Preface

In 1992 we published a book entitled *Fuzzy Measure Theory* (Plenum Press, New York), in which the term "fuzzy measure" was used for set functions obtained by replacing the additivity requirement of classical measures with weaker requirements of monotonicity with respect to set inclusion and continuity. That is, the book dealt with nonnegative set functions that were monotone, vanished at the empty set, and possessed appropriate continuity properties when defined on infinite sets.

It seems that *Fuzzy Measure Theory* was the only book available on the market at that time devoted to this emerging new mathematical theory. Some ten years after its publication we began to see that the subject had expanded so much that a second edition of the book, or even a new book on the subject, was needed. We eventually decided to write a new book because the new material we wished to include was too extensive for—and far beyond the usual scope—of a second edition. More importantly, we felt that some fundamental changes regarding this topic's scope and terminology would be desirable and timely.

As far as the scope of the new book, *Generalized Measure Theory*, is concerned, we felt, on the basis of recent developments in the literature, that the material should not be restricted to set functions that had to be nonnegative and monotone. Rather, it needed to capture a broader class of set functions; a function in this class would have only one requirement to qualify as a "measure": it would vanish at the empty set. Then, various special requirements could be introduced as needed to restrict this broad class of set functions to specialized subclasses. One of these subclasses would consist of nonnegative, monotone, and continuous set functions that vanish at the empty set—or fuzzy measures—the subject of our previous book.

Regarding terminology, it was obvious that we needed to revise it completely in view of the expanded scope of the book. First, we had to introduce a name for the most general measures. We did so by referring to nonnegative set functions that vanish at the empty set as *general measures* and referring to those that are not required to be nonnegative as *signed general measures*. Second, we needed to introduce appropriate names of the various subclasses of general measures or signed general measures. This we did in Chapters 3 and 4, where we followed, by and large, the terminology established in the literature. However, it should be emphasized that we made a deliberate decision to abandon the central term of our previous book, the term "fuzzy measure." We judge this term to be highly misleading. Indeed, the so-called fuzzy measures do not involve any fuzziness. They are just special set functions that are defined on specified classes of classical sets, not on classes of fuzzy sets. Since the primary characteristic of such functions is monotonicity, we deemed it reasonable to call these set functions *monotone measures* rather than fuzzy measures.

However, contrary to the concept of fuzzy measures in our previous book, monotone measures as understood in *Generalized Measure Theory* need not be continuous. If, in fact, they are continuous then they are here specifically referred to as *continuous monotone measures*. Moreover, if they are only semicontinuous from below or from above, then they are called, respectively, *lower-semicontinuous* or *upper-semicontinuous monotone measures*. Clearly, any continuous monotone measure is both lower-semicontinuous and upper-semicontinuous.

There is another reason why abandoning the term "fuzzy measure" is justified: It is certainly meaningful to fuzzify any class of measures, as we show in Chapter 14. A given class of measures is "fuzzified" when it is defined on fuzzy sets rather than on classical sets. However, the resulting term—"fuzzified fuzzy measures" we find awkward, not properly descriptive, and quite confusing. For all these reasons, we decided to replace the term "fuzzy measure" with "continuous monotone measure" and to use the term "monotone measure" when continuity or even semicontinuity is not required. When they *are* fuzzified we refer to these measures as "fuzzified monotone measures." When measures of any other type are defined on classes of fuzzy sets we refer to them as *fuzzified measures* of the respective type. We thus use names such as *fuzzified general measures, fuzzified monotone measures, fuzzified continuous monotone measures,* and the like.

We realize it is not likely that the confusing term "fuzzy measures" for "measures defined on classes of crisp sets" will soon disappear in the literature. However, we are confident that the time is ripe to stop using it. In a sense we have joined some major contributors to generalized measure theory who have already abandoned this ill-descriptive term.

We have made in this book a few additional terminological changes with respect to our previous book. However, all these changes affect special concepts, so we explain our rationale for making these changes as we introduce each concept.

Our previous book contains, in addition to its original material, six of our reprinted papers. In this book, no reprinted papers are included. Instead the original material is substantially expanded. Major expansions are in the area of integration, methods for constructing generalized measures, fuzzification of generalized measures, and applications of generalized measure theory.

Much like our previous book, this book is primarily a text for a one-semester graduate or upper division course. Such a course is suitable not only for programs in mathematics, where it might be offered at the junior or senior level, but also for programs in numerous other areas. These would include systems science, computer science, information science, and cognitive sciences, as well as artificial intelligence, quantitative management, mathematical social sciences, and virtually all areas of engineering and natural sciences. The book may also be useful for researchers in these areas.

Although a solid background in mathematical analysis is required for understanding the material presented, the book is otherwise self-contained. This is achieved by the inclusion of needed prerequisites regarding classical sets, classical measures, and fuzzy sets, as given in Chapter 2. In general, the book is written in the textbook style, characterized by generous use of examples and exercises. Each chapter concludes with notes containing relevant historical, bibliographical, and other remarks relating to the covered material, which are useful for further study of generalized measure theory and its applications. Compared with our previous book, the bibliography of *Generalized Measure Theory* is substantially expanded. Two glossaries are included for convenience of the reader, Glossary of Key Concepts (Appendix A) and Glossary of Symbols (Appendix B).

Omaha, Nebraska, USA Zhenyuan Wang
Binghamton, New York, USA George J.Klir

Chapter 2
Preliminaries

2.1 Classical Sets

2.1.1 Set Inclusion and Characteristic Function

Let X be a nonempty set. Unless otherwise stated, all sets that we consider are subsets of X. Set X is called a *universe of discourse* or a *universal set*. The elements of X are called *points*. Universal set X may contain finite, countably infinite, or uncountably infinite number of points. A set that consists of a finite number of points $x_1, x_2, \ldots, x_n$ (or, a countably infinite number of points $x_1, x_2, \ldots$) may be denoted by $\{x_1, x_2, \ldots, x_n\}(\{x_1, x_2, \ldots\}$, respectively). A set containing no point is called the *empty set* and is denoted by Ø.

If x is a point of X and E is a subset of X, the notation

$$x \in E$$

means that x belongs to E, i.e., x is an element of E; and the statement that x does not belong to E is denoted by

$$x \notin E.$$

Thus, for every point x of X we have

$$x \in X$$

and

$$x \notin \emptyset.$$

A set of sets is called a *class*. If E is a set and $\mathbf{C}$ is a class, then

$$E \in \mathbf{C}$$

means that set E belongs to class $\mathbf{C}$.

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If, for each x, $\pi(x)$ is a proposition concerning x, then the symbol

$$\{x|\pi(x)\}$$

denotes the set of all those points x for which $\pi(x)$ is true; that is,

$$x_0 \in \{x|\pi(x)\} \Leftrightarrow \pi(x_0)\text{is true.}$$

If the point x is replaced with set E, such a symbol may be used to indicate a class. For example,

$$\{E|x \in E\}$$

denotes the class of those sets that contain the point x.

Example 2.1. Let $X = \{1, 2, \ldots\}$. Then, $A = \{x|x \text{ is odd and less than} 10\} = \{1, 3, 5, 7, 9\}$.

Example 2.2. Let X be the set of all real numbers, which is often referred to as the real line or one-dimensional Euclidean space. The class $\{(a, b)| -\infty < a < b < \infty\}$ is the class consisting of all open intervals on the real line.

If E and F are sets, the notation

$$E \subset F \text{ or } F \supset E$$

means that E is a subset of F, i.e., every point of E belongs to F. In this case, we say that F *includes* E, or that E *is included* by F. For every set E we have

$$\emptyset \subset E \subset X.$$

Two sets E and F are called *equal* iff

$$E \subset F \text{ and } F \subset E;$$

that is, they contain exactly the same points. This is denoted by

$$E = F.$$

The symbols $\subset$ or $\supset$ also may be used for classes. If $\mathbf{E}$ and $\mathbf{F}$ are classes, then

$$\mathbf{E} \subset \mathbf{F}$$

means that every set of $\mathbf{E}$ belongs to $\mathbf{F}$, that is, $\mathbf{E}$ is a subclass of $\mathbf{F}$.

If $E_1, E_2, \ldots, E_n$ are nonempty sets, then

$$E = \{(x_1, x_2, \ldots, x_n)|x_i \in E_i,\ i = 1, 2, \ldots, n\}$$

is called an *n-dimensional product set* and is denoted by

$$E = E_1 \times E_2 \times \ldots \times E_n.$$

Similarly, if $\{E_t | t \in T\}$ is a family of nonempty sets, where T is an infinite index set, then

$$E = \{x_t, t \in T \,|\, x_t \in E_t \text{ for each } t \in T\}$$

is called an *infinite-dimensional product set*.

Example 2.3. Let X_1 and X_2 be one-dimensional Euclidean spaces. Then $X = X_1 \times X_2 = \{(x_1, x_2) | x_1 \in (-\infty, \infty), x_2 \in (-\infty, \infty)\}$ is the two-dimensional Euclidean space. The set $\{(x_1, x_2) | x_1 > x_2\}$ is a half (open) plane under the line $x_2 = x_1$, while the set $\{(x_1, x_2) | x_1^2 + x_2^2 < r^2\}$ is the open circle centering at the origin with a radius r, where $r > 0$.

Example 2.4. Let $X_t = \{0, 1\}, t \in \{1, 2, \ldots\}$. The space

$$\begin{aligned} X &= X_1 \times X_2 \times \ldots \times X_n \times \ldots \\ &= \{(x_1, x_2, \ldots, x_n, \ldots) | x_t \in \{0, 1\} \text{ for each } t \in \{1, 2, \ldots\}\} \end{aligned}$$

is an infinite-dimensional product space. Each point $(x_1, x_2, \ldots, x_n, \ldots)$ in this space corresponds to the binary number $0.\, x_1 x_2 \ldots x_n \ldots$ in $[0, 1]$. Such a correspondence is not one to one, but it is onto.

If E is a set, the function χ_E, defined for all $x \in X$ by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E, \end{cases}$$

is called the *characteristic function* of set E. The correspondence between sets and their characteristic functions is one to one, that is,

$$E = F \Leftrightarrow \chi_E(x) = \chi_F(x), \; \forall x \in X.$$

It is easy to see that

$$E \subset F \Leftrightarrow \chi_E(x) \le \chi_F(x), \; \forall x \in X,$$

and that

$$\chi_X \equiv 1, \chi_\emptyset \equiv 0.$$

2.1.2 Operations on Sets

Let **C** be any class of subsets of X. The set of all those points of X that belong to at least one set of the class **C** is called the *union* of the sets of **C**. This is denoted by

$$\bigcup \mathbf{C}.$$

If to every t of a certain index set T there corresponds a set E_t, then the union of the sets of class

$$\{E_t | t \in T\}$$

may be also denoted by

$$\bigcup_{t \in T} E_t \quad \text{or} \quad \bigcup_t E_t.$$

Especially, when

$$\mathbf{C} = \{E_1, E_2\},$$

then $\bigcup \mathbf{C}$ is denoted by

$$E_1 \cup E_2;$$

and if

$$\mathbf{C} = \{E_1, E_2, \ldots, E_n\} \qquad (\mathbf{C} = \{E_1, E_2, \ldots\})$$

then $\bigcup \mathbf{C}$ is denoted by

$$E_1 \cup E_2 \cup \ldots \cup E_n \quad \text{or} \quad \bigcup_{i=1}^{n} E_i \quad \left(\bigcup_{i=1}^{\infty} E_i, \ \text{respectively} \right).$$

The set of all those points of X which belong to every set of the class **C** is called the *intersection* of the sets of **C**. This is denoted by $\bigcap \mathbf{C}$. Symbols similar to those used for unions are available, such as $\bigcap_{t \in T} E_t$ (or $\bigcap_t E_t$), $E_1 \cap E_2$, $E_1 \cap E_2 \cap \ldots \cap E_n$ (or $\bigcap_{i=1}^{n} E_i$), and $\bigcap_{i=1}^{\infty} E_i$. If F is a set, the class $\{E \cap F | E \in \mathbf{C}\}$ is denoted by $\mathbf{C} \cap F$.

Example 2.5. Let $X = \{a, b, c, d\}, \mathbf{C} = \{\{a\}, \{b, c\}, \{b, d\}, \{c, d\}\}, F = \{a, b\}$. Then $\mathbf{C} \cap F = \{\{a\}, \{b\}, \emptyset\}$.

Example 2.6. Let $X = (-\infty, \infty)$, $\mathbf{C} = \{[a, b] | -\infty < a \leq b < \infty\}$, $F = [0, 1]$. Then, $\mathbf{C} \cap F = \{[a, b] | 0 \leq a \leq b \leq 1\}$, that is, the class of all closed subintervals of the unit closed interval.

It is convenient to adopt the conventions that

$$\bigcup_{t \in T} E_t = \emptyset$$

and

$$\bigcap_{t \in T} E_t = X$$

when T is empty.

Proposition 2.1. *The following statements are equivalent:*

(1) $E \subset F$;
(2) $E \cup F = F$;
(3) $E \cap F = E$.

Two sets E and F are called *disjoint* iff

$$E \cap F = \emptyset.$$

A class **C** is called *disjoint* iff every two distinct sets of **C** are disjoint; in this case we refer to the union of the sets of **C** as a disjoint union.

If E is a set, the set of all those points of X that do not belong to E is called the *complement* of E. This is denoted by $\overline{E}$.

Proposition 2.2. *The set operations union, intersection, and complement have the following properties:*

Involution: $\overline{\overline{E}} = E$

Commutativity: $E \cup F = F \cup E$

$E \cap F = F \cap E$

Associativity: $$\bigcup_{t \in T}\left(\bigcup_{s \in S_t} E_s\right) = \bigcup_{s \in \cup_{t \in T} S_t} E_s$$

$$\bigcap_{t \in T}\left(\bigcap_{s \in S_t} E_s\right) = \bigcap_{s \in \cup_{t \in T} S_t} E_s$$

Distributivity: $$F \cap \left(\bigcup_{t \in T} E_t\right) = \bigcup_{t \in T} (F \cap E_t)$$

$$F \cup \left(\bigcap_{t \in T} E_t\right) = \bigcap_{t \in T} (F \cup E_t)$$

Idempotence: $E \cup E = E$

$E \cap E = E$

Absorption:	$E \cup (E \cap F) = E$
	$E \cap (E \cup F) = E$
Absorption of complement:	$E \cup (\overline{E} \cap F) = E \cup F$
	$E \cap (\overline{E} \cup F) = E \cap F$
Absorption by X and Ø:	$E \cup X = X$
	$E \cap \emptyset = \emptyset$
Identity:	$E \cup \emptyset = E$
	$E \cap X = E$
Law of contradiction:	$E \cap \overline{E} = \emptyset$
Law of excluded middle:	$E \cup \overline{E} = X$
DeMorgan's laws:	$\overline{\bigcup_{t \in T} E_t} = \bigcap_{t \in T} \overline{E_t}$
	$\overline{\bigcap_{t \in T} E_t} = \bigcup_{t \in T} \overline{E_t}$

where S_t, T are index sets.

From the above a duality is suggestive. In general, we have the following principle of duality: Any valid identity among sets obtained by unions, intersections, and complements, remains valid if the symbols

$$\cap, \subset, \text{ and } \emptyset$$

are interchanged with

$$\cup, \supset, \text{ and } X,$$

respectively (and if the equality and complementation are left unchanged).

If E and F are sets, the set of all those points of E that do not belong to F is called the difference of E and F. This is denoted by

$$E - F.$$

If $E \supset F$, the difference $E - F$ is called *proper*. Clearly,

$$E - F = E \cap \overline{F}.$$

The *symmetric difference* of E and F, in symbols

$$E \,\Delta\, F,$$

is defined by

$$E \,\Delta\, F = (E - F) \cup (F - E).$$

Let $\{E_1, E_2, \ldots\}$ (or $\{E_n\}$, briefly) be a sequence of sets. The set of all those points of X that belong to E_n for infinitely many values of n is called the *superior limit* of $\{E_n\}$, and is denoted by

$$\limsup_n E_n \text{ or } \overline{\lim_n}\, E_n;$$

the set of all points of X that belong to E_n for all but a finite number of values of n is called the *inferior limit* of $\{E_n\}$, and denoted by

$$\liminf_n E_n \text{ or } \underline{\lim}_n\, E_n.$$

Proposition 2.3.

$$\limsup_n E_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i;$$

$$\liminf_n E_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} E_i.$$

Example 2.7. Let $X = \{a, b\}$ and let a set sequence $\{E_n\}$ be defined as follows:

$$E_n = \begin{cases} \{a\} & \text{if } n \text{ is even} \\ \{b\} & \text{if } n \text{ is odd.} \end{cases}$$ Then, $\limsup_n E_n = X$ and $\liminf_n E_n = \emptyset$.

Example 2.8. Let $X = (-\infty, \infty)$ and let a set sequence $\{E_n\}$ be defined as follows: $E_1 = [0, 1)$, $E_2 = [0, 1/2)$, $E_3 = [1/2, 1)$, $E_4 = [0, 1/4)$, $E_5 = [1/4, 1/2)$, $E_6 = [1/2, 3/4)$, $E_7 = [3/4, 1)$, $E_8 = [0, 1/8)$, $\ldots$. Then, $\limsup_n E_n = [0, 1)$ and $\liminf_n E_n = \emptyset$.

Proposition 2.4. $\liminf_n E_n \subset \limsup_n E_n$.

If

$$\limsup_n E_n = \liminf_n E_n$$

we use the notation

$$\lim_n E_n$$

for this set and say that the limit of $\{E_n\}$ exists and that this set is the *limit* of $\{E_n\}$. Sometimes we write $E_n \to E$ when $\lim_n E_n = E$.

Example 2.9. Let $X = \{1, 2, \ldots\}$ and let $\{E_n\}$ be a set sequence in which $E_n = \{n\}, n = 1, 2, \ldots$. Then, we have

$$\limsup_n E_n = \liminf_n E_n = \emptyset.$$

Hence, the limit of $\{E_n\}$ exists, and $\lim_n E_n = \emptyset$.

We say that $\{E_n\}$ is *increasing* if

$$E_n \subset E_{n+1}, \quad \forall n = 1, 2, \ldots,$$

and $\{E_n\}$ is *decreasing* if

$$E_n \supset E_{n+1}, \quad \forall n = 1, 2, \ldots.$$

Both increasing and decreasing sequences are called *monotone*.

Proposition 2.5. *For any monotone sequence* $\{E_n\}$, $\lim_n E_n$ *exists and equals*

$$\bigcup_n E_n \quad \text{or} \quad \bigcap_n E_n$$

according as $\{E_n\}$ *is increasing or decreasing, respectively.*

Usually, we write $E_n \nearrow E$ when $\{E_n\}$ is increasing and $\lim_n E_n = E$, whereas we write $E_n \searrow E$ when $\{E_n\}$ is decreasing and $\lim_n E_n = E$.

Example 2.10. Let $X = (-\infty, \infty)$. If $\{E_n\}$ is a set sequence in which $E_n = [1/n, 1], n = 1, 2, \ldots$, then $\{E_n\}$ is increasing, and $E_n \nearrow \bigcup_n E_n = (0, 1]$. If $\{F_n\}$ is a set sequence in which $F_n = (-(1 + 1/n), 1 + 1/n), n = 1, 2, \ldots$, then $\{F_n\}$ is decreasing, and $F_n \searrow \bigcap_n F_n = [-1, 1]$.

The discussion of monotone sequences $\{E_n\}$ can be generalized to families of sets $\{E_t | t \in T\}$, where T is an interval (finite or infinite) of real numbers. If for any $t, t' \in T, E_t \subset E_{t'}$ whenever $t \leq t'$, then $\{E_t\}$ is increasing, and

$$\lim_{t \to t_0 -} E_t = \bigcup_{t < t_0, t \in T} E_t,$$
$$\lim_{t \to t_0 +} E_t = \bigcap_{t > t_0, t \in T} E_t;$$

if for any $t, t' \in T, E_t \supset E_{t'}$ whenever $t \leq t'$, then $\{E_t\}$ is decreasing, and

$$\lim_{t \to t_0 -} E_t = \bigcap_{t < t_0, t \in T} E_t,$$
$$\lim_{t \to t_0 +} E_t = \bigcup_{t > t_0, t \in T} E_t,$$

where symbols $\lim_{t\to t_0-}$ and $\lim_{t\to t_0+}$ denote the left limit at t_0 and the right limit at t_0, respectively.

The following proposition gives the correspondence between the operations of sets and the operations of characteristic functions of sets.

Proposition 2.6.

(1) $$\chi_E = \sup_{t\in T} \chi_{E_t}, \textit{ where } E = \bigcup_{t\in T} E_t;$$

in particular,

$$\chi_{E\cup F} = \max(\chi_E, \chi_F);$$

(2) $$\chi_E = \inf_{t\in T} \chi_{E_t}, \textit{ where } E = \bigcap_{t\in T} E_t;$$

in particular,

$$\chi_{E\cap F} = \min(\chi_E, \chi_F);$$

(3) $$\chi_{\overline{E}} = 1 - \chi_E;$$
(4) $$\chi_{E-F} = \chi_E - \min(\chi_E, \chi_F) = \min(\chi_E, 1-\chi_F) = \max(0, \chi_E - \chi_F);$$
(5) $$\chi_{E\Delta F} = |\chi_E - \chi_F|;$$
(6) $$\chi_{\limsup_n E_n} = \limsup_n \chi_{E_n},$$

$$\chi_{\liminf_n E_n} = \liminf_n \chi_{E_n},$$

and if $\lim_n E_n$ *exists, then*

$$\chi_{\lim_n E_n} = \lim_n \chi_{E_n}.$$

2.1.3 Classes of Sets

Definition 2.1. The class of all subsets of X is called the *power set* of X, and is denoted by

$$\mathbf{P}(X).$$

Definition 2.2. A nonempty class $\mathbf{R}$ is called a *ring*, iff $\forall E, F \in \mathbf{R}$,

$$E \cup F \in \mathbf{R} \text{ and } E - F \in \mathbf{R}.$$

In other words, a ring is a nonempty class that is closed under the formation of unions and differences. Because of the associativity of the set union a ring is also closed under the formation of finite unions.

Proposition 2.7. *The empty set Ø belongs to every ring.*

Theorem 2.1. *Any ring is closed under the formation of symmetric differences and intersections; and, conversely, a nonempty class that is closed under the formation of symmetric differences and intersections is a ring.*

Proof. From

$$E\,\Delta\,F = (E - F) \cup (F - E)$$

and

$$E \cap F = (E \cup F) - (E\,\Delta\,F),$$

we obtain the first conclusion. The converse conclusion issues from

$$E \cup F = (E\,\Delta\,F)\ \Delta\ (F \cap F)$$

and

$$E - F = (E\,\Delta\,F) \cap E. \qquad \square$$

Theorem 2.2. *A nonempty class that is closed under the formation of intersections, proper differences, and disjoint unions is a ring.*

Proof. The conclusion follows from

$$E\Delta F = [E - (E \cap F)] \cup [F - (E \cap F)]$$

and Theorem 2.1. □

Example 2.11. The class of all finite subsets of X is a ring.

Example 2.12. Let X be the real line, that is

$$X = (-\infty, \infty) = \{x| - \infty < x < \infty\}.$$

The class of all finite unions of bounded, left closed, and right open intervals, that is, the class of all sets which have the form

$$\bigcup_{i=1}^{n} \{x| - \infty < a_i \leq x < b_i < \infty\},$$

is a ring.

Definition 2.3. A nonempty class **R** is called an *algebra* iff

(1) $\forall E, F \in \mathbf{R}$,

$$E \cup F \in \mathbf{R};$$

(2) $\forall E \in \mathbf{R}$,

$$\overline{E} \in \mathbf{R}.$$

In other words, an algebra is a nonempty class that is closed under the formation of unions and complements. Obviously, in this definition, "$\cup$" can be replaced by "$\cap$".

Theorem 2.3. *An algebra is a ring containing X and, conversely, a ring that contains X is an algebra.*

Proof. Let $\mathbf{R}$ be an algebra. Since

$$E - F = E \cap \overline{F} = (\overline{\overline{E} \cup F}),$$

and, if $E \in \mathbf{R}$, then

$$X = E \cup \overline{E} \in \mathbf{R},$$

we have the first part of the theorem. Conversely, if $\mathbf{R}$ is a ring containing X, then $\forall E \in \mathbf{R}$,

$$\overline{E} = X - E \in \mathbf{R},$$

and the second part follows. □

Example 2.13. The class of all finite sets and their complements is an algebra.

The property described by this example can be generalized into the following proposition.

Proposition 2.8. *If* $\mathbf{R}$ *is a ring, then* $\mathbf{R} \cup \{E | \overline{E} \in \mathbf{R}\}$ *is an algebra.*

Definition 2.4. A nonempty class $\mathbf{S}$ is called a *semiring* iff

(1) $\forall E,\ F \in \mathbf{S}$,

$$E \cap F \in \mathbf{S};$$

(2) $\forall E,\ F \in \mathbf{S}$ satisfying $E \subset F$, there exists a finite class $\{C_0, C_1, \ldots, C_n\}$ of sets in $\mathbf{S}$, such that

$$E = C_0 \subset C_1 \subset \ldots \subset C_n = F$$

and

$$D_i = C_i - C_{i-1} \in \mathbf{S} \qquad \text{for } i = 1, 2, \ldots, n.$$

Every ring is a semiring, and the empty set belongs to any semiring.

Example 2.14. The class consisting of all singletons of X and the empty set is a semiring.

Example 2.15. Let X be the real line. The class of all bounded, left closed, and right open intervals is a semiring.

Definition 2.5. A nonempty class $\mathbf{F}$ is called a *σ-ring* iff

(1) $\forall E,\ F \in \mathbf{F}$,

$$E - F \in \mathbf{F};$$

(2) $\forall E_i \in \mathbf{F}, i = 1, 2, \ldots,$

$$\bigcup_{i=1}^{\infty} E_i \in \mathbf{F}.$$

Any σ-ring is a ring which is closed under the formation of countable unions.

Proposition 2.9. *Any σ-ring is closed under the formation of countable intersections; and, therefore, if* $\mathbf{F}$ *is a σ-ring and a set sequence* $\{E_n\} \subset \mathbf{F}$, *then*

$$\limsup_n E_n \in \mathbf{F} \text{ and } \liminf_n E_n \in \mathbf{F}.$$

Example 2.16. The class of all countable sets is a σ-ring.

Definition 2.6. A *σ-algebra* (or say, *σ-field*) is a *σ-ring* that contains X.

Example 2.17. The class of all countable sets and their complements is a σ-algebra.

Proposition 2.10. *If* $\mathbf{F}$ *is a σ-ring, then* $\mathbf{F} \cup \{E | \overline{E} \in \mathbf{F}\}$ *is a σ-algebra.*

Definition 2.7. A nonempty class $\mathbf{M}$ is called a *monotone class* iff, for every monotone sequence $\{E_n\} \subset \mathbf{M}$, we have

$$\lim_n E_n \in \mathbf{M}.$$

Proposition 2.11. *Any σ-ring is a monotone class.*

Proposition 2.12. *If a ring is also a monotone class, then it is a σ-ring.*

Example 2.18. Let X be the real line. The class of all intervals (the empty set and singletons may be regarded as intervals: $\emptyset = (a, a], \{a\} = [a, a]$) is a monotone class.

Definition 2.8. A nonempty class $\mathbf{F}_p$ is called a *plump class* iff $\forall \{E_t | t \in T\} \subset \mathbf{F}_p$,

$$\bigcup_t t \in \mathbf{F}_p \quad \text{and} \quad \bigcap_t E_t \in \mathbf{F}_p,$$

where T is an arbitrary index set.

Proposition 2.13. *Any plump class is a monotone class.*

Example 2.19. Let X be the unit closed interval [0,1]. The class of all sets that have the form $[0, a)$, or the form $[0, a]$, where $a \in [0, 1]$, is a plump class.

The relations among the above-mentioned concepts of classes are illustrated in Fig. 2.1.

Proposition 2.14. *Let E be a fixed set. If $\mathbf{C}$ is a σ-ring (respectively, ring, semiring, monotone class, plump class), then so is $\mathbf{C} \cap E$.*

Theorem 2.4. *Let $\mathbf{C}$ be a class. There exists a unique ring $\mathbf{R}_0$ such that it is the smallest ring including $\mathbf{C}$; that is,*

$$\mathbf{R}_0 \supset \mathbf{C}$$

and for any ring $\mathbf{R}$,

$$\mathbf{R} \supset \mathbf{C} \Rightarrow \mathbf{R} \supset \mathbf{R}_0.$$

$\mathbf{R}_0$ *is called the ring generated by $\mathbf{C}$ and is denoted by $\mathbf{R}(\mathbf{C})$.*

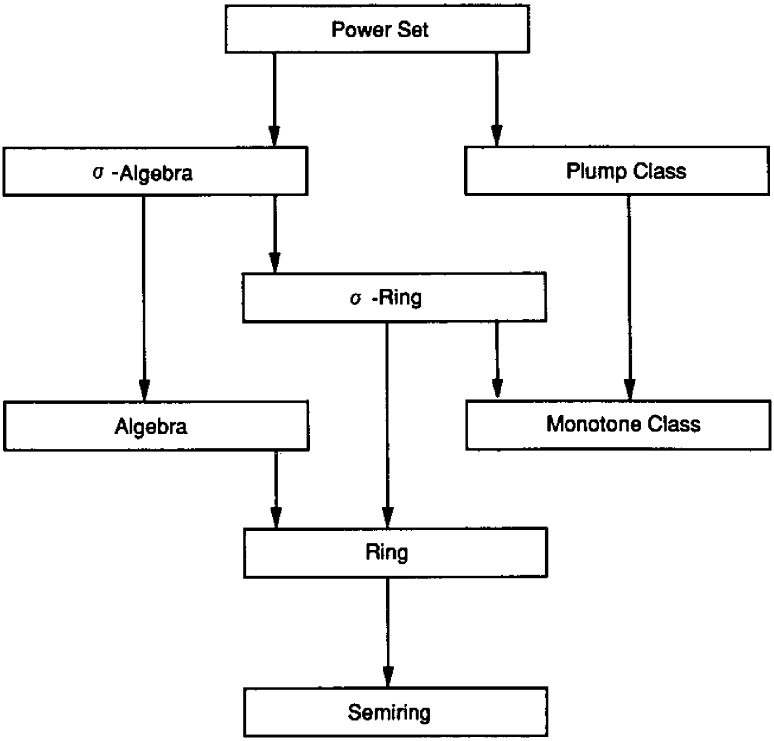

Fig. 2.1 The ordering of classes of sets

Proof. $\mathbf{P}(X)$ is a ring including $\mathbf{C}$. The intersection of all rings including $\mathbf{C}$ is also a ring including $\mathbf{C}$, and it is the desired ring $\mathbf{R}_0$. The uniqueness is evident. □

In the same way, we can also give the concepts of σ-ring, monotone class, and plump class generated by $\mathbf{C}$, and use $\mathbf{F}(\mathbf{C})$, $\mathbf{M}(\mathbf{C})$, and $\mathbf{F}_p(\mathbf{C})$ to denote them, respectively.

Example 2.20. Let X be an infinite set. If $\mathbf{C}$ is the class of all singletons, then $\mathbf{R}(\mathbf{C})$ is the class of all finite sets, and $\mathbf{F}(\mathbf{C})$ is the class of all countable sets.

Example 2.21. Let X be the real line. If $\mathbf{C}$ is the class of all finite open intervals, then $\mathbf{M}(\mathbf{C})$ is the class of all intervals, and $\mathbf{F}_p(\mathbf{C}) = \mathbf{P}(X)$.

Proposition 2.15. *If* $\mathbf{C}_1 \subset \mathbf{C}_2$, *then* $\mathbf{K}(\mathbf{C}_1) \subset \mathbf{K}(\mathbf{C}_2)$, *where* $\mathbf{K}$ *may be taken as* $\mathbf{R}$, $\mathbf{F}$, $\mathbf{M}$, *or* $\mathbf{F}_p$.

Theorem 2.5. *Let* $\mathbf{S}$ *be a semiring. Then,* $\mathbf{R}(\mathbf{S})$ *is the class of all finite, disjoint unions of sets in* $\mathbf{S}$.

Proof. Denote the class of all finite, disjoint unions of sets in $\mathbf{S}$ by $\mathbf{R}_0$. Clearly,

$$\mathbf{R}_0 \supset \mathbf{S}.$$

What follows is a proof that $\mathbf{R}_0$ is a ring.

(1) $\mathbf{R}_0$ is closed under the formation of intersections: $\forall E,\ F \in \mathbf{R}_0$ with

$$E = \bigcup_{i=1}^{n} E_i \text{ and } F = \bigcup_{j=1}^{m} F_j,$$

where $\{E_1, \ldots, E_n\}$ and $\{F_1, \ldots, F_m\}$ are disjoint classes of sets in $\mathbf{S}$, we have

$$E \cap F = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_i \cap F_j$$

and, moreover, we know that

$$\{E_j \cap F_j | i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\}$$

is a disjoint class. Since $\mathbf{S}$ is closed under the formation of intersections,

$$E_i \cap F_j \in \mathrm{S} \text{ for any } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m.$$

Hence, we have

$$E \cap F \in \mathbf{R}_0.$$

(2) $\mathbf{R}_0$ is closed under the formation of proper differences: For any E and F given in (1), if $F \subset E$, the difference $E - F$ may be expressed by a finite, disjoint union of sets having the form

$$E_i - \bigcup_{j=1}^{m} F_j.$$

Each $E_i - \bigcup_{j=1}^{m} F_j$ may also be expressed by a finite, disjoint union of the sets in $\mathbf{S}$. Thus, we have

$$E - F \in \mathbf{R}_0.$$

(3) It is evident that $\mathbf{R}_0$ is closed under the formation of disjoint unions. By Theorem 2.2, we know that $\mathbf{R}_0$ is a ring.

Finally, since $\mathbf{R}$ is closed under the formation of finite unions, if $\mathbf{R}$ is a ring containing $\mathbf{S}$, it should contain every finite union of sets in $\mathbf{S}$. Hence, $\mathbf{R} \supset \mathbf{R}_0$. This completes the proof. □

Theorem 2.6. $\mathbf{F}(\mathbf{S}) = \mathbf{F}(\mathbf{R}(\mathbf{S}))$.

Proof. On the one hand, since $\mathbf{S} \subset \mathbf{R}(\mathbf{S})$, by Proposition 2.15, we have

$$\mathbf{F}(\mathbf{S}) \subset \mathbf{F}(\mathbf{R}(\mathbf{S})).$$

On the other hand, since $\mathbf{F}(\mathbf{S}) \supset \mathbf{S}$ and $\mathbf{F}(\mathbf{S})$ is a ring, we have $\mathbf{F}(\mathbf{S}) \supset \mathbf{R}(\mathbf{S})$. Furthermore, since $\mathbf{F}(\mathbf{S})$ is a σ-ring, we have

$$\mathbf{F}(\mathbf{S}) \supset \mathbf{F}(\mathbf{R}(\mathbf{S})).$$

Consequently, we have

$$\mathbf{F}(\mathbf{S}) = \mathbf{F}(\mathbf{R}(\mathbf{S})).$$ □

Example 2.22. Let X be the real line and let $\mathbf{S}$ be the semiring given in Example 2.15. Then $\mathbf{F}(\mathbf{S})$ is called the *Borel field* on the real line, and it is usually denoted by $\mathbf{B}$. The sets in $\mathbf{B}$ are called *Borel sets*. We have seen the process of constructing $\mathbf{R}(\mathbf{S})$ from $\mathbf{S}$ by Theorem 2.5, and $\mathbf{R}(\mathbf{S})$ is just the ring given in Example 2.12. But the process for constructing $\mathbf{B}$ from $\mathbf{R}(\mathbf{S})$ is quite complex. $\mathbf{B}$ is also the σ-ring generated by the class of all open intervals, by the class of all closed intervals, by the class of all left open and right closed intervals, by the class of all left closed and right open intervals, or by the class of all intervals, respectively.

Theorem 2.7. *If* **C** *is a class, then*

$$\mathbf{F}_p(\mathbf{C}) = \left\{ \bigcup_{t \in T} \bigcap_{s \in S_t} E_S | E_S \in \mathbf{C}, S_t \text{ and } T \text{ are arbitrary index sets} \right\}.$$

Proof. Denote the right part of this equality by **E**.

(1) $\mathbf{E} \supset \mathbf{C}$ because S_t and T may be taken as singletons.
(2) By an application of the associativity of the set union, we know that **E** is closed under the formation of arbitrary unions.
(3) Because an arbitrary intersection of arbitrary unions of sets in a class **C** may be expressed by an arbitrary union of arbitrary intersections of sets in that class **C**, and because arbitrary intersections are associative, **E** is closed under the formation of arbitrary intersections.

Thus, **E** is a plump class including **C** and, therefore, $\mathbf{E} \supset \mathbf{F}_p(\mathbf{C})$. Conversely, any plump class including **C** includes **E**; hence, $\mathbf{F}_p(\mathbf{C}) \supset \mathbf{E}$. Consequently, $\mathbf{F}_p(\mathbf{C}) = \mathbf{E}$. □

Theorem 2.8. *For any class* **C** *and any set* A*,*

$$\mathbf{F}(\mathbf{C}) \cap A = \mathbf{F}(\mathbf{C} \cap A).$$

Similar conclusions about rings, monotone classes, and plump classes are true, as well.

Proof.

(1) $\mathbf{F}(\mathbf{C}) \cap A$ is a σ-ring and includes $\mathbf{C} \cap A$; so

$$\mathbf{F}(\mathbf{C}) \cap A \supset \mathbf{F}(\mathbf{C} \cap A) .$$

(2) Let

$$\mathbf{E} = \{E | E \cap A \in \mathbf{F}(\mathbf{C} \cap A), E \in \mathbf{F}(\mathbf{C})\}.$$

E is a σ-ring, and $\mathbf{E} \supset \mathbf{C}$. So $\mathbf{E} \supset \mathbf{F}(\mathbf{C})$, that is, $\forall E \in \mathbf{F}(\mathbf{C})$,

$$E \cap A \in \mathbf{F}(\mathbf{C} \cap A).$$

This shows that

$$\mathbf{F}(\mathbf{C}) \cap A \subset \mathbf{F}(\mathbf{C} \cap A).$$

Consequently,

$$\mathbf{F}(\mathbf{C}) \cap A = \mathbf{F}(\mathbf{C} \cap A).$$

The rest may be proved in the same way. □

Example 2.23. Let **B** be the Borel field on the real line. $\mathbf{B} \cap [0, 1]$ is called the Borel field on the unit interval. It is the σ-ring generated by the class of all intervals in [0, 1].

Theorem 2.9. *If* **R** *is a ring, then*

$$\mathbf{M}(\mathbf{R}) = \mathbf{F}(\mathbf{R}).$$

Proof. From Proposition 2.11, we know that **F(R)** is a monotone class. Since $\mathbf{F}(\mathbf{R}) \supset \mathbf{R}$, we have

$$\mathbf{F}(\mathbf{R}) \supset \mathbf{M}(\mathbf{R}).$$

If **M(R)** is a σ-ring, then we have

$$\mathbf{M}(\mathbf{R}) \supset \mathbf{F}(\mathbf{R}),$$

and, therefore, the proof would be complete.

To complete the proof, we need to prove that **M(R)** is a σ-ring. For any set F, let **K**(F) be the class of all those sets E for which $E - F$, $F - E$, and $E \cup F$ are all in **M(R)**. It is easy to see, by the symmetry of the positions of E and F in the definition of $\mathbf{K}(F)$, that

$$E \in \mathbf{K}(F) \Leftrightarrow F \in \mathbf{K}(E).$$

If $\{E_n\}$ is a monotone sequence of sets in $\mathbf{K}(F)$, then we have

$$\lim_n E_n - F = \lim_n (E_n - F) \in \mathbf{M}(\mathbf{R}),$$

$$F - \lim_n E_n = \lim_n (F - E_n) \in \mathbf{M}(\mathbf{R}),$$

$$F \cup \lim_n E_n = \lim_n (F \cup E_n) \in \mathbf{M}(\mathbf{R}),$$

that is, $\lim_n E_n \in \mathbf{K}(F)$. So, if $\mathbf{K}(F)$ is not empty, then it is a monotone class. $\forall F \in \mathbf{R}$, if $E \in \mathbf{R}$, then $E \in \mathbf{K}(F)$; that is, $\mathbf{R} \subset \mathbf{K}(F)$. It follows that

$$\mathbf{M}(\mathbf{R}) \subset \mathbf{K}(F), \quad \forall F \in \mathbf{R}.$$

Hence, $\forall E \in \mathbf{M}(\mathbf{R}), \forall F \in \mathbf{R}$, we have $E \in \mathbf{K}(F)$; therefore, by symmetry, $F \in \mathbf{K}(E)$; that is,

$$\mathbf{R} \subset \mathbf{K}(E),$$

for any $E \in \mathbf{M}(\mathbf{R})$. Noting again that $\mathbf{K}(E)$ is a monotone class, we have

$$\mathbf{M}(\mathbf{R}) \subset \mathbf{K}(E), \qquad \forall E \in \mathbf{M}(\mathbf{R}).$$

This shows that $\mathbf{M(R)}$ is a ring. From Proposition 2.12, we know that $\mathbf{M(R)}$ is a σ-ring. □

Corollary 2.1. *A monotone class including a ring includes the σ-ring generated by this ring.*

2.1.4 Atoms and Holes

Let $\mathbf{C}$ be an arbitrary nonempty class of subsets of X.

Definition 2.9. For any point $x \in X$, the set $\bigcap\{E|x \in E \in \mathbf{C}\}$ is called the *atom* of $\mathbf{C}$ at x, and denoted by $A(x/\mathbf{C})$. If there is no confusion, it will be called the atom at x, or atom for short, and denoted by $A(x)$. The class of all atoms of $\mathbf{C}$ is denoted by $\mathbf{A}\,[\mathbf{C}]$, that is,

$$\mathbf{A}[\mathbf{C}] = \{A(x/\mathbf{C})|x \in X\}.$$

Clearly, for every $x \in X, x \in A(x)$. So, every atom is nonempty.
When $\bigcup \mathbf{C} \neq X$, then $A(x/\mathbf{C}) = X$ for any $x \notin \bigcup \mathbf{C}$. Thus, if we write

$$\mathbf{A}^{-}[\mathbf{C}] = \{A(x/\mathbf{C})|x \in \bigcup \mathbf{C}\},$$

then we have

$$\mathbf{A}[\mathbf{C}] - \mathbf{A}^{-}[\mathbf{C}] \subset \{X\}.$$

Proposition 2.16. *If $x \in E \in \mathbf{C}$, then $A(x) \subset E$.*

Example 2.24. Let $X = \{a, b, c\}, \mathbf{C} = \{A, B, C\}$, where $A = \{a\}$, $B = \{a, b\}$, $C = \{b, c\}$. Then, A, $\{b\}$, and C are atoms. That is, $A = A(a)$, $\{b\} = A(b)$, $C = A(c)$. From this example, we can see that it is not necessary that all sets in $\mathbf{C}$ be atoms of $\mathbf{C}$, and that all atoms of $\mathbf{C}$ belong to $\mathbf{C}$. But, if $\mathbf{C}$ is closed under the formation of arbitrary intersections, then we have

$$\mathbf{A}[\mathbf{C}] \subset \mathbf{C};$$

that is, in this case, $\forall A(x)$,

$$A(x) = \bigcap\{E|x \in E \in \mathbf{C}\} \in \mathbf{C}.$$

Example 2.25. If $\mathbf{C} = \mathbf{P}(X)$, then $\mathbf{A}[\mathbf{C}] = \{\{x\}|x \in X\}$.

Proposition 2.17. $\bigcup \mathbf{A}^{-}[\mathbf{C}] = \bigcup \mathbf{C}$.

Theorem 2.10. *Any set in $\mathbf{C}$ may be expressed by a union of atoms of $\mathbf{C}$; moreover, any intersection of sets in $\mathbf{C}$ may be expressed by a union of atoms of $\mathbf{C}$.*

Proof. It is sufficient to prove the second conclusion.

Let $\{E_t | t \in T\}$ be a family of sets in **C**. We have

$$\bigcap_{t \in T} E_t = \bigcup \left\{ A(x) | x \in \bigcap_{t \in T} E_t \right\}$$

In fact, on the one hand, by Proposition 2.16, for any $x \in \bigcap_{t \in T} E_t$, and any $t \in T$,

$$A(x) \subset E_t.$$

So, for any $x \in \bigcap_{t \in T} E_t$,

$$A(x) \subset \bigcap_{t \in T} E_t,$$

and it follows that

$$\bigcup \left\{ A(x) | x \in \bigcap_{t \in T} E_t \right\} \subset \bigcap_{t \in T} E_t.$$

On the other hand, since $x \in A(x)$, we have

$$\bigcap_{t \in T} E_t = \left\{ x | x \in \bigcap_{t \in T} E_t \right\} \subset \bigcup \left\{ A(x) | x \in \bigcap_{t \in T} E_t \right\}.$$

The proof is thus complete. □

Theorem 2.11. *Any intersection of atoms may be expressed by a union of atoms.*

Proof. Since any atom of **C** is an intersection of sets in **C**, by Theorem 2.10 and the associativity of intersections, we obtain the conclusion. □

Example 2.26. Let $X = \{a, b, c, d\}$, $\mathbf{C} = \{A, B, C, D\}$, where $A = \{a, c, d\}$, $B = \{b, c, d\}$, $C = \{c\}$, $D = \{d\}$. Then, $A(a) = A$, $A(b) = B$, $A(c) = C$, $A(d) = D$. We have

$$A(a) \cap A(b) = A(c) \cup A(d).$$

Theorem 2.12. *If* $A' \in \mathbf{A}[\mathbf{C}], x \in A'$, *then* $A(x) \subset A'$

Proof. Let

$$A' = A(x') = \bigcap \{E | x' \in E \in \mathbf{C}\} = \bigcap_{t \in T} E_t,$$

where $E_t \in \mathbf{C}, T$ is an index set. Since $x \in A'$, we have $x \in E_t$ for all $t \in T$. Therefore, by Proposition 2.16, $A(x) \subset E_t$ for all $t \in T$. Consequently, we have

$$A(x) \subset A'. \qquad \square$$

Theorem 2.13. $A(x/\mathbf{C}) = A(x/\mathbf{A}[\mathbf{C}])$ *for any* $x \in X$, *and* $\mathbf{A}[\mathbf{C}] = \mathbf{A}[\mathbf{A}[\mathbf{C}]]$.

Proof. $\forall x \in X,$ if $x \in B$ for some $B \in \mathbf{A}[\mathbf{C}]$, we have, by Theorem 2.12,

$$A(x/\mathbf{C}) \subset B,$$

and, therefore,

$$A(x/\mathbf{C}) \subset \bigcap\{B|x \in B \in \mathbf{A}[\mathbf{C}]\}.$$

Reviewing $x \in A(x/\mathbf{C}) \in \mathbf{A}[\mathbf{C}]$, we have

$$A(x/\mathbf{C}) \supset \bigcap\{B|x \in B \in \mathbf{A}[\mathbf{C}]\}.$$

Thus,

$$A(x/\mathbf{C}) = \bigcap\{B|x \in B \in \mathbf{A}[\mathbf{C}]\} = A(x/\mathbf{A}[\mathbf{C}]).$$

Consequently, we have

$$\mathbf{A}[\mathbf{C}] = \mathbf{A}[\mathbf{A}[\mathbf{C}]]. \qquad \square$$

Theorem 2.14. $\mathbf{A}[\mathbf{C} \cup \mathbf{A}[\mathbf{C}]] = \mathbf{A}[\mathbf{C}]$.

Proof. $\forall x \in X,$

$$\begin{aligned} A(x/\mathbf{C} \cup \mathbf{A}[\mathbf{C}]) &= \bigcap\{E|x \in E \in \mathbf{C} \cup \mathbf{A}[\mathbf{C}]\} \\ &= \left(\bigcap\{E|x \in E \in \mathbf{C}\}\right) \cap \left(\bigcap\{E|x \in E \in \mathbf{A}[\mathbf{C}]\}\right). \\ &= A(x/\mathbf{C}) \cap A(x/\mathbf{A}[\mathbf{C}]) = A(x/\mathbf{C}) \end{aligned}$$

Thus,

$$\mathbf{A}[\mathbf{C} \cup \mathbf{A}[\mathbf{C}]] = \mathbf{A}[\mathbf{C}]. \qquad \square$$

Theorem 2.15. *If* $\mathbf{C}' = \{\bigcup_{t \in T} E_t | E_t \in \mathbf{C}, t \in T,\ T \text{ is an arbitrary index set}\}$, *then* $\mathbf{A}[\mathbf{C}'] = \mathbf{A}[\mathbf{C}]$.

Proof. $\bigcup \mathbf{C}' = \bigcup \mathbf{C}$. $\forall x \in \bigcup \mathbf{C}$, by absorption, we have

$$\begin{aligned} A(x/\mathbf{C}') &= \bigcap \{E \mid x \in E \in \mathbf{C}'\} \\ &= \bigcup \left\{ \bigcup_{t \in T} E_t \middle| x \in \bigcup_{t \in T} E_t, E_t \in \mathbf{C}, t \in T, T \text{ is an arbitrary set} \right\}. \\ &= \bigcap \{E \mid x \in E \in \mathbf{C}\} = A(x/\mathbf{C}) \end{aligned}$$

Thus, we have

$$\mathbf{A}[\mathbf{C}'] = \mathbf{A}[\mathbf{C}].$$

□

Theorem 2.16. *If* $\mathbf{C}$ *is closed under the formation of difference, then* $\mathbf{A}^{-}[\mathbf{C}]$ *is a partition of* $\bigcup \mathbf{C}$ (*Definition 2.18*).

Proof. Since $\bigcup \mathbf{A}^{-}[\mathbf{C}] = \bigcup \mathbf{C}$, we only need to prove that the different atoms in $\mathbf{A}^{-}[\mathbf{C}]$ must be disjoint, that is, $\forall A(x), A(y) \in \mathbf{A}^{-}[\mathbf{C}]$,

$$A(x) \neq A(y) \Rightarrow A(x) \cap A(y) = \emptyset.$$

If both $x \in A(y)$ and $y \in A(x)$, then, by Theorem 2.12, we have $A(x) = A(y)$. So, when $A(x) \neq A(y)$, we can suppose $x \notin A(y)$ without any loss of generality. In this case, if there exists $z \in A(x) \cap A(y)$, we have the result that, from $x \notin A(y)$ and $z \in A(y)$, there exists $E \in \mathbf{C}$ such that $x \notin E$, but $z \in E$. Thus, if we take $F \in \mathbf{C}$, satisfying $x \in F$ and set $G = F - E$, then $x \in G \in \mathbf{C}$, but $z \notin G$. This contradicts the fact that $z \in A(x)$. Therefore, we have $A(x) \cap A(y) = \emptyset$. □

Corollary 2.2. *If* $\mathbf{F}$ *is an algebra, then* $\mathbf{A[F]}$ *is a partition of* X.

The following theorem provides an expression of $\mathbf{F}_p(\mathbf{C})$ by the atoms of $\mathbf{C}$.

Theorem 2.17. $\mathbf{F}_p(\mathbf{C}) = \{\bigcup_{t \in T} A_t \mid A_t \in \mathbf{A}[\mathbf{C}], T \text{ is an arbitrary index set}\}$.

Proof. By Theorem 2.7, Theorem 2.10, and the associativity of set unions, the conclusion immediately follows. □

Theorem 2.18. $\mathbf{A}[\mathbf{F}_p(\mathbf{C})] = \mathbf{A}[\mathbf{C}]$.

Proof. It follows directly from Theorems 2.13, 2.15, and 2.17. □

Theorem 2.19. $\mathbf{F}_p(\mathbf{C}) = \mathbf{F}_p(\mathbf{A}[\mathbf{C}])$.

Proof. From the definition of the atom and Theorem 2.10, the equality is easily obtained. □

A concept of AU-class is interrelated closely with the concept of the atom.

Definition 2.10. The *AU-class* is a nonempty class $\mathbf{C}$ with anticlosedness under the formation of unions, that is, $\forall \mathbf{C}' \subset \mathbf{C}$,

$$\bigcup \mathbf{C}' \in \mathbf{C} \Rightarrow \bigcup \mathbf{C}' \in \mathbf{C}'.$$

By the convention for operations of union and intersection (introduced in Section 2.1.2), if $\mathbf{C}'$ is an empty class, then $\bigcup \mathbf{C}' = \emptyset$. Hence, if $\emptyset \in \mathbf{C}$, and $\mathbf{C}$ is an AU-class, it should follow that $\emptyset \in \mathbf{C}'$. This is a contradiction. So, no AU-class contains the empty set $\emptyset$.

Proposition 2.18. *If* $\mathbf{C}$ *is an AU-class, then all nonempty subclasses of* $\mathbf{C}$ *are AU-classes as well.*

Theorem 2.20. $\mathbf{A}[\mathbf{C}]$ *is an AU-class.*

Proof. Let $\{A(x)|x \in D\}$ be a family of atoms of $\mathbf{C}$. Denote

$$B = \bigcup \{A(x)|x \in D\} = \bigcup_{x \in D} A(x).$$

If $B \in \mathbf{A}[\mathbf{C}]$, then $\exists x_0 \in B$ such that $B = A(x_0)$. From $x_0 \in \bigcup_{x \in D} A(x)$, we have $x_0 \in A(x_0')$ for some $x_0' \in D$. By applying Theorem 2.12, it follows that

$$A(x_0') \supset A(x_0) = B.$$

The inverse inclusion relation is evident. Consequently, we have

$$B = A(x_0') \in \{A(x)|x \in D \subset \bigcup \mathbf{C}\}.$$

This shows that $\mathbf{A}[\mathbf{C}]$ is an AU-class. □

In general, if $\mathbf{C}$ is an AU-class, a set in $\mathbf{C}$ may not be an atom of $\mathbf{C}$.

Example 2.27. X and $\mathbf{C}$ are given as in Example 2.24. It is easy to verify that $\mathbf{C}$ is an AU-class, but B is not an atom of $\mathbf{C}$.

However, we have the following property.

Theorem 2.21. *Let* $\mathbf{C}$ *be an AU-class. If* $\mathbf{C} \supset \mathbf{A}[\mathbf{C}]$, *then we have*

$$\mathbf{C} = \mathbf{A}[\mathbf{C}].$$

Proof. If $\mathbf{C} \neq \mathbf{A}[\mathbf{C}]$, then there exists a nonempty set $E \in \mathbf{C}$, but $E \notin \mathbf{A}[\mathbf{C}]$. By Theorem 2.10 there exists a family of atoms $\{A_t|\, t \in T\}$ such that $E = \bigcup_{t \in T} A_t$. Since $\mathbf{C}$ is an AU-class, $\exists t_0 \in T$ such that $E = A_{t_0} \in \mathbf{A}[\mathbf{C}]$. This contradicts $E \notin \mathbf{A}[\mathbf{C}]$. □

A dual concept to the "atom" is the "hole."

Definition 2.11. Let $\hat{\mathbf{C}} = \{\overline{E} | E \in \mathbf{C}\}$. For any point $x \in X$, the set

$$\bigcup \{E | x \in \overline{E} \in \hat{\mathbf{C}}\}$$

is called the *hole* of **C** at x, denoted by $H(x/\mathbf{C})$, or $H(x)$ for short. The class of all holes of **C** is denoted by **H**[**C**].

We can also write

$$H(x/\mathbf{C}) = \bigcup \{E | x \notin E \in \mathbf{C}\}.$$

It is evident that, for any $x \in X, x \notin H(x/\mathbf{C})$. So, X is not a hole.

The relation between hole and atom is given in the following proposition.

Proposition 2.19. $H(x/\mathbf{C}) = \overline{A(x/\hat{\mathbf{C}})}$.

Example 2.28. We use X and **C** given in Example 2.24. In this case, $\overline{A} = \{b, c\} = C, \overline{B} = \{c\},\ \overline{C} = \{a\} = A$. Consequently, $H(a) = C,\ H(b) = A,\ H(c) = B$.

Example 2.29. If $\mathbf{C} = \mathbf{P}(X)$, then $\mathbf{H}[\mathbf{C}] = \{\overline{\{x\}} | x \in X\}$.

Definition 2.12. The *AI-class* is a nonempty class **C** with anticlosedness under the formation of intersections, that is, $\forall \mathbf{C}' \subset \mathbf{C}$,

$$\bigcap \mathbf{C}' \in \mathbf{C} \Rightarrow \bigcap \mathbf{C}' \in \mathbf{C}'.$$

All properties of the AU-class can be easily converted into analogous properties of the AI-class by replacing atoms with holes [Liu and Wang 1985, 1987].

2.1.5 S-Compact Space

Let **C** be a nonempty class of subsets of X. Usually, we also use the term "space" to mean (X, **C**). Especially, when **C** is a σ-algebra (or σ-ring), denoted by **F**, we call (X, **F**) a *measurable space*, and the sets in **F** are called *measurable sets*. We say (X, **C**) or (X, **F**) is to be finite, countable, or uncountable if X is finite, countable, or uncountable, respectively.

Definition 2.13. (X, **C**) is said to be *S-precompact* iff for any sequence of sets in **C** there exists some convergent subsequence, that is, $\forall \{E_n\} \subset \mathbf{C}, \exists \{E_{n_i}\} \subset \{E_n\}$ such that,

$$\limsup_i E_{n_i} = \liminf_i E_{n_i};$$

(X, **C**) is said to be *S-compact* iff it is S-precompact and the limit of the above-mentioned subsequence belongs to **C**, that is, $\forall \{E_n\} \subset \mathbf{C}, \exists \{E_{n_i}\} \subset \{E_n\}$ such that $\lim_i E_{n_i}$ exists and

$$\lim_i E_{n_i} \in \mathbf{C}.$$

Obviously, any S-precompact measurable space is S-compact.

Example 2.30. Any finite space is S-compact. In fact, if $(X, \mathbf{C})$ is a finite space, then $\mathbf{C}$ is finite too. So, from any sequence of sets in $\mathbf{C}$, we can always pick out a subsequence in which all sets are identical; therefore, this subsequence converges to the same set as that in the subsequence.

From the above example we can also see that, although X is not finite, $(X, \mathbf{C})$ is S-compact so long as $\mathbf{C}$ is finite.

Example 2.31. If $\mathbf{C}$ is a nest (or, say, a chain; in this case it is fully ordered by the inclusion relation between sets), then $(X, \mathbf{C})$ is S-precompact. To show this, it is sufficient to prove the following lemma.

Lemma 2.1. *If* $\mathbf{C}$ *is an infinite nest, then there exists a monotone subsequence of sets in* $\mathbf{C}$.

Proof. According to the order given by the inclusion relation, if there exists $\mathbf{D} \subset \mathbf{C}$ that does not have the greatest element, then we can pick out an increasing sequence of elements (that is, sets) in $\mathbf{D}$ (and therefore, in $\mathbf{C}$). Otherwise, any subset of $\mathbf{C}$ has its greatest element. Thus, we take the greatest element of $\mathbf{C}$ as E_1, the greatest element of $\mathbf{C} - \{E_1\}$ as E_2, the greatest element of $\mathbf{C} - \{E_1, E_2\}$ as $E_3, \ldots$. Finally, we obtain a decreasing subsequence $\{E_n\}$ of $\mathbf{C}$. □

In the following, we give an example of the non-S-precompact space, in which the universe of discourse X is an uncountable set.

Example 2.32. Let X_0 be a set that contains at least two points, $X = X_1 \times X_2 \times \ldots \times X_n \times \ldots$ be an infinite-dimensional product space, where $X_i = X_0,\ i = 1, 2, \ldots$, and $\mathbf{C} = \mathbf{P}(X)$. Take $a \in X_0$ arbitrarily and denote

$$A_n = X_1 \times X_2 \times \ldots \times X_{n-1} \times \{a\} \times X_{n+1} \times \ldots.$$

A_n is an nth dimensional cylinder set based on $\{a\}$. Then, for such a set sequence $\{A_n\}$ there exists no subsequence that is convergent. In fact, for any given subsequence $\{A_{n_i}\} \subset \{A_n\}$, we take $b \in X_0 - \{a\}$ arbitrarily, and set

$$x_k = \begin{cases} a & \text{if } k = n_{2i}, i = 1, 2, \ldots \\ b & \text{else.} \end{cases}$$

Denote $x = (x_1, x_2, \ldots)$; then $x \in A_{n_{2i}}$, but $x \notin A_{n_{2i-1}}, i = 1, 2, \ldots$.
So,

$$x \in \limsup_i A_{n_i},$$

but

$$x \quad \liminf_i A_{n_i}.$$

That is, the subsequence $\{A_{n_i}\}$ does not converge. Therefore, $(X, \mathbf{C})$ is not S-precompact.

For a countable space we have an affirmative conclusion.

Theorem 2.22. *If X is countable, then $(X, \mathbf{C})$ is S-precompact.*

Proof. Denote $X = \{x_1, x_2, \ldots\}$. Any subset E of X corresponds uniquely to a binary number

$$\begin{aligned} b(E) &= {}_1b \times (1/2) + {}_2b \times (1/2)^2 + \ldots + {}_nb \times (1/2)^n + \ldots \\ &= 0.{}_1b_2b \ldots {}_nb \ldots \end{aligned}$$

in [0,1], where

$${}_ib = \begin{cases} 1 & \text{if } x_i \in E \\ 0 & \text{if } x_i \notin E. \end{cases}$$

We should note that such a correspondence is not one to one; for example, $\{x_1\}$ corresponds to 0.1, $\overline{\{x_1\}}$ corresponds to $0.0111\ldots$, but $0.1 = 0.0111\ldots$.

For an arbitrarily given set sequence $\{E_n\} \subset \mathbf{C}$, $\{E_n\}$ corresponds to a number sequence $\{b_n\} \subset [0, 1]$ with $E_n \mapsto b_n$. Since $\{b_n\}$ is bounded, there exists a convergent subsequence $\{b_{n_i}\}$. If all $b_{n_i}, i = 1, 2, \ldots,$ are constant, then the conclusion of this theorem is obviously true. Otherwise, we can suppose, with no loss of generality, that $\{b_{n_i}\}$ is strictly decreasing, and $b_{n_i} \to b \in [0, 1]$. If we adopt the restriction that b is represented by a binary number with infinitely many zeros after its decimal point, then b corresponds uniquely to a set E by the converse of the above-mentioned correspondence. It is not difficult to see that $\overline{E}$ must be an infinite set. Arbitrarily fixing a bit ${}_jb$ of b, we have ${}_jb_{n_i} = {}_jb$ when i is large enough. That is to say, there exist at most finitely many sets in $\{E_{n_i}\}$ that do not contain x_j when $x_j \in E$; and there exist at most finitely many sets in $\{E_{n_i}\}$ that contain x_j when $x_j \in \overline{E}$. This shows that $x_j \in \liminf_i E_{n_i}$ when $x_j \in E$ and $x_j \in \liminf_i \overline{E_{n_i}} = \overline{\limsup_i E_{n_i}}$ when $x_j \in \overline{E}$, namely,

$$\liminf_i E_{n_i} \supset E \text{ and } \overline{\limsup_i E_{n_i}} \supset \overline{E}.$$

The latter implies that

$$\limsup_i E_{n_i} \subset E.$$

So,

$$\limsup_i E_{n_i} \subset E \subset \liminf_i E_{n_i}.$$

This means that $\lim_i E_{n_i}$ exists.

Thus, we have proved that $(X, \mathbf{C})$ is S-precompact. □

If we consider a measurable space $(X, \mathbf{F})$ with $X \in \mathbf{F}$, then, by Theorem 2.16, $\mathbf{A[F]}$ is a partition of X. The quotient space $(X_{\mathbf{A}}, \mathbf{F_A})$ induced by $\mathbf{A[F]}$ from $(X, \mathbf{F})$

(Definition 2.19) is called the *reduced space* of (X, **F**). $\mathbf{F_A}$ and **F** are isomorphic. So, we can get a further theorem as follows.

Theorem 2.23. *If the reduced space of* (X, **F**) *is countable, then* (X, **F**) *is S-compact.*

Proof. The conclusion of this theorem follows from Theorem 2.22 and the fact that the S-precompact measurable space is S-compact. □

Theorem 2.24. *If* **F** *is a* σ*-algebra containing only countably many sets* (*that is,* **F** *is a countable class*), *then* (X, **F**) *is S-compact.*

Proof. Since F is a countable class of sets and **F** is closed under the formation of countable intersections, every atom $A(x/\ \mathbf{F}\ ) \in\ \mathbf{F}$. So, A[**F**] is a countable class, too. This shows that the reduced space $(X_A, \mathbf{F}_A)$ of (X, **F**) is countable. Therefore, by Theorem 2.23, (X, **F**) is S-compact. □

2.1.6 Relations, Posets, and Lattices

Definition 2.14. Let E and F be nonempty sets. A *relation* R from E to F is a subset of $E \times F$. If $(a, b) \in R$, we say "a is related to b" and write aRb; if $(a, b) \notin R$, we say "a is not related to b" and write $a\not{R}b$. In the special case when $R \subset E \times E$, we use "on E" instead of "from E to E."

Example 2.33. Let $X = \{a, b, c\}, E = \{a, b\},$ and $B = \{0, 1\}$. The characteristic function χ_E of E is a relation (denoted by R_E) from X to B. We have $aR_E1, bR_E1, cR_E0, a\not{R}_E0, b\not{R}_E0, c\not{R}_E1$.

Example 2.34. Let $X = (-\infty, \infty)$. The symbol $<$ with the common meaning "less than" is a relation on X, and it is a subset of $X \times X : R = \{(x, y)|x<y\}$. We have, for example $(1, 2) \in R, (-5, 5) \in R, (2, 1) \notin R,$ and $(1, 1) \notin R$.

Example 2.35. Let X be a nonempty set. The inclusion of sets $\subset$ is a relation on $\mathbf{P}(X)$; that is, $\{(E, F)|E \subset F\}$ is a subset of $\mathbf{P}(X) \times \mathbf{P}(X)$.

Example 2.36. Let E be any nonempty set. The *identity relation* on E, denoted by Δ_E, is the set of all pairs in $E \times E$ with equal elements:

$$\Delta_E = \{(a, a)|a \in E\}.$$

Example 2.37. Let $X = \{0, 1, 2, \ldots\}$. We can define a relation R_3 on X as follows: aR_3b iff $a = b$ (mod 3); that is, a and b have the same remainder when they are divided by 3.

Definition 2.15. Let R be a relation from E to F. The *inverse* of R, denoted by R^{-1}, is the relation from F to E which consists of those ordered pairs (b, a) for which aRb; that is $R^{-1} = \{(b, a)|(a, b) \in R\}$.

It is easy to see that

$$aRb \Leftrightarrow bR^{-1}a$$

and, therefore, we have the following proposition.

Proposition 2.20. $(R^{-1})^{-1} = R$.

Example 2.38. Let R be the relation given in Example 2.34. Its inverse, $R^{-1} = \{(x,y)|y<x\} = \{(x,y)|x>y\}$, has the meaning "greater than" and is denoted by the symbol $>$.

Definition 2.16. A relation R on a set E is called:

(a) *reflexive* iff aRa for each $a \in E$;
(b) *symmetric* iff aRb implies bRa for any $a, b \in E$;
(c) *transitive* iff aRb and bRc implies aRc for any $a, b, c \in E$.

Definition 2.17. A relation R on a set E is called an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Example 2.39. The identity relation Δ, as defined in Example 2.36, is reflexive, symmetric, and transitive; hence, it is an equivalence relation.

Example 2.40. The relation defined in Example 2.34 ("less than," $<$) is neither reflexive nor symmetric, but it is transitive.

Example 2.41. Let $X = (-\infty, \infty)$. The relation described by the phrase "less than or equal to," which is usually denoted by the symbol $\leq$, is reflexive and transitive but it is not symmetric.

Example 2.42. The relation R_3 defined in Example 2.37 is reflexive, symmetric, and transitive; consequently, it is an equivalence relation.

Definition 2.18. A disjoint class $\{E_1, E_2, \ldots, E_n\}$ of nonempty subsets of E is called a *partition* of E iff $\bigcup_{i=1}^{n} E_i = E$.

Example 2.43. Let $X = \{a, b, c, d, e, f, g\}$, and let

(1) $A_1 = \{a, c, e\}, A_2 = \{b\}, A_3 = \{d, g\}$
(2) $B_1 = \{a, e, g\}, B_2 = \{c, d\}, B_3 = \{b, e, f\}$
(3) $C_1 = \{a, b, e, g\}, C_2 = \{c\}, C_3 = \{d, f\}$
(4) $D_1 = X$
(5) $E_1 = \{a\}, E_2 = \{b\}, E_3 = \{c\}, E_4 = \{d\}, E_5 = \{e\}, E_6 = \{f\}, E_7 = \{g\}$.

Then, classes $\{C_1, C_2, C_3\}$, $\{D_1\}$, and $\{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ are partitions of X, but $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ are not.

Example 2.44. Let $X = [0, \infty)$. The class $\{[n-1, n)|n = 1, 2, \ldots\}$ is a partition of X.

Definition 2.19. Let R be an equivalence relation on E. For each $x \in E$, the set $[x] = \{y|xRy\}$ is called an *equivalence class* of E (in fact, it is a subset of E). The class of all equivalence classes of E induced by R, denoted by E/R, is called the *quotient* of E by R, that is, $E/R = \{[x]|x \in E\}$.

Proposition 2.21. *Let R be an equivalence relation on a set E. Then*

$$[x] = [y] \Leftrightarrow xRy$$

for any $x, y \in E$, and E/R is a partition of E.

Example 2.45. For the relation R_3 defined in Example 2.37, the quotient X/R_3 is formed by the following three distinct equivalence classes:

$$E_0 = \{0, 3, 6, 9, \ldots\}$$
$$E_1 = \{1, 4, 7, 10, \ldots\}$$
$$E_2 = \{2, 5, 8, 11, \ldots\}$$

$\{E_0, E_1, E_2\}$ is a partition of $X = \{0, 1, 2, \ldots\}$.

Definition 2.20. A relation R on set E is called *antisymmetric* iff aRb and bRa imply $a = b$ for any $a, b \in E$.

Example 2.46. The relations given in Example 2.34, 2.35, and 2.41 are antisymmetric.

Definition 2.21. Let R be a relation on a set E. If R is reflexive, antisymmetric, and transitive, then R is called a *partial ordering on* E, and (E, R) is called a *partially ordered set* (or, *poset*).

Example 2.47. Referring to Example 2.35, the pair $(\mathbf{P}(X), \subset)$ is a partially ordered set.

Example 2.48. Referring to Example 2.41, the pair $(X, \leq)$ is a partially ordered set.

Example 2.49. Let $\overline{F}$ be the set of all generalized real-valued functions on $(-\infty, \infty)$. We define a relation $\leq$ on $\overline{F}$ as follows: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in (-\infty, \infty)$. The relation $\leq$ is a partial ordering on $\overline{F}$ and, therefore, $(\overline{F}, \leq)$ is a partially ordered set.

From now on we use $(P, \leq)$ to denote a partially ordered set.

Definition 2.22. Let $(P, \leq)$ be a partially ordered set and let $E \subset P$. An element a in P is called an *upper bound* of E iff $x \leq a$ for all $x \in E$. An upper bound a of E is called the *least upper bound* of E (or *supremum* of E) iff $a \leq b$ for any upper bound b of E. The least upper bound of E is denoted by $\sup E$ or $\vee E$. An element a in P is called a *lower bound* of E iff $a \leq x$ for all $x \in E$. A lower bound a of E is called the *greatest lower bound* of E (or *infimum* of E) iff $b \leq a$ for any lower bound b of E. The greatest lower bound of E is denoted by $\inf E$ or $\wedge E$.

When E consists of only two elements, say x and y, we may write $x \vee y$ instead of $\vee \{x, y\}$ and $x \wedge y$ instead of $\wedge \{x, y\}$.

Proposition 2.22. *If the least upper bound (or the greatest lower bound) of a set $E \subset P$ exists, then it is unique.*

Definition 2.23. A partially ordered set $(P, \leq)$ is called an *upper semilattice* (or *lower semilattice*) iff $x \vee y$ (or $x \wedge y$, respectively) exists for any $x, y \in P$. $(P, \leq)$. is called a *lattice* iff it is both upper semilattice and lower semilattice.

Example 2.50. The partially ordered set $(\mathbf{P}(X), \subset)$ is a lattice. For any sets $E, F \subset X, E \cup F = \sup\{E, F\}$ and $E \cap F = \inf\{E, F\}$.

Definition 2.24. A partially ordered set $(P, \leq)$ is called a *fully ordered set* or a *chain* iff either $x \leq y$ or $y \leq x$ for any $x, y \in P$.

Example 2.51. The partially ordered set $(X, \leq)$ of Example 2.41 is a fully ordered set.

Example 2.52. The partially ordered set $(\overline{F}, \leq)$, of Example 2.49 is not a fully ordered set.

Example 2.53. The partially ordered set $(\mathbf{P}(X), \subset)$ is not a fully ordered set if X consists of more than one point.

The fully ordered set $((-\infty, \infty), \leq)$ has many convenient properties. One of them, which is often used in this text, is expressed by the following proposition.

Proposition 2.23. *Let E be a set of real numbers. If E has an upper bound (or a lower bound), then sup E (or inf E) exists; furthermore, for any given $\varepsilon > 0$, there exists $x = x(\varepsilon) \in E$ such that sup $E \leq x + \varepsilon$* (or $x - \varepsilon \leq \inf E$, *respectively).*

2.2 Classical Measures

Let X be a nonempty set, $\mathbf{C}$ be a nonempty class of subsets of X, and $\mu : \mathbf{C} \to [0, \infty]$ be a nonnegative, extended real valued set function defined on $\mathbf{C}$.

Definition 2.25. A set E in $\mathbf{C}$ is called the *null set* (with respect to μ) iff $\mu(E) = 0$.

Definition 2.26. μ is *additive* iff

$$\mu(E \cup F) = \mu(E) + \mu(F)$$

whenever

$$E \in \mathbf{C}, F \in \mathbf{C}, E \cup F \in \mathbf{C}, \text{ and } , E \cap F = \emptyset.$$

Definition 2.27. μ is *finitely additive* iff

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$$

for any finite, disjoint class $\{E_1, E_2, \ldots, E_n\}$ of sets in $\mathbf{C}$ whose union is also in $\mathbf{C}$.

Definition 2.28. μ is *countably additive* iff

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for any disjoint sequence $\{E_n\}$ of sets in $\mathbf{C}$ whose union is also in $\mathbf{C}$.

Definition 2.29. μ is *subtractive* iff

$$E \in \mathbf{C}, F \in \mathbf{C}, E \subset F, F - E \in \mathbf{C}, \text{ and } \mu(E) < \infty$$

imply

$$\mu(F - E) = \mu(F) - \mu(E).$$

Theorem 2.25. *If μ is additive, then it is subtractive.*

Definition 2.30. μ is called a *measure* on $\mathbf{C}$ iff it is countably additive and there exists $E \in \mathbf{C}$ such that $\mu(E) < \infty$.

Example 2.54. If $\mu(E) = 0, \forall E \in \mathbf{C}$, then μ is a measure on $\mathbf{C}$.

Example 2.55. Let $\mathbf{C}$ contain at least one finite set. If $\mu(E) = |E|, \forall E \in \mathbf{C}$, where $|E|$ is the number of those points that belong to E, then μ is a measure on $\mathbf{C}$.

Theorem 2.26. *If μ is a measure on $\mathbf{C}$ and $\emptyset \in \mathbf{C}$, then $\mu(\emptyset) = 0$. Moreover, μ is finitely additive.*

Definition 2.31. Let μ be a measure on $\mathbf{C}$. A set E in $\mathbf{C}$ is said to have a *finite measure* iff $\mu(E) < \infty$; E is said to have a *σ-finite measure* iff there exists a sequence $\{E_n\}$ of sets in $\mathbf{C}$ such that

$$E \subset \bigcup_{n=1}^{\infty} E_n \text{ and } \mu(E_n) < \infty, n = 1, 2, \ldots.$$

μ is *finite* (or *σ-finite*) on $\mathbf{C}$ iff every $\mu(E)$ is finite (or σ-finite, respectively) for every $E \in \mathbf{C}$.

Definition 2.32. Let μ be a measure on $\mathbf{C}$. μ is *complete* iff

$$E \in \mathbf{C}, F \subset E, \text{and } \mu(E) = 0$$

imply

$$F \in \mathbf{C}.$$

In other words, a measure on $\mathbf{C}$ is complete if and only if any subset of a null set belongs to $\mathbf{C}$.

Definition 2.33. μ is *monotone* iff

$$E \in \mathbf{C}, F \in \mathbf{C}, \textit{and}, E \subset F$$

imply

$$\mu(E) \leq \mu(F).$$

In the following, we take a semiring $\mathbf{S}$, a ring $\mathbf{R}$, and a σ-ring $\mathbf{F}$, respectively, as the class $\mathbf{C}$, and μ is always a nonnegative, extended real-valued set function on this class.

Theorem 2.27. *Let* $\mathbf{S}$ *be a semiring. If* μ *is additive on* $\mathbf{S}$*, then it is finitely additive and monotone.*

Definition 2.34. μ is *subadditive* iff

$$\mu(E) \leq \mu(E_1) + \mu(E_2)$$

whenever

$$E \in \mathbf{C},\ E_1 \in \mathbf{C}, \quad E_2 \in \mathbf{C},\ \text{and } E = E_1 \cup E_2.$$

Definition 2.35. μ is *finitely subadditive* iff

$$\mu(E) \leq \sum_{i=1}^{n} \mu(E_i)$$

for any finite class $\{E_1, E_2, \ldots, E_n\}$ of sets in $\mathbf{C}$ such that $E = \bigcup_{i=1}^{n} E_i \in \mathbf{C}$.

Definition 3.36. μ is *countably subadditive* iff

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

for any sequence $\{E_i\}$ of sets in $\mathbf{C}$ such that $E = \bigcup_{i=1}^{\infty} E_i \in \mathbf{C}$.

Theorem 2.28. *If* μ *is countably subadditive and* $\mu(\emptyset) = 0$*, then* μ *is finitely subadditive.*

Definition 2.37. Let $E \in \mathbf{C}$. μ is *continuous from below at* E iff

$$\{E_n\} \subset \mathbf{C},\ E_1 \subset E_2 \subset \ldots,\ \text{and } \lim_n E_n = E$$

imply

$$\lim_n \mu(E_n) = \mu(E);$$

μ is *continuous from above at* E iff

$$\{E_n\} \subset \mathbf{C},\ E_1 \supset E_2 \supset \ldots, \mu(E_1) < \infty,$$

and

$$\lim_n E_n = E$$

imply

$$\lim_n \mu(E_n) = \mu(E).$$

μ is *continuous from below* (*on* $\mathbf{C}$) iff it is continuous from below at every set in $\mathbf{C}$; μ is *continuous from above* (*on* $\mathbf{C}$) iff it is continuous from above at every set in $\mathbf{C}$; μ is *continuous* iff it is both continuous from below and continuous from above.

Theorem 2.29. *If μ is a measure on a semiring $\mathbf{S}$, then μ is countably subadditive and continuous.*

Definition 2.38. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be classes of subsets of X, $\mathbf{C}_1 \subset \mathbf{C}_2$, and μ_1 and μ_2 be set functions on $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively. μ_2 is called an *extension* of μ_1 iff $\mu_1(E) = \mu_2(E)$ whenever $E \in \mathbf{C}_1$.

Let $\mathbf{S}$ be a semiring, $\mathbf{R}(\mathbf{S})$ be the ring generated by $\mathbf{S}$. Since any set in $\mathbf{R}(\mathbf{S})$ can be expressed by a disjoint finite union of sets in $\mathbf{S}$, we have the following extension theorem for a measure on $\mathbf{S}$.

Theorem 2.30. *If μ is a measure on $\mathbf{S}$, then there is a unique measure $\overline{\mu}$ on $\mathbf{R}(\mathbf{S})$ such that $\overline{\mu}$ is an extension of μ. If μ is finite or σ-finite, then so is $\overline{\mu}$.*

The extension of μ (on $\mathbf{S}$) may also be denoted by $\overline{\mu}$ [on $\mathbf{R}(\mathbf{S})$] without any confusion.

Example 2.56. Let $X = (-\infty, \infty)$. $\mathbf{S} = \{[a, b) | -\infty < a \le b < \infty\}$ is a semiring. Define a set function μ on $\mathbf{S}$ by

$$\mu([a, b)) = b - a.$$

μ is countably additive, and, therefore, μ is a finite measure on **S**. μ can be extended to a finite measure on **R**(**S**), the class of all finite, disjoint unions of bounded, left closed, and right open intervals. More generally, if g is a finite, increasing, and left continuous real-valued function of a real variable, then

$$\mu_g([a,b)) = g(b) - g(a), \forall [a,b) \in \mathbf{S}$$

determines a finite measure μ_g on **S**, and it can be extended onto **R**(**S**).

Example 2.57. Let the ring **R** consist of all finite subsets of X and f be an extended real-valued, nonnegative function on X. If we define μ by

$$\mu(\{x_1, x_2, \ldots, x_n\}) = \sum_{i=1}^{n} f(x_i) \text{ for any } \{x_1, x_2, \ldots, x_m\} \in \mathbf{R} \text{ and } \mu(\emptyset) = 0,$$

then μ is a measure on **R**. In fact, the class **S** consisting of all singletons of X and the empty set Ø is a semiring. If we define μ on **S** by

$$\mu(\{x\}) = f(x) \text{ for any } x \in X \text{ and } \mu(\emptyset) = 0,$$

then μ is a measure on **S**, and the above-mentioned measure μ on **R** is just the extension of this measure on **S**.

Theorem 2.31. *If μ is a measure on a ring* **R**, *then it is continuous.*

Theorem 2.32. *Let μ be additive on a ring* **R** *and $\mu(\emptyset) = 0$. If μ is either continuous from below, or continuous from above at the empty set Ø and finite, then it is σ-additive on* **R**.

It should be noted that, on a semiring, an analogous conclusion of Theorem 2.32 is not true.

Example 2.58. Let $X = \{x | 0 \leq x \leq 1, x \text{ is a rational number}\}$, $\mathbf{S} = \{\{x | a \leq x \leq b,\ x \text{ is a rational number}\} \,|\, 0 \leq a \leq b \leq 1,\ a \text{ and } b \text{ are rational numbers}\}$. If we define μ on **S** by

$$\mu(\{x | a \leq x \leq b, x \text{ is a rational number}\}) = b - a,$$

then μ is finitely additive and continuous, but it is not countably additive.

Definition 2.39. A nonempty class **C** is *hereditary* iff

$$F \in \mathbf{C}$$

whenever

$$E \in \mathbf{C} \text{ and } F \subset E.$$

A hereditary class is a σ-ring if and only if it is closed under the formation of countable unions.

Example 2.59. The classes given in Examples 2.11, 2.14, and 2.16 are hereditary, and the last one is a hereditary σ-ring.

The hereditary σ-ring generated by a class $\mathbf{C}$, i.e., the smallest hereditary σ-ring containing $\mathbf{C}$, is denoted by $\mathbf{H}_\sigma(\mathbf{C})$

Theorem 2.33. $\mathbf{H}_\sigma(\mathbf{C})$ *is the class of all sets that can be covered by countably many sets in* $\mathbf{C}$.

Example 2.60. Let $X = (-\infty, \infty)$ and $\mathbf{C}$ be the class of all bounded intervals in X. Then $\mathbf{H}_\sigma(\mathbf{C}) = \mathbf{P}(X)$.

If $\mathbf{C}$ is a nonempty class closed under the formation of countable unions, then $\mathbf{H}_\sigma(\mathbf{C})$ is just the class of all sets that are subsets of some set in $\mathbf{C}$.

Definition 2.40. Let $\mathbf{H}_\sigma$ be a hereditary σ-ring, μ^* be an extended, real-valued, nonnegative set function on $\mathbf{H}_\sigma$. μ^* is called an *outer measure* iff it is monotone, countably subadditive, and such that $\mu^*(\emptyset) = 0$.

The same terminology concerning finiteness, σ-finiteness, and extension is used for outer measures as for measures.

Example 2.61. Let X be a finite set and $X \times X$ be a product space. $\mathbf{P}(X \times X)$ is a hereditary σ-ring. Define μ^* on $\mathbf{P}(X \times X)$ by

$$\mu^*(E) = |\text{Proj}(E)|, \forall E \in \mathbf{P}(X \times X),$$

where $\text{Proj}(E) = \{x|(x, y) \in E\}$. Then μ^* is a finite outer measure on $\mathbf{P}(X \times X)$.

Theorem 2.34. *If* μ *is a measure on a ring* $\mathbf{R}$*, then the set function* μ^* *on* $\mathbf{H}_\sigma(\mathbf{R})$ *defined by*

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) | E_n \in \mathbf{R},\ n = 1, 2, \ldots; E \subset \bigcup_{n=1}^{\infty} E_n\right\}$$

is an extension of μ *to an outer measure on* $\mathbf{H}_\sigma(\mathbf{R})$*; if* μ *is* σ*-finite, then so is* μ^*.

This outer measure μ^* is called the outer measure induced by the measure μ.

Definition 2.41. Let μ^* be an outer measure on a hereditary σ-ring $\mathbf{H}_\sigma$. A set $E \in \mathbf{H}_\sigma$ is μ^**-measurable* iff

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \overline{E}), \forall A \in \mathbf{H}_\sigma.$$

Theorem 2.35. *If* μ^* *is an outer measure on a hereditary* σ*-ring* $\mathbf{H}_\sigma$ *and if* $\overline{\mathbf{F}}$ *is the class of all* μ^* *-measurable sets, then* $\overline{\mathbf{F}}$ *is a* σ*-ring, and the set function* $\overline{\mu}$ *defined for every* $E \in \overline{\mathbf{F}}$ *by* $\overline{\mu}(E) = \mu^*(E)$ *is a complete measure on* $\overline{\mathbf{F}}$.

This measure $\overline{\mu}$ is called the measure induced by the outer measure μ^*.

Theorem 2.36. *Let μ be a measure on a ring* $\mathbf{R}$, *μ^* be the outer measure induced by μ. Then every set in* $\mathbf{R}$ *is μ^*-measurable, and therefore* $\mathbf{F}(\mathbf{R}) \subset \overline{\mathbf{F}}$

Theorem 2.37. *If μ is a σ-finite measure on a ring* $\mathbf{R}$, *then so is the measure $\overline{\mu}$ on* $\mathbf{F}(\mathbf{R})$, *and $\overline{\mu}$ is the unique extension of μ on* $\mathbf{R}$ *to* $\mathbf{F}(\mathbf{R})$.

Theorem 2.38. *If μ is a measure on a σ-ring* $\mathbf{F}$,

$$\mathbf{F}' = \{E\Delta N | E \in \mathbf{F},\ N \subset F \text{ for some } F \in \mathbf{F} \text{ with } \mu(F) = 0\},$$

then $\mathbf{F}'$ *is a σ-ring, and set function μ' defined for every $E \in F$* by $\mu'(E\Delta N) = \mu(E)$ *is a complete measure on* $\mathbf{F}'$.

This measure μ' is called the completion of μ.

Theorem 2.39. *If μ is a σ-finite measure on a ring* $\mathbf{R}$, *then* $\mathbf{F}' = \overline{\mathbf{F}}$, *and μ' is just identical with $\overline{\mu}$*

Theorem 2.40. *If μ is a σ-finite measure on a ring* $\mathbf{R}$, *then for every $\varepsilon > 0$ and every set $E \in \mathbf{F}(\mathbf{R})$ that has finite measure there exists a set $E_0 \in \mathbf{R}$ such that*

$$\overline{\mu}(E \,\Delta\, E_0) \leq \varepsilon.$$

Example 2.62. In Example 2.56, a finite measure μ on $\mathbf{R}(\mathbf{S})$ satisfying $\mu([a,b)) = b - a$ for any $[a - b) \in \mathbf{S} = \{[a,b)| -\infty < a \leq b < \infty\}$ is obtained. This measure μ can be extended uniquely to a σ-finite measure on a σ-ring $\mathbf{B} = \mathbf{F}(\mathbf{R}(\mathbf{S})) = \mathbf{F}(\mathbf{S})$, the class of all Borel sets (this class is also a σ-field, so-called Borel field on the real line). The complete measure $\overline{\mu}$ on $\overline{\mathbf{B}}$ is called a *Lebesgue measure* (the incomplete measure μ on $\mathbf{B}$ is usually called a Lebesgue measure as well), and the sets in $\overline{\mathbf{B}}$ are called Lebesgue measurable sets of the real line. More generally, if g is a finite, increasing, and left continuous real-valued function of a real variable, the measure μ_g on $\mathbf{R}(\mathbf{S})$ obtained in Example 2.56 can be extended uniquely to a complete measure $\overline{\mu}_g$ on a σ-field $\overline{\mathbf{F}}_g$ containing the Borel field, and the measure $\overline{\mu}_g$ is called a *Lebesgue-Stieltjes measure* induced by g. In particular, if g is a probability distribution function, then g can uniquely determine a probability measure on the Borel field $\mathbf{B}$ on the real line. At last, it should be noted that not all subsets of $X = (-\infty,\ \infty)$ are Lebesgue measurable.

2.3 Fuzzy Sets

Let X be a nonempty set considered as the universe of discourse. A *standard fuzzy set* in X (that is, in fact, a standard fuzzy subset of X) is characterized by a membership function $m : X \to [0, 1]$. A standard fuzzy set is called *normalized* if

$$\sup_{x \in X} m(x) = 1.$$

We use the same symbols, capital letters $A, B, C, \ldots$, to denote both standard fuzzy sets and ordinary (crisp) sets in classical set theory. Membership functions of standard fuzzy sets $A, B, C, \ldots$, are denoted by $m_A, m_B, m_C, \ldots$. If A denotes a standard fuzzy set, then $m_A(x)$ is called the *grade of membership* of x in A. The class of all standard fuzzy sets is denoted by $\tilde{\mathbf{P}}(X)$. Since any ordinary set E in $\mathbf{P}(X)$ can be defined by its characteristic function $\chi_E : X \to \{0, 1\}$, it is a special standard fuzzy set and, therefore, $\mathbf{P}(X) \subset \tilde{\mathbf{P}}(X)$. In this book, only standard fuzzy sets are considered, and we refer to them from now on as, simply, fuzzy sets.

Definition 2.42. If $m_A(x) \leq m_B(x)$ for any $x \in X$, we say that fuzzy set A is *included* in fuzzy set B, and we write $A \subset B$ or, equivalently, $B \supset A$. If $A \subset B$ and $B \subset A$, we say that A and B are *equal* (or, A equals B), which we write as $A = B$.

Definition 2.43. Let A and B be fuzzy sets. The *standard union* of A and B, $A \cup B$, is defined by

$$m_{A \cup B}(x) = m_A(x) \vee m_B(x), \quad \forall x \in X,$$

where $\vee$ denotes the maximum operator.

Definition 2.44. Let A and B be fuzzy sets. The *standard intersection* of A and B, $A \cap B$, is defined by

$$m_{A \cap B}(x) = m_A(x) \wedge m_B(x), \quad \forall x \in X,$$

where $\wedge$ denotes the minimum operator.

Similar to the way operations on ordinary sets are treated, we can generalize the standard union and the standard intersection for an arbitrary class of fuzzy sets: If $\{A_t | t \in T\}$ is a class of fuzzy sets, where T is an arbitrary index set, then $\cup_{t \in T} A_t$ is the fuzzy set having membership function $\sup_{t \in T} \; m_{A_t}(x), x \in X$, and $\cap_{t \in T} A_t$ is the fuzzy set having membership function $\inf_{t \in T} \; m_{A_t}(x), x \in X$.

Definition 2.45. Let A be a fuzzy set. The *standard complement* of A, $\overline{A}$, is defined by

$$m_{\overline{A}}(x) = 1 - m_A(x), \; \forall x \in X.$$

Two or more of the three basic operations can also be combined. For example, the difference $A - B$ of fuzzy sets A and B can be expressed as $A \cap \overline{B}$, so that

$$m_{A-B}(x) = \min[m_A(x), 1 - m_B(x)]$$

for all $x \in X$. Since only standard operations on fuzzy sets are used in this book, we omit from now on the adjective "standard" if there is no confusion.

Example 2.63. Let $X = \{a, b, c\}$ and let fuzzy sets *A, B*, and *C* be defined by the following membership functions

$$m_A(x) = \begin{cases} 0.4 & \text{if } x = a \\ 0.7 & \text{if } x = b \\ 0 & \text{if } x = c, \end{cases}$$

$$m_B(x) = \begin{cases} 0.6 & \text{if } x = a \\ 1 & \text{if } x = b \\ 0.2 & \text{if } x = c, \end{cases}$$

$$m_C(x) = \begin{cases} 0.1 & \text{if } x = a \\ 0 & \text{if } x = b \\ 1 & \text{if } x = c. \end{cases}$$

Then, $A \subset B$,

$$m_{A \cup C}(x) = \begin{cases} 0.4 & \text{if } x = a \\ 0.7 & \text{if } x = b \\ 1 & \text{if } x = c, \end{cases}$$

$$m_{A \cap C}(x) = \begin{cases} 0.1 & \text{if } x = a \\ 0 & \text{if } x = b \\ 0 & \text{if } x = c, \end{cases}$$

and

$$m_{\overline{B}}(x) = \begin{cases} 0.4 & \text{if } x = a \\ 0 & \text{if } x = b \\ 0.8 & \text{if } x = c. \end{cases}$$

Example 2.64. Fuzzy sets can be used to represent fuzzy concepts. Let *X* be a reasonable age interval of human beings: [0, 100]. Assume that the concept of "young" is represented by a fuzzy set *Y* whose membership function is

$$m_Y(x) = \begin{cases} 1 & \text{if } x \leq 25 \\ (40 - x)/15 & \text{if } 25 < x < 40 \\ 0 & \text{if } x \geq 40 \end{cases}$$

and the concept of "old" is represented by a fuzzy set *O* whose membership function is

$$m_O(x) = \begin{cases} 0 & \text{if } x \le 50 \\ (x-50)/15 & \text{if } 50 < x < 65 \\ 1 & \text{if } x \ge 65. \end{cases}$$

Then, the grade of membership of 28 years of age in Y is 0.8 while that of 45 years of age in O is 0. Consider now the complement of Y and O whose membership functions are

$$m_{\overline{Y}}(x) = \begin{cases} 0 & \text{if } x \le 25 \\ (x-25)/15 & \text{if } 25 < x < 40 \\ 1 & \text{if } x \ge 40, \end{cases}$$

and

$$m_{\overline{O}}(x) = \begin{cases} 1 & \text{if } x \le 50 \\ (65-x)/15 & \text{if } 50 < x < 65 \\ 0 & \text{if } x \ge 65. \end{cases}$$

These fuzzy sets represent the concepts of "not young" and "not old," respectively. Fuzzy set $\overline{Y} \cap \overline{O}$ whose membership function is

$$m_{\overline{Y} \cap \overline{O}}(x) = \begin{cases} 0 & \text{if } x \le 25 \\ (x-25)/15 & \text{if } 25 < x < 40 \\ 1 & \text{if } 40 \le x \le 50 \\ (65-x)/15 & \text{if } 50 < x < 65 \\ 0 & \text{if } x \ge 65, \end{cases}$$

represents the concept of "neither young nor old," that is, "middle-aged" (Fig. 2.2a–2.2e). Furthermore, we have $O \subset \overline{Y}$ that is, "old" implies "not young."

Theorem 2.41. *The standard operations of union, intersection, and complement of fuzzy sets have the following properties, where S_t and T are index sets:*

Involution: $\overline{\overline{A}} = A$

Commutativity: $A \cup B = B \cup A$

$A \cap B = B \cap A$

Associativity:

$$\bigcup_{t \in T} \left(\bigcup_{s \in S_t} A_s \right) = \bigcup_{s \in \cup_{t \in T} S_t} A_s$$

$$\bigcap_{t \in T} \left(\bigcap_{s \in S_t} A_s \right) = \bigcap_{s \in \cup_{t \in T} S_t} A_s$$

Distributivity:

$$B \cap \left(\bigcup_{t \in T} A_t \right) = \bigcup_{t \in T} (B \cap A_t)$$

$$B \cup \left(\bigcap_{t \in T} A_t \right) = \bigcap_{t \in T} (B \cup A_t)$$

Idempotence:

$$A \cup A = A$$

$$A \cap A = A$$

Absorption:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Absorption by X *and* Ø*:*

$$A \cup X = X$$

$$A \cap \emptyset = \emptyset$$

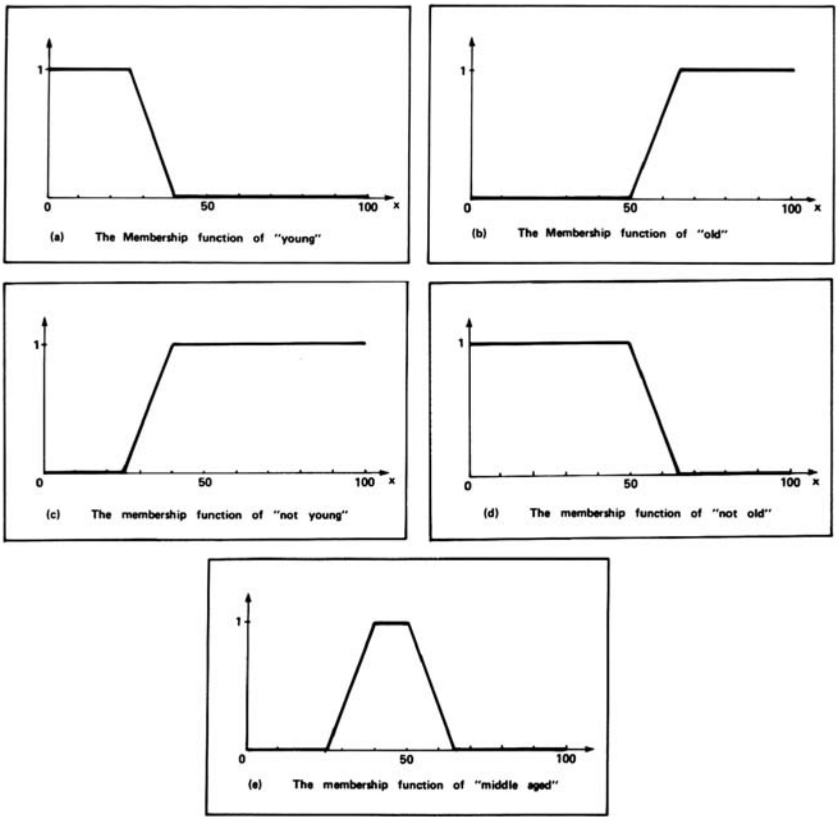

Fig. 2.2 Membership functions of fuzzy sets defined on the interval [0,100] and representing linguistic terms pertaining to age of human beings

Identity:

$$A \cup \emptyset = A$$

$$A \cap X = A$$

De Morgan's laws:

$$\overline{\bigcup_{t \in T} A_t} = \bigcap_{t \in T} \overline{A}_t$$

$$\overline{\bigcap_{t \in T} A_t} = \bigcup_{t \in T} \overline{A}_t$$

In comparison with crisp set operations (see Proposition 2.2), the law of contradiction and the law of excluded middle are not true for fuzzy sets. This is illustrated by the following example.

Example 2.65. X and A are given in Example 2.63. The complement $\overline{A}$ of A has a membership function

$$m_{\overline{A}}(x) = \begin{cases} 0.6 & \text{if } x = a \\ 0.3 & \text{if } x = b \\ 1 & \text{if } x = c. \end{cases}$$

We have

$$m_{A \cap \overline{A}}(x) = \begin{cases} 0.4 & \text{if } x = a \\ 0.3 & \text{if } x = b \\ 0 & \text{if } x = c \end{cases}$$
$$\neq m_{\emptyset}(x).$$

Similarly,

$$m_{A \cup \overline{A}}(x) = \begin{cases} 0.6 & \text{if } x = a \\ 0.7 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$
$$\neq m_X(x).$$

Definition 2.46. Let $A \in \tilde{\mathbf{P}}(X)$. The (crisp) set $\{x | m_A(x) > 0\}$ is called the *support* of A, and denoted by supp A.

Definition 2.47. Let $A \in \tilde{\mathbf{P}}(X)$. For any $\alpha \in [0, 1]$, the (crisp) sets $\{x | m_A(x) \geq \alpha\}$ and $\{x | m_A(x) > \alpha\}$ are called the α-*cut* and the *strong* α-cut of A, denoted by A_α and $A_{\alpha+}$, respectively.

Obviously, both A_α and $A_{\alpha+}$ are nonincreasing with respect to α. Clearly, the classes $\{A_\alpha | \alpha \in [0, 1]\}$ and $\{A_{\alpha+} | \alpha \in [0, 1]\}$ are nested.

Example 2.66. The fuzzy set Y is as given in Example 2.64. We have $Y_{0.2} = [0, 37]$ and $Y_{0.6} = [0, 31]$.

Theorem 2.42. *Let* $\{A_t | t \in T\} \subset \tilde{\mathbf{P}}(X)$. *Then, for any* $\alpha \in [0, 1]$,

$$\left(\bigcup_{t \in T} A_t\right)_{\alpha+} = \bigcup_{t \in T} (A_t)_{\alpha+}$$

and

$$\left(\bigcap_{t \in T} A_t\right)_{\alpha} = \bigcap_{t \in T} (A_t)_{\alpha}.$$

Theorem 2.43. *Let* $A \in \tilde{\mathbf{P}}(X)$. *Then*

$$\overline{A}_\alpha = \overline{(A_{(1-\alpha)+})}.$$

If we use a symbol $\alpha \cdot E$ to denote the fuzzy set whose membership function is

$$m(x) = \begin{cases} \alpha & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

for any $\alpha \in [0, 1]$ and any crisp set $E \in \mathbf{P}(X)$, then each fuzzy set can be fully characterized by its α-cuts, as expressed by the following theorem.

Theorem 2.44. (Decomposition Theorem). *For any* $A \in \tilde{\mathbf{P}}(X)$,

$$A = \bigcup_{\alpha \in [0,1]} \alpha \cdot A_\alpha.$$

Definition 2.48. Let $X = (-\infty, \infty)$. A normalized fuzzy set $A \in \tilde{\mathbf{P}}(X)$ is called a *fuzzy number* if A_α is a finite closed interval for each $\alpha \in (0, 1]$.

Definition 2.49. A *rectangular fuzzy number* is a fuzzy number with membership function having a form as

$$m(x) = \begin{cases} 1 & \text{if } x \in [a_l, a_r] \\ 0 & \text{otherwise,} \end{cases}$$

where $a_l, a_r \in R$ with $a_l \le a_r$.

A rectangular fuzzy number is identified with the corresponding vector $\langle a_l, a_r \rangle$ and is an interval number, essentially. Any crisp real number a can be regarded as a special rectangular fuzzy number with $a_l = a_r = a$.

Definition 2.50. A *triangular fuzzy number* is a fuzzy number with membership function

$$m(x) = \begin{cases} 1 & \text{if } x = a_0 \\ \dfrac{x - a_l}{a_0 - a_l} & \text{if } x \in [a_l, a_0) \\ \dfrac{x - a_r}{a_0 - a_r} & \text{if } x \in (a_0, a_r] \\ 0 & \text{otherwise,} \end{cases}$$

where $a_l, a_0, a_r \in R$, with $a_l \leq a_0 \leq a_r$.

A triangular fuzzy number is identified with the corresponding vector $\langle a_l, a_0, a_r \rangle$. A triangular fuzzy number is called *symmetric* if $a_0 - a_l = a_r - a_0$. Any crisp real number a can be regarded as a special triangular fuzzy number with $a_l = a_0 = a_r = a$.

Example 2.67. Let $X = (-\infty, \infty)$. Fuzzy sets A and B with

$$m_A(x) = \begin{cases} 0 & \text{if } x < 6 \text{ or } x > 12 \\ (x - 6)/3 & \text{if } 6 \leq x \leq 9 \\ (12 - x)/3 & \text{if } 9 < x \leq 12, \end{cases}$$

$$m_B(x) = \begin{cases} 0 & \text{if } x < 2 \text{ or } x > 4 \\ x - 2 & \text{if } 2 \leq x \leq 3 \\ 4 - x & \text{if } 3 < x \leq 4 \end{cases}$$

are triangular fuzzy numbers (Fig. 2.3).

Definition 2.51. A *trapezoidal fuzzy number* is a fuzzy number with membership function

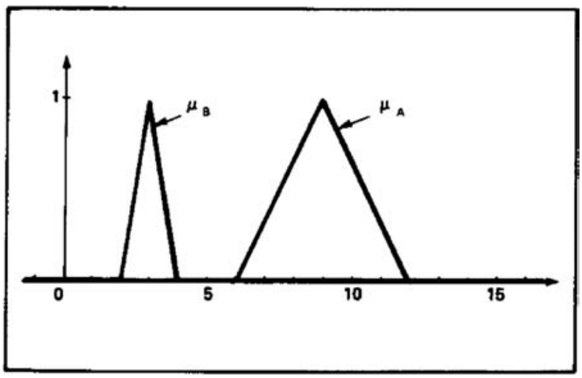

Fig. 2.3 Membership functions of triangular fuzzy numbers A and B

$$m(x) = \begin{cases} 1 & \text{if } x = [a_b, a_c] \\ \dfrac{x - a_l}{a_b - a_l} & \text{if } x \in [a_l, a_b) \\ \dfrac{x - a_r}{a_c - a_r} & \text{if } x \in (a_c, a_r] \\ 0 & \text{otherwise,} \end{cases}$$

where $a_l, a_b, a_c, a_r \in R$, with $a_l \leq a_b \leq a_c \leq a_r$.

A trapezoidal fuzzy number is identified with the corresponding vector $\langle a_l, a_b, a_c, a_r \rangle$. A trapezoidal fuzzy number is called *symmetric* if $a_b - a_l = a_r - a_c$. Any rectangular fuzzy number $\langle a_l, a_r \rangle$can be regarded as a special trapezoidal fuzzy number with $a_l = a_b$ and $a_c = a_r$. Similarly, any triangular fuzzy number $\langle a_l, a_0, a_r \rangle$ can be regarded as a special trapezoidal fuzzy number with $a_b = a_c = a_0$. Of course, any crisp real number a can be regarded as a special trapezoidal fuzzy number with $a_l = a_b = a_c = a_r = a$.

Example 2.68. Fuzzy sets Y, M, and O discussed in Examples 2.64 are trapezoidal fuzzy numbers.

Definition 2.52. Let $X = (-\infty,\ \infty)$. A fuzzy set $A \in \tilde{\mathbf{P}}(X)$ is called convex, if for any $x_1, x_2, x_3 \in X$,

$$m_A(x_2) \geq m_A(x_1) \wedge m_A(x_3)$$

where $x_1 \leq x_2 \leq x_3$.

Theorem 2.45. *Any fuzzy number is a convex fuzzy subset of* $(-\infty, \infty)$, *and its membership function is upper semicontinuous.*

The following extension principle introduced by Zadeh [1975] is a useful tool for extending nonfuzzy mathematical concepts to fuzzy sets (to fuzzify classical mathematical concepts).

Extension Principle. Let $X_1, X_2, \ldots, X_n$, and Y be nonempty (crisp) sets, $X = X_1 \times X_2 \times \cdots \times X_n$ be the product set of $X_1, X_2 \ldots, X_n$, and f be a mapping from X to Y. Then, for any given n fuzzy sets $A_i \in \tilde{\mathbf{P}}(X_i), i = 1, 2, \ldots, n$, we can induce a fuzzy set $B \in \tilde{\mathbf{P}}(Y)$ through f such that

$$m_B(y) = \sup_{y=f(x_1,x_2,\ldots,x_n)} \min[m_{A_1}(x_1), m_{A_2}(x_2), \ldots, m_{A_n}(x_n)],$$

where we use the convention that

$$\sup_{x \in \emptyset}\{x | x \in [0, \infty]\} = 0$$

when $f^{-1}(y) = \emptyset$.

As a special case, if $*$ is a binary operator on points in the universe of discourse X, then, by using the extension principle, we can obtain a binary operator $*$ (we use the same symbol) on fuzzy sets in $\tilde{\mathbf{P}}(X)$:

$$m_{A*B}(z) = \sup_{x*y=z} [m_A(x) \wedge m_B(y)], \forall z \in X,$$

where $A, B \in \tilde{\mathbf{P}}(X)$.

Now, we can use the extension principle to define addition, subtraction, multiplication, and division operations on fuzzy numbers, which are generalizations of the corresponding operations on real numbers.

Definition 2.53. Let A and B be fuzzy numbers. Then $A + B$, $A - B$, $A \cdot B$ and A/B are defined by

$$m_{A+B}(z) = \sup_{x+y=z} [m_A(x) \wedge m_B(y)],$$

$$m_{A-B}(z) = \sup_{x-y=z} [m_A(x) \wedge m_B(y)],$$

$$m_{A\cdot B}(z) = \sup_{x\cdot y=z} [m_A(x) \wedge m_B(y)],$$

and

$$m_{A/B}(z) = \sup_{x/y=z, y\neq 0} [m_A(x) \wedge m_B(y)] \text{ (when } 0 \notin \text{supp } B)$$

for any $z \in X$, respectively.

Example 2.69. Fuzzy numbers A and B are given in Example 2.67. Then we have

$$m_{A+B}(x) = \begin{cases} 0 & \text{if } x < 8 \text{ or } x > 16 \\ (x-8)/4 & \text{if } 8 \le x \le 12 \\ (16-x)/4 & \text{if } 12 < x \le 16 \end{cases}$$

(Fig. 2.4), and

$$m_{A-B}(x) = \begin{cases} 0 & \text{if } x < 2 \text{ or } x > 10 \\ (x-2)/4 & \text{if } 2 \le x \le 16 \\ (10-x)/4 & \text{if } 6 < x \le 10 \end{cases}$$

(Fig. 2.5). Viewing the real number 3 as a fuzzy number, we have $A = 3 \cdot B$ and $B = A/3$.

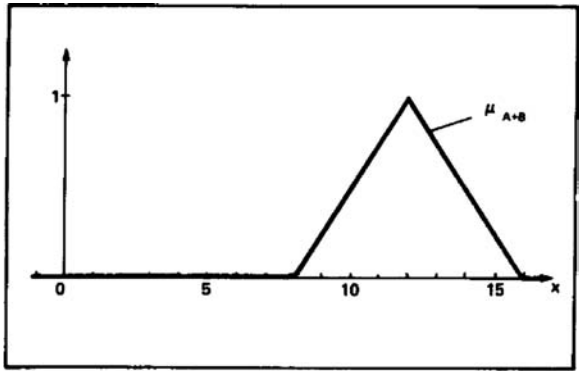

Fig. 2.4 Membership function of $A + B$

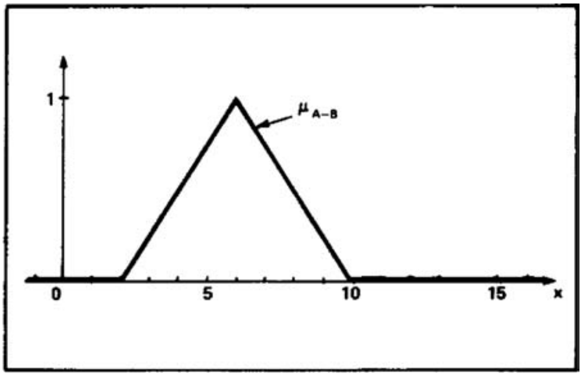

Fig. 2.5 Membership function of $A–B$

Definition 2.54. A *fuzzy partition* of X is a class of nonempty fuzzy sets defined on X, $\{A_i | i \in I\}$, such that

$$\sum_{i \in I} m_{A_i}(x) = 1$$

for all $x \in X$.

Clearly, any fuzzy set on X and its standard complement is a fuzzy partition of X. The three fuzzy sets that represent the concepts of *young, old,* and *middle-aged* in Example 2.64 form a fuzzy partition of the interval [0, 100].

Notes

2.1. The basic knowledge on sets and classes can be found in numerous books, including the classic book by Halmos [1950]. For a complete and up-to-date coverage of classical set theory, we recommend the book by Jech [2003].

2.2. The concepts of S-precompact and S-compact were introduced by Wang [1990b].
2.3. The concept of σ-algebra can be generalized to fuzzy σ-field, which is a class of fuzzy sets. This issue is discussed by Qiao [1990], as well as in Chapter 14.
2.4. Basic concepts of classical measure theory are introduced in Section 2.2 following the terminology and notation employed in the classic book on classical measure theory by Halmos [1950].
2.5. Standard fuzzy sets as well as standard operations on fuzzy sets were introduced in the seminal paper by Zadeh [1965]. Several other types of fuzzy sets were introduced later [Klir, 2006], but they are not considered in this book. Set theoretic operations on fuzzy sets are not unique. Intersections and unions of standard fuzzy sets are mathematically captured by operations known as triangular norms and conorms (or t-norms and t-conorms) [Klement et al., 2000, Klir and Yuan, 1995]. Complements of standard fuzzy sets are captured by monotone nonincreasing functions $\mathrm{c}\colon [0,1] \to [0,1]$ such that $\mathrm{c}(0) = 1$ and $\mathrm{c}(1) = 0$ [Klir and Yuan, 1995]. The standard intersection and union operations are the only *cutworthy* operations among the t-norms and t-conorms, which means that they are preserved in α-cuts for all $\alpha \in (0,1]$ in the classical sense. That is, $(A \cap B)_\alpha = A_\alpha \cap B_\alpha \text{and} (A \cup B)_\alpha = A_\alpha \cup B_\alpha$. No complements of fuzzy sets are cutworthy [Klir and Yuan, 1995].
2.6. In addition to operations of intersection, union, and complementation on fuzzy sets, it is perfectly meaningful to employ also *averaging operations* on fuzzy sets [Klir and Yuan, 1995].
2.7. For the sake of simplicity, we restrict in this book to triangular fuzzy numbers. A more general concept of a fuzzy number (sometimes called a fuzzy interval) involves nonlinear functions and its α-cut for $\alpha = 1$might be an interval of real numbers [Klir and Yuan, 1995].
2.8. Arithmetic operations on fuzzy numbers introduced in Definition 2.53 form a basis for the so-called *standard fuzzy arithmetic*, which is based on the assumption that there are no constraints among the fuzzy numbers involved. If this assumption is not warranted, the constraints must be taken into account. Principles of *constrained fuzzy arithmetic* are discussed in [Klir, 1997, 2006; Klir and Pan, 1998].
2.9. The literature on fuzzy set theory has been rapidly growing, especially during the last twenty years or so. Two important handbooks, edited by Ruspini et al. [1998] and Dubois and Prade [2000], are recommended as convenient sources of information on virtually any aspect of fuzzy set theory. From among the growing number of textbooks on fuzzy set theory, any of the following general textbooks is recommended for further study: [Klir and Yuan, 1995a], [Lin and Lee, 1996], [Nguyen and Walker, 1997], [Pedrycz and Gomide, 1998], and Zimmermann [1996]. Another valuable resource is the following pair of books that contain classical papers on fuzzy set theory by Lotfi A. Zadeh, the founder of fuzzy set theory: [Yager et al., 1987] and [Klir and Yuan, 1996].

Exercises

2.1. Let $X = (-\infty,\ \infty)$. Explain the following sets and classes in natural language:

(a) $\{X|0 \leq x \leq 1\}$;
(b) $\{X|x < 0\}$;
(c) $\{\{x\}|x \in X\}$;
(d) $\{E|E \subset X\}$.

2.2. Let $X_1 = X_2 = (-\infty, \infty), X = X_1 \times X_2$. Use shading to indicate the following sets on the Euclidean plane:

(a) $\{(x_1, x_2)|x_1 + x_2 > 1\}$;
(b) $\{(x_1, x_2)|x_1^2 \leq x_2\}$;
(c) $\{(x_1, x_2)|x_2 > 5\}$.

2.3. Prove the following equalities:

(a) $(E - G) \cap (F - G) = (E \cap F) - G$;
(b) $(E - F) - G = E - (F \cup G)$;
(c) $E - (F - G) = (E - F) \cup (E \cap G)$;
(d) $(E - F) \cap (G - H) = (E \cap G) - (F \cup H)$.

2.4. Prove the following equalities:

(a) $E \Delta F = F \Delta E$;
(b) $E \Delta (F \Delta G) = (E \Delta F) \Delta G$;
(c) $E \cap (F \Delta G) = (E \cap F) \Delta (E \cap G)$.

2.5. Prove that $\overline{\limsup_n E_n} = \liminf_n \overline{E_n}$.

2.6. Indicate the superior limit and the inferior limit of the set sequence $\{E_n\}$ where E_n is given as follows:

(a) $E_n = (n, n + 3/2)$;
(b) $E_n = [a_n, b_n]$ with $a_n = \min(0, (-2)^n); b_n = \max\ (0, (-2)^n)$;
(c) $E_n = \{n, n + 1, \ldots\}$;
(d) $E_n = \{x|nx$ is a natural number$\}$;
(e) $E_n = [1/n, n]$.

2.7. Which set sequence in Exercise 2.6 is monotone and for which does the limit exist?

2.8. Prove:

$$\overline{\lim_n}(E \cup F_n) = E \cup \overline{\lim_n}\ F_n,$$
$$\underline{\lim_n}(E - F_n) = E - \overline{\lim_n}\ F_n.$$

2.9. Prove Proposition 2.6 (4), (5), and (6).

2.10. Let $X = (-\infty,\ \infty) \times (-\infty,\ \infty) = \{(x, y) | -\infty < x < \infty, -\infty < y < \infty\}$. Prove that the class of all sets that have the form

$$\{(x, y) | -\infty < a_1 \leq x < b_1 < \infty, -\infty < a_2 \leq y < b_2 < \infty\}$$

is a semiring.

2.11. Prove Proposition 2.11.

2.12. Is a monotone class closed under the formation of limit operations of set sequence? Why or why not?

2.13. Prove that

$$\mathbf{F}_p(\mathbf{C}) = \left\{ \bigcap_{t \in T} \bigcup_{s \in S_t} E_s | E_s \in \mathbf{C}, S_t \text{ and } T \text{ are arbitrary index sets} \right\}$$

2.14. Categorize the class **C** given in the following descriptions as either a ring, semiring, algebra, σ-ring, σ-algebra, monotone class, or a plump class:

(a) $X = (-\infty, \infty)$, **C** is the class of all bounded, left open, and right closed intervals
(b) $X = \{1, 2, \ldots\}, \mathbf{C} = \{\{n,\ n+1, \ldots\} | n = 1, 2, \ldots\} \cup \{\emptyset\}$
(c) X is a nonempty set, E is a nonempty subset of X, $\mathbf{C} = \{F | E \subset F \subset X\}$
(d) X is a nonempty set, E is a nonempty subset of $X, E \neq X$, $C = \{F | F \subset E\}$
(e) X is a nonempty set, E is a nonempty subset of X, $\mathbf{C} = \{E\}$.

2.15 What are the rings (algebras, σ-rings, σ-algebras, monotone classes, plump classes, respectively) generated by the classes **C** given in Exercise 2.14?

2.16. Indicate what A [**C**] is for each of the following classes **C**:

(a) $X = \{1, 2, 3, 4, 5\},\ \mathbf{C} = \{A, B, C, D, E\}$, where $A = \{1, 2, 3\}$, $B = \{1, 2, 4\},\ C = \{1\}, D = \{2, 4\},\ E = \emptyset$
(b) $X = \{1, 2, 3, 4, 5\}, \mathbf{C} = \{A, B, C, D, E\}$, where $A = \{1, 2, 3\}$, $B = \{1, 2, 4\},\ C = \{1\}, D = \{1, 5\}, E = \{4, 5\}$
(c) $X = \{1, 2, 3, 4, 5\}$, $\mathbf{C} = \{A, B, C, D, E\}$, where $A = \{1, 2,\ 3\}$, $B = \{1,\ 2,\ 4\}, C = \{1\}, D = \{1, 5\}, E = \{1, 2\}$
(d) $X = \{1,2,3,4, 5\}$, $\mathbf{C} = \{A, B, C, D, E\}$, where $A = \{4, 5\},\ B = \{3, 5\}$, $C = \{2, 3, 4, 5\}, D = \{2, 3, 4\}, E = \{3, 4, 5\}$
(e) $X = (-\infty, \infty), \mathbf{C} = \{\emptyset, B, \overline{B}, X\}$, where $B = [0, \infty)$
(f) $X = (-\infty, \infty)$, **C** is the class of all open intervals in X
(g) $X = (-\infty, \infty)$, **C** is the class of all closed intervals in X
(h) $X = (-\infty, \infty),\ \mathbf{C} = \{A_n |\ n = 1, 2, \ldots\}$, where $A_n = [1 - 1/n, n],\ n = 1, 2, \ldots$.
(i) $X = \{1,\ 2, \ldots\},\ \mathbf{C} = \{A_n |\ n = 1,\ 2, \ldots\}$, where $A_n = \{n, n+1, \ldots\}$, $n = 1, 2, \ldots$.

2.17. In Exercise 2.16, which classes are closed under the formation of arbitrary intersections? Verify that $\mathbf{A}[\mathbf{C}] \subset \mathbf{C}$ for these classes $\mathbf{C}$.
2.18. In Exercise 2.16, which classes are AU-classes? Referring to Exercise 2.17, observe that Theorem 2.21 is applicable in some of these cases.
2.19. Prove that

$$A(x/\mathbf{C}_1 \cup \mathbf{C}_2) = A(x/\mathbf{C}_1) \cap A(x/\mathbf{C}_2) \text{ for any } x \in X.$$

May we regard Theorem 2.14 as a special case of this statement?

2.20 Prove that if

$$\mathbf{C}' = \left\{ \bigcap_{t \in T} E_t | E_t \in \mathbf{C},\ t \in T, T \text{ is an arbitrary index sets} \right\},$$

then $\mathbf{A}[\mathbf{C}'] = [\mathbf{C}]$. Can you find a class larger than $\mathbf{C}'$ for which this result still holds?

2.21. Determine the class $\mathbf{H}[\mathbf{C}]$ based upon each of the classes given in Exercise 2.16.
2.22. Prove that any set in $\mathbf{C}$ may be expressed by an intersection of the holes of $\mathbf{C}$, moreover, prove that any union of sets in $\mathbf{C}$ may be expressed by an intersection of the holes of $\mathbf{C}$.
2.23. Use one of the classes given in Exercise 2.16 to verify the conclusion given in Exercise 2.21.
2.24. Prove that

$$\mathbf{F}_p(\mathbf{C}) = \left\{ \bigcap_{t \in T} H_t | H_t \in \mathbf{H}[\mathbf{C}], T \text{ is an arbitrary index set} \right\}.$$

2.25. Prove that if $\mathbf{C}$ is closed under the formation of unions then $\mathbf{H}[\mathbf{C}] \subset \mathbf{C}$.
2.26. In Exercise 2.16, which classes are closed under the formation of unions? Verify that $[\mathbf{H}[\mathbf{C}] \subset \mathbf{C}$ for these classes.
2.27. Prove that if $\mathbf{C}$ is an AI-class then $X \notin \mathbf{C}$.
2.28. Let $\mathbf{C}$ be an AI-class. Prove that if $\mathbf{C} \supset \mathbf{H}[\mathbf{C}]$ then $\mathbf{C} = H[\mathbf{C}]$.
2.29. In Exercise 2.16, which classes are AI-classes? Referring also to Exercise 2.26, verify, for some class(es) $\mathbf{C}$, the statement suggested in Exercise 2.28.
2.30. Let X be the set of all integers and $\mathbf{C} = \mathbf{P}(X)$. Is $(X, \mathbf{C})$ S-compact?

Take $E_n = \{x | 0 \leq (-1)^n x \leq n, x \in X\}, n = 1, 2, \ldots$. Can you find a convergent subsequence of $\{E_n\}$?

2.31. Prove that, if $(X, \mathbf{C})$ is S-precompact and $A \subset X$, then $(A, \mathbf{C} \cap A)$ is S-precompact.

2.32. Prove that if $(X, \mathbf{C})$ is S-precompact and $\mathbf{C}' \subset \mathbf{C}$, then $(X, \mathbf{C}')$ is S-precompact.

2.33. Let $X = \{1, 2, 3, 4\}$. Consider the following relation on X:
$R_1 = \{(1,1), (1,3)\}$
$R_2 = \{(2,2), (3,2), (4,1)\}$
$R_3 = \{(1,4), (2,3)\}$
$R_4 = \{(1,1), (4,4)\}$
$R_5 = \{(1,1), (2,2), (3,3), (4,4), (1,4)\}$
$R_6 = \{(1,2), (2,1), (2,3), (1,3), (3,1)\}$
$R_7 = X \times X$
$R_8 = \emptyset$.

Determine whether or not each relation is

(a) reflexive
(b) symmetric
(c) transitive.

2.34. Let $X = (-\infty, \infty)$ and $\cong$ be the relation on $X \times X$ defined by

$$(x_1, y_1) \cong (x_2, y_2) \text{ iff } x_1 - y_1 = x_2 - y_2.$$

(a) Prove that $\cong$ is an equivalence relation.
(b) Find the equivalence class of (2,1).
(c) Find the quotient $X/\cong$.

2.35. Let R be a relation on X. Prove that $R \subset \Delta$ iff R is both symmetric and antisymmetric.

2.36. Let $X = \{0, 1, 2, \ldots\}$. A relation $\leq$ on $X \times X$ is defined as follows:

$$(x_1, y_1) \leq (x_2, y_2) \text{ iff } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Prove that $(X \times X, \leq)$ is a lattice. Show that by replacing $X \times X$ with the two-dimensional Euclidean space $(-\infty, \infty) \times (-\infty, \infty)$ we still obtain a lattice.

2.37. Let $X = [0, 1]$ and let $\mathbf{C}$ consist of $\emptyset, X, A = [0, 0.25), B = [0, 0.5), C = [0, 0.75)$, and $D = [0.25, 0.75)$. Consider a set function μ defined on $\mathbf{C}$ as follows: $\mu(\emptyset) = 0, \mu(A) = 2, \mu(B) = 2, \mu(C) = 4, \mu(D) = 2, \mu(X) = 4$.

(a) Show that μ is additive on $\mathbf{C}$.
(b) Can μ be extended to an additive function on the ring generated by $\mathbf{C}$?

2.38. Assuming that a set function μ is finitely additive on a ring $\mathbf{R}$, show that

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

for all $A, B \in \mathbf{R}$.

2.39. Let $X = \{x_1, x_2, x_3\}$. μ is a set function defined for all singleton of X with $\mu(\{x_i\}) = 2^{-i}, i = 1, 2, 3$. Extend μ to be a measure on the power set of X.

2.40. Let (X, **F**) be a measurable space, μ be a measure on **F**. Show that

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} E_i\right)$$

and

$$\mu\left(\bigcap_{i=1}^{n} E_i\right) = \sum_{I \subset \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \mu\left(\bigcup_{i \in I} E_i\right),$$

where $\{E_1, E_2, \ldots, E_n\}$ is a finite subclass of **F**.

2.41 Prove Theorem 2.28.

2.42. Prove Theorem 2.31.

2.43. Let (X, **F**) be a measurable space, and let μ be a measure on **F**. For any $A \subset X$, define set function μ' by $\mu'(A) = \inf\{\mu(E) | A \subset E \subset X\}$.

Does μ' coincide with μ' on **F**? Furthermore, is μ a measure on **P**(X)? If yes, prove it; if not, construct a counterexample.

2.44. Consider the fuzzy sets A, B, and C defined on the set (interval) $X = [0, 10]$ by the following membership functions:

$$m_A(x) = \begin{cases} x^2 & \text{when } x \in [0,1] \\ (2-x)^2 & \text{when } x \in (1,2] \\ 0 & \text{otherwise} \end{cases}$$

$$m_B(x) = \begin{cases} x-2 & \text{when } x \in [2, 3] \\ 4-x & \text{when } x \in (3, 4] \\ 0 & \text{otherwise} \end{cases}$$

$$m_C(x) = \max\{0, 2(x-3) - (x-3)^2\}.$$

Determine:

(a) plots of the given membership functions and those representing standard complements of A, B, and C, and C;
(b) the standard intersection and standard union of B and C;
(c) the α-cut representations of A, B, and C.

2.45. Viewing fuzzy sets A, B, C in Exercise 2.44 as fuzzy numbers on **R**, determine:

(a) $A + B + C$
(b) $A - B - C$
(c) $AB + C$ and $AB - C$
(d) BC/A.

2.46. Show that under the standard operations fuzzy sets do not satisfy the law of excluded middle and the law of contradiction.
2.47. Show that under the standard operations fuzzy sets satisfy DeMorgan's laws.
2.48. Considering arithmetic operations on triangular fuzzy numbers, show that their:

(a) additions and subtractions are again triangular fuzzy numbers;
(b) multiplications and divisions may not be triangular fuzzy numbers.

2.49. Show that for any pair of fuzzy sets A and B on X, the concepts of set inclusion, standard intersection, and standard union are cutworthy (see Note 2.5).
2.50. Prove Theorem 2.42, which states that the operation of standard intersection and standard union on fuzzy sets are cutworthy and strong cutworthy, respectively.
2.51. Prove Theorem 2.43, which demonstrates that the standard complement of fuzzy sets is not cutworthy.
2.52. Explain why averaging operations are meaningful for fuzzy sets (even when they degenerate to crisp sets), while they are not meaningful for classical sets.