1

Introduction

1.1 Fluids, Structures, and Acoustics

Sound and structural vibrations are an inevitable product of unsteady flow and of its interactions with structures immersed in the flow. These interactions, and the manner in which they affect the production and dissipation of sound, are the subject of this book. This first chapter is a review of several fundamental topics in fluid mechanics, acoustics, and the theory of elasticity.

1.1.1 Notation

The mathematical notation is developed in this chapter. For future reference, however, it is convenient to note here the following conventions.

Latin suffixes i, j, and so forth denote vector or tensor components in the direction of the $i-$, $j-$, ... axis; $\mathbf{i}_j$ is a unit vector in the j-direction.

Green's Functions

The frequency-domain Green's function $G(\mathbf{x}, \mathbf{y}; \omega)$ and the time-domain Green's function $G(\mathbf{x}, \mathbf{y}, t - \tau)$ are defined such that

$$G(\mathbf{x}, \mathbf{y}, t - \tau) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} G(\mathbf{x}, \mathbf{y}; \omega) e^{-i\omega(t-\tau)} \, d\omega.$$

However, the Fourier time transform $f(\mathbf{x}, \omega)$ of a function $f(\mathbf{x}, t)$ of position $\mathbf{x}$ and time t are related by

$$f(\mathbf{x}, t) = \int_{-\infty}^{\infty} f(\mathbf{x}, \omega) e^{-i\omega t} \, d\omega.$$

1

For problems in the frequency domain, the notation $f(\mathbf{x}, \omega)$ implies a time dependence $e^{-i\omega t}$.

1.2 Equations of Motion of a Fluid

The analysis of fluid–structure interactions involves a consideration of equations governing the motion of fluids and solids and the boundary conditions that relate the solutions of these equations at the fluid–structure interface. A complete description of the state of a moving fluid is given at each time t and position $\mathbf{x} = (x_1, x_2, x_3)$ when the velocity $\mathbf{v}$ and any two thermodynamic variables are specified. These quantities generally vary with $\mathbf{x}$ and t, and five scalar equations are required to determine the motion in terms of the boundary conditions and a prescribed initial state. The equations are statements of the conservation of mass, momentum, and energy.

1.2.1 Equation of Continuity

Conservation of mass requires the rate of increase of the fluid mass within a fixed region of space V to be equal to the net influx due to convection across the boundaries of V. This is expressed in terms of $\mathbf{v}$ and the fluid density ρ by the equation of continuity

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{v}) = 0, \tag{1.2.1}$$

which may also be written

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \operatorname{div}\mathbf{v} = 0, \tag{1.2.2}$$

where $D/Dt = \partial/\partial t + \mathbf{v}\cdot\nabla \equiv \partial/\partial t + v_j\partial/\partial x_j$ is the material derivative, which measures rate of change following the motion of a fluid particle, and a repeated suffix j implies summation over $j = 1, 2, 3$. By writing Equation (1.2.2) in the form $\operatorname{div}\mathbf{v} = \rho D(1/\rho)/Dt$, we see that $\operatorname{div}\mathbf{v}$ is just the rate of change of fluid volume per unit volume following the motion of the fluid. This vanishes when a fluid is incompressible, when the equation of continuity becomes

$$\operatorname{div}\mathbf{v} = 0. \tag{1.2.3}$$

1.2.2 Momentum Equation

The momentum equation for a viscous fluid is called the *Navier–Stokes equation* and expresses the rate of change of momentum of a fluid particle in terms of

the pressure p, the *viscous stress tensor* σ_{ij}, and body forces (such as gravity) **F** per unit volume:

$$\rho Dv_i/Dt = -\partial p/\partial x_i + \partial \sigma_{ij}/\partial x_j + F_i$$
$$\equiv -\partial p_{ij}/\partial x_j + F_i, \qquad (1.2.4)$$

where

$$p_{ij} = p\delta_{ij} - \sigma_{ij} \qquad (1.2.5)$$

is the *compressive stress tensor*. Viscous stresses are caused by frictional forces between neighboring fluid elements moving at different velocities. For an isotropic, *Newtonian* fluid (such as dry air and water)

$$\sigma_{ij} = 2\eta_{ij} + \left(\eta' - \frac{2}{3}\eta\right)\delta_{ij}e_{kk}, \qquad (1.2.6)$$

where

$$e_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i) \qquad (1.2.7)$$

is the rate of strain tensor and η and η' are, respectively, the *shear* and *bulk* coefficients of viscosity. These coefficients are functions of the pressure and temperature, T, and in general vary throughout the flow, although in many cases the variations are sufficiently small that they can be neglected. Equation (1.2.6) is a phenomenological relation that is applicable provided the time scales of unsteady fluid motions are much larger than the relaxation times of the internal degrees of freedom of the fluid molecules [1].

The bulk coefficient of viscosity η' vanishes for monatomic gases, and in this case (and for most liquids, such as water), the fluid is said to be Stokesian, with

$$\sigma_{ij} = 2\eta\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right). \qquad (1.2.8)$$

When the variation of η is neglected, the momentum equation for a Stokesian fluid is

$$D\mathbf{v}/Dt = (-1/\rho)\nabla p + \nu\left[\nabla^2\mathbf{v} + \frac{1}{3}\nabla(\text{div }\mathbf{v})\right] + F/\rho, \qquad (1.2.9)$$

where

$$\nu = \eta/\rho \qquad (1.2.10)$$

is the kinematic viscosity.

Table 1.2.1. *Air and water values*

Element	ρ kg/m^3	η kg/ms	ν m^2/s
Air	1.23	$1.764\ 10^{-5}$	$1.433\ 10^{-5}$
Water	1,000	$1.284\ 10^{-3}$	$1.284\ 10^{-6}$

Values of ρ, η, and ν for air and water at 10°C and one-atmosphere pressure are given in the Table 1.2.1.

When viscous stresses are ignored, equations (1.2.1) and (1.2.4) describe the motion of an ideal fluid and are called *Euler's equations*. These are of lower order in the spatial derivatives of velocity than the full Navier–Stokes equations, and this implies that it will not normally be possible to satisfy all of the conditions that a real fluid must satisfy at the physical boundaries of the flow (see Section 1.5).

1.2.3 Thermodynamic Relations

Apart from the pressure, density, and temperature, we shall also need to introduce the specific internal energy e, specific enthalpy $w \equiv e + p/\rho$, and specific entropy s, where a *specific* quantity denotes the value per unit mass of the fluid. For a single-component system in a state of equilibrium, any thermodynamic variable can be expressed in terms of any two given variables by making use of the equation of state and certain thermodynamic identities.

When a quantity of heat dQ is supplied to unit mass of fluid, as a result of which its volume changes by $d(1/\rho)$, the internal energy e increases by $de = dQ - pd(1/\rho)$, the second term on the right being the work done on the fluid element by the pressure because of the volume change. If the change occurs sufficiently slowly, the fluid element is always in thermodynamic equilibrium, and the heat input and the change in specific entropy are related by $dQ = T\,ds$, which leads to the fundamental identity

$$de = T\,ds - pd(1/\rho). \tag{1.2.11}$$

This "quasi-static" relation does not necessarily imply that the motion is thermodynamically *reversible* because the fluid element, although internally in equilibrium, need not be in equilibrium with its surroundings. When it is written in terms of the enthalpy w, we have

$$dw = T\,ds + dp/\rho. \tag{1.2.12}$$

The specific heats at constant volume c_v and constant pressure c_p are equal to the amounts of heat absorbed per unit mass of fluid per unit rise in temperature when the heating occurs at constant volume and constant pressure, respectively. Equations (1.2.11) and (1.2.12) supply

$$c_v = T(\partial s/\partial T)_\rho = (\partial e/\partial T)_\rho, \qquad c_p = T(\partial s/\partial T)_p = (\partial w/\partial T)_p.$$

(1.2.13)

The specific heats are related by

$$c_p - c_v = T\beta^2/\rho \mathrm{K}_T,$$

(1.2.14)

where β and K_T are, respectively, the coefficients of expansion and isothermal compressibility of the fluid, defined by

$$\beta = (-1/\rho)(\partial \rho/\partial T)_p, \quad \mathrm{K}_T = (1/\rho)(\partial \rho/\partial p)_T.$$

(1.2.15)

The equation of state of an ideal gas is usually expressed in the form

$$p = \rho \mathrm{R} T,$$

(1.2.16)

where the gas constant $\mathrm{R} = 8{,}314/$(mean molecular weight) in SI units ($\mathrm{R} \approx 287$ Joules/kg °K for dry air at normal temperatures and pressures). For a nonreacting mixture of gases in equilibrium at temperature T, Dalton's law of partial pressures states that the total pressure $p = \sum_j p_j$, where $p_j = \rho_j \mathrm{R}_j T$ is the pressure of the jth constituent gas of density ρ_j and gas constant R_j.

The specific heats of an ideal gas composed of rigid molecules each with N rotational degrees of freedom, satisfy $c_p - c_v = \mathrm{R}$, and $\gamma = c_p/c_v = (N+5)/(N+3)$. When c_p, c_v are *constant*, we also have

$$e = c_v T = p/(\gamma - 1)\rho, \quad w = c_p T = \gamma p/(\gamma - 1)\rho,$$
$$s = c_v \ln(p/\rho^\gamma).$$

(1.2.17)

Equation (1.2.11) implies that the excess thermal energy required to raise the temperature of a fluid particle at constant pressure, as opposed to constant volume, is expended as work $\int p\,d(1/\rho)$ performed on its environment during expansion and that the relevant thermodynamic function for describing such processes is the enthalpy w.

1.2.4 Thermodynamic Variables

When velocity gradients are present the fluid cannot be in strict thermodynamic equilibrium, and thermodynamic variables require special interpretation. For a

fluid in nonuniform motion it is customary to define the density ρ and internal energy e in the usual way, that is, such that ρ and ρe are the mass and internal energy per unit volume. The pressure and all other thermodynamic variables are then defined by means of the same functions of ρ and e as would be the case in thermal equilibrium. The "thermodynamic pressure" $p = p(\rho, e)$ so defined is no longer the sole source of normal stress on a surface drawn in the fluid. The mean normal stress is obtained by averaging $p_{ij} n_i n_j$ (see (1.2.5)) over all possible directions of the unit vector $\mathbf{n}$ defining the normal to a surface element. The average is obtained by evaluating $(1/4\pi)\oint n_i n_j \, dS$ over the surface of a sphere of unit radius. The integral is equal to $\frac{4}{3}\pi \delta_{ij}$, and the mean normal stress is therefore equal to $\frac{1}{3} p_{ii} = p - \eta' \, \mathrm{div}\, \mathbf{v}$, which differs from the thermodynamic pressure p if the bulk coefficient of viscosity η' is nonzero. This is the case for a fluid whose molecules possess rotational degrees of freedom, whose relaxation time (during which thermal equilibrium is reestablished after, say, an expansion of the fluid) is large relative to the equilibration time of the translational degrees of freedom. For example, in an expanding diatomic gas ($\mathrm{div}\, \mathbf{v} > 0$) the temperature must decrease, but the reduction in the rotational energy lags slightly behind that of the translational energy; the thermodynamic pressure $p = (\gamma - 1)\rho e$ accordingly exceeds the actual pressure $\frac{1}{3} p_{ii}$ by an amount equal to $\eta' \, \mathrm{div}\, \mathbf{v}$. The rate of adjustment of the rotational degrees of freedom is such that in practice η' is of the same order as η (for dry air $\eta' \sim 0.6\eta$).

It may be shown [2] that whereas the thermodynamic pressure differs from the mean normal stress by a term that is linear in the velocity gradient, the corresponding departure of the thermodynamic specific entropy s from the true entropy is proportional at least to the square of such gradients, and the difference is usually small in practice.

1.2.5 Crocco's Equation and Bernoulli's Equation

The momentum equation (1.2.4) can also be expressed in *Crocco's* form

$$\partial v_i / \partial t + \partial B / \partial x_i = -(\boldsymbol{\omega} \wedge \mathbf{v})_i + T \partial s / \partial x_i + (1/\rho)\partial \sigma_{ij}/\partial x_j + F_i/\rho, \tag{1.2.18}$$

where B is the *total enthalpy*,

$$B = w + \frac{1}{2} v^2, \tag{1.2.19}$$

and $\boldsymbol{\omega} = \mathrm{curl}\, \mathbf{v}$ is the *vorticity*. It is derived from (1.2.4) by making use of (1.2.12) and the identity $D\mathbf{v}/Dt \equiv \partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \partial v/\partial t + \nabla(\frac{1}{2}v^2) + \boldsymbol{\omega} \wedge \mathbf{v}$.

In an *ideal fluid* viscosity and thermal conduction are negligible. If, in addition, the motion is *homentropic* (s = constant throughout the flow), irrotational ($\boldsymbol{\omega} = \mathbf{0}$), and the body force $\mathbf{F}$ is conservative, that is, there is a scalar function of position Φ such that $\mathbf{F} = \rho\nabla\Phi$, Crocco's equation implies the existence of a velocity potential ϕ, such that

$$\mathbf{v} = \nabla\phi. \tag{1.2.20}$$

A first integral of the momentum equation then yields *Bernoulli's equation*, which may be expressed in either of the forms:

$$\partial\phi/\partial t + B - \Phi = f(t), \quad \partial\phi/\partial t + w + \frac{1}{2}v^2 - \Phi = f(t). \tag{1.2.21}$$

The velocity potential ϕ is undefined to within an arbitrary function of time so that the function $f(t)$ may be set equal to a constant or zero. For steady flow $\partial\phi/\partial t = 0$, $f(t)$ = constant, and Bernoulli's equation reduces to $w + \frac{1}{2}v^2 - \Phi$ = constant. It is clear from (1.2.21), and from Crocco's equation (1.2.18), that the potential Φ of the body force may be absorbed into the total enthalpy B.

A flow is said to be *isentropic* when the specific entropy of each fluid particle is constant, that is, when $Ds/Dt = 0$. For steady flow of an ideal fluid subject to conservative body forces, Crocco's equation, $\nabla B = -\boldsymbol{\omega}\wedge\mathbf{v} + T\nabla s$, implies that $DB/Dt \equiv (\mathbf{v}\cdot\nabla)B = -\mathbf{v}\cdot\boldsymbol{\omega}\wedge\mathbf{v} + (\mathbf{v}.\nabla)s = 0$ so that B is constant along a streamline. If the flow is homentropic, B is constant over the fixed fluid surfaces spanned by intersecting streamlines and vortex lines (because $\mathbf{v}\cdot\boldsymbol{\omega}\wedge\mathbf{v} \equiv \mathbf{0}$).

1.2.6 Reynolds Number

An equation for the rate of change of the momentum density $\rho\mathbf{v}$ is obtained by adding the continuity equation (1.2.1) multiplied by $\mathbf{v}$ to the momentum equation (1.2.4) and writing the result in the form

$$\frac{\partial(\rho v_i)}{\partial t} = -\frac{\partial}{\partial x_j}(p\delta_{ij} + \rho v_i v_j - \sigma_{ij}) + F_i. \tag{1.2.22}$$

The quantity $\rho v_i v_j$ is called the *Reynolds stress* and accounts for changes in the momentum ρv_i by convection at velocity v_j. The order of magnitude of the Reynolds stress relative to the viscous stress σ_{ij} at any given point in the flow

is determined by the value of the *Reynolds number*

$$\mathrm{Re} = u\ell/\nu, \qquad (1.2.23)$$

in which u and ℓ are the respective characteristic values of the velocity and length scale of variation of the flow in the neighborhood of the point in question.

In regions where $\mathrm{Re} \gg 1$ viscosity plays a minor role in the local development of the flow. This is the case, for example, in high-speed jets, where $\ell \sim$ jet diameter and $u \approx$ mean jet velocity. In turbulent flow over a wall, the motion in a thin layer of fluid adjacent to the wall (the "viscous sublayer," where u and ℓ are small and $\mathrm{Re} \to 0$) is dominated by the viscous diffusion of momentum across the layer. Further away from the wall, in the region of fully developed turbulence, $\ell \approx$ boundary layer thickness and $u \approx$ velocity of the main stream, and Re typically exceeds 10^5 or more. The viscous (i.e., molecular) transport of momentum is then negligible compared with that due to turbulence convection.

1.2.7 Vorticity

The motion of a spherical particle of fluid may be decomposed into a uniform translation at the velocity $\mathbf{v}$, say, of its center, a straining motion determined by the rate of strain tensor e_{ij} and a rotation at angular velocity $\frac{1}{2}\boldsymbol{\omega} = \frac{1}{2}\,\mathrm{curl}\,\mathbf{v}$ [3]. The vorticity is therefore a measure of the angular momentum of a fluid particle. Conservation of angular momentum suggests that vorticity may be regarded as a quantity characterizing the *intrinsic* kinetic energy of a flow, inasmuch as it determines the motion that would persist in an incompressible fluid at rest at infinity if all the boundaries were brought to rest. A vortex line in the fluid is tangential to the vorticity vector at all points along its length. Because $\mathrm{div}\,\boldsymbol{\omega} = \mathrm{div}(\mathrm{curl}\,\mathbf{v}) \equiv 0$, a vortex line cannot begin or end within the fluid. The "no-slip" condition requires the fluid velocity at the boundary to be the same as that of the boundary. A vortex line must therefore form a closed loop or end on a rotating surface bounding the flow [3].

Vorticity is transported by convection and molecular diffusion, and for this reason it is often permissible to assume that an initially confined region of vorticity remains localized. The vorticity equation is obtained from the curl of Crocco's equation (1.2.18) and by using the vector identity $\mathrm{curl}(\boldsymbol{\omega} \wedge \mathbf{v}) = (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\,\mathrm{div}\,\mathbf{v} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$ and the continuity equation (1.2.1):

$$\rho D(\boldsymbol{\omega}/\rho)/Dt = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} + \nabla T \wedge \nabla s$$
$$+ \mathrm{curl}((1/\rho)\partial\rho_{ij}/\partial x_j) + \mathrm{curl}(\mathbf{F}/\rho). \qquad (1.2.24)$$

The terms on the right-hand side account for changes in ω/ρ following the motion of a fluid particle. They are interpreted as follows:

1. $(\omega \cdot \nabla)\mathbf{v}$: rotation and stretching of vortex lines by the velocity field. The magnitude of ω/ρ increases in direct proportion to the stretching of vortex lines by the flow. When the remaining terms on the right of (1.2.24) are absent, the motion satisfies *Kelvin's circulation theorem* [2–4]. This asserts that, in an inviscid, homentropic fluid subject only to conservative body forces, the *circulation C*, equal to the line integral $\oint \mathbf{v} \cdot d\mathbf{x}$ taken around a closed material contour in the fluid, does not change with time ($DC/Dt = 0$). It follows that vortex lines move with the fluid, and in particular that the vorticity of a material element of the fluid remains zero if it is initially zero. In other words, vorticity cannot be created in a body of fluid if it is initially irrotational and the motion is inviscid and homentropic.

2. $\nabla T \wedge \nabla s$: production of vorticity when temperature and entropy gradients are not parallel, or equivalently (because, from (1.2.12), $\nabla T \wedge \nabla s = -\nabla(1/\rho) \wedge \nabla p$), when density and pressure gradients are not parallel.

3. $\mathrm{curl}((1/\rho)\partial\sigma_{ij}/\partial x_j)$: molecular diffusion of vorticity. This term is important only in regions of high shear. When variations in the kinematic viscosity ν are negligible, we recover the more familiar expression

$$\mathrm{curl}((1/\rho)\partial\sigma_{ij}/\partial x_j) = \nu\nabla^2\omega. \qquad (1.2.25)$$

If local effects of compressibility are unimportant the vorticity equation (1.2.24) reduces to

$$D\omega/Dt = (\omega \cdot \nabla)\mathbf{v} + \nu\nabla^2\omega + \mathrm{curl}(\mathbf{F}/\rho). \qquad (1.2.26)$$

4. $\mathrm{curl}(\mathbf{F}/\rho)$: production of vorticity by nonconservative body forces.

Vorticity is generated at solid boundaries, and viscosity is responsible for its diffusion into the body of the fluid, where it can subsequently be convected by the flow. When body forces and density variations are neglected, integration of (1.2.26) over a fluid region V contained within a control surface Σ moving with the fluid, and bounded internally by a moving solid surface S, yields

$$\frac{\partial}{\partial t} \int \omega \, d^3\mathbf{x} = - \oint_{S+\Sigma} ((\mathbf{n} \cdot \omega)\mathbf{v} + \nu\mathbf{n} \wedge \nabla^2\mathbf{v}) \, dS, \qquad (1.2.27)$$

where in the surface integrals over S and Σ, $\mathbf{n}$ is a unit normal directed *into* V. Vorticity changes in V are therefore caused by the rotation and stretching of vortex tubes at the boundaries and diffusion across the boundaries by viscosity.

The momentum equation (1.2.9) and the no-slip condition imply that $(\mathbf{n} \cdot \boldsymbol{\omega})\mathbf{v} = 2(\boldsymbol{\Omega} \cdot \mathbf{n})\mathbf{U}$ and $\nu\mathbf{n} \wedge \nabla^2\mathbf{v} = (1/\rho)\mathbf{n} \wedge \nabla p + D(\mathbf{n} \wedge \mathbf{U})/Dt - (\boldsymbol{\Omega} \wedge n) \wedge \mathbf{U}$ on S, where $\mathbf{U}$ and $\boldsymbol{\Omega}$ are, respectively, the velocity and angular velocity of the surface element dS. Equation (1.2.27) is therefore equivalent to

$$\frac{\partial}{\partial t} \int \boldsymbol{\omega} \, d^3\mathbf{x} = - \oint_S (2(\boldsymbol{\Omega}\cdot\mathbf{n})\mathbf{U} - (\boldsymbol{\Omega} \wedge \mathbf{n}) \wedge \mathbf{U} + (1/\rho)\mathbf{n} \wedge \nabla p$$
$$+ \, D(\mathbf{n} \wedge \mathbf{U})/Dt) \, d\mathrm{S} - \oint_{\Sigma} ((\mathbf{n} \cdot \boldsymbol{\omega})\mathbf{v} + \nu\mathbf{n} \wedge \nabla^2\mathbf{v}) \, d\mathrm{S}.$$

(1.2.28)

This shows that the mechanism of vorticity production on S is independent of the value of the viscosity ν. Viscosity merely serves to diffuse the vorticity into the fluid from the surface [3, 5, 6]. The terms in the first integral on the right-hand side may be regarded as sources of vorticity on S. For a rigid solid in translational motion, vorticity production requires that either the tangential surface acceleration, $D(\mathbf{n} \wedge \mathbf{U})/Dt$, or surface pressure gradient, $\mathbf{n} \wedge \nabla p$, should be nonzero.

1.2.8 The Energy Equation

In homentropic flow, the equation $s = $ constant provides a relation between the thermodynamic variables that permits the motion to be determined from the equations of continuity and momentum together with the equation of state. In more general situations, where s is variable, it is necessary to introduce an energy equation to account for coupling between macroscopic motions and the internal energy of the fluid. This equation governs the conservation of the total energy of the system (i.e., the dissipation of mechanical energy) and is derived by analyzing the energy balance for a moving particle of fluid.

Consider a small element of fluid of volume V, bounded by a surface S. The kinetic and internal energies per unit volume are equal respectively to $1/2\rho v^2$ and ρe, and the energy of the fluid in V is $E = \rho \mathrm{V}(\frac{1}{2}v^2 + e)$. Changes in E are produced by the work done by the compressive stress p_{ij} on the boundary S, by the flux of heat energy through S by molecular conduction, and by the work performed by the body force $\mathbf{F}$ within V. If thermal conduction is expressed in terms of the *heat flux vector* $\mathbf{Q}$, energy conservation requires

$$\frac{DE}{Dt} = \oint_S (p_{ij}v_i + Q_j) \, d\mathrm{S}_j + \mathrm{V}v_j F_j,$$

(1.2.29)

where the surface element dS is directed *into* V. According to the divergence