Preface

When I started giving talks on regularity theory for degenerate and singular parabolic equations, a fixed-point in the conversation during the coffee-break that usually followed the seminar was the apparent contrast between the beauty of the subject and its technical difficulty. I could not agree more on the beauty part but, most of the times, overwhelmingly failed to convince my audience that the technicalities were not all that hard to follow. As in many other instances, it was the fact that the results in the literature were eventually stated and proved in their most possible generality that made the whole subject seem inexpugnable.

So when I had the chance of preparing a short course on the method of intrinsic scaling, I decided to present the theory from scratch for the simplest model case of the degenerate p-Laplace equation and to leave aside technical refinements needed to deal with more general situations. The first part of the notes you are about to read is the result of that effort: an introductory and self-contained approach to intrinsic scaling, aiming at bringing to light what is really essential in this powerful tool in the analysis of degenerate and singular equations. As another striking feature of the method is its pervasiveness in terms of the applications, in the second part of the book, intrinsic scaling is applied to several models arising from flows in porous media, chemotaxis and phase transitions. The aim is to convince the reader of the strength of the method as a systematic approach to regularity for an important and relevant class of nonlinear partial differential equations.

The analysis of degenerate and singular parabolic equations is an extremely vast and active research topic and in this contribution there is, by no means, any intention to exhaust the theory. On the contrary, the focus is on a particular subject – the (Hölder) continuity of solutions – and a unifying set of ideas. We hope that the careful study of theses notes will enable the reader to master the essential features of the method of intrinsic scaling, which is instrumental in dealing with more elaborate aspects of the theory, like the boundedness of solutions, Harnack inequalities or systems of equations.

The first four chapters contain material that would fit well in an advanced graduate course on regularity theory for partial differential equations. Each chapter corresponds roughly, with the exception of the first one, to two 90 min. classes. Chapters 5–7 are independent from one another and each could be chosen to complement the course, according to individual preferences. I would probably suggest choosing chapter 5 for that purpose.

These lecture notes had its origin in a minicourse I delivered at the 2005 Summer Program of IMPA in Rio de Janeiro. Later that year, I taught a shorter version of the course at the University of Florence. I would like to thank Marcelo Viana and Vincenzo Vespri for their kind invitations and for the wonderful hospitality. I am also indebted to all the colleagues and students who took the course for their interest and input and, in particular, to my former PhD student Eurica Henriques.

Finally, I warmly thank Emmanuele DiBenedetto for his continuing support and advice.

Coimbra, November 2007 *José Miguel Urbano*

2

Weak Solutions and *a Priori* Estimates

We will concentrate on the parabolic $p-$Laplace equation

$$u_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0 \,, \qquad p > 1, \tag{2.1}$$

a quasilinear second-order partial differential equation, with principal part in divergence form. If $p > 2$, the equation is degenerate in the space part, due to the vanishing of its modulus of ellipticity $|\nabla u|^{p-2}$ at points where $|\nabla u| = 0$. The singular case corresponds to $1 < p < 2$: the modulus of ellipticity becomes unbounded at points where $|\nabla u| = 0$.

In this chapter we place no restriction on the values of $p > 1$. The theory is markedly different in the degenerate and singular cases and we will later restrict our attention to $p > 2$. The results extend to a variety of equations and, in particular, to equations with general principal parts satisfying appropriate structure assumptions and with lower order terms. We have chosen to present the results and the proofs for the particular model case (2.1) to bring to light what we feel are the essential features of the theory. Remarks on generalizations, which in some way or another correspond to more or less sophisticated technical improvements, are left to a later section.

2.1 Definition of Weak Solution

Let Ω be a bounded domain in $\mathbb{R}^d$, with smooth boundary $\partial\Omega$. Let

$$\Omega_T = \Omega \times (0, T] \,, \quad T > 0,$$

be the space-time domain, with lateral boundary $\Sigma = \partial\Omega \times (0, T)$ and parabolic boundary

$$\partial_p \Omega_T = \Sigma \cup (\Omega \times \{0\}) \,.$$

We start with the precise definition of local weak solution for (2.1).

Definition 2.1. *A local weak solution of* (2.1) *is a measurable function*

$$u \in C_{\text{loc}}\left(0,T; L^2_{\text{loc}}(\Omega)\right) \cap L^p_{\text{loc}}\Big(0,T; W^{1,p}_{\text{loc}}(\Omega)\Big)$$

such that, for every compact $K \subset \Omega$ *and for every subinterval* $[t_1,t_2]$ *of* $(0,T]$,

$$\int_K u\varphi \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left\{-u\varphi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\right\} dx\, dt = 0, \tag{2.2}$$

for all $\varphi \in H^1_{\text{loc}}\left(0,T; L^2(K)\right) \cap L^p_{\text{loc}}\Big(0,T; W^{1,p}_0(K)\Big)$.

It would be technically convenient to have at hand a formulation of weak solution involving the time derivative u_t. Unfortunately, solutions of (2.1), whenever they exist, possess a modest degree of time-regularity and, in general, u_t has a meaning only in the sense of distributions. To overcome this limitation, we introduce the Steklov average of a function $v \in L^1(\Omega_T)$, defined, for $0 < h < T$, by

$$v_h := \begin{cases} \frac{1}{h}\displaystyle\int_t^{t+h} v(\cdot,\tau)\, d\tau & \text{if } \ t \in (0, T-h] \\ \\ \qquad 0 & \text{if } \ t \in (T-h, T]. \end{cases} \tag{2.3}$$

The proof of the following lemma follows from the general theory of L^p spaces.

Lemma 2.2. *If* $v \in L^{q,r}(\Omega_T)$ *then, as* $h \to 0$, *the Steklov average* v_h *converges to* v *in* $L^{q,r}(\Omega_{T-\epsilon})$, *for every* $\epsilon \in (0,T)$. *If* $v \in C\left(0,T; L^q(\Omega)\right)$ *then, as* $h \to 0$, *the Steklov average* $v_h(\cdot,t)$ *converges to* $v(\cdot,t)$ *in* $L^q(\Omega)$, *for every* $t \in (0, T-\epsilon)$ *and every* $\epsilon \in (0,T)$.

It is a simple exercise to show that the definition of local weak solution previously introduced is equivalent to the following one.

Definition 2.3. *A local weak solution of (2.1) is a measurable function*

$$u \in C_{\text{loc}}\left(0,T; L^2_{\text{loc}}(\Omega)\right) \cap L^p_{\text{loc}}\Big(0,T; W^{1,p}_{\text{loc}}(\Omega)\Big)$$

such that, for every compact $K \subset \Omega$ *and for every* $0 < t < T-h$,

$$\int_{K\times\{t\}} \left\{(u_h)_t\, \varphi + \left(|\nabla u|^{p-2}\nabla u\right)_h \cdot \nabla\varphi\right\} dx = 0, \tag{2.4}$$

for all $\varphi \in W^{1,p}_0(K)$.

We will show that locally bounded solutions of (2.1) are locally Hölder continuous within their domain of definition. No specific boundary or initial values need to be prescribed for u. A theory of boundedness of weak solutions of (2.1) is quite different from the linear theory (*cf.* [14]): weak solutions are locally bounded only if $d(p-2)+p>0$. It can be shown by counterexample that this condition is sharp. Although the arguments below are of local nature, to simplify the presentation we assume that u is a.e. defined and bounded in Ω_T and set

$$M := \|u\|_{L^\infty(\Omega_T)}.$$

2.2 Local Energy Estimates: The Building Blocks of the Theory

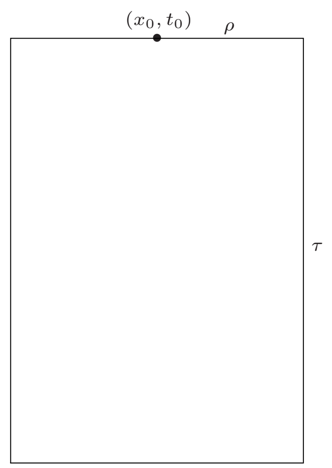
The building blocks of the method of intrinsic scaling are *a priori* estimates for weak solutions. Once these estimates are obtained, we can forget the equation and the problem becomes, purely, a problem in analysis: showing that functions that satisfy certain integral inequalities belong to a certain regularity class (*e.g.*, are locally Hölder continuous). These estimates are integral inequalities on level sets that *measure* the behaviour of the function near its infimum and its supremum in the interior of an appropriate cylinder.

Given a point $x_0 \in \mathbb{R}^d$, denote by $K_\rho(x_0)$ the d-dimensional cube with centre at x_0 and wedge 2ρ:

$$K_\rho(x_0) := \left\{ x \in \mathbb{R}^d : \max_{1\le i\le d} |x_i - x_{0i}| < \rho \right\}$$

and put $K_\rho := K_\rho(0)$; given a point $(x_0,t_0) \in \mathbb{R}^{d+1}$, the cylinder of radius ρ and height $\tau > 0$ with vertex at (x_0,t_0) is

$$(x_0,t_0) + Q(\tau,\rho) := K_\rho(x_0) \times (t_0-\tau, t_0).$$



We write $Q(\tau,\rho)$ to denote $(0,0)+Q(\tau,\rho)$. We use the usual notations for the positive and negative parts of a function:

$$v_+ = \max(v,0) \qquad \text{and} \qquad v_- = (-v)_+ .$$

We now deduce the energy estimates. Without loss of generality, we restrict to cylinders with vertex at the origin $(0,0)$, the changes being obvious for cylinders with vertex at a generic (x_0,t_0). Consider a cylinder $Q(\tau,\rho)\subset\Omega_T$ and let $0\le\zeta\le 1$ be a piecewise smooth cutoff function in $Q(\tau,\rho)$ such that

$$|\nabla\zeta|<\infty \qquad \text{and} \qquad \zeta(x,t)=0\ , \quad x\notin K_\rho . \tag{2.5}$$

Proposition 2.4. *Let u be a local weak solution of (2.1) and $k\in\mathbb{R}$. There exists a constant $C\equiv C(p)>0$ such that, for every cylinder $Q(\tau,\rho)\subset\Omega_T$,*

$$\begin{aligned}\sup_{-\tau<t<0}\int_{K_\rho\times\{t\}}(u-k)_\pm^2\,\zeta^p\,dx+\int_{-\tau}^0\int_{K_\rho}|\nabla(u-k)_\pm\zeta|^p\,dx\,dt\\ \le\int_{K_\rho\times\{-\tau\}}(u-k)_\pm^2\,\zeta^p\,dx+C\int_{-\tau}^0\int_{K_\rho}(u-k)_\pm^p\,|\nabla\zeta|^p\,dx\,dt\\ +p\int_{-\tau}^0\int_{K_\rho}(u-k)_\pm^2\,\zeta^{p-1}\,\zeta_t\,dx\,dt.\end{aligned} \tag{2.6}$$

Proof. Let $\varphi=\pm(u_h-k)_\pm\zeta^p$ in (2.4) and integrate in time over $(-\tau,t)$ for $t\in(-\tau,0)$. The first term gives

$$\begin{aligned}\int_{-\tau}^t\int_{K_\rho}(u_h)_t\,\varphi\,dx\,d\theta&=\frac12\int_{-\tau}^t\int_{K_\rho}\left[(u_h-k)_\pm^2\right]_t\,\zeta^p\,dx\,d\theta\\ \longrightarrow\frac12\int_{K_\rho\times\{t\}}(u-k)_\pm^2\,\zeta^p\,dx&-\frac12\int_{K_\rho\times\{-\tau\}}(u-k)_\pm^2\,\zeta^p\,dx\\ &-\frac{p}{2}\int_{-\tau}^t\int_{K_\rho}(u-k)_\pm^2\,\zeta^{p-1}\,\zeta_t\,dx\,d\theta,\end{aligned}$$

after integrating by parts and passing to the limit in $h\to0$ (using Lemma 2.2). Concerning the other term, letting first $h\to0$, we obtain

$$\begin{aligned}&\int_{-\tau}^t\int_{K_\rho}\left[|\nabla u|^{p-2}\nabla u\right]_h\cdot\nabla\varphi\,dx\,d\theta\\ &\longrightarrow\int_{-\tau}^t\int_{K_\rho}|\nabla u|^{p-2}\nabla u\cdot\left[\pm\nabla(u-k)_\pm\zeta^p\pm p(u-k)_\pm\zeta^{p-1}\nabla\zeta\right]\,dx\,d\theta\\ &\quad\ge\int_{-\tau}^t\int_{K_\rho}|\nabla(u-k)_\pm|^p\,\zeta^p\,dx\,d\theta\end{aligned}$$

$$
\begin{aligned}
&-p \int_{-\tau}^{t} \int_{K_\rho} |\nabla (u-k)_\pm|^{p-1} (u-k)_\pm \, \zeta^{p-1} \, |\nabla \zeta| \, dx \, d\theta \\
\geq\ & \frac{1}{2} \int_{-\tau}^{t} \int_{K_\rho} |\nabla (u-k)_\pm \, \zeta|^p \, dx \, d\theta \\
&-C(p) \int_{-\tau}^{t} \int_{K_\rho} (u-k)_\pm^p \, |\nabla \zeta|^p \, dx \, d\theta,
\end{aligned}
$$

using the inequality of Young

$$
ab \leq \frac{\varepsilon^p}{p} a^p + \frac{1}{p' \varepsilon^{p'}} b^{p'},
$$

with the choices

$$
a = (u-k)_\pm \, |\nabla \zeta| \, , \quad b = |\nabla (u-k)_\pm \, \zeta|^{p-1} \, , \quad \text{and} \quad \varepsilon = [2(p-1)]^{\frac{1}{p'}} .
$$

Since $t \in (-\tau, 0)$ is arbitrary, we can combine both estimates to obtain (2.6). □

Remark 2.5. In (2.6), there is an intentional ambiguity in the way we wrote $|\nabla (u-k)_\pm \zeta|^p$. The gradient can either affect only $(u-k)_\pm$ (as follows directly from the estimates in the proof) or the product $(u-k)_\pm \zeta$ (as the extra term can clearly be absorbed into the right hand side of the estimate).

2.3 Local Logarithmic Estimates

We now introduce a logarithmic function for which we obtain further local estimates. These are the subsidiary building blocks of the theory but nevertheless play a crucial role in the proof, allowing for the expansion in time to a full cylinder $Q(\tau, \rho)$ of certain results obtained for sub-cylinders of $Q(\tau, \rho)$.

Given constants a, b, c, with $0 < c < a$, define the nonnegative function

$$
\psi^\pm_{\{a,b,c\}}(s) := \left(\ln \left\{ \frac{a}{(a+c) - (s-b)_\pm} \right\} \right)_+
$$

$$
= \begin{cases} \ln \left\{ \frac{a}{(a+c) \pm (b-s)} \right\} & \text{if } b \pm c \ \lessgtr \ s \ \lessgtr \ b \pm (a+c) \\ \\ 0 & \text{if } s \ \lesseqqgtr \ b \pm c \end{cases}
$$

whose first derivative is

$$
\left(\psi^\pm_{\{a,b,c\}} \right)' (s) = \begin{cases} \dfrac{1}{(b-s) \pm (a+c)} & \text{if } b \pm c \ \lessgtr \ s \ \lessgtr \ b \pm (a+c) \\ \\ 0 & \text{if } s \ \lessgtr \ b \pm c \end{cases} \qquad \gtreqless \ 0,
$$

and second derivative, off $s = b \pm c$, is

$$\left(\psi^{\pm}_{\{a,b,c\}}\right)'' = \left\{\left(\psi^{\pm}_{\{a,b,c\}}\right)'\right\}^2 \geq 0.$$

Now, given a bounded function u in a cylinder $(x_0, t_0) + Q(\tau, \rho)$ and a number k, define the constant

$$H^{\pm}_{u,k} := \operatorname*{ess\,sup}_{(x_0,t_0)+Q(\tau,\rho)} |(u-k)_{\pm}| .$$

The following function was introduced in [11] and since then has been used as a recurrent tool in the proof of results concerning the local behaviour of solutions of degenerate and singular equations:

$$\Psi^{\pm}\left(H^{\pm}_{u,k}, (u-k)_{\pm}, c\right) \equiv \psi^{\pm}_{\left\{H^{\pm}_{u,k}, k, c\right\}}(u) , \qquad 0 < c < H^{\pm}_{u,k}. \tag{2.7}$$

From now on, when referring to this function we will write it as $\psi^{\pm}(u)$, omitting the subscripts, whose meaning will be clear from the context.

Let $x \mapsto \zeta(x)$ be a time-independent cutoff function in $K_\rho(x_0)$ satisfying (2.5). The logarithmic estimates in cylinders $Q(\tau, \rho)$ with vertex at the origin read as follows.

Proposition 2.6. *Let u be a local weak solution of (2.1) and $k \in \mathbb{R}$. There exists a constant $C \equiv C(p) > 0$ such that, for every cylinder $Q(\tau, \rho) \subset \Omega_T$,*

$$\begin{aligned} \sup_{-\tau<t<0} \int_{K_\rho \times \{t\}} \left[\psi^{\pm}(u)\right]^2 \zeta^p \, dx &\leq \int_{K_\rho \times \{-\tau\}} \left[\psi^{\pm}(u)\right]^2 \zeta^p \, dx \\ &\quad + C \int_{-\tau}^{0} \int_{K_\rho} \psi^{\pm}(u) \left|(\psi^{\pm})'(u)\right|^{2-p} |\nabla \zeta|^p \, dx \, dt. \end{aligned} \tag{2.8}$$

Proof. Take $\varphi = 2\,\psi^{\pm}(u_h) \left[(\psi^{\pm})'(u_h)\right] \zeta^p$ as a testing function in (2.4) and integrate in time over $(-\tau, t)$ for $t \in (-\tau, 0)$. Since $\zeta_t \equiv 0$,

$$\begin{aligned} &\int_{-\tau}^{t} \int_{K_\rho} (u_h)_t \left\{2\,\psi^{\pm}(u_h) \left[(\psi^{\pm})'(u_h)\right] \zeta^p\right\} dx \, d\theta \\ &\qquad = \int_{-\tau}^{t} \int_{K_\rho} \left(\left[\psi^{\pm}(u_h)\right]^2\right)_t \zeta^p \, dx \, d\theta \\ &\qquad = \int_{K_\rho \times \{t\}} \left[\psi^{\pm}(u_h)\right]^2 \zeta^p \, dx - \int_{K_\rho \times \{-\tau\}} \left[\psi^{\pm}(u_h)\right]^2 \zeta^p \, dx. \end{aligned}$$

From this, letting $h \to 0$,

$$\int_{-\tau}^{t}\int_{K_\rho} (u_h)_t \left\{2\,\psi^{\pm}(u_h)\ \left[(\psi^{\pm})'(u_h)\right]\ \zeta^p\right\} dx\,d\theta$$
$$\longrightarrow \int_{K_\rho\times\{t\}} \left[\psi^{\pm}(u)\right]^2\ \zeta^p\,dx - \int_{K_\rho\times\{-\tau\}} \left[\psi^{\pm}(u)\right]^2\ \zeta^p\,dx.$$

As for the remaining term, we first let $h \to 0$, to obtain

$$\begin{aligned}
&\int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^{p-2}\nabla u\cdot\nabla\left\{2\,\psi^{\pm}(u)\ \left[(\psi^{\pm})'(u)\right]\ \zeta^p\right\} dx\,d\theta\\
&=\int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^{p}\left\{2\ \left(1+\psi^{\pm}(u)\right)\left[(\psi^{\pm})'(u)\right]^2\ \zeta^p\right\} dx\,d\theta\\
&\quad + p\int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^{p-2}\nabla u\cdot\nabla\zeta\ \left\{2\,\psi^{\pm}(u)\ \left[(\psi^{\pm})'(u)\right]\ \zeta^{p-1}\right\} dx\,d\theta\\
&\\
&\ge\int_{-\tau}^{t}\int_{K_\rho} |\nabla u|^{p}\left\{2\ \left(1+\psi^{\pm}(u)-\psi^{\pm}(u)\right)\ \left[(\psi^{\pm})'(u)\right]^2\ \zeta^p\right\} dx\,d\theta\\
&\quad -2(p-1)^{p-1}\int_{-\tau}^{t}\int_{K_\rho} \psi^{\pm}(u)\ \left|(\psi^{\pm})'(u)\right|^{2-p}|\nabla\zeta|^p\ dx\,d\theta\\
&\ge -C\int_{-\tau}^{t}\int_{K_\rho} \psi^{\pm}(u)\ \left|(\psi^{\pm})'(u)\right|^{2-p}|\nabla\zeta|^p\ dx\,d\theta.
\end{aligned}$$

Since $t \in (-\tau, 0)$ is arbitrary, we can combine both estimates to obtain (2.8). $\square$

2.4 Some Technical Tools

We gather in this section a few technical facts that, although marginal to the theory, are essential in establishing its main results.

1. A Lemma of De Giorgi

Given a continuous function $v : \Omega \to \mathbb{R}$ and two real numbers $k_1 < k_2$, we define

$$\begin{aligned}
[v > k_2] &:= \{x\in\Omega\ :\ v(x) > k_2\}\,,\\
[v < k_1] &:= \{x\in\Omega\ :\ v(x) < k_1\}\,,\\
[k_1 < v < k_2] &:= \{x\in\Omega\ :\ k_1 < v(x) < k_2\}\,.
\end{aligned}\tag{2.9}$$

Lemma 2.7 (De Giorgi, [10]). *Let* $v \in W^{1,1}(B_\rho(x_0)) \cap C(B_\rho(x_0))$, *with* $\rho > 0$ *and* $x_0 \in \mathbb{R}^d$, *and let* $k_1 < k_2 \in \mathbb{R}$. *There exists a constant* C, *depending only on* d *(and thus independent of* ρ, x_0, v, k_1 *and* k_2*), such that*

$$(k_2 - k_1)\ |[v > k_2]| \leq C\ \frac{\rho^{d+1}}{|[v < k_1]|} \int_{[k_1 < v < k_2]} |\nabla v|\, dx.$$

Remark 2.8. The conclusion of the lemma remains valid, provided Ω is convex, for functions $v \in W^{1,1}(\Omega) \cap C(\Omega)$. We will use it in the case Ω is a cube. In fact, the continuity is not essential for the result to hold. For a function merely in $W^{1,1}(\Omega)$, we define the sets in (2.9) through any representative in the equivalence class. It can be shown that the conclusion of the lemma is independent of that choice.

2. Geometric Convergence of Sequences

The following lemmas concern the geometric convergence of sequences and are instrumental in the iterative schemes that will be derived along the proofs.

Lemma 2.9. *Let* (X_n), $n = 0, 1, 2, \ldots$, *be a sequence of positive real numbers satisfying the recurrence relation*

$$X_{n+1} \leq C\, b^n\, X_n^{1+\alpha}$$

where $C, b > 1$ *and* $\alpha > 0$ *are given. If*

$$X_0 \leq C^{-1/\alpha}\, b^{-1/\alpha^2}$$

then $X_n \to 0$ *as* $n \to \infty$.

Lemma 2.10. *Let* (X_n) *and* (Z_n), $n = 0, 1, 2, \ldots$, *be sequences of positive real numbers satisfying the recurrence relations*

$$\begin{cases} X_{n+1} \leq C\, b^n \left(X_n^{1+\alpha} + X_n^\alpha Z_n^{1+\kappa} \right) \\[2ex] Z_{n+1} \leq C\, b^n \left(X_n + Z_n^{1+\kappa} \right) \end{cases}$$

where $C, b > 1$ *and* $\alpha, \kappa > 0$ *are given. If*

$$X_0 + Z_0^{1+\kappa} \leq (2C)^{-\frac{1+\kappa}{\sigma}}\, b^{-\frac{1+\kappa}{\sigma^2}}, \qquad \textit{with} \quad \sigma = \min\{\alpha, \kappa\},$$

then $X_n, Z_n \to 0$ *as* $n \to \infty$.

3. An Embedding Theorem

Let $V_0^p(\Omega_T)$ denote the space

$$V_0^p(\Omega_T) = L^\infty\left(0,T;L^p(\Omega)\right) \cap L^p\left(0,T;W_0^{1,p}(\Omega)\right)$$

endowed with the norm

$$\|u\|^p_{V^p(\Omega_T)} = \operatorname*{ess\,sup}_{0\le t\le T} \|u(\cdot,t)\|^p_{p,\Omega} + \|\nabla u\|^p_{p,\Omega_T}.$$

The following embedding theorem holds (cf. [14, page 9]).

Theorem 2.11. *Let $p > 1$. There exists a constant γ, depending only on d and p, such that for every $v \in V_0^p(\Omega_T)$,*

$$\|v\|^p_{p,\Omega_T} \le \gamma \mid |v| > 0 \mid^{\frac{p}{d+p}} \|v\|^p_{V^p(\Omega_T)}.$$

4. A Poincaré-type Inequality

Let $\Omega \subset \mathbb{R}^d$ be a bounded and convex set. Consider a function $\varphi \in C(\overline{\Omega})$, $0 \le \varphi \le 1$, such that the sets

$$[\varphi > k], \quad 0 < k < 1$$

are all convex.
The following theorem holds (cf. [14, page 5]).

Theorem 2.12. *Let $v \in W^{1,p}(\Omega)$, $p \ge 1$ and assume that the set*

$$\Xi := [v = 0] \cap [\varphi = 1]$$

has positive measure. There exists a constant C, depending only on d and p, and independent of v and φ, such that

$$\int_\Omega |v|^p\,\varphi\,dx \le C\,\frac{|\operatorname{diam}\Omega|^{dp}}{|\Xi|^{\frac{(d-1)p}{d}}} \int_\Omega |\nabla v|^p\,\varphi\,dx.$$

5. Constants

With C we denote constants that depend only on d and p; these constants may be different if they appear in different lines.