Preface

Noncommutative geometry, which can rightfully claim the role of a philosophy in mathematical studies, undertakes to replace good old notions of classical geometry (such as manifolds, vector bundles, metrics, differentiable structures, etc.) by their abstract operator-algebraic analogs and then to study the latter by methods of the theory of operator algebras. At first sight, this pursuit of maximum possible generality harbors the danger of completely forgetting the classical beginnings, so that not only the answers but also the questions would defy stating in traditional terms. Noncommutative geometry itself would become not only a method but also the main subject of investigation according to the capacious but not too practical formula: "Know thyself." Fortunately, this is not completely true (or even is completely untrue) in reality: there are numerous problems that are quite classical in their statement (or at least admit an equivalent classical statement) but can be solved only in the framework of noncommutative geometry. One of such problems is the subject of the present book.

The classical elliptic theory developed in the well-known work of Atiyah and Singer on the index problem relates an analytic invariant of an elliptic pseudodifferential operator on a smooth compact manifold, namely, its index, to topological invariants of the manifold itself. The index problem for *nonlocal* (and hence nonpseudodifferential) elliptic operators is much more complicated and requires the use of substantially more powerful methods than those used in the classical case. It should be noted that although elliptic theory (more precisely, its analytic branch) for nonlocal operators has been studied sufficiently well, meaningful results concerning the index problem for nonlocal elliptic operators have until recently been rather spare. It is only very recently that several important results (some of which are due to the authors) have appeared, suggesting that the solution of this problem is eventually within reach. Therefore, there is a need to gather together the facts already known on the topic.

That is why the present book has been written. Methods of K-theory of operator algebras and noncommutative geometry are used here to solve the index problem for nonlocal elliptic operators associated with a countable group of diffeomorphisms of a manifold.

Furthermore, to make the presentation self-contained and hence the book understandable for readers with standard university education in mathematics, we have decided to include the Appendix, which contains some material used in noncommutative elliptic theory, namely, C^*-algebras and their K-theory as well as basics of the theory of cyclic homology and cohomology.

The main results contained in the book were obtained during the authors' stay as guests researchers at Institut für Analysis, Leibniz Universität Hannover (Hannover, Germany).

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Finally, we especially wish to thank Professor G. V. Rozenblyum of Göteborg University, Sweden, who was the first to acquaint us with the beautiful world of noncommutative geometry.

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Introduction

Differential equations containing values of unknown functions and their derivatives at different points of a manifold are called *nonlocal differential equations*. The simplest equation of this type has the form

$$D_1 u(x) + D_2 u(g(x)) = f(x), \qquad x \in \Omega,$$

where D_1 and D_2 are some differential operators, u is the unknown function, and $g\colon \Omega \to \Omega$ is a self-mapping of the domain where the equation is considered. We shall consider only equations in which the mapping g is invertible.

Such equations arise in numerous physical and mathematical problems, in particular, in problems related to noncommutative geometry. We present only some of them:

1. Elliptic theory on the noncommutative torus and the quantum Hall effect. Differential operators on the noncommutative torus were studied by Connes in [24, 27], who, in particular, obtained an index formula for such operators. The coefficients of these operators contain shift operators generated by irrational rotations.

2. More general nonlocal operators related to deformations of function algebras on toric manifolds, in particular, to quantum spheres obtained by noncommutative isospectral deformations. (See Connes–Landi [29], Connes–Dubois-Violette [28], Landi–van Suijlekom [50], etc.)

3. Nonlocal boundary value problems[1] (Carleman [23], Antonevich [3], Bitsadze [21], Dezin [34], Skubachevskii [71], etc.).

These examples naturally justify interest in general nonlocal elliptic operators, i.e., in differential or pseudodifferential operators whose coefficients include not only operators of multiplication by functions but also shift operators induced by a discrete group Γ of diffeomorphisms of the manifold. Finiteness theorems for such operators were obtained by Antonevich and Lebedev (e.g., see [1,2,4] and references therein). The present book deals mainly with the topological

[1] Note, however, that we do not consider nonlocal boundary value problems in this book.

(or, if you like, noncommutative-geometric) aspects of the theory. Namely, for general nonlocal operators we obtain a cohomological index formula.

Let us explain the main results of the book in more detail. We consider differential operators whose coefficients contain shift operators corresponding to the action of a discrete group Γ on a smooth closed manifold. Under the assumption that the group is of polynomial growth and the action is embedded in an action of a compact Lie group of diffeomorphisms, we show that to a nonlocal elliptic operator one can assign a Fredholm operator in Hilbert modules over the group C^*-algebra $C^*(\Gamma)$. The latter operator has a well-defined index that is an element of the K-group of this algebra:

$$\operatorname{ind}_{C^*(\Gamma)} D \in K_0(C^*(\Gamma)). \tag{0.1}$$

The Fredholm index of the original operator can be obtained as the image of the index (0.1) under the mapping induced by the trivial representation $C^*(\Gamma) \to \mathbb{C}$.

We present formulas that allow us to calculate the index (0.1) in terms of the symbol of the operator. First, we derive an index formula in K-theory. To this end, we establish the stable homotopy classification of nonlocal elliptic operators, construct the direct image mapping for nonlocal elliptic symbols under an embedding of manifolds, and generalize the Bott periodicity theorem to the case of infinite discrete groups. Then, in Chaps. 9 and 10, we obtain cohomological formulas for the coupling of the index (0.1) with cyclic cocycles over a smooth local subalgebra in $C^*(\Gamma)$. The simplest of these formulas (Chap. 9) leads to formulas for the Fredholm index. Cohomological formulas are given in terms of the Chern character determined here for the symbol and the Todd class modified in the spirit of [15].

Finally, we construct formulas for the Λ-index for elliptic nonlocal operators acting in Hilbert modules over a C^*-algebra Λ in a sufficiently wide class of algebras. The result is obtained by combining the methods developed in the present book and the classical approach in [57] (where index formulas were obtained for local elliptic operators over C^*-algebras).