Preface

We organized this CIME Course with the aim to bring together a group of top leaders on the fields of *calculus of variations* and *nonlinear partial differential equations*. The list of speakers and the titles of lectures have been the following:

- Luigi Ambrosio, *Transport equation and Cauchy problem for non-smooth vector fields.*
- Luis A. Caffarelli, *Homogenization methods for non divergence equations.*
- Michael Crandall, *The infinity-Laplace equation and elements of the calculus of variations in L-infinity.*
- Gianni Dal Maso, *Rate-independent evolution problems in elasto-plasticity: a variational approach.*
- Lawrence C. Evans, *Weak KAM theory and partial differential equations.*
- Nicola Fusco, *Geometrical aspects of symmetrization.*

In the original list of invited speakers the name of Pierre Louis Lions was also included, but he, at the very last moment, could not participate.

The Course, just looking at the number of participants (more than 140, one of the largest in the history of the CIME courses), was a great success; most of them were young researchers, some others were well known mathematicians, experts in the field. The high level of the Course is clearly proved by the quality of notes that the speakers presented for this Springer Lecture Notes.

We also invited Elvira Mascolo, the CIME scientific secretary, to write in the present book an overview of the history of CIME (which she presented at Cetraro) with special emphasis in calculus of variations and partial differential equations.

Most of the speakers are among the world leaders in the field of *viscosity solutions* of partial differential equations, in particular nonlinear pde's of *implicit type*. Our choice has not been random; in fact we and other mathematicians have recently pointed out a theory of *almost everywhere solutions* of pde's of *implicit type*, which is an approach to solve *nonlinear systems* of pde's. Thus this Course has been an opportunity to bring together experts of viscosity solutions and to see some recent developments in the field.

We briefly describe here the articles presented in this Lecture Notes.

Starting from the lecture by Luigi Ambrosio, where the author studies the well-posedness of the Cauchy problem for the homogeneous conservative continuity equation

$$\frac{d}{dt}\mu_t + D_x \cdot (b\mu_t) = 0\,, \qquad (t,x) \in I \times \mathbb{R}^d$$

and for the transport equation

$$\frac{d}{dt}w_t + b \cdot \nabla w_t = c_t\,,$$

where $b(t,x) = b_t(x)$ is a given time-dependent vector field in $\mathbb{R}^d$. The interesting case is when $b_t(\cdot)$ is not necessarily Lipschitz and has, for instance, a Sobolev or BV regularity. Vector fields with this "low" regularity show up, for instance, in several PDE's describing the motion of fluids, and in the theory of conservation laws.

The lecture of Luis Caffarelli gave rise to a joint paper with Luis Silvestre; we quote from their introduction:

"When we look at a differential equation in a very irregular media (composite material, mixed solutions, etc.) from very close, we may see a very complicated problem. However, if we look from far away we may not see the details and the problem may look simpler. The study of this effect in partial differential equations is known as *homogenization.* The effect of the inhomogeneities oscillating at small scales is often not a simple average and may be hard to predict: a geodesic in an irregular medium will try to avoid the bad areas, the roughness of a surface may affect in nontrivial way the shapes of drops laying on it, etc... The purpose of these notes is to discuss three problems in homogenization and their interplay.

In the first problem, we consider the homogenization of a free boundary problem. We study the shape of a drop lying on a rough surface. We discuss in what case the homogenization limit converges to a perfectly round drop. It is taken mostly from the joint work with Antoine Mellet *(see the precise references in the article by Caffarelli and Silvestre in this lecture notes).* The second problem concerns the construction of plane like solutions to the minimal surface equation in periodic media. This is related to homogenization of minimal surfaces. The details can be found in the joint paper with Rafael de la Llave. The third problem concerns existence of homogenization limits for solutions to fully nonlinear equations in ergodic random media. It is mainly based on the joint paper with Panagiotis Souganidis and Lihe Wang.

We will try to point out the main techniques and the common aspects. The focus has been set to the basic ideas. The main purpose is to make this advanced topics as readable as possible."

Michael Crandall presents in his lecture an outline of the theory of the archetypal L^∞ variational problem in the calculus of variations. Namely, given an open $U \subset \mathbb{R}^n$ and $b \in C(\partial U)$, find $u \in C(\overline{U})$ which agrees with the boundary function b on ∂U and minimizes

$$\mathcal{F}_\infty(u, U) := \| |Du| \|_{L^\infty(U)}$$

among all such functions. Here $|Du|$ is the Euclidean length of the gradient Du of u. He is also interested in the "Lipschitz constant" functional as well: if K is any subset of $\mathbb{R}^n$ and $u : K \to \mathbb{R}$, its least Lipschitz constant is denoted by

$$\mathrm{Lip}\,(u, K) := \inf\left\{L \in \mathbb{R} : \ |u\,(x) - u\,(y)| \leq L\,|x - y|\,,\ \forall x, y \in K\right\}.$$

One has $\mathcal{F}_\infty(u, U) = \mathrm{Lip}\,(u, U)$ if U is *convex*, but equality does not hold in general.

The author shows that a function which is absolutely minimizing for Lip is also absolutely minimizing for $\mathcal{F}_\infty$ and conversely. It turns out that the absolutely minimizing functions for Lip and $\mathcal{F}_\infty$ are precisely the viscosity solutions of the famous partial differential equation

$$\Delta_\infty u \ = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0\,.$$

The operator Δ_∞ is called the "∞-Laplacian" and *"viscosity solutions"* of the above equation are said to be $\infty-$*harmonic*.

In his lecture Lawrence C. Evans introduces some new PDE methods developed over the past 6 years in so-called *"weak KAM theory"*, a subject pioneered by J. Mather and A. Fathi. Succinctly put, the goal of this subject is the employing of dynamical systems, variational and PDE methods to find "integrable structures" within general Hamiltonian dynamics. Main references *(see the precise references in the article by Evans in this lecture notes)* are Fathi's forthcoming book and an article by Evans and Gomes.

Nicola Fusco in his lecture presented in this book considers two model functionals: the *perimeter* of a set E in $\mathbb{R}^n$ and the *Dirichlet integral* of a scalar function u. It is well known that on replacing E or u by its *Steiner symmetral* or its *spherical symmetrization*, respectively, both these quantities decrease. This fact is classical when E is a smooth open set and u is a C^1 function. On approximating a set of finite perimeter with smooth open sets or a Sobolev function by C^1 functions, these inequalities can be extended by lower semicontinuity to the general setting. However, an approximation argument gives no information about the equality case. Thus, if one is interested in understanding when equality occurs, one has to carry on a deeper analysis, based on fine properties of sets of finite perimeter and Sobolev functions. Briefly, this is the subject of Fusco's lecture.

Finally, as an appendix to this CIME Lecture Notes, as we said Elvira Mascolo, the CIME scientific secretary, wrote an interesting overview of the history of CIME having in mind in particular *calculus of variations* and PDES.

We are pleased to express our appreciation to the speakers for their excellent lectures and to the participants for contributing to the success of the Summer School. We had at Cetraro an interesting, rich, nice, friendly atmosphere, created by the speakers, the participants and by the CIME organizers; also for this reason we like to thank the Scientific Committee of CIME, and in particular Pietro Zecca (CIME Director) and Elvira Mascolo (CIME Secretary). We also thank Carla Dionisi, Irene Benedetti and Francesco Mugelli, who took care of the day to day organization with great efficiency.

Bernard Dacorogna *and* Paolo Marcellini

Issues in Homogenization for Problems with Non Divergence Structure

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1 Introduction

When we look at a differential equation in a very irregular media (composite material, mixed solutions, etc.) from very close, we may see a very complicated problem. However, if we look from far away we may not see the details and the problem may look simpler. The study of this effect in partial differential equations is known as homogenization. The effect of the inhomogeneities oscillating at small scales is often not a simple average and may be hard to predict: a geodesic in an irregular medium will try to avoid the bad areas, the roughness of a surface may affect in nontrivial way the shapes of drops laying on it, etc...

The purpose of these notes is to discuss three problems in homogenization and their interplay.

In the first problem, we consider the homogenization of a free boundary problem. We study the shape of a drop lying on a rough surface. We discuss in what case the homogenization limit converges to a perfectly round drop. It is taken mostly from the joint work with Antoine Mellet [5].

The second problem concerns the construction of plane like solutions to the minimal surface equation in periodic media. This is related to homogenization of minimal surfaces. The details can be found in the joint paper with Rafael de la Llave [2].

The third problem concerns existence of homogenization limits for solutions to fully nonlinear equations in ergodic random media. It is mainly based on the joint paper with Panagiotis Souganidis and Lihe Wang [7].

We will try to point out the main techniques and the common aspects. The focus has been set to the basic ideas. The main purpose is to make this advanced topics as readable as possible. In every case, the original papers are referenced.

2 Homogenization of a Free Boundary Problem: Capillary Drops

The shape of a drop lying on a surface tries to minimize its energy for a given volume. The energy has a term proportional to the capillary area A between the water and the air, another term related to the contact area W between the drop and the surface, and a third term related to the gravitational potential energy.

$$\text{Energy} = \sigma A - \sigma\beta W + \Gamma \text{ Gravitational Energy}$$

For the time being, we will neglect the effect of gravity ($\Gamma = 0$) and consider $\sigma = 1$.

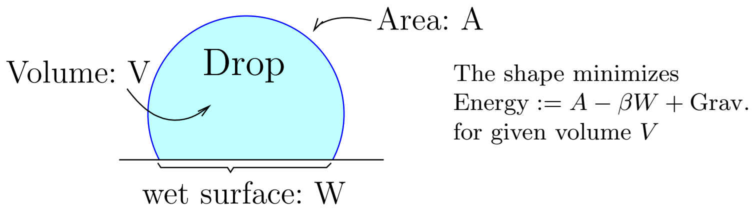


Fig. 1. A drop lying on a plane surface

The surface of the drop that is not in contact with the floor will have a constant mean curvature. We can see this perturbing its shape in a way that we preserve volume. If we add a bit of volume around a point and we subtract the same amount around another point, we obtain another admissible shape and so the corresponding area must increase. This implies that the mean curvature at both points must coincide.

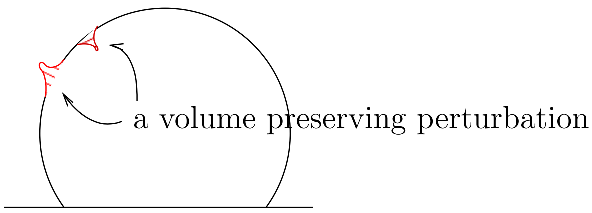


Fig. 2. Suitable perturbations show that the free surface has a constant mean curvature

The parameter β is a real number between -1 and 1 that depends on the surface and is the relative adhesion coefficient between the fluid and the surface. Its effect on the shape of the drop is to prescribe the contact angle at the free boundary: $\cos\gamma = \beta$.

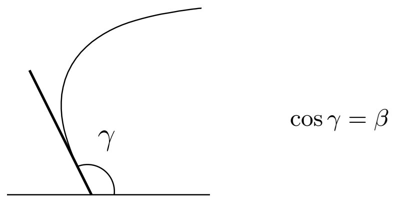


Fig. 3. The contact angle depends on β

A value $\beta > 0$ will cause the shape of the drop to expand trying to span a larger wet surface. When $\beta < 0$ (hydrophobic surface) on the other hand, the wet surface will tend to shrink. In the limit case $\beta = 1$ the wet surface would try to cover the whole plane, whereas for $\beta = -1$, the optimal shape would be a sphere that does not touch the floor at all.

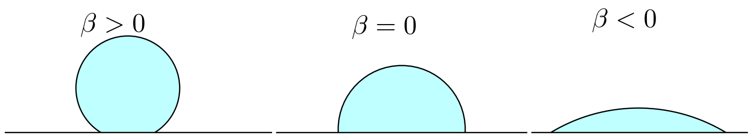


Fig. 4. Different shapes depending on the value of β

Under these conditions, it can be shown that there is a minimizer for the energy, and the shape of the corresponding drop is given by a sphere cap. The case we are interested however is when the drop rests on an irregular surface. Namely, we will consider a variable $\beta(x)$, oscillating fast and bounded so that $|\beta(x)| \leq \lambda < 1$. To capture the effect of a very oscillating adhesion coefficient, we fix a periodic function β and consider $\beta(x/\varepsilon)$ for a small ε. The energy is then given by

$$J_\varepsilon = A - \int_{\text{wet surface}} \beta\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x \tag{1}$$

Our purpose is to study the existence and regularity for a given $\varepsilon > 0$ of a shape that minimizes the energy. And we want to understand the way it behaves as $\varepsilon \to 0$. We will see that the absolute minimizers of J_ε converge uniformly to a spherical cap that corresponds to the minimizer of

$$J_0 = A - \langle\beta\rangle W$$

where $\langle\beta\rangle = \fint \beta\,\mathrm{d}x$ is the average of β.

However, the same conclusion cannot be taken for other critical points of J_ε. In general, the shapes of drops will not achieve an absolute minimizer. Any local minimum of J_ε would be a stable shape for a drop. The limits of these other solutions behave in a way that is harder to predict and many interesting phenomena can be observed. The most spectacular effect is the hysteresis: the contact angle depends on how the drop was formed. If the drop was formed by advancing the liquid, the final contact angle is greater than $\langle\beta\rangle$. If the equilibrium was achieved by receding the liquid (like in a process of evaporation), the angle obtained is less than $\langle\beta\rangle$.

The existence of a minimizer for each ε can be done in a very classical way in the framework of sets of finite perimeter. We will study some regularity properties of the minimizers. First we will show that the surface of the drop separates volume in a more or less balanced way. Secondly, we will see that the boundary of the contact set has a finite $n-1$ Hausdorff measure. Then we will use those estimates together with a stability result to show that the minimizers of J_ε converge to spherical caps as $\varepsilon \to 0$. To conclude this part, we will discuss the phenomena of Hysteresis.

2.1 Existence of a Minimizer

In order to prove existence, we have to work in the framework of boundaries of sets of finite perimeter.

Roughly, a set of finite perimeter Ω is the limit of polyhedra, Ω_k, of finite area, i.e.

$$|\Omega \Delta \Omega_k| \to 0$$

and $Area(\partial\Omega_k) \leq C$ for all k.

Sets of finite perimeter are defined up to sets of measure zero. We normalize E so that

$$0 < \left|\bar{E} \cap B_r(x)\right| < |B_r(x)| \quad \text{for all } x \in E \text{ and } r > 0$$

There is a well established theory for such sets. The classical reference is [13].

We will consider a set $E \subset \mathbb{R}^n \times [0, +\infty)$ that represents the shape of the drop. We denote (x, z) an arbitrary point with $x \in \mathbb{R}^n$ and $z \in [0, +\infty)$. Our energy functional reads

$$J_\varepsilon(E) = Area(\partial E \cap \{z > 0\}) - \int_{z=0} \beta\left(\frac{x}{\varepsilon}\right) \chi_E \, \mathrm{d}x \tag{2}$$

(In the following, we will omit the ε in J_ε unless it is necessary to stress it out).

The theory of finite perimeter set provides the necessary compactness results to show existence of a minimizer, as long as we restrict E to be a subset of a bounded set $\Gamma_{RT} := \{|x| < R, z < T\}$. Of course, we must take R and

T large enough so that we can fit at least one set E of volume V inside. To obtain an unrestricted minimizer of (2), we must prove that for R and T large enough, there is one corresponding minimizer E_{RT} that does not touch the boundary of Γ_{RT}. Since β is periodic, it is enough to show that E_{RT} remains bounded independently of R and T. If the diameter of E_{RT} is less than $R/2$, we can translate it an integer multiple of ε inside of Γ_{RT} to obtain an unrestricted minimizer. The detailed proof can be found in [5]

2.2 Positive Density Lemmas

The first regularity results we obtain for minimizers are related to the nondegenerate way the surface of the drop separates volume. All the proofs of these lemmas follow the same idea. An ordinary differential equation is constructed that exploits the nonlinearity of the isoperimetric inequality.

But before, we will make a few simple observations. Let E be a minimizer for a volume V_0, and let A be its free perimeter ($A = Area(\partial E \cap \{z > 0\})$). Above every point on the wet surface $E \cap \{z = 0\}$, there must be a point in the free surface: ∂E. Then

$$A \geq \int_{z=0} \chi_E \, \mathrm{d}x \geq \frac{1}{\lambda} \left| \int_{z=0} \beta\left(\frac{x}{\varepsilon}\right) \, \mathrm{d}x \right|$$

And therefore

$$(1-\lambda)A \leq J(E)$$

From the isoperimetric inequality we have $A \geq w_{n+1} V_0^{\frac{n}{n+1}}$. Since a sphere B with volume V_0 that does not touch the floor $\{z = 0\}$ is an admissible set, we also have:

$$(1-\lambda)A \leq J(E) \leq J(B) = w_{n+1} V_0^{\frac{n}{n+1}}$$

And thus we have both estimates:

$$c_0 V_0^{\frac{n}{n+1}} \leq A \leq C_1 V_0^{\frac{n}{n+1}}$$

Now we want to compare the minimum energy for two different volumes.

$$\min_{volume=V_0} J \leq \min_{volume=V_0+\delta} J \leq \min_{volume=V_0} J + C_1 V_0^{-\frac{1}{n+1}} \delta \tag{3}$$

The first inequality can be obtained simply taking the minimizer for $volume = V_0 + \delta$ and chopping a piece at the top of volume δ. Thus we obtain an admissible set of volume V_0 for which the energy J decreased.

For the second inequality, we consider the set E with volume V_0 that minimizes J and take a vertical dilation $E_t = \{(x,t) : (x,(1+t)^{-1}z) \in E\}$. Then for $t = \delta/V_0$, E_t is an admissible set of volume $V_0 + \delta$. The contact surface did not change, so its only difference in the energy is given by the free surface. Let A be the free perimeter of E, then the perimeter of E_t is less than $(1+t)A$, so their respective energies differ at most by tA. Then

$$\begin{aligned}\min_{volume=V_0+\delta} J - \min_{volume=V_0} J &\leq tA \\ &\leq \frac{\delta}{V_0}\, c_1 V_0^{\frac{n}{n+1}} \\ &\leq c_1 V_0^{-\frac{1}{n+1}} \delta\end{aligned}$$

The first lemma we want to prove is actually a classical result in minimal surfaces adapted to this case. We will come back to this lemma again when we study plane like minimal surfaces in periodic media in the second part of these notes.

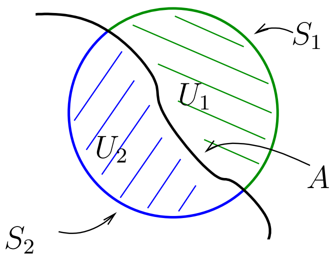
Before starting with the lemmas it is worth to point out an elementary fact of calculus that will come handy. If we have a nonnegative function u such that $u' \geq c u^{\frac{n+1}{n}}$ then u is a nondecreasing function that can stay equal to zero for any amount of time. But if $t_0 = \sup\{t : u(t) = 0\}$, then $u(t) \geq c(t - t_0)^n$ for any $t > t_0$.

Lemma 2.1. *Let $(x_0, z_0) \in \partial E$ with $z_0 > 0$. There exists a universal constant c such that for all $r < z_0$ we have*

$$|B_r(x_0, z_0) \cap E| \geq c r^{n+1}$$
$$|B_r(x_0, z_0) \setminus E| \geq c r^{n+1}$$

Proof. We define

$$U_1(r) = |B_r(x_0, z_0) \setminus E| \qquad S_1(r) = Area(\partial B_r(x_0, z_0) \setminus E)$$
$$U_2(r) = |B_r(x_0, z_0) \cap E| \qquad S_2(r) = Area(\partial B_r(x_0, z_0) \cap E)$$
$$A(r) = Area(B_r \cap \partial E)$$



By estimating $J(E \cup B_r)$ and $J(E \setminus B_r)$ and using (3), we can compare S_1 and S_2 to A.

$$J(E \cup B_r) \geq \min_{volume=V_0+U_1} J \geq J(E)$$
$$J(E) + S_1 - A \geq J(E)$$
$$S_1 - A \geq 0$$

We also know by the isoperimetrical inequality that $U_1^{\frac{n}{n+1}} \le C(A+S_1)$. If we combine this with the above inequality we obtain

$$U_1^{\frac{n}{n+1}} \le CS_1$$

But now we observe that $S_1(r) = U_1'(r)$, so we obtain the ODE: $U_1'(r) \ge cU_1^{\frac{n}{n+1}}$. Moreover, we know $U_1(0) = 0$ and $U_1(r) > 0$ for any $r > 0$. This implies the result of the lemma.

For U_2, a similar argument is done using the other inequality in (3).

With almost the same proof, we can also obtain a similar lemma for (x_0, z_0) in the boundary of the wet surface $E \cap \{z = 0\}$.

Lemma 2.2. *Given $x_0 \in \mathbb{R}^n$, let $\Gamma_{rt} = \{(x,z) : |x - x_0| \le r \wedge 0 \le z \le t\}$.*

There exist two universal constants $c_0, c_1 > 0$ such that for any minimizer E of J with volume V_0 such that

$$\{(x,t) : |x - x_0| \le r_0\} \subset E \quad (\textit{resp.} \ \subset \mathcal{C}E)$$
$$\exists z \in (0,t) \ \textit{such that} \ (x,z) \in \partial E$$

then

$$|\mathcal{C}E \cap \Gamma_{rt}| \ge c_0 r^{n+1} \quad (\textit{resp.}\, |E \cap \Gamma_{rt}| \ge c_0 r^{n+1})$$

for all $r < r_0$ (resp. for all $r < r_0$ such that $|E \cap \Gamma_{rt}| \le c_1 V_0$).

Remark. When we say $\{(x,t) : |x - x_0| \le r\} \subset E$, we actually mean that the trace of E on $\{(x,t) : |x - x_0| \le r\}$ is constant 1. Sets of finite perimeter have a well defined trace in L^1.

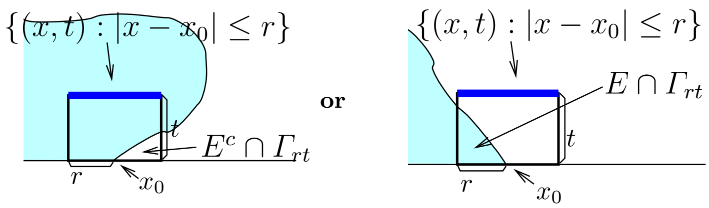
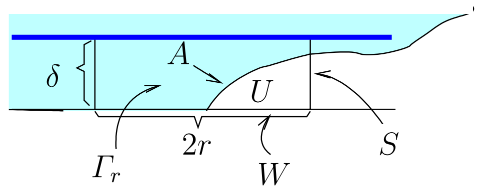


Fig. 5. Lemma 2.2

Proof. We proceed in a similar fashion as in the proof of Lemma 2.1. Let

$$U(r) = |\Gamma_{rt} \setminus E|$$
$$S(r) = Area(\partial\Gamma_{rt} \setminus E)$$
$$A(r) = Area(\Gamma_{rt} \cap \partial E)$$
$$W(r) = Area(\{z=0\} \cap \Gamma_{rt} \setminus E) = \int_{z=0 \wedge |x-x_0|<r} (1 - \chi_E)\, \mathrm{d}x$$

$$w(r) = \int_{z=0 \wedge |x-x_0|<r} \beta\left(\frac{x}{\varepsilon}\right)(1-\chi_E)\,\mathrm{d}x$$



Since above any point in the wet surface, there is a point in $\partial E \cap \Gamma_{rt}$, $W \le A$. Therefore, also $|w| \le \lambda A$.

By comparing $J(E)$ with $J(E \cup \Gamma_{rt})$, we get

$$\begin{aligned} J(E) &\le J(E \cup \Gamma_r) \\ J(E) &\le J(E) + S(r) - A(r) + w(r) \\ 0 &\le S(r) - A(r) + w(r) \\ 0 &\le S(r) - (1-\lambda)A(r) \end{aligned}$$

By the isoperimetric inequality we know that

$$U^{\frac{n}{n+1}} \le c(A + S + W)$$

Combining the above inequalities we obtain:

$$U^{\frac{n}{n+1}} \le C\, S(r)$$

And we observe that $S(r) = U'(r)$ to obtain the nonlinear ODE: $U'(r) \ge cU^{\frac{n}{n+1}}$. Moreover, $U(0) = 0$ and $U(r) > 0$ for any $r > 0$, then $U(r) > cr^{n+1}$.

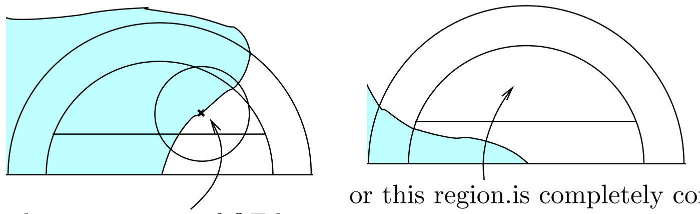
This proves the first case of the lemma. The other case follows almost in the same way but exchanging E and $\mathcal{C}E$. Since in that case we have to use the other inequality in (3), we must use that $|R \cap \Gamma_{rt}| \le c_1 V_0$ to control the extra term.

Corollary 2.1. *If* $(x_0, 0) \in \partial E$, *then*

$$\begin{aligned} \left|E \cap B_r^+(x_0,0)\right| &\ge c\, r^{n+1} \\ \left|\mathcal{C}E \cap B_r^+(x_0,0)\right| &\ge c\, r^{n+1} \end{aligned}$$

for every r *such that* $|E \cap B_r^+(x_0,0)| \le c_1 V_0$.

Proof. The set $B_{r/2}^+(x_0,0) \setminus \{z < \delta_0 r/2\}$ is either completely contained in E or $\mathcal{C}E$, or the set $B_{r/2}^+(x_0,0) \setminus \{z < \delta_0 r/2\} \cap \partial E$ is not empty.



In the first case, we apply Lemma 2.2 to obtain that both

$$\left|B_r^+(x_0,0) \cap E\right| \geq cr^{n+1}$$
$$\left|B_r^+(x_0,0) \setminus E\right| \geq cr^{n+1}$$

In the second case, there is a $(x_0, z_0) \in B_{r/2}^+(x_0,0) \setminus \{z < \delta_0 r/2\} \cap \partial E$, then we use 2.1 for a ball centered at (x_0, z_0) with radius $r/4$ to obtain also

$$\left|B_r^+(x_0,0) \cap E\right| \geq cr^{n+1}$$
$$\left|B_r^+(x_0,0) \setminus E\right| \geq cr^{n+1}$$

Corollary 2.2. *If* $(x_0, 0) \in \partial E$*, then*

$$Area(\partial E \cap B_r^+(x_0,0)) \geq c\, r^n$$

for every r *such that* $|E \cap B_r^+(x_0,0)| \leq c_1 V_0$.

Proof. This is a consequence of Corollary 2.1 combined with the isoperimetric inequality.

2.3 Measure of the Free Boundary

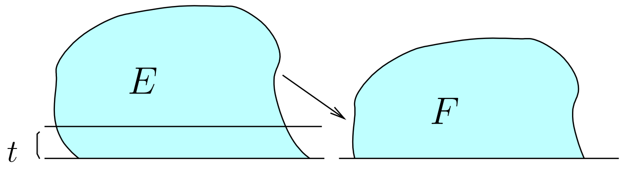
Our goal now is to show that the boundary of the wet surface $\partial(E \cap \{z = 0\})$ in $\mathbb{R}^n$ has a finite $n-1$ Hausdorff measure. We will do it by estimating the area of the drop close to it.

Now we will estimate the area of the drop that is close to the floor, and then we will obtain an estimate on the $n-1$ Hausdorff measure of the free boundary by a covering argument using the previous lemma.

Lemma 2.3. *There exists a constant* C *such that*

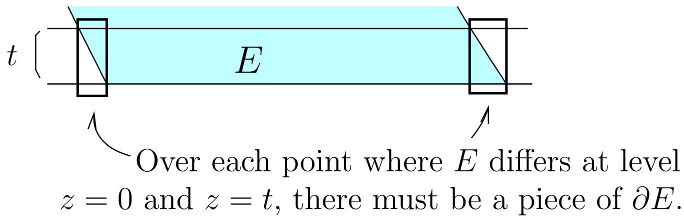
$$Area(\partial E \cap \{0 < z < t\}) \leq C V^{\frac{n-1}{n+1}} t$$

Proof. We will cut from E all the points for which $z < t$ and lower it to touch the floor again. We call F the set that we obtain (i.e. $F = \{(x,z) : (x, z+t) \in E\}$).



Since E is bounded, $|F| \leq |E| - Ct$, and thanks to (3) we have $J(E) \leq J(F) + Ct$. Moreover

$$J(E) - J(F) = Area(\partial E \cap \{0 < z < t\}) - \int_{z=0} \beta\left(\frac{x}{\varepsilon}\right)(\chi_E(x,0) - \chi_F(x,0))\,\mathrm{d}x$$



But if x belongs to the difference between $E \cap \{z = 0\}$ and $E \cap \{z = t\}$, then there must be a $z \in (0, t)$ such that $(x, z) \in \partial E$. Therefore

$$\int_{z=0} |\chi_E(x,0) - \chi_F(x,0)|\ \mathrm{d}x \leq Area(\partial E \cap \{0 < z < t\})$$

Thus we obtain

$$\min(1, 1-\lambda) Area(\partial E \cap \{0 < z < t\}) \leq C V^{\frac{n-1}{n+1}} t$$

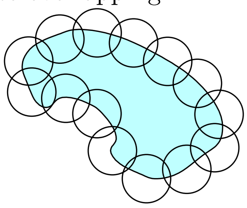
which concludes the proof.

We are now ready to establish the $n-1$ Hausdorff estimate on the free boundary.

Theorem 2.1. *The contact line $\partial(E \cap \{z = 0\})$ in $\mathbb{R}^n$ has finite $n-1$ Hausdorff measure and*

$$\mathcal{H}^{\frac{n-1}{n+1}}(\partial(E \cap \{z = 0\})) \leq C V^{\frac{n-1}{n+1}}$$

Proof. We consider a covering of $\partial(E \cap \{z = 0\})$ with balls of radius r and finite overlapping.



From Lemma 2.2, in each ball there is at least cr^n area. But by Lemma 2.3, the total area does not exceed $CV^{\frac{n-1}{n+1}} r$. Thus, the number of balls cannot exceed $CV^{\frac{n-1}{n+1}} r^{-(n-1)}$. Which proves the result.

2.4 Limit as $\varepsilon \to 0$

The $n-1$ Hausdorff estimate of the free boundary will help us prove that the minimizers E converge uniformly to a spherical cap as $\varepsilon \to 0$.

Let $\langle\beta\rangle$ be the average of β in the unit cube: $\langle\beta\rangle = \fint_{Q_1} \beta \, \mathrm{d}x$ and

$$J_0(E) = Area(\partial E \cap \{z > 0\}) + \langle\beta\rangle Area(E \cap \{z = 0\}) \tag{4}$$

As it was mentioned before, the minimizer of J_0 from all the sets with a given volume V is a spherical cap B_ρ^+ such that $\left|B_\rho^+\right| = V$ and the cosine of its contact angle is $\langle\beta\rangle$.

Let us check how different $J(E)$ and $J_0(E)$ are. Their only difference is in the term related to the wet surface. Recall that $\beta\left(\frac{x}{\varepsilon}\right)$ is periodic in cubes of size ε. For every such cube that is completely contained inside the wet surface of E it is the same to integrate $\beta\left(\frac{x}{\varepsilon}\right)$ or to integrate the average of β. The difference of $J(E)$ and $J_0(E)$ is then given only by the cells that intersect the boundary of $(E \cap \{z = 0\})$.

But according the the $n-1$ Hausdorff estimate of the free boundary, the number of such cells cannot exceed $CV^{\frac{n-1}{n+1}}\varepsilon^{1-n}$. Since the volume of each cell is ε^n we deduce:

$$|J_0(E) - J(E)| \le C\lambda V^{\frac{n-1}{n+1}}\varepsilon$$

The same conclusion can be taken for B_ρ^+:

$$\left|J_0(B_\rho^+) - J(B_\rho^+)\right| \le C\lambda V^{\frac{n-1}{n+1}}\varepsilon$$

And noticing that $J(E) \le J(B_\rho^+)$ and $J_0(B_\rho^+) \le J_0(E)$ we obtain

$$\left|J_0(E) - J_0(B_\rho^+)\right| \le C\lambda V^{\frac{n-1}{n+1}}\varepsilon$$

The convergence of E to B_ρ^+ is then a consequence of the following stability theorem whose proof we omit.

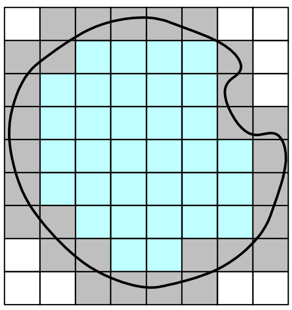


Fig. 6. In the inner cubes, it is the same to integrate $\beta(x/\varepsilon)$ or its average. The difference between J_0 and J is concentrated in the cells that intersect the free boundary

Theorem 2.2. *Let $E \subset B_R \times [0, R)$ such that $J_0(E) \leq J_0(B_\rho^+) + \delta$. Then there exists a universal $\alpha > 0$ and a constant C (depending on R) such that*

$$\left|E \triangle B_\rho^+\right| \leq C\delta^\alpha$$

Since E is bounded, this stability theorem tells us that $\left|E \triangle B_\rho^+\right|$ becomes smaller and smaller as $\varepsilon \to 0$. To obtain uniform convergence we have to use the regularity properties of E. By Lemma 2.1 or 2.2, if there was one point of ∂E far from ∂B_ρ^+, then there would be a fixed amount of volume of $E \triangle B_\rho^+$ around it, arriving to a contradiction. We state the theorem:

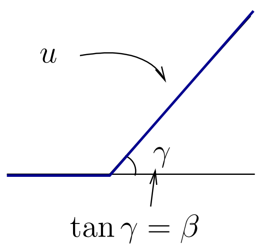
Theorem 2.3. *Given any $\eta > 0$, for ε small enough*

$$B_{(1-\eta)\rho}^+ \subset E \subset B_{(1+\eta)\rho}^+$$

2.5 Hysteresis

Although when we consider absolute minimizers of J_ε there are no surprises in the homogenization limit, in reality this behavior is almost never observed. When a drop is formed, its shape does not necessarily achieve an absolute minimum of the energy, but it stabilizes in any local minimum of J_ε. That is why to fully understand the possible shapes of drops lying on a rough surface, we must study the limits as $\varepsilon \to 0$ of all the critical points of J_ε.

Let us see a simplified equation in 1 dimension. Let u be the solution of the following free boundary problem:



$$\begin{aligned} u &\geq 0 && \text{in } [0,1] \\ u(0) &= 0 && \\ u(1) &= 1 && \\ u''(x) &= 0 && \text{if } u(x) > 0 \\ \frac{du}{dx^+} &= \beta\left(\frac{x}{\varepsilon}\right) && \text{for } x \in \partial\{u > 0\} \end{aligned}$$

This problem comes from minimizing the functional

$$J(u) = \int_0^1 |u'|^2 + \beta\left(\frac{x}{\varepsilon}\right)^2 \chi_{u>0} \, \mathrm{d}x$$

If β is constant, it is clear that there is only one solution, because only one line from $(1,1)$ hits the x axis with an angle $\gamma = \arctan \beta$. However, if β oscillates, there must be several solutions that correspond to several critical points of J_ε. There will be a solution hitting the x axis at the point x_0 as long as $\frac{1}{1-x_0} = \beta\left(\frac{x_0}{\varepsilon}\right)$. For a small ε this may happen at many points, as we can see in Figure 7.

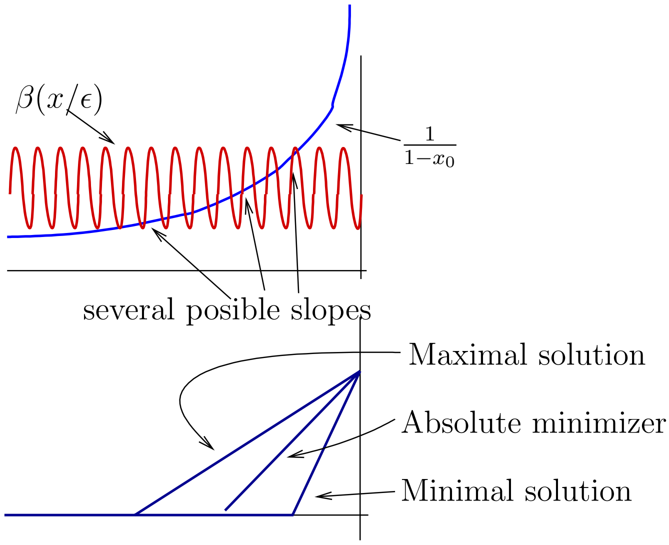


Fig. 7. Different solutions for a nonconstant β

Moreover, the set of possible slopes for the solutions gets more and more dense in the interval $[\min\beta, \max\beta]$ as ε gets small. As $\varepsilon \to 0$, we can get a sequence of solutions converging to a segment with any slope in that interval.

This example shows that the situation is not so simple. When we go back to our problem of the drop in more than one dimension, the expected possible slopes as $\varepsilon \to 0$ must be in the interval $[\arccos\max\beta, \arccos\min\beta]$. Exactly what they are depends on the particular geometry of the problem. If for example β depends only on one variable, let us say x_1, then when the free boundary aligns with the direction of x_1 we would expect to obtain a whole range of admissible slopes as in the $1D$ case. Let us sketch a proof in this case that there is a sequence of critical points of the functional that do not converge to a sphere cap as $\varepsilon \to 0$. We will construct a couple of barriers, and then find solutions that stay below them.

Suppose that β depends on only one variable and it is not constant. As we have shown in the previous section, the absolute minimizers converge to a sphere cap B_ρ^+ as $\varepsilon \to 0$. Let $S(x_1)$ be a function that touches B_ρ^+ at one end point $x_1 = -R$, but has a steeper slope at that point. Let us choose this slope $S'(-R) = \tan\alpha$ such that $\cos\alpha < \max\beta$, we can do this from the extra room that we have since β is not constant. Now let us continue $S(x_1)$ from that point first with a constant curvature larger than the curvature of B_ρ^+, and then continued as linear. Since S starts off with a steeper slope than B_ρ^+, we can make S so that $S > B_\rho^+$ for $x_1 > -R$. Now we translate S a tiny bit in the direction of x_1 to obtain S_1 so that $S_1 \leq B_\rho^+$ only in the set where S_1 has a positive curvature that is larger than the one of B_ρ^+. We construct a similar function S_2 in the other side of B_ρ^+. See Figure 8.

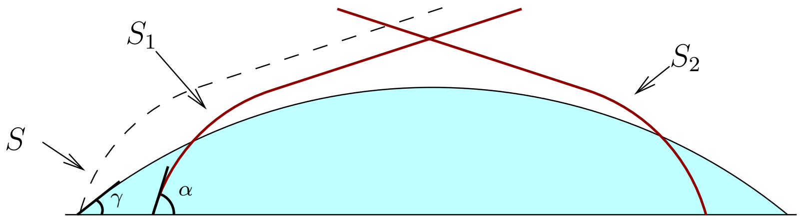


Fig. 8. Barrier functions

We will see that we can find a sequence of solutions for $\varepsilon \to 0$ that remains under S_1 and S_2. For suitable choices of ε, $\cos\alpha < \beta(-R)$ and also $\cos\alpha < \beta(R)$. For such ε, we minimize the energy J_ε constrained to remain below S_1 and S_2. In other words, we minimize J_ε from all the sets E subsets of

$$D = \{(x_1, x') : -R \le x_1 \le R \wedge z \le S_1(x_1) \wedge z \le S_2(x_1)\}$$

If E is the constrained minimizer, it will be a critical point (unconstrained) of J_ε as long as it does not touch the graphs of S_1 or S_2. Since only a tiny bit of B_ρ^+ is outside of D, $J_\varepsilon(E)$ will not differ from $J(B_\rho^+)$ much when ε is small. We can then apply the stability result of section 2.4 to deduce that ∂E remains in a neighborhood of ∂B_ρ^+. The curvature of ∂E will be constant where it is a free surface, and no larger than that value where it touches the boundary of D. Since ∂E is close to ∂B_ρ^+ everywhere, the curvature of the free part of ∂E cannot be very different from the curvature of ∂B_ρ^+. Therefore E cannot touch S_1 or S_2 in the part where these barriers are curved. The part where these barriers are straight is too far away from B_ρ^+, so E cannot reach that part either. It is only left to check the boundary $x_1 = \pm R$ and $z = 0$. But the contact angle of S_1 is smaller than $\arccos\beta(x_1)$ at those points, and then E cannot reach those points either. Thus, E must be a free minimizer. Since we can do this for ε arbitrarily small, when $\varepsilon \to 0$ we obtain limits of the homogenization problem that cannot be the sphere cap B_ρ^+ because they are trapped in a narrower strip $\{-R + \delta \le x_1 \le R + \delta\}$.

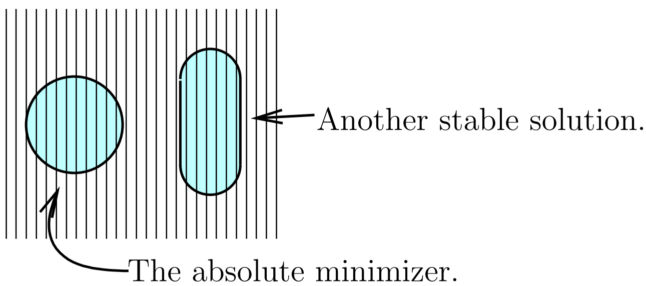


Fig. 9. Different drops can be formed on irregular surfaces

Other geometries may produce different variations. It is hard to predict what can be expected.

We may ask at this point what is then the shape that we will observe in a real physical drop. The answer is that it depends on how it was formed. If the equilibrium was reached after an expansion, then we can expect to see the largest possible contact angle. If on the other hand, the equilibrium was obtained after for example evaporation, then we can expect to see the least possible contact angle.

An interesting case is the drop lying on an inclined surface. If we consider gravity, there is no absolute minimizer for the energy, because we can slide down the drop all the way down and make the energy tend to $-\infty$. However, we see drops sitting on inclined surfaces all the time. The reason is that they stabilize in critical points for the energy. On the side that points down, we can see a larger contact angle than the one in the other side. This effect would not be possible in a ideal perfectly smooth surface.

2.6 References

The equations of capillarity can be found in [12]. The case of constant β is studied in [G].

The proof of Theorem 2.2, as well as the existence of a minimizer for each ε and a comprehensive development of the topic can be found in [5].

Lemma 2.2 is not as in [5]. There a different approach is taken that also leads to Corollaries 2.1 and 2.2. This modification was suggested by several people.

The phenomena of Hysteresis, and in particular the case of the drop on an inclined surface is discussed in [6]. Previous references for hysteresis are [17], [16] and [15].

Related methods are used for the problem of flame propagation in periodic media [3], [4].

3 The Construction of Plane Like Solutions to Periodic Minimal Surface Equations

The second homogenization problem that we would like to discuss is related to minimal surfaces in a periodic medium.

In two dimensions, minimal surfaces are just geodesics. Suppose we are given a differential of length $a(x, \boldsymbol{\nu})$ in $\mathbb{R}^2$, and given two points x, y we want to find the curve joining them with the minimum possible length. In other words, we want to minimize

$$d(x,y) = \inf L(\gamma) = \int_\gamma a(z, \boldsymbol{\sigma})\, ds$$

among all curves γ joining x to y.

Here s is the usual differential of length and σ the unit tangent vector. We consider a function $a(x,\sigma)$ that is strictly positive $(0<\lambda\le a(x,\sigma)\le\Lambda)$ and, to avoid the formation of Young measures (that is: oscillatory zig-zags) when trying to construct geodesics, it must satisfy

$$|v|a\left(x,\frac{v}{|v|}\right)\ \text{is a strictly convex cone.}$$

We assume that a is periodic in unit cubes. By that we mean that a is invariant under integer translations, i.e. $a(x+h,\sigma)=a(x,\sigma)$ for any vector h with integer coordinates. Let us also assume that a is smooth although this property is not needed. Due to the periodicity, at large distances $d(x,y)$ becomes almost translation invariant, since for any vector z there is a vector $\tilde z$ with integer coordinates such that $|z-\tilde z|\le\frac{\sqrt{n}}{2}$ and

$$\begin{aligned}|d(x+z,y+z)-d(x,y)| &= |d(x+z,y+z)-d(x+\tilde z,y+\tilde z)|\\ &\le \sqrt{n}\,\Lambda\end{aligned}$$

Another way of saying the same thing is to look at the geodesics from very far away, that is to rescale the medium by a very small ε,

$$a_\varepsilon(x,\boldsymbol{\sigma})=a\left(\frac{x}{\varepsilon},\boldsymbol{\sigma}\right)\ .$$

The distance becomes almost translation invariant

$$|d_\varepsilon(x,y)-d_\varepsilon(x+z,y+z)|\le\varepsilon\sqrt{n}\,\Lambda$$

and as ε goes to zero we obtain an effective norm $\|x\|=\lim_{\varepsilon\to0}d_\varepsilon(x,0)$.

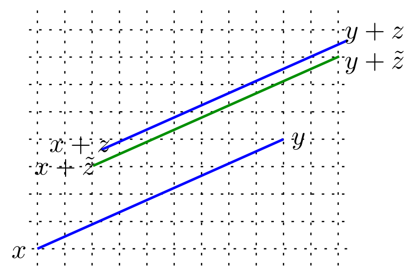


Fig. 10. The distance is almost translation invariant

The question we are interested to study is the following: given any line

$$L=\{\lambda\sigma,\lambda\in R\}$$

Can we construct a global geodesic S that stays at a finite distance from L? That is S remains trapped in a strip, around L whose width depends only on λ,Λ.

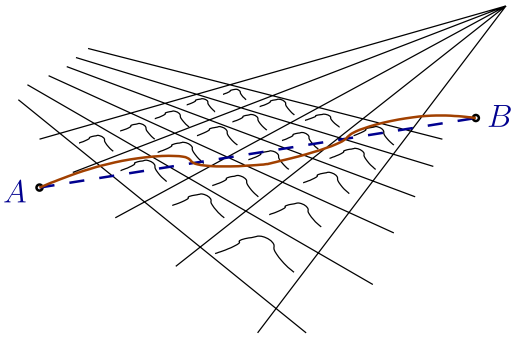


Fig. 11. Line like geodesic

The answer is *yes* in $2D$ (Morse) and *no* in $3D$ (Hevlund). An inspection of Hevlund counterexample shows that, unlike classical homogenization, where diffusion processes tend to *average the medium*, geodesics try to *beat the medium* by choosing specific paths, and leaving *bad areas* untouched.

In the 80's Moser suggested that in $\mathbb{R}^n$, unlike geodesics, minimal hypersurfaces should be forced to *average* the medium, and given any plane π, it should be possible to construct *plane like* minimal surfaces for the periodic medium.

More precisely given a *differential of area* form, we would like to consider surfaces S that locally minimize

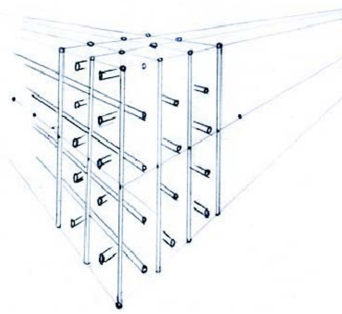


Fig. 12. Hevlund Counterexample: It costs *one* to travel inside narrow pipes, a large K outside. Then, the best strategy is to jump only once from pipe to pipe, i.e., the effective norm is $\|x\| = |x| + |y| + |z|$

$$A^*(S) = \int_S a(x, \nu)\, dA$$

where dA is the usual differential of area, ν the normal vector to A, and a, as before satisfies,

i. $0 < \lambda \leq a(x,\nu) \leq \Lambda$
ii. $|v|a(x,v/|v|)$ is a strictly convex cone.
iii. a is periodic in x.

These conditions for a, translate in the following properties of A^*.

i. $\lambda Area(S) \leq A^*(S) \leq \Lambda Area(S)$.
ii. $A^*(S) = A^*(\tau_z S)$, for any translation τ_z with integer coordinates.

By a local minimizer of A^*, we mean a surface S such that if another surface S_1 coincides with S everywhere but in a bounded set B, then $A^*(S \cap B) \leq A^*(S_1 \cap B)$.

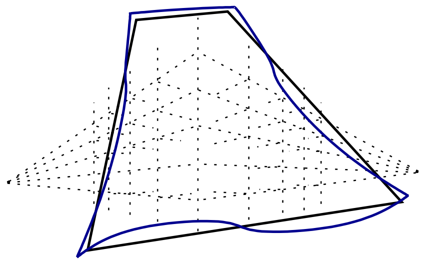


Fig. 13. Plane-like minimal surface in a periodic medium (for instance a medium with a periodic Riemman metric)

The main theorem is the following:

Theorem 3.1. *There exists a universal constant $M(\lambda, \Lambda, n)$ such that: for any unit vector ν_0 there exists an A^* local area minimizer S contained in the strip $\pi_M = \{x : |\langle x, \nu_0\rangle| < M\}$.*

A first attempt to construct such local area minimizer is to look at surfaces that are obtained by adding a periodic perturbation to the plane $\pi = \{x : \langle x, \nu_0 \rangle = 0\}$. This will be possible if π has a rational slope, or equivalently that π can be generated by a set of $n-1$ vectors $e_1, \dots, e_{n-1}$ with integer coordinates. The advantage of this case is that a translation in the direction of each e_j fixes π as well as the metric, so we can expect that we can find a local A^* minimizer that is also fixed by the same set of translations. If we can prove Theorem 3.1 in this context and the constant M does not depend on the vectors $e_1, \dots, e_{n-1}$ but only on λ, Λ and dimension, then the general case (irrational slope) follows by a limiting process.

We will work in the framework of boundaries of sets of locally finite perimeter.

A set of locally finite perimeter Ω is a set such that for any ball B, $B \cap \Omega$ has a finite perimeter (as in the first part of these notes, see [13]) For such sets, differential of area of $\partial\Omega$, and unit normal vectors are well defined, under our hypothesis A^* makes sense and is lower semicontinuous under convergence in measure for sets.

Main Steps of the Proof

We will consider the family of sets $\mathbb{D}$ such that $\Omega \in \mathbb{D}$ if Ω is a set of locally finite perimeter, $\tau_{e_j}(\Omega) = \Omega$ for every j (where $\tau_{e_j}(\Omega) := \Omega + e_j$), and

$$\pi_M^- = \{x : \langle x, \nu_0\rangle \geq -M\} \subset \Omega \subset \pi_M^+ = \{x : \langle x, \nu_0\rangle \leq M\}$$

And within $\mathbb{D}$, we will consider those sets Ω_0 that are local A^*-minimizers among sets $\Omega \in \mathbb{D}$. Since we are in the context of periodic perturbations of a plane, a local A^*-minimizer is simply a minimizer of A^* of the portion of $\partial\Omega$ inside the fundamental cube given by all the points of the form $\lambda_1 e_1 + \cdots + \lambda_{n-1}e_{n-1} + \lambda_n \nu_0$ where $\lambda_j \in [0,1]$ for $j = 1, \ldots, n-1$ and $\lambda_n \in [-M, M]$.

Of course, such an Ω_0 is not a *free* local minimizer since whenever $\partial\Omega_0$ touches the boundary of π^- or π^+ we are not free to perturb it outwards.

Our objective is to show that if M is large enough $S_0 = \partial\Omega_0$ does not see this restriction. In other words, Ω_0 would be a local A^*-minimizer not only among the sets in $\mathbb{D}$ but also among all sets of locally finite perimeter.

The main ingredients are:

a) A positive density property
b) An area estimate for $\partial\Omega$

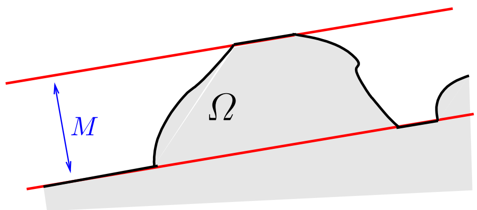


Fig. 14. Restricted Minimizer

c) Minimizers are *ordered*

Lemma 3.1 (Positive density). *There are two universal constants $c_0, C_1 > 0$ such that a minimizer $\partial\Omega_0$ of A^* satisfies*

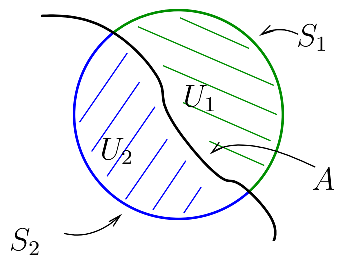
$$c_0 r^n \leq \frac{|\Omega_0 \cap B_r(x_0)|}{|B_r|} \leq C_1 r^n$$

for any $x_0 \in \partial\Omega_0$.

Proof. This lemma is actually the same as Lemma 2.1 in a slightly different context. The only difference is that instead of (3), we must use now that $\partial\Omega_0$ is a minimal surface. We include the proof here for completeness.

We define

$$U_1(r) = |B_r(x_0,z_0) \setminus \Omega_0| \qquad S_1(r) = Area(\partial B_r(x_0,z_0) \setminus \Omega_0)$$
$$U_2(r) = |B_r(x_0,z_0) \cap \Omega_0| \qquad S_2(r) = Area(\partial B_r(x_0,z_0) \cap \Omega_0)$$
$$A(r) = Area(B_r \cap \partial\Omega_0)$$



Since $\partial\Omega_0$ is a minimal surface,

$$A(r) \le \frac{1}{\lambda} A^*(B_r \cap \partial\Omega_0) \le \frac{1}{\lambda} A^*(\partial B_r(x_0,z_0) \setminus \Omega_0) \le \frac{\Lambda}{\lambda} S_1(r)$$

Similarly $A(r) \le \frac{\Lambda}{\lambda} S_2(r)$.

We also know by the isoperimetrical inequality that $U_1^{\frac{n}{n+1}} \le C(A + S_1)$. If we combine this with the above inequality we obtain

$$U_1^{\frac{n}{n+1}} \le C S_1$$

But now we observe that $S_1(r) = U_1'(r)$, so we obtain the ODE: $U_1'(r) \ge cU_1^{\frac{n}{n+1}}$. Moreover, we know $U_1(0) = 0$ and $U_1(r) > 0$ for any $r > 0$. This implies the result of the lemma.

In the same way, we obtain the result for U_2.

Lemma 3.2. *There are two universal constants $c_0, C_1 > 0$ such that a minimizer $\partial\Omega_0$ of A^* satisfies*

$$c_0 R^{n-1} \le \mathcal{H}^{n-1}(\partial\Omega_0 \cap B_R) \le C_1 R^{n-1}$$

for large values of R.

Proof. Notice that the set $\Omega_1 = \{x : \langle x, \nu\rangle < 0\}$ is an admissible set in $\mathbb{D}$. Then $A^*(\partial\Omega_0 \cap \text{fundamental cube}) \le A^*(\partial\Omega_1 \cap \text{fundamental cube})$. Besides, $Area(\partial\Omega_1 \cap \text{fundamental cube}) \le Area(\partial\Omega_0 \cap \text{fundamental cube})$. Thus, $Area(\partial\Omega_0 \cap B_R)$ and $Area(\partial\Omega_1 \cap B_R)$ are comparable when R is large.

This would be the same as the result of the lemma if it was true that the area of the boundary of a set of finite perimeter coincides with its $n-1$ Hausdorff measure. Unfortunately, that is not always true. In general we can say that the $n-1$ Hausdorff measure is only greater or equal to the area. But in this case we can compare them thanks to Lemma 3.1. If we take a finite overlapping covering with balls of radius r centered at $\partial\Omega_0 \cap B_R$, by Lemma 3.1 plus the isoperimetric inequality, the surface of $\partial\Omega_0$ inside each ball cannot be less than $c_0 r^{n-1}$. Then, there cannot be more than CR^{n-1}/r^{n-1} such balls, and the Hausdorff estimate follows.

Lemma 3.3. *Minimizers are* ordered*, that is if Ω_0 and Ω_1 are minimizers, then so are $\Omega_0 \cup \Omega_1$ and $\Omega_0 \cap \Omega_1$.*

Proof. $\partial\Omega_0 \cup \partial\Omega_1 = \partial(\Omega_0 \cup \Omega_1) \cup \partial(\Omega_0 \cap \Omega_1)$ and thus if we add the areas (A^*) inside the fundamental cube of $\partial\Omega_0$ and $\partial\Omega_1$, it is the same as adding the corresponding ones for $\partial(\Omega_0 \cap \Omega_1)$ and $\partial(\Omega_0 \cup \Omega_1)$. But since Ω_0 and Ω_1 are A^* area minimizers, necessarily all those areas are the same and then both $\Omega_0 \cup \Omega_1$ and $\Omega_0 \cap \Omega_1$ must be minimizers too.

Using Lemma 3.3, we can construct the smallest minimizer $\overline{\Omega}$ in $\mathbb{D}$ by taking the intersection of all minimizers in $\mathbb{D}$. We point out the similarity with Perron's method.

$\overline{\Omega}$ recuperates an important property, the Birkhoff property: If τ_z is an integer translation with $\langle z, \nu_0 \rangle \le 0$ (resp. ≥ 0) then

$$\tau_z(\overline{\Omega}) \subset \overline{\Omega} \quad (\text{resp. } \supset \overline{\Omega})$$

Indeed $\tau_z(\overline{\Omega}) \cap \overline{\Omega}$ and $\tau_z(\overline{\Omega}) \cup \overline{\Omega}$ are minimizers respectively for $\tau_z(\pi_M)$ and π_M, while $\overline{\Omega}$ and $\tau_z(\overline{\Omega})$ are the actual smallest minimizers.

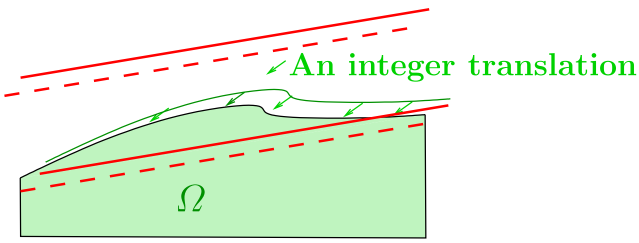


Fig. 15. Birkhoff Property. Integer translations send $\tau_z(\Omega)$ inside Ω or Ω inside $\tau_z(\Omega)$ depending on whether $\langle z, \nu_0 \rangle \le 0$ or $\langle z, \nu_0 \rangle \ge 0$

Lemma 3.2 tells us that for large balls $B_R(0)$, the number N of disjoint unit cubes intersecting $\partial\Omega_0$ must be of order $N \sim C_1 R^{n-1}$ independently of M. Since the strip $\pi_M \cap B_R$ has roughly MR^{n-1} cubes, many cubes in $\pi_M \cap B_R$ must be contained in Ω_0 or $\mathcal{C}\Omega_0$.

Combining the above properties we see the following:

i) There are many *clean cubes* that do not intersect $\partial\overline{\Omega}$, and thus they are contained in either $\overline{\Omega}$ or its complement. Moreover, there are many such cubes that are not too close to the boundary of π_M.
ii) Any integer translation $\tau_z(Q)$ of a cube $Q \subset \overline{\Omega}$ with $\langle z, \nu_0\rangle \leq 0$ is contained in $\overline{\Omega}$. Conversely for a cube $Q \subset \mathcal{C}\overline{\Omega}$, if $\langle z, \nu_0\rangle \geq 0$ then $\tau_z(Q) \subset \mathcal{C}\overline{\Omega}$.

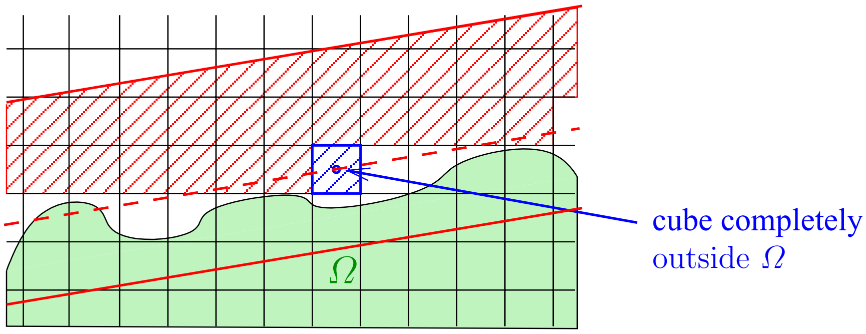


Fig. 16. If one cube is outside of $\overline{\Omega}$, then any cube whose center is above the dotted line is outside of $\overline{\Omega}$

From i), we can find a *clean* cube Q that is not too close to the boundary of π_M. If this cube Q is contained in $\overline{\Omega}$ and M is large, then the union of all the translations $\tau_z(Q)$ for z with integer coordinates and $\langle z, \nu_0\rangle \leq 0$ covers a strip around the bottom of π_M (see Figure 16 *upside down*). But then we have a thick *clean* strip, which means that we could translate $\overline{\Omega}$ a unit distance down and still have a local minimizer, which would contradict the fact that $\overline{\Omega}$ is the minimum of them.

Therefore, we must be able to find a *clean* cube contained in $\mathcal{C}\overline{\Omega}$. Arguing as above, this implies that there is a complete *clean* strip around the top of π_M (like in Figure 16). Thus, we are free to perturb upwards. Moreover, we can lift the whole set $\overline{\Omega}$ by an integer amount and obtain another minimizer that does not touch the boundary of π_M, and then $\overline{\Omega}$ is a *free* minimizer.

In this way we prove the theorem when π has a rational slope. Since M depends only on λ, Λ and dimension, we approximate a general π by planes with rational slopes and prove the theorem by taking the limit of the respective minimizers (or a subsequence of them).

3.1 References

The content of this part is based on the joint paper with Rafael de la Llave [2].

The problem had been proposed by Moser in another C.I.M.E. course [M1] (See also [M2], [18]). The interest of constructing line like geodesics was related to foliating the torus with them or at least laminate it.

4 Existence of Homogenization Limits for Fully Nonlinear Equations

Let us start the third part of these notes with a review on the definitions of fully nonlinear elliptic equations.

A second order fully nonlinear equation is given by an expression of the form

$$F(D^2u, Du, u, x) = 0 \tag{5}$$

for a general nonlinear function $F : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. For simplicity, we will consider equations that do not depend on Du or u. So they have the form

$$F(D^2u, x) = 0 \tag{6}$$

The equation (6) is said to be *elliptic* when $F(M+N, x) \geq F(M, x)$ every time N is a positive definite matrix. Moreover, (6) is said to be *uniformly elliptic* when we have $\lambda\,|N| \leq F(M+N, x) - F(M, x) \leq \Lambda\,|N|$ for two positive constants $0 < \lambda \leq \Lambda$ and where $|N|$ denotes the norm of the matrix N. The simplest example of a uniformly elliptic equation is the laplacian, for which $F(M, x) = \operatorname{tr} M$.

Existence, uniqueness and regularity theory for uniformly elliptic equations is a well developed subjet. It is studied in the framework of viscosity solutions that is a concept that was first introduced by Crandall and Lions for Hamilton Jacobi equations. We will consider only uniformly elliptic equations thoughout this section.

A continuous function u is said to be a *viscosity* subsolution of (6) in an open set Ω, and we write $F(D^2u, x) \geq 0$, when each time a second order polynomial P touches u from above at a point $x_0 \in \Omega$ (i.e. $P(x_0) = u(x_0)$ and $P(x) > u(x)$ for x in a neighborhood of x_0), then $F(D^2P(x_0), x_0) \geq 0$. Respectively, u is a supersolution ($F(D^2u, x) \leq 0$) if every time P touches u from below at x_0 then $F(D^2P(x_0), x_0) \leq 0$. For the general theory of viscosity solutions see [8] or [1].

In the same way as for subharmonic and superharmonic functions, sub- and supersolutions of uniformly elliptic equations satisfy the comparison principle: if u and v are respectively a sub- and supersolution of an equation like (6) and $u \leq v$ on the boundary of a bounded domain Ω, then also $u \leq v$ in the interior of Ω.

Suppose now that we have a family of uniformly elliptic equations (with the same λ and Λ) that do not depend on x (are translation invariant): $F_j(D^2u) = 0$ for $j = 1, \dots, k$. Let us suppose that at every point in space we choose one of these equations with some probability. To fix ideas, let us divide $\mathbb{R}^n$ into unit cubes with integer corners and in each cube we pick one of these equations at random with some given probability. The equation that we obtain for the whole space will change on each cube, it will not look homogeneous, it will not be translation invariant, and it will strongly depend on the random choice at every cube. However if we look at the equation from far away, somehow the differences from point to point should average out and we should obtain a translation invariant equation.

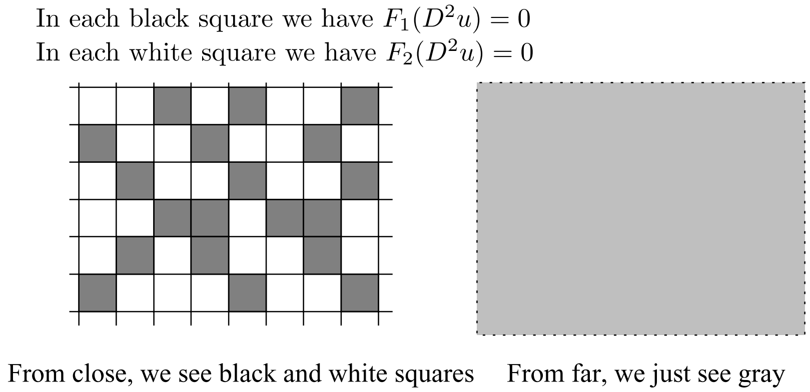


Fig. 17. A chessboard like configuration

Let $(\mathbb{S}, \mu)$ be the probability space of all the possible configuration. For each $\omega \in \mathbb{S}$ we have an x-dependent equation

$$F(D^2u, x, w) = 0$$

What we would expect is that if we consider solutions u_ω^ε of the equation (with same given boundary values)

$$F(D^2u_\omega^\varepsilon, \frac{x}{\varepsilon}, w) = 0 \tag{7}$$

with probability 1, they would converge to solutions u_0 of a translation invariant (constant coefficients) equation

$$F(D^2u_0) = 0$$

thus, in this limiting process that corresponds to looking at the medium from far away, the differences from point to point should dissapear. An moreover, it should lead to the same uniform equation for almost all ω.

Our purpose is to prove the existence of this limiting equation.

The appropriate setting for the idea of mixed media that from far away looks homogeneous is ergodic theory. Out assumptions are:

1. For each ω in the probability space $\mathbb{S}, \mu$ we have a uniformly elliptic equation
$$F(D^2u, x, \omega) = 0$$
defined in all $\mathbb{R}^n$.
2. Translating the equation in any direction z with integer coordinates is the same as shifting the configuration ω, i.e.
$$F(M, x - z, \omega) = F(M, x, \tau_z(\omega))$$
and we ask this transformation $\omega \mapsto \tau_z(\omega)$ to preserve probability.
3. *Ergodicity assumption*: For any set $S \subset \mathbb{S}$ of positive measure, the union of all the integer translations of S covers almost all $\mathbb{S}$
$$\mu\left(\bigcup_{z\in\mathbb{Z}^n} \tau_z(S)\right) = 1$$

Under these conditions, we obtain the following theorem:

Theorem 4.1. *There exists an homogenization limit equation*
$$\tilde{F}(D^2u_0) = 0$$
to which solutions of the problem (7) *converge almost surely.*

4.1 Main Ideas of the Proof

When we have a translation invariant equation $F(D^2u) = 0$, if u is a solution of such equation, that means that for each point x, the matrix $D^2u(x)$ lies on the zero level set $\{M \in \mathbb{R}^{n\times n} : F(M) = 0\}$. We can describe the equation completely if we are able to classify all quadratic polynomials P as solutions, subsolutions or supersolutions, because that would tell us for what matrices M, $F(M)$ is equal, greater or less than zero.

Let us choose a polynomial P_0 in a large cube Q_R and let us compare $P_0 + t\,|x|^2$ with the solution of
$$\begin{aligned} F(D^2u, x, \omega) &= 0 && \text{in } Q_R \\ u &= P_0 + t\,|x|^2 && \text{in } \partial Q_R \end{aligned}$$

If t is very large, $P_0 + t\,|x|^2$ will be a subsolution of the equation and thus $P_0 + t\,|x|^2 \le u$ in Q_R. Equally, if λ is very negative then $P_0 + t\,|x|^2 \ge u$ in Q_R. For some intermediate values of t, $P_0 + t\,|x|^2$ and u cross each other, so for these values it is not so clear at this point if $P_0 + t\,|x|^2$ is going to be a sub or supersolution of the homogenization limit equation.

Let us forget about the term $t\,|x|^2$ for a moment. Given a quadratic polynomial $P = \sum_{ij} M_{ij} x_i x_j$, we want to solve the equation

$$\begin{aligned} F(D^2 u^\varepsilon, \frac{x}{\varepsilon}, w) &= 0 \quad \text{in } Q_1 \\ u^\varepsilon &= P \quad \text{on } \partial Q_1 \end{aligned} \tag{8}$$

for a unit cube Q_1. Subsolutions of our homogenized equations are those polynomials for which u^ε tends to lie above P as $\varepsilon \to 0$. Similarly, supersolutions are those for which u^ε tends to be below P. If the polynomial P is borderline between these two behaviors, then it would be a solution of the homogenization limit equation.

It is important to notice that we can either think of the problem at scale ε in a unit cube (with u^ε) or we can keep unit scale and consider a large cube. To look at the equation (8) for $\varepsilon \to 0$ is equivalent to keep the same scale and consider larger cubes. Indeed, if we consider $u(x) = \frac{1}{\varepsilon^2} u^\varepsilon(\varepsilon x)$, then for $R = \varepsilon^{-1}$, we have

$$\begin{aligned} F(D^2 u, x, w) &= 0 \quad \text{in } Q_R \\ u &= P \quad \text{in } \partial Q_R \end{aligned} \tag{9}$$

For a cube Q_R of side R. It is convenient to choose R to be integer, in order to fit an integer number of whole unit cubes in Q_R. Now instead of taking $\varepsilon \to 0$, we can take $R \to +\infty$. We will be switching between these two points of view constantly.

Let v be the solution of the corresponding obstacle problem. The function v is the least supersolution of the equation (9) such that $v \geq P$:

$$\begin{aligned} F(D^2 v, x, w) &\leq 0 \quad \text{in } Q_R \\ v &= P \quad \text{in } \partial Q_R \\ v &\geq P \quad \text{in } Q_R \\ F(D^2 v, x, w) &= 0 \quad \text{in the set } \{v > P\} \end{aligned} \tag{10}$$

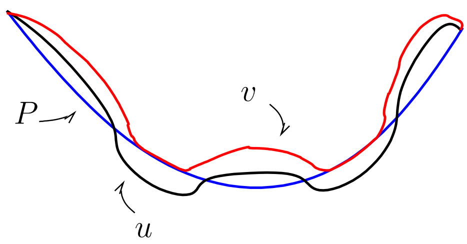


Fig. 18. The polynomial P, the free solution u and the least supersolution above the polynomial v

We also call $v^\varepsilon = \varepsilon^2 v(x/\varepsilon)$, the solution of the obstacle problem at scale ε. Let ρ be the measure of the contact set $\{v = P\}$ in Q_R:

$$\rho(Q_R) = |\{v = P\}|$$

The value of ρ controls the difference between u and v. A small value of ρ means that v touches P at very few points, and thus it is *almost* a free solution. The idea is that if ρ remains small compared to $|Q_R|$ as $R \to +\infty$, then P would be a subsolution of the homogenized equation. A large value of ρ means that v touches P in many points. If $\frac{\rho}{|Q_R|} \to 1$ as $R \to \infty$, that would mean that P is a supersolution. Moreover, we will show that every time $\frac{\rho}{|Q_R|}$ converges to a positive value, then $u^\varepsilon \to P$.

The first thing we must prove is that $\frac{\rho}{|Q_R|}$ indeed converges to some value as $R \to +\infty$ (or $\varepsilon \to 0$). Notice that $\frac{\rho}{|Q_R|}$ is the measure of the contact set at scale ε: $|\{v^\varepsilon = P\}|$.

In this problem, what plays the role of the Birkhoff property is a subadditivity condition for ρ, as the following lemma says.

Lemma 4.1. *If a cube Q is the disjoint union of a sequence of cubes Q_j, then*

$$\rho(Q) \le \sum_j \rho(Q_j)$$

Proof. Let v be the solution of the obstacle problem in the cube Q that coincides with P on ∂Q. Let v_j be the corresponding ones for the cubes Q_j. Since $v \ge P$ in Q, $v \ge v_j$ on ∂Q_j. Then by comparison principle $v \ge v_j$ in Q_j. Therefore the contact set $\{x \in Q : v(x) = P(x)\}$ is contained in the union of the contact sets $\{x \in Q_j : v_j(x) = P(x)\}$, and the lemma follows.

This subadditivity condition plus the ergodicity condition and

$$\rho(Q_R(x - z), \omega) = \rho(Q_R(x), \tau_z(\omega))$$

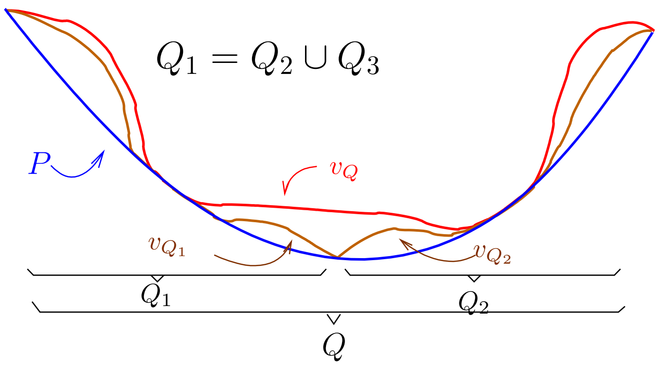


Fig. 19. Pay attention to the contact sets: ρ is subadditive

are the conditions for a subadditive ergodic theorem (which can be found in [9]) that says that as R go to infinity $\frac{\rho(Q_R(x_0)}{|Q_R(x_0)|}$ converges to a constant h_0 with probability 1. We will characterize polynomials P as sub- or supersolutions according to whether $h_0 = 0$ or $h_0 > 0$.

Lemma 4.2. *If* $h_0 = 0$*, then*

$$\liminf_{\varepsilon \to 0} u^\varepsilon \geq P$$

Proof. Using the Alexandrov-Backelman-Pucci inequality (See for example [1]), we can obtain a precise estimate of $v^\varepsilon - u^\varepsilon$ depending on ρ:

$$\sup_{Q_R} v^\varepsilon - u^\varepsilon \leq C R \rho^{1/n}$$

where C is a universal constant.

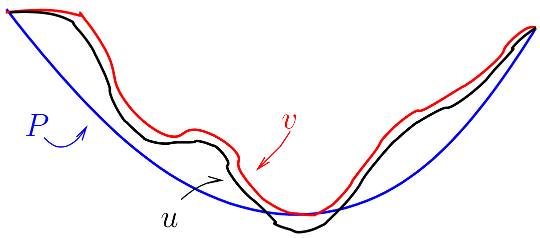


Fig. 20. If the contact set if small, then u and v are close

If $h_0 = 0$, as ε goes to zero we have:

$$\begin{aligned} u^\varepsilon(x) &\geq v^\varepsilon(x) - C\varepsilon\rho^{1/n} \\ &\geq v^\varepsilon(x) - C\left(\frac{\rho}{|Q_R|}\right)^{1/n} \\ &\geq P - o(1) \end{aligned}$$

Then, as $\varepsilon \to 0$, u^ε tends to be above P, and we finish the proof of the lemma.

The last lemma suggests that P is a subsolution of the homogenization limit equation if $h_0 = 0$. Now we will consider the case $h_0 > 0$. In order to show that in that case u_ε tends to be below P, we have to use that v^ε separates from P by a universal quadratic speed depending only on the ellipticity of the equation.

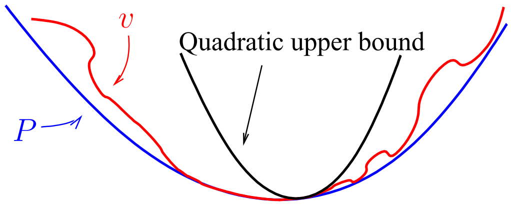


Fig. 21. Quadratic separation

The quadratic separation from the contact set is a general characteristic of the obstacle problem. What it means is that if $v^\varepsilon(x_0) = P(x_0)$, then

$$v^\varepsilon(x) - P(x) \le C\,|x - x_0|^2$$

for a constant C depending only on λ, Λ and dimension.

The quadratic separation in this problem plays the role of the positive density in the previous ones.

Lemma 4.3. *If* $h_0 > 0$*, then*

$$\limsup_{\varepsilon \to 0} u^\varepsilon \le P$$

Proof. We will show that the contact set $\{v^\varepsilon = P\}$ spreads all over the unit cube. Then, using the quadratic separation we show that $v^\varepsilon \to P$ as $\varepsilon \to 0$.

We want to show that if we split the unit cube in m smaller cubes of equal size, for any value of m, then for ε small enough there is a piece of the contact set in each small cube. We know that the measure of the contact set $|\{x \in Q_1 : v^\varepsilon(x) = P(x)\}|$ converges to $h_0 > 0$. The unit cube Q_1 is split into m smaller cubes. Let Q be any of these cubes, we have $v^\varepsilon \ge P$ on ∂Q, so v^ε is a supersolution of the corresponding obstacle problem in Q and $\frac{|\{x \in Q : v^\varepsilon = P(x)\}|}{|Q|}$ cannot converge to any value larger than h_0 as $\varepsilon \to 0$. If in some cube the contact set is empty $\{x \in Q : v^\varepsilon(x) = P(x)\} = \varnothing$, then, since the whole contact set covers a proportion h_0 of the measure of the unit cube, there must be one of the smaller cubes where the contact set covers more than h_0 times the measure of this cube (at least for a sequence $\varepsilon_k \to 0$). And that is a contradiction, which means that the contact set $\{v^\varepsilon = P\}$ must spread all over.

But if $\{v^\varepsilon = P\}$ spreads all over the unit cube, then v^ε converges to P uniformly due to the universal quadratic separation. Since $v^\varepsilon \ge u^\varepsilon$, $\limsup_{\varepsilon \to 0} u^\varepsilon \le P$.

So, now we have a way to classify every polynomial as subsolution to the homogenization limit equation ($\tilde{F}(D^2 P) \ge 0$) if $h_0 = 0$ or supersolution ($\tilde{F}(D^2 P) \le 0$) if $h_0 > 0$. There is still a little bit of ambiguity because a polynomial could be both things at a time (if it is precisely a solution). That is easily solved by considering $P_0 + t\,|x|^2$ for small values of t. We say that P_0

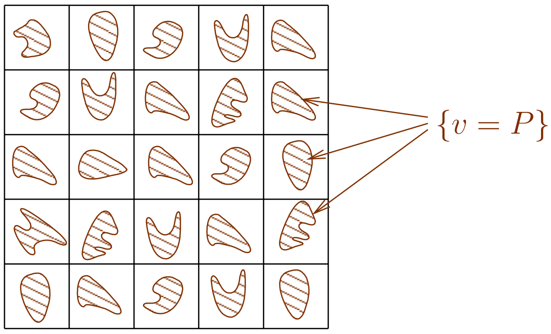


Fig. 22. Each small cube must contain about the same amount of contact set when $\varepsilon << 1$

is a sub or supersolution if we can check it for $P_0 + t\,|x|^2$ for arbitrarily small values of t.

In this way we are able to completely characterize the zero level set of $\tilde{F}$.

Moreover, if we want to construct the complete function $\tilde{F}$, then we have to identify all its level sets, not only the zero level set. To do that we just consider the problem:

$$F(D^2u^\varepsilon, \frac{x}{\varepsilon}, w) - t = 0$$

to describe the level set $\tilde{F}(M) = t$. And we recover $\tilde{F}$ completely.

Now, based on our construction of $\tilde{F}$, it is easy to show that for any boundary data, problem (7) will converge with probability 1 to a function u_0 that satisfies comparison with polynomials in the right way to be a viscosity solution of $\tilde{F}(D^2u_0) = 0$. We finish with the theorem:

Theorem 4.2. *Let u^ε be the solutions to*

$$\begin{aligned} F(D^2u^\varepsilon, \frac{x}{\varepsilon}, w) &= 0 \quad \textit{in } \Omega \\ u^\varepsilon &= g \quad \textit{in } \partial\Omega \end{aligned} \tag{11}$$

for a domain Ω and a continuous function g on $\partial\Omega$. Then as $\varepsilon \to 0$, almost surely u^ε converge uniformly to a function u that solves

$$\begin{aligned} \tilde{F}(D^2u) &= 0 \quad \textit{in } \Omega \\ u &= g \quad \textit{in } \partial\Omega \end{aligned} \tag{12}$$

Proof. Due to the uniform ellipticity of F, the functions u^ε are uniformly continuous, and therefore by Arzela-Ascoli there is a subsequence u^{ε_k} that converges uniformly to a continuous function u.

Let us suppose that a quadratic polynomial P touches u from above at a point x_0. Then we can lower P a little bit by subtracting a small constant δ_1

such that $P(x_0) < u(x_0)$ and $P(x) > u(x)$ for x in the boundary of a small cube $Q_{\delta_2}(x_0)$ centered at x_0.

Since u^{ε_k} converge to u uniformly, the same property holds for them. Namely, for large enough k

$$\begin{aligned} P(x_0) &\leq u^{\varepsilon_k}(x_0) - \delta_1 \\ P(x) &> u^{\varepsilon_k}(x) \quad \text{for } x \in \partial Q_{\delta_2}(x_0) \end{aligned}$$

Let w_k be the solutions to

$$\begin{aligned} F(D^2 w_k, \frac{x}{\varepsilon}, w) &= 0 \quad \text{in } Q_{\delta_2}(x_0) \\ w_k &= P \quad \text{in } \partial Q_{\delta_2}(x_0) \end{aligned} \tag{13}$$

By comparison principle, $w_k \leq u^{\varepsilon_k}$, then $w_k(x_0) \leq P(x_0) - \delta_1$ for large k. So, we can apply Lemma 4.3 to obtain that the value of h_0 corresponding to P cannot be positive. Then $\tilde{F}(D^2 P) \geq 0$.

In a similar way, we can show that if a quadratic polynomial touches u from below then it must be a supersolution of $\tilde{F}$.

Therefore u must be a viscosity solution of (12). Since (12) has a unique solution, all the convergent subsequences of u^ε must converge to the same limit. Thus the whole sequence u^ε converges uniformly to u.

4.2 References

This part in homogenization is based on the joint work with Panagiotis Souganidis and Lihe Wang [7], where actually a more complete theorem is proved. The fact that equations that depend on ∇u are considered in that paper adds some extra complications.

Some of the ideas have their roots in the work of Dal Maso and Modica ([9] and [10]) for the variational case.

Periodic homogenization for second order elliptic equations was considered in [11].

Acknowledgements

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