Preface

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts (geodesics, connections, curvature, ...) and objectives, in particular to understand certain classes of (compact) Riemannian manifolds defined by curvature conditions (constant or positive or negative curvature, ...). By way of contrast, geometric analysis is a perhaps somewhat less systematic collection of techniques, for solving extremal problems naturally arising in geometry and for investigating and characterizing their solutions. It turns out that the two fields complement each other very well; geometric analysis offers tools for solving difficult problems in geometry, and Riemannian geometry stimulates progress in geometric analysis by setting ambitious goals.

It is the aim of this book to be a systematic and comprehensive introduction to Riemannian geometry and a representative introduction to the methods of geometric analysis. It attempts a synthesis of geometric and analytic methods in the study of Riemannian manifolds.

The present work is the fifth edition of my textbook on Riemannian geometry and geometric analysis. It has developed on the basis of several graduate courses I taught at the Ruhr-University Bochum and the University of Leipzig. The main new features of the present edition are the systematic inclusion of flow equations and a mathematical treatment of the nonlinear sigma model of quantum field theory. These new topics also led to a systematic reorganization of the other material. Naturally, I have also included several smaller additions and minor corrections (for which I am grateful to several readers).

Let me now briefly describe the contents:

In the first chapter, we introduce the basic geometric concepts, like differentiable manifolds, tangent spaces, vector bundles, vector fields and one-parameter groups of diffeomorphisms, Lie algebras and groups and in particular Riemannian metrics. We also treat the existence of geodesics with two different methods, both of which are quite important in geometric analysis in general. Thus, the reader has the opportunity to understand the basic ideas of those methods in an elementary context before moving on to more difficult versions in subsequent chapters. The first method is based on the local existence and uniqueness of geodesics and will be applied again in Chapter 8 for two-dimensional harmonic maps. The second method is the heat flow method that gained prominence through Perelman's solution of the Poincaré conjecture by the Ricci flow method.

The second chapter introduces de Rham cohomology groups and the essential tools from elliptic PDE for treating these groups. We prove the existence of harmonic forms representing cohomology classes both by a variational method, thereby introducing another of the basic schemes of geometric analysis, and by the heat flow method. The linear setting of cohomology classes allows us to understand some key ideas without the technical difficulties of nonlinear problems.

The third chapter treats the general theory of connections and curvature.

In the fourth chapter, we introduce Jacobi fields, prove the Rauch comparison theorems for Jacobi fields and apply these results to geodesics. We also develop the global geometry of spaces of nonpositive curvature.

These first four chapters treat the more elementary and basic aspects of the subject. Their results will be used in the remaining, more advanced chapters.

The fifth chapter treats Kähler manifolds symmetric spaces as important examples of Riemannian manifolds in detail.

The sixth chapter is devoted to Morse theory and Floer homology.

In the seventh chapter, we treat harmonic maps between Riemannian manifolds. We prove several existence theorems and apply them to Riemannian geometry. The treatment uses an abstract approach based on convexity that should bring out the fundamental structures. We also display a representative sample of techniques from geometric analysis.

In the eighth chapter, we treat harmonic maps from Riemann surfaces. We encounter here the phenomenon of conformal invariance which makes this two-dimensional case distinctively different from the higher dimensional one.

The ninth chapter treats variational problems from quantum field theory, in particular the Ginzburg-Landau, Seiberg-Witten equations, and a mathematical version of the nonlinear supersymmetric sigma model. In mathematical terms, the two-dimensional harmonic map problem is coupled with a Dirac field. The background material on spin geometry and Dirac operators is already developed in earlier chapters. The connections between geometry and physics will be further explored in a forthcoming monograph [144].

A guiding principle for this textbook was that the material in the main body should be self contained. The essential exception is that we use material about Sobolev spaces and linear elliptic an parabolic PDEs without giving proofs. This material is collected in Appendix A. Appendix B collects some elementary topological results about fundamental groups and covering spaces.

Also, in certain places in Chapter 6, we do not present all technical details, but rather explain some points in a more informal manner, in order to keep the size of that chapter within reasonable limits and not to loose the patience of the readers.

We employ both coordinate free intrinsic notations and tensor notations depending on local coordinates. We usually develop a concept in both notations while we sometimes alternate in the proofs. Besides not being a methodological purist, reasons for often prefering the tensor calculus to the more elegant and concise intrinsic one are the following. For the analytic aspects, one often has to employ results about (elliptic) partial differential equations (PDEs), and in order to check that the relevant assumptions like ellipticity hold and in order to make contact with the notations usually employed in PDE theory, one has to write down the differential equation in local coordinates. Also, manifold and important connections have been established between theoretical physics and our subject. In the physical literature, usually the tensor notation is employed, and therefore, familiarity with that notation is necessary for exploring those connections that have been found to be stimulating for the development of mathematics, or promise to be so in the future.

As appendices to most of the paragraphs, we have written sections with the title "Perspectives". The aim of those sections is to place the material in a broader context and explain further results and directions without detailed proofs. The material of these Perspectives will not be used in the main body of the text. Similarly, after Chapter 4, we have inserted a section entitled "A short survey on curvature and topology" that presents an account of many global results of Riemannian geometry not covered in the main text. – At the end of each chapter, some exercises for the reader are given. We assume of the reader sufficient perspicacity to understand our system of numbering and cross-references without further explanations.

The development of the mathematical subject of Geometric Analysis, namely the investigation of analytical questions arising from a geometric context and in turn the application of analytical techniques to geometric problems, is to a large extent due to the work and the influence of Shing-Tung Yau. This book, like its previous editions, is dedicated to him.

I am also grateful to Minjie Chen for dedicated help with the Tex file.

Jürgen Jost

Chapter 2

De Rham Cohomology and Harmonic Differential Forms

2.1 The Laplace Operator

We need some preparations from linear algebra. Let V be a real vector space with a scalar product $\langle \cdot, \cdot \rangle$, and let $\Lambda^p V$ be the p-fold exterior product of V. We then obtain a scalar product on $\Lambda^p V$ by

$$\langle v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p \rangle = \det(\langle v_i, w_j \rangle) \tag{2.1.1}$$

and bilinear extension to $\Lambda^p(V)$. If $e_1, \ldots, e_d$ is an orthonormal basis of V,

$$e_{i_1} \wedge \ldots \wedge e_{i_p} \quad \text{with } 1 \leq i_1 < i_2 < \ldots < i_p \leq d \tag{2.1.2}$$

constitute an orthonormal basis of $\Lambda^p V$.

An *orientation* on V is obtained by distinguishing a basis of V as positive. Any other basis that is obtained from this basis by a base change with positive determinant then is likewise called positive, and the remaining bases are called negative.

Let now V carry an orientation. We define the linear star operator

$$* : \Lambda^p(V) \to \Lambda^{d-p}(V) \quad (0 \leq p \leq d)$$

by

$$*(e_{i_1} \wedge \ldots \wedge e_{i_p}) = e_{j_1} \wedge \ldots \wedge e_{j_{d-p}}, \tag{2.1.3}$$

where $j_1, \ldots, j_{d-p}$ is selected such that $e_{i_1}, \ldots, e_{i_p}, e_{j_1}, \ldots, e_{j_{d-p}}$ is a positive basis of V. Since the star operator is supposed to be linear, it is determined by its values on some basis (2.1.3).

In particular,

$$*(1) = e_1 \wedge \ldots \wedge e_d \tag{2.1.4}$$
$$*(e_1 \wedge \ldots \wedge e_d) = 1, \tag{2.1.5}$$

if $e_1, \ldots, e_d$ is a positive basis.

From the rules of multilinear algebra, it easily follows that if A is a $d \times d$-matrix, and if $f_1, \ldots, f_p \in V$, then

$$*(Af_1 \wedge \ldots \wedge Af_p) = (\det A) * (f_1 \wedge \ldots \wedge f_p).$$

In particular, this implies that the star operator does not depend on the choice of positive orthonormal basis in V, as any two such bases are related by a linear transformation with determinant 1.

For a negative basis instead of a positive one, one gets a minus sign on the right hand sides of (2.1.3), (2.1.4), (2.1.5).

Lemma 2.1.1. $** = (-1)^{p(d-p)} : \Lambda^p(V) \to \Lambda^p(V)$.

Proof. $**$ maps $\Lambda^p(V)$ onto itself. Suppose

$$*(e_{i_1} \wedge \ldots \wedge e_{i_p}) = e_{j_1} \wedge \ldots \wedge e_{j_{d-p}} \quad \text{(cf. (2.1.3))}.$$

Then

$$** (e_{i_1} \wedge \ldots \wedge e_{i_p}) = \pm e_{i_1} \wedge \ldots \wedge e_{i_p},$$

depending on whether $e_{j_1}, \ldots, e_{j_{d-p}}, e_{i_1}, \ldots, e_{i_p}$ is a positive or negative basis of V. Now

$$\begin{aligned} &e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge e_{j_1} \wedge \ldots \wedge e_{j_{d-p}} \\ &\quad = (-1)^{p(d-p)} e_{j_1} \wedge \ldots \wedge e_{j_{d-p}} \wedge e_{i_1} \wedge \ldots \wedge e_{i_p}, \end{aligned}$$

and $(-1)^{p(d-p)}$ thus is the determinant of the base change from $e_{i_1}, \ldots, e_{j_{d-p}}$ to $e_{j_1}, \ldots, e_{i_p}$. □

Lemma 2.1.2. *For $v, w \in \Lambda^p(V)$*

$$\langle v, w \rangle = *(w \wedge *v) = *(v \wedge *w). \tag{2.1.6}$$

Proof. It suffices to show (2.1.6) for elements of the basis (2.1.2). For any two different such basis vectors, $w \wedge *v = 0$, whereas

$$\begin{aligned} *(e_{i_1} \wedge \ldots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \ldots \wedge e_{i_p})) &= *(e_1 \wedge \ldots \wedge e_d), \quad \text{where } e_1, \ldots, e_d \\ &\qquad \text{is an orthonormal basis (2.1.3)} \\ &= 1 \quad \text{by (2.1.5)}, \end{aligned}$$

and the claim follows. □

Remark. We may consider $\langle \cdot, \cdot \rangle$ as a scalar product on

$$\Lambda(V) := \bigoplus_{p=0}^{d} \Lambda^p(V)$$

with $\Lambda^p(V)$ and $\Lambda^q(V)$ being orthogonal for $p \neq q$.

Lemma 2.1.3. *Let $v_1, \ldots, v_d$ be an arbitrary positive basis of V. Then*

$$*(1) = \frac{1}{\sqrt{\det(\langle v_i, v_j \rangle)}} v_1 \wedge \ldots \wedge v_d. \tag{2.1.7}$$

Proof. Let $e_1, \ldots, e_d$ be a positive orthonormal basis as before. Then

$$v_1 \wedge \ldots \wedge v_d = (\det(\langle v_i, v_j \rangle))^{\frac{1}{2}} e_1 \wedge \ldots \wedge e_d,$$

and the claim follows from (2.1.4). □

Let now M be an oriented Riemannian manifold of dimension d. Since M is oriented, we may select an orientation on all tangent spaces $T_x M$, hence also on all cotangent spaces $T_x^* M$ in a consistent manner. We simply choose the Euclidean orthonormal basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}$ of $\mathbb{R}^d$ as being positive. Since all chart transitions of an oriented manifold have positive functional determinant, calling the basis $d\varphi^{-1}(\frac{\partial}{\partial x^1}), \ldots, d\varphi^{-1}(\frac{\partial}{\partial x^d})$ of $T_x M$ positive, will not depend on the choice of the chart.

Since M carries a Riemannian structure, we have a scalar product on each $T_x^* M$. We thus obtain a star operator

$$* : \Lambda^p(T_x^* M) \to \Lambda^{d-p}(T_x^* M),$$

i.e. a base point preserving operator

$$* : \Omega^p(M) \to \Omega^{d-p}(M) \quad (\Omega^p(M) = \Gamma(\Lambda^p(M))).$$

We recall that the metric on $T_x^* M$ is given by $(g^{ij}(x)) = (g_{ij}(x))^{-1}$. Therefore, by Lemma 2.1.3 we have in local coordinates

$$*(1) = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^d. \tag{2.1.8}$$

This expression is called the *volume form.*
In particular

$$\mathrm{Vol}\,(M) := \int_M *(1) \tag{2.1.9}$$

(provided this is finite).

For $\alpha, \beta \in \Omega^p(M)$ with compact support, we define the L^2-product as

$$\begin{aligned}(\alpha, \beta) :&= \int_M \langle \alpha, \beta \rangle * (1) \\ &= \int_M \alpha \wedge * \beta \quad \text{by Lemma 2.1.2}.\end{aligned}$$

This product on $\Omega^p(M)$ is obviously bilinear and positive definite.
We shall also use the L^2-norm

$$\|\alpha\| := (\alpha, \alpha)^{1/2}. \tag{2.1.10}$$

(In 2.2 below, we shall also introduce another norm, the Sobolov norm $\| \cdot \|_{H^{1,2}}$.) So far, we have considered only smooth sections of vector bundles, in particular only smooth p-forms. For later purposes, we shall also need L^p- and Sobolev spaces of sections of vector bundles. For this aim, from now on, we deviate from Definition 1.8.3 and don't require sections to be smooth anymore. We let E be a vector bundle over M, $s : M \to E$ a section of E with compact support. We say that s is contained in the *Sobolev space* $H^{k,r}(E)$, if for any bundle atlas with the property that on compact sets all coordinate changes and all their derivatives are bounded (it is not difficult to obtain such an atlas, by making coordinate neighborhoods smaller if necessary), and for any bundle chart from such an atlas,

$$\varphi : E_{|U} \to U \times \mathbb{R}^n$$

we have that $\varphi \circ s_{|U}$ is contained in $H^{k,r}(U)$. We note the following consistency property: If $\varphi_1 : E_{|U_1} \to U_1 \times \mathbb{R}^n$, $\varphi_2 : E_{|U_2} \to U_2 \times \mathbb{R}^n$ are two such bundle charts, then $\varphi_1 \circ s_{|U_1 \cap U_2}$ is contained in $H^{k,r}(U_1 \cap U_2)$ if and only if $\varphi_2 \circ s_{|U_1 \cap U_2}$ is contained in this space. The reason is that the coordinate change $\varphi_2 \circ \varphi_1^{-1}$ is of class C^∞, and all derivatives are bounded on the support of s which was assumed to be compact.

We can extend our product $(\cdot, \cdot)$ to $L^2(\Omega^p(M))$. It remains bilinear, and also positive definite, because as usual, in the definition of L^2, functions that differ only on a set of measure zero are identified.

We now make the assumption that M is compact, in order not to always have to restrict our considerations to compactly supported forms.

Definition 2.1.1. d^* is the operator which is (formally) adjoint to d on $\bigoplus_{p=0}^{d} \Omega^p(M)$ w.r.t. $(\cdot, \cdot)$. This means that for $\alpha \in \Omega^{p-1}(M), \beta \in \Omega^p(M)$

$$(d\alpha, \beta) = (\alpha, d^* \beta); \tag{2.1.11}$$

d^* therefore maps $\Omega^p(M)$ to $\Omega^{p-1}(M)$.

Lemma 2.1.4. $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ *satisfies*

$$d^* = (-1)^{d(p+1)+1} * d * . \tag{2.1.12}$$

Proof. For $\alpha \in \Omega^{p-1}(M), \beta \in \Omega^p(M)$

$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta \\ &= d\alpha \wedge *\beta + (-1)^{p-1}(-1)^{(p-1)(d-p+1)}\alpha \wedge * * (d * \beta) \\ &\qquad \text{by Lemma 2.1.1 } (d * \beta \text{ is a } (d-p+1)\text{-form}) \\ &= d\alpha \wedge *\beta - (-1)^{d(p+1)+1}\alpha \wedge * * d * \beta \\ &= \pm * (\langle d\alpha, \beta \rangle - (-1)^{d(p+1)+1}\langle \alpha, *d * \beta \rangle). \end{aligned}$$

We integrate this formula. By Stokes' theorem, the integral of the left hand side vanishes, and the claim results. □

Definition 2.1.2. The *Laplace(-Beltrami) operator* on $\Omega^p(M)$ is

$$\Delta = dd^* + d^*d : \Omega^p(M) \to \Omega^p(M).$$

$\omega \in \Omega^p(M)$ is called *harmonic* if

$$\Delta\omega = 0\,.$$

Remark. Since two stars appear on the right hand side of (2.1.12), d^* and hence also Δ may also be defined by (2.1.12) on nonorientable Riemannian manifolds. We just define it locally, hence globally up to a choice of sign which then cancels in (2.1.12). Similarly, the L^2-product can be defined on nonorientable Riemannian manifolds, because the ambiguity of sign of the $*$ involved cancels with the one coming from the integration.

More precisely, one should write

$$\begin{aligned} d^p &: \Omega^p(M) \to \Omega^{p+1}(M) \\ d^* &: \Omega^p(M) \to \Omega^{p-1}(M). \end{aligned}$$

Then

$$\Delta^p = d^{p-1}d^* + d^*d^p : \Omega^p(M) \to \Omega^p(M).$$

Nevertheless, we shall usually omit the index p.

Corollary 2.1.1. Δ *is (formally) selfadjoint, i.e.*

$$(\Delta\alpha, \beta) = (\alpha, \Delta\beta) \quad \textit{for } \alpha, \beta \in \Omega^p(M).$$

Proof. Directly from the definition of Δ. □

Lemma 2.1.5. $\Delta\alpha = 0 \iff d\alpha = 0$ *and* $d^*\alpha = 0$.

Proof.
" $\Leftarrow$ ": obvious.
" $\Rightarrow$ ": $(\Delta\alpha, \alpha) = (dd^*\alpha, \alpha) + (d^*d\alpha, \alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha)$.
Since both terms on the right hand side are nonnegative and vanish only if $d\alpha = 0 = d^*\alpha$, $\Delta\alpha = 0$ implies $d\alpha = 0 = d^*\alpha$. □

Corollary 2.1.2. *On a compact Riemannian manifold, every harmonic function is constant.* □

Lemma 2.1.6. $*\Delta = \Delta *$.

Proof. Direct computation. □

We want to compare the Laplace operator as defined here with the standard one on $\mathbb{R}^d$. For this purpose, let $f : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. We have

$$df = \frac{\partial f}{\partial x^i} dx^i$$

and for $\varphi = \varphi_i dx^i$ with compact support, and $*\varphi = \sigma_{i=1}^d (-1)^{i-1} \varphi_i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^d$

$$\begin{aligned} (df, \varphi) &= \int_{\mathbb{R}^d} \frac{\partial f}{\partial x^i} \varphi_i dx^1 \wedge \ldots \wedge dx^d \\ &= -\int_{\mathbb{R}^d} f \frac{\partial \varphi^i}{\partial x^i} dx^1 \wedge \ldots \wedge dx^d, \text{ since } \varphi \text{ is compactly supported.} \end{aligned}$$

It follows that $d^*\varphi = -\frac{\partial \varphi^i}{\partial x^i} = -\operatorname{div} \varphi$, and

$$\Delta f = d^* df = -\sum_{i=1}^d \frac{\partial^2 f}{(\partial x^i)^2} = -\operatorname{div}(\operatorname{grad} f).$$

This Laplace operator therefore differs from the usual one on $\mathbb{R}^d$ by a minus sign. This is regrettable, but cannot be changed any more since the notation has been established too thoroughly. With our definition above, Δ is a positive operator.

More generally, for a differentiable function, the Laplace-Beltrami operator is $f : M \to \mathbb{R}$

$$\Delta f = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \Big(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \Big), \tag{2.1.13}$$

with $g := \det(g_{ij})$. This is seen as follows:

Since for functions, i.e. 0-forms, we have $d^* = 0$, we get for $\varphi : M \to \mathbb{R}$

(differentiable with compact support)

$$\begin{aligned}\int \Delta f \cdot \varphi \sqrt{g} dx^1 \wedge \ldots \wedge dx^d &= (\Delta f, \varphi) = (df, d\varphi)\\ &= \int \langle df, d\varphi \rangle * (1)\\ &= \int g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \sqrt{g} dx^1 \ldots dx^d\\ &= - \int \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^i} \right) \varphi \sqrt{g} dx^1 \ldots dx^d,\end{aligned}$$

and since this holds for all $\varphi \in C_0^\infty(M, \mathbb{R})$, (2.1.13) follows.

For a function f, we may define its gradient as

$$\nabla f := \operatorname{grad} f := g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{2.1.14}$$

We thus have for any vector field X

$$\langle \operatorname{grad} f, X \rangle = X(f) = df(X). \tag{2.1.15}$$

The divergence of a vector field $Z = Z^i \frac{\partial}{\partial x^i}$ is defined as

$$\operatorname{div} Z := \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} Z^j) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \Big(\sqrt{g} g^{ij} \Big\langle Z, \frac{\partial}{\partial x^i} \Big\rangle \Big). \tag{2.1.16}$$

(2.1.13) then becomes

$$\Delta f = -\operatorname{div} \operatorname{grad} f. \tag{2.1.17}$$

In particular, if M is compact, and $f : M \to \mathbb{R}$ is a smooth function, then as a consequence of (2.1.17) and (2.1.16) or (2.1.13) and the Gauss theorem, we have

$$\int_M \Delta f * (1) = 0. \tag{2.1.18}$$

We now want to compute the Euclidean Laplace operator for p-forms. It is denoted by Δ_e; likewise, the star operator w.r.t. the Euclidean metric is denoted by $*_e$, and d^* is the operator adjoint to d w.r.t. the Euclidean scalar product.

Let now

$$\omega = \omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

be a p-form on an open subset of $\mathbb{R}^d$, as usual with an increasing p-tuple $1 \leq i_1 < i_2 < \ldots < i_p \leq d$. We choose $j_1, \ldots, j_{d-p}$ such that $\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_p}}, \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{i_{d-p}}}$ is a positive orthonormal basis of $\mathbb{R}^d$. In the sequel always

$$\ell \in \{1, \ldots, p\}, k \in \{1, \ldots, d-p\}.$$

Now

$$
\begin{aligned}
d\omega &= \sum_{k=1}^{d-p} \frac{\partial \omega_{i_1\ldots i_p}}{\partial x^{j_k}} dx^{j_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\
*_e d\omega &= \sum_{k=1}^{d-p} (-1)^{p+k-1} \frac{\partial \omega_{i_1\ldots i_p}}{\partial x^{j_k}} dx^{j_1} \wedge \ldots \wedge \widehat{dx^{j_k}} \wedge \ldots \wedge dx^{j_{d-p}} \qquad (2.1.19)\\
d *_e d\omega &= \sum_{k=1}^{d-p} (-1)^{p+k-1} \frac{\partial^2 \omega_{i_1\ldots i_p}}{(\partial x^{j_k})^2} dx^{j_k} \wedge dx^{j_1} \wedge \ldots \wedge \widehat{dx^{j_k}} \wedge \ldots \wedge dx^{i_{d-p}} \\
&\quad + \sum_{k=1}^{d-p} \sum_{\ell=1}^{p} (-1)^{p+k-1} \frac{\partial^2 \omega_{i_1\ldots i_p}}{\partial x^{j_k} \partial x^{i_\ell}} dx^{i_\ell} \wedge dx^{j_1} \wedge \ldots \wedge \widehat{dx^{j_k}} \wedge \ldots \wedge dx^{i_{d-p}} \qquad (2.1.20)\\
*_e d *_e d\omega &= \sum_{k=1}^{d-p} (-1)^{p+p(d-p)} \frac{\partial^2 \omega_{i_1\ldots i_p}}{(\partial x^{j_k})^2} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\
&\quad + \sum_{k=1}^{d-p} \sum_{\ell=1}^{p} (-1)^{pd+\ell} \frac{\partial^2 \omega_{i_1\ldots i_p}}{\partial x^{j_k} \partial x^{i_\ell}} dx^{j_k} \wedge dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\ell}} \wedge \ldots \wedge dx^{i_p}. \qquad (2.1.21)
\end{aligned}
$$

Hence with (2.1.12)

$$
\begin{aligned}
d^* d\omega &= \sum_{k=1}^{d-p} (-1) \frac{\partial^2 \omega_{i_1\ldots i_p}}{(\partial x^{j_k})^2} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\
&\quad + \sum_{k=1}^{d-p} \sum_{\ell=1}^{p} (-1)^{\ell+1} \frac{\partial^2 \omega_{i_1\ldots i_p}}{\partial x^{j_k} \partial x^{i_\ell}} dx^{j_k} \wedge dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\ell}} \wedge \ldots \wedge dx^{i_p} . \qquad (2.1.22)
\end{aligned}
$$

Analogously

$$
\begin{aligned}
*_e \omega &= \omega_{i_1\ldots i_p} dx^{j_1} \wedge \ldots \wedge dx^{j_{d-p}} \qquad (2.1.23)\\
d *_e \omega &= \sum_{\ell=1}^{p} \frac{\partial \omega_{i_1\ldots i_p}}{\partial x^{i_\ell}} dx^{i_\ell} \wedge dx^{j_1} \wedge \ldots \wedge dx^{i_{d-p}} \qquad (2.1.24)\\
*_e d *_e \omega &= \sum_{\ell=1}^{p} (-1)^{p(d-p)+d-p+\ell-1} \frac{\partial \omega_{i_1\ldots i_p}}{\partial x^{i_\ell}} dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\ell}} \wedge \ldots \wedge dx^{i_p} \qquad (2.1.25)\\
d *_e d *_e \omega &= \sum_{\ell=1}^{p} (-1)^{p(d-p)+d-p+\ell-1} \frac{\partial^2 \omega_{i_1\ldots i_p}}{(\partial x^{i_\ell})^2} dx^{i_\ell} \wedge dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\ell}} \wedge \ldots \wedge dx^{i_p} \\
&\quad + \sum_{\ell=1}^{p} \sum_{k=1}^{d-p} (-1)^{p(d-p)+d-p+\ell-1} \frac{\partial^2 \omega_{i_1\ldots i_p}}{\partial x^{i_\ell} \partial x^{j_k}} dx^{j_k} \wedge dx^{i_1} \wedge \ldots \wedge \widehat{dx^{i_\ell}} \wedge \ldots \wedge dx^{i_p}, \qquad (2.1.26)
\end{aligned}
$$

hence with (2.1.25)

$$dd^*\omega = \sum_{\ell=1}^{p}(-1)\frac{\partial^2\omega_{i_1\ldots i_p}}{(\partial x^{i_\ell})^2}dx^{i_1}\wedge\ldots\wedge dx^{i_p}$$
$$+\sum_{\ell=1}^{p}\sum_{k=1}^{d-p}(-1)^\ell\frac{\partial^2\omega_{i_1\ldots i_p}}{\partial x^{i_\ell}\partial x^{j_k}}dx^{j_k}\wedge dx^{i_1}\wedge\ldots\wedge\widehat{dx^{i_\ell}}\wedge\ldots\wedge dx^{i_p}. \tag{2.1.27}$$

(2.1.22) and (2.1.27) yield

$$\Delta_e\omega = d^*d\omega + dd^*\omega = (-1)\sum_{m=1}^{d}\frac{\partial^2\omega_{i_1\ldots i_p}}{(\partial x^m)^2}dx^{i_1}\wedge\ldots\wedge dx^{i_p}\,. \tag{2.1.28}$$

Some more formulae:

We write

$$\eta := \sqrt{g}dx^1\wedge\ldots\wedge dx^d =: \eta_{i_1\ldots i_d}dx^{i_1}\wedge\ldots\wedge dx^{i_d}\,. \tag{2.1.29}$$

For $\beta = \beta_{j_1\ldots j_p}dx^{j_1}\wedge\ldots\wedge dx^{j_p}$

$$\beta^{i_1\ldots i_p} := g^{i_1j_1}g^{i_2j_2}\ldots g^{i_pj_p}\beta_{j_1\ldots j_p}. \tag{2.1.30}$$

With these conventions, for $\alpha = \alpha_{i_1\ldots i_p}dx^{i_1}\wedge\ldots\wedge dx^{i_p}$

$$(*\alpha)_{i_{p+1}\ldots i_d} = \frac{1}{p!}\eta_{i_1\ldots i_p}\alpha^{i_1\ldots i_p} \tag{2.1.31}$$

and

$$(d^*\alpha)_{i_1\ldots i_{p-1}} = -g^{k\ell}\Big(\frac{\partial\alpha_{ki_1\ldots i_{p-1}}}{\partial x^\ell} - \Gamma^j_{k\ell}\alpha_{ji_1\ldots i_{p-1}}\Big). \tag{2.1.32}$$

Further

$$(\alpha,\beta) = \alpha_{i_1\ldots i_p}\beta^{i_1\ldots i_p} \tag{2.1.33}$$

$$(d\alpha,d\beta) = \frac{\partial\alpha_{i_1\ldots i_p}}{\partial x^k}\frac{\partial\beta_{j_1\ldots j_p}}{\partial x^\ell}g^{k\ell}g^{i_1j_1}\ldots g^{i_pj_p} \tag{2.1.34}$$

$$\begin{aligned}(d^*\alpha,d^*\beta) &= \Bigg(g^{k\ell}\Big(\frac{\partial\alpha_{ki_1\ldots i_{p-1}}}{\partial x^\ell} - \Gamma^j_{k\ell}\alpha_{ji_1\ldots i_{p-1}}\Big)e_{i_1}\wedge\ldots\wedge e_{i_{p-1}},\\ &\qquad g^{mn}\Big(\frac{\partial\beta_{mj_1\ldots j_{p-1}}}{\partial x^n} - \Gamma^r_{mn}\beta_{rj_1\ldots j_{p-1}}\Big)e_{j_1}\wedge\ldots\wedge e_{j_{p-1}}\Bigg) \qquad (2.1.35)\\ &= \frac{\partial\alpha_{ki_1\ldots i_{p-1}}}{\partial x^\ell}\frac{\partial\beta_{mj_1\ldots j_{p-1}}}{\partial x^n}g^{k\ell}g^{mn}g^{i_1j_1}\ldots g^{i_{p-1}j_{p-1}}\\ &\quad - \frac{\partial\alpha_{ki_1\ldots i_{p-1}}}{\partial x^\ell}\Gamma^i_{mn}\beta_{ij_1\ldots j_{p-1}}g^{k\ell}\ldots g^{i_{p-1}j_{p-1}}\\ &\quad - \frac{\partial\beta_{mj_1\ldots j_{p-1}}}{\partial x^n}\Gamma^j_{mn}\alpha_{ji_1\ldots i_{p-1}}g^{k\ell}g^{mn}g^{i_1j_1}\ldots g^{i_{p-1}j_{p-1}}.\end{aligned}$$

Formula (2.1.31) is clear. (2.1.32) may be verified by a straightforward, but somewhat lengthy computation. We shall see a different proof in §3.3 as a consequence of Lemma 3.3.4. The remaining formulae then are clear again.

2.2 Representing Cohomology Classes by Harmonic Forms

We first recall the definition of the de Rham cohomology groups. Let M be a differentiable manifold. The operator $d : \Omega^p(M) \to \Omega^{p+1}(M)$ satisfies (Theorem 1.8.5)

$$d \circ d = 0 \quad (d \circ d : \Omega^p(M) \to \Omega^{p+2}(M)). \tag{2.2.1}$$

$\alpha \in \Omega^p(M)$ is called *closed* if $d\alpha = 0$, *exact,* if there exists $\eta \in \Omega^{p-1}(M)$ with $d\eta = \alpha$. Because of (2.2.1), exact forms are always closed. Two closed forms $\alpha, \beta \in \Omega^p(M)$ are called *cohomologous* if $\alpha - \beta$ is exact. This property determines an equivalence relation on the space of closed forms in $\Omega^p(M)$, and the set of equivalence classes is a vector space over $\mathbb{R}$, called the p-th *de Rham cohomology group* and denoted by

$$H^p_{dR}(M, \mathbb{R}).$$

Usually, however, we shall simply write

$$H^p(M).$$

In this Paragraph, we want to show the following fundamental result:

Theorem 2.2.1 (**Hodge**). *Let M be a compact Riemannian manifold. Then every cohomology class in $H^p(M)$ $(0 \le p \le d = \dim M)$ contains precisely one harmonic form.*

Here, we shall demonstrate the Hodge theorem by a variational method. An alternative proof, by the heat flow method, as well as some important extensions, will be given in 2.4 below.

Proof. Uniqueness is easy: Let $\omega_1, \omega_2 \in \Omega^p(M)$ be cohomologous and both harmonic.

Then either $p = 0$ (in which case $\omega_1 = \omega_2$ anyway) or

$$\begin{aligned}(\omega_1 - \omega_2, \omega_1 - \omega_2) &= (\omega_1 - \omega_2, d\eta) \\ &\qquad \text{for some } \eta \in \Omega^{p-1}(M), \text{ since} \\ &\qquad \omega_1 \text{ and } \omega_2 \text{ are cohomologous} \\ &= (d^*(\omega_1 - \omega_2), \eta) \\ &= 0, \text{ since } \omega_1 \text{ and } \omega_2 \text{ are harmonic,} \\ &\qquad \text{hence satisfy } d^*\omega_1 = 0 = d^*\omega_2\,.\end{aligned}$$

Since $(\cdot,\cdot)$ is positive definite, we conclude $\omega_1 = \omega_2$, hence uniqueness. □

For the proof of existence, which is much harder, we shall use Dirichlet's principle.

Let ω_0 be a (closed) differential form, representing the given cohomology class in $H^p(M)$.

All forms cohomologous to ω_0 then are of the form

$$\omega = \omega_0 + d\alpha \quad (\alpha \in \Omega^{p-1}(M)).$$

We now minimize the L^2-norm

$$D(\omega) := (\omega, \omega)$$

in the class of all such forms.

The essential step consists in showing that the infimum is achieved by a smooth form η. Such an η then has to satisfy the Euler-Lagrange equations for D, i.e.

$$\begin{aligned} 0 &= \frac{d}{dt}(\eta + td\beta, \eta + td\beta)_{|t=0} \quad \text{for all}\, \beta \in \Omega^{p-1}(M) \\ &= 2(\eta, d\beta). \end{aligned} \tag{2.2.2}$$

This implies $\delta\eta = 0$. Since $d\eta = 0$ anyway, η is harmonic.

In order to make Dirichlet's principle precise, we shall need some results and constructions from the calculus of variations. Some of them will be merely sketched (see §A.1, A.2 of the Appendices), and for details, we refer to our textbook [143]. First of all, we have to work with the space of L^2-forms instead of the one of C^∞-forms, since we want to minimize the L^2-norm and therefore certainly need a space that is complete w.r.t. L^2-convergence. For technical purposes, we shall also need Sobolev spaces which we now want to define in the present context (see also §A.1).

On $\Omega^p(M)$, we introduce a new scalar product

$$((\omega, \omega)) := (d\omega, d\omega) + (\delta\omega, \delta\omega) + (\omega, \omega) \tag{2.2.3}$$

and put

$$\|\omega\|_{H^{1,2}(M)} := ((\omega, \omega))^{\frac{1}{2}}. \tag{2.2.4}$$

(This norm is to be distinguished from the L^2-norm of (2.1.10).) We complete the space $\Omega^p(M)$ of smooth p-forms w.r.t. the $\|\cdot\|_{H^{1,2}(M)}$-norm. The resulting Hilbert space will be denoted by $H^{1,2}_p(M)$ or simply by $H^{1,2}(M)$, if the index p is clear from the context.

Let now $V \subset \mathbb{R}^d$ be open. For a smooth map $f : V \to \mathbb{R}^n$, the Euclidean Sobolev norm is given by

$$\|f\|_{H^{1,2}_{\text{eucl.}}(V)} := \left(\int_V f \cdot f + \int_V \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^i} \right)^{\frac{1}{2}},$$

the dot $\cdot$ denoting the Euclidean scalar product.

With the help of charts for M and bundle charts for $\Lambda^p(M)$ for every $x_0 \in M$, there exist an open neighborhood U and a diffeomorphism

$$\varphi : \Lambda^p(M)_{|U} \to V \times \mathbb{R}^n$$

where V is open in $\mathbb{R}^d$, $n = \binom{d}{p}$ is the dimension of the fibers of $\Lambda^p(M)$, and the fiber over $x \in U$ is mapped to a fiber $\{\pi(\varphi(x))\} \times \mathbb{R}^n$, where $\pi : V \times \mathbb{R}^n \to V$ is the projection onto the first factor.

Lemma 2.2.1. *On any $U' \Subset U$, the norms*

$$\|\omega\|_{H^{1,2}(U')} \quad \textit{and} \quad \|\varphi(\omega)\|_{H^{1,2}_{\text{eucl.}}(V')}$$

(with $V' := \pi(\varphi(U'))$) are equivalent.

Proof. As long as we restrict ourselves to relatively compact subsets of U, all coordinate changes lead to equivalent norms. Furthermore, by a covering argument, it suffices to find for every x in the closure of U' a neighborhood U'' on which the claimed equivalence of norms holds.

After these remarks, we may assume that first of all $\pi \circ \varphi$ is the map onto normal coordinates with center x_0, and that secondly for the metric in our neighborhood of x_0, we have

$$|g_{ij}(x) - \delta_{ij}| < \varepsilon \text{ and } |\Gamma^i_{jk}(x)| < \varepsilon \text{ for } i,j,k = 1,\dots,d. \tag{2.2.5}$$

The formulae (2.1.33) - (2.1.35) then imply that the claim holds for sufficiently small $\varepsilon > 0$, i.e. for a sufficiently small neighborhood of x_0. Since $\bar{U}' \subset U$ is compact by assumption, the claim for U' follows by a covering argument. □

Lemma 2.2.1 implies that the Sobolev spaces defined by the norms $\|\cdot\|_{H^{1,2}(M)}$ and $\|\cdot\|_{H^{1,2}_{\text{eucl.}}}$ coincide. Hence all results for Sobolev spaces in the Euclidean setting may be carried over to the Riemannian situation. In particular, we have Rellich's theorem (cf. Theorem A.1.8):

Lemma 2.2.2. *Let $(\omega_n)_{n \in \mathbb{N}} \subset H^{1,2}_p(M)$ be bounded, i.e.*

$$\|\omega_n\|_{H^{1,2}(M)} \le K.$$

Then a subsequence of (ω_n) converges w.r.t. the L^2-norm

$$\|\omega\|_{L^2(M)} := (\omega, \omega)^{\frac{1}{2}}$$

to some $\omega \in H^{1,2}_p(M)$. □

Corollary 2.2.1. *There exists a constant c, depending only on the Riemannian metric of M, with the property that for all closed forms β that are orthogonal to the kernel of d^*,*

$$(\beta, \beta) \le c(d^*\beta, d^*\beta). \tag{2.2.6}$$

Proof. Otherwise, there would exist a sequence of closed forms β_n orthogonal to the kernel of d^*, with

$$(\beta_n, \beta_n) \geq n(d^*\beta_n, d^*\beta_n). \tag{2.2.7}$$

We put

$$\lambda_n := (\beta_n, \beta_n)^{-\frac{1}{2}}.$$

Then

$$1 = (\lambda_n\beta_n, \lambda_n\beta_n) \geq n(d^*(\lambda_n\beta_n), d^*(\lambda_n\beta_n)). \tag{2.2.8}$$

Since $d\beta_n = 0$, we have

$$\|\lambda_n\beta_n\|_{H^{1,2}} \leq 1 + \frac{1}{n}.$$

By Lemma 2.2.2, after selection of a subsequence, $\lambda_n\beta_n$ converges in L^2 to some form ψ. By (2.2.8), $d^*(\lambda_n\beta_n)$ converges to 0 in L^2. Hence $d^*\psi = 0$; this is seen as follows:

For all φ

$$\begin{aligned} 0 &= \lim_{n\to\infty}(d^*(\lambda_n\beta_n), \varphi) = \lim(\lambda_n\beta_n, d\varphi) \\ &= (\psi, d\varphi) = (d^*\psi, \varphi) \,\text{and hence}\, d^*\psi = 0. \end{aligned}$$

(With the same argument, $d\beta_n = 0$ for all n implies $d\psi = 0$.)

Now, since $d^*\psi = 0$ and β_n is orthogonal to the kernel of d^*,

$$(\psi, \lambda_n\beta_n) = 0. \tag{2.2.9}$$

On the other hand, $(\lambda_n\beta_n, \lambda_n\beta_n) = 1$ and the L^2-convergence of $\lambda_n\beta_n$ to ψ imply

$$\lim_{n\to\infty}(\psi, \lambda_n\beta_n) = 1.$$

This is a contradiction, and (2.2.7) is impossible. □

We can now complete the *proof of Theorem 2.2.1:*

Let $(\omega_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $D(\omega)$ in the given cohomology class, i.e.

$$\begin{aligned} \omega_n &= \omega_0 + d\alpha_n \\ D(\omega_n) &\to \inf_{\omega=\omega_0+d\alpha} D(\omega) =: \kappa. \end{aligned} \tag{2.2.10}$$

By (2.2.10), w.l.o.g.

$$(\omega_n, \omega_n) = D(\omega_n) \leq \kappa + 1. \tag{2.2.11}$$

As with Dirichlet's principle in $\mathbb{R}^d$, ω_n converges weakly to some ω, after selection of a subsequence.

We have

$$(\omega - \omega_0, \varphi) = 0 \,\text{for all}\, \varphi \in \Omega^p(M) \,\text{with}\, d^*\varphi = 0, \tag{2.2.12}$$

because

$$(\omega_n - \omega_0, \varphi) = (d\alpha_n, \varphi) = (\alpha_n, d^*\varphi) = 0 \text{ for all such } \varphi.$$

(2.2.12) means that $\omega - \omega_0$ is weakly exact.

We want to study this condition more closely and put

$$\eta := \omega - \omega_0.$$

We define a linear functional on $d^*(\Omega^p(M))$ by

$$\ell(\delta\varphi) := (\eta, \varphi), \tag{2.2.13}$$

ℓ is well defined; namely if $d^*\varphi_1 = d^*\varphi_2$, then

$$(\eta, \varphi_1 - \varphi_2) = 0 \text{ by (2.2.12)}.$$

For $\varphi \in \Omega^p(M)$ let $\pi(\varphi)$ be the orthogonal projection onto the kernel of d^*, and $\psi := \varphi - \pi(\varphi)$; in particular $d^*\psi = d^*\varphi$.

Then

$$\ell(d^*\varphi) = \ell(d^*\psi) = (\eta, \psi). \tag{2.2.14}$$

Since ψ is orthogonal to the kernel of δ, by Corollary 2.2.1,

$$\|\psi\|_{L^2} \le c\|d^*\psi\|_{L^2} = c\|d^*\varphi\|_{L^2}. \tag{2.2.15}$$

(2.2.14) and (2.2.15) imply

$$|\ell(d^*\varphi)| \le c\|\eta\|_{L^2}\|d^*\varphi\|_{L^2}.$$

Therefore, the function ℓ on $d^*(\Omega^p(M))$ is bounded and can be extended to the L^2-closure of $d^*(\Omega^p(M))$. By the Riesz representation theorem, any bounded linear functional on a Hilbert space is representable as the scalar product with an element of the space itself. Consequently, there exists α with

$$(\alpha, d^*\varphi) = (\eta, \varphi) \tag{2.2.16}$$

for all $\varphi \in \Omega^p(M)$.

Thus, we have weakly

$$d\alpha = \eta. \tag{2.2.17}$$

Therefore, $\omega = \omega_0 + \eta$ is contained in the closure of the considered class. Instead of minimizing among the ω cohomologous to ω_0, we could have minimized as well in the closure of this class, i.e., in the space of all ω for which there exists some α with

$$(\alpha, d^*\varphi) = (\omega - \omega_0, \varphi) \text{ for all } \varphi \in \Omega^p(M).$$

Then ω, as weak limit of a minimizing sequence, is contained in this class. Namely, suppose $\omega_n = \omega_0 + d\alpha_n$ weakly, i.e.

$$\ell_n(d^*\varphi) := (\alpha_n, d^*\varphi) = (\omega_n - \omega_0, \varphi)\ \forall\, \varphi \in \Omega^p(M).$$

By the same estimate as above, the linear functionals ℓ_n converge to some functional ℓ, again represented by some α. Since D also is weakly lower semicontinuous w.r.t. weak convergence, it follows that

$$\kappa \leq D(\omega) \leq \lim_{n\to\infty} \inf D(\omega_n) = \kappa,$$

hence

$$D(\omega) = \kappa.$$

Furthermore, by (2.2.2),

$$0 = (\omega, d\beta) \text{ for all } \beta \in \Omega^{p-1}(M). \tag{2.2.18}$$

In this sense, ω is weakly harmonic.

We still need the regularity theorem implying that solutions of (2.2.18) are smooth. This can be carried out as in the Euclidean case. If one would be allowed to insert $\beta = d^*\omega$ in (2.2.18) and integrate by parts, it would follow that

$$0 = (d^*\omega, d^*\omega),$$

i.e. $d^*\omega = 0$.

Iteratively, also higher derivatives would vanish, and the Sobolev embedding theorem would imply regularity. However, we cannot yet insert $\beta = d^*\omega$, since we do not know yet whether $dd^*\omega$ exists. This difficulty, however, may be overcome as usual by replacing derivatives by difference quotients (See §A.2 of the Appendix.). In this manner, one obtains regularity and completes the proof. □

Corollary 2.2.2. *Let M be a compact, oriented, differentiable manifold. Then all cohomology groups $H^p_{dR}(M,\mathbb{R})$ $(0 \leq p \leq d := \dim M)$ are finite dimensional.*

Proof. By Theorem 1.4.1, a Riemannian metric may be introduced on M. By Theorem 2.2.1 any cohomology class may be represented by a form which is harmonic w.r.t. this metric. We now assume that $H^p(M)$ is infinite dimensional. Then, there exists an orthonormal sequence of harmonic forms $(\eta_n)_{n\in\mathbb{N}} \subset H^p(M)$, i.e.

$$(\eta_n, \eta_m) = \delta_{nm} \text{ for } n, m \in \mathbb{N}. \tag{2.2.19}$$

Since the η_n are harmonic, $d^*\eta_n = 0$, and $d\eta_n = 0$. By Rellich's theorem (Lemma 2.2.2), after selection of a subsequence, (η_n) converges in L^2 to some η. This, however, is not compatible with (2.2.19), because (2.2.19) implies

$$\|\eta_n - \eta_m\|_{L^2} \geq 1 \text{ for } n \neq m,$$

so that (η_n) cannot be a Cauchy sequence in L^2.

This contradiction proves the finite dimensionality. □

Let now M be a compact, oriented, differentiable manifold of dimension d. We define a bilinear map

$$H^p_{dR}(M,\mathbb{R}) \times H^{d-p}_{dR}(M,\mathbb{R}) \to \mathbb{R}$$

by

$$(\omega,\eta) \mapsto \int_M \omega \wedge \eta \tag{2.2.20}$$

for representatives ω, η of the cohomology classes considered. It remains to show that (2.2.20) depends only on the cohomology classes of ω and η, in order that the map is indeed defined on the cohomology groups. If, however, ω' and ω are cohomologous, there exists a $(p-1)$ form α with $\omega' = \omega + d\alpha$, and

$$\begin{aligned}\int_M \omega' \wedge \eta &= \int_M (\omega + d\alpha) \wedge \eta \\ &= \int_M \omega \wedge \eta + \int_M d(\alpha \wedge \eta) \quad \text{since } \eta \text{ is closed} \\ &= \int_M \omega \wedge \eta \quad \text{by Stokes' theorem.}\end{aligned}$$

Therefore, (2.2.20) indeed depends only on the cohomology class of ω, and likewise only on the cohomology class of η.

Let us now recall a simple result of linear algebra. Let V and W be finite dimensional real vector spaces, and let

$$(\cdot,\cdot) : V \times W \to \mathbb{R}$$

be bilinear and nondegenerate in the sense that for any $v \in V, v \neq 0$, there exists $w \in W$ with $(v,w) \neq 0$, and conversely. Then V can be identified with the dual space W^* of W, and W may be identified with V^*. Namely,

$$\begin{aligned} i_1 &: V \to W^* \qquad \text{with } i_1(v)(w) := (v,w), \\ i_2 &: W \to V^* \qquad \text{with } i_2(w)(v) := (v,w), \end{aligned}$$

are two injective linear maps. Then V and W must be of the same dimension, and i_1 and i_2 are isomorphisms.

Theorem 2.2.2. *Let M be a compact, oriented, differentiable manifold of dimension d. The bilinear form (2.2.20) is nondegenerate, and hence $H^p_{dR}(M,\mathbb{R})$ is isomorphic to $(H^{d-p}_{dR}(M,\mathbb{R}))^*$.*

Proof. For each nontrivial cohomology class in $H^p(M)$, represented by some ω (i.e. $d\omega = 0$, but not $\omega = d\alpha$ for any $(p-1)$-form α), we have to find some cohomology class in $H^{d-p}(M)$ represented by some η, such that

$$\int_M \omega \wedge \eta \neq 0.$$

For this purpose, we introduce a Riemannian metric on M which is possible by Theorem 1.4.1. By Theorem 2.2.1, we may assume that ω is harmonic (w.r.t. this metric). By Lemma 2.1.6

$$\Delta * \omega = *\Delta\omega,$$

and therefore, $*\omega$ is harmonic together with ω. Now

$$\int_M \omega \wedge *\omega = (\omega, \omega) \neq 0, \text{ since } \ \omega \ \text{ does not vanish identically.}$$

Therefore, $*\omega$ represents a cohomology class in $H^{d-p}(M)$ with the desired property. Thus the bilinear form is nondegenerate, and the claim follows. □

Definition 2.2.1. The p-th *homology group* $H_p(M, \mathbb{R})$ of a compact, differentiable manifold M is defined to be $(H^p_{dR}(M, \mathbb{R}))^*$. The p-th *Betti number* of M is $b_p(M) := \dim H^p(M, \mathbb{R})$.

With this definition, Theorem 2.2.2 becomes

$$H_p(M, \mathbb{R}) \cong H^{d-p}_{dR}(M, \mathbb{R}). \tag{2.2.21}$$

This statement is called *Poincaré duality.*

Corollary 2.2.3. *Let M be a compact, oriented, differentiable manifold of dimension d. Then*

$$H^d_{dR}(M, \mathbb{R}) \cong \mathbb{R}. \tag{2.2.22}$$

and

$$b_p(M) = b_{d-p}(M) \quad \text{for} \quad 0 \leq p \leq d. \tag{2.2.23}$$

Proof. $H^0_{dR}(M, \mathbb{R}) \cong \mathbb{R}$. This follows e.g. from Corollary 2.1.2 and Theorem 2.2.1, but can also be seen in an elementary fashion.
Theorem 2.2.2 then implies (2.2.22), as well as (2.2.23). □

As an example, let us consider an n-dimensional torus T^n. As shown in §1.4, it can be equipped with a Euclidean metric for which the covering $\pi : \mathbb{R}^n \to T^n$ is a local isometry.

By (2.1.28), we have for the Laplace operator of the Euclidean metric

$$\Delta(\omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}) = (-1) \sum_{m=1}^{n} \frac{\partial^2 \omega_{i_1 \ldots i_p}}{(\partial x^m)^2} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

($x^1, \ldots, x^n$ Euclidean coordinates of $\mathbb{R}^n$.) Thus, a p-form is harmonic if and only if all coefficients w.r.t. the basis $dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ are harmonic. Since T^n is compact, by Corollary 2.1.2, they then have to be constant. Consequently

$$b_p(T^n) = \dim H^p(T^n) = \dim \Lambda^p(\mathbb{R}^n) = \binom{n}{p} \qquad (0 \leq p \leq n).$$

Perspectives. The results of this Paragraph were found in the 1940s by Weyl, Hodge, de Rham and Kodaira.

2.3 Generalizations

The constructions of this chapter may easily be generalized. Here, we only want to indicate some such generalizations.

Let E and F be vector bundles over the compact, oriented, differentiable manifold M. Let $\Gamma(E)$ and $\Gamma(F)$ be the spaces of differentiable sections. Sobolev spaces of sections can be defined with the help of bundle charts: Let (f, U) be a bundle chart for E, f then identifies $E_{|U}$ with $U \times \mathbb{R}^n$. A section s of E is then contained in the Sobolev space $H^{k,p}(E)$ if for any such bundle chart and any $U' \Subset U$, we have $p_2 \circ f \circ s_{|U'} \in H^{k,p}(U', \mathbb{R}^n)$, where $p_2 : U' \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the second factor.

A linear map $L : \Gamma(E) \to \Gamma(F)$ is called (linear) differential operator of order ℓ from E to F if in any bundle chart, L defines such an operator. For the Laplace operator, of course $E = F = \Lambda^p(T^*M)$, $\ell = 2$.

In a bundle chart, we write L as

$$L = P_\ell(D) + \ldots + P_0(D),$$

where each $P_j(D)$ is an $(m \times n)$-matrix (m, n = fiber dimensions of E and F, resp.), whose components are differential operators of the form

$$\sum_{|\alpha|=j} a_\alpha(x) D^\alpha$$

where α is a multi index, and D^α is a homogeneous differential operator of degree $|\alpha| = j$. Let us assume that the $a_\alpha(x)$ are differentiable.

For $\xi = (\xi^1, \ldots, \xi^m) \in \mathbb{R}^m$, let $P_j(\xi)$ be the matrix obtained for $P_j(D)$ by replacing D^α by ξ^α.
$P_j(\xi)$ thus has components

$$\sum_{|\alpha|=j} a_\alpha(x) \xi^\alpha.$$

L is called *elliptic* at the point x, if $P_\ell(\xi)$ (ℓ = degree of L) is nonsingular at x for all $\xi \in \mathbb{R}^m \backslash \{0\}$. Note that in this case necessarily $n = m$.

L is called *elliptic* if it is elliptic at every point. Let now $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_F$ be bundle metrics on E and F, resp. (those always exist by Theorem 1.8.3), let M carry a Riemannian metric (existing by Theorem 1.4.1) and an orientation. Integrating the bundle metrics, for example

$$(\cdot, \cdot)_E := \int_M \langle \cdot, \cdot \rangle_E d\mathrm{Vol}_g \quad (d\mathrm{Vol}_g = \sqrt{\det(g_{ij})} dx^1 \wedge \ldots \wedge dx^d),$$

we obtain L^2-metrics on $\Gamma(E)$ and $\Gamma(F)$. Let L^* be the operator formally adjoint to L, i.e.

$$(Lv, w)_F = (v, L^* w)_E \quad \text{for } v \in \Gamma(E), w \in \Gamma(F).$$

L is elliptic if L^* is.

The importance of the ellipticity condition rests on the fact that solutions of elliptic differential equations are regular, and the space of solutions has finite dimension.

Here, however, this shall not be pursued any further.

2.4 The Heat Flow and Harmonic Forms

In this section, we shall present an alternative proof of Theorem 2.2.1. This proof will procede by solving a parabolic equation, the so-called heat flow. The idea is to let the objects involved, here p-forms, depend not only on the position x in the manifold M, but also on another variable, the "time" $t \in [0, \infty)$, and to replace the elliptic equation that one wishes to solve by a parabolic equation that one can solve for given starting values at time $t = 0$. In our case of differential forms, this heat equation is

$$\frac{\partial \beta(x,t)}{\partial t} + \Delta \beta(x,t) = 0 \tag{2.4.1}$$

$$\beta(x, 0) = \beta_0(x) \tag{2.4.2}$$

where β_0 is a p-form in the cohomology class that we wish to study.
The strategy then consists in showing that (2.4.1) can be uniquely solved for all positive t (this is called global or long time existence) and that, as $t \to \infty$, the solution $\beta(x,t)$ converges to a harmonic p-form in the same cohomology class.
(2.4.1) is a linear parabolic differential equation (or more precisely, a system of linear differential equations since the dimension of the fibers Λ^p is larger than 1 except for trivial cases). Therefore, the global existence and existence of solutions follows from the general theory of linear parabolic differential equations. Since we consider this equation as a prototype of other, typically nonlinear, parabolic differential equations arising in geometric analysis, we shall only use the short time existence here (which also holds for nonlinear equations by linearization) and deduce the long time existence from differential inequalities for the geometric objects involved.
The short time existence is contained in

Lemma 2.4.1. *Let $\beta_0 \in \Omega^p$ be of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. Then, for some $0 < \epsilon$, (2.4.1) has a solution $\beta(x,t)$ for $0 \le t < \epsilon$, and this solution is also of class $C^{2,\alpha}$.*

In order to procede to the global existence, we shall consider the L^2-norm

$$\|\beta(\cdot, t)\|^2 = \int_M \beta(x,t) \wedge * \beta(x,t) \tag{2.4.3}$$

and the energy

$$E(\beta(\cdot,t)) := \frac{1}{2}\|d\beta(\cdot,t)\| + \frac{1}{2}\|d^*\beta(\cdot,t)\|. \tag{2.4.4}$$

(Note that $(\|\beta(\cdot,t)\|^2 + 2E(\beta(\cdot,t)))^{1/2}$ is the Sobolev norm of $\beta(\cdot,t)$ as introduced in (2.2.4).)

Lemma 2.4.2.

$$\frac{d}{dt}\|\beta(\cdot,t)\|^2 \leq 0 \tag{2.4.5}$$

$$\frac{d^2}{dt^2}\|\beta(\cdot,t)\|^2 \geq 0 \tag{2.4.6}$$

$$\frac{d}{dt}E(\beta(\cdot,t)) \leq 0. \tag{2.4.7}$$

Proof.

$$\begin{aligned}\frac{d}{dt}\|\beta(\cdot,t)\|^2 &= 2(\tfrac{\partial}{\partial t}\beta(\cdot,t),\beta(\cdot,t))\\ &= -2(\Delta\beta(\cdot,t),\beta(\cdot,t))\\ &= -2(d\beta(\cdot,t),d\beta(\cdot,t)) - 2(d^*\beta(\cdot,t),d^*\beta(\cdot,t))\\ &= -4E(\beta(\cdot,t))\\ &\leq 0\end{aligned} \tag{2.4.8}$$

which shows (2.4.5). Next

$$\begin{aligned}\frac{d}{dt}E(\beta(\cdot,t)) &= (d\tfrac{\partial}{\partial t}\beta(\cdot,t),d\beta(\cdot,t)) + (d^*\tfrac{\partial}{\partial t}\beta(\cdot,t),d^*\beta(\cdot,t))\\ &= (\tfrac{\partial}{\partial t}\beta(\cdot,t),\Delta\beta(\cdot,t))\\ &= -(\tfrac{\partial}{\partial t}\beta(\cdot,t),\tfrac{\partial}{\partial t}\beta(\cdot,t))\\ &\leq 0\end{aligned}$$

which shows (2.4.7). (2.4.6) follows from this and (2.4.8). □

In particular, when $\beta(x,0) \equiv 0$, then, by (2.4.5), $\beta(x,t) \equiv 0$ for all t for which the solution exists. From this, we deduce

Corollary 2.4.1. *Solutions of (2.4.1) are unique*
(if $\beta_1(x,t)$ and $\beta_2(x,t)$ are solutions of (2.4.1) for $0 \leq t \leq T$ with the same initial values, i.e., $\beta_1(x,0) = \beta_2(x,0)$, then they also coincide for $0 \leq t \leq T$)
and satisfy a semigroup property
(if $\beta(\cdot,t)$ solves (2.4.1), then $\beta(\cdot,t+s) = \beta_s(\cdot,t)$ where $\beta_s(\cdot,t)$ is the solution of (2.4.1) with initial values $\beta_s(\cdot,0) = \beta(\cdot,s)$).

In fact, we have a more general stability result

Corollary 2.4.2. *For a family $\beta(x,t,s)$ of solutions of (2.4.1) that depends differentiably on the parameter $s \in \mathbb{R}$,*

$$\frac{d}{dt}\|\frac{\partial}{\partial s}\beta(\cdot,t,s)\|^2 \leq 0. \tag{2.4.9}$$

Proof. $\frac{\partial}{\partial s}\beta(x,t,s)$ also solves (2.4.1), and (2.4.9) therefore follows from (2.4.5). □

We now need some apriori estimates:

Lemma 2.4.3. *A solution $\beta(x,t)$ of (2.4.1) defined for $0 \leq t \leq T$ with initial values $\beta_0(x) \in L^2$ satisfies for $\tau \leq t \leq T$, for any $\tau > 0$, estimates of the form*

$$\|\beta(\cdot,t)\|_{C^{2,\alpha}(M)} + \|\frac{\partial}{\partial t}\beta(\cdot,t)\|_{C^{\alpha}(M)} \leq c_1 \tag{2.4.10}$$

with a constant c_1 depending only on $\|\beta_0\|_{L^2(M)}$, τ and the geometry of M (but not on the particular solution $\beta(x,t)$).

Remark. An important consequence of this lemma that we shall use repeatedly in the sequel is that from the estimates we can infer convergence results. In fact, the Arzela-Ascoli Theorem implies that any sequence (f_n) that is bounded in the Hölder space $C^{\alpha}(M)$ for some $0 < \alpha < 1$ contains a subsequence that converges in $C^{\alpha'}(M)$, for any $\alpha' < \alpha$. See [143] for details.
Proof. From (2.4.5),

$$\|\beta(\cdot,t)\|_{L^2(M)} \leq \|\beta_0\|_{L^2(M)} \tag{2.4.11}$$

See ... □

We can now deduce the global existence of solutions of (2.4.1):

Corollary 2.4.3. *Let $\beta_0 \in C^{2,\alpha}$ for some $0 < \alpha < 1$. Then the solution $\beta(x,t)$ of (2.4.1) with those initial values exists for all $t \geq 0$.*

Proof. By local existence (Lemma 2.4.1), the solution exists on some positive time interval $0 \leq t < \epsilon$. Whenever it exists on some interval $0 \leq t \leq T$, for $t \to T$, by Lemma 2.4.3, $\beta(x,t)$ converges to some form $beta(x,T)$ in $C^{2,\alpha'}$ for $0 < \alpha' < \alpha$. Applying the semigroup property (Corollary 2.4.1) and local existence (Lemma 2.4.1) again, the solution can be continued to some time interval beyond T, that is, it exists for $0 \leq t < T + \epsilon$. Thus, the existence interval is open and closed and nonempty and therefore consists of the entire positive real line. □

The final step in the program is the asymptotic behavior of solutions as $t \to \infty$. With this, we shall complete the proof of

Theorem 2.4.1 (Milgram-Rosenbloom). *Given a p-form $\beta_0(x)$ on M of class $C^{2,\alpha}$, for some $0<\alpha<1$, there exists a unique solution of*

$$\frac{\partial\beta(x,t)}{\partial t}+\Delta\beta(x,t)=0 \text{ for all } 0\le t<\infty \tag{2.4.12}$$

$$\text{with } \beta(x,0)=\beta_0(x) \tag{2.4.13}$$

As $t\to\infty$, $\beta(\cdot,t)$ converges in $C^{2,\alpha}$ to a harmonic form $H\beta$.
If β_0 is closed, i.e., $d\beta_0=0$, then all the forms $\beta(\cdot,t)$ are closed as well, $d\beta(\cdot,t)=0$. Also, in this case, if ω is a coclosed $(d-p)$-form, i.e. $d^*\omega=0$, then $\int_M\beta(x,t)\wedge\omega(x)$ does not depend on t, and we have $\int_M H\beta(x)\wedge\omega(x)=\int_M\beta_0(x)\wedge\omega(x)$.

This result obviously contains the Hodge Theorem 2.2.1 and provides an alternative proof of it.
Proof. Since $E(\beta(\cdot,t))\ge 0$, (2.4.5) implies that there exists at least some sequence $t_n\to\infty$ for which

$$\|\frac{\partial}{\partial t}\beta(\cdot,t_n)\|\to 0. \tag{2.4.14}$$

The control of the higher norms of $\beta(\cdot,t_n)$ of Lemma 2.4.3 then implies that $\Delta\beta(\cdot,t_n)=-\frac{\partial}{\partial t}\beta(\cdot,t_n)$ converges to 0 in some Hölder space $C^{2,\alpha'}$, that is, $\beta(\cdot,t_n)$ converges in $C^{2,\alpha'}$ to a harmonic form $H\beta$. The difference

$$\beta_1(x,t):=\beta(x,t)-H\beta(x)$$

then also solves (2.4.12). Using (2.4.14) and (2.4.5) once more, we see that $\|\beta(\cdot,t)-H\beta(\cdot)\|\to 0$ as $t\to\infty$, and by Lemma 2.4.3, $\beta(x,t)$ converges to $H\beta(x)$ in $C^{2,\alpha'}$.
Uniqueness was already deduced in Corollary 2.4.1.
Since the exterior derivative d commutes with the Laplacian Δ as is clear from the definition of the latter and obviously also with $\frac{\partial}{\partial t}$, if $\beta(x,t)$ solves (2.4.12), then so does $d\beta(x,t)$. Thus, using e.g. (2.4.5) again, if $d\beta_0=0$, then also $d\beta(\cdot,t)=0$. Finally, if also $d^*\omega=0$, then

$$\begin{aligned}\frac{\partial}{\partial t}\int_M\beta(x,t)\wedge\omega(x)&=-\int_M\Delta\beta(x,t)\wedge\omega(x)\\ &=-\int_M dd^*\beta(x,t)\wedge\omega(x)=-\int_M d^*\beta(x,t)\wedge d^*\omega(x)=0.\end{aligned}$$

□

The heat flow method can also conveniently deduce some refinements of this theorem. We observe

Lemma 2.4.4. *Under the assumptions of Theorem 2.4.1, the solution $\beta(x,t)$ of (2.4.1) converges exponentially towards the harmonic form $H\beta_0(x)$, that is,*

$$\|\beta(\cdot,t)-H\beta_0(\cdot)\|\le ce^{-\lambda t} \tag{2.4.15}$$

for some positive constants c,λ. Here, λ is independent of β.

Proof. Given $t > 0$, we seek β with $\|\beta\| = 1$ and $H\beta = 0$ for which for the solution $\beta(x,t)$ of (2.4.1) with initial values $\beta(x,0) = \beta(x)$,

$$\|\beta(\cdot,t)\|$$

is maximal. Since, by Lemma 2.4.3, the $C^{1,\alpha}$-norm of $\beta(\cdot,t)$ is bounded in terms of $\|\beta(\cdot,0)\|$, this maximum is attained. Let this maximal value be $b(t)$. Since $H\beta = 0$, (2.4.5) must be strictly negative. This implies $b(t) < 1$. The semigroup property of Corollary 2.4.1 then implies

$$b(nt) \le b(t)^n \text{ for } n \in \mathbb{N},$$

from which

$$b(t) \le e^{-\lambda t} \text{ for some } \lambda > 0.$$

Therefore, for general $\beta(x,0) \in L^2$, we obtain (2.4.15). □

We can then show

Corollary 2.4.4. *The equation*

$$\Delta\nu = \eta \tag{2.4.16}$$

for a p-form η of class L^2 is solvable iff

$$(\eta,\omega) = 0 \textit{ for all } \omega \textit{ with } \Delta\omega = 0. \tag{2.4.17}$$

This solution then is unique up to addition of a harmonic form.
Therefore, the space of p-forms of class L^2 admits the decomposition

$$\Omega^p_{L^2}(M) = \ker\Delta \bigoplus \operatorname{image}\Delta \tag{2.4.18}$$

(note that the first summand, the kernel of Δ, is finite dimensional).

Proof. We consider

$$\begin{aligned} \frac{\partial}{\partial t}\mu + \Delta\mu &= \gamma \\ \mu(\cdot,t) &= \mu_0. \end{aligned} \tag{2.4.19}$$

We put

$$T_t\mu_0 = \beta(\cdot,t)$$

for the solution of

$$\begin{aligned} \frac{\partial}{\partial t}\beta + \Delta\beta &= 0 \\ \beta(\cdot,t) &= \mu_0. \end{aligned} \tag{2.4.20}$$

We then have

$$\mu(x,t) = T_t\mu_0(x) + \int_0^t T_{t-s}\gamma(x)ds = T_t\mu_0(x) + \int_0^t T_s\gamma(x)ds \tag{2.4.21}$$

as γ does not depend on t.
By (2.4.15), we have

$$\|T_s\gamma - H\gamma\| \le e^{-\lambda s}$$

whence

$$\|\mu - tH\gamma - T_t\mu_0\| \le \int_0^t e^{-\lambda s} ds.$$

We conclude that

$$\nu(x) := \lim_{t\to\infty} (\mu(x,t) - tH\gamma(x))$$

exists, in L^2 and then also in $C^{2,\alpha}$, by the estimates. Since $\Delta H\gamma = 0$, we have

$$(\frac{\partial}{\partial t} + \Delta)(\mu(x,t) - tH\gamma(x)) = \eta(x) - H\eta(x).$$

Therefore,

$$\Delta\nu = \eta - H\eta.$$

This implies the solvability of (2.4.16) under the condition (2.4.17) because $\eta - H\eta$ is the projection onto the L^2-orthogonal complement of the kernel of Δ. □

Exercises for Chapter 2

1. Compute the Laplace operator of S^n on p-forms ($0 \le p \le n$) in the coordinates given in §1.1.

2. Let $\omega \in \Omega^1(S^2)$ be a 1-form on S^2. Suppose

$$\varphi^*\omega = \omega$$

for all $\varphi \in \mathrm{SO}(3)$. Show that $\omega \equiv 0$.

Formulate and prove a general result for invariant differential forms on S^n.

3. Give a detailed proof of the formula

$$*\Delta = \Delta * .$$

4. Let M be a two dimensional Riemannian manifold. Let the metric be given by $g_{ij}(x)dx^i \otimes dx^j$ in local coordinates (x^1, x^2). Compute the Laplace operator on 1-forms in these coordinates. Discuss the case where

$$g_{ij}(x) = \lambda^2(x)\delta_{ij}$$

with a positive function $\lambda^2(x)$.

5. Suppose that $\alpha \in H_p^{1,2}(M)$ satisfies

$$(d^*\alpha, d^*\varphi) + (d\alpha, d\varphi) = (\eta, \varphi) \quad \text{for all} \quad \varphi \in \Omega^p(M),$$

with some given $\eta \in \Omega^p(M)$. Show $\alpha \in \Omega^p(M)$, i.e. smoothness of α.

6. Compute a relation between the Laplace operators on functions on $\mathbb{R}^{n+1}$ and the one on $S^n \subset \mathbb{R}^{n+1}$.

7. Eigenvalues of the Laplace operator:

Let M be a compact oriented Riemannian manifold, and let Δ be the Laplace operator on $\Omega^p(M)$. $\lambda \in \mathbb{R}$ is called eigenvalue if there exists some $u \in \Omega^p(M), u \neq 0$, with

$$\Delta u = \lambda u.$$

Such a u is called eigenform or eigenvector corresponding to λ. The vector space spanned by the eigenforms for λ is denoted by V_λ and called eigenspace for λ.

Show:

a: All eigenvalues of Δ are nonnegative.

b: All eigenspaces are finite dimensional.

c: The eigenvalues have no finite accumulation point.

d: Eigenvectors for different eigenvalues are orthogonal.

The next results need a little more analysis (cf. e.g. [143])

e: There exist infinitely many eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$$

f: All eigenvectors of Δ are smooth.

g: The eigenvectors of Δ constitute an L^2-orthonormal basis for the space of p-forms of class L^2.

8. Here is another long exercise:

Let M be a compact oriented Riemannian manifold with boundary $\partial M \neq \emptyset$. For $x \in \partial M, V \in T_xM$ is called tangential if it is contained in $T_x\partial M \subset T_xM$ and $W \in T_xM$ is called normal if

$$\langle V, W \rangle = 0 \quad \text{for all tangential} \quad V.$$

An arbitrary $Z \in T_xM$ can then be decomposed into a tangential and a normal component:

$$Z = Z_{\tan} + Z_{\text{nor}}.$$

Analogously, $\eta \in \Gamma^p(T^x, M)$ can be decomposed into

$$\eta = \eta_{\tan} + \eta_{\text{nor}}$$

where η_{tan} operates on tangential p-vectors and η_{nor} on normal ones. For p-forms ω on M, we may impose the so-called absolute boundary conditions

$$\begin{array}{rcl} \omega_{\mathrm{tan}} & = & 0, \\ (\delta\omega)_{\mathrm{nor}} & = & 0, \end{array} \quad \text{on } \partial M,$$

or the relative boundary conditions

$$\begin{array}{rcl} \omega_{\mathrm{nor}} & = & 0, \\ (d\omega)_{\mathrm{nor}} & = & 0, \end{array} \quad \text{on } \partial M.$$

(These two boundary conditions are interchanged by the $*$-operator.)

Develop a Hodge theory under either set of boundary conditions.