Preface

The text of this book has its origins more than twenty-five years ago. In the seminar of the Dutch Singularity Theory project in 1982 and 1983, the second-named author gave a series of lectures on Mixed Hodge Structures and Singularities, accompanied by a set of hand-written notes. The publication of these notes was prevented by a revolution in the subject due to Morihiko Saito: the introduction of the theory of Mixed Hodge Modules around 1985. Understanding this theory was at the same time of great importance and very hard, due to the fact that it unifies many different theories which are quite complicated themselves: algebraic D-modules and perverse sheaves.

The present book intends to provide a comprehensive text about Mixed Hodge Theory with a view towards Mixed Hodge Modules. The approach to Hodge theory for singular spaces is due to Navarro and his collaborators, whose results provide stronger vanishing results than Deligne's original theory. Navarro and Guillén also filled a gap in the proof that the weight filtration on the nearby cohomology is the right one. In that sense the present book corrects and completes the second-named author's thesis.

Many suggestions and corrections to this manuscript were made by several colleagues: Benoît Audoubert, Alex Dimca, Alan Durfee, Alexey Gorinov, Dick Hain, Theo de Jong, Rainer Kaenders, Morihiko Saito, Vasudevan Srinivas, Duco van Straten, to mention a few. Thanks to all of you!

During the preparation of the manuscript the authors received hospitality and support from the universities of Grenoble and Nijmegen. Moreover, we thank Annie for providing us excellent working conditions at Veldhoven.

Grenoble/Nijmegen, August 2007   Chris Peters, Joseph Steenbrink

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Pure Hodge Structures

The Hodge decomposition of the n-th cohomology group of a Kähler manifold is the prototype of a Hodge structure of weight n. In this chapter we study these from a more abstract point of view. In § 2.1 and § 2.2 the foundations are laid. Hodge theoretic considerations for various sorts of fundamental classes associated to a subvariety are given in § 2.4.

In § 2.3 some important concepts are developed which play a central role in the remainder of this book, in particular the concept of a Hodge complex, which is introduced in § 2.3. The motivating example comes from the holomorphic De Rham complex on a compact Kähler manifold and is called the *Hodge-De Rham complex*. However, to show that this indeed gives an example of a Hodge complex follows only after a strong form of the Hodge decomposition is shown to hold. This also allows one to put a Hodge structure on the cohomology of any compact complex manifold which is bimeromorphic to a Kähler manifold, in particular algebraic manifolds that are not necessarily projective.

In Chapter 3 we shall extend the notion of a Hodge complex of sheaves to that of a mixed Hodge complex of sheaves.

We finally show in § 2.5 that the cohomology of varieties with quotient singularities also admits a pure rational Hodge structure.

2.1 Hodge Structures

2.1.1 Basic Definitions

We place the definition of a weight k Hodge structure (Def. 1.12) in a wider context. Let V be a finite dimensional real vector space and let $V_{\mathbb{C}} = V \otimes \mathbb{C}$ be its complexification.

Definition 2.1. A **real Hodge structure** on V is a direct sum decomposition

$$V_{\mathbb{C}} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}, \text{with } V^{p,q} = \overline{V^{q,p}} \quad \textbf{(the Hodge decomposition}.)$$

The numbers

$$h^{p,q}(V) := \dim V^{p,q}$$

are **Hodge numbers** of the Hodge structure. The polynomial

$$P_{\mathrm{hn}}(V) = \sum_{p,q \in \mathbb{Z}} h^{p,q}(V) u^p v^q \qquad \text{(II–1)}$$

its associated **Hodge number polynomial**.

If the real Hodge structure V is of the form $V = V_R \otimes_R \mathbb{R}$ where R is a subring of $\mathbb{R}$ and V_R is an R-module of finite type we say that V_R carries carries an R-**Hodge structure**.

A **morphism of R-Hodge structures** is a morphism $f : V_R \to W_R$ of R-modules whose complexification maps $V^{p,q}$ to $W^{p,q}$.

If V is real Hodge structure, the **weight k part** $V^{(k)}$ is the real vector space underlying $\bigoplus_{p+q=k} V^{p,q}$. If $V = V^{(k)}$, we say that V is a weight k real Hodge structure and if $V = V_R \otimes_R \mathbb{R}$ we speak of a weight k R-Hodge structure. Usually, if $R = \mathbb{Z}$ we simply say that V or $V_{\mathbb{Z}}$ **carries a weight k Hodge structure**.

Examples 2.2. i) The De Rham group $H^k_{\mathrm{DR}}(X)$ of a compact Kähler manifold has canonical real Hodge structure of weight k defined by the classical Hodge decomposition. We have seen (Corr. 1.13) that it is in fact an integral Hodge structure.

ii) The Hodge structure $\mathbb{Z}(1)$ of Tate (I–3) has variants over any subring R of $\mathbb{R}$: we put $R(k) := R \otimes_{\mathbb{Z}} \mathbb{Z}(k)$.

iii) The top cohomology of a compact complex manifold X of dimension say n, can be identified with a certain Tate structure. Indeed, the **trace map** is the isomorphism given by

$$\mathrm{tr} : H^{2n}(X; \mathbb{R}) \overset{\sim}{\longrightarrow} \mathbb{R}(-n), \quad \omega \mapsto \left(\frac{1}{2\pi \mathrm{i}}\right)^n \int_X \omega. \qquad \text{(II–2)}$$

Let $V = V^{(k)}$ be a weight k Hodge structure. The **Hodge filtration** associated to this Hodge structure is given by

$$F^p(V) = \bigoplus_{r \geq p} V^{r,s}.$$

Conversely, a decreasing filtration

$$V_{\mathbb{C}} \supset \cdots \supset F^p(V) \supset F^{p+1}(V) \cdots$$

on the complexification $V_{\mathbb{C}}$ with the property that $F^p \cap \overline{F^q} = 0$ whenever $p + q = k + 1$ defines a weight k Hodge structure by putting

$$V^{p,q} = F^q \cap \overline{F^q}.$$

The condition that $F^p \cap \overline{F^q} = 0$ whenever $p + q = k + 1$ is equivalent to $F^p \oplus \overline{F^{k-p+1}} = V_{\mathbb{C}}$ and we say that the *filtration $F^\bullet$ is k-opposed to its complex conjugate filtration.*

Definition 2.3 (Multi-linear algebra constructions). Suppose that V, W are real vector spaces with a Hodge structure of weight k, respectively ℓ, the Hodge filtration on $V \otimes W$ is given by

$$F^p(V \otimes W)_{\mathbb{C}} = \sum_m F^m(V_{\mathbb{C}}) \otimes F^{p-m}(W_{\mathbb{C}}) \subset V_{\mathbb{C}} \otimes_{\mathbb{C}} W_{\mathbb{C}}.$$

This gives $V \otimes W$ a Hodge structure of weight $k + \ell$ with Hodge number polynomial given by

$$P_{\mathrm{hn}}(V \otimes W) = P_{\mathrm{hn}}(V)P_{\mathrm{hn}}(W). \tag{II--3}$$

Similarly, the multiplicative extension of the Hodge filtration to the tensor algebra

$$TV = \bigoplus_a T_a V \quad \text{with } T_a V := \bigotimes^a V$$

of V is defined by

$$F^p T_a V = \sum_{k_1 + \cdots k_a = p} F^{k_1} V_{\mathbb{C}} \otimes \ldots F^{k_a} V_{\mathbb{C}}$$

and gives a Hodge structure of weight ak on $T_a V$. It induces a Hodge structure of the same weight on the degree a-piece of the symmetric algebra SV of V and the exterior algebra ΛV of V. We can also put a Hodge structure on duals, or, more generally spaces of homomorphisms as follows:

$$F^p \operatorname{Hom}(V, W)_{\mathbb{C}} = \{f : V_{\mathbb{C}} \to W_{\mathbb{C}} \mid f F^n(V_{\mathbb{C}}) \subset F^{n+p}(W_{\mathbb{C}}) \quad \forall n\}$$

This defines a Hodge structure of weight $\ell - k$ on $\operatorname{Hom}(V, W)$ with Hodge number polynomial

$$P_{\mathrm{hn}}(\operatorname{Hom}(V, W)(u, v) = P_{\mathrm{hn}}(V)(u^{-1}, v^{-1})P_{\mathrm{hn}}(W)(u, v). \tag{II--4}$$

In particular, taking $W = \mathbb{R}$ with $W_{\mathbb{C}} = W^{0,0}$ we get a Hodge structure of weight $-k$ on the dual $V^\vee$ of V with Hodge number polynomial

$$P_{\mathrm{hn}}(V^\vee)(u, v) = P_{\mathrm{hn}}(V)(u^{-1}, v^{-1}). \tag{II--5}$$

Finally, we can define a Hodge structure of weight $ak - b\ell$ on $T_a V \otimes T_b V^\vee = V^{\otimes a} \otimes (V^\vee)^{\otimes b}$ using the multiplicative extension of F to the tensor algebra $TV \otimes TV^\vee$. The multiplication in each of the algebras TV, SV, ΛV, $TV \otimes TV^\vee$ is a morphism of Hodge structures.

Given any R-Hodge structure V, define its r-**th Tate twist** by

$$V(r) := V \otimes_R R(r).$$

If V has weight m, $V(r)$ has weight $m - 2r$ and

$$V(r)^{p,q} = V^{p+r,q+r}.$$

Note that one has:

$$P_{\mathrm{hn}}(V(r)) = P_{\mathrm{hn}}(V)(uv)^{-r}. \tag{II–6}$$

If W is another R-Hodge structure, giving A morphism $V(-r) \to W$ is also called a **morphism of Hodge structures** $V \to W$ **of type** (r,r). Morphisms of Hodge structures preserve the Hodge filtration. The converse is also true:

Proposition 2.4. *Let V, W be R-Hodge structures of weight k. Suppose that $f : V \to W$ is an $\mathbb{R}$-linear map preserving the R-structures and such that*

$$f_{\mathbb{C}}(F^p V) \hookrightarrow F^p W.$$

Then f is a morphism of R-Hodge structures.

Proof. One has $f_{\mathbb{C}}(\overline{F^q V}) \hookrightarrow \overline{F^q W}$, so, if $p + q = k$, we have

$$f_{\mathbb{C}}(V^{p,q}) = f_{\mathbb{C}}(F^p V) \cap \overline{F^q V} \hookrightarrow F^p W \cap \overline{F^q W} = W^{p,q}. \qquad \square$$

Clearly, the image of a morphism of Hodge structures is again a Hodge structure. By the above constructions the duality operation preserves Hodge structures, and so the kernel of a morphism of Hodge structures is a Hodge structure. Using the preceding multi-linear algebra constructions, it is not hard to see that we in fact have:

Corollary 2.5. *The category of R-Hodge structures is an abelian category which we denote $\mathfrak{hs}_R$. If $R = \mathbb{Z}$ we simply write $\mathfrak{hs}$.*

Hodge structures can also be defined through group representations and this is useful in the context of Mumford-Tate groups (see § 2.2). Introduce the algebraic group

$$\mathbb{S} := \{\text{the restriction of scalars from } \mathbb{C} \text{ to } \mathbb{R} \text{ à la Weil of the group } \mathbb{G}_m\}.$$

By definition, the complex points of $\mathbb{S}$ correspond to pairs of points $z, z' \in \mathbb{C}^*$. The point z corresponds to the standard embedding $\mathbb{C}^\times \hookrightarrow \mathbb{C}$ while z' corresponds to the complex conjugate embedding. Hence complex conjugation sends (z, z') to $(\bar{z}', \bar{z})$ and the real points $\mathbb{S}(\mathbb{R})$ consists of $\mathbb{C}^\times$ embedded into the group $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ of complex points through $a \mapsto (a, \bar{a})$. So $\mathbb{S}$ is just the group $\mathbb{C}^\times$ considered as a *real* algebraic group.

Note that there is a natural embedding $w : \mathbb{G}_m \to \mathbb{S}$ of algebraic groups which on complex points is the diagonal embedding $a \mapsto (a, a)$ and on real points is the embedding of $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$. Note that $\mathbb{C}^\times = \mathbb{R}^\times \cdot S^1$ where S^1 are the real points of the unitary group $U(1)$. We can extend the embedding $S^1 \hookrightarrow \mathbb{C}^\times$ to an embedding $U(1) \hookrightarrow \mathbb{S}$ and then

$$\mathbb{S} = U(1) \cdot w(\mathbb{G}_m).$$

Definition 2.6. A **complex Hodge structure on a complex vector space** W is a representation of $\mathbb{S}(\mathbb{C})$ on W. This amounts to a bigrading

$$W = \bigoplus_{p,q} W^{p,q}, \quad W^{p,q} = \{w \in W \mid (a,b)w = a^{-p}b^{-q}w, \ (a,b) \in \mathbb{S}(\mathbb{C})\}.$$

Now suppose that $W = V_{\mathbb{C}}$, where V is a real vector space. Then the above representation is a real representation if and only if the action of $\mathbb{S}(\mathbb{C})$ on the complex conjugate of any of the above summands is the summand on which the action is the conjugate action. This means precisely that the complex conjugate of $W^{p,q}$ is $W^{q,p}$. Looking at the action of the subgroup $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^{\times}$ we obtain the decomposition of V into weight spaces

$$V^{(k)} = \{v \in V \mid av = a^{-k}v, \ a \in \mathbb{R}^{\times}\},$$

i.e. $V^{(k)}$ is a real Hodge structure of weight k. If the representation is defined over a subring R of $\mathbb{R}$, these are weight k R-Hodge structures and conversely.

Suppose that we only have an $U(1)$-action on V. Then W splits into eigenspaces W^{ℓ} on which u acts via the character u^{ℓ}. Again W^{ℓ} is the conjugate of $W^{-\ell}$ and we would have a weight k Hodge structure if we declare its weight to be k: just put $W^{p,q} = W^{k-2q} = W^{-k+2p}$. Conversely, a real Hodge structure of weight k is an $U(1)$-action on W defined over $\mathbb{R}$ plus the specification of the number k. In fact, the argument shows:

Lemma 2.7. *let V_R be an R-module of finite rank. Then V_R admits the structure of an R-Hodge structure if and only if there is a homomorphism*

$$h : \mathbb{S} \to \mathrm{GL}(V \otimes_R \mathbb{R})$$

defined over $\mathbb{R}$, such that $h \circ w : \mathbb{G}_m \to \mathrm{GL}(V \otimes_R \mathbb{R})$ is defined over R.

Equivalently, an R-Hodge structure consists of an R-space V_R equipped with an action of $U(1)$ defined over R.

As an example, consider the one-dimensional Hodge structures. These are exactly the Hodge structures of Tate. The group $U(1)$ acts trivially on these. So the action of $U(1)$ defined by a Hodge structure F on V is the same as the one given by $F(\ell)$ on $V(\ell)$. This illustrates the fact that $\mathbb{S} = \mathbb{G}_m \cdot U(1)$ where the action of the subgroup $\mathbb{G}_m$ registers the weight and this gives another interpretation of the preceding weight shift as the multiplication with a character of $\mathbb{G}_m$. In this setting we have the **Weil operator**

$$C|W^{p,q} = \mathrm{i}^{p-q}, \tag{II–7}$$

the image of $\mathrm{i} \in \mathbb{S}(\mathbb{R})$ under the representation (recall that i is identified with $(\mathrm{i}, -\mathrm{i}) \in \mathbb{S}(\mathbb{C})$).

Recall the construction of the Grothendieck group (Def. A.4) 3). It is defined for any abelian category such as the category $\mathfrak{hs}_R$ of R-Hodge structures:

it is the free group on the isomorphism classes $[V]$ of Hodge structures V modulo the subgroup generated by $[V] - [V'] - [V'']$ where $0 \to V' \to V \to V'' \to 0$ is an exact sequence of R-Hodge structures. It carries a *ring structure* coming from the tensor product. Because the Hodge number polynomial (II–1) is clearly additive and by (II–3) behaves well on products, we have:

Lemma 2.8. *The Hodge number polynomial defines a ring homomorphism*

$$P_{\mathrm{hn}} : K_0(\mathfrak{hs}_R) \to \mathbb{Z}[u, v, u^{-1}, v^{-1}].$$

Inside $K_0(\mathfrak{hs}_R)$ Tate twisting r-times can be expressed as $[H] \mapsto [H] \cdot \mathbb{L}^{-r}$ where

$$\mathbb{L} = H^2(\mathbb{P}^1) \in K_0(\mathfrak{hs}_R). \tag{II–8}$$

2.1.2 Polarized Hodge Structures

The classical example of polarized Hodge structures is given by the primitive cohomology groups on a compact Kähler manifold (X, ω). If the Kähler class $[\omega]$ belongs to $H^2(X; R)$ for some subring R of $\mathbb{R}$, the Hodge-Riemann form Q restricts to an R-valued form on

$$H_R = H^k_{\mathrm{prim}}(X; R) := \mathrm{Im}\left[H^k(X; R) \to H^k(X; \mathbb{C})\right] \cap H^k_{\mathrm{prim}}(X)$$

where the homomorphism is the coefficient homomorphism. Recall the Hodge-Riemann bilinear relations with respect to the Hodge-Riemann form Q (see Definition 1.33). The first of these relations states that the primitive (p, q)-classes are Q-orthogonal to (r, s)-classes as long as $(p, q) \neq (s, r)$. This can be conveniently reformulated in terms of the Hodge filtration $F^m = \bigoplus_{p \geq m} H^{p,q}_{\mathrm{prim}}$ as follows. Note that F^m is Q-orthogonal to F^{k-m+1} since in the latter only (r, s)-forms occur with $r \geq k - m + 1$ while in the first (p, q)-forms occur with $q \leq k - m$. The dimension of F^m being complementary to $\dim F^{k-m+1}$, we therefore have that the Q-orthogonal complement of F^m equals F^{k-m+1}.

The second Hodge-Riemann relation can be reformulated using the Weil operator C, which – as we saw before (II–7)– acts as multiplication by i^{p-q} on (p, q)-forms. We find that in writing $\mathrm{i}^{p-q}Q(u, \bar{u}) = Q(Cu, \bar{u})$, u a primitive (p, q)-form, the right hand side makes sense for any k-form. In this way we arrive at the following

Definition 2.9. A **polarization** of an R-Hodge structure V of weight k is an R-valued bilinear form

$$Q : V \otimes V \longrightarrow R$$

which is $(-1)^k$-symmetric and such that

1) The orthogonal complement of F^m is F^{k-m+1};

2) The hermitian form on $V \otimes \mathbb{C}$ given by

$$Q(Cu, \bar{v})$$

is positive-definite.

Any R-Hodge structure that admits a polarization is said to be **polarizable**.

Example 2.10. The m-th cohomology of a compact Kähler manifold is an integral Hodge structure of weight m. If R is a *field*, this Hodge structure is R-polarizable if there exists a Kähler class in $H^2(X; R)$. In fact, since R is a field, the Lefschetz decomposition (I–12) yields a direct splitting of Hodge structures

$$H^m(X; R) \simeq \bigoplus_{r \geq (k-n)_+} H^{m-2r}_{\mathrm{prim}}(X; R)(-r)$$

and each of the summands carries a polarization. The Tate twist arises naturally: instead of the Kähler class we take $1/(2\pi \mathrm{i})$ times this class, which is represented by the curvature form (Def. B.39) of the Kähler metric. It belongs to $H^2(X; R)(-1)$ and cup product with it defines the modified Lefschetz-operator, say $\tilde{L} : H^k(X; R) \to H^{k+2}(X; R)(-1)$. To have a polarization on all of $H^m(X; R)$ we demand that the direct sum splitting be orthogonal and we *change signs on the summands* (see [Weil, p. 77]):

$$Q(\textstyle\sum_r L^r a_r, \sum_s L^s b_s) := \epsilon(k) \sum_r (-1)^r \int_X \tilde{L}^{n-m+2r}(a_r \wedge b_r),$$
$$a_r, b_r \quad \in \ H^{m-2r}(X; R).$$

Now there is a particularly concise reformulation of Definition 2.9 if we consider $S = (2\pi \mathrm{i})^{-k} Q$ as a *morphism* of Hodge structures $V \otimes V \to R(-k)$. Since $F^m(V \otimes V) = \sum_{r+s=m} F^r V \otimes F^s V$, this demand is equivalent to the first relation. For the second, note that it follows as soon as we know that the real-valued symmetric form $Q(Cu, v)$ is positive definite on the *real* primitive cohomology. This then leads to the following

Definition 2.9 (bis). A **polarization** of an R-Hodge structure V of weight k is a homomorphism of Hodge structures

$$S : V \otimes V \longrightarrow R(-k)$$

which is $(-1)^k$-symmetric and such that the real-valued symmetric bilinear form

$$Q(u, v) := (2\pi \mathrm{i})^k S(Cu, v) \tag{II–9}$$

is positive-definite on $V \otimes_R \mathbb{R}$.

Corollary 2.11. *Let V be an R-polarizable weight k Hodge structure. Any choice of a polarization on V induces an isomorphism R-Hodge structures $V \xrightarrow{\sim} V^{\vee}(-k)$ of weight k.*

We finish this section with an important principle:

Corollary 2.12 (SEMI-SIMPLICITY). *Let (V, Q) be an R-polarized Hodge structure and let W be a Hodge substructure. Then the form Q restricts to an R-polarization on W. Its orthogonal complement $W^\perp$ likewise inherits the structure of an R-polarized Hodge structure and V decomposes into an orthogonal direct sum $V = W \oplus W^\perp$. Hence, the category of R-polarized Hodge structures is semi-simple.*

Proof. Since W is stable under the action of the Weil operator, the form S given by (II–9) restricts to a positive definite form on $W \otimes_R \mathbb{R}$ and so we have an orthogonal sum decomposition as stated.

2.2 Mumford-Tate Groups of Hodge Structures

In this section (V, F) denotes a finite dimensional $\mathbb{Q}$-Hodge structure of weight k. We have seen in § 2.1 that this means that we have a homomorphism

$$h_F : \mathbb{S} \to \mathrm{GL}(V)$$

of algebraic groups such that $t \in w(\mathbb{G}_m(\mathbb{R}))$ acts as $v \mapsto t^{-k}v$. Recall also that $\mathbb{S} = \mathrm{U}(1) \cdot w(\mathbb{G}_m)$. Restricting h_F to the subgroup $\mathrm{U}(1)$ gives the homomorphism of algebraic groups

$$h_F|\mathrm{U}(1) : \mathrm{U}(1) \to \mathrm{GL}(V).$$

The group $\mathbb{S}$ has two characters z and $\bar{z}$ which on complex points $\mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*$ correspond to the two projections and hence on $\mathbb{S}(\mathbb{R})$ give the identity, respectively the complex conjugation, which explains the notation.

Definition 2.13. 1) The **Mumford-Tate group** $\mathrm{MT}(V, F)$ of the Hodge structure (V, F) is the Zariski-closure of the image of h_F in $\mathrm{GL}(V)$ over $\mathbb{Q}$, i.e. the smallest algebraic subgroup G of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that $G(\mathbb{C})$ contains $h_F(\mathbb{S}(\mathbb{C}))$.
  2) The **extended Mumford-Tate group** $\widetilde{\mathrm{MT}}(V, F)$ is the Zariski-closure of the image of $[h_F \times z]$ in $\mathrm{GL}(V) \times \mathbb{G}_m$, i.e. the smallest subgroup $\tilde{G}$ of $\mathrm{GL}(V) \times \mathbb{G}_m$ defined over $\mathbb{Q}$ and such that $\tilde{G}(\mathbb{C})$ contains $(h_F \times z)\mathbb{S}(\mathbb{C})$.
  3) The **Hodge group** or **special Mumford-Tate group** $\mathrm{HG}(V, F)$ is the Zariski-closure of the image of $h_F|\mathrm{U}(1)$.

Remark 2.14. Projection onto the first factor identifies $\widetilde{MT}(V, F)$ up to isogeny with $\mathrm{MT}(V, F)$, unless V has weight 0 and then it equals $\mathrm{MT}(V, F) \times \mathbb{G}_m$. As an illustration, consider $V = \mathbb{Q}(p)$. Then for $(u, v) \in \mathbb{C}^* \times \mathbb{C}^* = \mathbb{S}(\mathbb{C})$, $h_F(u, v)t = (uv)^{-p}t$ and the extended Mumford-Tate group equals $\mathbb{G}_m$ embedded in $\mathbb{G}_m \times \mathbb{G}_m$ via $u \mapsto (u^{-2p}, u)$ where the situation with respect to projection onto the first factor differs for the cases $p = 0$ and $p \neq 0$.

To have a more practical way of determining the Mumford-Tate group, we use as a motivation that all representations of $\mathrm{GL}(V)$ can be found from looking at the induced action on tensors

$$T^{m,n}V = V^{\otimes m} \otimes (V^\vee)^{\otimes n}.$$

Indeed, this is a property of reductive algebraic groups as we shall see below. Together with the action of $\mathbb{G}_m$ on the Hodge structure of Tate $\mathbb{Q}(p)$ this defines a natural action of $\mathrm{GL}(V) \times \mathbb{G}_m$ on $T^{m,n}V(p)$ and hence an action of the Mumford-Tate group $\widetilde{\mathrm{MT}}(V,F)$ on $T^{m,n}V(p)$. The induced Hodge structure on $T^{m,n}V(p)$ has weight $(m-n)k - 2p$. Assume it is even, say $w = 2q$. Then $\mathrm{HG}(V,F)$ acts trivially on Hodge vectors (i.e. rational type (q,q)-vectors) inside $T^{m,n}V(p)$, while any $t \in w(\mathbb{G}_m(\mathbb{R}))$ multiplies an element in $T^{m,n}V(p)$ of pure type (q,q) by $|t|^{2q}$. Hence, if the weight of $T^{m,n}V(p)$ is zero, the Hodge vectors inside $T^{m,n}V(p)$ are fixed by the entire Mumford-Tate group. The content of the following theorem is the main result of this section.

Theorem 2.15. *The Mumford-Tate group $\widetilde{\mathrm{MT}}(V,F)$ is exactly the (largest) algebraic subgroup of $\mathrm{GL}(V) \times \mathbb{G}_m$ which fixes all Hodge vectors inside $T^{m,n}V(p)$ for all (m,n,p) such that $(m-n)k - 2p = 0$. The Hodge group is the subgroup of $\mathrm{GL}(V)$ which fixes all Hodge vectors in all tensor representations $T^{m,n}V$.*

Before embarking on the proof let us recall that an algebraic group is **reductive** if it is the product of an algebraic torus and a (Zariski-connected) semi-simple group, both of which are normal subgroups. A group is **semi-simple** if it has no closed connected commutative normal subgroups except the identity. The groups $\mathrm{SL}(n), \mathrm{SO}(n), \mathrm{SU}(n), \mathrm{Sp}(n)$ are examples of semi-simple groups. The group $\mathrm{GL}(n)$ itself is reductive. In the sequel we use at several points (see [Sata80, I.3]:

Theorem 2.16. *An algebraic group over a field of characteristic zero is reductive if and only if all its finite-dimensional representations decompose into a direct product of irreducible ones.*

We need now a general result about the behaviour of tensor representations for reductive groups G with respect to algebraic subgroups H. For simplicity, assume that $G \subset \mathrm{GL}(V)$. and consider $T^{m,n}V$ as a G-representation. For any subgroup H of G, the set of vectors inside $T^{m,n}V$ fixed by H is as usual denoted by $(T^{m,n}V)^H$. We then put

$$\tilde{H} := \{g \in G \mid \text{there is some } (m,n) \text{ such that } g|(T^{m,n}V)^H = \mathrm{id}\}.$$

If g fixes $(T^{m,n})^H$ and g' fixes $(T^{m',n'}V)^H$, then gg'^{-1} fixes $(T^{m-m',n-n'}V)^H$ so that $\tilde{H}$ is a subgroup of G. This group obviously contains H and we want to know when the two groups coincide. This is the criterion:

Lemma 2.17. *In the above notation $H = \tilde{H}$ if H is reductive or if every character of H lifts to a character of G.*

Proof. The crucial remark is that any representation of G is contained in a direct sum of representations of type $T^{m,n}V$ (see [DMOS, I, Prop 3.1]). Also, by Chevalley's theorem (loc. cit.) the subgroup H is the stabilizer of a line L in some finite dimensional representation V, which we may assume to be such a direct sum. If H is reductive, $V = V' \oplus L$ for some H-stable V' and $V^\vee = (V')^\vee \oplus L^\vee$ so that H is exactly the group fixing a generator of $L \otimes L^\vee$ in $V \otimes V^\vee$ and so $H = \tilde{H}$. If all characters of H extend to G, the one-dimensional representation of H given by L comes from a representation of G. Then H is the group fixing a generator of $L \otimes L^\vee$ inside $V \otimes V^\vee$, a tensor representation of the desired type. $\quad\square$

Proof (of the Theorem): We apply the preceding with $G = \mathrm{GL}(V) \times \mathbb{G}_m$ and H the extended Mumford-Tate group. By definition, the largest algebraic subgroup of $\mathrm{GL}(V) \times \mathbb{G}_m$ which fixes all Hodge vectors inside $T^{m,n}V(p)$, $(m-n)k - 2p = 0$ is the group $\tilde{H}$. We must show that $\tilde{H} = H$. To do this, we use the criterion that any rationally defined character $\chi : \mathrm{MT}(V) \to \mathbb{G}_m$ should extend to all of $\mathrm{GL}(V) \times \mathbb{G}_m$. Look at the restriction of this character to the diagonal matrices $\mathbb{G}_m \subset \mathrm{MT}(V, F)$. By Example 2.2 2), it defines a Hodge structure of Tate $\mathbb{Q}(k)$ and so, after twisting W by $\mathbb{Q}(-k)$ the character becomes trivial and so extends to $\mathrm{GL}(V) \times \mathbb{G}_m$ as the trivial character. Then also the original character extends to $\mathrm{GL}(V) \times \mathbb{G}_m$. $\quad\square$

The importance of the previous theorem stems from the following

Observation 2.18. *The rational Hodge substructures of $T^{m,n}V$ are exactly the rational sub-representations of the Mumford-Tate group acting on $T^{m,n}V$.*

Proof. Suppose that $W \subset T^{m,n}V$ is a rational sub-representation of the Mumford-Tate group. Then the composition $h : \mathbb{S} \hookrightarrow \mathrm{MT}(V, F) \to \mathrm{GL}(W)$ defines a rational Hodge structure on W. The converse can be seen in a similar fashion. $\quad\square$

Next, suppose that we have a *polarized* Hodge structure. Almost by definition of a polarization (Def. 2.9-bis) the Hodge group preserves the polarization: for all $t \in \mathrm{U}(1)$ and $u, v \in V$ one has $S(t \cdot u, t \cdot v) = S(u, v)$. Using this one shows:

Theorem 2.19. *The Mumford-Tate group of a Hodge structure which admits a polarization is a reductive algebraic group.*

Proof. It suffices to prove this for the Hodge group. The Weil element $C = h_F(i)$ is a real point of this group. The square acts as $(-1)^k$ on V and hence lies in the centre of $\mathrm{MT}(V, F)$. The inner automorphism $\sigma := \mathrm{ad}(C)$ of $\mathrm{HG}(V, F)$ defined by C is therefore an involution. Such an involution defines a real-form

G_σ of the special Mumford-Tate group. By definition this is the real algebraic group G_σ whose real points are

$$G_\sigma(\mathbb{R}) = \{g \in \mathrm{HG}(V, F)(\mathbb{C}) \mid \sigma(g) = g\}.$$

There is an isomorphism

$$G_\sigma(\mathbb{C}) \overset{\cong}{\longrightarrow} \mathrm{HG}(V, F)(\mathbb{C})$$

such that complex conjugation on $G_\sigma(\mathbb{C})$ followed by σ corresponds to complex conjugation on $\mathrm{HG}(V, F)(\mathbb{C})$. This means that

$$\sigma(\bar{g}) = \mathrm{ad}(C)(\bar{g}) = g. \tag{II--10}$$

If the Hodge structure (V, F) admits a polarization Q, the following computation shows that G_σ admits a positive definite form and hence is compact. For $u, v \in V_\mathbb{C}$ and $g \in \mathrm{HG}(V, F)(\mathbb{C})$ we have, applying (II–10)

$$Q(Cu, \bar{v}) = (\bar{g}Cu, \bar{g}\bar{v}) = Q(CC^{-1}\bar{g}Cu) = Q(C\,\mathrm{ad}(C)(\bar{g})u, \overline{gv}) = Q(Cgu, \overline{gv}).$$

It follows that the positive definite form on $V_\mathbb{R}$ given by $Q(C-, -)$ is invariant under G_σ.

The compactness of G_σ implies that any finite dimensional representation of it completely decomposes into a direct product of irreducible ones and so, by the characterization of reductive groups, G_σ and also the special Mumford-Tate group is reductive. $\square$

Since $\mathrm{MT}(V, F)$ is the product of the Hodge group and the diagonal matrices and since a group is semi-simple if and only if the identity is the only normal closed connected abelian subgroup, the previous theorem implies:

Corollary 2.20. *The Hodge group is semi-simple precisely when the centre of the Mumford Tate group consists of the scalar matrices.*

2.3 Hodge Filtration and Hodge Complexes

2.3.1 Hodge to De Rham Spectral Sequence

Recall (Theorem 1.8) that for a Kähler manifold X, we have a Hodge decomposition and an associated Hodge filtration

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{r,s}(X), \quad F^p H^k(X, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X).$$

Let us first explain how to define a putative Hodge filtration on De Rham cohomology of any compact complex manifold X in terms of a spectral sequence relating the holomorphic and differentiable aspects. First embed the holomorphic De Rham complex into the complexified De Rham complex

$$\Omega_X^\bullet \xrightarrow{j} \mathcal{E}_X^\bullet(\mathbb{C}).$$

The decomposition into types of the sheaf complex $\mathcal{E}^\bullet(\mathbb{C})$ gives the filtered complex

$$F^p(\mathcal{E}^\bullet(\mathbb{C})) = \bigoplus_{r \geq p} \mathcal{E}_X^{r, \bullet - r}$$

and the homomorphism j becomes a filtered homomorphism provided we put the trivial filtration

$$\sigma^{\geq p} \Omega_X^\bullet = \{0 \to \cdots \to 0 \to \Omega_X^p \to \Omega_X^{p+1} \cdots \to \Omega_X^n\} \quad (n = \dim X).$$

on the De Rham complex. Then $\mathrm{Gr}^p(j)$ gives the Dolbeault complex

$$0 \to \Omega_X^p \to \mathcal{E}_X^{p,0} \xrightarrow{\bar\partial} \to \mathcal{E}_X^{p,1} \xrightarrow{\bar\partial} \cdots$$

By Dolbeault's lemma this is exact and so j induces a quasi-isomorphism on the level of graded complexes. So the E_1-terms of the first spectral sequence, which computes the hypercohomology of the graded complex (see equation (A–29)) is just the De Rham-cohomology of the preceding complex, i.e. $'E_1^{p,q} = H^q(X, \Omega_X^p)$. The first spectral sequence of hypercohomology (viewed as coming from the trivial filtration) reads therefore

$$'E_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet) = H_{\mathrm{DR}}^{p+q}(X; \mathbb{C})$$
$$(\textbf{Hodge to De Rham spectral sequence}).$$

Consider now the filtration on the abutment:

Definition 2.21. The **putative Hodge filtration** on $H_{\mathrm{DR}}^k(X; \mathbb{C})$ is given by
$$F^p H_{\mathrm{DR}}^k(X; \mathbb{C}) = \mathrm{Im}\left(\mathbb{H}^k(X, \sigma^{\geq p}\Omega^\bullet) \xrightarrow{\alpha_p} \mathbb{H}^k(X, \Omega^\bullet)\right).$$

The **Hodge subspaces** are given by

$$H^{p,q}(X) = F^p H_{\mathrm{DR}}^{p+q}(X; \mathbb{C}) \cap \overline{F^q H_{\mathrm{DR}}^{p+q}(X; \mathbb{C})}.$$

The terminology is justified by considering a Kähler manifold.

Proposition 2.22. *Let X be a compact Kähler manifold. Then the Hodge to De Rham spectral sequence degenerates at E_1; the putative Hodge filtration coincides with the actual Hodge filtration, and the Hodge subspaces $H^{p,q}(X)$ coincide with the subspace of the De Rham classes having a harmonic representative of type (p, q).*

Proof. As seen before (see the discussion following Theorem B.18), we have a canonical isomorphism $H^{r,s}(X) \cong H^s(X, \Omega_X^r)$ (Dolbeault's theorem) and so

$$\sum_{p+q=k} \dim {}'E_1^{p,q} = \sum_{p+q=k} \dim H^{p,q} = \dim H^k(X; \mathbb{C}) = \sum_{p+q=k} \dim E_\infty^{p,q}$$

which implies that the spectral sequence degenerates at E_1 (since E_{r+1} is a subquotient of E_r). Hence the map α_p is injective and $h^{p,k-p}(X) = \dim \mathbb{H}^k(X, \sigma^{\geq p}\Omega^\bullet)) - \dim \mathbb{H}^k(X, \sigma^{\geq p+1}\Omega^\bullet))$ and so

$$\dim \mathbb{H}^k(X, \sigma^{\geq p}\Omega^\bullet)) = \sum_{r \geq p} \dim H^{r,k-r}(X) = \dim F^p H^k(X;\mathbb{C})$$

which means that the image of j_p^* is $F^p H^k(X;\mathbb{C})$. Also $\mathrm{Gr}^p(j)$ induces an isomorphism $H^q(\Omega_X^p) \to H^{p,q}(X)$ and so

$$F^p H^k(X;\mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X). \square$$

Remark 2.23. The proof of the degeneration of the Hodge to De Rham spectral sequence hints at an algebraic approach to the Hodge decomposition. In fact Faltings [Falt] and Deligne-Illusie [Del-Ill] found a purely algebraic proof for the degeneracy of the Hodge to De Rham which works in any characteristic. The De Rham cohomology in this setting by definition is the hypercohomology of the algebraic De Rham complex, the algebraic variant of the holomorphic De Rham complex. The Hodge filtration is again induced by the trivial filtration on the De Rham complex. The proof then proceeds by first showing it first in characteristic p for smooth varieties of dimension $> p$ which can be lifted to the ring of Witt vectors of length 2. Since this can be arranged for if the variety is obtained from a variety in characteristic zero by reduction modulo p the result then follows in characteristic zero. In passing we note that there are many examples of surfaces in characteristic p for which the Hodge to De Rham spectral sequence does *not* degenerate. See [Del-Ill, 2.6 and 2.10] for a bibliography.

2.3.2 Strong Hodge Decompositions

Since by Corollary 1.10 the space $H^{p,q}(X)$ can be characterized as the subspace of $H_{\mathrm{DR}}^{p+q}(X;\mathbb{C})$ of classes representable by closed (p,q)-forms, the previous proposition motivates the following definition.

Definition 2.24. Let X be a compact complex manifold. We say that $H^k(X;\mathbb{C})$ admits a **Hodge decomposition in the strong sense** if

1) For all p and q with $p + q = k$ the Hodge (p,q)-subspace $H^{p,q}(X)$ as defined above can be identified with the subspace of $H^k(X;\mathbb{C})$ consisting of classes representable by closed forms of type (p,q). The resulting map

$$H^{p,q}(X) \to H_{\bar{\partial}}^{p,q}(X) \cong H^q(\Omega_X^p)$$

is required to be an isomorphism.

2) There is direct decomposition

$$H_{\mathrm{DR}}^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

3) The natural morphism from Bott-Chern cohomology to De Rham cohomology

$$H_{\mathrm{BC}}^{p,q}(X) = \frac{d\text{-closed forms of type } (p,q)}{\partial\bar\partial\Gamma\mathcal{E}_X^{p-1,q-1}} \to H_{\mathrm{DR}}^{p+q}(X)\otimes\mathbb{C}$$

which sends the class of a d-closed (p,q)-form to its De Rham class is injective with image $H^{p,q}(X)$.

Example 2.25. For any compact Kähler manifold X the Hodge decomposition on $H^k(X;\mathbb{C})$ is a Hodge decomposition in the strong sense.

By definition, the graded pieces of the putative Hodge filtration sequence give the $'E_\infty$-terms of the spectral sequence. If the Hodge to De Rham spectral sequence degenerates at E_1 it follows therefore that these graded pieces are canonically isomorphic to the Dolbeault groups. It does *not* imply that the putative Hodge filtration defines a Hodge structure on the De Rham groups. It is for instance not true in general that the graded pieces are isomorphic to the Hodge subspaces, even when the spectral sequence degenerates at E_1.

Example 2.26. As is well known (see e.g. [B-H-P-V, Chapter IV]), for surfaces the Hodge to De Rham spectral sequence, also called the Fröhlicher spectral sequence, degenerates at E_1 whereas there is no Hodge decomposition on $H^1(X)$ for a non-Kähler surface X since $b_1(X)$ is odd for those. This is for example the case of a Hopf surface which is the quotient of $\mathbb{C}^2 - \{0\}$ by the cyclic group of dilatations $z \to 2^k z$, $k \in \mathbb{Z}$. Such a surface is indeed diffeomorphic to $S^1 \times S^3$ and its first Betti number is 1 and so H^1 can never admit a Hodge decomposition. In fact the two Hodge subspaces are equal and hence equal to $F^1 = F^0$, the Dolbeault group $H^1(\mathcal{O}_X)$ maps isomorphically onto these, whereas the other Dolbeault group $H^0(\Omega_X)$ is zero and maps to $F^0/F^1 = 0$.

The following proposition summarizes what one can say in general. We first introduce some terminology. We say that a filtration F on a the complexification of a real vector space V is *k-transverse* if $F^p \cap \overline{F^{q+1}} = \{0\}$ whenever $p+q=k$. Note that this is automatic when F defines a real Hodge structure of weight k on V and a k-transverse filtration is a Hodge filtration if $\dim F^p + \dim F^{q+1} = \dim V$ whenever $p+q=k$.

Proposition 2.27. *Suppose that the Hodge to De Rham spectral sequence degenerates. Then the Dolbeault group $H^q(X,\Omega_X^p)$ is canonically isomorphic to $\mathrm{Gr}_F^p H_{\mathrm{DR}}^{p+q}(X;\mathbb{C})$ and one has the equality*

$$b_k := \dim H^k(X;\mathbb{C}) = \sum_{p+q=k} \dim H^q(\Omega_X^p).$$

Suppose that the putative Hodge filtration on $H^k(X;\mathbb{C})$ is k-transverse, and that it is $(2n-k)$-transverse on $H^{2n-k}(X;\mathbb{C})$. Then the putative Hodge filtrations on $H^k(X;\mathbb{C})$ and $H^{2n-k}(X;\mathbb{C})$ are both Hodge filtrations. For $p+q=k$ and $p+q=2n-k$ the spaces $H^q(X,\Omega_X^p) \cong \mathrm{Gr}_F^p H_{\mathrm{DR}}^{p+q}(X;\mathbb{C})$ get canonically identified with $H^{p,q}(X)$.

Proof. We only need to prove the statements about the putative Hodge filtration. For this, we provisionally set $h^{p,q} = \dim H^q(\Omega_X^p)$ so that $b_k = \sum_{p+q=k} h^{p,q}$. Now for any t we have $\dim F^t = \dim \overline{F^t} = \sum_{r \geq t} h^{r,k-r}$. The assumption on the putative Hodge filtration then implies

$$\sum_{r \geq p} h^{r,k-r} + \sum_{r \geq k-p+1} h^{r,k-r} \leq b_k = \sum_r h^{r,k-r}$$

and hence

$$\sum_{r \geq p} h^{r,k-r} \leq \sum_{r \leq k-p} h^{r,k-r}.$$

This inequality for $2n-k$-cohomology with p replaced by $n-k-p$, together with Serre duality ($h^{p,q} = h^{n-p,n-q}$) yields the reverse inequality. So we have equality and hence the dimensions of F^p and $\overline{F^{q+1}}$ add up to $\dim H^{p+q}(X;\mathbb{C})$ when $p+q=k$ or $p+q=2n-k$. So we get Hodge structures and $F^p H^k(X;\mathbb{C}) = \bigoplus_{r \geq p} H^{r,s}(X)$. Since $H^q(\Omega_X^p)$ is canonically isomorphic to $\mathrm{Gr}_F^p H^{p+q}(X;\mathbb{C}) \cong H^{p,q}(X)$, the last assertion follows as well.   □

In fact, we can even show that the assumptions of the preceding Proposition guarantee a Hodge decomposition in the strong sense on $H^k(X)$ and $H^{2m-k}(X)$. Indeed, we have the following statement which is an algebraic version of the $\partial\overline{\partial}$-Lemma (1.9). For a proof see [B-H-P-V, I, Lemma 13.6].

Corollary 2.28. 1) *Under the assumptions of Proposition 2.27, any cohomology class in degree k or in degree $(2n-k)$ can be represented by a form which is ∂- as well as $\overline{\partial}$-closed.*

2) *For a d-closed (p,q)-form α, $p+q=k$ or $p+q=2n-k$ the following statements are equivalent:*
  a) *$\alpha = d\beta$ for some $p+q-1$-form β;*
  b) *$\alpha = \overline{\partial}\beta''$ for some $(p,q-1)$-form β'';*
  c) *$\alpha = \partial\overline{\partial}\gamma$ for some $(p-1,q-1)$-form γ;*

4) *The natural morphism*

$$H_{\mathrm{BC}}^{p,q}(X) = \frac{d\text{-closed forms of type }(p,q)}{\partial\overline{\partial}\Gamma\mathcal{E}_X^{p-1,q-1}} \to H_{\mathrm{DR}}^{p+q}(X) \otimes \mathbb{C}$$

which sends the class of a d-closed (p,q)-form to its De Rham class is injective with image $H^{p,q}(X)$. In particular, the latter space consists precisely of the De Rham classes representable by a closed form of type (p,q).

5) *For $p + q = k$ or $p + q = 2n - k$ the natural map*

$$H^{p,q}(X) \to H^{p,q}_{\bar\partial}(X) \cong H^q(\Omega^p_X)$$

resulting from the identification of $H^{p,q}(X)$ as the space consisting of the De Rham classes representable by a closed form of type (p, q) is an isomorphism.

Despite the fact that holomorphic images of Kähler manifolds of the same dimension are not always Kähler [Hart70, p. 443] we can show:

Theorem 2.29. *Let X, Y be compact complex manifolds. Suppose that X is Kähler and that $f : X \to Y$ is a surjective holomorphic map. Then $H^k(Y)$ admits a Hodge decomposition in the strong sense. In fact $f^* : H^k(Y; \mathbb{R}) \to H^k(X; \mathbb{R})$ is injective and $f^* H^k(Y; \mathbb{R})$ is a real Hodge substructure of $H^k(X; \mathbb{R})$.*

Proof. We first show that f^* is injective. In fact, this holds for any surjective differentiable map $f : X \to Y$ between compact differentiable manifolds. To see this, first reduce to the equi-dimensional case by choosing a submanifold $Z \subset X$ to which f restricts as a generically finite map, say of degree d. With $f_!$ Poincaré dual to f_*, the composition $f_! \circ f^*$ is multiplication with d and so f^* is injective.

Next, we observe that for $m = \dim Y$, a generator of $H^{2m}(Y; \mathbb{C}) = H^{m,m}(Y) = H^m(\Omega^m_Y)$ is represented by the volume form vol_h with respect to some hermitian metric h on Y. If ω is the Kähler form on X, the form ω^c, $c = \dim X - \dim Y$ restricts to a volume form on the generic fibre F of f and hence

$$\int_X f^*(\mathrm{vol}_h) \wedge \omega^c = \int_Y \mathrm{vol}_h \int_F \omega^c \neq 0.$$

So $f^* : H^m(\Omega^m_Y) \to H^m(\Omega^m_X)$ is non-zero. Now one uses Serre duality to prove injectivity on $H^q(\Omega^p_Y)$ for all p and q. Indeed, given any non-zero class $\alpha \in H^q(\Omega^p_Y)$ choose $\beta \in H^{m-q}(\Omega^{m-p}_Y)$ such that $\alpha \wedge \beta \neq 0$. Then $f^*\alpha \wedge f^*\beta = f^*(\alpha \wedge \beta) \neq 0$ and hence $f^*\alpha \neq 0$.

Now compare the Hodge to De Rham spectral sequence for Y with that for X. What we just said shows that the E_1-term of the first injects into the E_1-term of the latter. For X the Hodge to De Rham spectral sequence degenerates and so $d_r = 0, r \geq 1$ and $E_1 = E_2 = \cdots$. It follows recursively that the same holds for the Hodge to De Rham spectral sequence for Y. In particular, it degenerates. But more is true. The map f^* on the level of spectral sequences induces an injection $F^p H^k(Y) \hookrightarrow F^p H^k(X)$ and since f^* commutes with complex conjugation, we conclude that $F^p H^k(Y)$ meets $\overline{F^{k-p+1} H^k(Y; \mathbb{C})}$ only in $\{0\}$ and so the hypothesis of Prop. 2.27 is satisfied and the result follows upon applying Corollary 2.28.   □

By Hironaka's theorem [Hir64] the indeterminacy locus of a meromorphic map $X \dashrightarrow Y$ can be eliminated by blowing up. Since the blow up of a Kähler

manifold is again Kähler (see [Kod54, Sect. 2, Lemma 1]) we can apply the previous theorem to a manifold bimeromorphic to a Kähler manifold.

Corollary 2.30. *Let X be a compact complex manifold bimeromorphic to a Kähler manifold. Then $H^k(X; \mathbb{C})$ admits a strong Hodge decomposition. This is in particular true for a (not necessarily projective) compact algebraic manifold. In particular, the previous theorem remains true when X is only bimeromorphic to a Kähler manifold.*

2.3.3 Hodge Complexes and Hodge Complexes of Sheaves

Comparison between complexes should take place in suitable derived categories. We prefer however to give explicit morphisms realizing these comparison morphisms. To fix ideas we introduce the following definitions.

Definition 2.31. Let $K^\bullet$, $L^\bullet$ two bounded below complexes in an abelian category. A **pseudo-morphism** between $K^\bullet$ and $L^\bullet$ is a chain of morphisms of complexes

$$K^\bullet \xrightarrow{f} K_1^\bullet \overset{\text{qis}}{\underset{\sim}{\longleftarrow}} K_2^\bullet \overset{\text{qis}}{\underset{\sim}{\longrightarrow}} \cdots \overset{\text{qis}}{\underset{\sim}{\longrightarrow}} K_{n+1}^\bullet = L^\bullet.$$

It induces a morphism in the derived category. We shall denote such a pseudo-morphism by

$$f : K^\bullet {\dashrightarrow} L^\bullet.$$

If also f is a quasi-isomorphism we speak of a **pseudo-isomorphism**. It becomes invertible in the derived category. We denote these by

$$f : K^\bullet \overset{\text{qis}}{\underset{\sim}{\dashrightarrow}} L^\bullet.$$

A **morphism** between two pseudo-morphisms $K^\bullet \xrightarrow{f} K_1^\bullet \overset{\text{qis}}{\underset{\sim}{\longleftarrow}} \cdots \overset{\text{qis}}{\underset{\sim}{\longrightarrow}} K_m^\bullet$ and $L^\bullet \xrightarrow{g} L_1^\bullet \overset{\text{qis}}{\underset{\sim}{\longleftarrow}} \cdots \overset{\text{qis}}{\underset{\sim}{\longrightarrow}} L_m^\bullet$ consists of a sequence of morphism $K^j \to L^j$, $j = 1, \ldots, m$ such that the obvious diagrams commute. Note that such morphisms are only possible between sequences of equal length.

Definition 2.32.   1) Let R a noetherian subring of $\mathbb{C}$ such that $R \otimes \mathbb{Q}$ is a field (mostly R will be $\mathbb{Z}$ or $\mathbb{Q}$). An R-**Hodge complex $K^\bullet$ of weight** m consist of
 – A bounded below complex of R-modules $K_R^\bullet$ such that the cohomology groups $H^k(K_R^\bullet)$ are R-modules of finite type,
 – A bounded below filtered complex $(K_{\mathbb{C}}^\bullet, F)$ of complex vector spaces with differential **strictly compatible** with F and a
 – **comparison morphism** $\alpha : K_R^\bullet {\dashrightarrow} K_{\mathbb{C}}^\bullet$, which is a pseudo-morphism in the category of bounded below complexes of R-modules and becomes a pseudo-isomorphism after tensoring with $\mathbb{C}$.

$$\alpha \otimes \mathrm{id} : K_R^\bullet \otimes \mathbb{C} \overset{\text{qis}}{\underset{\sim}{\dashrightarrow}} K_{\mathbb{C}}^\bullet,$$

and such that the induced filtration on $H^k(K_\mathbb{C}^\bullet)$ determines an R-Hodge structure of weight $m + k$ on $H^k(K_R^\bullet)$.

Its associated **Hodge-Grothendieck characteristic** is

$$\chi_{\mathrm{Hdg}}(K^\bullet) := \sum_{k \in \mathbb{Z}} (-1)^k \left[H^k(K^\bullet) \right] \in K_0(\mathfrak{hs}_R).$$

2) Let X be a topological space. An R-**Hodge complex of sheaves** of weight m on X consists of the following data

– A bounded below complex of sheaves of R-modules $\mathcal{K}_R^\bullet$ such that the hypercohomology groups $\mathbb{H}^k(X, \mathcal{K}^\bullet)$ are finitely generated as R-modules,

– A filtered complex of sheaves of complex vector spaces $\{\mathcal{K}_\mathbb{C}^\bullet, F\}$ and a pseudo-morphism $\alpha : \mathcal{K}_R^\bullet \dashrightarrow \mathcal{K}_\mathbb{C}^\bullet$ in the category of sheaves of R-modules on X inducing a pseudo-isomorphism (of sheaves of $\mathbb{C}$-vector spaces)

$$\alpha \otimes \mathrm{id} : \mathcal{K}_R^\bullet \otimes \mathbb{C} \overset{\mathrm{qis}}{\underset{\sim}{\dashrightarrow}} \mathcal{K}_\mathbb{C}^\bullet,$$

and such that the R-structure on $\mathbb{H}^k(\mathcal{K}_\mathbb{C}^\bullet)$ induced by α and the filtration induced by F determine an R-Hodge structure of weight $k + m$ for all k. Moreover, one requires that the spectral sequence for the derived complex $R\Gamma(X, \mathcal{K}_\mathbb{C}^\bullet)$ (see (B–12)) with the induced filtration

$$\mathbb{H}^{p+q}(X, \mathrm{Gr}_F^p \mathcal{K}_\mathbb{C}^\bullet) \Longrightarrow \mathbb{H}^{p+q}(X, \mathcal{K}_\mathbb{C}^\bullet)$$

degenerates at E_1 (by Lemma A.42 this is equivalent to saying that the differentials of the derived complex are strict).

3) A **morphism** of Hodge complexes (of sheaves) of weight m, consists of a triple$(h_R, h_\mathbb{C}, \kappa)$ where h_R is a morphism of (of sheaves of) R-modules, $h_\mathbb{C}$ a homomorphism of (sheaves of) $\mathbb{C}$-vector spaces and $\kappa : \alpha \to \beta$ is a morphism of pseudo-morphisms.

The notions of a Hodge complex and that of a Hodge complex of sheaves are related in the following way.

Proposition 2.33. *Given an R-Hodge complex of sheaves on X of weight m, say*

$$\mathcal{K}^\bullet = \left(\mathcal{K}_R^\bullet, (\mathcal{K}_\mathbb{C}^\bullet, F), \alpha \right),$$

any choice of representatives for the triple

$$R\Gamma \mathcal{K}^\bullet = \left(R\Gamma(\mathcal{K}_R^\bullet), (R\Gamma(\mathcal{K}_\mathbb{C}^\bullet, F), R\Gamma(\alpha) \right)$$

yields an R-Hodge complex. With $a_X : X \to \mathrm{pt}$ the constant map, we have

$$\chi_{\mathrm{Hdg}}(R\Gamma(\mathcal{K}^\bullet)) = [R(a_X)_* \mathcal{K}^\bullet] \in K_0(\mathfrak{hs}_R).$$

Here we view $R(a_X)_* \mathcal{K}^\bullet$, a complex of sheaves over the point pt, as a complex of R-modules whose (finite rank) cohomology groups $\mathbb{H}^k(X, \mathcal{K}^\bullet)$ are R-Hodge structures so that the right hand side makes sense in $K_0(\mathfrak{hs}_R)$.

Example 2.34. The existence of a strong Hodge decomposition for Kähler manifolds (Example 2.25) in fact tells us that for X a compact Kähler manifold, the constant sheaf $\underline{\mathbb{Z}}_X$, the holomorphic De Rham complex $\varOmega_X^\bullet$ with the trivial filtration σ together with the inclusion $\underline{\mathbb{Z}}_X \hookrightarrow \varOmega_X^\bullet$ (which gives the pseudo-isomorphism $\underline{\mathbb{C}}_X \to \varOmega_X^\bullet$) is an integral Hodge complex of sheaves of weight 0. The same is true for any complex manifold bimeromorphic to a Kähler manifold. This complex will be called the **Hodge-De Rham complex of sheaves** on X and be denoted by

$$\mathcal{H}dg^\bullet(X) = (\underline{\mathbb{Z}}_X, (\varOmega_X^\bullet, \sigma), \underline{\mathbb{Z}}_X \hookrightarrow \varOmega_X^\bullet).$$

Taking global sections on the Godement resolution gives $R\varGamma\mathcal{H}dg^\bullet(X)$, the canonically associated **De Rham complex** of X with Hodge-Grothendieck characteristic

$$\chi_{\mathrm{Hdg}}(X) = \sum_{k \in \mathbb{Z}} (-1)^k \left[H^k(X) \right] = [R(a_X)_* \underline{\mathbb{Z}}_X^{\mathrm{Hdg}}] \in K_0(\mathfrak{hs}). \quad \text{(II–11)}$$

Lemma-Definition 2.35. 1) For an R-Hodge complex of sheaves $\mathcal{K}^\bullet = (\mathcal{K}_R^\bullet, (\mathcal{K}_\mathbb{C}^\bullet, F), \alpha)$ of weight m, and $k \in \mathbb{Z}$ we define the k-**th Tate-twist** by

$$\mathcal{K}^\bullet(k) := (\mathcal{K}_R^\bullet \otimes \mathbb{Z}(2\pi\mathrm{i})^k, (\mathcal{K}_\mathbb{C}^\bullet, F[k]), \alpha \cdot (2\pi\mathrm{i})^k).$$

It is an R-Hodge sheaf of weight $m - 2k$. This operation induces the Tate-twist in hypercohomology

$$\mathbb{H}^\ell(X, \mathcal{K}_\mathbb{C}^\bullet(k)) = \mathbb{H}^\ell(X, \mathcal{K}_\mathbb{C}^\bullet)(k).$$

A similar definition holds for $K^\bullet(k)$ where $K^\bullet$ is an R-Hodge complex.

2) We define the **shifted complex** by

$$\mathcal{K}^\bullet[r] := (\mathcal{K}_R^\bullet[r], (\mathcal{K}_\mathbb{C}^\bullet[r], F[r]), \alpha[r]).$$

It is a Hodge complex of sheaves of weight $m + r$ A similar definition holds for $K^\bullet[r]$ where $K^\bullet$ is an R-Hodge complex.

2.4 Refined Fundamental Classes

We recall (Proposition 1.14) that for any irreducible subvariety Y of codimension d in a compact algebraic manifold X the integral fundamental class $\mathrm{cl}(Y) \in H^{2d}(X)$ has pure type (d, d). This means that the fundamental class belongs to the d-th Hodge filtration level. So we can also define a **fundamental Hodge cohomology class**

$$\mathrm{cl}_{\mathrm{Hdg}}(Y) \in F^d H^{2d}(X; \mathbb{C}) = \mathbb{H}^{2d}(X, F^d \varOmega_X^\bullet)$$

and the integral class maps to it under the inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$. To keep track of various powers of $2\pi i$ introduced when integrating forms, it is better to replace this inclusion by

$$\epsilon^d : \mathbb{Z}(d) \hookrightarrow \mathbb{C} \tag{II–12}$$

and we consider the fundamental class as a class $\mathrm{cl}(Y) \in H^{2d}(X, \mathbb{Z}(d))$ which, under ϵ^d, maps to the image of the Hodge class in $H^{2d}(X; \mathbb{C})$. This is summarized in the following diagram

$$
\begin{array}{ccc}
\mathrm{cl}_{\mathrm{Hdg}}(Y) & \longmapsto & \mathrm{cl}_{\mathbb{C}}(Y) \\
\Cap & & \Cap \ \uparrow \\
\mathbb{H}^{2d}(X, F^d \Omega_X^\bullet) \hookrightarrow \mathbb{H}^{2d}(X, \Omega_X^\bullet) = H^{2d}(X; \mathbb{C}) & & \\
& \uparrow {\scriptstyle (\epsilon^d)_*} & \\
& H^{2d}(X; \mathbb{Z}(d)) \ni \mathrm{cl}(Y) &
\end{array}
$$

Remark. There is a much more intrinsic reason to consider $\mathrm{cl}(Y)$ as a class inside $H^{2d}(X, \mathbb{Z}(d))$ rather than as an integral class. The reason is that the only algebraically defined resolution of $\mathbb{C}$ is the holomorphic De Rham complex $\Omega_X^\bullet$ and the only algebraically defined fundamental class is coming from Grothendieck's theory of Chern classes. To *algebraically* relate the first Chern class which is naturally living in $H^1(\mathcal{O}_X^*) = \mathbb{H}^2(X, 0 \to \mathcal{O}_X^* \to 0)$ to a class in $H^2(X, \mathbb{C}) = \mathbb{H}^2(X, \Omega_X^\bullet)$ one uses $d\log : \mathcal{O}_X^* \to \Omega_X^1$ and zero else. This misses out the factor $2\pi i$ which is inserted in the C^∞ De Rham theory. It follows that $\mathrm{cl}(Y)$ as defined in this way is no longer integral, but has values in $\mathbb{Z}(d)$. See [DMOS, I.1] where this is carefully explained. This remark becomes relevant when one wants to compare fundamental classes for algebraic varieties defined over fields $k \subset \mathbb{C}$ when one changes the embedding of k in $\mathbb{C}$.

Remark 2.36. Contininuing the preceding Remark, suppose that X is a nonsingular algebraic variety defined over a field k of finite transcendence degree over $\mathbb{Q}$. Any embedding $\sigma : k \hookrightarrow \mathbb{C}$ defines a complex manifold $X^{(\sigma)}$ and a codimension d cycle Z on X defines a fundamental class $\mathrm{cl}^{(\sigma)}(Z) \in H^{2d}(X^{(\sigma)}; \mathbb{C})$ which is rational in the sense that it belongs to $H^{2d}(X^{(\sigma)}; 2\pi i \mathbb{Q})$. On the other hand, we have the algebraic De Rham groups $H_{\mathrm{DR}}^m(X/k)$ which are k-spaces, they are the hypercohomology groups of the algebraic De Rham complex $\Omega_{X/k}^\bullet$. These compare to complex cohomology through a canonical comparison isomorphism

$$\iota_\sigma : H_{\mathrm{DR}}^m(X/k) \otimes_{\sigma,k} \mathbb{C} \xrightarrow{\sim} H^m(X^{(\sigma)}; \mathbb{C})$$

and under this isomorphism for $m = 2d$ the class $\mathrm{cl}(Z)$ on the right corresponds to a class

$$\mathrm{cl}_{\mathrm{B}}(Z) \in H_{\mathrm{DR}}^{2d}(X/k) \otimes (2\pi i)^d := H^{2d}(X)(d).$$

Then the class $\iota_\sigma \, \mathrm{cl}_{\mathrm{B}}(Z)$ is rational in the above sense. This motivates the definition of an absolute Hodge class:

Definition 2.37. Let X be a non-singular algebraic variety defined over a field k of finite transcendence degree over $\mathbb{Q}$. A class $\beta \in H^{2d}(X)(d)$ is **absolute Hodge** if for all embeddings $\sigma : k \hookrightarrow \mathbb{C}$ the image $\iota_\sigma(\beta) \in H^{2d}(X^{(\sigma)}; \mathbb{C})$ is rational.

If such a class β has the property that $\iota_\sigma(\beta)$ is rational for just one embedding we speak of a **Hodge class**. These come up in the Hodge conjecture 1.16 for a complex projective variety. To explain this, note that such a variety is of course defined over a given subfield k of $\mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$ and there is a preferred embedding $k \hookrightarrow \mathbb{C}$.

Deligne's "hope" is that like the algebraic cycle classes, all such Hodge classes are absolute Hodge. This has been verified only for abelian varieties [DMOS].

We now continue our study of refined cycle classes in the setting of local cohomology, the main result being as follows.

Theorem 2.38. *Let X be a compact algebraic manifold and let $Y \subset X$ be an irreducible d-dimensional subvariety. Then the following variants of interrelated fundamental classes exist:*

1) *There is a* **refined Thom class**

$$\tau_{\mathrm{Hdg}}(Y) \in \mathbb{H}_Y^{2d}(X, F^d \Omega_X^\bullet)$$

   whose image under the map $\mathbb{H}_Y^{2d}(X, F^d \Omega_X^\bullet) \to \mathbb{H}_Y^{2d}(X, \Omega_X^\bullet) = H_Y^{2d}(X; \mathbb{C})$ coincides with the image under the map (II–12) of the Thom class $\tau(Y) \in H_Y^{2d}(X, \mathbb{Z}(d))$.
2) *There is a class $\tau^{d,d} \in H_Y^d(X, \Omega_X^d)$ which is the projection of the refined Thom class.*
3) *Forgetting supports, the class $\tau_{\mathrm{Hdg}}(Y)$ maps to $\mathrm{cl}_{\mathrm{Hdg}}(Y)$.*
4) *The various classes in this construction are related as follows*

$$
\begin{array}{ccc}
\tau^{d,d}(Y)) \longleftarrow\!\shortmid\ \tau_{\mathrm{Hdg}}(Y) \longmapsto & \tau(Y) \\
\Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad & \Big\downarrow \\
\mathrm{cl}^{d,d}(Y) \longleftarrow\!\shortmid\ \mathrm{cl}_{\mathrm{Hdg}}(Y) \longmapsto & \mathrm{cl}_{\mathbb{C}}(Y)
\end{array}
$$

   and where the elements come from the commutative diagram

$$
\begin{array}{ccc}
H_Y^d(\Omega_Y^d) \longleftarrow & \mathbb{H}_Y^{2d}(X, F^d \Omega_X^\bullet) & \longrightarrow H_Y^{2d}(X; \mathbb{C}) \\
\Big\downarrow & \Big\downarrow & \Big\downarrow \\
H^d(\Omega_X^d) \longleftarrow \mathbb{H}^{2d}(X, F^d \Omega_X^\bullet) = F^d H^{2d}(X; \mathbb{C}) \longrightarrow & H^{2d}(X; \mathbb{C})
\end{array}
$$

We start with a localizing tool. Let $\mathcal{F}$ be any sheaf on X. The assignment $U \mapsto H_Y^k(U, \mathcal{F})$ defines a presheaf on X whose associated sheaf is denoted by $\mathcal{H}_Y^k(\mathcal{F})$. These sheaves are related to the local cohomology groups through a spectral sequence

$$E_2^{r,s} = H^r(X, H_Y^s(\mathcal{F})) \Longrightarrow H_Y^{r+s}(X, \mathcal{F}) \qquad \text{(II--13)}$$

which is the second spectral sequence associated to the functor of taking sections with support in Y.

Lemma 2.39. *Let X be a complex manifold, $Y \subset X$ a codimension c subvariety and $\mathcal{E}$ a locally free sheaf on X. Then*

1) *the cohomology sheaf satisfies*

$$H_Y^q(\mathcal{E}) = 0, \quad q < c;$$

2) *there is an isomorphism*

$$H_Y^c(X, \mathcal{E}) \xrightarrow{\sim} H^0(X, H_Y^c(\mathcal{E})).$$

Proof. For a proof of the first assertion see [S-T, Prop. 1.12]. The second assertion then follows from the spectral sequence (II--13). $\square$

We state a consequence for hypercohomology. We assume that we have a complex $\mathcal{K}^\bullet$ of locally free sheaves on X and we consider the first spectral sequence with respect to the trivial filtration $\sigma^{\geq p} = F^p$ for the functor of hypercohomology with supports in Y whose E_1-terms are

$$E_1^{q,r} = H_Y^q(X, F^s\mathcal{K}^r) \Longrightarrow \mathbb{H}_Y^{q+r}(F^s\mathcal{K}_X^\bullet), \quad F^s\mathcal{K}^r = \begin{cases} \mathcal{K}^r \text{ if } r \geq s \\ 0 \text{ if } r < s. \end{cases}$$

We find:

Corollary 2.40. *For a codimension c subvariety $Y \subset X$, we have*

$$\mathbb{H}_Y^m(X, F^s\mathcal{K}^\bullet) = 0, \quad m < s + c$$

and

$$\mathbb{H}_Y^{s+c}(X, F^s\mathcal{K}^\bullet) \cong H^0(X, H_Y^c(X, \mathcal{K}^s)).$$

Proof of Theorem 2.38. Step 1: Reduction to the case where Y is a smooth subvariety.

We let $Y_{\text{reg}}, Y_{\text{sing}}$ be the regular locus, respectively the singular locus of Y and we put

$$X^0 := X - Y_{\text{sing}}$$

Let us combine the usual exact sequences for cohomology with support together with the excision exact sequences (B--36) to a commutative diagram

$$
\begin{array}{ccccccc}
\mathbb{H}_{Y_{\text{sing}}}^{2d}(X, F^d\Omega_X^\bullet) & \to & \mathbb{H}_Y^{2d}(X, F^d\Omega_X^\bullet) & \to & \mathbb{H}_{Y_{\text{reg}}}^{2d}(X^0, F^d\Omega_X^\bullet) & \to & \mathbb{H}_{Y_{\text{sing}}}^{2d+1}(X, F^d\Omega_X^\bullet) \\
\| & & \downarrow r & & \downarrow r_{\text{reg}} & & \| \\
\mathbb{H}_{Y_{\text{sing}}}^{2d}(X, F^d\Omega_X^\bullet) & \to & \mathbb{H}^{2d}(X, F^d\Omega_X^\bullet) & \to & \mathbb{H}^{2d}(X^0, F^d\Omega_X^\bullet) & \to & \mathbb{H}_{Y_{\text{sing}}}^{2d+1}(X, F^d\Omega_X^\bullet).
\end{array}
$$

In this diagram the first terms on the left vanish by Prop. 2.40. So one can define a unique Hodge class $\mathrm{cl}_{\mathrm{Hdg}}(Y_{\mathrm{reg}}) \in \mathbb{H}^{2d}(X^0, F^d \Omega_X^\bullet)$ which comes from the Hodge class of the pair (X, Y). A diagram chase then shows that one can reduce the construction of a Thom class to the smooth case (X^0, Y_{reg}).

In what follows we are going to construct a refined Thom class for (X^0, Y_{reg}) which maps to the usual Thom class for this pair. This suffices to complete the proof, in view of the commutative diagram

$$
\begin{array}{ccc}
\mathbb{H}_{Y_{\mathrm{reg}}}^{2d}(X^0, F^d \Omega_X^\bullet) & \longrightarrow & \mathbb{H}_{Y_{\mathrm{reg}}}^{2d}(X^0, \Omega_{X^0}^\bullet) = H_{Y_{\mathrm{reg}}}^{2d}(X^0; \mathbb{C}) \\
\downarrow & & \downarrow \\
\mathbb{H}^{2d}(X^0, F^d \Omega_{X^0}^\bullet) & \longrightarrow & \mathbb{H}^{2d}(X^0, \Omega_{X^0}^\bullet) = H^{2d}(X^0; \mathbb{C})
\end{array}
$$

Step 2: Construction of $\tau^{d,d}(Y) \in H_Y^d(X, \Omega_X^d)$ for Y a complete intersection in a smooth (not necessarily compact) algebraic manifold X.

Let us cover X by Stein open sets $\{U_\alpha\}, \alpha \in I$. Suppose that $U_\alpha \cap Y$ is given by $f_\alpha^{(k)} = 0$, $k = 1, \ldots, d$. The open sets $U_\alpha^k := U_\alpha - \{f_\alpha^{(k)} = 0\}$, $k = 1, \ldots, d$ form an acyclic covering of $U_\alpha - Y \cap U_\alpha$. Consider the Čech $(d-1)$-cocycle

$$
(U_\alpha^1 \cap \cdots \cap U_\alpha^d) \longmapsto \eta_\alpha := \left[d \log f_\alpha^{(1)} \wedge \cdots \wedge d \log f_\alpha^{(d)} \right].
$$

If we take other equations it is easy to write down a $(d-2)$ co-chain whose coboundary gives the difference. Under the isomorphism

$$
H^{d-1}(U_\alpha - (Y \cap U_\alpha), \Omega_X^d) \xrightarrow{\sim} H_Y^d(U_\alpha, \Omega_X^d)
$$

its class maps to a class $c_\alpha \in H_Y^d(U_\alpha, \Omega_X^d)$ which is therefore independent of the choice of equations for Y. Hence the c_α patch together to a section of the sheaf $\mathcal{H}_Y^d(\Omega_Y^d)$. We then apply Lemma 2.39.

Step 3: Lifting of the class $\tau^{d,d}(Y)$ to a class $\tau_{\mathrm{Hodge}}(Y) \in \mathbb{H}_Y^{2d}(X, F^d \Omega_X^\bullet)$.

To do this, we consider the long exact sequence in hypercohomology with supports in Y associated to the exact sequence of complexes

$$
0 \to F^{d+1} \Omega_X^\bullet \to F^d \Omega_X^\bullet \to \Omega_X^d[-d] \to 0.
$$

It reads

$$
\begin{array}{ccccccc}
\mathbb{H}_Y^{2d}(X, F^{d+1} \Omega_X^\bullet) & \to & \mathbb{H}_Y^{2d}(X, F^d \Omega_X^\bullet) & \to & H_Y^d(X, \Omega_X^d) & \xrightarrow{\partial} & \mathbb{H}_Y^{2d+1}(X, F^{d+1} \Omega_X^\bullet) \\
\| & & & & & & \| \wr \\
0 & & & & & & H^0(X, \mathcal{H}_Y^d(X, \Omega_X^{d+1})).
\end{array}
$$

Here we use Cor. 2.40. It follows that to calculate $\partial(\tau^{d,d}(Y))$, it suffices to do this locally. We use the same notation as in the previous step. So $\partial \tau^{d,d}(Y)|_{U_\alpha}$ is represented by the co-cycle

$$
(U_\alpha^1 \cap \cdots \cap U_\alpha^d) \mapsto d\eta_\alpha = d \left[d \log f_\alpha^{(1)} \wedge \cdots \wedge d \log f_\alpha^{(d)} \right] = 0.
$$

So $\partial(\tau^{d,d}(Y)) = 0$ and there is a unique lift of this class to $\tau_{\text{Hodge}}(Y) \in \mathbb{H}^{2d}_Y(X, F^d \Omega^\bullet_X)$.

Step 4: Proof that the class $\tau_{\text{Hodge}}(Y) \in \mathbb{H}^{2d}_Y(X, F^d \Omega^\bullet_X)$ maps to the Thom class $\tau_{\mathbb{C}}(Y) \in H^{2d}_Y(X; \mathbb{C})$.

Recall (B.2.9) that Poincaré-duality implies that $\tau_{\mathbb{C}}(Y)$ generates local cohomology. Suppose that $\tau_{\text{Hodge}}(Y)$ maps to $m\tau_{\mathbb{C}}(Y)$. To show that $m = 1$ a local computation suffices. Hence, by functoriality, we can reduce to the case of the origin in $\mathbb{C}^d$. Again, by functoriality we can further restrict down to a complex line passing through the origin. Next, we look at the closed 1-form dz/z on $\mathbb{C} - \{0\}$. It defines a De Rham class in $H^1(\mathbb{C} - \{0\})$ which generates the first integral cohomology of $H^1(\mathbb{C} - \{0\})$ under the embedding $\epsilon : \mathbb{Z}(1) \to \mathbb{C}$. This is simply the residue formula. The corresponding image $\partial(dz/z) \in H^2_0(\mathbb{C})$ generates integral cohomology with support in 0. It follows that $m = 1$. $\square$

Remark 2.41. This construction also provides us with refined Thom classes for cycles $Y = \sum n_i Y_i$ of codimension d with support in $|Y| = \bigcup_i Y_i$. Indeed, one merely uses the isomorphism

$$H^{2d}_{|Y|}(X; F^d \Omega^\bullet_X) \cong \bigoplus_i H^{2d}_{Y_i}(X; F^d \Omega^\bullet_X)$$

coming from restriction and puts

$$\tau_{\text{Hdg}}(Y) = \sum_i n_i \tau_{\text{Hdg}}(Y_i).$$

To verify that restriction induces an isomorphism, one first remarks that this is obvious if the Y_i are disjoint, while the general case can be reduced to this case by comparing cohomology with support in $|Y|$ with cohomology with support in $\bigcup_i Y_i - (Y_i \bigcap \bigcup_{j \neq i} Y_j)$ using the excision exact sequence and the previous vanishing results.

2.5 Almost Kähler V-Manifolds

In this section we shall see that the Hodge decomposition is valid for the cohomology groups of a class of varieties that are possibly singular.

A **V-manifold** of dimension n is a complex space which can be covered by charts of the form U_i/G_i, $i \in I$, with $U_i \subset \mathbb{C}^n$ open and G_i a finite group of holomorphic automorphisms of U_i.

An **almost Kähler V-manifold** is a V-manifold X for which there exists a manifold Y bimeromorphic to a Kähler manifold and a proper modification $f : Y \to X$ onto X. Here we recall that a **proper modification** is a proper holomorphic map which induces a biholomorphic map over the complement of a nowhere dense analytic subset.

Examples 2.42.   1) A global quotient of a complex manifold by a finite group of holomorphic automorphisms. An important example is the case of a **weighted projective space** $\mathbb{P}(q_0, \ldots, q_n)$, where the q_j are non-negative integers, the weights. It is defined as the quotient of $\mathbb{P}^n$ by the coordinate-wise action of the product $\mu_{q_0} \times \cdots \times \mu_{q_n}$ of the q_j-th roots of unity μ_j, $j = 0, \ldots, n$. It can also be described as the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^\times$ given by $t \cdot (z_0, \ldots, z_n) = (t^{q_0} z_0, \ldots, t^{q_n} z_n)$. The natural quotient map is denoted

$$p : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}(q_0, \ldots, q_n).$$

The subgroup $\mu(q_j) \subset \mathbb{C}^\times$ stabilizes $V_j = \{z_j = 1\}$ and p identifies $p(V_j)$ with the quotient $V_j = U_j/\mu(q_j)$. These together form the standard open affine covering of $\mathbb{P}(q_0, \ldots, q_n)$. Without loss of generality one may assume that the q_j have no factor in common and we may even assume that this is true for any $(n-2)$-tuple of weights.

A subvariety X of $\mathbb{P}(q_0, \ldots, q_n)$ is called **quasi-smooth** if the cone $p^{-1}X \subset \mathbb{C}^{n+1} - \{0\}$ is smooth. In other words, the only singularity of the corresponding affine cone is the vertex. It is not hard to see that a quasi-smooth subvariety of weighted projective space is a V-manifold.

2) The quotient of any torus by the cyclic group of order two generated by the involution $x \mapsto -x$, a **Kummer variety**.

3) A complete complex algebraic V-manifold admits a resolution of singularities Y and by Chow's lemma, Y is bimeromorphic to a smooth projective variety. It follows that a complete complex algebraic V-manifold is an almost Kähler V-manifold

4) Let us refer to [Oda] for the subject of **toric varieties**. We only say that to each convex polytope Π with integral vertices spanning $\mathbb{R}^n$ as a vector space there corresponds an n-dimensional toric variety X_Π and vice-versa. Each vertex v determines the cone $\bigcup_{n \geq 1} n\Pi_v$, where Π_v is the polytope Π translated over $-v$. If this cone has exactly n 1-dimensional faces it is called simplicial and Π is simplicial if all Π_v are simplicial. The singularities are in general rather bad, but if Π is simplicial, X_Π is a V-manifold.

The main result is

Theorem 2.43. *Let X be an almost Kähler V-manifold. Then $H^k(X; \mathbb{Q})$ admits a Hodge structure of weight k.*

Before we can prove this theorem, we need some preparations. First we note that locally a V-manifold is obtained as the quotient of a ball B by a finite group G of linear unitary automorphisms (see [Cart57, proof of Theorem 4]). The quotient B/G is smooth if and only if G is generated by generalized reflections (elements whose fixed locus is a hyperplane). In general, if we let G_{big} the subgroup of G generated by the generalized reflections and $G_{\text{small}} = G/G_{\text{big}}$, the smooth quotient $B' = B/G_{\text{big}}$ is acted upon by G_{small} with quotient B/G.

This description also shows that X is a rational homology manifold and hence Poincaré-duality holds with respect to rational coefficients.

Next, we need to digress on singularities. Recall that a module M over a local noetherian local ring $(R, \mathfrak{m})$ of Krull dimension n is called **Cohen-Macaulay** if it has a regular sequence of maximal length n (an ordered sequence $(t_1, \ldots, t_m)$ of elements $t_j \in \mathfrak{m}$ is called an M-**regular sequence** if each of the t_j is not a zero-divisor in $M/(t_1, \ldots, t_{j-1})M$). A local ring is called Cohen-Macaulay if it is Cohen-Macaulay as a module over itself.

A (germ of a) singularity (X, x) is called Cohen-Macaulay if $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring.

Examples 2.44. 1) Smooth points are of course Cohen-Macaulay.

2) Reduced curve singularities are Cohen-Macaulay.

3) Quotient singularities are quotients of a germ of smooth manifold (Y, y) by the action of a finite group G of holomorphic automorphisms. These are Cohen-Macaulay, since the local ring at the point $x \in X = Y/G$ corresponding to y is the ring of G-invariants $\mathcal{O}_{Y,y}^G$ of $\mathcal{O}_{Y,y}$ and hence a direct factor of the Cohen-Macaulay ring $\mathcal{O}_{Y,y}$ which itself is Cohen-Macaulay over $\mathcal{O}_{Y,y}^G$.

By [R-R-V], every equi-dimensional complex analytic space X of dimension n has a **dualizing complex** $\omega_X^\bullet$ which actually is an object in the derived category of bounded below complexes of $\mathcal{O}_X$-modules. It can be defined locally as follows. Suppose $U \subset X$ is an open subset embeddable into an open set $V \subset \mathbb{C}^N$, say $i : U \hookrightarrow V$. Then the complex

$$\omega_U^\bullet := R \operatorname{Hom}_{\mathcal{O}_V}(\mathcal{O}_U, \Omega_V^N[N])[-n]$$

is supported on U and is actually independent of the choice of V.

The dualizing complex intervenes in a duality statement of which we only need some special cases:

Theorem 2.45. 1) ***Serre-Grothendieck duality***: *Let X be a compact complex space. For any $\mathcal{O}_X$-coherent sheaf $\mathcal{F}$ we have*

$$H^q(X, \mathcal{F})^\vee = \operatorname{Ext}^{n-q}(\mathcal{F}, \omega_X^\bullet).$$

2) Let $f : Z \to X$ be a finite morphism between complex spaces. For any $\mathcal{O}_Z$-coherent sheaf $\mathcal{F}$ we have

$$f_* \mathcal{E}xt_{\mathcal{O}_Z}^i(\mathcal{F}, \omega_Z^\bullet) = \mathcal{E}xt_{\mathcal{O}_X}^i(f_* \mathcal{F}, \omega_X^\bullet).$$

It can be shown that for a normal Cohen-Macaulay space X with singular locus X_{sing} and inclusion $i : X_{\mathrm{reg}} = X - X_{\mathrm{sing}} \hookrightarrow X$ of the smooth locus, the dualizing complex is actually a *sheaf*

$$\omega_X := i_* \Omega_{X_{\mathrm{reg}}}^n$$

viewed as a complex placed in degree 0. In the special case of a V-manifold X, this sheaf, or more precisely, the complex $i_* \Omega_{X_{\mathrm{reg}}}^\bullet$ can be described in terms of the local geometry of X:

Lemma 2.46. *Let $B \subset \mathbb{C}^n$ be an open ball and let G be a finite unitary subgroup acting on B. Let $p : B \to X = B/G$ be the quotient map. Then we have an equality of complexes*

$$\tilde{\Omega}_X^\bullet := i_* \Omega_{X_{\mathrm{reg}}}^\bullet = (p_* \Omega_B^\bullet)^G.$$

In particular, $\tilde{\Omega}_X^\bullet$ is a resolution of the constant sheaf $\underline{\mathbb{C}}_X$.

Proof. If $G = G_{\mathrm{small}}$ the subvariety $p^{-1} X_{\mathrm{sing}}$ has codimension ≥ 2 in B and p induces the finite unramified cover $q : B' = B - p^{-1} X_{\mathrm{sing}} \to X_{\mathrm{reg}}$. Then $\Omega_{X_{\mathrm{reg}}}^\bullet = (q_* \Omega_B^\bullet)^G$. Let $j : B' \hookrightarrow B$ be the inclusion. The assertion follows from

$$i_* \Omega_{X_{\mathrm{reg}}}^\bullet = (i_* q_* \Omega_B^\bullet)^G = (p_* j_* \Omega_{B'}^\bullet)^G = (p_* \Omega_B^\bullet)^G,$$

where the last equality follows since $q^{-1} X_{\mathrm{sing}}$ has codimension ≥ 2 in B.

If $G = G_{\mathrm{big}}$ the map p is ramified along hypersurfaces and locally on B, the map is given by $(z_1, z_2, \ldots, z_n) \mapsto (z_1^e, z_2, \ldots, z_n)$. Remembering that $X = X_{\mathrm{reg}}$, as before we have $\Omega_X^\bullet = (p_* \Omega_B^\bullet)^G$ and the result follows in this case as well.

In the general case, we factor the map p into $B \xrightarrow{p'} B/G_{\mathrm{big}} \xrightarrow{p''} B/G$ and we use that

$$(p_* \Omega_B^\bullet)^G = \left(p_*'' (p_*' \Omega_B^\bullet)^{G_{\mathrm{big}}} \right)^{G_{\mathrm{small}}}.$$

The last assertion follows from the corresponding assertion on B upon taking G-invariants. $\square$

If we apply the relative duality statement above to the quotient map p, we find

Corollary 2.47. *Let X be an n-dimensional V-manifold. Then*

1) $\mathcal{H}om_{\mathcal{O}_X}(\tilde{\Omega}_X^p, \omega_X) = \tilde{\Omega}_X^{n-p}$ *for all p;*
2) $\mathcal{E}xt_{\mathcal{O}_X}^i(\tilde{\Omega}_X^p, \omega_X) = 0$ *for all p and all $i > 0$.*

Using the local to global spectral sequence for Ext we conclude from this that

$$\mathrm{Ext}_{\mathcal{O}_X}^p(\tilde{\Omega}_X^q, \omega_X) = H^p(X, \tilde{\Omega}_X^{n-p}).$$

Combining this with Serre-Grothendieck duality this shows

Corollary 2.48. $H^q(X, \tilde{\Omega}_X^p)$ *is dual to $H^{n-q}(X, \tilde{\Omega}_X^{n-p})$.*

Proof of Theorem 2.43:. Since $\tilde{\Omega}_X^\bullet$ is a resolution of the constant sheaf $\underline{\mathbb{C}}_X$, the spectral sequence in hypercohomology now reads

$$E_1^{pq} = H^q(X, \tilde{\Omega}_X^q) \Longrightarrow H^{p+q}(X; \mathbb{C}).$$

Let $f : Y \to X$ be a proper modification with Y bimeromorphic to a Kähler manifold. There is a natural morphism of sheaf complexes

$$\tilde{\Omega}^{\bullet}_X \to f_* \Omega^{\bullet}_Y$$

which can be seen to be an isomorphism. The local calculation showing this can be found in [Ste77a, Lemma 1.11]. It follows that there is a morphism f^* between the above spectral sequence and the Hodge-to De Rham spectral sequence for Y. We claim that f^* is already injective on the level of the E_1-terms. To see this, we use the previous Corollary: for every non-zero $\alpha \in E_1^{p,q}$, there exists a $\beta \in E_1^{n-p,n-q}$ with $\alpha \wedge \beta \neq 0$. Then $f^*\alpha \wedge f^*\beta = f^*(\alpha \wedge \beta) \neq 0$, since f^* is an isomorphism in the top cohomology. It follows that α is non-zero and so f^* is injective. But then the spectral sequence we started with degenerates at E_1 as well and f^* induces an isomorphism

$$H^q(X, \tilde{\Omega}^p_X) \xrightarrow{\sim} H^{p,q}(Y) \cap f^* H^{p+q}(X;\mathbb{C}).$$

We thus obtain a Hodge decomposition on $H^k(X;\mathbb{C})$ making f^* a morphism of Hodge structures.

Historical Remarks. The group theoretic point of view of the notion of Hodge structure is due to Mumford and has been exploited by Deligne in his study of absolute Hodge cycles (see the monograph [DMOS]). It has been used as a tool in approaching the Hodge conjecture on abelian varieties. See also the Appendix by Brent Gordon in [Lewis].

The Hodge complexes of sheaves are one of the basic building blocks for later constructions of mixed Hodge structures in geometric situations. This notion is inspired by Deligne [Del71], [Del74] but is different from his in that we prefer working with (filtered) complexes of sheaves instead of classes of these up to quasi-isomorphism. The algebraic version of the $\partial\bar{\partial}$-Lemma is a variation of an argument due to Deligne [Del71, Prop. 4.3.1]. The Hodge theoretic study of V-manifolds has been carried out in [Ste77b]. The notion of V-manifold is due to Satake [Sata56].

The Hodge theoretic aspects of the fundamental class have been extensively studied by El Zein in [ElZ].