex along the W-P geodesics).

Recall that, G being a finite group, its action can be linearized at the fixed points, in particular $\mathrm{Fix}(G)$ is a smooth submanifold.

The idea is to use Morse theory for the function f which is strictly convex, and proper, thus it always has a minimum when restricted to a submanifold of $\mathcal{T}_g$

(1) There is a unique critical point x_o for f on $\mathcal{T}_g$, which is an absolute minimum on $\mathcal{T}_g$ (thus $\mathcal{T}_g$ is diffeomorphic to an euclidean space).

(2) If we are given a connected component M of $\mathrm{Fix}(G)$, then a critical point y_o for the restriction of f to M is also a critical point for f on $\mathcal{T}_g$: in fact f is G invariant, thus df vanishes on the normal space to M at y_o.
(3) Thus every connected component M of $\mathrm{Fix}(G)$ contains x_o, and, $\mathrm{Fix}(G)$ being smooth, it is connected. $\mathrm{Fix}(G)$ is nonempty since x_0, being the unique minimum, belongs to $\mathrm{Fix}(G)$.
(4) Since f is strictly convex, and proper on $\mathrm{Fix}(G)$, then by Morse theory $\mathrm{Fix}(G)$ is diffeomorphic to an euclidean space.

□

In the mixed case there is a subgroup G^o of index 2 consisting of transformations which do not mix the two factors, and a corresponding subgroup π^o of $\pi = \pi_1(S)$ of index 2, corresponding to an étale double cover S' yielding a surface of unmixed type. By the first part of the proof, it will suffice to show that, once we have found a lifting isomorphism of π^o with a subgroup Γ^o of $Aut(\mathbb{H}) \times Aut(\mathbb{H})$, then the lifting isomorphism of π with a subgroup Γ of $Aut(\mathbb{H} \times \mathbb{H})$ is uniquely determined.

The transformations of Γ^o are of the form $(x, y) \to (\gamma_1(x), \gamma_2(y))$. Pick any transformation in $\Gamma \setminus \Gamma^o$: it will be a transformation of the form $(a(y), b(x))$. Since it normalizes Γ^o, for each $\delta \in \Gamma^o$ there is $\gamma \in \Gamma^o$ such that

$$a\gamma_2 = \delta_1 a, \quad b\gamma_1 = \delta_2 b.$$

We claim that a, b are uniquely determined. For instance, if a' would also satisfy $a'\gamma_2 = \delta_1 a'$, we would obtain

$$a'a^{-1} = \delta_1 (a'a^{-1}) \delta_1^{-1}.$$

This would hold in particular for every $\delta_1 \in \Pi_{g_1}$, but since only the identity centralizes such a Fuchsian group, we conclude that $a' = a$.

Remark 5.21 *A completely similar result holds in higher dimension, but the Zeuthen-Segre theorem allows an easier formulation in dimension two.*

One can moreover weaken the hypothesis on the fundamental group, see Theorem B of [Cat00].

5.3 Complex Conjugation and Real Structures

The interest of Theorem 5.19 lies in its constructive aspect.

Theorem 5.19 shows that in order to construct a whole connected component of the moduli space of surfaces of general type, given by surfaces isogenous to a product, it suffices, in the unmixed type, to provide the following data:

(i) A finite group G
(ii) Two orbifold fundamental groups $A_1 := \pi_1(b_1, m_1, \dots m_r)$, $A_2 := \pi_1 (b_2, n_1, \dots n_h)$

(iii) Respective surjections $\rho_1 : A_1 \to G$, $\rho_2 : A_2 \to G$ such that
(iv) If we denote by Σ_i the image under ρ_i of the conjugates of the powers of the generators of A_i of finite order, then

$$\Sigma_1 \cap \Sigma_2 = \{1_G\}$$

(v) Each surjection ρ_i is order preserving, in the sense for instance that a generator of $A_1 := \pi_1(b_1, m_1, \dots m_r)$ of finite order m_i has as image an element of the same order m_i.

In fact, if we take a curve C_1' of genus b_1, and r points on it, to ρ_1 corresponds a Galois covering $C_i \to C_i'$ with group G, and the elements of G which have a fixed point on C_i are exactly the elements of Σ_i. Therefore we have a diagonal action of G on $C_1 \times C_2$ (i.e., such that $g(x, y) = (\rho_1(g)(x), \rho_2(g)(y))$, and condition iv) is exactly the condition that G acts freely on $C_1 \times C_2$.

There is some arbitrariness in the above choice, namely, in the choice of the isomorphism of the respective orbifold fundamental groups with A_1, A_2, and moreover one can compose each ρ_i simultaneously with the same automorphism of G (i.e., changing G up to isomorphism). Condition (v) is technical, but important in order to calculate the genus of the respective curves C_i.

In order to pass to the complex conjugate surface (this is an important issue in Theorem 5.19), it is clear that we take the conjugate curve of each C_i', and the conjugate points of the branch points, but we have to be more careful in looking at what happens with the homomorphisms ρ_i.

For this reason, it is worthwhile to recall some basic facts about complex conjugate structures and real structures.

Definition 5.22 *Let X be an almost complex manifold, i.e., the pair of a differentiable manifold M and an almost complex structure J: then the complex conjugate almost complex manifold $\bar{X}$ is given by the pair $(M, -J)$. Assume now that X is a* complex manifold, *i.e., that the almost complex structure is integrable. Then the same occurs for $-J$, because, if $\chi : U \to \mathbb{C}^n$ is a local chart for X, then $\overline{\chi} : U \to \mathbb{C}^n$ is a local chart for $\bar{X}$.*

In the case where X is a projective variety $X \subset \mathbb{P}^N$, then we easily see that $\bar{X}$ equals $\sigma(X)$, where $\sigma : \mathbb{P}^N \to \mathbb{P}^N$ is given by complex conjugation, and the homogeneous ideal of $\bar{X} = \sigma(X)$ is the complex conjugate of the homogeneous ideal I_X of X, namely:

$$I_{\bar{X}} = \{P \in \mathbb{C}[z_0, \dots z_N] | \overline{P(\bar{z})} \in I_X\}.$$

Definition 5.23 *Given complex manifolds X, Y let $\phi\colon X \to \bar{Y}$ be a holomorphic map. Then the same map of differentiable manifolds defines an* antiholomorphic *map $\bar{\phi}\colon X \to Y$ (also, equivalently, an* antiholomorphic *map $\bar{\phi}^{**}\colon \bar{X} \to \bar{Y}$).*

A map $\phi\colon X \to Y$ is said to be dianalytic *if it is either holomorphic or antiholomorphic. ϕ determines also a dianalytic map $\phi^{**}\colon \bar{X} \to \bar{Y}$ which is holomorphic iff ϕ is holomorphic.*

The reason to distinguish between the maps ϕ, $\bar{\phi}$ and $\bar{\phi}^{**}$ in the above definition lies in the fact that maps between manifolds are expressed locally as maps in local coordinates, so in these terms $\bar{\phi}(x)$ is indeed the antiholomorphic function $\overline{\phi(x)}$, while $\bar{\phi}^{**}(x) = \phi(\bar{x})$.

With this setup notation, we can further proceed to define the concept of a real structure on a complex manifold.

Definition 5.24 *Let X be a complex manifold.*

(1) The Klein Group of X, denoted by $\mathcal{K}l(X)$ or by $Dian(X)$, is the group of dianalytic automorphisms of X.

(2) A real structure on X is an antiholomorphic automorphism $\sigma : X \to X$ such that $\sigma^2 = Id_X$.

Remark 5.25 *We have a sequence*

$$0 \to Bihol(X) := Aut(X) \to Dian(X) := \mathcal{K}l(X) \to \mathbb{Z}/2 \to 0$$

which is exact if and only if X is biholomorphic to $\bar{X}$, and splits if and only if X admits a real structure.

Example 5.26 *Consider the anharmonic elliptic curve corresponding to the Gaussian integers: $X := \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$.*

Obviously X is real, since $z \to \bar{z}$ is an antiholomorphic involution.

But there are infinitely many other real structures, since if we take an antiholomorphism σ we can write $\sigma(z) = i^r \bar{z} + \mu$, $\mu = a + ib$, with $a, b \in \mathbb{R}/\mathbb{Z}$ and the condition $\sigma(\sigma(z)) \equiv z (mod\ \mathbb{Z} \oplus i\mathbb{Z})$ is equivalent to

$$i^r \bar{\mu} + \mu = n + im,\ n, m \in \mathbb{Z} \Leftrightarrow a + ib + i^r a - i^{r+1} b = n + im$$

and has the following solutions:

- $r = 0, a \in \{0, 1/2\}, b$ arbitrary
- $r = 1, a = -b$ arbitrary
- $r = 2, a$ arbitrary, $b \in \{0, 1/2\}$
- $r = 3, a = b$ arbitrary.

In the above example the group of biholomorphisms is infinite, and we have an infinite number of real structures, but many of these are isomorphic, as the number of isomorphism classes of real structures is equal to the number of conjugacy classes (for $Aut(X)$) of such splitting involutions.

For instance, in the genus 0 case, there are only two conjugacy classes of real structures on $\mathbb{P}^1_{\mathbb{C}}$:

$$\sigma(z) = \bar{z},\ \sigma(z) = -\frac{1}{\bar{z}}.$$

They are obviously distinguished by the fact that in the first case the set of real points $X(\mathbb{R}) = Fix(\sigma)$ equals $\mathbb{P}^1_{\mathbb{R}}$, while in the second case we have an empty set. The sign is important, because the real structure $\sigma(z) = \frac{1}{\bar{z}}$, which has $\{z||z|=1\}$ as set of real points, is conjugated to the first. Geometrically, in the first case we have the circle of radius 1, $\{(x,y,z)\in\mathbb{P}^2_{\mathbb{C}}|x^2+y^2+z^2=1\}$, in the second the imaginary circle of radius -1, $\{(x,y,z)\in\mathbb{P}^2_{\mathbb{C}}|x^2+y^2+z^2=-1\}$.

It is clear from the above discussion that there can be curves C which are isomorphic to their conjugate, yet do not need to be real: this fact was discovered by C. Earle, and shows that the set of real curves is only a semialgebraic set of the complex moduli space, because it does not coincide with the set $\mathfrak{M}_g(\mathbb{R})$ of real points of $\mathfrak{M}_g$.

We want now to give some further easy example of this situation.

We observe preliminarily that C is isomorphic to $\bar{C}$ if and only in there is a finite group G of automorphisms such that C/G has a real structure which lifts to an antiholomorphism of C (in fact, if $C\cong\bar{C}$ it suffices to take $Aut(C)=G$ if $g(C)\geq 2$).

We shall denote this situation by saying that the covering $C\to C/G$ is real.

Definition 5.27 *We shall say that the covering $C\to C/G$ is an* n-angle covering *if $C/G\cong\mathbb{P}^1$ and the branch points set consists of n points.*

We shall say that C is an n-angle curve *if $C\to C/Aut(C)$ is an n-angle covering.*

Remark 5.28 *(a) Triangle coverings furnish an example of a moduli space (C,G), of the type discussed above, which consists of a single point.*

(b) If $C\to C/G$ is an n-angle covering *with n odd, then the induced real structure on $C/G\cong\mathbb{P}^1$ has a non empty set of real points (the branch locus B is indeed invariant), thus we may assume it to be the standard complex conjugation $z\mapsto\bar{z}$.*

Example 5.29 *We construct here examples of families of real quadrangle covers $C\to C/G$ such that $(C/G)(\mathbb{R})=\emptyset$, and such that, for a general curve in the family, $G=Aut(C)$, and the curve C is not real. The induced real structure on $C/G\cong\mathbb{P}^1$ is then $\sigma(z)=-\frac{1}{\bar{z}}$, and the quotient $(C/G)/\sigma\cong\mathbb{P}^2_{\mathbb{R}}$.*

We choose then as branch set $B\subset\mathbb{P}^1_{\mathbb{C}}$ the set $\{\infty,0,w,-\frac{1}{\bar{w}}\}$, and denote by $0,u$ the corresponding image points in $\mathbb{P}^2_{\mathbb{R}}$.

Observe now that

$$\pi_1(\mathbb{P}^2_{\mathbb{R}}\setminus\{0,u\})=\langle a,b,x|ab=x^2\rangle\cong\langle a,x\rangle$$

and the étale double covering $\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^2_{\mathbb{R}}$ corresponds to the quotient obtained by setting $a=b=1$, thus $\pi_1(\mathbb{P}^1_{\mathbb{C}}\setminus B)$ is the free group of rank 3

$$\pi_1(\mathbb{P}^1_{\mathbb{C}}\setminus B)=\langle a,x^2,x^{-1}ax\rangle\cong\langle a,b,a':=x^{-1}ax,b':=x^{-1}bx|ab=a'b'\rangle.$$

We let G' be the group $(\mathbb{Z}/2n)\oplus(\mathbb{Z}/m)$, and let C be the Galois cover of $\mathbb{P}^2_{\mathbb{R}}$ branched in $\{0,u\}$ corresponding to the epimorphisms such that $x\mapsto(1,0),a\mapsto(0,1)$. It follows that C is a 4-angle covering with group $G\cong(2\mathbb{Z}/2n\mathbb{Z})\oplus(\mathbb{Z}/m\mathbb{Z})$. It is straightforward to verify the following

Claim: G' contains no antiholomorphism of order 2, if n is even.

Thus it follows that C is not real, provided that $G = Aut(C)$. To simplify things, let $n = 4, m = 2$. By Hurwitz' formula C has genus 3, and $8 = |G| = 4(g-1)$. Assume that $Aut(C) \neq G$. If G has index 2, then we get an involution on $\mathbb{P}^1$ preserving the branch set B. But the cross-ratio of the four points equals exactly $-\frac{1}{|w|^2}$, and this is not anharmonic for w general (i.e., $\neq 2, -1, 1/2$). If instead $C \to C/Aut(C)$ is a triangle curve, then we get only a finite number of curves, and again a finite set of values of w, which we can exclude.

Now, since $|Aut(C)| > 8(g-1)$, if $C \to C/Aut(C)$ is not a triangle curve, then the only possibility, by the Hurwitz' formula, is that we have a quadrangle cover with branching indices $(2, 2, 2, 3)$. But this is absurd, since a ramification point of order 4 for $C \to C/G$ must have a higher order of ramification for the map $C \to C/Aut(C)$.

There is however one important special case when a curve isomorphic to its conjugate must be real, we have namely the following

Proposition 5.30 *Let $C \to C/G$ be a triangle cover which is real and has distinct branching indices ($m_1 < m_2 < m_3$) : then C is real (i.e., C has a real structure).*

Proof. Let σ be the real structure on $C/G \cong \mathbb{P}^1$. The three branch points of the covering must be left fixed by σ, since the branching indices are distinct (observe that $\mu(\sigma_* \gamma_i)$ is conjugate to $\mu(\gamma_i)$, whence it has the same order). Thus, without loss of generality we may assume that the three branch points are real, and indeed equal to $\{0, 1, \infty\}$, while $\sigma(z) = \bar{z}$.

Choose 2 as base point, and a basis of the fundamental group as in Fig. 4:

$$\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}, 2) = \langle \alpha, \beta, \gamma | \alpha\beta\gamma = 1 \rangle, \ \sigma_* \alpha = \alpha^{-1}, \sigma_* \gamma = \gamma^{-1}.$$

Now, σ lifts if and only if the monodromy μ of the G-covering is equivalent to the one of $\mu \circ \sigma_*$ by an inner automorphism $Int(\phi)$ of the symmetric group which yields a group automorphism $\psi \colon G \to G$. Set $a := \mu(\alpha), b := \mu(\beta)$. Then these two elements generate G, and since $\psi(a) = a^{-1}, \psi(b) = b^{-1}$ it follows that ψ has order 2, as well as the corresponding covering transformation. We have shown the existence of the desired real structure. $\square$

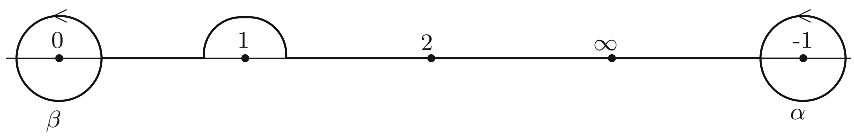

Fig. 4. The loops α and β

We shall now give a simple example of a nonreal triangle cover, based on the following

Lemma 5.31 *Let G be the symmetric group $\mathfrak{S}_n$ in $n \geq 7$ letters, let $a := (5,4,1)(2,6)$, $c := (1,2,3)(4,5,6,\dots,n)$.*
Assume that n is not divisible by 3: then
(1) There is no automorphism ψ of G carrying $a \to a^{-1}$, $c \to c^{-1}$
(2) $\mathfrak{S}_n =< a, c >$
(3) The corresponding triangle cover is not real

Proof. (1) Since $n \neq 6$, every automorphism of G is an inner one. If there is a permutation g conjugating a to a^{-1}, c to c^{-1}, g would leave each of the sets $\{1,2,3\}$, $\{4,5,\dots,n\}$, $\{1,4,5\}$, $\{2,6\}$ invariant. By looking at their intersections we conclude that g leaves the elements $1,2,3,6$ fixed. But then $gcg^{-1} \neq c^{-1}$.

(2) Observe that a^3 is a transposition: hence, it suffices to show that the group generated by a and c is 2-transitive. Transitivity being obvious, let us consider the stabilizer of 3. Since n is not divisible by 3, the stabilizer of 3 contains the cycle $(4,5,6,\dots,n)$; since it contains the transposition $(2,6)$ as well as $(5,4,1)$, this stabilizer is transitive on $\{1,2,4,\dots,n\}$.

(3) We have $ord(a) = 6$, $ord(c) = 3(n-3)$, $ord(b) = ord(ca) = ord((1,6,3)(4,2,7,\dots n)) = LCM(3,(n-4))$. Thus the orders are distinct and the nonexistence of such a ψ implies that the triangle cover is not real. □

We can now go back to Theorem 5.19, where the surfaces homeomorphic to a given surface isogenous to a product were forming one or two connected components in the moduli space. The case of products of curves is an easy example where we get one irreducible component, which is self conjugate. We show now the existence of countably many cases where there are two distinct connected components.

Theorem 5.32 *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product of unmixed type, with $g_1 \neq g_2$. Then S is deformation equivalent to $\bar{S}$ if and only if (C_j, G) is deformation equivalent to $(\overline{C_j}, G)$ for $j = 1,2$. In particular, if (C_1, G) is rigid,i.e., $C_1 \to C_1/G$ is a triangle cover, S is deformation equivalent to $\bar{S}$ only if (C_1, G) is isomorphic to $(\overline{C_1}, G)$. There are infinitely many connected components of the moduli space of surfaces of general type for which S is not deformation equivalent to $\bar{S}$.*

Proof. $\bar{S} = \overline{(C_1 \times C_2)}/G = ((\overline{C_1} \times \overline{C_2})/G$ and since $g_1 \neq g_2$ the normal subgroups Π_{g_j} of the fundamental group $\pi_1(\bar{S})$ are uniquely determined. Hence $\bar{S}$ belongs to the same irreducible connected component containing S (according to the key Proposition) if and only if $(\overline{C_j}, G)$ belongs to the same irreducible connected component containing (C_j, G).

We consider now cases where $C_1 \to C_1/G$ is a triangle cover, but not isomorphic to $(\overline{C_1}, G)$: then clearly S is not deformation equivalent to $\bar{S}$.

We let , for $n \geq 7, n \neq 0(mod3)$, $C_1 \to C_1/G$ be the nonreal triangle cover provided by Lemma 5.31. Let g_1 be the genus of C_1, observe that $2g_1 - 2 \leq (5/6)\ n!$ and consider an arbitrary integer $g \geq 2$ and a surjection $\Pi_g \to \mathcal{S}_n$ (this always exists since Π_g surjects onto a free group with g generators).

The corresponding étale covering of a curve C of genus g is a curve C_2 with genus $g_2 > g_1$ since $2g_2 - 2 \geq (2g-2)\ n! \geq 2\ n!$. The surfaces $S = C_1 \times C_2$ are our desired examples, the action of $G = \mathcal{S}_n$ on the product is free since the action on the second factor is free. □

Kharlamov and Kulikov gave [KK02, KK02-b] rigid examples of surfaces S which are not isomorphic to their complex conjugate, for instance they considered a $(\mathbb{Z}/5)^2$ covering of the plane branched on the nine lines in the plane $\mathbb{P}^2$ dual to the nine flexes of a cubic, the Fermat cubic for example. These examples have étale coverings which were constructed by Hirzebruch ([Hirz83], see also [BHH87]) in order to produce simple examples of surfaces on the Bogomolov Miyaoka Yau line $K^2 = 3c_2$, which, by results of Yau and Miyaoka [Yau77,Miya83] have the unit ball in $\mathbb{C}^2$ as universal covering, whence they are strongly rigid according to a theorem of Mostow [Most73]: this means that any surface homotopically equivalent to them is either biholomorphic or antibiholomorphic to them.

Kharlamov and Kulikov prove that the Klein group of such a surface S consists only of the above group $(\mathbb{Z}/5)^2$ of biholomorphic transformations, for an appropriate choice of the $(\mathbb{Z}/5)^2$ covering, such that to pairs of conjugate lines correspond pairs of elements of the group which cannot be obtained from each other by the action of a single automorphism of the group $(\mathbb{Z}/5)^2$.

In the next section we shall show how to obtain rigid examples with surfaces isogenous to a product.

5.4 Beauville Surfaces

Definition 5.33 *A surface S isogenous to a higher product is called a* Beauville surface *if and only if S is rigid.*

This definition is motivated by the fact that Beauville constructed such a surface in [Bea78] , as a quotient $F \times F$ of two Fermat curves of degree 5 (and genus 6). Rigidity was observed in [Cat00].

Example 5.34 ('The' Beauville surfaces) *Let F be the plane Fermat 5-ic $\{x^5 + y^5 + z^5 = 0\}$. The group $(\mathbb{Z}/5)^2$ has a projective action obtained by multiplying the coordinates by fifth roots of unity. The set of stabilizers is given by the multiples of $a := e_1, b := e_2, c := e_1 + e_2$, where $e_1(x,y,z) = (\epsilon x, y, z)$, $e_2(x,y,z) = (x, \epsilon y, z), \epsilon := exp(2\pi i/5)$. In other words, F is a triangle cover of $\mathbb{P}^1$ with group $(\mathbb{Z}/5)^2$ and generators $e_1, e_2, -(e_1+e_2)$. The set σ of stabilizers* *is the union of three lines in the vector space* $(\mathbb{Z}/5)^2$*, corresponding to three points in* $\mathbb{P}^1_{\mathbb{Z}/5}$*. Hence, there is an automorphism* ψ *of* $(\mathbb{Z}/5)^2$ *such that* $\psi(\Sigma) \cap \Sigma = \{0\}$*. Beauville lets then* $(\mathbb{Z}/5)^2$ *act on* $F \times F$ *by the action* $g(P,Q) := (gP, \psi(g)Q)$*, which is free and yields a surface* S *with* $K_S^2 = 8$*,* $p_g = q = 0$*. It is easy to see that such a surfaces is not only real, but defined over* $\mathbb{Q}$*. It was pointed out in [BaCa04] that there are exactly two isomorphism classes of such Beauville surfaces.*

Let us now construct some Beauville surfaces which are not isomorphic to their complex conjugate.

To do so, we observe that the datum of an unmixed Beauville surface amounts to a purely group theoretical datum, of two systems of generators $\{a, c\}$ and $\{a', c'\}$ for a finite group G such that, defining b through the equation $abc = 1$, and the stabilizer set $\Sigma(a,c)$ as

$$\cup_{i \in \mathbb{N}, g \in G} \{g a^i g^{-1}, g b^i g^{-1}, g c^i g^{-1}\}$$

the following condition must be satisfied, assuring that the diagonal action on the product of the two corresponding triangle curves is free

$$\Sigma(a,c) \cap \Sigma(a',c') = \{1_G\}.$$

Example 5.35 *Consider the symmetric group* $\mathfrak{S}_n$ *for* $n \equiv 2 (mod\ 3)$*, define elements* $a, c \in \mathfrak{S}_n$ *as in Lemma 5.31, and define further* $a' := \sigma^{-1}$*,* $c' := \tau\sigma^2$*, where* $\tau := (1,2)$ *and* $\sigma := (1, 2, \dots, n)$*. It is obvious that* $\mathfrak{S}_n = < a', c' >$*. Assuming* $n \geq 8$ *and* $n \equiv 2(3)$*, it is easy to verify that* $\Sigma(a,c) \cap \Sigma(a',c') = \{1\}$*, since one observes that elements which are conjugate in* $\mathfrak{S}_n$ *have the same type of cycle decomposition. The types in* $\Sigma(a,c)$ *are derived from* (6)*,* $(3n-9)$*,* $(3n-12)$*, (as for instance* (3)*,* (2)*,* $(n-4)$ *and* $(n-3)$*) since we assume that* 3 *does neither divide* n *nor* $n-1$*, whereas the types in* $\Sigma(a',c')$ *are derived from* (n)*,* $(n-1)$*, or* $(\frac{n-1}{2}, \frac{n+1}{2})$*.*

One sees therefore (since $g_1 \neq g_2$*) that the pairs* $(a,c), (a',c')$ *determine Beauville surfaces which are not isomorphic to their complex conjugates.*

Our knowledge of Beauville surfaces is still rather unsatisfactory, for instance the following question is not yet completely answered.

Question 5.36 *Which groups* G *can occur?*

It is easy to see (cf. [BCG05]) that if the group G is abelian, then it can only be $(\mathbb{Z}/n)^2$, where $G.C.D.(n, 6) = 1$.

Together with I. Bauer and F. Grunewald, we proved in [BCG05] (see also [BCG06]) the following results:

Theorem 5.37 *(1) The following groups admit unmixed Beauville structures*

(a) $\mathfrak{A}_n$ *for large* n

(b) $\mathfrak{S}_n$ *for* $n \in \mathbf{N}$ *with* $n \geq 7$

(c) $\mathbf{SL}(2, \mathbb{F}_p)$*,* $\mathbf{PSL}(2, \mathbb{F}_p)$ *for* $p \neq 2, 3, 5$

After checking that all finite simple nonabelian groups of order ≤ 50000, with the exception of $\mathfrak{A}_5$, admit unmixed Beauville structures, we were led to the following

Conjecture 5.38 *All finite simple nonabelian groups except* $\mathfrak{A}_5$ *admit an unmixed Beauville structure.*

Beauville surfaces were extensively studied in [BCG05] (cf. also [BCG06]) with special regard to the effect of complex conjugation on them.

Theorem 5.39 *There are Beauville surfaces* S *not biholomorphic to* $\bar{S}$ *with group*

(1) The symmetric group $\mathfrak{S}_n$ *for any* $n \geq 7$
(2) The alternating group $\mathfrak{A}_n$ *for* $n \geq 16$ *and* $n \equiv 0$ *mod* 4, $n \equiv 1$ *mod* 3, $n \not\equiv 3, 4$ *mod* 7

We got also examples of isolated real points in the moduli space which do not correspond to real surfaces:

Theorem 5.40 *Let* $p > 5$ *be a prime with* $p \equiv 1$ *mod* 4, $p \not\equiv 2, 4$ *mod 5*, $p \not\equiv 5$ *mod* 13 *and* $p \not\equiv 4$ *mod* 11. *Set* $n := 3p + 1$. *Then there is a Beauville surface* S *with group* $\mathfrak{A}_n$ *which is biholomorphic to its conjugate* $\bar{S}$, *but is not real.*

Beauville surfaces of the mixed type also exist, but their construction turns out to be quite more complicated (see [BCG05]). Indeed (cf. [BCG06-b]) the group of smallest order has order 512.

6 Lecture 5: Lefschetz Pencils, Braid and Mapping Class Groups, and Diffeomorphism of ABC-Surfaces

6.1 Surgeries

The most common surgery is the connected sum, which we now describe.

Let M be a manifold of real dimension m, thus for each point $p \in M$ there is an open set U_p containing p and a homeomorphism (local coordinate chart) $\psi_p : U_p \to V_p \subset \mathbb{R}^m$ onto an open set V_p of $\mathbb{R}^m$ such that (on its domain of definition)

$\psi_{p'} \circ \psi_p^{-1}$ is a:

- Homeomorphism (onto its image) if M is a *topological manifold*
- Diffeomorphism (onto its image) if M is a *differentiable manifold*
- Biholomorpism (onto its image) if M is a *complex manifold* (in this last case $m = 2n$, $\mathbb{R}^m = \mathbb{C}^n$).

Definition 6.1 *The operation of connected sum $M_1 \sharp M_2$ can be done for two differentiable or topological manifolds of the same dimension.*

Choose respective points $p_i \in M_i$ and local charts

$$\psi_{p_i} : U_{p_i} \xrightarrow{\cong} B(0, \epsilon_i) := \{x \in \mathbb{R}^m | |x| < \epsilon_i\}.$$

Fix positive real numbers $r_i < R_i < \epsilon_i$ such that

$$(**)\ R_2/r_2 = R_1/r_1$$

and set $M_i^* := M_i \backslash \psi_{p_i}^{-1}(\overline{B(0, r_i)})$: then M_1^* and M_2^* are glued together through the diffeomorphism $\psi : N_1 := B(0, R_1) \backslash \overline{B(0, r_1)} \to N_2 := B(0, R_2) \backslash \overline{B(0, r_2)}$ such that $\psi(x_1) = \frac{R_2 r_1}{|x_1|} \tau(x_1)$ where either $\tau(x) = x$, or $\tau(x)$ is an orientation reversing linear isometry (in the case where the manifolds M_i are oriented, we might prefer, in order to furnish the connected sum $M_1 \sharp M_2$ of a compatible orientation, to have that ψ be orientation preserving).

In other words the connected sum $M_1 \sharp M_2$ is the quotient space of the disjoint union $(M_1^*) \cup^o (M_2^*)$ through the equivalence relation which identifies $y \in \psi_{p_1}^{-1}(N_1))$ to $w \in \psi_{p_2}^{-1}(N_2))$ iff

$$w = \psi_{p_2}^{-1} \circ \psi \circ \psi_{p_1}(y).$$

We have the following

Theorem *The result of the operation of connected sum is independent of the choices made.*

An elementary and detailed proof in the differentiable case (the one in which we are more interested) can be found in [B-J90], pages 101–110.

Example 6.2 *The most intuitive example (see Fig. 5) is the one of two compact orientable Riemann surfaces M_1, M_2 of respective genera g_1, g_2: $M_1 \sharp M_2$ has then genus $g_1 + g_2$. In this case, however, if M_1, M_2 are endowed of a complex structure, we can even define a connected sum as complex manifolds, setting $\psi(z_1) = e^{2\pi i\theta}\ \frac{R_2 r_1}{z_1}$.*

Here, however, the complex structure is heavily dependent on the parameters p_1, p_2, $e^{2\pi i\theta}$, and $R_2 r_1 = R_1 r_2$.

In fact, if we set $t := R_2 r_1 e^{2\pi i\theta} \in \mathbb{C}$, we see that $z_1 z_2 = t$, and if $t \to 0$ then it is not difficult to see that the limit of $M_1 \sharp M_2$ is the singular curve obtained from M_1, M_2 by glueing the points p_1, p_2 to obtain the node $z_1 z_2 = 0$.

This interpretation shows that we get in this way all the curves near the boundary of the moduli space $\mathfrak{M}_g$. It is not clear to us in this moment how big a subset of the moduli space one gets through iterated connected sum operations. One should however point out that many of the conjectures made about the stable cohomology ring $H^*(\mathfrak{M}_g, \mathbb{Z})$ were suggested by the possibility of interpreting the connected sum as a sort of H-space structure on the union of all the moduli spaces $\mathfrak{M}_g$ (cf. [Mum83]).

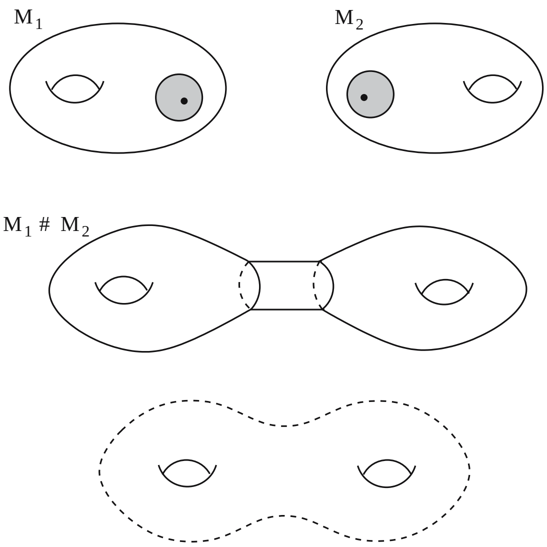

Fig. 5. The connected sum

Remark 6.3 *(1) One cannot perform a connected sum operation for complex manifolds of dimension > 1. The major point is that there is no biholomorphism bringing the inside boundary of the ring domain N_1 to the outside boundary of N_2. The reason for this goes under the name of holomorphic convexity: if $n \geq 2$ every holomorphic function on N_1 has, by Hartogs' theorem, a holomorphic continuation to the ball $B(0, R_1)$. While, for each point p in the outer boundary, there is a holomorphic function f on N_1 such that $lim_{z \to p}|f(z)| = \infty$.*

(2) The operation of connected sum makes the diffeomorphism classes of manifolds of the same dimension m a semigroup: associativity holds, and as neutral element we have the sphere $S^m := \{x \in \mathbb{R}^{m+1} ||x| = 1\}$.

(3) A manifold M is said to be irreducible *if $M \cong M_1 \sharp M_2$ implies that either M_1 or M_2 is homotopically equivalent to a sphere S^m.*

A further example is the more general concept of

Definition 6.4 (SURGERY) *For $i = 1, 2$, let $N_i \subset M_i$ be a differentiable submanifold.*

Then there exists (if $M_i = \mathbb{R}^N$ this is an easy consequence of the implicit function theorem) an open set $U_i \supset N_i$ which is diffeomorphic to the normal bundle ν_{N_i} of the embedding $N_i \to M_i$, and through a diffeomorphism which carries N_i onto the zero section of ν_{N_i}.

Suppose now that we have diffeomorphisms $\phi : N_1 \to N_2$, and $\psi : (\nu_{N_1} - N_1) \to (\nu_{N_2} - N_2)$, the latter compatible with the projections $p_i : \nu_{N_i} \to N_i$ (i.e., $p_2 \circ \psi = \phi \circ p_1$), and with the property of being orientation reversing on the *fibres. We can then define as before a manifold $M_1 \sharp_\psi M_2$, the quotient of the disjoint union $(M_1 - N_1) \cup^o (M_2 - N_2)$ by the equivalence relation identifying $(U_1 - N_1)$ with $(U_2 - N_2)$ through the diffeomorphism induced by ψ.*

Remark 6.5 *This time the result of the operation depends upon the choice of ϕ and ψ.*

The two surgeries described above combine together in the special situation of the fibre sum.

Definition 6.6 (FIBRE SUM) *For $i = 1, 2$, let $f_i : M_i \to B_i$ be a proper surjective differentiable map between differentiable manifolds, let $p_i \in B_i$ be a noncritical value, and let $N_i \subset M_i$ be the corresponding smooth fibre $N_i := f_i^{-1}(p_i)$.*

Then there exists a natural trivialization (up to a constant matrix) of the normal bundle ν_{N_i} of the embedding $N_i \to M_i$, and if we assume as before that we have a diffeomorphism $\phi : N_1 \to N_2$ we can perform a surgery $M := M_1 \sharp_\phi M_2$, and the new manifold M admits a proper surjective differentiable map onto the connected sum $B := B_1 \sharp B_2$.

The possibility of variations on the same theme is large: for instance, given $f_i : M_i \to B_i$ $(i = 1, 2)$ proper surjective differentiable maps between differentiable manifolds with boundary, assume that $\partial M_i \to \partial B_i$ is a fibre bundle, and there are compatible diffeomorphisms $\phi : \partial B_1 \to \partial B_2$ and $\psi : \partial M_1 \to \partial M_2$: then we can again define the fibre sum $M := M_1 \sharp_\psi M_2$ which admits a proper surjective differentiable map onto $B := B_1 \sharp_\phi B_2$.

In the case where $(B_2, \partial B_2)$ is an euclidean ball with a standard sphere as boundary, and $M_2 = F \times B_2$, the question about unicity (up to diffeomorphism) of the surgery procedure is provided by a homotopy class. Assume in fact that we have two attaching diffeomorphisms $\psi, \psi' : \partial M_1 \to F \times \partial B_2$. Then from them we construct $\Psi := \psi' \circ \psi^{-1} : F \times \partial B_2 \to F \times \partial B_2$, and we notice that $\Psi(x, t) = (\Psi_1(x, t), \Psi_2(t))$, where $\Psi_2(t) = \phi' \circ \phi^{-1}$. We can then construct a classifying map $\chi : \partial B_2 \cong S^{n-1} \to Diff(F)$ such that

$$\Psi_1(x, t) = \chi(\Psi_2(t))(x).$$

We get in this way a free homotopy class $[\chi]$, on which the diffeomorphism class of the surgery depends. If this homotopy class is a priori trivial, then the result is independent of the choices made: this is the case for instance if F is a compact complex curve of genus $g \geq 1$.

In order to understand better the unicity of these surgery operations, and of their compositions, we therefore see the necessity of a good understanding of isotopies of diffeomorphisms. To this topic is devoted the next subsection.

6.2 Braid and Mapping Class Groups

E. Artin introduced the definition of the *braid group* (cf. [Art26, Art65]), thus allowing a remarkable extension of Riemann's concept of monodromy of algebraic functions. Braids are a powerful tool, even if not so easy to handle, and especially appropriate for the study of the differential topology of algebraic varieties, in particular of algebraic surfaces.

Remark 6.7 *We observe that the subsets* $\{w_1, \dots, w_n\} \subset \mathbb{C}$ *of* n *distinct points in* $\mathbb{C}$ *are in one to one correspondence with monic polynomials* $P(z) \in \mathbb{C}[z]$ *of degree* n *with non vanishing discriminant* $\delta(P)$*.*

Definition 6.8 *Let* $\mathbb{C}[z]^1_n$ *be the affine space of monic polynomials of degree* n*. Then the group*

$$\mathcal{B}_n := \pi_1(\mathbb{C}[z]^1_n \setminus \{P | \delta(P) = 0\}),$$

i.e., the fundamental group of the space of monic polynomials of degree n *having* n *distinct roots, is called* Artin's braid group.

Usually, one takes as base point the polynomial $P(z) = (\prod_{i=1}^n (z - i)) \in \mathbb{C}[z]^1_n$ (or the set $\{1, \dots, n\}$).

To a closed (continuous) path $\alpha : [0, 1] \to (\mathbb{C}[z]^1_n \setminus \{P | \delta(P) = 0\})$ one can associate the subset $B_\alpha := \{(z, t) \in \mathbb{C} \times \mathbb{R} \ \mid \ \alpha_t(z) := \alpha(t)(z) = 0\}$ of $\mathbb{R}^3$, which gives a visually suggestive representation of the associated braid.

It is however customary to view a braid as moving from up to down, that is, to associate to α the set $B'_\alpha := \{(z, t) | (z, -t) \in B_\alpha\}$.

Figure 6 below shows two realizations of the same braid.

Remark 6.9 *There is a lifting of* α *to* $\mathbb{C}^n$*, the space of ordered* n*-tuples of roots of monic polynomials of degree* n*, hence there are (continuous) functions* $w_i(t)$ *such that* $w_i(0) = i$ *and* $\alpha_t(z) = \prod_{i=1}^n (z - w_i(t))$*.*

It follows that to each braid is naturally associated a permutation $\tau \in \mathfrak{S}_n$ *given by* $\tau(i) := w_i(1)$*.*

Even if it is not a priori evident, a very powerful generalization of Artin's braid group was given by M. Dehn (cf. [Dehn38], we refer also to the book [Bir74]).

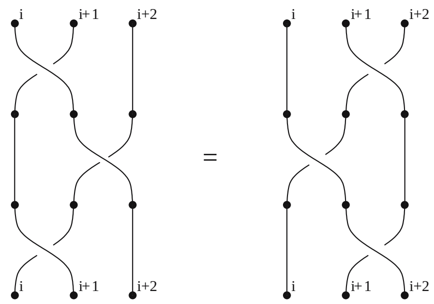

Fig. 6. Relation $aba = bab$ on braids

Definition 6.10 *Let M be a differentiable manifold, then the* mapping class group *(or* Dehn group*) of M is the group*

$$Map(\mathrm{M}) := \pi_0(Diff(\mathrm{M})) = (Diff(\mathrm{M})/Diff^0(\mathrm{M})),$$

where $Diff^0(\mathrm{M})$, the connected component of the identity, is the subgroup of diffeomorphisms of M isotopic *to the identity (i.e., they are connected to the identity by a path in $Diff(\mathrm{M})$).*

Remark 6.11 *If M is oriented then we often tacitly take $Diff^+(\mathrm{M})$, the group of orientation preserving diffeomorphisms of M instead of $Diff(\mathrm{M})$, in the definition of the mapping class group. But it is more accurate to distinguish in this case between $Map^+(\mathrm{M})$ and $Map(\mathrm{M})$.*

If M is a compact complex curve of genus g, then its mapping class group is denoted by Map_g. The representation of $M = C_g$ as the $K(\pi, 1)$ space $\mathbb{H}/\Pi_g$, i.e., as a quotient of the (contractible) upper halfplane $\mathbb{H}$ by the free action of a Fuchsian group isomorphic to $\Pi_g \cong \pi_1(C_g)$, immediately yields the isomorphism $Map_g \cong Out(\Pi_g) = Aut(\Pi_g)/Int(\Pi_g)$.

In this way the orbifold exact sequences considered in the previous lecture

$$1 \to \Pi_{g_1} \to \pi_1^{orb} \to G \to 1$$

determine the topological action of G since the homomorphism $G \to Map_g$ is obtain by considering, for $g \in G$, the automorphisms obtained via conjugation by a lift $\tilde{g} \in \pi_1^{orb}$ of g.

The relation between Artin's and Dehn's definition is the following:

Theorem 6.12 *The braid group $\mathcal{B}_n$ is isomorphic to the group*

$$\pi_0(Map^\infty(\mathbb{C}\backslash\{1, \dots n\})),$$

where $Map^\infty(\mathbb{C}\backslash\{1, \dots n\})$ is the group of diffeomorphisms which are the identity outside the disk with centre 0 and radius $2n$.

In this way Artin's standard generators σ_i of $\mathcal{B}_n$ $(i = 1, \dots n-1)$ can be represented by the so-called half-twists.

Definition 6.13 *The* half-twist *σ_j is the diffeomorphism of $\mathbb{C}\backslash\{1, \dots n\}$ isotopic to the homeomorphism given by:*

– Rotation of $180°$ on the disk with centre $j + \frac{1}{2}$ and radius $\frac{1}{2}$

– On a circle with the same centre and radius $\frac{2+t}{4}$ the map σ_j is the identity if $t \geq 1$ and rotation of $180(1-t)$ degrees, if $t \leq 1$

Now, it is obvious from Theorem 6.12 that $\mathcal{B}_n$ acts on the free group $\pi_1(\mathbb{C}\backslash\{1, \dots n\})$, which has a geometric basis (we take as base point the complex number $p := -2ni$) $\gamma_1, \dots \gamma_n$ as illustrated in Fig. 7.

This action is called the *Hurwitz action of the braid group* and has the following algebraic description

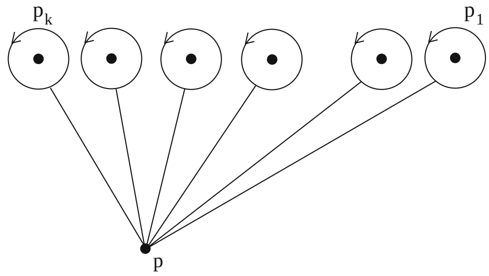

Fig. 7. A geometric basis of $\pi_1(\mathbb{C} - \{1, \dots n\})$

- $\sigma_i(\gamma_i) = \gamma_{i+1}$
- $\sigma_i(\gamma_i\gamma_{i+1}) = \gamma_i\gamma_{i+1}$, whence $\sigma_i(\gamma_{i+1}) = \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}$
- $\sigma_i(\gamma_j) = \gamma_j$ for $j \neq i, i+1$

Observe that the product $\gamma_1\gamma_2 \dots \gamma_n$ is left invariant under this action.

Definition 6.14 *Let us consider a group G and its cartesian product G^n. The map associating to each $(g_1, g_2, \dots, g_n)$ the product $g := g_1 g_2 \dots, g_n \in G$ gives a partition of G^n, whose subsets are called* factorizations *of an element $g \in G$.*

$\mathcal{B}_n$ acts on G^n leaving invariant the partition, and its orbits are called the Hurwitz equivalence classes of factorizations*.*

We shall use the following notation for a factorization: $g_1 \circ g_2 \circ \cdots \circ g_n$, which should be carefully distinguished from the product $g_1 g_2 \dots g_n$, which yields an element of G.

Remark 6.15 *A broader equivalence relation for the set of factorizations is obtained considering the equivalence relation generated by Hurwitz equivalence and by* simultaneous conjugation*. The latter, using the following notation $a_b := b^{-1}ab$, corresponds to the action of G on G^n which carries $g_1 \circ g_2 \circ \cdots \circ g_n$ to $(g_1)_b \circ (g_2)_b \circ \cdots \circ (g_n)_b$.*

Observe that the latter action carries a factorization of g to a factorization of the conjugate g_b of g, hence we get equivalence classes of factorizations for conjugacy classes of elements of G.

The above equivalence relation plays an important role in several questions concerning plane curves and algebraic surfaces, as we shall soon see.

Let us proceed for the meantime considering another interesting relation between the braid groups and the Mapping class groups.

This relation is based on the topological model provided by the hyperelliptic curve C_g of equation

$$w^2 = \prod_{i=1}^{2g+2} (z - i)$$

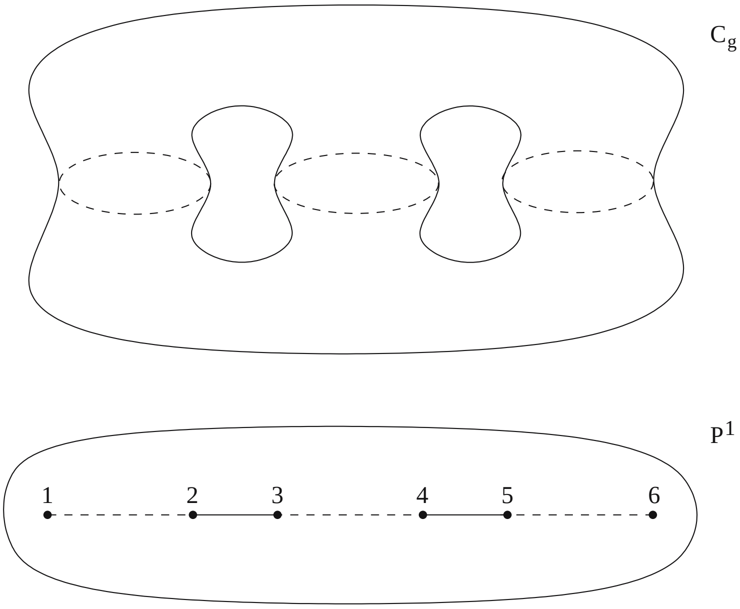

Fig. 8. Hyperelliptic curve of genus 2

(see Fig. 8 describing a hyperelliptic curve of genus $g = 2$).

Observe that, if Y is the double unramified covering of $(\mathbb{P}^1 - \{1, \dots 2g+2\})$, inverse image of $(\mathbb{P}^1 - \{1, \dots 2g+2\})$ in C_g, C_g is the natural compactification of Y obtained by adding to Y the *ends* of Y (i.e., in such a compactification one adds to Y the following $lim_{K \subset\subset Y} \pi_0(Y - K)$).

This description makes it clear that every homeomorphism of $(\mathbb{P}^1 - \{1, \dots 2g+2\})$ which leaves invariant the subgroup associated to the covering Y admits a lifting to a homeomorphism of Y, whence also to a homeomorphism of its natural compactification C_g.

Such a lifting is not unique, since we can always compose with the nontrivial automorphism of the covering.

We obtain in this way a central extension

$$1 \to \mathbb{Z}/2 =< H >\to \mathcal{M}ap_g^h \to \mathcal{M}ap_{0,2g+2} \to 1$$

where

- H is the hyperelliptic involution $w \to -w$ (the nontrivial automorphism of the covering)
- $\mathcal{M}ap_{0,2g+2}$ is the Dehn group of $(\mathbb{P}^1 - \{1, \dots 2g+2\})$
- $\mathcal{M}ap_g^h$ is called the hyperelliptic subgroup of the mapping class group $\mathcal{M}ap_g$, which consists of all the possible liftings.
  If $g \geq 3$, it is a proper subgroup of $\mathcal{M}ap_g$.

While Artin's braid group $\mathcal{B}_{2g+2}$ has the following presentation:

$$\langle \sigma_1, \dots \sigma_{2g+1} | \sigma_i \sigma_j = \sigma_j \sigma_i \ for |i-j| \geq 2,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle,$$

Dehn's group of $(\mathbb{P}^1 - \{1, \dots 2g+2\})$ $\mathcal{M}ap_{0,2g+2}$ has the presentation:

$$\langle \sigma_1, \dots \sigma_{2g+1} | \sigma_1 \dots \sigma_{2g+1} \sigma_{2g+1} \dots \sigma_1 = 1, (\sigma_1 \dots \sigma_{2g+1})^{2g+2} = 1,$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \ for |i-j| \geq 2,\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle,$$

finally the hyperelliptic mapping class group $\mathcal{M}ap^h_g$ has the presentation:

$$\langle \xi_1, \dots \xi_{2g+1}, H | \xi_1 \dots \xi_{2g+1} \xi_{2g+1} \dots \xi_1 = H, H^2 = 1, (\xi_1 \dots \xi_{2g+1})^{2g+2} = 1,$$

$$H \xi_i = \xi_i H \ \forall i, \xi_i \xi_j = \xi_j \xi_i \ for |i-j| \geq 2,\ \xi_i \xi_{i+1} \xi_i = \xi_{i+1} \xi_i \xi_{i+1} \rangle.$$

We want to illustrate the geometry underlying these important formulae. Observe that σ_j yields a homeomorphism of the disk U with centre $j + 1/2$ and radius $3/4$, which permutes the two points $j, j+1$.

Therefore there are two liftings of σ_j to homeomorphisms of the inverse image V of U in C_g: one defines then ξ_j as the one of the two liftings which acts as the identity on the boundary ∂V, which is a union of two loops (see Fig. 9).

ξ_j is called the *Dehn twist* and corresponds geometrically to the diffeomorphism of a truncated cylinder which is the identity on the boundary, a rotation by 180° on the equator, and on each parallel at height t is a rotation by t 360° (where $t \in [0, 1]$).

One can define in the same way a Dehn twist for each loop in C_g (i.e., a subvariety diffeomorphic to S^1):

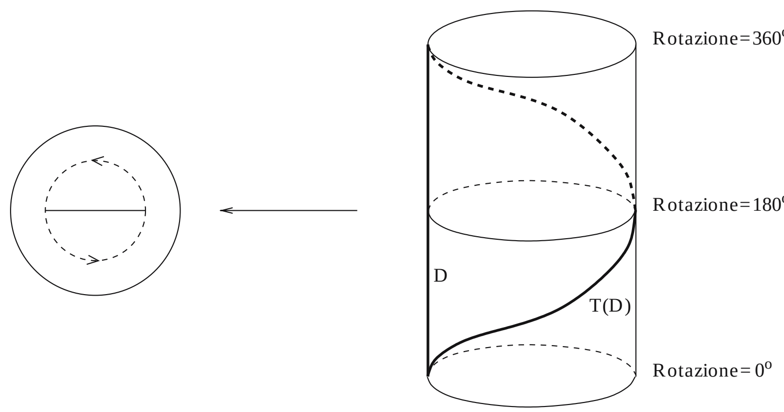

Fig. 9. At the *left*, a half twist; at the *right*: its lift, the Dehn-Twist-T, and its action on the segment D

Definition 6.16 *Let C be an oriented Riemann surface. Then a* positive Dehn twist T_α *with respect to a simple closed curve α on C is an isotopy class of a diffeomorphism h of C which is equal to the identity outside a neighbourhood of α orientedly homeomorphic to an annulus in the plane, while inside the annulus h rotates the inner boundary of the annulus by $360°$ to the right and damps the rotation down to the identity at the outer boundary.*

Dehn's fundamental result [Dehn38] was the following

Theorem 6.17 *The mapping class group $\mathcal{M}ap_g$ is generated by Dehn twists.*

Explicit presentations of $\mathcal{M}ap_g$ have later been given by Hatcher and Thurston [HT80], and an improvement of the method lead to the simplest available presentation, due to Wajnryb ([Waj83], see also [Waj99]).

We shall see in the next subsection how the Dehn twists are related to the theory of Lefschetz fibrations.

6.3 Lefschetz Pencils and Lefschetz Fibrations

The method introduced by Lefschetz for the study of the topology of algebraic varieties is the topological analogue of the method of hyperplane sections and projections of the classical italian algebraic geometers.

An excellent exposition of the theory of Lefschetz pencils is the article by Andreotti and Frankel [A-F69], that we try to briefly summarize here.

Let $X \subset \mathbb{P}^N$ be projective variety, which for simplicity we assume to be smooth, and let $L \cong \mathbb{P}^{N-2} \subset \mathbb{P}^N$ be a general linear subspace of codimension 2. L is the base locus of a pencil of hyperplanes $H_t, t \in \mathbb{P}^1$, and the indeterminacy locus of a rational map $\phi : \mathbb{P}^N \backslash L \to \mathbb{P}^1$.

The intersection $Z : X \cap L$ is smooth, and the blow up of X with centre Z yields a smooth variety X' with a morphism $f \colon X' \to \mathbb{P}^1$ whose fibres are isomorphic to the hyperplane sections $Y_t := X \cap H_t$, while the exceptional divisor is isomorphic to the product $Z \times \mathbb{P}^1$ and on it the morphism f corresponds to the second projection.

Definition 6.18 *The dual variety $W^\vee \subset \mathbb{P}^{N^\vee}$ of a projective variety W is defined as the closure of the set of hyperplanes which contain the tangent space TW_p at a smooth point $p \in W$. A pencil of hyperplanes $H_t, t \in \mathbb{P}^1$, is said to be a* Lefschetz pencil *if the line L' dual to the subspace L*

(1) Does not intersect $W^\vee$ if $W^\vee$ is not a hypersurface

(2) Intersects $W^\vee$ transversally in $\mu := deg(W^\vee)$ points otherwise

An important theorem is the

Biduality theorem: $(W^\vee)^\vee = W$.

It follows from the above theorem and the previous definition that if $W^\vee$ is not a hypersurface, f is a differentiable fibre bundle, while in case (2) all the fibres are smooth, except μ fibres which correspond to tangent hyperplanes

H_{t_j}. And for these Y_{t_j} has only one singular point p_j, which has an ordinary quadratic singularity as a hypersurface in X (i.e., there are local holomorphic coordinates $(z_1, \dots z_n)$ for X such that locally at p_h

$$Y_{t_h} = \{z | \sum_j z_j^2 = 0\}).$$

Writing $z_j = u_j + iv_j$, the equation $\sum_j z_j^2 = \rho$ for $\rho \in \mathbb{R}$ reads out as $\sum_j u_j v_j = 0, \sum_j (u_j^2 - v_j^2) = \rho$. In vector notation, and assuming $\rho \in \mathbb{R}_{\geq 0}$, we may rewrite as

$$\langle u, v \rangle = 0, |u|^2 = \rho + |v|^2.$$

Definition 6.19 *The* vanishing cycle *is the sphere $\Sigma_{t_h+\rho}$ of $Y_{t_h+\rho}$ given, for $\rho \in \mathbb{R}_{>0}$, by $\{u + iv| \ |u|^2 = \rho, v = 0\}$.*

The normal bundle of the vanishing cycle Σ_t in Y_t is easily seen, in view of the above equations, to be isomorphic to the tangent bundle to the sphere S^{n-1}, whence we can identify a tubular neighbourhood of Σ_t in Y_t to the unit ball in the tangent bundle of the sphere S^{n-1}. We follow now the definition given in [Kas80] of the corresponding Dehn twist.

Definition 6.20 *Identify the sphere $\Sigma = S^{n-1} = \{u||u| = 1\}$ to the zero section of its unit tangent bundle $Y = \{(u,v)|\langle u, v\rangle = 0, |u| = 1, |v| \leq 1\}$.*

Then the Dehn twist $T := T_\Sigma$ is the diffeomorphism of Y such that, if we let $\gamma_{u,v}(t)$ be the geodesic on S^{n-1} with initial point u, initial velocity v, then

$$T(u,v) := -(\gamma_{u,v}(\pi|v|), \frac{d}{dt}\gamma_{u,v}(\pi|v|)).$$

We have then: (1) T is the antipodal map on Σ

(2) T is the identity on the boundary $\partial Y = \{(u,v)|\langle u, v\rangle = 0, |u| = 1 = |v|\}$.

One has the

Picard-Lefschetz Theorem *The Dehn twist T is the local monodromy of the family Y_t (given by the level sets of the function $\sum_j z_j^2$).*

Moreover, by the classical Ehresmann theorem, one sees that a singular fibre Y_{t_j} is obtained from a smooth fibre by substituting a neighbourhood of the vanishing cycle Σ with the contractible intersection of the complex quadratic cone$\sum_j z_j^2 = 0$ with a ball around p_j. Hence

Theorem 6.21 (Generalized Zeuthen Segre formula) *The number μ of singular fibres in a Lefschetz pencil, i.e., the degree of the dual variety $X^\vee$, is expressed as a sum of topological Euler numbers*

$$e(X) + e(Z) = 2e(Y) + (-1)^n \mu,$$

where Y is a smooth hyperplane section, and $Z = L \cap X$ is the base locus of the pencil.

Proof. (idea) Replacing Z by $Z \times \mathbb{P}^1$ we see that we replace $e(Z)$ by $2e(Z)$, hence the left hand side expresses the Euler number of the blow up X'.

This number can however be computed from the mapping f: since the Euler number is multiplicative for fibre bundles, we would have that this number were $2e(Y)$ if there were no singular fibre. Since however for each singular fibre we replace something homotopically equivalent to the sphere S^{n-1} by a contractible set, we have to subtract $(-1)^{n-1}$ for each singular fibre. □

Lefschetz pencils were classically used to describe the homotopy and homology groups of algebraic varieties.

The main point is that the finite part of X', i.e., $X' - Y_\infty$, has the socalled 'Lefschetz spine' as homotopy retract.

In order to explain what this means, assume, without loss of generality from the differentiable viewpoint, that the fibres Y_0 and Y_∞ are smooth fibres, and that the singular fibres occur for some roots of unity t_j, which we can order in counterclockwise order.

Definition 6.22 *Notation being as before, define the relative vanishing cycle Δ_j as the union, over t in the segment $[0, t_j]$, of the vanishing cycles $\Sigma_{t,j}$: these are defined, for t far away from t_j, using a trivialization of the fibre bundle obtained restricting $f: X' \to \mathbb{P}^1$ to the half open segment $[0, t_j)$.*

The Lefschetz spine *of the Lefschetz pencil is the union of the fibre Y_0 with the μ relative vanishing cycles Δ_j.*

Theorem 6.23 (Lefschetz' theorems I and II) *(1) The Lefschetz spine is a deformation retract of $X' - Y_\infty$.*

(2) The affine part $X - Y_\infty$ has the homotopy type of a cell complex of dimension n.

(3) The inclusion $\iota: Y_0 \to X$ induces homology homomorphisms $H_i(\iota): H_i(Y_0, \mathbb{Z}) \to H_i(X, \mathbb{Z})$ which are

(3i) Bijective for $i < n-1$

(3ii) Surjective if $i = n-1$; moreover

(4) The kernel of $H_{n-1}(\iota)$ is generated by the vanishing cycles, i.e., by the images of $H_{n-1}(\Sigma_{0,j}, \mathbb{Z})$.

Comment on the proof:

(1) Follows by using the Ehresmann's theorem outside of the singularities, and by retracting locally a neighbourhood of the singularities partly on a smooth fibre Y_t, with $t \in (0, t_j)$, and partly on the union of the vanishing cycles. Then one goes back all the way to Y_0.

For (2) we simply observe that $X - Y_\infty$ has $(Y_0 \backslash Z) \cup (\cup_j \Delta_j)$ as deformation retract. Hence, it is homotopically equivalent to a cell complex obtained by attaching μ n-cells to $Y_0 \backslash Z$, and (2) follows then by induction on n.

(3) and (4) are more delicate and require some diagram chasing, which can be found in [A-F69], and which we do not reproduce here. □

In the 1970s Moishezon and Kas realized (see e.g. [Moi77] and [Kas80]), after the work of Smale on the smoothing of handle attachments, that Lefschetz fibrations could be used to investigate the differential topology of algebraic varieties, and especially of algebraic surfaces.

For instance, they give a theoretical method, which we shall now explain, for the extremely difficult problem to decide whether two algebraic surfaces which are not deformation equivalent are in fact diffeomorphic [Kas80].

Definition 6.24 *Let M be a compact differentiable (or even symplectic) manifold of real even dimension $2n$*

A Lefschetz fibration *is a differentiable map $f : M \to \mathbb{P}^1_{\mathbb{C}}$ which*

(a) is of maximal rank except for a finite number of critical points $p_1, \dots p_m$ which have distinct critical values $b_1, \dots b_m \in \mathbb{P}^1_{\mathbb{C}}$.

(b) has the property that around p_i there are complex coordinates $(z_1, \dots z_n) \in \mathbb{C}^n$ such that locally $f = \sum_j z_j^2 + const.$ (in the symplectic case, we require the given coordinates to be Darboux coordinates, i.e., such that the symplectic form ω of M corresponds to the natural constant coefficients symplectic structure on $\mathbb{C}^n$).

Remark 6.25 *(1) A similar definition can be given if M is a manifold with boundary, replacing $\mathbb{P}^1_{\mathbb{C}}$ by a disc $D \subset \mathbb{C}$.*

(2) An important theorem of Donaldson [Don99] asserts that for symplectic manifolds there exists (as for the case of projective manifolds) a Lefschetz pencil, i.e., a Lefschetz fibration $f : M' \to \mathbb{P}^1_{\mathbb{C}}$ on a symplectic blow up M' of M (see [MS98] for the definition of symplectic blow-up).

(3) A Lefschetz fibration with smooth fibre $F_0 = f^{-1}(b_0)$ and with critical values $b_1, \dots b_m \in \mathbb{P}^1_{\mathbb{C}}$, once a geometric basis $\gamma_1, \gamma_2, \dots, \gamma_m$ of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{b_1, \dots, b_m\}, b_0)$ is chosen, determines a factorization of the identity in the mapping class group $\mathcal{M}ap(F_0)$

$$\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m = Id$$

as a product of Dehn twists.

(4) Assume further that $b_0, b_1, \dots b_m \in \mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$: then the Lefschetz fibration determines also a homotopy class of an arc λ between $\tau_1 \tau_2 \dots \tau_m$ and the identity in $Diff^0(F_0)$. This class is trivial when $F_0 = C_g$, a compact Riemann surface of genus $g \geq 1$.

(5) More precisely, the Lefschetz fibration f determines isotopy classes of embeddings $\phi_j : S^{n-1} \to F_0$ and of bundle isomorphisms ψ_j between the tangent bundle of S^{n-1} and the normal bundle of the embedding ϕ_j; τ_j corresponds then to the Dehn twist for the embedding ϕ_j.

We are now ready to state the theorem of Kas (cf. [Kas80]).

Theorem 6.26 *Two Lefschetz fibrations (M, f), (M', f') are equivalent (i.e., there are two diffeomorphisms $u : M \to M', v : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f' \circ u = v \circ f$) if and only if*

(1) The corresponding invariants

$$(\phi_1, \dots \phi_m), (\psi_1, \dots \psi_m); (\phi'_1, \dots \phi'_m), (\psi'_1, \dots \psi'_m)$$

correspond to each other via a diffeomorphism of F_0 *and a diffeomorphism* v *of* $\mathbb{P}^1$. *This implies in particular*

(1′) The two corresponding factorizations of the identity in the mapping class group are equivalent (under the equivalence relation generated by Hurwitz equivalence and by simultaneous conjugation).

(2) The respective homotopy classes λ, λ' *correspond to each other under the above equivalence.*

Conversely, given $(\phi_1, \dots \phi_m)(\psi_1, \dots \psi_m)$ *such that the corresponding Dehn twists* $\tau_1, \tau_2, \dots \tau_m$ *yield a factorization of the identity, and given a homotopy class* λ *of a path connecting* $\tau_1 \tau_2 \dots \tau_m$ *to the identity in* $Diff(F_0)$, *there exists an associated Lefschetz fibration.*

If the fibre F_0 *is a Riemann surface of genus* $g \geq 2$ *then the Lefschetz fibration is uniquely determined by the equivalence class of a factorization of the identity*

$$\tau_1 \circ \tau_2 \circ \dots \circ \tau_m = Id$$

as a product of Dehn twists.

Remark 6.27 *(1) A similar result holds for Lefschetz fibrations over the disc and we get a factorization*

$$\tau_1 \circ \tau_2 \circ \dots \circ \tau_m = \phi$$

of the monodromy ϕ *of the fibration over the boundary of the disc* D.

(2) A Lefschetz fibration with fibre C_g *admits a symplectic structure if each Dehn twist in the factorization is positively oriented (see Sect. 2 of [A-B-K-P-00]).*

Assume that we are given two Lefschetz fibrations over $\mathbb{P}^1_{\mathbb{C}}$: then we can consider the fibre sum of these two fibrations, which depends as we saw on a diffeomorphism chosen between two respective smooth fibers (cf. [G-S99] for more details).

This operation translates (in view of the above quoted theorem of Kas) into the following definition of ‘conjugated composition’ of factorization:

Definition 6.28 *Let* $\tau_1 \circ \tau_2 \circ \dots \circ \tau_m = \phi$ *and* $\tau'_1 \circ \tau'_2 \circ \dots \circ \tau'_r = \phi'$ *be two factorizations: then their* composition conjugated by ψ *is the factorization*

$$\tau_1 \circ \tau_2 \circ \dots \tau_m \circ (\tau'_1)_\psi \circ (\tau'_2)_\psi \circ \dots \circ (\tau'_r)_\psi = \phi \circ (\phi')_\psi.$$

Remark 6.29 *(1) If* ψ *and* ϕ' *commute, we obtain a factorization of* $\phi\phi'$.

(2) A particular case is the one where $\phi = \phi' = id$ *and it corresponds to Lefschetz fibrations over* $\mathbb{P}^1$.

No matter how beautiful the above results are, for a general X projective or M symplectic, one has Lefschetz pencils, and not Lefschetz fibrations, and a natural question is to which extent the surgery corresponding to the blowup does indeed simplify the differentiable structure of the manifold. In the next subsection we shall consider results by Moishezon somehow related to this question.

6.4 Simply Connected Algebraic Surfaces: Topology Versus Differential Topology

In the case of compact topological manifolds of real dimension 4 the methods of Morse theory and of simplification of cobordisms turned out to encounter overwhelming difficulties, and only in 1982 M. Freedman [Free82], using new ideas in order to show the (topological) triviality of certain handles introduced by Casson, was able to obtain a complete classification of the simply connected compact topological 4-manifolds.

Let M be such a manifold, fix an orientation of M, and let

$$q_M \colon H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$$

be the intersection form, which is unimodular by Poincaré duality.

Theorem 6.30 (Freedman's theorem) *Let M be an oriented compact simply connected topological manifold: then M is determined by its intersection form and by the Kirby-Siebenmann invariant $\alpha(M) \in \mathbb{Z}/2$, which vanishes if and only if $M \times [0,1]$ admits a differentiable structure.*

The basic invariants of q_m are its signature $\sigma(M) := b^+(M) - b^-(M)$, and its parity ($q_m$ is said to be *even* iff $q_m(x,x) \equiv 0 \ (mod 2) \ \forall x \in H_2(M,\mathbb{Z})$).

A basic result by Serre [Ser64] says that if q_M is indefinite then it is determined by its rank, signature and parity.

The corollary of Freedman's theorem for complex surfaces is the following

Theorem 6.31 *Let S be a compact simply connected complex surface, and let r be the divisibility index of the canonical class $c_1(K_X) \in H^2(X,\mathbb{Z})$.*

S is said to be EVEN if q_S is EVEN, and this holds iff $r \equiv 0 (mod 2)$, else S is said to be ODD. Then

- *(EVEN) If S is EVEN, then S is topologically a connected sum of copies of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and of a $K3$ surface if the signature of the intersection form is negative, and of copies of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and of a $K3$ surface with opposed orientation in the case where the signature is positive.*
- *(ODD) S is ODD: then S is topologically a connected sum of copies of $\mathbb{P}^2_{\mathbb{C}}$ and of ${\mathbb{P}^2_{\mathbb{C}}}^{opp}$.*

Proof. S has a differentiable structure, whence $\alpha(S) = 0$, and the corollary follows from Serre's result if the intersection form is indefinite.

We shall now show that if the intersection form is definite, then $S \cong \mathbb{P}^2_{\mathbb{C}}$. Observe that $q = 0$, since S is simply connected, and therefore $b^+(S) = 2p_g + 1$, in particular the intersection form is positive, $b_2 = 2p_g + 1$, hence $e(S) = 2\chi(S) + 1$, and $K_S^2 = 10\chi(S) - 1$ by Noether's formula.

By the Yau Miyaoka inequality $q = 0$ implies $K_S^2 \leq 9\chi(S)$, whence $\chi(S) \leq 1$ and $p_g = 0$.

Therefore $\chi(S) = 1$, and $K_S^2 = 9$. Applying again Yau's theorem [Yau77] we see that $S = \mathbb{P}^2_{\mathbb{C}}$. In fact, if S were of general type its universal cover would be the unit ball in $\mathbb{C}^2$, contradicting simple connectivity. □

Remark 6.32 *$\mathbb{P}^{2\,opp}_{\mathbb{C}}$ is the manifold $\mathbb{P}^2_{\mathbb{C}}$ with opposed orientation.*

A K3 surface is (as we already mentioned) a surface S orientedly diffeomorphic to a nonsingular surface X of degree 4 in $\mathbb{P}^3_{\mathbb{C}}$, for instance

$$X = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3_{\mathbb{C}} | x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

(by a theorem of Kodaira, cf. [Kod63], S is also deformation equivalent to such a surface X).

Not only $\mathbb{P}^2_{\mathbb{C}}$ is the only algebraic surface with a definite intersection form, but Donaldson showed that a result of a similar flavour holds for differentiable manifolds, i.e., if we have a positive definite intersection form, then we have topologically a connected sum of copies of $\mathbb{P}^2_{\mathbb{C}}$.

There are several restrictions for the intersection forms of differentiable manifolds, the oldest one being Rokhlin's theorem stating that the intersection form in the even case is divisible by 16. Donaldson gave other restrictions for the intersection forms of differentiable 4-manifolds (see [D-K90]), but the socalled 11/8 conjecture is still unproven: it states that if the intersection form is even, then we have topologically a connected sum as in the case (EVEN) of Theorem 6.31.

More important is the fact that Donaldson's theory has made clear in the 1980s [Don83, Don86, Don90, Don92] how drastically homeomorphism and diffeomorphism differ in dimension 4, and especially for algebraic surfaces.

Later on, the Seiberg-Witten theory showed with simpler methods the following result (cf. [Wit94] o [Mor96]):

Theorem 6.33 *Any diffeomorphism between minimal surfaces (a fortiori, an even surface is necessarily minimal) S, S' carries $c_1(K_S)$ either to $c_1(K_{S'})$ or to $-c_1(K_{S'})$*

Corollary 6.34 *The divisibility index r of the canonical class $c_1(K_S) \in H^2(S, \mathbb{Z})$ is a differentiable invariant of S.*

Since only the parity $r(mod 2)$ of the canonical class is a topological invariant it is then not difficult to construct examples of simply connected algebraic surfaces which are homeomorphic but not diffeomorphic (see [Cat86]).

Let us illustrate these examples, obtained as simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$.

These surfaces are contained in the geometric vector bundle whose sheaf of holomorphic sections is $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b) \bigoplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c,d)$ and are described there by the following two equations:

$$z^2 = f(x,y),$$

$$w^2 = g(x,y),$$

where f and g are bihomogeneous polynomials of respective bidegrees $(2a,2b)$, $(2c,2d)$ (f is a section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a,2b)$, g is a section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2c,2d)$).

These Galois covers of $\mathbb{P}^1 \times \mathbb{P}^1$, with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$, are smooth if and only if the two curves $C := \{f = 0\}$ and $D := \{g = 0\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ are smooth and intersect transversally.

The holomorphic invariants can be easily calculated, since, if $p \colon X \to \mathbb{P} := \mathbb{P}^1 \times \mathbb{P}^1$ is the finite Galois cover, then

$$p_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus z\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a,-b) \oplus w\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-c,-d) \oplus zw\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a-c,-b-d).$$

Hence $h^1(\mathcal{O}_X) = 0$, whereas $h^2(\mathcal{O}_X) = (a-1)(b-1) + (c-1)(d-1) + (a+c-1)(b+d-1)$. Assume that X is smooth: then the ramification formula yields

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(p^*K_{\mathbb{P}^1\times\mathbb{P}^1} + R) = p^*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a+c-2, b+d-2))$$

since $R = div(z) + div(w)$. In particular, $K_X^2 = 8(a+c-2)(b+d-2)$ and the holomorphic invariants of such coverings depend only upon the numbers $(a+b-2)(c+d-2)$ and $ab+cd$.

Theorem 6.35 *Let S, S' be smooth bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types $(a,b)(c,d), (a',b')(c',d')$.*

Then S is of general type for $a+c \geq 3, b+d \geq 3$, and is simply connected. Moreover, the divisibility $r(S)$ of the canonical class K_S is equal to $G.C.D.((a+c-2),(b+d-2))$.

S and S' are (orientedly) homeomorphic if and only if $r(S) \equiv r(S')(mod 2)$ and

$$(a+b-2)(c+d-2) = (a'+b'-2)(c'+d'-2) \text{ and } ab+cd = a'b'+c'd'.$$

S and S' are not diffeomorphic if $r(S) \neq r(S')$, and for each integer h, we can find such surfaces $S_1, \dots S_h$ which are pairwise homeomorphic but not diffeomorphic.

Idea of the proof. Set for simplicity $u := (a+c-2), v := (b+d-2)$ so that $\mathcal{O}_S(K_S) = p^*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(u,v))$ is ample whenever $u, v \geq 1$.

The property that S is simply connected (cf. [Cat84] for details) follows once one establishes that the fundamental group $\pi_1((\mathbb{P}^1 \times \mathbb{P}^1) \backslash (C \cup D))$ is abelian. To establish that the group is abelian, since it is generated by a product of simple geometric loops winding once around a smooth point of $C \cup D$, it suffices to show that these loops are central. But this follows from considering a Lefschetz pencil having C (respectively, D) as a fibre (in fact, an S^1 bundle over a punctured Riemann surface is trivial).

Since this group is abelian, it is generated by two elements γ_C, γ_D which are simple geometric loops winding once around C, resp. D. The fundamental group $\pi_1(S \backslash R)$ is then generated by $2\gamma_C$ and $2\gamma_D$, but these two elements lie in the kernel of the surjection $\pi_1(S \backslash R) \to \pi_1(S)$ and we conclude the triviality of this latter group.

The argument for the divisibility of K_S is more delicate, and we refer to [Cat86] for the proof of the key lemma asserting that $p^*(H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})) = H^2(S, \mathbb{Z})^G$ where G is the Galois group $G = (\mathbb{Z}/2)^2$ (the proof uses arguments of group cohomology and is valid in greater generality). Thus, the divisibility of K_S equals the one of $c_1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(u, v))$, i.e., $G.C.D.(u, v)$.

Now, resorting to Freedman's theorem, it suffices to observe that rank and signature of the intersection form are given by $e(S) - 2, \sigma(S)$, and these, as we saw in the first lecture, equal $12\chi(S) - K_S^2, K_S^2 - 8\chi(S)$. In this case $K_S^2 = 8uv$, $\chi(S) = uv + (ab + cd)$.

There remain to find h such surfaces, and for this purpose, we use Bombieri's argument (appendix to [Cat84]): namely, let $u'_i v'_i = 6^n$ be h distinct factorizations and, for a positive number T, set $u_i := Tu'_i, v_i := Tv'_i$. It is clear that $G.C.D.(u_i, v_i) = T(G.C.D.(u'_i, v'_i))$ and these G.C.D.'s are distinct since the given factorizations are distinct (as unordered factorizations), and they are even integers if each u'_i, v'_i is even.

It suffices to show that there are integers w_i, z_i such that, setting $a_i := (u_i + w_i)/2 + 1, c_i := (u_i - w_i)/2 + 1, b_i := (v_i - z_i)/2 + 1, d_i := (v_i + z_i)/2 + 1$, then $a_i b_i + c_i d_i = constant$ and the required inequalities $a_i, b_i, c_i, d_i \geq 3$ are verified.

This can be done by the box principle.

It is important to contrast the existence of homeomorphic but not diffeomorphic algebraic surfaces to an important theorem established at the beginning of the study of 4-manifolds by C.T.C. Wall [Wall62]:

Theorem 6.36 (C.T.C. Wall) *Given two simply connected differentiable 4-manifolds M, M' with isomorphic intersection forms, then there exists an integer k such that the iterated connected sums $M \sharp k(\mathbb{P}^1 \times \mathbb{P}^1)$ and $M' \sharp k(\mathbb{P}^1 \times \mathbb{P}^1)$ are diffeomorphic.*

Remark 6.37 *(1) If we take $\mathbb{P}^{2\ opp}_{\mathbb{C}}$, i.e., $\mathbb{P}^2$ with opposite orientation, then the selfintersection of a line equals -1, just as for the exceptional curve of a blow up. It is easy to see that blowing up a point of a smooth complex surface S is the same differentiable operation as taking the connected sum $S \sharp \mathbb{P}^{2\ opp}_{\mathbb{C}}$.*

(2) Recall that the blowup of the plane $\mathbb{P}^2$ in two points is isomorphic to the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in a point. Whence, for the connected sum

calculus, $M\sharp(\mathbb{P}^1 \times \mathbb{P}^1)\ \sharp{\mathbb{P}^2_{\mathbb{C}}}^{opp} \cong M\sharp(\mathbb{P}^2)\ \sharp 2{\mathbb{P}^2_{\mathbb{C}}}^{opp}$*. From Wall's theorem follows then (consider Wall's theorem for* M^{opp}*) that for any simply connected 4-manifold* M *there are integers* k, p, q *such that* $M\sharp(k+1)(\mathbb{P}^2)\ \sharp(k){\mathbb{P}^2_{\mathbb{C}}}^{opp} \cong p(\mathbb{P}^2)\ \sharp(q){\mathbb{P}^2_{\mathbb{C}}}^{opp}$*.*

The moral of Wall's theorem was that homeomorphism of simply connected 4-manifolds implies *stable* diffeomorphism (i.e., after iterated connected sum with some basic manifolds as $(\mathbb{P}^1 \times \mathbb{P}^1)$ or, with both $\mathbb{P}^2, {\mathbb{P}^2_{\mathbb{C}}}^{opp}$).

The natural question was then how many such connected sums were indeed needed, and if there were needed at all. As we saw, the Donaldson and Seiberg Witten invariants show that some connected sum is indeed needed.

Boris Moishezon, in collaboration with Mandelbaum, studied the question in detail [Moi77, M-M76, M-M80] for many concrete examples of algebraic surfaces, and gave the following

Definition 6.38 *A differentiable simply connected 4-manifold* M *is* completely decomposable *if there are integers* p, q *with* $M \cong p(\mathbb{P}^2)\ \sharp(q){\mathbb{P}^2_{\mathbb{C}}}^{opp}$*, and* almost completely decomposable *if* $M\sharp(\mathbb{P}^2)$ *is completely decomposable (note that the operation yields a manifold with odd intersection form, and if* M *is an algebraic surface* $\neq \mathbb{P}^2$*, then we get an indefinite intersection form.*

Moishezon and Mandelbaum [M-M76] proved almost complete decomposability for smooth hypersurfaces in $\mathbb{P}^3$, and Moishezon proved [Moi77] almost complete decomposability for simply connected elliptic surfaces. Observe that rational surfaces are obviously completely decomposable, and therefore one is only left with simply connected surfaces of general type, for which as far as I know the question of almost complete decomposability is still unresolved.

Donaldson's work clarified the importance of the connected sum with $\mathbb{P}^2$, showing the following results (cf. [D-K90] pages 26–27).

Theorem 6.39 (Donaldson) *If* M_1, M_2 *are simply connected differentiable 4-manifolds with* $b^+(M_i) > 0$*, then the Donaldson polynomial invariants* $q_k \in S^d(H^2(M, \mathbb{Z})$ *are all zero for* $M = M_1\sharp M_2$*. If instead* M *is an algebraic surfaces, then the Donaldson polynomials* q_k *are* $\neq 0$ *for large* k*. In particular, an algebraic surface cannot be diffeomorphic to a connected sum* $M_1\sharp M_2$ *with* M_1, M_2 *as above (i.e., with* $b^+(M_i) > 0$*).*

6.5 ABC Surfaces

This subsection is devoted to the diffeomorphism type of certain series of families of bidouble covers, depending on three integer parameters (a,b,c) (cf. [Cat02, CW04]).

Let us make some elementary remark, which will be useful in order to understand concretely the last part of the forthcoming definition.

Consider the projective line $\mathbb{P}^1$ with homogeneous coordinates (x_0, x_1) and with nonhomogeneous coordinate $x := x_1/x_0$. Then the homogeneous polynomials of degree m $F(x_0, x_1)$ are exactly the space of holomorphic sections of

$\mathcal{O}_{\mathbb{P}^1}(m)$: in fact to such an F corresponds the pair of holomorphic functions $f_0(x) := \frac{F(x_0,x_1)}{x_0^m}$ on $U_0 := \mathbb{P}^1\backslash\{\infty\}$, and $f_1(1/x) := \frac{F(x_0,x_1)}{x_1^m}$ on $U_1 := \mathbb{P}^1\backslash\{0\}$. They satisfy the cocycle condition $f_0(x)x^m = f_1(1/x)$.

We assumed here m to be a positive integer, because $\mathcal{O}_{\mathbb{P}^1}(-m)$ has no holomorphic sections, if $m > 0$. On the other hand, sheaf theory (the exponential sequence and the partition of unity argument) teaches us that the cocycle x^{-m} for $\mathcal{O}_{\mathbb{P}^1}(-m)$ is cohomologous, if we use differentiable functions, to $\bar{x}^m$ (indeed $x^{-m} = \frac{\bar{x}^m}{|x|^{2m}}$, a formula which hints at the homotopy $\frac{\bar{x}^m}{|x|^{2mt}}$ of the two cocycles).

This shows in particular that the polynomials $F(\bar{x}_0, \bar{x}_1)$ which are homogeneous of degree m are differentiable sections of $\mathcal{O}_{\mathbb{P}^1}(-m)$.

Since sometimes we shall need to multiply together sections of $\mathcal{O}_{\mathbb{P}^1}(-m)$ with sections of $\mathcal{O}_{\mathbb{P}^1}(m)$,and get a global function, we need the cocycles to be the inverses of each other. This is not a big problem, since on a circle of radius R we have $\bar{x}x = R^2$. Hence to a polynomial $F(\bar{x}_0, \bar{x}_1)$ we associate the two functions

$$f_0(\bar{x}) := \frac{F(\bar{x}_0, \bar{x}_1)}{\bar{x}_0^m} \text{ on } \{x||x| \le R\}$$

$$f_1(1/\bar{x}) := R^{2m} \frac{F(\bar{x}_0, \bar{x}_1)}{\bar{x}_1^m} \text{ on } \{x||x| \ge R\}$$

and this trick allows to carry out local computations comfortably.

Let us go now to the main definition:

Definition 6.40 *An (a, b, c) surface is the minimal resolution of singularities of a simple bidouble cover S of $(\mathbb{P}^1 \times \mathbb{P}^1)$ of type $((2a, 2b), (2c, 2b)$ having at most Rational Double Points as singularities.*

An $(a, b, c)^{nd}$ surface is defined more generally as (the minimal resolution of singularities of) a natural deformation *of an (a, b, c) surface with R.D.P.'s : i.e., the canonical model of an $(a, b, c)^{nd}$ surface is embedded in the total space of the direct sum of two line bundles L_1, L_2 (whose corresponding sheaves of sections are $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, b), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(c, b)$), and defined there by a pair of equations*

$$(***) \qquad z_{a,b}^2 = f_{2a,2b}(x, y) + w_{c,b}\phi_{2a-c,b}(x, y)$$
$$w_{c,b}^2 = g_{2c,2b}(x, y) + z_{a,b}\psi_{2c-a,b}(x, y)$$

where f,g ,ϕ, ψ, are bihomogeneous polynomials , belonging to respective vector spaces of sections of line bundles: $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a, 2b)), \phi \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a - c, b))$ and $g \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2c, 2d)), \psi \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2c - a, b))$.

A perturbation *of an (a, b, c) surface is an oriented smooth 4-manifold defined by equations as $(***)$, but where the sections ϕ, ψ are differentiable, and we have a* dianalytic perturbation *if ϕ, ψ are polynomials in the variables $x_i, y_j, \overline{x_i}, \overline{y_j}$, according to the respective positivity or negativity of the entries of the bidegree.*

Remark 6.41 *By the previous formulae,*

(1)(a,b,c) surfaces have the same invariants $\chi(S) = 2(a+c-2)(b-1) + b(a+c), K_S^2 = 16(a+c-2)(b-1)$

(2) the divisibility of their canonical class is $G.C.D.((a+c-2), 2(b-1))$

(3) Moreover, we saw that (a,b,c) surfaces are simply connected, thus

(4) Once we fix b and the sum $(a+c) = s$, the corresponding (a,b,c) surfaces are all homeomorphic

As a matter of fact, once we fix b and the sum $(a+c)$, the surfaces in the respective families are homeomorphic by a homeomorphism carrying the canonical class to the canonical class. This fact is a consequence of the following proposition, which we learnt from [Man96]

Proposition 6.42 *Let S, S' be simply connected minimal surfaces of general type such that $\chi(S) = \chi(S') \geq 2$, $K_S^2 = K_{S'}^2$, and moreover such that the divisibility indices of K_S and $K_{S'}$ are the same.*

Then there exists a homeomorphism F between S and S', unique up to isotopy, carrying $K_{S'}$ to K_S.

Proof. By Freedman's theorem ([Free82], cf. especially [F-Q90], page 162) for each isometry $h : H_2(S,\mathbb{Z}) \to H_2(S',\mathbb{Z})$ there exists a homeomorphism F between S and S', unique up to isotopy, such that $F_* = h$. In fact, S and S' are smooth 4-manifolds, whence the Kirby-Siebenmann invariant vanishes.

Our hypotheses that $\chi(S) = \chi(S')$, $K_S^2 = K_{S'}^2$ and that $K_S, K_{S'}$ have the same divisibility imply that the two lattices $H_2(S,\mathbb{Z})$, $H_2(S',\mathbb{Z})$ have the same rank, signature and parity, whence they are isometric since S, S' are algebraic surfaces. Finally, by Wall's theorem [Wall62] (cf. also [Man96], page 93) such isometry h exists since the vectors corresponding to the respective canonical classes have the same divisibility and by Wu's theorem they are characteristic: in fact Wall's condition $b_2 - |\sigma| \geq 4$ (σ being the signature of the intersection form) is equivalent to $\chi \geq 2$. □

We come now to the main result of this section (see [CW04] for details)

Theorem 6.43 *Let S be an (a,b,c)-surface and S' be an $(a+1,b,c-1)$-surface. Moreover, assume that $a,b,c-1 \geq 2$. Then S and S' are diffeomorphic.*

Idea of the Proof.

Before we dwell into the proof, let us explain the geometric argument which led me to conjecture the above theorem in 1997.

Assume that the polynomials f, g define curves C, D which are union of vertical and horizontal lines. Fix for simplicity affine coordinates in $\mathbb{P}^1$. Then we may assume, without loss of generality, that the curve C is constituted by the horizontal lines $y = 1, \dots y = 2b$, and by the vertical lines $x = 2, \dots x = 2a+1$, while the curve D is formed by the horizontal lines $y = -1, \dots y = -2b$, and by the vertical lines $x = 0$, $x = 1/4$, $x = 2a + 2, \ldots x = 2a + 2c - 1$. The corresponding surface X has double points as singularities, and its minimal resolution is a deformation of a smooth (a, b, c)-surface (by the cited results of Brieskorn and Tjurina).

Likewise, we let X' be the singular surface corresponding to the curve C' constituted by the horizontal lines $y = 1, \ldots y = 2b$, and by the vertical lines $x = 0$, $x = 1/4$, $x = 2, \ldots x = 2a + 1$, and to the curve D' formed by the horizontal lines $y = -1, \ldots y = -2b$, and by the $(2c - 2)$ vertical lines $x = 2a + 2, \ldots x = 2a + 2c - 1$.

We can split X as the union $X_0 \cup X_\infty$, where $X_0 := \{(x, y, z, w) | \; |x| \leq 1\}$, $X_\infty := \{(x, y, z, w) | \; |x| \geq 1\}$, and similarly $X' = X'_0 \cup X'_\infty$.

By our construction, we see immediately that $X'_\infty = X_\infty$, while there is a natural diffeomorphism Φ of $X_0 \cong X'_0$.

It suffices in fact to set $\Phi(x, y, z, w) = (x, -y, w, z)$.

The conclusion is that both S and S' are obtained glueing the same two 4-manifolds with boundary S_0, S_∞ glueing the boundary $\partial X_0 = \partial X_\infty$ once through the identity, and another time through the diffeomorphism Φ. It will follow that the two 4-manifolds are diffeomorphic if the diffeomorphism $\Phi|_{\partial S_0}$ admits an extension to a diffeomorphism of S_0.

(1) The relation with Lefschetz fibrations comes from the form of Φ, since Φ does not affect the variable x, but it is essentially given by a diffeomorphism Ψ of the fibre over $x = 1$,

$$\Psi(y, z, w) = (-y, w, z).$$

Now, the projection of an (a, b, c) surface onto $\mathbb{P}^1$ via the coordinate x is not a Lefschetz fibration, even if f, g are general, since each time one of the two curves C, D has a vertical tangent, we shall have two nodes on the corresponding fibre. But a smooth general natural deformation

$$\begin{aligned} z^2 &= f(x, y) + w\phi(x, y) \\ w^2 &= g(x, y) + z\psi(x, y), \end{aligned} \tag{1}$$

would do the game if $\phi \neq 0$ (i.e., $2a - c > 0$) and $\psi \neq 0$ (i.e., $2c - a > 0$).

Otherwise, it is enough to take a perturbation as in the previous definition (a dianalytic one suffices), and we can realize both surfaces S and S' as symplectic Lefschetz fibrations (cf. also [Don99, G-S99]).

(2) The above argument about S, S' being the glueing of the same two manifolds with boundary S_0, S_∞ translates directly into the property that the corresponding Lefschetz fibrations over $\mathbb{P}^1$ are fibre sums of the same pair of Lefschetz fibrations over the respective complex discs $\{x| \; |x| \leq 1\}$,$\{x| \; |x| \geq 1\}$.

(3) Once the first fibre sum is presented as composition of two factorizations and the second as twisted by the 'rotation' Ψ, (i.e., as we saw, the same composition of factorizations, where the second is conjugated by Ψ), in order to prove that the two fibre sums are equivalent, it suffices to apply a very

simple lemma, which can be found in [Aur02], and that we reproduce here because of its beauty

Lemma 6.44 (Auroux) *Let τ be a Dehn twist and let F be a factorization of a central element $\phi \in \mathcal{M}ap_g$, $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m = \phi$.*

If there is a factorization F' such that F is Hurwitz equivalent to $\tau \circ F'$, then $(F)_\tau$ is Hurwitz equivalent to F.

In particular, if F is a factorization of the identity, $\Psi = \Pi_h \tau'_h$, and $\forall h\ \exists F'_h$ such that $F \cong \tau'_h \circ F'_h$, then the fibre sum with the Lefschetz pencil associated with F yields the same Lefschetz pencil as the fibre sum twisted by Ψ.

Proof.

If $\cong$ denotes Hurwitz equivalence, then

$$(F)_\tau \cong \tau \circ (F')_\tau \cong F' \circ \tau \cong (\tau)_{(F')^{-1}} \circ F' = \tau \circ F' \cong F.$$

□

Corollary 6.45 *Notation as above, assume that F: $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_m = \phi$ is a factorization of the Identity and that Ψ is a product of some Dehn twists τ_i appearing in F. Then the fibre sum with the Lefschetz pencil associated with F yields the same result as the same fibre sum twisted by Ψ.*

Proof. We need only to verify that for each h, there is F'_h such that $F \cong \tau_h \circ F'_h$.

But this is immediately obtained by applying $h-1$ Hurwitz moves, the first one between τ_{h-1} and τ_h, and proceeding further to the left till we obtain τ_h as first factor. □

(4) It suffices now to show that the diffeomorphism Ψ is in the subgroup of the mapping class group generated by the Dehn twists which appear in the first factorization.

Figure 10 below shows the fibre C of the fibration in the case $2b = 6$: it is a bidouble cover of $\mathbb{P}^1$, which we can assume to be given by the equations $z^2 = F(y)$, $w^2 = F(-y)$, where the roots of F are the integers $1, \dots, 2b$.

Moreover, one sees that the monodromy of the fibration at the boundary of the disc is trivial, and we saw that the map Ψ is the diffeomorphism of order 2 given by $y \mapsto -y$, $z \mapsto w$, $w \mapsto z$, which in our figure is given as a rotation of 180° around an axis inclined in direction north-west.

The figure shows a dihedral symmetry, where the automorphism of order 4 is given by $y \mapsto -y$, $z \mapsto -w$, $w \mapsto z$.

(5) A first part of the proof, which we skip here, consists in identifying the Dehn twists which appear in the first factorization.

It turns out that, among the Dehn twists which appear in the first factorization, there are those which correspond to the inverse images of the segments between two consecutive integers (cf. Fig. 10). These circles can be organized on the curve C in six chains (not disjoint) and finally one reduces oneself to the computational heart of the proof: showing that the isotopy class of Ψ is

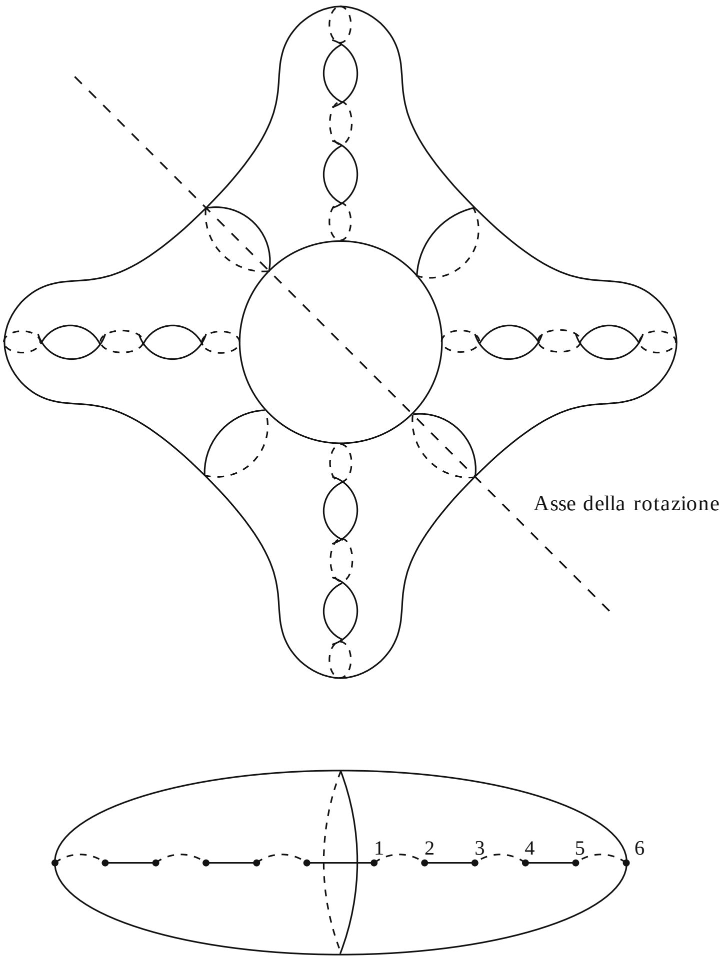

Fig. 10. The curve C with a dihedral symmetry

the same as the product Ψ' of the six Coxeter elements associated to such chains.

We recall here that, given a chain of curves $\alpha_1, \dots \alpha_n$ on a Riemann surface, the *Coxeter element associated to the chain* is the product

$$\Delta := (T_{\alpha_1})(T_{\alpha_2} T_{\alpha_1}) \dots (T_{\alpha_n} T_{\alpha_{n-1}} \dots T_{\alpha_1})$$

of the Dehn twists associated to the curves of the chain.

In order to finally prove that Ψ' (the product of such Coxeter elements) and Ψ are isotopic, one observes that if one removes the above cited chains of circles from the curve C, one obtains 4 connected components which are diffeomorphic to circles. By a result of Epstein it is then sufficient to verify that Ψ and Ψ' send each such curve to a pair of isotopic curves: this last step needs a list of lengthy (though easy) verifications, for which it is necessary to have explicit drawings.

For details we refer to the original paper [CW04].

7 Epilogue: Deformation, Diffeomorphism and Symplectomorphism Type of Surfaces of General Type

As we repeatedly said, one of the fundamental problems in the theory of complex algebraic surfaces is to understand the moduli spaces of surfaces of general type, and in particular their connected components, which, as we saw in the third lecture, parametrize the deformation equivalence classes of minimal surfaces of general type, or equivalently of their canonical models.

We remarked that deformation equivalence of two minimal models S, S' implies their canonical symplectomorphism and a fortiori an oriented diffeomorphism preserving the canonical class (a fortiori, a homeomorphism with such a property).

In the late eighties Friedman and Morgan (cf. [F-M94]) made the bold conjecture that two algebraic surfaces are diffeomorphic if and only if they are deformation equivalent. We will abbreviate this conjecture by the acronym *def = diff*. Indeed, I should point out that I had made the opposite conjecture in the early eighties (cf. [Katata83]).

Later in this section we shall briefly describe the first counterexamples, due to M. Manetti (cf. [Man01]): these have the small disadvantage of providing nonsimplyconnected surfaces, but the great advantage of yielding non deformation equivalent surfaces which are canonically symplectomorphic (see [Cat02, Cat06] for more details).

We already described in Lecture 4 some easy counterexamples to this conjecture (cf. [Cat03, KK02, BCG05]), given by pairs of complex conjugate surfaces, which are not deformation equivalent to their complex conjugate surface.

We might say that, although describing some interesting phenomena, the counterexamples contained in the cited papers by Catanese, Kharlamov–Kulikov, Bauer–Catanese–Grunewald are 'cheap', since the diffeomorphism carries the canonical class to its opposite. I was recently informed [Fried05] by R. Friedman that also he and Morgan were aware of such 'complex conjugate' counterexamples, but for the case of some elliptic surfaces having an infinite fundamental group.

After the examples by Manetti it was however still possible to weaken the conjecture def = diff in the following way.

Question 7.1 *Is the speculation* def = diff *true if one requires the diffeomorphism $\phi : S \to S'$ to send the first Chern class $c_1(K_S) \in H^2(S, \mathbb{Z})$ in $c_1(K_{S'})$ and moreover one requires the surfaces to be simply connected?*

But even this weaker question turned out to have a negative answer, as it was shown in our joint work with Wajnryb [CW04].

Theorem 7.2 ([CW04]) *For each natural number h there are simply connected surfaces $S_1, \dots, S_h$ which are pairwise diffeomorphic, but not deformation equivalent.*

The following remark shows that the statement of the theorem implies a negative answer to the above question.

Remark 7.3 *If two surfaces are deformation equivalent, then there exists a diffeomorphism sending the canonical class $c_1(K_S) \in H^2(S, \mathbb{Z})$ to the canonical class $c_1(K_{S'})$. On the other hand, by the cited result of Seiberg-Witten theory we know that a diffeomorphism sends the canonical class of a minimal surface S to $\pm c_1(K_{S'})$. Therefore, if one gives at least three surfaces, which are pairwise diffeomorphic, one finds at least two surfaces with the property that there exists a diffeomorphism between them sending the canonical class of one to the canonical class of the other.*

7.1 Deformations in the Large of ABC Surfaces

The above surfaces $S_1, \dots, S_h$ in Theorem 7.2 belong to the class of the so-called (a, b, c)-surfaces, whose diffeomorphism type was shown in the previous Lecture to depend only upon the integers $(a + c)$ and b.

The above Theorem 7.2 is thus implied by the following result:

Theorem 7.4 *Let S, S' be simple bidouble covers of $\mathbf{P}^1 \times \mathbf{P}^1$ of respective types ((2a, 2b),(2c,2b), and (2a + 2k, 2b),(2c - 2k,2b), and assume*

- *(I) a, b, c, k are strictly positive even integers with $a, b, c - k \geq 4$*
- *(II) $a \geq 2c + 1$*
- *(III) $b \geq c + 2$ and either*
- *(IV1) $b \geq 2a + 2k - 1$ or (IV2) $a \geq b + 2$*

Then S and S' are not deformation equivalent.

The theorem uses techniques which have been developed in a series of papers by the author and by Manetti [Cat84,Cat87,Cat86,Man94,Man97]. They use essentially the local deformation theory a' la Kuranishi for the canonical models, normal degenerations of smooth surfaces and a study of quotient singularities of rational double points and of their smoothings (this method was used in [Cat87] in order to study the closure in the moduli space of a class of bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying other types of inequalities).

Although the proof can be found in [Cat02, CW04], and in the Lecture Notes by Manetti in this volume, I believe it worthwhile to sketch the main ideas and arguments of the proof.

Main arguments of the Proof.

These are the three main steps of the proof:

Step I: determination of a subset $\mathfrak{N}_{a,b,c}$ of the moduli space

Step II: proof that $\mathfrak{N}_{a,b,c}$ is an open set

Step III: proof that $\mathfrak{N}_{a,b,c}$ is a closed set

Let us first of all explain the relevance of hypothesis (2) for step III. If we consider the natural deformations of (a, b, c) surfaces, which are parametrized by a quadruple of polynomials (f, g, ϕ, ψ) and given by the two equations

$$z^2 = f(x, y) + w\phi(x, y),$$

$$w^2 = g(x, y) + z\psi(x, y),$$

we observe that f and g are polynomials of respective bidegrees $(2a, 2b)$, $(2c, 2b)$, while ϕ and ψ have respective bidegrees $(2a - c, b)$, $(2c - a, b)$. Hence $a \geq 2c+1$, implies that $\psi \equiv 0$, therefore every small deformation preserves the structure of an iterated double cover. This means that the quotient Y of our canonical model X by the involution $z \mapsto -z$ admits an involution $w \mapsto -w$, whose quotient is indeed $\mathbb{P}^1 \times \mathbb{P}^1$.

This fact will play a special role in the study of limits of such $(a, b, c)^{nd}$ surfaces, showing that this iterated double cover structure passes in a suitable way to the limit, hence $\mathfrak{N}_{a,b,c}$ is a closed subset of the moduli space.

Step I.

The family $(\mathfrak{N}_{a,b,c})$ consists of all the (minimal resolutions of the) natural deformations of simple bidouble covers of the Segre-Hirzebruch surfaces $\mathbb{F}_{2h}$ which have only Rational Double Points as singularities and are of type ((2a, 2b),(2c,2b).

In order to explain what this means, let us recall, as in [Cat82] pages 105–111, that a basis of the Picard group of $\mathbb{F}_{2h}$ is provided, for $h \geq 1$, by the fibre F of the projection to $\mathbb{P}^1$, and by $F' := \sigma_\infty + hF$, where σ_∞ is the unique section with negative self-intersection $= -2h$. Observe that $F^2 = {F'}^2 = 0, FF' = 1$, and that F is nef, while $F' \cdot \sigma_\infty = -h$.

We set $\sigma_0 := \sigma_\infty + 2hF$, so that $\sigma_\infty \sigma_0 = 0$, and we observe (cf. Lemma 2.7 of [Cat82]) that $|m\sigma_0 + nF|$ has no base point if and only if $m, n \geq 0$. Moreover, $|m\sigma_0 + nF|$ contains σ_∞ with multiplicity ≥ 2 if $n < -2h$.

At this moment, the above remarks and the inequalities (II), (III), (IV) can be used to imply that all natural deformations have the structure of an iterated double covering, since their canonical models are defined by the following two equations:

$$z^2 = f(x, y) + w\phi(x, y),$$
$$w^2 = g(x, y).$$

Step II.

A key point here is to look only at the deformation theory of the canonical models.

To prove that the family of canonical models $(\mathfrak{N}_{a,b,c})$ yields an open set in the moduli space it suffices to show that, for each surface X, the Kodaira Spencer map is surjective.

In fact, one can see as in in [Cat82] that the family $(\mathfrak{N}_{a,b,c})$ is parametrized by a smooth variety which surjects onto $H^1(\Theta_{\mathbb{F}})$.

Observe that the tangent space to the Deformations of X is provided by $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$.

Denoting by $\pi : X \to \mathbb{F} := \mathbb{F}_{2h}$ the projection map and differentiating equations (7) we get an exact sequence for Ω^1_X

$$o \to \pi^*(\Omega^1_{\mathbb{F}}) \to \Omega^1_X \to \mathcal{O}_{R_z}(-R_z) \oplus \mathcal{O}_{R_w}(-R_w) \to 0$$

as in (1.7) of [Man94], where $R_z = div(z), R_w = div(w)$.

Applying the derived exact sequence for $\mathrm{Hom}_{\mathcal{O}_X}(\ldots, \mathcal{O}_X)$ we obtain the same exact sequence as Theorem (2.7) of [Cat82], and (1.9) of [Man94], namely:

$$(**)\ 0 \to H^0(\Theta_X) \to H^0(\pi^*\Theta_{\mathbb{F}}) \to H^0(\mathcal{O}_{R_z}(2R_z)) \oplus H^0(\mathcal{O}_{R_w}(2R_w)) \to$$
$$\to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to H^1(\pi^*\Theta_{\mathbb{F}}).$$

There is now some technical argument, quite similar to the one given in [Cat82], and where our inequalities are used in order to show that $H^1(\pi^*\Theta_{\mathbb{F}}) = H^1(\Theta_{\mathbb{F}} \otimes \pi_*(\mathcal{O}_X))$ equals $H^1(\Theta_{\mathbb{F}})$: we refer to [CW04] for details.

Summarizing the proof of step II, we observe that the smooth parameter space of our family surjects onto $H^1(\Theta_{\mathbb{F}})$, and its kernel, provided by the natural deformations with fixed base $\mathbb{F}_{2h}$, surjects onto $H^0(\mathcal{O}_{R_z}(2R_z)) \oplus H^0(\mathcal{O}_{R_w}(2R_w))$. Thus the Kodaira Spencer is onto and we get an open set in the moduli space.

Step III.

We want now to show that our family $\mathfrak{N}_{a,b,c}$ yields a closed set in the moduli space.

It is clear at this moment that we obtained an irreducible component of the moduli space. Let us consider the surface over the generic point of the base space of our family: then it has $\mathbb{Z}/2$ in the automorphism group (sending $z \to -z$, as already mentioned).

As shown in [Cat82], this automorphism acts then biregularly on the canonical model X_0 of each surface corresponding to a point in the closure of our open set. This holds in fact more generally for the action of any finite group G: the representation of G on $H^0(S, \mathcal{O}(5K_S))$ depends on discrete data, whence it is fixed in a family, and then the set of fixed points in the pseudomoduli space $\{X | g(X) = X\ \forall g \in G\}$ is a closed set.

We use now the methods of [Cat87, Man97], and more specifically we can apply Theorem 4.1 of [Man97] to conclude with

Claim III .1 If X_0 is a canonical model which is a limit of canonical models X_t of surfaces S_t in our family, then the quotient Y_0 of X_0 by the subgroup $\mathbb{Z}/2 \subset Aut(X_0)$ mentioned above is a surface with Rational Double Points.

Claim III .2 The family of such quotients Y_t has a $\mathbb{Z}/2$-action over the generic point, and dividing by it we get (cf. [Man97, Theorem 4.10]) as quotient Z_0 a Hirzebruch surface. Thus our surface X_0 is also an iterated double cover of some $\mathbb{F}_{2h}$, hence it belongs to the family we constructed.

Argument for claim III.1 Since smooth canonical models are dense, we may assume that X_0 is a limit of a 1-parameter family X_t of smooth canonical models; for the same reason we may assume that the quotient Y_0 is the limit of smooth surfaces $Y_t = X_t/(\mathbb{Z}/2)$ (of general type if $c, b \geq 3$).

Whence,

(1) Y_0 has singularities which are quotient of Rational Double Points by $(\mathbb{Z}/2)$.

(2) Y_t is a smoothing of Y_0, and since we assume the integers c, b to be even, the canonical divisor of Y_t is 2-divisible.

Now, using Theorem 3.6, the involutions acting on RDP's can be classified (cf. [Cat87] for this and the following), and it turns out that the quotient singularities are again RDP's, with two possible exceptions:

Type (c): the singularity of Y_0 is a quotient singularity of type $\frac{1}{4k+2}(1, 2k)$, and X_0 is the A_{2k} singularity, quotient by the subgroup $2\mathbb{Z}/(4k+2)\mathbb{Z}$.

Type (e): the singularity of Y_0 is a quotient singularity of type $\frac{1}{4k+4}(1, 2k+1)$, and X_0 is the A_{2k+1} singularity, quotient by the subgroup $2\mathbb{Z}/(4k+4)\mathbb{Z}$.

The versal families of deformations of the above singularities have been described by Riemenschneider in [Riem74], who showed:

(C) In the case of type (c), the base space is smooth, and it yields a smoothing admitting a simultaneous resolution.

(E) In the case of type (e), the base space consists of two smooth components intersecting transversally, $T_1 \cup T_2$. T_1 yields a smoothing admitting a simultaneous resolution (we denote this case by 'case (E1)').

Hypothesis (2), of 2-divisibility of the canonical divisor of Y_t, is used in two ways. The first consequence is that the intersection form on $H^2(Y_t, \mathbb{Z})$ is even; since however the Milnor fibre of the smoothing is contained in Y_t, it follows that no 2-cycle in the Milnor fibre can have odd selfintersection number. This then excludes case (C), and also case (E1) for $k \geq 1$.

In case (E2) we have a socalled $\mathbb{Z}$-Gorenstein smoothing, namely, the T_2 family is the quotient of the hypersurface

$$(***) \quad uv - z^{2n} = \Sigma_{h=0}^{1} t_h z^{hn}$$

by the involution sending $(u, v, z) \mapsto (-u, -v, -z)$.

The result is that the Milnor fibre has a double étale cover which is the Milnor fibre of A_{n-1} ($n = k+1$), in particular its fundamental group equals $\mathbb{Z}/2$. The universal cover corresponds to the cohomology class of the canonical divisor. This however contradicts condition (2), and case (E2) is excluded too.

For case (E1) $k = 0$ we argue similarly: the involution acts trivially on the parameter t, and in the central fibre it has an isolated fixed point. Because of simultaneous resolution, the total space $\cup_t X_t$ may be taken to be smooth, and then the set of fixed points for the involution is a curve mapping isomorphically on the parameter space $\{t\}$. Then the Milnor fibre should have a double cover ramified exactly in one point, but this is absurd since by van Kampen's theorem the point complement is simply connected.

Argument for claim III.2

Here, $Z_t := (Y_t/\mathbb{Z}/2) \cong \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ and again the canonical divisor is 2-divisible. Whence, the same argument as before applies, showing that Z_0 has necessarily Rational Double Points as singularities. But again, since the Milnor fibre embeds in $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, the intersection form must have negativity at most 1, and be even. This leaves only the possibility of an A_1 singularity. This case can be again excluded by the same argument given for the case (E1) $k = 0$ above.

Proof that Theorem 7.4 implies Theorem 7.2.

It suffices to show what we took up to now for granted: the irreducible component $\mathfrak{N}_{a,b,c}$ uniquely determines the numbers a, b, c up to the obvious permutations: $a \leftrightarrow c$, and , if $a = c$, the possibilities of exchanging a with b.

It was shown more generally in [Cat84] Theorem 3.8 that the natural deformations of bidouble covers of type $(2a, 2b)(2c, 2d)$ yield an irreducible component of the moduli space, and that these are distinct modulo the obvious permutations (exchange type $(2a, 2b)(2c, 2d)$ with type $(2c, 2d)(2a, 2b)$ and with type $(2b, 2a)(2d, 2c)$). This follows from geometrical properties of the canonical map at the generic point.

However, the easiest way to see that the irreducible component $\mathfrak{N}_{a,b,c}$ determines the numbers a, b, c, under the given inequalities (II0, III), (IV) is to observe that the dimension of $\mathfrak{N}_{a,b,c}$ equals $M := (b + 1)(4a + c + 3) + 2b(a + c + 1) - 8$. Recall in fact that $K^2/16 = (a + c - 2)(b - 1)$, and $(8\chi - K^2)/8 = b(a + c)$: setting $\alpha = a + c, \beta = 2b$, we get that α, β are then the roots of a quadratic equation, so they are determined up to exchange, and uniquely if we restrict our numbers either to the inequality $a \geq 2b$ or to the inequality $b \geq a$.

Finally $M = (\frac{\beta}{2} + 1)(\alpha + 3) + \beta(\alpha + 1) - 8 + 3a(\frac{\beta}{2} + 1)$ then determines a, whence the ordered triple (a, b, c). $\square$

Remark 7.5 *If, as in [Cat02], we assume*

(IV2) $a \geq b + 2$,

then the connected component $\mathfrak{N}_{a,b,c}$ *of the moduli space contains only iterated double covers of* $\mathbb{P}^1 \times \mathbb{P}^1$.

7.2 Manetti Surfaces

Manetti in [Man01] considers surfaces which are desingularization of certain $(\mathbb{Z}/2)^r$ covers X of rational surfaces Y which are blowup of the quadric $Q := \mathbb{P}^1 \times \mathbb{P}^1$ at n points $P_1, \ldots P_n$.

His construction is made rather complicated, not only by the desire to construct an arbitrarily high number of surfaces which are pairwise diffeomorphic but not deformation equivalent, but also by the crucial target to obtain that every small deformation is again such a Galois $(\mathbb{Z}/2)^r$ cover. This requirement makes the construction not very explicit (Lemma 3.6 ibidem).

Let us briefly recall the structure of normal finite $(\mathbb{Z}/2)^r$ covers with smooth base Y (compare [Par91, Man01], and also [BC06] for a description in terms of the monodromy homomorphism).

We denote by $G = (\mathbb{Z}/2)^r$ the Galois group, and by σ an element of G. We denote by $G^\vee := Hom(G, \mathbb{C}^*)$ the dual group of characters, $G^\vee \cong (\mathbb{Z}/2)^r$, and by χ an element of $G^\vee$. As for any flat finite abelian covering $f\colon X \to Y$ we have

$$f_*\mathcal{O}_X = \bigoplus_{\chi \in G^\vee} \mathcal{O}_Y(-L_\chi) = \mathcal{O}_Y \oplus (\bigoplus_{\chi \in G^\vee \setminus \{0\}} \mathcal{O}_Y(-L_\chi)).$$

To each element of the Galois group $\sigma \in G$ one associates a divisor D_σ,such that $2D_\sigma$ is the direct image divisor $f_*(R_\sigma)$, R_σ being the divisorial part of the set of fixed points for σ.

Let x_σ be a section such that $div(x_\sigma) = D_\sigma$: then the algebra structure on $f_*\mathcal{O}_X$ is given by the following symmetric bilinear multiplication maps:

$$\mathcal{O}_Y(-L_\chi) \otimes \mathcal{O}_Y(-L_\eta) \to \mathcal{O}_Y(-L_{\chi+\eta})$$

associated to the section

$$x_{\chi,\eta} \in H^0(Y, \mathcal{O}_Y(L_\eta + L_\chi - L_{\chi+\eta})),\ x_{\chi,\eta} := \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

Associativity follows since, given characters χ, η, θ, $\{\sigma | (\chi+\eta)(\sigma) = \theta(\sigma) = 1\}$ is the disjoint union of $\{\sigma | \chi(\sigma) = \theta(\sigma) = 1, \eta(\sigma) = 0\}$ and of $\{\sigma | \eta(\sigma) = \theta(\sigma) = 1, \chi(\sigma) = 0\}$, so that

$$\mathcal{O}_Y(-L_\chi) \otimes \mathcal{O}_Y(-L_\eta) \otimes \mathcal{O}_Y(-L_\theta) \to \mathcal{O}_Y(-L_{\chi+\eta+\theta})$$

is given by the section $\prod_{\sigma \in \Sigma} x_\sigma$, where

$$\Sigma := \{\sigma | \chi(\sigma) = \eta(\sigma) = 1, \text{or } \chi(\sigma) = \theta(\sigma) = 1, \text{or } \eta(\sigma) = \theta(\sigma) = 1\}.$$

In particular, the covering $f\colon X \to Y$ is embedded in the vector bundle $\mathbb{V}$ which is the direct sum of the line bundles whose sheaves of sections are the $\mathcal{O}_Y(-L_\chi)$, and is there defined by equations

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

Noteworthy is the special case $\chi = \eta$, where $\chi + \eta$ is the trivial character 1, and $z_1 = 1$.

In particular, let $\chi_1, \dots \chi_r$ be a basis of $G^\vee \cong (\mathbb{Z}/2)^r$, and set $z_i := z_{\chi_i}$. We get then the r equations

$$(\sharp)\ z_i^2 = \prod_{\chi_i(\sigma)=1} x_\sigma.$$

These equations determine the field extension, hence one gets X as the normalization of the Galois cover given by $(\sharp)$.

We can summarize the above discussion in the following

Proposition 7.6 *A normal finite $G \cong (\mathbb{Z}/2)^r$ covering of smooth variety Y is completely determined by the datum of*

(1) reduced effective divisors D_σ, $\forall \sigma \in G$, which have no common components

(2) divisor classes $L_1, \dots L_r$, for $\chi_1, \dots \chi_r$ a basis of $G^\vee$, such that we have the following linear equivalence

(3)

$$2L_i \equiv \sum_{\chi_i(\sigma)=1} D_\sigma.$$

Conversely, given the datum of (1) and (2), if (3) holds, we obtain a normal scheme X with a finite $G \cong (\mathbb{Z}/2)^r$ covering $f\colon X \to Y$.

Idea of the proof

It suffices to determine the divisors L_χ for the other elements of $G^\vee$. But since any χ is a sum of basis elements, it suffices to exploit the fact that the linear equivalences

$$L_{\chi+\eta} \equiv L_\eta + L_\chi \ - \sum_{\chi(\sigma)=\eta(\sigma)=1} D_\sigma$$

must hold, and apply induction. Since the covering is well defined as the normalization of the Galois cover given by $(\sharp)$, each L_χ is well defined. Then the above formulae determine explicitly the ring structure of $f_*\mathcal{O}_X$, hence X. $\square$

A natural question is of course when the scheme X is a variety, i.e., X being normal, when X is connected, or equivalently irreducible. The obvious answer is that X is irreducible if and only if the monodromy homomorphism

$$\mu\colon H_1(Y \backslash (\cup_\sigma D_\sigma), \mathbb{Z}) \to G$$

is surjective.

Remark 7.7 *As a matter of fact, we know, from the cited theorem of Grauert and Remmert, that μ determines the covering. It is therefore worthwhile to see how μ determines the datum of (1) and (2).*

Write for this purpose the branch locus $D := \sum_\sigma D_\sigma$ as a sum of irreducible components D_i. To each D_i corresponds a simple geometric loop γ_i around

D_i, and we set $\sigma_i := \mu(\gamma_i)$. Then we have that $D_\sigma := \sum_{\sigma_i=\sigma} D_i$. For each character χ, yielding a double covering associated to the composition $\chi \circ \mu$, we must find a divisor class L_χ such that $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$.

Consider the exact sequence

$$H^{2n-2}(Y,\mathbb{Z}) \to H^{2n-2}(D,\mathbb{Z}) = \oplus_i \mathbb{Z}[D_i] \to H_1(Y \backslash D, \mathbb{Z}) \to H_1(Y,\mathbb{Z}) \to 0$$

and the similar one with $\mathbb{Z}$ replaced by $\mathbb{Z}/2$. Denote by Δ the subgroup image of $\oplus_i \mathbb{Z}/2[D_i]$. The restriction of μ to Δ is completely determined by the knowledge of the σ_i 's, and we have

$$0 \to \Delta \to H_1(Y \backslash D, \mathbb{Z}/2) \to H_1(Y, \mathbb{Z}/2) \to 0.$$

Dualizing, we get

$$0 \to H^1(Y, \mathbb{Z}/2) \to H^1(Y \backslash D, \mathbb{Z}/2) \to Hom(\Delta, \mathbb{Z}/2) \to 0.$$

The datum of μ, extending $\mu|_\Delta$ is then seen to correspond to an affine space over the vector space $H^1(Y,\mathbb{Z}/2)$: and since $H^1(Y,\mathbb{Z}/2)$ classifies divisor classes of 2-torsion on Y, we infer that the different choices of L_χ such that $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$ correspond bijectively to all the possible choices for μ.

Corollary 7.8 *Same notation as in proposition 7.6. Then the scheme X is irreducible if $\{\sigma | D_\sigma > 0\}$ generates G.*

Proof. We have seen that if $D_\sigma \geq D_i \neq 0$, then $\mu(\gamma_i) = \sigma$, whence we infer that μ is surjective. □

An important role plays again here the concept of *natural deformations.* This concept was introduced for bidouble covers in [Cat84], Definition 2.8, and extended to the case of abelian covers in [Par91], Definition 5.1. However, the two definitions do not coincide, because Pardini takes a much larger parameter space. We propose therefore to call Pardini's case the case of *extended natural deformations.*

Definition 7.9 *Let $f\colon X \to Y$ be a finite $G \cong (\mathbb{Z}/2)^r$ covering with Y smooth and X normal, so that X is embedded in the vector bundle $\mathbb{V}$ defined above and is defined by equations*

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} x_\sigma.$$

Let $\psi_{\sigma,\chi}$ be a section $\psi_{\sigma,\chi} \in H^0(Y, \mathcal{O}_Y(D_\sigma - L_\chi))$, given $\forall \sigma \in G, \chi \in G^\vee$. To such a collection we associate an extended natural deformation*, namely, the subscheme of $\mathbb{V}$ defined by equations*

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} (\sum_\theta \psi_{\sigma,\theta} \cdot z_\theta).$$

We have instead a (restricted) natural deformation *if we restrict ourselves to the θ's such that $\theta(\sigma) = 0$, and we consider only an equation of the form*

$$z_\chi z_\eta = z_{\chi+\eta} \prod_{\chi(\sigma)=\eta(\sigma)=1} \left(\sum_{\theta(\sigma)=0} \psi_{\sigma,\theta} \cdot z_\theta \right).$$

The deformation results which we explained in the last lecture for simple bidouble covers work out also for $G \cong (\mathbb{Z}/2)^r$ which are locally simple, i.e., enjoy the property that for each point $y \in Y$ the σ's such that $y \in D_\sigma$ are a linear independent set. This is a good notion since (compare [Cat84], Proposition 1.1) if also X is smooth the covering is indeed locally simple.

One has the following result (see [Man01], Sect. 3)

Proposition 7.10 *Let $f : X \to Y$ be a locally simple $G \cong (\mathbb{Z}/2)^r$ covering with Y smooth and X normal. Then we have the exact sequence*

$$\oplus_{\chi(\sigma)=0}(H^0(\mathcal{O}_{D_\sigma}(D_\sigma - L_\chi))) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to \mathrm{Ext}^1_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X).$$

In particular, every small deformation of X is a natural deformation if

(i) $H^1(\mathcal{O}_Y(-L_\chi)) = 0$

(ii) $\mathrm{Ext}^1_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X) = 0$

If moreover

(iii) $H^0(\mathcal{O}_Y(D_\sigma - L_\chi)) = 0\ \forall \sigma \in G, \chi \in G^\vee$,

every small deformation of X is again a $G \cong (\mathbb{Z}/2)^r$ covering.

Comment on the proof.

In the above proposition condition (i) ensures that $H^0(\mathcal{O}_Y(D_\sigma - L_\chi)) \to H^0(\mathcal{O}_{D_\sigma}(D_\sigma - L_\chi))$ is surjective.

Condition (ii) and the above diagram imply then that the natural deformations are parametrized by a smooth manifold and have surjective Kodaira Spencer map, whence they induce all the infinitesimal deformations. □

In Manetti's application one needs an extension of the above result. In fact (ii) does not hold, since the manifold Y is not rigid (one can move the points $P_1, \dots P_n$ which are blown up in the quadric Q). But the moral is the same, in the sense that one can show that all the small deformations of X are G-coverings of a small deformation of Y.

Before we proceed to the description of the Manetti surfaces, we consider some simpler surfaces, which however clearly illustrate one of the features of Manetti's construction.

Definition 7.11 *A singular bidouble Manetti surface of type (a, b) and triple of order n is a singular bidouble cover of $Q := \mathbb{P}^1 \times \mathbb{P}^1$ branched on three smooth curves C_1, C_2, C_3 belonging to the linear system of sections of the sheaf $\mathcal{O}_Q(a, b)$ and which intersect in n points $p_1, \dots p_n$, with distinct tangents.*

A smooth bidouble Manetti surface of type (a, b) and triple of order n is the minimal resolution of singularities S of such a surface X as above.

Remark 7.12 *(1) With such a branch locus, a Galois group of type* $G = (\mathbb{Z}/2)^r$ *can be only* $G = (\mathbb{Z}/2)^3$ *or* $G = (\mathbb{Z}/2)^2$ *(we can exclude the uninteresting case* $G = (\mathbb{Z}/2)$*). The case* $r = 3$ *can only occur if the class of the three curves* (a, b) *is divisible by two since, as we said, the homology group of the complement* $Q \backslash (\cup_i C_i)$ *is the cokernel of the map* $H^2(Q, \mathbb{Z}) \to \oplus_1^3(\mathbb{Z}C_i)$*. The case* $r = 3$ *is however uninteresting, since in this case the elements* $\phi(\gamma_i)$ *are a basis, thus over each point* p_i *we have a nodal singularity of the covering surface, which obviously makes us remain in the same moduli space as the one where the three curves have no intersection points whatsoever.*

(2) Assume that $r = 2$*, and consider the case where the monodromy* μ *is such that the* $\mu(\gamma_i)$*'s are the three nontrivial elements of the group* $G = (\mathbb{Z}/2)^2$*.*

Let $p = p_i$ *be a point where the three smooth curves* C_1, C_2, C_3 *intersect with distinct tangents: then over the point* p *there is a singularity* (X, x) *of the type considered in Example 3.3, namely, a quotient singularity which is analytically the cone over a rational curve of degree 4.*

If we blow up the point p*, and get an exceptional divisor* E*, the loop* γ *around the exceptional divisor* E *is homologous to the sum of the three loops* $\gamma_1, \gamma_2, \gamma_3$ *around the respective three curves* C_1, C_2, C_3*. Hence it must hold* $\mu(\gamma) = \sum_i \mu(\gamma_i) = 0$*, and the pull back of the covering does not have* E *in the branch locus. The inverse image* A *of* E *is a* $(\mathbb{Z}/2)^2$ *covering of* E *branched in three points, and we conclude that* A *is a smooth rational curve of self-intersection* -4*.*

One sees (compare [Cat99]) that

Proposition 7.13 *Let* X *be a singular bidouble Manetti surface of type* (a, b) *and triple of order* n*: then if* S *is the minimal resolution of the singularities* $x_1, \dots x_n$ *of* X*, then* S *has the following invariants:*

$K_S^2 = 18ab - 24(a + b) + 32 - n$

$\chi(S) = 4 + 3(ab - a - b)$.

Moreover S *is simply connected if* (a, b) *is not divisible by 2.*

Idea of the proof For $n = 0$ these are the standard formulae since $2K_S = f^*(3a - 4, 3b - 4)$, and $\chi(\mathcal{O}_Q(-a, -b)) = 1 + 1/2(a(b - 2) + b(a - 2))$.

For $n > 0$, each singular point x_n lowers K_S^2 by 1, but leaves $\chi(S)$ invariant. In fact again we have $2K_X = f^*(3a - 4, 3b - 4)$, but $2K_S = 2K_X - \sum_i A_i$. For $\chi(S)$, one observes that x_i is a rational singularity, whence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S)$.

It was proven in [Cat84] that S is simply connected for $n = 0$ when (a, b) is not divisible by 2 (in the contrary case the fundamental group equals $\mathbb{Z}/2$.) Let us then assume that $n \geq 1$.

Consider now a 1-parameter family $C_{3,t}$, $t \in T$, such that for $t \neq 0$ $C_{3,t}$ intersects C_1, C_2 transversally, while $C_{3,0} = C_3$. We get a corresponding family X_t of bidouble covers such that X_t is smooth for $t \neq 0$ and, as we just saw, simply connected. Then S is obtained from $X_t, t \neq 0$ replacing the Milnor fibres by tubular neighbourhoods of the exceptional divisors A_i, $i = 1, \dots n$. Since A_i is smooth rational, these neighbourhoods are simply connected, and the result follows then easily by the first van Kampen theorem, which implies that $\pi_1(S)$ is a quotient of $\pi_1(X_t), t \neq 0$. □

The important fact is that the above smooth bidouble Manetti surfaces of type (a, b) and triple of order n are parametrized, for $b = la, l \geq 2, n = la(2a - c), 0 < 2c < a$, by a disconnected parameter space ([Man01], Corollary 2.12: observe that we treat here only the case of $k = 3$ curves).

We cannot discuss here the method of proof, which relies on the socalled Brill Noether theory of special divisors: we only mention that Manetti considers the two components arising form the respective cases where $\mathcal{O}_{C_1}(p_1 + \dots p_n) \cong \mathcal{O}_{C_1}(a - c, b)$, $\mathcal{O}_{C_1}(p_1 + \dots p_n) \cong \mathcal{O}_{C_1}(a, b - lc)$, and shows that the closures of these loci yield two distinct connected components.

Unfortunately, one sees easily that smooth bidouble Manetti surfaces admit natural deformations which are not Galois coverings of the blowup Y of Q in the points $p_1, \dots p_n$, hence Manetti is forced to take more complicated $G \cong (\mathbb{Z}/2)^r$ coverings (compare Sect. 6 of [Man01], especially page 68, but compare also the crucial Lemma 3.6).

The Galois group is chosen as $G = (\mathbb{Z}/2)^r$,where $r := 2 + n + 5$ (once more we make the simplifying choice $k = 1$ in 6.1 and foll. of [Man01]).

Definition 7.14 *(1) Let* $G_1 := (\mathbb{Z}/2)^2$,$G_2 := (\mathbb{Z}/2)^n$, $G' := G_1 \oplus G_2 \oplus (\mathbb{Z}/2)^4$, $G := G' \oplus (\mathbb{Z}/2)$.

(2) Let $D : G' \to Pic(Y)$ *be the mapping sending*

- *The three nonzero elements of* G_1 *to the classes of the proper transforms of the curves* C_i*, i.e., of* $\pi^*(C_j) - \sum_i A_i$
- *The canonical basis of* G_2 *to the classes of the exceptional divisors* A_i
- *The first two elements of the canonical basis of* $(\mathbb{Z}/2)^4$ *to the pull back of the class of* $\mathcal{O}_Q(1, 0)$*, the last two to the pull back of the class of* $\mathcal{O}_Q(0, 1)$
- *The other elements of* G' *to the zero class.*

With the above setting one has (Lemma 3.6 of [Man01])

Proposition 7.15 *There is an extension of the map* $D : G' \to Pic(Y)$ *to* $D : G \to Pic(Y)$*, and a map* $L : G^\vee \to Pic(Y)$*,* $\chi \mapsto L_\chi$ *such that*

(i) The cover conditions $2L_\chi \equiv \sum_{\chi(\sigma)=1} D_\sigma$ *are satisfied*

(ii) $-D_\sigma + L_\chi$ *is an ample divisor*

(iii) D_σ *is an ample divisor for* $\sigma \in G \backslash G'$

Definition 7.16 *Let now* S *be a* G*-covering of* Y *associated to the choice of some effective divisors* D_σ *in the given classes.* S *is said to be a Manetti surface.*

For simplicity we assume now that these divisors D_σ are smooth and intersect transversally, so that S is smooth.

Condition (iii) guarantees that S is connected, while condition (ii) and an extension of the argument of Proposition 7.10 shows that all the small deformations are G-coverings of such a rational surface Y, blowup of Q at n points.

We are going now only to very briefly sketch the rest of the arguments:

Step A It is possible to choose one of the D_σ's to be so positive that the group of automorphisms of a generic such surface S is just the group G.

Step B Using the natural action of G on any such surface, and using again arguments similar to the ones described in Step III of the last lecture, one sees that we get a closed set of the moduli space.

Step C The families of surfaces thus described fibre over the corresponding families of smooth bidouble Manetti surfaces: since for the latter one has more than one connected component, the same holds for the Manetti surfaces.

In the next section we shall show that the Manetti surfaces corresponding to a fixed choice of the extension D are canonically symplectomorphic.

In particular, they are a strong counterexample to the Def=Diff question.

7.3 Deformation and Canonical Symplectomorphism

We start discussing a simpler case:

Theorem 7.17 *Let S and S' be the respective minimal resolutions of the singularities of two singular bidouble Manetti surfaces X, X' of type (a, b), both triple of the same order n: then S and S' are diffeomorphic, and indeed symplectomorphic for their canonical symplectic structure.*

Proof. In order to set up our notation, we denote by C_1, C_2, C_3 the three smooth branch curves for $p : X \to Q$, and denote by $p_1, .., p_n$ the points where these three curves intersect (with distinct tangents): similarly the covering $p' : X' \to Q$ determines C'_1, C'_2, C'_3 and $p'_1, .., p'_n$. Let Y be the blow up of the quadric Q at the n points $p_1, .., p_n$, so that S is a smooth bidouble cover of Y, similarly S' of Y'.

Without loss of generality we may assume that C_1, C_2 intersect transversally in $2ab$ points, and similarly C'_1, C'_2.

We want to apply Theorem 4.9 to S, S' (i.e., the X, X' of Theorem 4.9 are our S, S'). Let $\hat{C}_3$ be a general curve in the pencil spanned by C_1, C_2, and consider the pencil $C(t) = tC_3 + (1 - t)\hat{C}_3$. For each value of t, $C_1, C_2, C(t)$ meet in $p_1, .., p_n$, while for $t = 0$ they meet in $2ab$ points, again with distinct tangents by our generality assumption. We omit the other finitely many t's for which the intersection points are more than n, or the tangents are not distinct. After blowing up $p_1, .., p_n$ and taking the corresponding bidouble covers, we obtain a family S_t with $S_1 = S$, and such that S_0 has exactly $2ab - n := h$ singular points, quadruple of the type considered in Example 3.3.

Similarly, we have a family S'_t, and we must find an equisingular family $Z_u, u \in U$, containing S_0 and S'_0.

Let $\mathbb{P}$ be the linear system $\mathbb{P}(H^0(Q, \mathcal{O}_Q(a, b))$, and consider a general curve in the Grassmannian $Gr(1, \mathbb{P})$, giving a one dimensional family $C_1[w], C_2[w]$,

$w \in W$, of pairs of points of $\mathbb{P}$ such that $C_1[w]$ and $C_2[w]$ intersect transversally in $2ab$ points of Q.

Now, the covering of W given by

$$\{(w, p_1(w), \dots p_n(w)) | p_1(w), \dots p_n(w) \in C_1[w] \cap C_2[w],\ p_i(w) \neq p_j(w) \text{for } i \neq j\}$$

is irreducible. This is a consequence of the General Position Theorem (see [ACGH85], page 112) stating that if C is a smooth projective curve, then for each integer n the subset $C^n_{dep} \subset C^n$,

$$C^n_{dep} := \{(p_1, \dots p_n) | p_i \neq p_j \text{for } i \neq j,\ p_1, \dots p_n \text{ are linearly dependent}\}$$

is smooth and irreducible.

We obtain then a one dimensional family with irreducible basis U of rational surfaces $Y(u)$, obtained blowing up Q in the n points $p_1(w(u))$,...$p_n(w(u))$, and a corresponding family Z_u of singular bidouble covers of $Y(u)$, each with $2ab - n$ singularities of the same type described above.

We have then the situation of Theorem 4.9, whence it follows that S, S', endowed with their canonical symplectic structures, are symplectomorphic. □

The same argument , mutatis mutandis, shows (compare [Cat02, Cat06])

Theorem 7.18 *Manetti surfaces of the same type (same integers $a, b, n, r = 2n + 7$, same divisor classes $[D_\sigma]$) are canonically symplectomorphic.*

Manetti indeed gave the following counterexample to the Def= Diff question:

Theorem 7.19 (Manetti) *For each integer $h > 0$ there exists a surface of general type S with first Betti number $b_1(S) = 0$, such that the subset of the moduli space corresponding to surfaces which are orientedly diffeomorphic to S contains at least h connected components.*

Remark 7.20 *Manetti proved the diffeomorphism of the surfaces which are here called Manetti surfaces using some results of Bonahon ([Bon83]) on the diffeotopies of lens spaces.*

We have given a more direct proof also because of the application to canonical symplectomorphism.

Corollary 7.21 *For each integer $h > 0$ there exist surfaces of general type $S_1, \dots S_h$ with first Betti number $b_1(S_j) = 0$, socalled* Manetti surfaces*, which are canonically symplectomorphic, but which belong to h distinct connected components of the moduli space of surfaces diffeomorphic to S_1.*

In spite of the fact that we begin to have quite a variety of examples and counterexamples, there are quite a few interesting open questions, the first one concerns the existence of simply connected surfaces which are canonically symplectomorphic, but not deformation equivalent:

Question 7.22 *Are the diffeomorphic (a,b,c)-surfaces of Theorem 6.43, endowed with their canonical symplectic structure, indeed symplectomorphic?*

Remark 7.23 *A possible way of showing that the answer to the question above is yes (and therefore exhibiting symplectomorphic simply connected surfaces which are not deformation equivalent) goes through the analysis of the braid monodromy of the branch curve of the 'perturbed' quadruple covering of $\mathbb{P}^1 \times \mathbb{P}^1$ (the composition of the perturbed covering with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ yields the Lefschetz fibration). One would like to see whether the involution ι on $\mathbb{P}^1$, $\iota(y) = -y$ can be written as the product of braids which show up in the factorization.*

This approach turns out to be more difficult than the corresponding analysis which has been made in the mapping class group, because the braid monodromy contains very many 'tangency' factors which do not come from local contributions to the regeneration of the branch curve from the union of the curves $f = 0, g = 0$ counted twice.

Question 7.24 *Are there (minimal) surfaces of general type which are orientedly diffeomorphic through a diffeomorphism carrying the canonical class to the canonical class, but, endowed with their canonical symplectic structure, are not canonically symplectomorphic?*

Are there such examples in the simply connected case?

The difficult question is then: how to show that diffeomorphic surfaces (diffeomorphic through a diffeomorphism carrying the canonical class to the canonical class) are not symplectomorphic?

We shall briefly comment on this in the next section, referring the reader to the other Lecture Notes in this volume (for instance, the one by Auroux and Smith) for more details.

7.4 Braid Monodromy and Chisini' Problem

Let $B \subset \mathbb{P}^2_{\mathbb{C}}$ be a plane algebraic curve of degree d, and let P be a general point not on B. Then the pencil of lines L_t passing through P determines a one parameter family of d-uples of points of $\mathbb{C} \cong L_t \backslash \{P\}$, namely, $L_t \cap B$.

Blowing up the point P we get the projection $\mathbb{F}_1 \to \mathbb{P}^1$, whence the braid at infinity is a full rotation, corresponding to the generator of the (infinite cyclic) centre of the braid group $\mathcal{B}_d$,

$$(\Delta_d^2) := (\sigma_{d-1}\sigma_{d-2}\ldots\sigma_1)^d.$$

Therefore one gets a factorization of Δ_d^2 in the braid group $\mathcal{B}_d$, and the equivalence class of the factorization does neither depend on the point P (if P is chosen to be general), nor does it depend on B, if B varies in an equisingular family of curves.

Chisini was mainly interested in the case of *cuspidal* curves (compare [Chi44, Chi55]), mainly because these are the branch curves of a generic projection $f : S \to \mathbb{P}^2_{\mathbb{C}}$, for any smooth projective surface $S \subset \mathbb{P}^r$.

More precisely, a *generic projection* $f : S \to \mathbb{P}^2_{\mathbb{C}}$ has the following properties:

- It is a finite morphism whose branch curve B has only nodes and cusps as singularities, and moreover
- The local monodromy around a smooth point of the branch curve is a transposition

Maps with those properties are called *generic coverings*: for these the local monodromies are only $\mathbb{Z}/2 = \mathfrak{S}_2$ (at the smooth points of the branch curve B), $\mathfrak{S}_3$ at the cusps, and $\mathbb{Z}/2 \times \mathbb{Z}/2$ at the nodes.

In such a case we have a *cuspidal* factorization, i.e., all factors are powers of a half twist, with respective exponents $1, 2, 3$.

Chisini posed the following daring

Conjecture 7.25 (Chisini's conjecture) *Given two generic coverings $f : S \to \mathbb{P}^2_{\mathbf{C}}$, $f' : S' \to \mathbb{P}^2_{\mathbf{C}}$, both of degree at least 5, assume that they have the same branch curve B. Is it then true that f and f' are equivalent?*

Observe that the condition on the degree is necessary, since a counterexample for $d \leq 4$ is furnished by the dual curve B of a smooth plane cubic (as already known to Chisini, cf. [Chi44]). Chisini in fact observed that there are two generic coverings, of respective degrees 3 and 4, and with the given branch curve. Combinatorially, we have a triple of transpositions corresponding in one case to the sides of a triangle ($d = 3$, and the monodromy permutes the vertices of the triangle), and in the other case to the three medians of the triangle ($d = 4$, and the monodromy permutes the vertices of the triangle plus the barycentre).

While establishing a very weak form of the conjecture [Cat86]. I remarked that the dual curve B of a smooth plane cubic is also the branch curve for three nonequivalent generic covers of the plane from the Veronese surface (they are distinct since they determine three distinct divisors of 2-torsion on the cubic).

The conjecture seems now to have been almost proven (i.e., it is not yet proven in the strongest possible form), after that it was first proven by Kulikov (cf. [Kul99]) under a rather complicated assumption, and that shortly later Nemirovskii [Nem01] noticed (just by using the Miyaoka-Yau inequality) that Kulikov's complicated assumption was implied by the simpler assumption $d \geq 12$.

Kulikov proved now [Kul06] the following

Theorem 7.26 (Kulikov) *Two generic projections with the same cuspidal branch curve B are isomorphic unless if the projection $p : S \to \mathbb{P}^2$ of one of them is just a linear projection of the Veronese surface.*

Chisini's conjecture concerns a fundamental property of the fundamental group of the complement $\mathbb{P}^2 \backslash B$, namely to admit only one conjugacy class of surjections onto a symmetric group $\mathcal{S}_n$, satisfying the properties of a generic covering.

In turn, the fundamental group $\pi_1(\mathbb{P}^2 \backslash B)$ is completely determined by the braid monodromy of B, i.e., the above equivalence class (modulo Hurwitz equivalence and simultaneous conjugation) of the above factorization of Δ_d^2. So, a classical question was: which are the braid monodromies of cuspidal curves?

Chisini found some necessary conditions, and proposed some argument in order to show the sufficiency of these conditions, which can be reformulated as

Chisini's problem: (cf. [Chi55]).

Given a cuspidal factorization, which is regenerable to the factorization of a smooth plane curve, is there a cuspidal curve which induces the given factorization?

Regenerable means that there is a factorization (in the equivalence class) such that, after replacing each factor σ^i $(i = 2, 3)$ by the i corresponding factors (e.g. , σ^3 is replaced by $\sigma \circ \sigma \circ \sigma$) one obtains the factorization belonging to a non singular plane curve.

A negative answer to the problem of Chisini was given by B. Moishezon in [Moi94].

Remark 7.27 *(1) Moishezon proves that there exist infinitely many non equivalent cuspidal factorizations observing that $\pi_1(\mathbb{P}^2_{\mathbb{C}} \backslash B)$ is an invariant defined in terms of the factorization alone and constructing infinitely many non isomorphic such groups. On the other hand, the family of cuspidal curves of a fixed degree is an algebraic set, hence it has a finite number of connected components. These two statements together give a negative answer to the above cited problem of Chisini.*

The examples of Moishezon have been recently reinterpreted in [ADK03], with a simpler treatment, in terms of symplectic surgeries.

Now, as conjectured by Moishezon, a cuspidal factorization together with a generic monodromy with values in $\mathfrak{S}_n$ induces a covering $M \to \mathbb{P}^2_{\mathbf{C}}$, where the fourmanifold M has a unique symplectic structure (up to symplectomorphism) with class equal to the pull back of the Fubini Study form on $\mathbb{P}^2$ (see for instance [A-K00]).

What is more interesting (and much more difficult) is however the converse.

Extending Donaldson's techniques (for proving the existence of symplectic Lefschetz fibrations) Auroux and Katzarkov [A-K00] proved that each symplectic 4-manifold is in a natural way 'asymptotically' realized by such a generic covering.

They show that, given a symplectic fourmanifold (M, ω) with $[\omega] \in H^2(M, \mathbb{Z})$, there exists a multiple m of a line bundle L with $c_1(L) = [\omega]$ and three general sections s_0, s_1, s_2 of $L^{\otimes m}$, which are ϵ-holomorphic with many of their derivatives (that a section s is ϵ-holomorphic means, very roughly speaking, that once one has chosen a compatible almost complex structure, $|\bar{\partial} s| < \epsilon \; |\partial s|$) yielding a finite covering of the plane $\mathbb{P}^2$ which is generic and with branch curve a symplectic subvariety whose singularities are only nodes and cusps.

The only price they have to pay is to allow also negative nodes, i.e., nodes which in local holomorphic coordinates are defined by the equation

$$(y - \bar{x})(y + \bar{x}) = 0.$$

The corresponding factorization in the braid group contains then only factors which are conjugates of σ_1^j, with $j = -2, 1, 2, 3$.

Moreover, the factorization is not unique, because it may happen that two consecutive nodes, one positive and one negative, may disappear, and the corresponding two factors disappear from the factorization. In particular, $\pi_1(\mathbb{P}^2_{\mathbb{C}} \backslash B)$ is no longer an invariant and the authors propose to use an appropriate quotient of $\pi_1(\mathbb{P}^2_{\mathbb{C}} \backslash B)$ in order to produce invariants of symplectic structures.

It seems however that, in the computations done up to now, even the groups $\pi_1(\mathbb{P}^2_{\mathbb{C}} \backslash B)$ allow only to detect homology invariants of the projected fourmanifold [ADKY04].

Let us now return to the world of surfaces of general type.

Suppose we have a surface S of general type and a pluricanonical embedding $\psi_m \colon X \to \mathbb{P}^N$ of the canonical model X of S. Then a generic linear projection of the pluricanonical image to $\mathbb{P}^2_{\mathbb{C}}$ yields, if $S \cong X$, a generic covering $S \to \mathbb{P}^2_{\mathbb{C}}$ (else the singularities of X create further singularities for the branch curve B and other local coverings).

By the positive solution of Chisini's conjecture, the branch curve B determines the surface S uniquely (up to isomorphism). We get moreover the equivalence class of the braid monodromy factorization, and this does not change if S varies in a connected family of surfaces with K_S ample (i.e., the surfaces equal their canonical models).

Motivated by this observation of Moishezon, Moishezon and Teicher in a series of technically difficult papers (see e.g. [MT92]) tried to calculate fundamental groups of complements $\pi_1(\mathbb{P}^2_{\mathbb{C}} \backslash B)$, with the intention of distinguishing connected components of the moduli spaces of surfaces of general type.

Indeed, it is clear that these groups are invariants of the connected components of the open set of the moduli space corresponding to surfaces with ample canonical divisor K_S. Whether one could even distinguish connected components of moduli spaces would in my opinion deserve a detailed argument, in view of the fact that several irreducible components consist of surfaces whose canonical divisor is not ample (see for instance [Cat89] for several series of examples).

But it may be that the information carried by $\pi_1(\mathbb{P}^2_{\mathbb{C}} \backslash B)$ be too scanty, so one could look at further combinatorial invariants, rather than the class of the braid monodromy factorization for B.

In fact a generic linear projection of the pluricanonical image to $\mathbb{P}^3_{\mathbb{C}}$ gives a surface Σ with a double curve Γ'. Now, projecting further to $\mathbb{P}^2_C$ we do not only get the branch curve B, but also a plane curve Γ, image of Γ'.

Even if Chisini's conjecture tells us that from the holomorphic point of view B determines the surface S and therefore the curve Γ, it does not follow that the fundamental group $\pi_1(\mathbb{P}^2_{\mathbf{C}} \backslash B)$ determines the group $\pi_1(\mathbb{P}^2_{\mathbf{C}} \backslash (B \cup \Gamma))$.

It would be interesting to calculate this second fundamental group, even in special cases.

Moreover, generalizing a proposal done by Moishezon in [Moi83], one can observe that the monodromy of the restriction of the covering $\Sigma \to \mathbb{P}^2$ to $\mathbb{P}^2_{\mathbf{C}} \backslash (B \cup \Gamma))$ is more refined, since it takes values in a braid group $\mathcal{B}_n$, rather than in a symmetric group $\mathcal{S}_n$.

One could proceed similarly also for the generic projections of symplectic fourmanifolds.

But in the symplectic case one does not have the advantage of knowing a priori an explicit number $m \leq 5$ such that ψ_m is a pluricanonical embedding for the general surface S in the moduli space.

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